June 22, 1991

We have seen that a lifting hom.

$$A \longrightarrow RA/IA^2 \qquad a \longmapsto a + \varphi a$$

$$0 = (a_1 + \varphi a_1)\circ(a_2 + \varphi a_2) - (a_1 a_2 + \varphi(a_1 a_2))$$
$$= (a_1 \circ a_2 - a_1 a_2) + (\delta\varphi)(a_1, a_2)$$

$$\boxed{\therefore\ \delta\varphi(a_1,a_2) = da_1\, da_2}$$

gives rise in a natural way to a lifting homomorphism $A \longrightarrow \hat{R}A$, and more generally, to a SDR equivalence $\hat{R}A \overset{\longleftarrow}{\longrightarrow} A$. The idea is to consider the derivation D on RA defined by $Da = \varphi a$. Then $D(RA) \subset IA$, $D(IA^n) \subset IA^n$, so D induces a derivation on the associated graded algebra $\operatorname{gr} RA = \bigoplus IA^n/IA^{n+1}$ such that $D = 0$ on gr^0. Also

$$D(da_1 da_2) = D(a_1 a_2 - a_1 \circ a_2) = \varphi(a_1 a_2) - \varphi a_1 \circ a_2 - a_1 \circ \varphi a_2$$
$$\equiv -(\delta\varphi)(a_1, a_2) = -da_1 da_2 \qquad \text{mod } IA^2.$$

So $D = -n$ on gr^n, and the various eigenspaces of D on $\varprojlim RA/IA^n$ give [redacted] an isomorphism of $(\operatorname{gr} RA)^\wedge$ with $\hat{R}A$.

Now we would like to discuss the following relative situation. Suppose A is an algebra under S, and consider the obvious surjection $RA \longrightarrow R_S A$. We would like a SDR equivalence

$$\hat{R}A \longrightarrow \hat{R}_S A$$

Assume we have a lifting homomorphism

$$R_S A / I_S A^2 \longrightarrow RA/IA^2$$

Then we have an A-bimodule lifting $\Omega_S^2 A \longrightarrow \Omega^2 A$. Also the cocycle $d_S a_1\, d_S a_2$ becomes cohomologous to to $da_1 da_2$ in $\Omega^2 A$. More precisely, if the lifting homomorphism (+ obvious $\Omega_S^2 A \hookrightarrow \Omega^2 A$) is $a \longmapsto a + \varphi a$, then

$$0 = (a_1 + \varphi a_1) \circ (a_2 + \varphi a_2) - \left(a_1 a_2 + \varphi(a_1 a_2) - d_S a_1 d_S a_2\right)$$

$$= -da_1 da_2 + (\delta\varphi)(a_1, a_2) + d_S a_1 d_S a_2$$

$$\Longrightarrow \quad (\delta\varphi)(a_1, a_2) = + \big(\underbrace{da_1 da_2 - d_S a_1 d_S a_2}_{\substack{\text{projection of } da_1 da_2 \text{ onto} \\ \text{the Kernel of } \Omega^2 A \twoheadrightarrow \Omega_S^2 A.}}\big)$$

Then if we define D on RA by $Da = \varphi a$ we have

$$D(da_1 da_2) = -(\delta\varphi)(a_1, a_2) = -(da_1 da_2 - d_S a_1 d_S a_2)$$

$$= \begin{cases} 0 & \text{on } \Omega_S^2 A \subset \Omega^2 A \\ -1 & \text{on the complement.} \end{cases}$$

and so we have the desired SDR equivalence.

Thus we want to analyze when such a φ exists. There are two cases we want to handle: First of all, when $A \otimes_S A$ is a projective bimodule. Secondly, when $S = A$ and A is quasi-free. The natural hypotheses suggested by these two cases are to assume the bimodule sequence

$$(*) \qquad 0 \longrightarrow A dS A \longrightarrow \Omega^1 A \longrightarrow \Omega_S^1 A \longrightarrow 0$$

splits and that $AdSA$ is a projective bimodule. Let's show these hypotheses work. First of all the splitting of $(*)$ means that we have a derivation $d' : A \longrightarrow AdSA$ such that if we put $d'' = d - d'$, then $d''(S) = 0$. (I recall that $A \otimes_S A$ projective is equivalent to the existence of such a splitting $d = d' + d''$ and further such that d' is an inner derivation.)

So we have

$$\Omega^1 A = \Omega^1_s A \oplus A\,dSA$$
$$d = d' + d''$$

Thus the basic 2-cocycle $(d \cup d)(a_1, a_2) = da_1\,da_2$ splits into 4 parts

$$d \cup d = d' \cup d' + \underbrace{(d' \cup d'' + d'' \cup d' + d'' \cup d'')}$$

and we want to write ⌣ as a coboundary. This will follow from the hypothesis that $A\,dSA$ is projective, once we recall facts about cup products.

In general an element of $H^p(A, X)$ is a map $A \longrightarrow \Sigma^p X$ in the derived category of bimodules. The cup product

$$H^p(A, X) \otimes H^q(A, Y) \longrightarrow H^{p+q}(A, X \otimes_A Y)$$

can be [redacted] realized by three maps.

$$\begin{tikzcd}
A = A \otimes_A A \arrow[r, "1 \otimes v"] \arrow[d, "u \otimes 1"'] \arrow[dr, "u \otimes v"] \arrow[dr, phantom, "(-1)^{pq}"', near start] & A \otimes_A \Sigma^q Y \arrow[d, "u \otimes 1"] \\
\Sigma^p X \otimes_A A \arrow[r, "1 \otimes v"'] & \Sigma^p X \otimes_A \Sigma^q Y
\end{tikzcd}$$

If X is projective, $q > 0$, (and X, Y are bimodules concentrated in degree zero), then $1 \otimes v$ has to be the zero map.

(June 23)

What this means is that because $A\,dSA$ is projective, the 2-cocycles $d' \cup d''$, $d'' \cup d'$, $d'' \cup d''$ will be coboundaries. Here's how to do this explicitly.

First we need a splitting of

$$0 \to AdSA \otimes_A \Omega^1 A \longrightarrow AdSA \otimes A \longrightarrow AdSA \to 0$$

which is the same as a connection ∇ in the bimodule $AdSA$. Notice that because $AdSA$ is a direct summand of $\Omega^1 A$ which is a free left A-module, the existence of a bimodule splitting of the above exact sequence is equivalent to $AdSA$ being a projective bimodule.

Given ∇ consider $\varphi = \nabla d'' : \bar{A} \longrightarrow AdSA \otimes_A \Omega^1 A$

We have

$$(\delta\varphi)(a_1, a_2) = (\nabla d'' a_1) a_2 - \underbrace{\nabla(d'' a_1 a_2 + a_1 d'' a_2)}_{(\nabla d'' a_1) a_2 + d'' a_1 da_2 + a_1 \nabla d'' a_2} + a_1 (\nabla d'' a_2)$$

$$= -d'' a_1 \, da_2$$

This expresses $d'' \circ d$ as a 1-coboundary. Similarly taking the components of ∇ with respect to the decomposition $AdSA \otimes_A \Omega^1 A = AdSA \otimes_A \Omega^1_S A \oplus AdSA \otimes_A AdSA$ we express $d'' \circ d'$ and $d'' \circ d''$ as coboundaries.

On the other hand a bimodule splitting of

$$0 \longrightarrow \Omega^1 A \otimes_A AdSA \longrightarrow A \otimes AdSA \longrightarrow AdSA \to 0$$

is equivalent to an operator $\nabla^l : AdSA \to \Omega^1 A \otimes_A AdSA$ satisfying

$$\nabla^l(a\xi) = a \nabla^l \xi + da\, \xi$$
$$\nabla^l(\xi a) = (\nabla^l \xi) a$$

We have if $\psi(a) = \nabla^l d'' : \bar{A} \to \Omega^1 A \otimes_A AdSA$

$$(\delta\psi)(a_1, a_2) = (\nabla^l d'' a_1) a_2 - \underbrace{\nabla^l(d'' a_1 a_2 + a_1 d'' a_2)}_{(\nabla^l d'' a_1) a_2 + a_1 (\nabla^l d'' a_2) + da_1 d'' a_2} + a_1 (\nabla d'' a_2)$$

$$= -da_1 \, d'' a_2$$

Thus $d \circ d''$, and also $d' \circ d''$ and $d'' \circ d''$ are coboundaries.

Thus we conclude that assuming

1) The exact sequence of A-bimodules

$$0 \longrightarrow A\,dS\,A \longrightarrow \Omega^1 A \longrightarrow \Omega^1_S A \longrightarrow 0$$

splits. (equiv. $\exists\ d'' : A \to A\,dS\,A$ derivation such that $(d - d'')(S) = 0$).

2) $A\,dS\,A$ is a projective A-bimodule.

there exists φ such that

$$\delta\varphi = d \circ d - d' \circ d',$$

and hence there exists a lifting homomorphism

$$\underset{\text{under } \circ}{A \oplus \Omega^2_S A} \longrightarrow \underset{\text{under } \circ}{A \oplus \Omega^2 A}$$

and then a derivation D on $\hat{R}A$ ~~with~~ eigenvalues $-n$, $n \in \mathbb{N}$, which gives a SDR equivalence

$$\hat{R}A \rightleftarrows \hat{R}_S A .$$

<u>Remark</u>: $\{R_S A / I_S^n A\}$ is a retract of $\{RA/IA^n\}$ so it is a quasi-free adic, inverse system of algebras, which is not necessarily the completion of ~~as~~ quasi-free algebra. ~~So~~ the generalization to such adic algebras is not vacuous.

June 23, 1991

Let us now consider the case where $S = \mathbb{C}[F]$, $F^2 = 1$. Here

$$A\,dS\,A = A\,dF\,A = A \otimes_S \Omega^1 S \otimes_S A$$

so a connection in $A\,dS\,A$ should be obtained from a connection in $\Omega^1 S$.

Recall we have a connection in the bimodule S over itself with $\nabla 1 = \frac{1}{2} F dF$. In fact this connection is unique. To see this directly, notice first that as $\Omega^1 S = S\,dF\,S$, we have $F\xi = -\xi F$ for all $\xi \in \Omega^1 S$. Then

$$\left.\begin{aligned} \nabla F &= \nabla(1 \cdot F) = (\nabla 1)F + dF \\ \nabla F &= \nabla(F \cdot 1) = F(\nabla 1) \end{aligned}\right\} \Longrightarrow \underbrace{[F, \nabla 1]}_{2F\nabla 1} = dF$$

so $\nabla 1 = \frac{1}{2} F dF$.

This connection ∇ on S induces one ∇ on $\Omega^1 S$ by $\nabla(dF) = \nabla(dF \cdot 1) = dF\,\nabla(1) = dF \frac{1}{2} F dF = -\frac{1}{2} F dF^2$.

(Notice this is the obvious extension of ∇ to $\Omega^1 S \to \Omega^2 S$).

Check

$$\begin{tikzcd} 0 \arrow[r] & \Omega^2 S \arrow[r, "j"] & \Omega^1 S \otimes S \arrow[r] & \Omega^1 S \arrow[l, bend right, dashed] \arrow[r] & 0 \end{tikzcd}$$

$$\tfrac{1}{2}(dF \otimes 1 - F dF \otimes F) \longleftarrow\!\shortmid\; dF$$

Now
$$\begin{aligned} j(\nabla dF) &= dF \otimes 1 - \tfrac{1}{2}(dF \otimes 1 - F dF \otimes F) \\ &= \tfrac{1}{2}(dF \otimes 1 + F dF \otimes F) \\ &= -\tfrac{1}{2} F dF (F \otimes 1 - 1 \otimes F) \end{aligned}$$

$$\therefore \nabla(dF) = -\tfrac{1}{2} F dF^2.$$

Let us put $\boxed{\begin{aligned} Y &= \tfrac{1}{2} F dF \\ \nabla(Y) &= -\tfrac{1}{2} dY \end{aligned}}$ so that

We have

$$d = \underbrace{d + \mathrm{ad}(Y)}_{d'} + \underbrace{(-\mathrm{ad}\, Y)}_{d''}$$

$$(d + \mathrm{ad}(Y))F = dF + [\tfrac{1}{2}FdF, F] = 0.$$

and we have seen there is a connection ∇ in AdS A given by

$$\nabla(a_0 Y a_1) = a_0(-\tfrac{1}{2}dY)a_1 + a_0 Y da_1$$

Put $\varphi(a) = \nabla(d''a) = \nabla[a, Y] = \nabla(aY - Ya)$

$$= a(-\tfrac{1}{2}dY) - (-\tfrac{1}{2}dY)a - Yda$$

$$\varphi(a) = -Yda + \underbrace{\tfrac{1}{2}[dY, a]}_{\text{killed by } \delta}$$

Then $(\delta\varphi)(a_1, a_2) = -(Yda_1)a_2 + Y(da_1 a_2 + a_1 da_2) - a_1 Y da_2$

$$= [Y, a_1]da_2$$

$$= -d''a_1 da_2$$

Notice that since $d''a = [a, Y]$, the actual choice of ∇Y doesn't seem to matter much, since

$$\varphi(a) = \nabla d''a = \nabla(aY - Ya) = -Yda + \underbrace{[a, \nabla Y]}_{\delta\text{-cocycle}}.$$

Similarly for the left connection we found

$$\psi(a) = \nabla^l d''a = \nabla^l(aY - Ya) = \underline{daY} + \underbrace{[a, \nabla^l Y]}_{\delta\text{-cocycle}}$$

(In fact we computed $\nabla^l Y = -\tfrac{1}{2}dY$). And

$$(\delta\psi)(a_1, a_2) = da_1 Y a_2 - (da_1 a_2 + a_1 da_2)Y + a_1 da_2 Y$$

$$= da_1[Y, a_2] = -da_1 d''a_2$$

Let's summarize what seems to be the important output. We have (normalized) 1-cochains

$a \mapsto -Y\,da$ w. coboundary $-d''\cup d$

$a \mapsto da\,Y$ ———— $-d\cup d''$

We want a cochain with coboundary $-(d''\cup d' + d'\cup d'' + d''\cup d'')$. Consider the components of the above cochains

Recall $da = \underbrace{da-[a,Y]}_{d'a} + \underbrace{[a,Y]}_{d''a}$

$$-Y\,da = -Y\,d'a - Y[a,Y]$$

$$da\,Y = d'a\,Y + [a,Y]\,Y$$

Observe that $[a,Y]Y - (-Y[a,Y]) = [a,Y^2]$ is a coboundary, hence the right terms are cohomologous. So let's average them to get the cochain

$$\tfrac{1}{2}\big([a,Y]Y - Y[a,Y]\big) = \tfrac{1}{2}[[a,Y],Y]$$

whose coboundary should be $-d''\cup d''$.

~~Thus the coboundary of~~

~~$\Phi(a) = [da,Y] - \frac{1}{2}[Y,[a,Y]]$~~

~~should be $-(d''\cup d' + d'\cup d'' + d''\cup d'')$~~

Thus the coboundary of

$$\Phi(a) = d'a\,Y - Y\,d'a + \tfrac{1}{2}[[a,Y],Y]$$
$$= [da,Y] - \tfrac{1}{2}[[a,Y],Y]$$

should be $-(d''\cup d' + d'\cup d'' + d''\cup d'')$. Check

$$-(\delta\Phi)(a_1,a_2) = da_1[a_2,Y] + [a_1,Y]\,da_2$$
$$-\tfrac{1}{2}[a_1,Y][a_2,Y] - \tfrac{1}{2}[a_1,Y][a_2,Y]$$
$$= (da_1 - [a_1,Y])[a_2,Y] + [a_1,Y](da_2 - [a_2,Y]) + [a_1,Y][a_2,Y]$$

Summarize (with [redacted] the goal of straightening out the signs). We define the derivation D on RA by $Da = \varphi a$, where $\varphi : \bar{A} \to \Omega^2 A$.

Then $$D(da_1 da_2) = -(\delta\varphi)(a_1, a_2) \quad \text{mod } IA^2$$

In the quasi-free case $\varphi a = \nabla da$ and
$$-(\delta\varphi)(a_1, a_2) = \nabla(da_1 a_2 + a_1 da_2) - (\nabla da_1) a_2 - a_1 \nabla da_2$$
$$= da_1 da_2$$

Thus D has the eigenvalues $n = 0, 1, 2, \ldots$

~~[illegible]~~ The lifting homom. is $a - \varphi(a)$:

$$(a_1 - \varphi a_1) \circ (a_2 - \varphi a_2) = a_1 a_2 - da_1 da_2 - \varphi a_1 a_2 - a_1 \varphi a_2$$
$$\overset{?}{=} a_1 a_2 - \varphi(a_1 a_2)$$
$$\iff da_1 da_2 + \varphi a_1 a_2 + a_1 \varphi a_2 = \varphi(a_1 a_2)$$
$$\iff -(\delta\varphi)(a_1, a_2) = da_1 da_2$$

Check that $D(a - \varphi(a)) = \varphi(a) - \varphi(a) = 0 \quad \text{mod } IA^2$.

<u>Example</u>: $A = \mathbb{C}[F]$.
$$(F + cFdF^2) \circ (F + cFdF^2) = 1 - dF^2 + 2c\, dF^2 = 1$$
$\iff c = \frac{1}{2}$. Thus $\tilde{F} = F + \frac{1}{2} FdF^2$ is the idempotent lifting F and $\varphi(F) = -\frac{1}{2} FdF^2$ which checks with $\nabla(dF) = -\frac{1}{2} FdF^2$

Program: Take $S = \mathbb{C}[F]$. You want to construct a lifting homomorphism $R_S A / I_S A^2 \to RA/IA^2$. The original idea for obtaining a lifting homom. $\hat{R}_S A \to \hat{R}A$ was to lift F to $\tilde{F}$, then take the centralizer of $\tilde{F}$ which maps onto $\hat{R}_S A$ and then lifts $R_S A$ using the fact ~~that~~ A/S is a projective

S-bimodule. I have to compare this method with the one obtained from

$$\varphi(a) = [da, Y] + \tfrac{1}{2}[[a,Y],Y]$$
$$= [d'a, Y] + \tfrac{1}{2}[[a,Y],Y]$$

which we know satisfies

$$-(\delta\varphi)(a_1,a_2) = d'a_1[a_2,Y] + [a_1,Y]d'a_2 + [a_1,Y][a_2,Y]$$

June 24, 1991

Consider the case where $S = \mathbb{C}[e] \subset A$ and e is a central idempotent. We have

$$d(a) = d(eae + e^{\perp} a e^{\perp})$$
$$= edae + e^{\perp} da e^{\perp}$$
$$+ deae + eade - deae^{\perp} - e^{\perp} a de$$
$$= (edae + e^{\perp} da e^{\perp}) + [a, \underbrace{ede - e^{\perp} de}_{(2e-1)de = \frac{1}{2} F dF.}]$$

We have a direct sum decomposition

$$\Omega^1 A = e\Omega^1 A e \oplus e^{\perp}\Omega^1 A e^{\perp} \oplus e\Omega^1 A e^{\perp} \oplus e^{\perp}\Omega^1 A e$$
$$\cong \Omega^1(eA) \oplus \Omega^1(e^{\perp}A) \oplus \underbrace{(eA \otimes e^{\perp}A)}_{A(ede\,e^{\perp})A} \oplus \underbrace{(e^{\perp}A \otimes eA)}_{A(e^{\perp}de\,e)A}$$

Let us consider the case of $A = \tilde{\mathcal{A}} = \mathbb{C} \oplus \mathcal{A}$ where $\mathcal{A}$ is a unital algebra. Here e is the identity of $\mathcal{A}$ and $e^{\perp}A = \mathbb{C}e^{\perp}$. Then we have $\Omega^1(e^{\perp}A) = 0$, so

$$(*) \qquad \Omega^1 A \cong \Omega^1 \mathcal{A} \oplus \underbrace{\mathcal{A}}_{\mathcal{A}ede} \oplus \underbrace{\mathcal{A}}_{dee\mathcal{A}}$$

As a check recall

$$\Omega^1 A \simeq A \otimes \bar{A} = \mathcal{A}^{\otimes 2} \oplus \mathcal{A}$$
$$\Omega^1 \mathcal{A} \simeq \mathcal{A} \otimes \bar{\mathcal{A}}$$

and in passing from $\Omega^1 A$ to $\Omega^1 \mathcal{A}$ one kills two copies of $\mathcal{A}$.

The direct sum decomposition $(*)$ allows us to describe A, $\Omega^1 A$, (and maybe $\Omega^n A$) via matrices

$$A = \begin{pmatrix} \mathcal{a} & 0 \\ 0 & \mathbb{C} \end{pmatrix} \qquad \Omega^1 A = \begin{pmatrix} \Omega^1 \mathcal{a} & \mathcal{a} e d e \\ d e e \mathcal{a} & 0 \end{pmatrix}$$

$$\Omega^2 A = \begin{pmatrix} \Omega^1 \mathcal{a} & \mathcal{a} e d e \\ d e e \mathcal{a} & 0 \end{pmatrix} \otimes_{\begin{pmatrix} \mathcal{a} & 0 \\ 0 & \mathbb{C} \end{pmatrix}} \begin{pmatrix} \Omega^1 \mathcal{a} & \mathcal{a} e d e \\ d e e \mathcal{a} & 0 \end{pmatrix}$$

It would be nice to use this description to understand the canonical lifting homomorphism $\hat{R}_S A \longrightarrow \hat{R} A$ which we seem to have. First we ought to work what happens to first order, i.e. we want a lifting homomorphism

$$(\mathbb{C} + \mathcal{a}) \oplus \Omega^2 \mathcal{a} \longrightarrow A \oplus \Omega^2 A$$

with respect to the Fedosov product.

June 25, 1991

Let us return to

$$\varphi(a) = [da, Y] - \tfrac{1}{2}[[a,Y],Y]$$
$$= [d'a, Y] + \tfrac{1}{2}[[a,Y],Y]$$

Instead of φ consider

$$\psi(a) = [da, Y] \quad = [d'a, Y] + [[a,Y],Y]$$

and define D' to be the derivation of RA such that $D'a = \psi(a)$. Then

$$D'(da_1 da_2) = -(\delta\psi)(a_1, a_2)$$
$$= da_1 [a_2, Y] + [a_1, Y] da_2$$
$$= (d' \cup d'' + d'' \cup d' + 2 d'' \cup d'')(a_1, a_2)$$

so $$D' = \begin{cases} 0 & \text{on } \Omega_S^2 A \\ 1 & \text{on } \Omega_S^1 A \cdot AdSA + AdSA \cdot \Omega_S^1 A \\ 2 & \text{on } AdSA \cdot AdSA \end{cases}$$

and $$D'(a - \psi(a)) = \psi(a) - D'(\psi(a))$$
$$= [da, Y] - ([d'a, Y] + 2[[a,Y],Y])$$
$$= -[[a,Y],Y] = -\tfrac{1}{2} D'([[a,Y],Y])$$

$$\therefore \quad D'(a - \psi(a) + \tfrac{1}{2}[[a,Y],Y]) = 0$$

$$D'(a - \underbrace{([d'a, Y] + \tfrac{1}{2}[[a,Y],Y])}_{\varphi(a)}) = 0$$

Thus we get the same lifting to first order from D and D'. However it might be the case that the lifting homomorphisms associated to D and D' are different, which would be unfortunate.

Consider $A = R/I$ and assume both R, A quasi-free. Then we can construct an SDR equivalence

$$\hat{R} = \varprojlim R/I^n \rightleftarrows A$$

as follows. We [redacted] have the exact sequence

$$(*) \qquad 0 \longrightarrow I/I^2 \longrightarrow A \otimes_R \Omega^1 R \otimes_R A \longrightarrow \Omega^1 A \longrightarrow 0$$

and since $\Omega^1 R$, $\Omega^1 A$ are projective as bimodules, it follows that I/I^2 is a projective A-bimodule. We can choose a lifting hom. $A \longrightarrow \hat{R}$ [redacted], and then we have a surjection $\hat{I} \longrightarrow I/I^2$ of A-bimodules. As I/I^2 is projective, we can choose a bimodule lifting $I/I^2 \longrightarrow \hat{I}$. Then we obtain an algebra hom.

$$T_A(I/I^2) \longrightarrow \hat{R}$$

carrying $T_A^{>0}(I/I^2)$ to $\hat{I}$ such that the map on assoc. graded algebras is an isom. in degrees 0, 1, and hence surjective. Thus we have a surjection $\hat{T}_A(I/I^2) \longrightarrow \hat{R}$, and since R is quasi-free there is a lifting homomorphism, which has to be an isom. on gr^0, gr^1 hence this lifting hom is surjective and we conclude $\hat{T}_A(I/I^2) \xrightarrow{\sim} \hat{R}$.

But here's an easy way to do [redacted] this construction. Recall that $(*)$ corresponds to the square zero extension R/I^2 by pull-back:

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] & A \otimes_R \Omega^1 R \otimes_R A \arrow[r] & \Omega^1 A \arrow[r] & 0 \\
0 \arrow[r] & I/I^2 \arrow[r] \arrow[u, equal] & R/I^2 \arrow[r] \arrow[u, "d"] & A \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

So a bimodule splitting of ⊛ corresponds to a lifting homom. $R/I \to R/I^2$. Pick such a lifting homomorphism and consider the derivation D on R/I^2 which is 1 on I/I^2 and 0 on the lift of A. D gives a derivation of R with values in I/I^2 which corresponds to an R-bimodule map $\Omega^1 R \to I/I^2$. Since $\Omega^1 R$ is projective, we can ~~left~~ lift this to an R-bimodule map $\Omega^1 R \to I$. Thus we can extension D to a derivation on R carrying R into I. Then $D = 1$ on $I/I^2 \Longrightarrow D = n$ on I^n/I^{n+1} and so $\hat{R} \simeq \prod_n gr_I^n R$.

This construction has a very nice geometric interpretation: Given a submanifold $Z \subset M$, one chooses a splitting of

$$0 \longrightarrow TZ \longrightarrow TM|_Z \longrightarrow N \longrightarrow 0$$

and then one chooses a vector field on M vanishing appropriately on Z. (Recall that if a section of a vector bundle vanishes at a point, there is a canonical map from the tangent space at that point to the fibre. Thus a vector field vanishing along Z gives a map from the normal bundle to Z into $TM|_Z$; we [redacted] require this map to be the splitting.)

This raises the question as to what extent the map $\hat{Q}A \simeq$ [redacted] $\hat{\Omega}A$, which we have associated to a connection on $\Omega^1 A$, is the "exponential map.

Affine spaces. Let V be a vector space equipped with a distinguished non-zero element 1_V. Let

$$\bar{S}V = SV/(1-1_V)$$

Its variety is the affine space $\{f \in V^* \mid f(1_V)=1\}$. Given $(V, 1_V)$, $(W, 1_W)$ one can form the product $(V \oplus W,\ 1_V + 1_W)$. The corresponding affine space is the "join" of the affine spaces corresponding to V and W. For example take two non parallel non intersecting lines in $\mathbb{R}^3$; the join is the whole space $\mathbb{R}^3$. Consider a push-out

$$\begin{tikzcd} \mathbb{C}1_V \oplus \mathbb{C}1_W \arrow[r, two heads, "\alpha"] \arrow[d, hook] & \mathbb{C} \arrow[d, hook] \\ V \oplus W \arrow[r, two heads] & X \end{tikzcd}$$

where $\alpha(1_V) = 1-t$, $\alpha(1_W) = t$. For $t \neq 0,1$ the affine space corresponding to X is the product of the affine spaces belonging to V, W.

Actually it seems that to $(V, 1_V)$ belongs a family of affine spaces

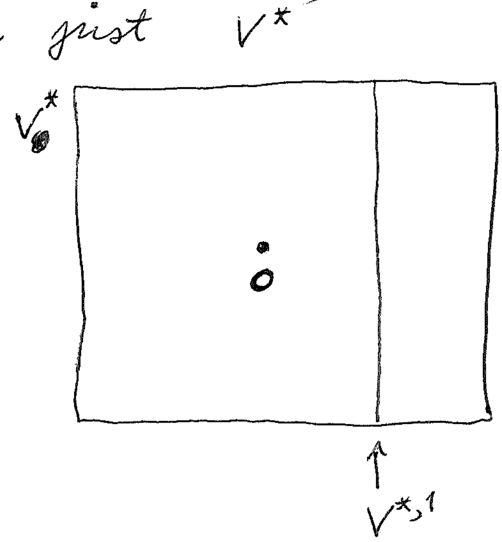
$$V^{*,c} = \{f \in V^* \mid f(1_V) = c\}$$

which are canonically isomorphic for $c \neq 0$. For $c=0$ we get $\bar{V}^*$, and to be complete we probably should add a projective space for $c = \infty$.

The join of two affine spaces contains each affine space as as subspace and this affine subspaces are disjoint. This picture corresponds to the obvious surjection

$$\bar{S}(V \oplus W) \longrightarrow \bar{S}V \times \bar{S}W$$

We are interested in the case where $(W, 1_W) = (\mathbb{C}, 1)$. Here the join is just V^*



We have omitted the product of two affine spaces. In general if P, P' are torsors under G, G' then $P \times P'$ is a torsor under $G \times G'$. This corresponds to taking tensor product of the corresp. algebras

$$\bar{S}V \otimes \bar{S}W = \bar{S}(V \oplus W/(1_V - 1_W)).$$

Perhaps things would be clearer if you worked dually with surjections $V^* \longrightarrow \mathbb{C}$. Then you have the product of affine spaces [redacted] corresponding to the fibre product $V^* \times_{\mathbb{C}} W^*$, and the join which corresponds to

$$V^* \oplus W^* \longrightarrow \mathbb{C} \oplus \mathbb{C} \xrightarrow{+} \mathbb{C}$$

In the category of affine spaces, the fibre product $V^* \times_{\mathbb{C}} W^*$ [redacted] is the product and the join is [redacted] the direct sum. There is some sort of flow on the join which is reminiscent of Cayley transforms + Morse theory of Grassmannians.

Formula for the flows. There [redacted] are two flows one might consider on the real line with fixpoints ± 1, which arise from the C.T. ~~for~~ the circle $= u(1)$. Use the derivation $t\partial_t$ for which t^n

is an eigenfunction with eigenvalue n.

The first flow has trajectories

$$g_t = \frac{1+ity}{1-ity} = \frac{1-t^2y^2}{1+t^2y^2} + i\,\frac{2ty}{1+t^2y^2}$$

y constant

Thus $$x = \frac{1-t^2y^2}{1+t^2y^2}$$

$$t\dot{x} = \frac{(1+t^2y^2)(-2t^2y^2) - (1-t^2y^2)(2t^2y^2)}{(1+t^2y^2)^2}$$

$$= -\frac{4t^2y^2}{(1+t^2y^2)^2} = -\left(\frac{2ty}{1+t^2y^2}\right)^2$$

$$= -(1-x^2)$$

$\therefore$ The first flow is $\boxed{t\dot{x} = -(1-x^2)}$

The second flow has trajectories

$$g_t = \frac{1+ity}{\sqrt{1+t^2y^2}}$$

$$x = \frac{1}{\sqrt{1+t^2y^2}}$$

$$t\dot{x} = \left(-\tfrac{1}{2}\right)(1+t^2y^2)^{-3/2}(2t^2y^2)$$

$$= -x\left(\frac{ty}{\sqrt{1+t^2y^2}}\right)^2 = -x(1-x^2)$$

The second flow is $\boxed{t\dot{x} = -x(1-x^2)}$

I don't know what to make of this. We encountered it earlier in the following way. Given a lifting x of an involution, think of x as a self-adjoint operator $-1 < x < 1$ and write it $x = \dfrac{\alpha}{\sqrt{1+\alpha^2}}$ with α self-adjoint. Then scaling $\alpha_t = t^{-1}\alpha$ gives

$$x_t = \frac{t^{-1}\alpha}{\sqrt{t^{-2}\alpha^2+1}} = \frac{\alpha}{\sqrt{\alpha^2+t^2}} = \frac{1}{\sqrt{1+t^2y^2}}, \quad y=\alpha^{-1}.$$

It's not clear whether any of this has any significance.

Summarize: When we take the join of two affine spaces we obtain a linear function x, the join parameter, which can be normalized to be ± 1 on the two affine subspaces. Then for each pair of points one from each affine subspace, we a line joining them and we can construct a flow from one to the other - this is the first type $t\dot{x} = -(1-x^2)$ - or a flow towards the midpoint - this is the second type $t\dot{x} = -x(1-x^2)$.

June 28, 1991

Let V be a vector space with a given non-zero element 1_V, let $R = T(V)$, let I be the kernel of the canonical surjection

$$R = T(V) \longrightarrow RV \times \mathbb{C}$$

where the first component is the canonical surjection onto $RV = TV/(1-1_V)$ and the second component is the augmentation $TV \to TV/(V) = \mathbb{C}$.
Thus

$$I = R(1-x)R \cap RVR$$

where we write x for 1_V.

Recall that for coprime ideals: $R = J+K$ we have $J \cap K = JK + KJ$. In effect the inclusion $\supset$ is clear, and

$$J \cap K = (J\cap K)R \subset (J\cap K)JK + (J\cap K)KJ$$
$$\subset J^2K + K^2J \subset JK + KJ.$$

Thus we have

$$I = R(1-x)RVR + RVR(1-x)R$$
$$= R(1-x)VR + RV(1-x)R$$

since $RV = VR = RVR$.

We next calculate I/I^2.

Let us put $A = R/I \cong RV \times \mathbb{C}$. The image of x in A is the idempotent $(1,0)$ which we denote e. Put $a = eA$ so that $a \cong RV$ with $e \leftrightarrow$ identity elt. We have

$$e^{\perp}A = \mathbb{C}e^{\perp} \subset A.$$

We now use the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow A \otimes_R \Omega^1 R \otimes_R A \xrightarrow{\pi} \Omega^1 A \longrightarrow 0$$
$$\begin{array}{c} \wr\| \\ A \otimes V \otimes A \end{array}$$

of A-bimodules. Recall

$$\begin{aligned}
\Omega^1 A = \; & e\Omega^1 A e && \cong \; a \otimes V \otimes a \\
\oplus \; & e^\perp \Omega^1 A e^\perp && \cong \; \Omega^1 \mathbb{C} = 0 \\
\oplus \; & e\,\Omega^1 A e^\perp && = \; a\,de \; \simeq a \\
\oplus \; & e^\perp \Omega^1 A e^\perp && = \; de\, a \; \simeq a
\end{aligned}$$

$$\begin{aligned}
A \otimes dV \otimes A = \; & a\,dV\,a && \simeq \; a \otimes \bar{V} \otimes a \\
\oplus \; & e^\perp dV e^\perp && \simeq \; V \\
\oplus \; & a\,dV e^\perp && \simeq \; a \otimes V \\
\oplus \; & e^\perp dV\,a && \simeq \; V \otimes a
\end{aligned}$$

We need to compute π, which is a bimodule maps. We have

$$\pi(dv) = edve + e^\perp dve + edve^\perp + e^\perp dve^\perp$$

$$e^\perp dve = e^\perp d(ev)e = e^\perp deve = (e^\perp de)v$$

$$edve^\perp = ed(ve)e^\perp = ev\,dee^\perp = v(ede)$$

$$e^\perp dve^\perp = e^\perp d(ve)e^\perp = e^\perp v\,dee^\perp = 0.$$

$$\therefore \quad \pi(dv) = edve + (e^\perp de)v + v(ede)$$
$$= edve + (de)v + v(de)$$

Let's review the steps so far. We have V with the non-zero elt $x = 1_V \in V$ and we consider $T(V) \longrightarrow RV \times \mathbb{C} = TV/I$

where $\quad I = R(1-x)R \cap RVR$

$$= R(1-x)VR + RV(1-x)R.$$

where $R = T(V)$.

Our problem is to construct a lifting homomorphism $R/I \longrightarrow \varprojlim_n R/I^n$. Another way of putting this is we want like to project any linear map $\rho : V \to B \ni$ to a linear map $\rho' : V \to B$ satisfying

$$\rho'(v) = \rho'(x)\rho'(v) = \rho'(v)\rho'(x)$$

for all $v \in V$, provided $\rho(v), \rho(x)\rho(v), \rho(v)\rho(x)$ are congruent modulo some nilpotent ideal.

The idea is first to study the lifting problem to first order: $R/I \longrightarrow R/I^2$. We can compute I/I^2, in fact the extension R/I^2, using the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow A \otimes_R \Omega^1 R \otimes_R A \xrightarrow{\pi} \Omega^1 A \longrightarrow 0$$

$$\begin{array}{c} A \otimes_R \Omega^1 R \otimes_R A \\ \| \\ A \otimes V \otimes A \end{array}$$

We have

$$\pi(dv) = e d\bar{v} e + e^\perp d\bar{v} e + e d\bar{v} e^\perp + e^\perp d\bar{v} e^\perp$$

$$= e d\bar{v} + de\,\bar{v} + \bar{v}\,de + 0$$

where e denotes the image of x in $A = R/I \cong RV \times \mathbb{C}$, i.e. $e \leftrightarrow (1,0) \in RV \times \mathbb{C}$, and where $\bar{v}$ denotes the image of v under $T(V) \to RV$. Note that $e\bar{v} = \bar{v}e = \bar{v}$.

We have for π the direct sum of the maps

1) $\quad A \otimes V \otimes A \longrightarrow A \otimes \bar{V} \otimes A$

2) $\quad e^\perp \otimes V \otimes A \longrightarrow de\,A \simeq A$

3) $\quad A \otimes V \otimes e^\perp \longrightarrow A\,de \simeq A$

4) $\quad e^\perp \otimes V \otimes e^\perp \longrightarrow 0$

1) the $\mathcal{A}$-bimodule map is induced by $v \longmapsto \bar{v}$; this corresponds to $dv \longmapsto e\,dv\,e = d\bar{v} \in \Omega^1\mathcal{A}$.
The kernel of 1) is $\mathcal{A} \otimes x \otimes \mathcal{A} \simeq \mathcal{A}\,dx\,\mathcal{A}$. It is a free $\mathcal{A}$-bimodule generated by $e\,dx\,e$.
1) has two obvious sections given by

$$\bar{v} \longmapsto e \otimes v \otimes e - \bar{v} \otimes x \otimes e$$

$$\bar{v} \longmapsto e \otimes v \otimes e - e \otimes x \otimes \bar{v}$$

These vanish when $v = x$ since $\bar{x} = e$.

The simplest section of 1) is perhaps given by averaging these

$$\bar{v} \longmapsto e \otimes v \otimes e - \tfrac{1}{2}(\bar{v} \otimes x \otimes e + e \otimes x \otimes \bar{v}).$$

However, one can take other linear combinations. It seems that there is a "join" of possibilities.

2) is the left $\mathbb{C}e^\perp$, right $\mathcal{A}$ bimodule map $e^\perp \otimes v \otimes e \longmapsto e^\perp d\bar{v}\,e = de\,\bar{v}$. It is therefore just the left multiplication map

$$V \otimes \mathcal{A} \longmapsto \mathcal{A} \qquad v \otimes \alpha \longmapsto \bar{v}\alpha$$

There's an obvious section sending e to $x \otimes e$

Similarly 3) is the right multiplication map $\mathcal{A} \otimes V \longrightarrow \mathcal{A}$, which has the obvious section (which is a map of left $\mathcal{A}$, right $\mathbb{C}$ modules) $e \longmapsto e \otimes x$.

4) has an obvious section, the zero map.

Putting these sections together we obtain an A-bimodule section of π, and we know this is equivalent to a lifting homom. $A \longrightarrow R/I^2$. Now we should work this out explicitly, using the cartesian square

$$\begin{tikzcd} R/I^2 \arrow[r] \arrow[d, "d"] & A \arrow[d, "d"] \\ A\otimes V\otimes A \arrow[r, "\pi"] & \Omega^1 A \end{tikzcd}$$

What this means is that we take an elt. of A, say $\bar{v}$, lift $d\bar{v}$ [redacted] using the section of π, and then represent the result by an element of R. Recall

$$d\bar{v} = e\,d\bar{v}\,e + e^{\perp}d\bar{v}\,e + e\,d\bar{v}\,e^{\perp} + \cancel{e^{\perp}d\bar{v}\,e^{\perp}}$$
$$= e\,d\bar{v}\,e + de\,\bar{v} + \bar{v}\,de$$

The lifting of $e\,d\bar{v}\,e$ we decided to use is

$$e\otimes v\otimes e - \tfrac{1}{2}(\bar{v}\otimes x\otimes e + e\otimes x\otimes \bar{v})$$

which should be written

$$e\,dv\,e - \tfrac{1}{2}(\bar{v}\,dx\,e + e\,dx\,\bar{v}) \in \mathcal{A}\otimes_R \Omega^1 R\otimes_R \mathcal{A}$$

The lifting of $de\,\bar{v}$ is [redacted] $e^{\perp}dx\,\bar{v} \in \mathcal{C}e^{\perp}\otimes_R \Omega^1 R\otimes_R \mathcal{A}$

The lifting of $\bar{v}\,de$ is $\bar{v}\,dx\,e^{\perp} \in \mathcal{A}\otimes_R \Omega^1 R\otimes_R \mathcal{A}$

Thus we have the element

[redacted]

$$\xi = \begin{matrix} e\,dv\,e - \tfrac{1}{2}(\bar{v}\,dx\,e + e\,dx\,\bar{v}) \\ + e^{\perp}dx\,\bar{v} + \bar{v}\,dx\,e^{\perp} \end{matrix} \in A\otimes_R \Omega^1 R\otimes_R A$$

such that $\pi(\xi) = d\bar{v} \in \Omega^1 A$. Now we wish to find $\underbrace{v + \text{suitable element of } I}_{\text{This should be } v-\varphi(v)}$ mapping to ξ.

June 29, 1991

We consider

$$R = \mathbb{C}\langle x,y\rangle \longrightarrow \mathbb{C}[y]\times\mathbb{C} = R/I$$

$$x \longmapsto (1,0) = e$$

$$y \longmapsto (y,0) = v$$

I generated by $x(1-x)$, $y(1-x)$, $(1-x)y$

Consider $\partial : R \longrightarrow A\otimes_R \Omega^1 R \otimes_R A$, $A = R/I$.

Then I/I^2 is the A subbimodule generated by

$$\partial(x(1-x)) = dx\, e^\perp - e\, dx = \begin{pmatrix} -e\,dx\,e & 0 \\ 0 & e^\perp dx\, e^\perp \end{pmatrix}$$

$$\partial(y(1-x)) = dy\, e^\perp - v\, dx = \begin{pmatrix} -v\,dx\,e & e\,dy\,e^\perp - v\,dx\,e^\perp \\ 0 & e^\perp dy\, e^\perp \end{pmatrix}$$

$$\partial((1-x)y) = e^\perp dy - dx\, v = \begin{pmatrix} -e\,dx\,v & 0 \\ e^\perp dy\, e - e^\perp dx\, v & e^\perp dy\, e^\perp \end{pmatrix}$$

Exact sequences: $A\otimes W\otimes A$ $\quad W = \mathbb{C}x + \mathbb{C}y$

$$\begin{tikzcd} & & A\otimes W\otimes A \arrow[d, equal] & & \\ 0 \arrow[r] & I/I^2 \arrow[r] & A\otimes_R \Omega^1 R\otimes_R A \arrow[r, "\pi"] & \Omega^1 A \arrow[r] & 0 \\ 0 \arrow[r] & I/I^2 \arrow[r] \arrow[u, equal] & R/I^2 \arrow[r] \arrow[u, "\partial"] & A \arrow[r] \arrow[u, "d"] & 0 \end{tikzcd}$$

Recall there's a fairly canonical bimodule lifting for π:

$$\begin{array}{ccc} A\otimes W\otimes A & \xrightarrow{\pi} & \Omega^1 A \\ \| & & \| \\ \begin{pmatrix} a\,dW\,a & a\,dW\,e^\perp \\ e^\perp dW\,a & e^\perp dW\,e^\perp \end{pmatrix} & & \begin{pmatrix} a\,d\mathring{V}\,a & a\,de \\ de\,a & 0 \end{pmatrix} \end{array}$$

Canonical liftings

$$e^\perp dy\,\alpha \longmapsto e^\perp d\sigma\,\alpha = de\,\sigma\,\alpha$$

$$e^\perp dW a \underset{\longleftarrow\text{-}\text{-}\text{-}}{\xrightarrow{\pi}} de\,a$$

$$e^\perp dx\,\alpha \longleftarrow\!\!\shortmid\; de\,\alpha$$

In effect

$$\begin{array}{ccc} e^\perp dW a & \longrightarrow & de\,a \\ \wr\| & & \wr\| \\ W\otimes a & \xrightarrow[\text{left mult}]{} & a. \end{array}$$

$V = W/\mathbb{C}x$. Let L denote the canonical lifting

Then $\boxed{L(de\,\alpha) = e^\perp dx\,\alpha}$

Similarly

$$a\,dW e^\perp \xrightarrow{\pi} a\,de$$

$$\alpha\,dy\,e^\perp \longmapsto \alpha\,d\sigma\,e^\perp = \alpha\,\sigma\,de$$

and the canonical lifting is

$$\boxed{L(\alpha\,de) = \alpha\,dx\,e^\perp}$$

Next we need

$$a\,dW\,a \longrightarrow \Omega^1 a = a\otimes V\otimes a = a\,dV\,a$$

which kills $a\,dx\,a$. Here we have the liftings

$$\alpha_0\,d\sigma\,\alpha_1 \longmapsto \alpha_0(e\,dy\,e - \sigma\,dx\,e)\alpha_1$$

$$\longmapsto \alpha_0(e\,dy\,e - e\,dx\,\sigma)\alpha_1$$

Suppose we take a suitable average

$$\boxed{L(\alpha_0\, dv\, \alpha_1) = \alpha_0\left(edye - t\, v dxe - (1-t)\, edxv\right)\alpha_1}$$

Now take

$$dy = \begin{pmatrix} e\,dy\,e & e\,dy\,e^{\perp} \\ e^{\perp}dy\,e & e^{\perp}dy\,e^{\perp} \end{pmatrix}$$

The lifting $A \longrightarrow R/I^2$ we seek sends v to $y - \varphi(y)$, where

$$\partial(y - \varphi(y)) = L(dy)$$

i.e. $\partial\varphi(y) = \partial y - L(dy)$.

$$L(dv) = L\begin{pmatrix} edve & vde \\ dev & 0 \end{pmatrix}$$

$$= \begin{pmatrix} edye - tvdxe - (1-t)edxv & vdxe^{\perp} \\ e^{\perp}dxv & 0 \end{pmatrix}$$

To find $\varphi(y)$?

$$\therefore\ \partial\varphi(y) = \begin{pmatrix} tvdxe + (1-t)edxv & edye^{\perp} - vdxe^{\perp} \\ e^{\perp}dye - e^{\perp}dxv & e^{\perp}dye^{\perp} \end{pmatrix}$$

But

$$\partial(xy(1-x)) = \begin{pmatrix} -vdxe & edye^{\perp} - vdxe^{\perp} \\ 0 & 0 \end{pmatrix}$$

$$\partial((1-x)yx) = \begin{pmatrix} -edxv & 0 \\ e^{\perp}dye - e^{\perp}dxv & 0 \end{pmatrix}$$

$$\partial\big((1-x)y(1-x)\big) = \begin{pmatrix} 0 & 0 \\ 0 & e^{\perp} dy\, e^{\perp} \end{pmatrix}$$

$$\partial\big(x(1-x)y\big) = \begin{pmatrix} -e\,dx\,v & 0 \\ 0 & 0 \end{pmatrix}$$

$$\partial\big(y\,x(1-x)\big) = \begin{pmatrix} -v\,dx\,e & 0 \\ 0 & 0 \end{pmatrix}$$

So $\varphi(y) =$

$$\begin{aligned} & xy(1-x) + (1-x)yx + (1-x)y(1-x) \\ & - x(1-x)y \quad - yx(1-x) \\ & - t\,x(1-x)y \quad - (1-t)yx(1-x) \end{aligned}$$

$$\boxed{\begin{aligned} \varphi(y) = {} & xy(1-x) + (1-x)yx + (1-x)y(1-x) \\ & - (1+t)\,x(1-x)y - (2-t)\,y(1-x)x \end{aligned}}$$

Check this by letting $y = x$.

$$\begin{aligned} \varphi(x) &= x^2(1-x)\big(2 - (1+t) - (2-t)\big) + x(1-x)^2 \\ &= x(1-x)\big(x(-1) + 1 - x\big) = -(2x-1)x(1-x) \end{aligned}$$

$$\therefore\ x - \varphi(x) = x + (2x-1)x(1-x)$$

which is the lifting of the idempotent e.

(Check: $x - \varphi(x) = x(\cancel{1} + 2x - 2x^2 - \cancel{1} + x) = x^2(3-2x)$

$$\begin{aligned} (x-\varphi x)^2 &= x^2\big(1+(2x-1)(1-x)\big)^2 \equiv x^2\big(1+2(2x-1)(1-x)\big) \\ &\equiv x^2\big(1+2(1-x)\big) = x^2(3-2x) \end{aligned}$$

(congruences mod $x^2(x-1)^2$)

This appears to be the correct formula. Let's check, again by considering the commutative case, where x, y commute. Then we get

$$y - \varphi(y) = \underline{y(1 + (2x-1)(1-x))}$$

General consideration. Since I^2 contains [redacted] $y(x-1)x(x-1)$, $(x-1)yx(x-1)$, $x(x-1)y(x-1)$, $x(x-1)(x-1)y$, whose highest degree monomials are yx^3, xyx^2, x^2y, x^3y it follows that mod I^2 we can reduce any monomial in x, y to one where there are [redacted] none of the monomials yx^3, xyx^2, x^2yx, x^3y inside. So for monomials where y occurs once, one has modulo I^2, only $y, xy, yx, x^2y, xyx, yx^2$, that is, degree ≤ 2 in x.

Let's check our results. Consider $R = T(V)$ and define a derivation D on R by

$$\begin{aligned} D(y) &= \varphi(y) \\ &= xy(1-x) + (1-x)yx + (1-x)y(1-x) \\ &\quad - \tfrac{3}{2}\left(x(1-x)y + y(1-x)x \right) \\ &= y - xyx - \tfrac{3}{2}\left(x(1-x)y + y(1-x)x \right). \end{aligned}$$

for all $y \in V$. Observe that

$$\begin{aligned} D(x) &= x - x^3 - 3x^2(1-x) \\ &= x - 3x^2 + 2x^3 \\ &= x(1-x)(1-2x) = -(2x-1)x(1-x) \end{aligned}$$

Note that as I is generated by the elements $(1-x)y$, $y(1-x)$ for $y \in V$, we have $D(R) \subset I$, so D is compatible with the I-adic filtration and $D = 0$ on R/I. Let's compute D on I/I^2:

$$D(y(1-x)) = Dy\,(1-x) - y\,Dx$$

$$= \left(y - xyx - \tfrac{3}{2}\cancel{(x(1-x)y + y(1-x)x)}\right)(1-x)$$

$$- yx(1-x)(1-2x)$$

$$\equiv (y - xyx)(1-x) + yx(1-x)(2(x-1)+1)$$

$$\equiv (y - xyx + yx)(1-x)$$

$$= y(1-x) + (1-x)yx(1-x)$$

$$\equiv y(1-x)$$

Similarly

$$D((1-x)y) = (1-x)\left(y - xyx - \tfrac{3}{2}\cancel{(x(1-x)y + y(1-x)x)}\right)$$

$$+ (2x-1)x(1-x)y$$

$$= (1-x)(y - xyx + (2x-1)xy)$$

$$\equiv (1-x)(y - xyx + xy)$$

$$= (1-x)y - (1-x)xy(1-x) \equiv (1-x)y$$

Thus $D = 1$ on I/I^2, and we get the required SDR equivalence of $\hat{R}$ with R/I.

Let's describe this flow in the commutative case. We have

$$\begin{aligned}\dot{y} &= y - yx^2 - 3yx(1-x) \\ &= y(1 - x^2 - 3x + 3x^2) \\ &= y(1 - 3x + 2x^2) = y(1-x)(1-2x) \\ \dot{x} &= x(1-x)(1-2x).\end{aligned}$$

Thus

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{y}{x}, \qquad \frac{dy}{y} = \frac{dx}{x}$$

$\log y = \log x + C$, $\quad y = Cx$. (Here we are thinking) in terms of a plane. Thus the flow takes place on the lines through the origin, and is essentially determined by what happens on the x line. We've seen that $x = 0, \frac{1}{2}, 1$ are fix pts, and the motion goes from 0 and 1 towards $\frac{1}{2}$.

Tomorrow we must work out the relation of the above derivation and the one constructed for $RA \to R_5 A$:

$$Da = [da, Y] - \frac{1}{2}[[a, Y], Y]$$

June 30, 1991

Consider now with a change of notation the map $RA \longrightarrow R_S A$ where $S = \mathbb{C}[e]$ and $A = \tilde{\mathcal{A}}$. Recall that we have the derivation D of $RA (= \Omega^+ A$ with $\circ)$ given by

$$Da = [da, Y] - \tfrac{1}{2}[[a, Y], Y]$$

where $Y = \frac{1}{2} F dF = (2e-1)de = ede - e^\perp de = ede - dee$. Let us calculate Da for $a \in \mathcal{A}$ i.e. $ae = ea = a$. Observe first that

1) $$dea = da - eda = e^\perp da$$

2) $$de^2 a = dee^\perp da = ededa$$

3) $$dae = da - ade$$

$$[da, Y] = da(2e-1)de - (2e-1)deda$$
$$= -dade + 2(da - ade)de + (1-2e)deda$$

$$[da, Y] = dade + (1-2e)deda - 2a\,de^2$$

$$[a, Y] = a(ede - \cancel{e^\perp} de) - (d\cancel{ee^\perp} - dee)a = ade + dea$$

$$[[a,Y],Y] = [ade + dea, ede - dee]$$

$$= \begin{matrix} (\cancel{ade} + dea)\,ede \\ -(ade + \cancel{dea})\,dee \\ -ede(\cancel{ade} + dea) \\ +dee(ade + \cancel{dea}) \end{matrix} = \begin{matrix} deade \\ -ade^2 \\ -de^2 a \\ deade \end{matrix} = \begin{matrix} 2deade \\ -ade^2 \\ -ededa \end{matrix}$$

$$= 2e^\perp dade - ededa - ade^2$$

$$[da, Y] = dade + (1-2e)deda - 2ade^2$$

$$\tfrac{1}{2}[[a,Y],Y] = e^{\perp}dade - \tfrac{1}{2}ededa - \tfrac{1}{2}ade^2$$

$$\boxed{\begin{aligned} \therefore\ Da &= [da, Y] - \tfrac{1}{2}[[a,Y],Y] \\ &= e\,dade + (1-\tfrac{3}{2}e)deda - \tfrac{3}{2}ade^2 \end{aligned}}$$

On the other hand recall the formula with $x = \rho e$, $y = \rho a$:

$$Dy = y - xyx - \tfrac{3}{2}\left((x-x^2)y + y(x-x^2)\right)$$

In this situation using $RA = \Omega^{+}A$ with $\circ$ we get

$$Da = a - (e\circ a\circ e) - \tfrac{3}{2}\Big(\underbrace{(e-e\circ e)}_{de^2}\circ a + a\circ\underbrace{(e-e\circ e)}_{de^2}\Big)$$

$$= a - (e\circ a\circ e) - \tfrac{3}{2}\big(\underbrace{de^2a}_{ededa} + ade^2\big)$$

$$e\circ a\circ e = (a - deda)\circ e = (a - dade) - de\underbrace{dae}_{da - ade}$$

$$= a - dade - deda + \underbrace{deade}_{e^{\perp}da}$$

$$= a - edade - deda$$

$$\therefore\ Da = edade + deda - \tfrac{3}{2}ededa - \tfrac{3}{2}ade^2$$

$$= edade + (1-\tfrac{3}{2}e)deda - \tfrac{3}{2}ade^2$$

which agrees with the above.

Observation: Recall that the derivation

$$Dy = y - xyx - \tfrac{3}{2}\left((x-x^2)y + y(x-x^2)\right)$$

is the special case $t = 1/2$ of the derivation

$$D_t y = y - xyx - (1+t)(x-x^2)y - (2-t)y(x-x^2)$$

which we found on page 432. These differs by

$$(D-D_t)y = (t-\tfrac{1}{2})\{(x-x^2)y - y(x-x^2)\}$$
$$= (t-\tfrac{1}{2})[x-x^2, y]$$

which is an inner derivation. Thus the liftings for different t should be conjugate up to inner automorphisms.

July 12, 1991

Connections on a bimodule appear to be more involved than I thought. Suppose we have an A-bimodule E. Think of E as corresponding to a vector bundle (non-commutative) whose algebra is $T_A(E)$. In the commutative situation $S_A(E)$ is the algebra corresponding to the vector bundle E^*. Now a connection in E allows one to lift vector fields on (the variety of) A to vector fields on E^*. In other words it lifts derivations on A to derivations on $S_A(E)$.

Let's study the non-commutative analogue. To simplify suppose A quasi-free and E a projective A-bimodule, so that we know $R = T_A(E)$ is quasi-free. We have the exact sequence of R-bimodules

$$* \quad 0 \longrightarrow R \otimes_A \Omega^1 A \otimes_A R \longrightarrow \Omega^1 R \longrightarrow \Omega^1_A R \longrightarrow 0$$

$$\Omega^1_A R \cong R \otimes_A E \otimes_A R$$

since the kernel at the left in general is $\mathrm{Tor}_1^A(R,R)$, which is zero in our case. [redacted] Let us [redacted] consider derivations with values in R-bimodules. To be able to extend such derivations on A to ones on R means the above sequence splits.

Notice that R is graded, hence so is the above sequence. We want to have a splitting of the sequence $*$, which means an A-bimodule lifting of M into $\Omega^1 R$. Supposing the grading preserved, we want to split the A-bimodule sequence which is the degree one part relative to the grading. This is

$$0 \longrightarrow \begin{matrix} E\otimes_A \Omega^1 A \\ \oplus \\ \Omega^1 A \otimes_A E \end{matrix} \longrightarrow (\Omega^1 R)_{(1)} \longrightarrow E \longrightarrow 0$$

We can check this by looking at the degree 1 part of

$$0 \longrightarrow R\otimes_A \Omega^1 A \otimes_A R \longrightarrow R \otimes R \longrightarrow R \otimes_A R \longrightarrow 0$$

which is

$$0 \longrightarrow \begin{matrix} E\otimes_A \Omega^1 A \\ \oplus \\ \Omega^1 A \otimes_A E \end{matrix} \longrightarrow \begin{matrix} E\otimes A \\ \oplus \\ A\otimes E \end{matrix} \longrightarrow \begin{matrix} E \\ \oplus \\ E \end{matrix} \longrightarrow 0$$

Thus $(\Omega^1 R)_{(1)} = \operatorname{Ker}\left\{ \begin{matrix} E\otimes A \\ \oplus \\ A\otimes E \end{matrix} \xrightarrow{m_r + m_\ell} E \right\}$

Let's check further by considering a derivation $D : A \longrightarrow M$, where M is an R-bimodule. To extend D to R we need to define $D\xi$ for ξ on E such that

$$D(a\xi) = Da\,\xi + a\,D\xi$$

$$D(\xi a) = \xi\, Da + D\xi\, a$$

or better

$$D(a_1 \xi a_2) = Da_1\, \xi a_2 + a_1 D\xi a_2 + a_1 \xi Da_2$$

Let's review. We have $R = T_A(E)$ and we are interested in the degree one part of

$$\begin{tikzcd}
R\otimes_A \Omega^1 A \otimes_A R \arrow[r] \arrow[d, equal] & R\otimes R \arrow[r] & R\otimes_A R \arrow[r] & 0 \\
R\otimes_A \Omega^1 A\otimes_A R \arrow[r] & \Omega^1 R \arrow[r] \arrow[u, phantom, sloped, "\subset"] & \Omega^1_A R \arrow[r] \arrow[u, phantom, sloped, "\subset"] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & & E & E \\
0 \arrow[r] & \begin{matrix} E\otimes_A \Omega^1 A \\ \oplus \\ \Omega^1 A\otimes_A E \end{matrix} \arrow[r, "\delta_r \oplus \delta_l"] \arrow[d, equal] & \begin{matrix} E\otimes A \\ \oplus \\ A\otimes E \end{matrix} \arrow[u, "m_r + m_l"] \arrow[r, "m_r \oplus m_l"] & \begin{matrix} E \\ \oplus \\ E \end{matrix} \arrow[u, "1+1"] \arrow[r] & 0 \\
0 \arrow[r] & (\ \bullet\ ) \arrow[r] & \Omega^1 R_{(1)} \arrow[u, phantom, sloped, "\subset"] \arrow[r] & E \arrow[u, phantom, sloped, "\subset\ \binom{1}{-1}"] \arrow[r] & 0
\end{tikzcd}$$

I am interested in splitting the bottom sequence of A-bimodules. Observe that the class of the bottom sequence lies in

$$\mathrm{Ext}^1\left(E, \begin{matrix} E\otimes_A \Omega^1 A \\ \oplus \\ \Omega^1 A\otimes_A E \end{matrix}\right) = \mathrm{Ext}^1(E, E\otimes_A \Omega^1 A) \oplus \mathrm{Ext}^1(E, \Omega^1 A\otimes_A E)$$

and it should correspond to the difference of the extensions $[m_r]$ $[m_l]$ in the Baer sense. This should mean that to split this extension is the same as splitting both the m_r and m_l extensions. Thus a lifting for $\Omega^1 R_{(1)} \twoheadrightarrow E$ should be equivalent to a pair ∇_r, ∇_l consisting of right and left connections on E.

Let's check this is so. We have $d : E \to \Omega^1 R_{(1)}$ given by $d\xi = \xi\otimes 1 - 1\otimes \xi$. Assume we have a lifting $\lambda : E \to \Omega^1 R_{(1)}$ and define ∇_r, ∇_l by

$$\circledast \qquad \xi\otimes 1 - 1\otimes\xi = \lambda(\xi) + j_\ell \nabla_\ell \xi + j_r \nabla_r \xi$$

Apply to $a\xi$ and subtract a times this:

$$a\xi\otimes 1 - 1\otimes a\xi = \lambda(a\xi) + j_\ell\nabla_\ell(a\xi) + j_r\nabla_r(a\xi)$$
$$a\xi\otimes 1 - a\otimes\xi = a\lambda(\xi) + a j_\ell\nabla_\ell(\xi) + a j_r\nabla_r(\xi)$$

$$\underbrace{(a\otimes 1 - 1\otimes a)\xi}_{j_\ell(da\,\xi)} = j_\ell\left(\nabla_\ell(a\xi) - a\nabla_\ell\xi\right) + j_r\left(\nabla_r(a\xi) - a\nabla_r\xi\right)$$

$$\Longrightarrow \qquad \begin{aligned} \nabla_\ell(a\xi) &= a\nabla_\ell\xi + da\,\xi \\ \nabla_r(a\xi) &= a\nabla_r\xi \end{aligned}$$

Similarly applying $\circledast$ to ξa and subtracting $\circledast a$ gives

$$\nabla_\ell(\xi a) = (\nabla_\ell\xi)a$$
$$\nabla_r(\xi a) = (\nabla_r\xi)a + \xi\,da$$

Thus ∇_ℓ is a left connection and ∇_r is a right connection.

Thus it would seem that the good definition of connection in the bimodule E is a pair consisting of a left and right connection. $\exists$ a left connection forces E ~~to~~ be left projective and then $\exists$ a right connection means that E must be a projective bimodule.

~~Consider now~~ ~~besides ∇_r, ∇_ℓ~~ ~~$\Omega^1 A$~~

Consider the bimodule $E = A$, and suppose given a connection on it consisting of

$$\nabla_r : A \longrightarrow A\otimes_A \Omega^1 A = \Omega^1 A$$
$$\nabla_\ell : A \longrightarrow \Omega^1 A\otimes_A A = \Omega^1 A$$

Besides these operators we also have $d: A \longrightarrow \Omega^1 A$.

Observe that if ∇_r is a right connection then $d - \nabla_r$ is a left conn:

$$(d-\nabla_r)(a a_1) = da\, a_1 + a\, da_1 - a \nabla_r a_1$$

$$= a(d-\nabla_r) a_1 + da\, a_1$$

$$(d-\nabla_r)(a_1 a) =$$

$$da_1 a + \cancel{a_1 da} - (\nabla_r a_1) a - \cancel{a_1 da}$$

$$= (d-\nabla_r) a$$

Notice also that the difference of two right connections is a central element of $\Omega^1 A$:

$$da = [a, \nabla_r 1] \qquad da = [a, \nabla_r' 1]$$

$$\Longrightarrow \quad [a, \nabla_r 1 - \nabla_r' 1] = 0.$$

Thus a connection (∇_r, ∇_ℓ) on A is equivalent to a right connection together with a central element ξ of $\Omega^1 A$, where

$$\nabla_\ell = d - \nabla_r - \xi$$

Next consider the bimodule $\Omega^1 A$. Given (∇_r, ∇_ℓ) we again have $\nabla_r, \nabla_\ell, d: \Omega^1 A \to \Omega^2 A$

In this case $d + \nabla_r$ is a left-connection

$$(d+\nabla_r)(a\xi)$$

$$= da\,\xi + a\, d\xi + a \nabla_r \xi = a(d+\nabla_r)\xi + da\,\xi$$

$$(d+\nabla_r)(\xi a) = d\xi\, a - \cancel{\xi\, da} + (\nabla_r \xi) a + \cancel{\xi\, da}$$

$$= (d+\nabla_r)(\xi)\, a$$

Thus the difference

$$d + \nabla_r - \nabla_\ell \ : \ \Omega^1 A \longrightarrow \Omega^2 A$$

is a bimodule map. Maybe this is the analogue of torsion.

Consider now what it means for the torsion to be zero. It means simply that

$$(\nabla_r - \nabla_\ell)(da) = 0$$

Thus torsion-free connections correspond to the lifting homomorphisms $A \longrightarrow RA/IA^2$.

~~[illegible] we want to understand the [illegible]~~

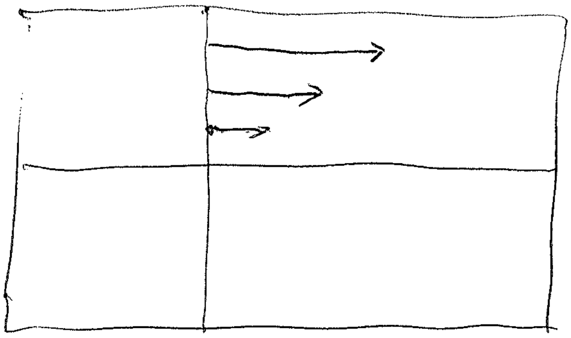
Actually we have worked before with $\Omega^1 R_{(1)}$ in the case $E = \Omega^1 A$. In this case $R = T_A(\Omega^1 A) = \Omega A$ and $\Omega^1 R_{(1)}$ we found to be isomorphic to $A \otimes \bar{A} \otimes A \oplus \Omega^2 A$. (Sept 1989 p 66). This maps onto $\Omega^1 A = E$ via the obvious surjection $A \otimes \bar{A} \otimes A \twoheadrightarrow \Omega^1 A$. So a lifting of $\Omega^1 A$ back into $\Omega^1 R_{(1)}$ is equivalent to a lifting into $A \otimes \bar{A} \otimes A$ and a bimodule map $\Omega^1 A \longrightarrow \Omega^2 A$. This checks the above.

Next we [redacted] want to consider the exponential map for A equipped with a (say torsion-free) connection. The algebra describing the tangent bundle TM is $S_A(\Omega^1_A)$ in the comm. case. In the non-comm. case we look at $T_A(\Omega^1 A) = \Omega A$. [redacted] A connection should give a flow on the tangent bundle, because a point of the tangent bundle is a tangent vector on M which can be lifted horizontally thus giving a vector field on the tangent bundle. So we are after a derivation on $T_A(\Omega^1 A) = \Omega A$ extending d on A. Why?

First note that the inclusion homom. $A \hookrightarrow \Omega A$ corresp. geometrically to the projection $TM \longrightarrow M$, and we know the vector field on TM when projected is a tautological thing. This is the tautological field of tangent vectors over M parametrized by TM we encountered with variation maps.

Next the augmentation $\Omega A \twoheadrightarrow$ corresponds to the inclusion given by the zero section $M \longrightarrow TM$, which is the fix pts for the "geodesic flow".

Also the geodesic flow is a "shear" flow



(parabolic as opposed to our derivations with eigenvalues $n \in \mathbb{N}$.)

So the project is to find the derivation on $T_A(\Omega^1 A) = \Omega A$ extending d. Call this geodesic flow derivation D. Then $Da = da$ and we need $D(da)$.

July 13, 1991

Let there be given a connection (∇_r, ∇_ℓ) in the A-bimodule E. We know that E is a projective bimodule, hence $R = T_A(E)$ is a projective A-bimodule except for the A in degree 0. So we have an exact sequence

$$0 \longrightarrow R \otimes_A \Omega^1 A \otimes_A R \longrightarrow \Omega^1 R \longrightarrow \Omega^1_A R \longrightarrow 0$$
$$\Omega^1_A R = R \otimes_A E \otimes_A R$$

I claim this sequence of R-bimodules has a splitting determined by (∇_r, ∇_ℓ). Define a derivation $D : R \longrightarrow R \otimes_A \Omega^1 A \otimes_A R$ by

$$D a = da \qquad a \in A$$
$$D \xi = (\nabla_r \xi) \otimes 1 + 1 \otimes (\nabla_\ell \xi)$$
$$\in \left(E \otimes_A \Omega^1 A \otimes_A A \oplus A \otimes_A \Omega^1 A \otimes_A E \right) \subset R \otimes_A \Omega^1 A \otimes_A R$$

To do calculations let us suppress the tensor signs. Thus we identify $R \otimes_A \Omega^1 A \otimes_A R$ with its image $R\, dA\, R \subset \Omega^1 R$. Now we check D is compatible with the A-bimodule structure on E:

$$D(a_1 \xi a_2) = \nabla_r (a_1 \xi a_2) + \nabla_\ell (a_1 \xi a_2)$$
$$= a_1 (\nabla_r \xi\, a_2 + \xi\, da_2) + (a_1 \nabla_\ell \xi + da_1\, \xi)\, a_2$$
$$= a_1 \xi \underbrace{da_2}_{Da_2} + a_1 \underbrace{(\nabla_r \xi + \nabla_\ell \xi)}_{D\xi} a_2 + \underbrace{da_1}_{Da_1} \xi a_2$$

Exponential map. Suppose we have (∇_r, ∇_ℓ) given for $E = \Omega^1 A$. Then we use this connection to lift the derivation $d : A \longrightarrow \Omega A = R$. Thus we have a derivation X of R with values in R such that $Xa = da$ and such that

$$X(\xi) = (\nabla_r + \nabla_\ell)(\xi)$$

Here ∇_r stands for

$$\underbrace{\Omega^1 A}_{E} \longrightarrow \underbrace{\Omega^1 A}_{E} \otimes_A \Omega^1 A = \Omega^2 A \subset R$$

$$\xi \otimes da \longmapsto \xi\, da$$

Recall from yester-day that the torsion is

$$d + \nabla_r - \nabla_\ell : \Omega^1 A \longrightarrow \Omega^2 A$$

which is a bimodule map determined by

$(d + \nabla_r - \nabla_\ell)(da) = \nabla_r da - \nabla_\ell da.$

The derivation X is supposed to give rise to the ~~expolette~~ geodesic flow on the tangent bundle. We have

$$Xa = da$$
$$X(da) = \nabla_r da + \nabla_\ell da$$

Thus it depends only on the sum $\nabla_r d + \nabla_\ell d : A \longrightarrow \Omega^2 A$. Let

$$\nabla_\ell = \nabla_r + d - \tau$$

where τ is the torsion. Then set

$$\nabla_r' = \nabla_r - \frac{\tau}{2} \qquad \nabla_\ell' = \nabla_\ell + \frac{\tau}{2}$$

and we have $\quad \nabla_\ell' = \nabla_r' + d \quad$ so $(\nabla_r', \nabla_\ell')$

is a connection with zero torsion. Moreover

$$\nabla'_r + \nabla'_\ell = \nabla_r + \nabla_\ell$$

Thus from the viewpoint of geodesics we can suppose the torsion is zero, whence ∇_r, ∇_ℓ are determined by the 1-cochain

$$\nabla_r da = \nabla_\ell da = \phi a$$

such that $-(\delta\phi)(a_1, a_2) = da_1 da_2$.

The geodesic flow is then

$$\boxed{\begin{aligned} Xa &= da \\ X(da) &= 2\phi a \end{aligned}}$$

Check this is a well-defined derivation on ΩA.

$$X(d(a_1 a_2)) = X(da_1 a_2 + a_1 da_2)$$

$$= Xda_1\, a_2 + da_1 da_2 + da_1 da_2 + a_1\, Xda_2$$

i.e. $\quad \phi(a_1a_2) = \phi a_1\, a_2 + da_1 da_2 + a_1 \phi a_2$.

The idea is now to use the 1-parameter group of automorphisms e^{tX} which should be defined on $\hat{\Omega} A$. This is clear because we have $X(A) \subset dA$ $\quad X(\Omega^1 A) = X(AdA) \subset dA^2 + A\phi A \subset \Omega^2 A$. Thus $X(\Omega^n A) \subset \Omega^{n+1} A$ for all n.

Let N be the derivation of ΩA given by $N\omega = |\omega|\,\omega$. Then

$$[N, X] = X$$

since X has degree 1.

I want to look at a symmetric

exponential map which associates to a tangent vector v at x the pair $\exp_x(\frac{1}{2}v)$ and $\exp_x(-\frac{1}{2}v)$.

Define a homom.

$$QA \xrightarrow{u_t} \hat{\Omega}A \qquad \text{by}$$

$$\theta a \longmapsto e^{tX} a$$

$$\theta^\gamma a \longmapsto e^{-tX} a$$

i.e.
$$pa \longmapsto \cosh(tX) a$$
$$qa \longmapsto \sinh(tX) a$$

These homomorphism for different t are related by the rescaling automorphism c^N of $\hat{\Omega}A$ where c is scalar:

$$c^N e^{tX} a = e^{ctX} a$$

so we might as well take $t = 1$. It seems that there is only one symmetric exponential map isomorphism $\hat{Q}A \xrightarrow{\sim} \hat{\Omega}A$ around.

A problem: On $\hat{\Omega}A$ we have the derivations N, X. Find what they correspond to on $\hat{Q}A$. Recall that we have a derivation D on QA, which might correspond to $\frac{1}{2}N$, given by

$$\boxed{\begin{aligned} Da &= \phi a \\ D(da) &= \tfrac{1}{2}da + d\phi a \end{aligned}}$$

The hope would be commutativity in

$$\begin{array}{ccc} QA & \xrightarrow{u} & \Omega A \\ {\scriptstyle 2D}\downarrow & & \downarrow{\scriptstyle N} \\ QA & \xrightarrow{u} & \Omega A \end{array}$$

Let's calculate for $A = \mathbb{C}[F]$ where

$$\phi(F) = -\frac{1}{2} F dF^2$$

We have $X(F) = dF$

$$X^2(F) = X(dF) = 2\phi(F) = -FdF^2$$

$$X(dF^2) = X(dF)\,dF + dF\,X(dF)$$
$$= -FdF^3 + dF(-FdF^2)$$
$$= -FdF^3 + FdF^3 = 0$$

$$X(FdF) = dF^2 + F(-FdF^2) = 0$$

So we have

$$XF = dF$$
$$X^2F = -FdF^2$$
$$X^3F = -dF^3$$
$$X^4F = FdF^4$$

$$\therefore \; e^X F = F + dF - \frac{1}{2} FdF^2 - \frac{1}{3!} dF^3 + \frac{1}{4!} FdF^4 + \cdots$$
$$= F\cos(dF) + \sin(dF).$$

So

$$u : \overbrace{F+dF}^{\theta F} \longmapsto F\cos(dF) + \sin(dF)$$
$$: F-dF \longmapsto F\cos(dF) - \sin(dF)$$
$$: F \longmapsto F\cos(dF)$$
$$: dF \longmapsto \sin(dF)$$

Check that $u(\theta F)$ is an involution

$$\big(F\cos(dF) + \sin(dF)\big)^2 = F\cos(dF)F\cos d\theta + (\sin dF)^2$$
$$+ F\cos(dF)\sin dF + \sin(dF)F\cos(dF)$$
$$= (\cos dF)^2 + (\sin dF)^2 = 1.$$

Now QA is generated by F, dF subject to

$$F \circ F = 1 - dF \circ dF \qquad F \circ dF + dF \circ F = 1$$

ΩA is generated by F, dF with relations

$$F^2 = 1, \qquad FdF + dFF = 0$$

Let's compute u^{-1}. We have

$$u(dF) = \sin(dF)$$

Now u^{-1} is a homomorphism from ordinary to Fedosov product, so

$$dF = u^{-1}\sin(dF) = \sin\left(u^{-1}(dF)\right)$$

$$\therefore \quad \boxed{u^{-1}(dF) = \arcsin(dF)}$$

We have also

$$u(F) = F\cos(dF)$$

so

$$\begin{aligned} F &= u^{-1}\left(F\cos(dF)\right) \\ &= u^{-1}(F)\circ u^{-1}\left(\cos(dF)\right) \\ &= u^{-1}(F)\circ \cos(\arcsin dF) \\ &= u^{-1}(F)\cdot\sqrt{1-dF^2} \end{aligned}$$

$\circ$ unnec.

giving $\boxed{u^{-1}(F) = F\left(1-dF^2\right)^{-1/2}}$.

This checks as it is $F\,(F\circ F)^{-1/2}$, which gives the involution lifting F in A.

Now find the derivation on $\hat{Q}A$ corresp. to N on $\hat{\Omega}A$.

$$\begin{tikzcd} F \arrow[r, mapsto, "u"] & F\cos(dF) \arrow[d, "N"] \\ F(1-dF^2)^{-1/2}\arcsin(dF)\times dF & -F\,dF\sin(dF) \arrow[l, mapsto, "u^{-1}"'] \end{tikzcd}$$

N like $dF\dfrac{\partial}{\partial(dF)}$

$$\begin{tikzcd} dF \arrow[r, mapsto] & \sin(dF) \arrow[d, "N"] \\ \arcsin(dF)\cos(\arcsin dF) & dF\cos dF \arrow[l, mapsto, "u^{-1}"'] \end{tikzcd}$$

$$= \arcsin(dF)\sqrt{1-dF^2}$$

Real Mess.

July 15, 1991

Calculations pertaining to isomorphisms
$\hat{R}A \simeq \hat{\Omega}^+ A$ for $A = \mathbb{C}[F]$.

There are three isomorphisms which we have constructed.

1) Symmetrize exponential map. The geodesic flow on ΩA associated to the connection with $\nabla(dF) = -\frac{1}{2} F dF^2$ is given by

$$X(F) = dF \qquad \qquad \begin{pmatrix} X(a) = da \\ X(da) = 2\phi a \end{pmatrix}$$
$$X(dF) = -FdF^2$$

and we have $e^X F = F\cos(dF) + \sin(dF)$

so the symmetrized exponential isomorphism is

$$\hat{Q}A \xrightarrow{\sim} \hat{\Omega}A$$
$$F \pm dF \longmapsto F\cos dF \pm \sin dF$$

2) Consider the derivation obtained from YM considerations

$$Da = \phi a \qquad\qquad DF = -\tfrac{1}{2} F dF^2$$
$$D(da) = \tfrac{1}{2} da + d\phi a \qquad\qquad D(dF) = \tfrac{1}{2} dF(1-dF^2)$$

Here

$$D\{(1-dF^2)^{-1/2}\} = \tfrac{1}{2} dF^2 (1-dF^2)^{-1/2}$$
$$D\{F(1-dF^2)^{-1/2}\} = 0$$
$$D\{dF(1-dF^2)^{-1/2}\} = \tfrac{1}{2}\left(dF(1-dF^2)^{-1/2}\right)$$

Thus

$$\hat{Q}A \xrightarrow{\sim} \hat{\Omega}A$$
$$F(1-dF^2)^{-1/2} \longleftrightarrow F$$
$$dF(1-dF^2)^{-1/2} \longleftrightarrow dF$$
$$F \longleftrightarrow F(1+dF^2)^{-1/2}$$
$$dF \longleftrightarrow dF(1+dF^2)^{-1/2}$$

3) The "affine" version (affine in the sense of our viewpoint about $R(A\times B)$ being a join). Here $RA = \mathbb{C}[z]$, $z \Leftrightarrow F$ and we use the vector field like $z\partial_z$ but such that $z=\pm 1$ are fix points. Thus $D = (z\mp 1)\partial_z$ near $z=\pm 1$ and eigenfn. for D corresponding to the eigenvalues $0, 1$ are

$$\tilde{z} = z\left(1-(1-z^2)\right)^{-1/2}$$

$$z - \tilde{z}$$

So $D = (z-\tilde{z})\partial_z$. Now use the Fedosov model for RA, and we have

$$DF = F - \tilde{F} = F\left(1-(1-dF^2)^{-1/2}\right)$$

and $D(F-\tilde{F}) = F - \tilde{F}$. Also

$$F - \tilde{F} \equiv F\left(-\tfrac{1}{2}dF^2\right) \equiv -\tfrac{1}{2}FdF^2 \qquad \text{mod } I\hat{A}^2$$

so we have the isom.

$$\hat{R}A \xrightarrow{\sim} \hat{\Omega}^+A$$

$$\tilde{F} = F(1-dF^2)^{-1/2} \longleftrightarrow F$$

$$F - \tilde{F} \longleftrightarrow -\tfrac{1}{2}FdF^2$$

$$F \longleftrightarrow F\left(1-\tfrac{1}{2}dF^2\right)$$

$$dF^2 = 1 - F\circ F \longleftrightarrow dF^2\left(1-\tfrac{1}{4}dF^2\right)$$

Notice that it would be natural to extend to $\hat{Q}A$ by setting

$$dF \longleftrightarrow dF\left(1-\tfrac{1}{4}dF^2\right)^{+1/2}$$

and then we have three different isomorphisms

$$\hat{Q}A \xrightarrow{\sim} \hat{\Omega}A$$

$$\begin{cases} F\,[redacted] \longmapsto F\cos(dF) \\ dF \longmapsto [redacted]\sin(dF) \end{cases}$$

$$\begin{cases} F \longmapsto F\left(1+dF^2\right)^{-1/2} \\ dF \longmapsto dF\left(1+dF^2\right)^{-1/2} \end{cases}$$

$$\begin{cases} F \longmapsto F\left(1-\tfrac{1}{2}dF^2\right) \\ dF \longmapsto dF\left(1-\tfrac{1}{4}dF^2\right)^{1/2} \end{cases}$$

July 18, 1991

Recall the result that for A separable there is a canonical bimodule lifting $A \to A \otimes A$. Let's review the "computational" proof. First suppose $A = \mathrm{End}(V) = V \otimes V^*$. Then

$$A \otimes A = V \otimes V^* \otimes V \otimes V^*$$

with left mult. given by the left mult. of A on ~~the~~ the left copy of V, and right mult. given by right mult. of A on the right copy of V^*. It is clear that

$$(A \otimes A)^{\natural} = \sum v_i \otimes (V^* \otimes V) \otimes v_i^*$$

The multiplication map is

$$v_i \otimes \sum_{j,k} \alpha_{jk} (v_j^* \otimes v_k) \otimes v_i^* \longmapsto \mathrm{tr}(\alpha) \cdot 1_A$$

So bimodule liftings are given by matrices α such that $\mathrm{tr}(\alpha) = 1$. Canonical choice is $\alpha_{jk} = \frac{1}{n} \delta_{jk}$, whence the central elt is

$$z = \frac{1}{n} \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji}$$

where $e_{ij} = v_i \otimes v_j^*$

Note that this $z \in A \otimes A$ is invariant under the flip σ of $A \otimes A$.

I recall also that the central $z \in (A \otimes A)^{\natural}$ with $m(z) = 1$ are described by $m\sigma(z) \in 1 + [A, A]$. So the canonical z above is the z with $m\sigma(z) = 1$. This means that if $z = \sum x_i \otimes y_i$, then $\sum y_i x_i = 1$.

Prop: A separable $\Rightarrow \exists!\ z \in (A \otimes A)^{\natural}$ such that $m(z) = 1$ and $\sigma z = z$.

Note that if [redacted] $z = \sum x_i \otimes y_i = \sum y_i \otimes x_i$ then we have $\sum x_i a \otimes y_i = \sum_i x_i \otimes a y_i$ as well as [redacted] $\sum a x_i \otimes y_i = \sum x_i \otimes a y_i$.

Proof. We have seen this is true for a matrix algebra, so we only have to see what happens with respect to a central idempotent e. We have $A = eA \oplus e^{\perp}A$. Now

$$(A \otimes A)^{\natural} = (eA \otimes eA)^{\natural} \oplus (e^{\perp}A \otimes e^{\perp}A)^{\natural}$$

so a central element in $A \otimes A$ is equivalent to central elements in $eA \otimes eA$, $e^{\perp}A \otimes e^{\perp}A$. The rest is clear.

I'm missing a non computational proof that [redacted] $z = \sum x_i \otimes y_i \in (A \otimes A)^{\natural}$ with $\sum x_i y_i = $ [redacted] 1 and $m\sigma z = \sum y_i x_i = 1$ necessarily satisfies $\sigma z = z$. [redacted]

An interesting point is that this canonical element z defines an isomorphism $A = A^*$ as [redacted] vector spaces firstly, and secondly as bimodules. Recall that if $z \in A \otimes A$ and if V, W are the smallest subspaces of A such that $z \in V \otimes W$, then V and W are naturally dual with z playing the role of the identity element in $V \otimes V^*$. Moreover $az = za$ [redacted] $\forall a \in A$ means

V is a left ideal, W is a right ideal and the isom $W = V^*$ is compatible with [redacted] the action of A. Similarly if σz is central [redacted] V is a right ideal, W is a left ideal and $W = V^*$ as A-modules. Thus when both $z, \sigma z$ are central W, V are ideals and $W = V^*$ as A-bimodules.

Now when $m(z) = 1$, we know from the calculations that [redacted] $V = W = A$, so we have a canonical isomorphism $A = A^*$ as A-bimodules.

Recall that A^* is the universal bimodule for traces. Thus there is a canonical trace on A. From the computation for matrices

$$z = \frac{1}{n} \sum e_{ij} \otimes e_{ji}$$

we conclude that the canonical trace is $\frac{1}{n}$ times the ordinary matrix trace.

Further comments. If $z = \sum x_i \otimes y_i \in (A \otimes A)^\natural$ such that $\sum x_i y_i = 1$, then

$$m \longmapsto \sum x_i m y_i$$

is a projection of M onto $M^\natural$, and

$$m \longmapsto \sum y_i m x_i$$

induces a lifting $M_\natural \longrightarrow M$. When $z = \sigma z$, these two maps coincide.

The next topic to discuss is [redacted] whether there is a canonical lifting $\Omega^1 A \longrightarrow A \otimes \bar{A} \otimes A$ when A is separable. Such a lifting is given by $\phi : \bar{A} \longrightarrow \Omega^2 A$ such that $-(\delta\phi)(a_1, a_2) = da_1 da_2$, i.e.

ϕ is a 1-cochain whose coboundary is $d \cup d$. There are two choices since we have $\delta Y = d$, ~~where~~ i.e. $[a, Y] = da$, namely

$$d \cup d = \delta(Y \cup d) = -\delta(d \cup Y)$$

Thus we have $\phi_1 a = -Y da$

or $\phi_2 a = da Y$

e.g. $-(\delta\phi_2)(a_1, a_2) = d(a_1 a_2)Y - da_1 Y a_2 - a_1 da_2 Y$

$= da_1 [a_2, Y] = da_1 da_2$

Notice that ϕ_1, ϕ_2 are cohomologous since

$$\delta(Y^2)(a) = [a, Y^2] = [a, Y]Y + Y[a, Y]$$
$$= (d \cup Y + Y \cup d)(a).$$

Here's an approach. We have on A the connection given by (∇_r, ∇_ℓ), where

$\nabla_r 1 = Y \qquad \nabla_\ell 1 = -Y.$

This has torsion zero, so $\nabla_\ell = d - \nabla_r$:

$$\nabla_\ell a = \nabla_\ell (a 1) = a\nabla_\ell 1 + da = da - aY$$
$$= (d - \nabla_r)(a)$$

We can extend ∇_r to a right connection on $\Omega^1 A$ by

$$\nabla_r (a_0 da_1) = -\nabla_r a_0 \, da_1$$

One can check this works, but note that it commutes with left multiplication, and also

$$\nabla_r (da) = -\nabla_r 1 \, da = -Y da$$

which we know is a 1-cochain whose coboundary is $da_1 da_2$

Similarly we can extend ∇_ℓ to a left connection on $\Omega^1 A$ by

$$\nabla_\ell (da_0 \, a_1) = - da_0 \nabla_\ell a_1$$

This commutes with right mult. and

$$\nabla_\ell (da) = - da \, \nabla_\ell 1 = da \, Y$$

which is a 1-cochain having coboundary $da_1 da_2$.

Thus
$$\nabla_r (da) = - Y da$$
$$\nabla_\ell (da) = da \, Y$$
are the two candidates we have. The torsion of (∇_r, ∇_ℓ), which is $\nabla_\ell - d - \nabla_r$ is the bimodule map such that

$$(\nabla_\ell - d - \nabla_r)(da) = da \, Y + Y da$$
$$= [a, Y] Y + Y [a, Y] = [a, Y^2]$$

which is a derivation as it should be.

Conclude: The canonical torsion-free connection on A for A separable seems not to give rise to a torsion-free connection on $\Omega^1 A$, but rather to a canonical connection with torsion.

In the example $\mathbb{C}[F]$ one has $Y = \frac{1}{2} F dF$, $Y^2 = -\frac{1}{4} dF^2$ and $[F, Y^2] = 0$, so in this case the torsion is zero.

Further comment. Observe

$$z = \frac{1}{n} \sum e_{ij} \otimes e_{ji}$$

gives $A \longrightarrow A \otimes A$, hence

$$A = A \otimes_A A \longrightarrow (A \otimes A) \otimes_A (A \otimes A) = A^{\otimes 3}$$

where $1 \longmapsto 1 \otimes_A 1 \longmapsto \frac{1}{n^2} \sum (e_{ij} \otimes e_{ji}) \otimes_A (e_{k\ell} \otimes e_{\ell k})$

$$= \frac{1}{n^2} \sum e_{ij} \otimes \underbrace{e_{ji} e_{k\ell}}_{|j\rangle \underbrace{\langle i|k\rangle}_{\delta_{ik}} \langle \ell|} \otimes e_{\ell k}$$

$$= \frac{1}{n^2} \sum_{i,j,\ell} e_{ij} \otimes e_{j\ell} \otimes e_{\ell i}$$

July 20, 1991

Assume $M^\natural \xrightarrow{\sim} M_\natural$ for all bimodules M. Consider $A\otimes A$ as a bimodule for the internal structure where left & right mult. by $a\in A$ are

$$a(a_1\otimes a_2) = a_1\otimes a a_2$$
$$(a_1\otimes a_2)a = a_1 a\otimes a_2$$

Then we have an isomorphism

$$A\overset{\natural\ \text{internal}}{\otimes} A \xrightarrow{\sim} (A\otimes A)_{\natural\ \text{internal}}$$

Denote the former $A\overset{\natural}{\otimes}A = \{\sum x_i\otimes y_i \mid \sum x_i a\otimes y_i = \sum x_i\otimes a y_i\}$. The latter is $A\otimes_A A = A$. So we get an isomorphism

$$\circledast \quad m : A\overset{\natural}{\otimes}A \xrightarrow{\sim} A$$

which is a bimodule isomorphism for the external structure.

In particular there is a unique $\sum x_i\otimes y_i \in A\overset{\natural}{\otimes}A$ such that $\sum x_i y_i = 1$, and since 1 is central in A, the fact that $\circledast$ is a bimodule isomorphism tells us that

$$\sum x_i\otimes y_i \in (A\overset{\natural}{\otimes}A)^\natural$$

i.e. this element is central for the external structure.

Now we want to show $\sum y_i x_i = 1$.

The follows because $\sum y_i x_i \in A^{\natural}$ and its image in $A_{\natural}$ is $\sum x_i y_i = 1$. Thus we win as $A^{\natural} \xrightarrow{\sim} A_{\natural}$.

Computationally, once we have $\sum x_i \otimes y_i \in (A \overset{\natural}{\otimes} A)^{\natural}$ with $\sum x_i y_i = 1$, then $m \longmapsto \sum x_i m y_i$ projects M onto $M^{\natural}$ and it kills $[A, M]$. Thus

$$\sum y_i x_i = \sum x_j (\sum y_i x_i) y_j \qquad \sum y_i x_i \in A^{\natural}$$

$$= \sum x_j (\sum x_i y_i) y_j \qquad (\text{kills } [A,M])$$

$$= \sum x_j y_j = 1.$$

It follows from this and the isom $(*)$ that we have

$$\sum x_i \otimes y_i = \sum y_i \otimes x_i$$

i.e. $\sigma z = z$ for $z = \sum x_i \otimes y_i$.

Thus it seems that separability in characteristic zero ~~...~~ is effectively stronger than in char p. The way to express this is via the lemma $M^{\natural} \xrightarrow{\sim} M_{\natural}$. In characteristic p this means I think that the matrix algebras have degree $\not\equiv 0 \mod p$.

Example: $A = \mathbb{C}[G]$. (G finite) Then $z = \frac{1}{|G|} \sum g \otimes g^{-1}$

Another point: Capping with z gives a bimodule map $A^* \longrightarrow A$ which is an isomorphism. Why: If we choose $z = \sum_{i=1}^{n} x_i \otimes y_i$

with n least, then we know x_i is a basis for a left ideal $V \subset A$ and y_i is a basis for a right ideal $W \subset A$, and that we have an isomorphism of A-modules $W = V^*$ such that z corresponds to the identity in $V \otimes W = V \otimes V^*$. But the identity $\sum y_i x_i = 1$ implies $1 \in V$, and V is a left ideal containing 1, etc.

So we have a canonical bimodule isomorphism $A \simeq A^*$. Recall A^* is the universal bimodule with a trace. Thus on A we have a canonical trace τ which is non degenerate (the bilinear form $\tau(xy)$ is non degenerate).

It turns out that τ is the trace on the regular representation. For example, consider $A = \mathbb{C}[G]$, where $z = \frac{1}{|G|} \sum g \otimes g^{-1}$. In general $\tau \in A^*$ is the unique linear functional such that $\sum x_i \tau(y_i) = 1$. This gives $\tau(g) = \begin{cases} 0 & g \neq 1 \\ |G| & g = 1 \end{cases}$

In the case of $A = M_n\mathbb{C}$, we have $z = \frac{1}{n} \sum e_{ij} \otimes e_{ji}$ so the dual basis to $\{e_{ij}\}$ is $\{\frac{1}{n} e_{ji}\}$, which means $\tau(e_{ij}) = n\delta_{ij} = n\, tr(e_{ij})$. Thus $\tau = n \times$ matrix trace, so $\tau =$ trace of left mult. on $M_n\mathbb{C}$.

In general suppose A finite diml and let $\tau(a) =$ tr of left mult by a on A. Assume that the bilinear form $\tau(a_1 a_2)$ is non-degenerate. Let $\{x_i\}$ be a basis for A and $\{y_i\}$ the dual basis so that $\tau(x_i y_j) = \delta_{ij}$.

Consider $z = \sum x_i \otimes y_i \in A \otimes A$.

Certainly $\sigma z = z$ because the quadratic form $\tau(a_1 a_2)$ is symmetric. Also $az = za$, $a(\sigma z) = (\sigma z)a$ should follow from the fact that τ gives a bimodule isom $A \xrightarrow{\sim} A^*$. Let's check directly. Suppose given $a \in A$, let's find the matrix relative to the basis x_i. We have $a x_i = \sum_j a_{ji} x_j$ so

$$\tau(y_j a x_i) = \tau(a x_i y_j) = a_{ji}$$

In particular $\tau(a \sum_i x_i y_i) = \sum a_{ii} = \tau(a)$ for all a, so by non-degeneracy $m(z) = \sum x_i y_i = 1$.

Note that in the case $A = \mathbb{C}[G]$ with

$$Y = \frac{1}{|G|} \sum_{g \in G} g^{-1} dg = -\frac{1}{|G|} \sum_{g \in G} dg^{-1} g = -\frac{1}{|G|} \sum_{g \in G} dg\, g^{-1}$$

we have for $h \in G$ that

$$[h, Y^2] = dh\, Y + Y dh$$

$$= \frac{1}{|G|} \sum_{g \neq 1, h} \left(dh\, g^{-1} dg - dg\, g^{-1} dh \right)$$

Since $\Omega^2 A = \Omega^1 A \otimes \bar{A} = \bar{A} \otimes A \otimes \bar{A}$, this will be nonzero once $|G| \geqslant 3$.

For A separable we know any two liftings in a nilpotent extension are conjugate. The universal situation is given by the two canonical maps $A \overset{\theta}{\underset{\theta^\gamma}{\rightrightarrows}} \hat{Q}A$, so there is an invertible g in $\hat{Q}A$ such that $\theta^\gamma = g\theta g^{-1}$. The question is whether it is possible to construct g starting from $Y \in \Omega^1 A$ such that $da = [a, Y]$ for all a.

Again let us consider pairs of homomorphisms $\theta, \bar{\theta} : A \longrightarrow R$ and try to construct a flow whose reverse tends to move them together. The flow is given by a vector field which at $\theta, \bar{\theta}$ gives a pair of derivations $\dot{\theta}, \dot{\bar{\theta}}$ with respect to $\theta, \bar{\theta}$ respectively. To first order (i.e. for square zero extensions) we have that $\theta - \bar{\theta}$ is a derivation relative to θ. ~~so~~ $\cancel{\theta - \bar{\theta} = [\theta, Y] - [\bar{\theta}, Y]}$ ~~and thus~~

$$\cancel{\bar{\theta} = \theta + [Y, \theta] = (1+Y)\theta(1+Y)^{-1}}$$

~~Thus we can take the flow~~

$\cancel{\dot{\theta}}$

Thus it is an inner derivation. In the universal first order case we have $(\theta - \bar{\theta})a = 2da = [a, 2Y]$, $= [\theta a, 2Y] = [\bar{\theta} a, 2Y]$ so we ought to be decreasing $\theta - \bar{\theta}$ using the reverse of a flow such that $\dot{\theta} - \dot{\bar{\theta}} = [\theta, Y(\theta, \bar{\theta})]$, where $Y(\theta, \bar{\theta})$ denotes the image of $Y \in \Omega^1 A \subset QA$ under the homom. $QA \longrightarrow R$ associated to $\theta, \bar{\theta}$.

One possibility then is

$$\dot{\theta} = 0$$
$$\dot{\bar{\theta}} = -[\bar{\theta}, Y(\theta,\bar{\theta})]$$

and a more symmetric possibility is

$$\dot{\theta} = [\theta, \tfrac{1}{2}Y(\theta,\bar{\theta})]$$
$$\dot{\bar{\theta}} = -[\bar{\theta}, \tfrac{1}{2}Y(\theta,\bar{\theta})]$$

This corresponds to the derivation D on QA given by

$$D(a+da) = [a+da, \tfrac{1}{2}Y]_0$$
$$D(a-da) = -[a-da, \tfrac{1}{2}Y]_0$$

i.e.

$$* \quad \begin{cases} Da = [da, \tfrac{1}{2}Y]_0 = [da, \tfrac{1}{2}Y]_- \\ D(da) = [a, \tfrac{1}{2}Y]_0 = \tfrac{1}{2}[a,Y] - \tfrac{1}{2}\overset{da\,dY + dY\,da}{\cancel{[da, dY]}} \end{cases}$$

see 468 for signs

Recall that we had 2 choices for a 1-cochain ϕ with $-(\delta\phi)(a_1,a_2) = da_1 da_2$ in the case of a separable algebra: $-Y da$ and $da\, Y$. The average is $[da, \tfrac{1}{2}Y]_- = \phi(a)$. Then we have $d\phi = -\tfrac{1}{2}[da, dY]_+$, so that the derivation $*$ has the form encountered before

$$Da = \phi a$$
$$D(da) = \tfrac{1}{2}da + d\phi a$$

Next we want to construct the inner automorphism conjugating $\theta, \bar{\theta}$. The derivation D gives rise to the 1-parameter group of automorphisms e^{tD} of $\hat{Q}A$.

The autom. e^{tD} corresponds to the ~~pair~~ pair of homoms.

$$\theta_t = e^{tD}\theta \quad , \quad \bar\theta_t = e^{tD}\bar\theta \; : \; A \longrightarrow \hat{Q}A$$

which represent the evolution of $\theta, \bar\theta$ in ~~time~~ time. We have

$$\partial_t \theta_t = e^{tD} D\theta = e^{tD}[\theta, \tfrac{1}{2}Y]$$
$$= [\theta_t, \tfrac{1}{2}Y_t] \qquad\qquad \partial_t\bar\theta_t = -[\bar\theta_t, \tfrac{1}{2}Y]$$

where $Y_t = e^{tD}Y$. $\therefore \; Y_t = Y_t(\theta_t, \bar\theta_t)$

Suppose we define g_t by

$$\cancel{\partial_t g_t = g_t \tfrac{1}{2}Y_t \, g_t \qquad g_0 = 1}$$

~~Then~~ $\cancel{\partial_t(g_t \theta_t g_t^{-1}) = g_t\left(-\tfrac{1}{2}Y_t\theta_t + \partial_t\theta_t + \theta_t\tfrac{Y_t}{2}\right)g_t}$

$$\partial_t g_t = -\frac{Y_t}{2} g_t \qquad\qquad g_t = 1$$

Then

$$\partial_t(g_t^{-1}\theta_t g_t) = g_t^{-1}\left(+\frac{Y_t}{2}\theta_t + \dot\theta_t - \theta_t\frac{Y_t}{2}\right)g_t = 0$$
$$\partial_t(g_t\bar\theta_t g_t^{-1}) = g_t\left(-\frac{Y_t}{2}\bar\theta_t + \dot{\bar\theta}_t + \bar\theta_t\frac{Y_t}{2}\right)g_t^{-1} = 0$$

which means

$$\theta_t = g_t\theta g_t^{-1} \qquad\qquad \bar\theta_t = g_t^{-1}\bar\theta g_t$$

Look again at $\quad \dot g_t = -e^{tD}\left(\frac{Y}{2}\right)g_t$

$$= -\left(e^{tD}\frac{Y}{2}e^{-tD}\right)e^{tD}g_t.$$ Thus

$$\left(e^{-tD} g_t\right)^{\bullet} = -D\, e^{-tD} g_t + e^{-tD}\left(-e^{tD}\left(\tfrac{Y}{2}\right) g_t\right)$$
$$= -D\, e^{-tD} g_t - \tfrac{Y}{2}\, e^{-tD} g_t$$

i.e. $$\left(e^{-tD} g_t\right)^{\bullet} = -\left(D + \tfrac{Y}{2}\right) e^{-tD} g_t$$

Thus $$g_t = e^{tD}\, e^{-t\left(D+\frac{Y}{2}\right)} 1$$

A natural question is whether g_t has a limit as $t \to -\infty$. Recall

$$\partial_t g_t = -\frac{Y_t}{2} g_t$$

where $Y_t = e^{tD}(Y)$. We know D has the eigenvalues $\frac{1}{n}n$, $n \in \mathbb{N}$, so $Y_t \to 0$ rapidly as $t \to -\infty$, expenentially in fact. So it would seem that $g_{-\infty}$ ~~exists~~ exists. Alternatively we can change to the variable $e^{\frac{1}{2}t} = z$. Then it becomes

$$z\,\partial_z\, g_z = -Y_z\, g_z$$

where $Y_z = z^{2D} Y$ should be a power series in z with 0 constant term since $Y \in \Omega \mathring{A}$ on which the eigenvalues of D are $\frac{1}{2}, 1, \frac{3}{2}, \ldots$ Thus $z = 0$ is a regular point of the D.E.

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* $$Da = da \circ \tfrac{1}{2} Y - \tfrac{1}{2} Y \circ da = \left[da, \tfrac{1}{2} Y\right].$$

$$D(da) = a \circ \tfrac{1}{2} Y - \tfrac{1}{2} Y \circ a = \tfrac{1}{2}\left([a, Y]_- - da\,dY - dY\,da\right)$$

since $$Y \circ a = Ya - (-1)^{|Y|} dY\,da$$
$$= Ya + dY\,da.$$

<u>July 24, 1991</u>

Review: Given ϕ with $-(\delta\phi)(a_1,a_2) = da_1 da_2$ we have the flow on $\bar{R}A$ given by

$$Da = \phi a$$

We know this extends to $\bar{Q}A$ by

$$D(da) = \tfrac{1}{2}da + d\phi a$$

In effect it suffices to verify that $D\theta : A \longrightarrow QA$ is a derivation relative to θ and $D\theta^{\gamma}$ is a derivation relative to θ^{γ}. This means checking

$$(D\theta)(a_1a_2) = (D\theta)(a_1)\circ\theta a_2 + \theta(a_1)\circ D\theta(a_2)$$

for $D\theta(a) = D(a+da) = \frac{1}{2}da + \phi a + d\phi a$ which is straightforward. Similarly for $\theta^{\gamma}a = a - da$

Next suppose A separable + let $Y \in \Omega^1 A$ satisfy $da = [a, Y]$. Then we can define a derivation D on QA by

$$D\theta = [\theta, Y] \qquad D\theta^{\gamma} = -[\theta^{\gamma}, Y]$$

This is equivalent to
$$\begin{cases} Da = [da, Y]^{\circ} = daY - Yda \\ D(da) = [a, Y]^{\circ} = [a, Y] - dadY - dYda \end{cases}$$

In effect $Y \circ a = Ya - (-1)^{|Y|} dY da = Ya + dYda$.

Recall that $\phi(a) = -Yda = -(Y\cup d)a$ and $daY = (d\cup Y)(a)$ satisfy $-\delta\phi = d\cup d$. $Da = daY - Yda = 2\phi(a)$ where $\phi = \frac{1}{2}(d\cup Y - Y\cup d)$ is the average. Note that $[a, Y] = da$ and $d(daY - Yda) = -(dadY + dYda)$.

Thus this derivation D of QA is

$$Da = 2\phi a$$
$$D(da) = da + 2d\phi a$$

which is twice the one considered before.

Now that we have the flow we define $\Theta_t : A \longrightarrow \hat{Q}A$ to be $e^{tD}\Theta_t$, and similarly define Θ_t^{δ}. Let g_t be the solution of

$$\partial_t g_t = -\underbrace{e^{tD}(Y)}_{Y_t} g_t \qquad g_0 = 1$$

Then $$\partial_t(g_t^{-1}\Theta_t g_t) = g_t^{-1}(+Y_t\Theta_t + \partial_t\Theta_t - \Theta_t Y_t)g_t = 0$$

Yesterday we argued that because the eigenvalues of D on $\hat{Q}A$ are $n \in \mathbb{N}$, we can define z^D where $z = e^t$. Then

$$z\partial_z g_z = -z^D(Y)\, g_z$$

and because $Y \in$ augmentation ideal of QA, $z^D(Y)$ is divisible by z. Thus

$$\partial_z g_z = -z^{D-1}(Y)\, g_z$$

has $z=0$ as regular point and $\lim_{t\to-\infty} g_t = g_z|_{z=0}$ exists.

I want to carry this all out in the case $A = \mathbb{C}[F]$. We have

$$dF\,Y = dF\,\tfrac{1}{2}F dF = -Y dF$$

so that there is a ~~~~ canonical ϕ around. We have

$$DF = 2\phi(F) = -FdF^2$$

$$D(dF) = dF + 2d\phi F = dF - dF^3$$

We calculated already

$$D\left((1-dF^2)^{-1/2}\right) = dF^2(1-dF^2)^{-1/2}$$

$$d\left(F(1-dF^2)^{-1/2}\right) = 0$$
$$d\left(dF(1-dF^2)^{-1/2}\right) = dF(1-dF^2)^{-1/2}$$

and further that the isomorphism $\hat{Q}A \simeq \hat{\Omega}A$ associated to D is

$\hat{Q}A$	$\sim$	$\hat{\Omega}A$
$F(1-dF^2)^{-1/2}$		F
$dF(1-dF^2)^{-1/2}$		dF
F		$F(1+dF^2)^{-1/2}$
dF		$dF(1+dF^2)^{-1/2}$

(From $dF(1-dF^2)^{-1/2} \Longleftrightarrow dF$ we obtain
$dF \longleftrightarrow dF(1+dF^2)^{-1/2}$: $\quad \frac{x}{\sqrt{1-x^2}} = y \Rightarrow x = \frac{y}{\sqrt{1+y^2}}$
Also $\frac{1}{\sqrt{1-x^2}} = \sqrt{1+y^2}$ i.e. $(1-dF^2)^{-1/2} \longleftrightarrow (1+dF^2)^{1/2}$)

Recall $\theta F = F + dF \longleftrightarrow (F+dF)(1+dF^2)^{-1/2}$
Apply $e^{tD} = z^D$ and use $D = n$ on $\Omega^n A$ in $\hat{\Omega}A$. This gives

$$\theta_t F \longleftrightarrow (F + z\,dF)(1+z^2 dF^2)^{-1/2}$$

We can check this by noting that the right side is the standard way to make $F + z\,dF$ into an involution:

$$(F + z\,dF)^2 = 1 + z\cancel{(FdF+FdF)} + z^2 dF^2$$

Thus we have a linear path $F + z\,dF$ which augments to an involution, but this is not perhaps a good viewpoint, since we didn't join

$$\theta F \longleftrightarrow F + dF/(1+dF^2)^{1/2}$$
$$\theta^{\gamma} F \longleftrightarrow F - dF/(1+dF^2)^{1/2}$$

by the linear path.

Observe that $\theta = \theta_z$ when $z = 1$ and $\theta^\gamma = \theta_z$ when $z = -1$ and this is obviously true in general.

The next thing to do is to find g_z. Recall that $g_t^{-1}\theta_t g_t$ is constant. We would like $g_t = 1$ at $t = -\infty$. This seems to be the most natural since we have the better parameter $z = e^t$ and we want to go directly between $z = 1$ and -1. So we want to find

$$\theta_z = g_z \theta_0 g_z^{-1} \qquad g_0 = 1.$$

where here θ_0 is the lifting $A \to \hat{R}A \subset \hat{Q}A$.

The diff'l equation defining g_z is

$$\partial_z g_z = - z^{D-1}(Y) g_z$$

I want to calculate the answer for $A = \mathbb{C}[F]$. Use the $\hat{\Omega}A$ model where $z^D = z^n$ on $\Omega^n A$.

$$Y = \tfrac{1}{2} FdF \longleftrightarrow \tfrac{1}{2} FdF\left(1+dF^2\right)^{-1}$$

$$z^{D-1}(Y) \longleftrightarrow \underbrace{\tfrac{1}{2} FdF\left(1+z^2dF^2\right)^{-1}}_{\tfrac{1}{2} FdF\left(1 - z^2(FdF)^2\right)^{-1}}$$

Our DE is

$$\partial_z g_z = \underbrace{\frac{-\tfrac{1}{2}FdF}{1-z^2(FdF)^2}}\, g_z$$

Observe the coefficient is a function of FdF, so the values for different z commute. Thus we have an abelian group situation and the answer is found by integrating.

$$\partial_z \log g_z = -\tfrac{1}{2} F dF \,\tfrac{1}{2}\left(\frac{1}{1+zFdF} + \frac{1}{1-zFdF}\right)$$

$$= -\tfrac{1}{4}\left(\frac{FdF}{1+zFdF} + \frac{FdF}{1-zFdF}\right)$$

$$= -\tfrac{1}{4}\left(\partial_z \log(1+zFdF) - \partial_z \log(1-zFdF)\right)$$

$$g_z^{-4} = \frac{1+zFdF}{1-zFdF}$$

We know how to take a square root of this namely

$$g_z^{-4} = \frac{(1+zFdF)^2}{1+z^2dF^2} \implies g_z^{-2} = \frac{1+zFdF}{\sqrt{1+z^2dF^2}}$$

But let's bring in what we know about involutions. Given

$$F_z = (F+zdF)(1+z^2dF^2)^{-1/2}$$

$$F_0 = F$$

The natural solution for $F_z = g_z F_0 g_z^{-1}$ is obtained by requiring $F_0 g_z F_0 = g_z^{-1}$. Then

$$F_z = g_z F_0 g_z^{-1} = g_z^2 F_0$$

and

$$g_z^{-2} = F_0 F_z = \frac{1+zFdF}{\sqrt{1+z^2dF^2}}$$

To understand derivations $D: QA \to QA$ which commute with γ. [redacted]

In general a derivation D on QA is equivalent to a derivation $D\theta : A \to QA$ rel θ [redacted] together with a derivation $D\theta^\gamma : A \to QA$ rel θ^γ. If $D\gamma = \gamma D$, then $D\theta^\gamma = D\gamma\theta$ [redacted] $= \gamma D \theta$. Thus a D commuting with γ should be equivalent to a derivation $D\theta: A \to QA$ relative to θ, i.e. to a A-bimodule morphism $\Omega^1 A \to QA$ where QA is regarded as A-bimodule via θ.

Let's assume $\Omega^1 A$ is a projective bimodule and see what we can construct. We have an obvious bimodule morphism

$$\Omega^1 A \subset QA/JA^2 = A \oplus \Omega^1 A$$

semi direct product

which we can lift to a bimod morphism $\Omega^1 A \to JA$. Better is to say that we have $JA/JA^2 = \Omega^1 A$ and we wish to find a lifting $\Omega^1 A \to JA$ for the θ bimodule structure on JA. The arbitrariness if we try to lift up to JA/JA^{n+1} ~~[illegible]~~ from JA/JA^n is a bimodule map $\Omega^1 A \to \Omega^n A$, i.e. a derivation $A \to \Omega^n A$.

Concretely we can proceed with the following notation (which may be awkward) suppose the derivation $D\theta$ we seek is written out in [redacted] components

$$Da = \phi_2 a + \phi_4 a + \cdots \qquad = \phi a$$

$$D(da) = \psi_1 a + \psi_3 a + \cdots \qquad = \psi a$$

The relations to preserve are

$$a_1 a_2 = a_1 \circ a_2 + da_1 \circ da_2$$

$$d(a_1 a_2) = a_1 \circ da_2 + da_1 \circ a_2$$

This means

$$\phi(a_1 a_2) = a_1 \circ \phi a_2 + \phi a_1 \circ a_2$$
$$+ da_1 \circ \psi a_2 + \psi a_1 \circ da_2$$

which becomes

$$\boxed{\begin{aligned} +\delta\phi &= d\phi \cup d + d \cup d\phi \\ &\quad - \psi \cup d - d \cup \psi \end{aligned}}$$

Similarly we have

$$\boxed{\begin{aligned} \delta\psi &= d \cup d\psi - d\psi \cup d \\ &\quad - \phi \cup d - d \cup \phi \end{aligned}}$$

Using $\boxed{\delta d f = d\delta f - d \cup f - (-1)^{|f|} f \cup d}$

we have checked that the above equations are consistent with $\delta^2\phi = 0$ and $\delta^2\psi = 0$.

Better approach is to work with

$$D\Theta a = f_1 a + f_2 a + f_3 a + \cdots \quad = fa$$

The derivation condition is

$$\begin{aligned} f(a_1 a_2) &= D\Theta(a_1 a_2) \\ &= \Theta a_1 \circ D\Theta a_2 + D\Theta a_1 \circ \Theta a_2 \\ &= (a_1 + da_1) \circ f a_2 + f a_1 \circ (a_2 + da_2) \end{aligned}$$

$$= a_1 f a_2 - da_1\, df a_2 + da_1\, f a_2$$
$$+ f a_1\, a_2 - (-1)^{|f|} df a_1\, da_2 + f a_1\, da_2$$

or simply

$$* \quad \boxed{\delta f = -d \cup f - f \cup d + d \cup df + (-1)^{|f|} df \cup d}$$

where $(-1)^{|f|} df$ stands for $\sum (-1)^n df_n$.

[redacted] Presumably using
$$\delta df = d\delta f - d \cup f - (-1)^{|f|} f \cup d$$
the above equation $*$ for δf is consistent with $\delta^2 f = 0$

Componentwise $*$ becomes

$$** \quad \boxed{\delta f_n = -d \cup f_{n-1} - f_{n-1} \cup d + d \cup df_{n-2} + (-1)^n df_{n-2} \cup d}$$

and should be able to solve this recursively, the point being that as $\Omega^1 A$ is projective 1-cocycles are 1-coboundaries [redacted] so it's enough to know the right side is killed by δ; this should follow from the consistency just mentioned.

Consider this recursion relation for $n=1$ assuming $f_0 = 0$. Then $\delta f_1 = 0$, so $f_1 : \bar{A} \to \Omega^1 A$ is a derivation. We take $f_1 = d$ as mentioned above. Then the relation for $n=2$ is

$$\delta f_2 = -2 d \cup d \quad \text{i.e} \quad -\delta f_2 = 2 d \cup d$$

which means $f_2 = 2\phi$, ϕ as usual. Then

$$\delta f_3 = -d \cup f_2 - f_2 \cup d \ \text{[redacted]}$$

We have $\delta df_2 = \underbrace{d\delta f_2}_{d(-2d \cup d) = 0} - d \cup f_2 - f_2 \cup d$

so the obvious solution is

$$f_3 = df_2 = 2d\phi$$ as we have

seen. Then

$$\delta f_4 = d \cup (-f_3 + df_2) + (-f_3 + df_2) \cup d$$

so ~~if~~ we take $f_3 = df_2$, we can take $f_4 = 0$

and then $f_5 = f_6 = \cdots = 0$.

Let's remove this solution and try to understand solutions ~~starting~~ with

$$\delta f_3 = 0 \qquad \text{and} \quad f_1 = f_2 = 0$$

$$\delta f_4 = - d \cup f_3 - f_3 \cup d$$

$$\delta f_5 = - d \cup f_4 - f_4 \cup d + d \cup df_3 - df_3 \cup d$$

It's not clear whether this provides any more finite support solutions. Wait: Note

$$d\delta f_4 = + d \cup df_3 - df_3 \cup d$$

so $$\delta f_5 = - d \cup f_4 - f_4 \cup d + d\delta f_4$$

and this is $\delta d f_4$. So we have

$$\delta f_5 = \delta d f_4 \qquad \text{or} \qquad \delta(f_5 - df_4) = 0$$

~~Then~~ Next

$$\delta f_6 = + d \cup (-f_5 + df_4) + (-f_5 + df_4) \cup d$$

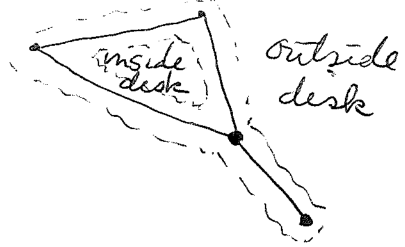
so if we take $f_5 = df_4$, then we can take $f_6 = 0$, and then $f_7 = 0$ since $f_6 = 0$ and $df_5 = 0$, etc.

This pattern ought to repeat, so we see that the construction of derivations of QA when A is quasi-free is basically a finite support business. There is a rather natural parametrization (depending on ϕ) of the possible choices for D in terms of sequences $\{g_n\}_{n\geq 1}$ where $g_n : A \longrightarrow \Omega^n A$ is a derivation.

July 30, 1991

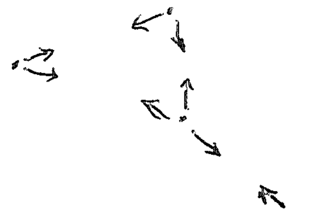
Graeme tells me about Kontsevich:

Ribbon graphs represent cells of moduli spaces. One glues in disks for each circuit compatible with the cyclic order



The parameters are the lengths of the edges - before glueing a disk one fixes the points on the boundary.

Given an associative algebra A with $\underbrace{a}_{\text{nondegenerate}}$ trace τ one gets a function on ribbon graphs which is unchanged under collapsing an edge (this corresponds to taking a face of the cell in moduli space - the dimension of the cell is the number of edges); hence an element of $H^0(\mathcal{M}_{g,n})$. Form $\bigotimes_I A$ where I is the set consisting of a vertex together with a direction of leaving the vertex.



Thus I = components of the graph with midpoints of edges and vertices removed.

Also I = ways of enriching the graph to a connected graph. I is a double cover of the set of edges. Because τ is a non degenerate trace it determines a symmetric element z in $A \otimes A$. Each edge determines a pair of elements of I

and the pairs are disjoint, hence we have $\prod_{e \text{ edge}} z_e \in \bigotimes_I A$

Next the cyclic order at each vertex v gives us a map

$$\tau_v : \bigotimes_{\substack{i \text{ assoc} \\ \text{to } v}} A \longrightarrow A_\natural \xrightarrow{\tau} \mathbb{C}$$

Thus we have $\prod_v \tau_v : \bigotimes_I A \longrightarrow \mathbb{C}$.

and so we get a number

$$\prod_v \tau_v \cdot \prod_e z_e$$

attached to the ribbon graph.

<u>Example</u>: If $A = \mathbb{C}[G]$ with the usual trace, then the number gives the number of principal G-bundles over the graph.

Supposedly the above construction apparently generalizes to A_∞ algebras and leads to elts of $H^i(\mathcal{M}_{g,n})$.

The marked points on the surfaces are the centers of the disks. As one moves over moduli space the boundary of the i-th disk is a sort of circle bundle. However because of finite automorphisms of the surface the structure is more complicated. Kontsevich says look at finite subsets of S^1 where points can coalesce but modulo rigid rotations. One then gets a two-form on this configuration space as follows. Let the angles starting from some point be $l_1, l_2, \ldots, l_n$ so that $\sum l_i = 2\pi$. Then the ~~two~~-form is

$$\sum_{i<j} dl_i\, dl_j \, .$$

The configuration space is a rational $K(\mathbb{Z}, 2)$ and this form represents the first Chern class.

August 16, 1991

Here are some ideas worth working on it seems.

1) You know that an element of $HC^2(A)$ can be represented by a square zero extension $A = R/I$, $I^2 = 0$ together with a trace on R. You also know that a square-zero extension of A is equivalent to a bimodule extension of $\Omega^1 A$. Can you describe what a trace on R is in terms of the bimodule extension? Hopefully you can find a categorical description of an element of $HC^2(A)$. Idea: Use that $HC_2(A)$ is computed by

$$R/[R,I] \rightleftarrows \Omega^1 R/[R,\Omega^1 R] + I\Omega^1 R$$

and that the odd space in this complex is the commutator quotient space of the extension of $\Omega^1 A$:

$$0 \longrightarrow I \longrightarrow A \otimes_R \Omega^1 R \otimes_R A \longrightarrow \Omega^1 A \longrightarrow 0$$

(The feeling I have is that trace on a nilpotent extension of A is too rigid a notion. One wants to replace it by bimodule data probably involving non linearity in some way.)

Important examples here are tensor products and Heisenberg-Weyl algebras.

2) Consider Morita self-equivalences of $\mathbb{C}$. This should be the Picard category of lines over $\mathbb{C}$. (at least if we stick to $\mathbb{C}$ linear functors). This Picard category has 1 object up to isom and ~~its~~ automorphisms are $\mathbb{C}^\times$. So

it is $B\mathbb{C}^\times$. However if I somehow take topology into account I get $BS^1 = \mathbb{C}P^\infty$.

Similarly in connection with Lundell one looks at the (non-unital) homomorphisms

$$M_n \longrightarrow M_N$$

such that $1 \longmapsto$ idempotent of rank n (these are the ones consistent with the Morita equivalence of both matrix algebra with $\mathbb{C}$). The topology of these homom. is the same as the space

$$U_N / S^1 \times U_{N-n}$$

which in the $N \to \infty$ limit is BS^1.

This suggests to me that the ultimate nature of the S-operation is to be found in the understanding of Morita equivalences.

1991