April 7, 1991

Let us consider square zero algebra extensions

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

where A is commutative and M is an A-module (= A-bimodule such that $am=ma$). Such extensions are classified up to isomorphism by $H^2(A,M)$. Let's discuss things first "geometrically" without reference to cohomology. Write $\mathrm{Exalg}(A,M)$ for the set of isom. classes of these extensions. This is an A-module in a natural way.

The first result is a decomposition

$$(1) \qquad \mathrm{Exalg}(A,M) = \mathrm{Exalgcomm}(A,M) \oplus \mathrm{Hom}_A(\Omega^2 A, M)$$

which can be described as follows. For any extension as above we have $[B,B] \subset M$, hence $[B,[B,B]] = 0$. This implies we can define a commutative product on B by $x \cdot y = \frac{1}{2}(xy+yx)$. Also the bracket $[x,y]$ is a skew symmetric biderivation from A to M, equivalently, an A-module map $\Omega^2 A \longrightarrow M$. We have a canonical extension R_A/I_A^2 of A by Ω^2_A where $[a_1,a_2] = da_1 da_2$, hence an A-module map $\Omega^2 A \longrightarrow M$ gives rise by pushout to an extension of A by M. I should have said that (1) is the correspondence with associates to $A = B/M$ the associated comm. alg extension and the bracket map. (In fact what we really have is an exact sequence

$$0 \longrightarrow \mathrm{Exalgcomm}(A,M) \longrightarrow \mathrm{Exalg}(A,M) \longrightarrow \mathrm{Hom}_A(\Omega^2 A, M)$$

which we have split canonically using $\frac{1}{2}$.)

The extensions coming from $\mathrm{Hom}_A(\Omega^2_A, M)$ are those for which the associated commutative algebra extension is trivial, i.e. ~~such~~ that we ~~can~~ find a $\rho : A \to B$ satisfying $\rho(a_1 a_2) = \frac{1}{2}(\rho a_1 \rho a_2 + \rho a_2 \rho a_1)$.

Corollary: A is formally smooth (nilpotent commutative algebra extensions split) iff

$$\mathrm{Exalg}(A, M) \xrightarrow{\sim} \mathrm{Hom}_A(\Omega^2 A, M).$$

Next let's consider the cohomological side. We have

$$\mathrm{Exalg}(A, M) = H^2(A, M) = \mathrm{Ext}^2_{A\otimes A}(A, M)$$

$$= \mathrm{Ext}^1_{A\otimes A}(\Omega^1 A, M).$$

In general if N is an $A\otimes A$-module we have a spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_A\left(\mathrm{Tor}_q^{A\otimes A}(N, A), M\right) \Rightarrow \mathrm{Ext}^{p+q}_{A\otimes A}(N, M)$$

Take $N = \Omega^1 A$ and use $\mathrm{Tor}_0^{A\otimes A}(\Omega^1 A, A) = \Omega^1_A$, $\mathrm{Tor}_1^{A\otimes A}(\Omega^1 A, A) = H_2(A, A)$. Then we have the following 5 term exact sequence associated to the spectral sequence.

$$0 \to \mathrm{Ext}^1_A(\Omega^1_A, M) \to H^2(A, M) \to \mathrm{Hom}_A(H_2(A,A), M)$$

(*)

$$\to \mathrm{Ext}^2_A(\Omega^1_A, M) \to .$$

Further, we know in general that there are canonical maps (A commutative)

$$\Omega^n_A \longrightarrow H_n(A, A) \longrightarrow \Omega^n_A$$

making Ω^n_A a direct summand of $H_n(A, A)$. The latter map is induced by the map $\Omega A \to \Omega_A$, and the former from the algebra ~~structure~~ on $H_\cdot(A,A)$ (shuffle product). (Remark that $\Omega^n_A \to H_n(A, A)$ is in Hochschild-Kostant-Rosenberg; it amounts to the fact that anti~~symmetrization~~ of a Hochschild cocycle gives a current and that antisymmetrization of a Hochschild coboundary gives 0.)

Thus Ω^2_A is a direct summand of $H_2(A,A)$. From the exact sequence ⊛ and the preceding corollary we obtain

<u>Corollary:</u> A is formally smooth iff Ω^1_A is a projective A-module and $\Omega^2_A = H_2(A,A)$.

April 11, 1991

Tensor products. Recall that we have a natural surjection

$$\alpha_{R,S} : X(R*S) \longrightarrow X(R) \otimes X(S)$$

obtained as follows. Let $I = I_{R,S}$ be the ideal in $R*S$ defined by

$$I_{R,S} = (R*S)[R,S](R*S)$$

that is, it is the ideal generated by $[x,y]$ for $x \in R$, $y \in S$. We have

$$(*) \qquad \begin{tikzcd} R*S/I^2 & R\otimes S \oplus \Omega^1R \otimes \Omega^1 S \arrow[l, "\sim"'] \\ xy & x \otimes y \arrow[l, mapsto] \\ x_0y_0[x_1,y_1] & x_0dx_1 \otimes y_0dy_1 \arrow[l, mapsto] \end{tikzcd}$$

$\alpha_{R,S}$ is then given by the canonical surjection

$$X(R*S) \longrightarrow X^I(R*S, I)$$

followed by a canonical identification of the latter with $X(R) \otimes X(S)$:

$$\begin{tikzcd} R*S/I^2 + [R*S, I^2] \arrow[r, shift left] \arrow[d, equal] & \left(\Omega^1(R*S)/I\Omega^1(R*S)\right)_\natural \arrow[l, shift left] \arrow[d, equal] \\ R\otimes S \oplus \Omega^1 R_\natural \otimes \Omega^1 S_\natural & \Omega^1 R_\natural \otimes S \oplus R \otimes \Omega^1 S_\natural \end{tikzcd}$$

What is important here are the isomorphism $(*)$ above and the $R\otimes S$-bimodule isomorphism

$$\Omega^1(R*S)/F^1_I\,\Omega^1(R*S) \xrightarrow{\sim} S\otimes\Omega^1R\otimes S \oplus R\otimes\Omega^1S\otimes R$$

Formula for $\alpha_{R,S}$. In degree 0 one uses

$$R*S = R\otimes S \oplus RS[R,S] \oplus I^2_{R,S}$$

to decompose an element of $R*S$ into elements $xy \in RS$, and $x_0y_0[x_1,y_1] \in RS[R,S]$, and elts of $I^2_{R,S}$.
Then $\alpha_{R,S}(xy) = x\otimes y \in R\otimes S$

$$\alpha_{R,S}\left(x_0y_0[x_1,y_1]\right) = x_0dx_1 \otimes y_0dy_1 \in \Omega^1R_\natural \otimes \Omega^1S_\natural$$

and $\alpha_{R,S}(I^2_{R,S}) = 0$. In degree 1 use

$$\Omega^1(R*S) = (R*S)\otimes_R \Omega^1R\otimes_R(R*S) \oplus (R*S)\otimes_S\Omega^1S\otimes_S(R*S)$$
$$= S\otimes\Omega^1R\otimes S \oplus R\otimes\Omega^1S\otimes R \oplus F^1_I(\Omega^1(R*S))$$

to decompose any element of $\Omega^1(R*S)$ into elements

$$y_1x_0dx_1y_2 \xmapsto{\alpha_{R,S}} x_0dx_1 \otimes y_2y_1 \in \Omega^1R_\natural\otimes S$$
$$x_1y_0dy_1x_2 \xmapsto{\;\;''\;\;} x_2x_1\otimes y_0dy_1 \in R\otimes\Omega^1S_\natural$$
$$F^1_I(\Omega^1(R*S)) \xmapsto{\;\;''\;\;} 0$$

Let us next consider $R*S*T$. The problem is associativity, i.e. whether

$$\begin{tikzcd}
X(R*S*T) \arrow[r, "\alpha_{R,S*T}"] \arrow[d, "\alpha_{R*S,T}"] & X(R)\otimes X(S\otimes T) \arrow[d, "1\otimes\alpha_{S,T}"] \\
X(R*S)\otimes X(T) \arrow[r, "\alpha_{R,S}\otimes 1"'] & X(R)\otimes X(S)\otimes X(T)
\end{tikzcd}$$

commutes. In other to check this we would to find a suitable quotient of $X(R*S*T)$ through which both maps factor. Then it's enough to check equality on enough elements of the quotient complex.

April 12, 1991

Calculations in $R*S/I^2_{R,S}$. Recall that $R*S \xrightarrow{\sim} \bigoplus_{n\ge 0} \Omega^n R \otimes \Omega^n S$

$$x_0 y_0 [x_1,y_1]\cdots[x_n,y_n] \longleftrightarrow x_0\, dx_1 \cdots dx_n \otimes y_0\, dy_1 \cdots dy_n$$

and that the product in $R*S$ corresponds under this isom. to

$$(\xi_1\otimes\eta_1)\circ(\xi_2\otimes\eta_2) = \xi_1\xi_2\otimes\eta_1\eta_2 - (-1)^{|\xi_1|}\xi_1\, d\xi_2 \otimes d\eta_1\,\eta_2$$

Thus left multiplication by x and y on $R*S$ are respectively the operators

$$(x\circ) = x\otimes 1 \qquad\qquad (y\circ) = 1\otimes y - d\otimes dy$$

on forms.

Let's calculate $x_1y_1\cdots x_ny_n$ in $R*S/I^2_{R,S}$. We have

$$x_1y_1\cdots x_ny_n \longleftrightarrow (x_1\otimes y_1 - x_1 d\otimes dy_1)(x_2\otimes y_2 - x_2 d\otimes dy_2) \cdots\cdots (x_n\otimes y_n - x_n d\otimes dy_n)(1\otimes 1)$$

$$= x_1\cdots x_n\otimes y_1\cdots y_n - \sum_{j=1}^{n} (x_1\otimes y_1)\cdots(x_{j-1}\otimes y_{j-1})(x_j d\otimes dy_j)(x_{j+1}\otimes y_{j+1})\cdots(x_n\otimes y_n)(1\otimes 1)$$

$$= x_1\cdots x_n\otimes y_1\cdots y_n - \underbrace{\sum_{j=1}^{n} x_1\cdots x_j\, d(x_{j+1}\cdots x_n)\otimes y_1\cdots y_{j-1}\,dy_j\, y_{j+1}\cdots y_n}_{\shortparallel}$$

$$\sum_{j<i} x_1\cdots x_{i-1}\,dx_i\, x_{i+1}\cdots x_n \otimes y_1\cdots y_{j-1}\,dy_j\, y_{j+1}\cdots y_n$$

Thus

$$\boxed{\begin{aligned} x_1y_1\cdots x_ny_n \equiv\ & x_1\cdots x_n\, y_1\cdots y_n \\ & - \sum_{1\le j<i\le n} x_1\cdots x_{i-1}\, y_1\cdots y_{j-1}\,[x_i,y_j]\,x_{i+1}\cdots x_n\, y_{j+1}\cdots y_n \\ & (\mathrm{mod}\ I^2_{R,S}) \end{aligned}}$$

Let's check that $I_{R,S}^2$ is the smallest ideal modulo which this calculate holds. First note that

$$\begin{aligned} I_{R,S} &= (R*S)[R,S](R*S) \\ &= (R*S)[R,S] \\ &= [R,S](R*S) \end{aligned}$$

because of $R[R,S] = [R,S]R$, $S[R,S] = [R,S]S$.

Thus

$$I_{R,S}^2 = \text{[redacted]}\ (R*S)[R,S]^2$$

which is clear also from our formula for $R*S$. On the other hand

$$y_1 x_2 y_2 x_3 \longleftrightarrow (1\otimes y_1 - d\otimes dy_1)(x_2\otimes y_2 - x_2 d\otimes dy_2)(x_3\otimes 1)$$

$$= (1\otimes y_1 - d\otimes dy_1)(x_2x_3\otimes y_2 - x_2dx_3\otimes dy_2)$$

$$\begin{aligned} = {} & x_2x_3\otimes y_1y_2 - x_2dx_3\otimes y_1dy_2 \\ & - d(x_2x_3)\otimes \boxed{dy_1y_2} + dx_2dx_3\otimes dy_1dy_2 \end{aligned}$$

$$d(y_1y_2) - y_1dy_2$$

~~$x_2x_3\otimes y_1y_2$~~

~~$x_2dx_3\otimes y_1dy_2 - d(x_2x_3)$~~

$$\begin{aligned} \therefore\quad y_1x_2y_2x_3 = {} & x_2x_3y_1y_2 - x_2y_1[x_3,y_2] \\ & - [x_2x_3, y_1y_2] + y_1[x_2x_3, y_2] \\ & + [x_2,y_1][x_3,y_2] \end{aligned}$$

$$\begin{aligned} \text{[redacted]} = {} & x_2x_3y_1y_2 - x_2y_1[x_3,y_2] + [x_2,y_1][x_3,y_2] \\ & - \boxed{[x_2x_3, y_1]y_2} \end{aligned}$$

$$\begin{aligned} & x_2[x_3,y_1]y_2 \\ & + [x_2,y_1]x_3y_2 \end{aligned}$$

$$\boxed{\begin{aligned} y_1x_2y_2x_3 = {} & x_2x_3y_1y_2 - \big([x_2,y_1]x_3y_2 + x_2[x_3,y_1]y_2 + x_2y_1[x_3,y_2]\big) \\ & + [x_2,y_1][x_3,y_2] \end{aligned}}$$

This shows that the calculation at the bottom of p336 holds modulo an ideal iff $[R,S]^2 = 0$ modulo that ideal.

Actually I ought to derive the above formula directly without reference to differential forms just to get some practice.

$$\begin{aligned} yxy'x' &= xyy'x' - [x,y]y'x' \\ &= xx'yy' - \left(x[x',yy'] + [x,y]x'y'\right) + [x,y][x',y'] \end{aligned}$$

Also

$$\begin{aligned} yxy'x' &= yxx'y' - yx[x',y'] \\ &= xx'yy' - \left([xx',y]y' + xy[x',y']\right) + [x,y][x',y'] \end{aligned}$$

In both cases we get

$$\begin{aligned} yxy'x' = xx'yy' &- \left([x,y]x'y' + x[x',y]y' + xy[x',y']\right) \\ &+ [x,y][x',y'] \end{aligned}$$

Recall the map

$$\Phi : \bigoplus_{n\geq 0} \Omega^n R \otimes \Omega^n S \xrightarrow{\sim} R * S$$

$$x_0 dx_1 \dots dx_n \otimes y_0 dy_1 \dots dy_n \longmapsto x_0 y_0 [x_1, y_1] \dots [x_n, y_n]$$

Let's compute for $n=1$ the consequences of changing order.

$$\begin{aligned} \Phi(x_0 dx_1 \otimes dy_1 y_2) &= \Phi\left(x_0 dx_1 \otimes (d(y_1 y_2) - y_1 dy_2)\right) \\ &= x_0[x_1, y_1 y_2] - x_0 y_1 [x_1, y_2] = x_0[x_1, y_1] y_2 \end{aligned}$$

$$\begin{aligned} \Phi(dx_1 x_2 \otimes dy_1 y_2) &= \Phi\left((d(x_1 x_2) - x_1 dx_2) \otimes dy_1 y_2\right) \\ &= [x_1 x_2, y_1] y_2 - x_1 [x_2, y_1] y_2 = [x_1, y_1] x_2 y_2 \end{aligned}$$

$$\begin{aligned} \Phi(dx_1 x_2 \otimes y_0 dy_1) &= \Phi\left((d(x_1 x_2) - x_1 dx_2) \otimes y_0 dy_1\right) \\ &= y_0[x_1 x_2, y_1] - x_1 y_0 [x_2, y_1] = y_0 [x_1, y_1] x_2 - [x_1, y_0][x_2, y_1] \end{aligned}$$

So we have

$$\boxed{\begin{aligned}
\Phi(x_0 dx_1 \otimes y_0 dy_1) &= x_0 y_0 [x_1, y_1] \\
\Phi(x_0 dx_1 \otimes dy_1 y_2) &= x_0 [x_1, y_1] y_2 \\
\Phi(dx_1 x_2 \otimes dy_1 y_2) &= [x_1, y_1] x_2 y_2 \\
\Phi(dx_1 x_2 \otimes y_0 dy_1) &= y_0 [x_1, y_1] x_2 - [x_1, y_0][x_2, y_1]
\end{aligned}}$$

Let's calculate $y_1 x_1 \cdots y_n x_n$ in $R * S$.

It corresponds to

$$\prod_{j=1}^{n} (1 \otimes y_j - d \otimes dy_j)(x_j \otimes 1) \cdot (1 \otimes 1)$$

$$= \prod_{j=1}^{n} (x_j \otimes y_j - d \cdot x_j \otimes dy_j) \quad (1 \otimes 1)$$

$$= \quad x_1 \cdots x_n \otimes y_1 \cdots y_n$$

$$- \sum_{a=1}^{n} x_1 \cdots x_{a-1}\, d(x_a x_{a+1} \cdots x_n) \otimes y_1 \cdots y_{a-1}\, dy_a\, y_{a+1} \cdots y_n$$

$$+ \sum_{1 \le a < b \le n} x_1 \cdots x_{a-1}\, d(x_a \cdots x_{b-1})\, d(x_b \cdots x_n) \otimes y_1 \cdots y_{a-1}\, dy_a\, y_{a+1} \cdots y_{b-1}\, dy_b\, y_{b+1} \cdots y_n$$

$$- \sum_{1 \le a < b < c \le n} x_1 \cdots x_{a-1}\, d(x_{a+1} \cdots x_{b-1})\, d(x_b \cdots x_{c-1})\, d(x_c \cdots x_n) \otimes y_1 \cdots y_{a-1}\, dy_a \cdots dy_b \cdots dy_c \cdots$$

. . . .

Now note

$$\Phi(dx_1 dx_2 \otimes y_0 dy_1 \overbrace{y' dy_2}^{d(y'y_2) - dy' y_2}) =$$

$$y_0 [x_1, y_1]\big([x_2, y'y_2] - [x_2, y'] y_2\big) = y_0 [x_1, y_1] y' [x_2, y_2]$$

Also $\Phi(dx_1 dx_2 \otimes y_0 \underbrace{dy_1\, y'}_{d(y_1 y') - y_1 dy'} dy_2)$

$$= y_0[x_1, y_1 y'][x_2, y_2] - y_0 y_1 [x_1, y'][x_2, y_2]$$

$$= y_0 [x_1, y_1] y' [x_2, y_2]$$

similarly it should be true that

$$\Phi(dx_1 dx_2 \cdots dx_n \otimes y_1' dy_1 y_2' dy_2 \cdots y_n' dy_n y_{n+1}')$$

$$= y_1' [x_1, y_1] y_2' [x_2, y_2] y_3' \cdots [x_n, y_n] y_{n+1}'$$

Thus we ought to have in $R * S$

$$y_1 x_1 \cdots y_n x_n = x_1 \cdots x_n y_1 \cdots y_n$$

$$- \sum_{1 \le a \le n} x_1 \cdots x_{a-1} y_1 \cdots y_{a-1} [x_a \cdots x_n, y_a] y_{a+1} \cdots y_n$$

$$+ \sum_{1 \le a_1 < a_2 \le n} x_1 \cdots x_{a_1-1} y_1 \cdots y_{a_1-1} [x_{a_1} \cdots x_{a_2-1}, y_{a_1}] y_{a_1+1} \cdots y_{a_2-1} [x_{a_2} \cdots x_n, y_{a_2}] y_{a_2+1} \cdots y_n$$

$$- \cdots$$

General term will be for $a_1 < \cdots < a_k$ and appears

$$(-1)^k x_1 \cdots x_{a_1-1} y_1 \cdots y_{a_1-1} [x_{a_1} \cdots x_{a_2-1}, y_{a_1}] y_{a_1+1} \cdots y_{a_2-1} [x_{a_2} \cdots x_{a_3-1}, y_{a_2}] y_{a_2+1} \cdots$$

Now we can expand these out to get a sum of terms indexed by $a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k \le n$, which appears

$$(-1)^k x_1 \cdots x_{a_1-1} y_1 \cdots y_{a_1-1} x_{a_1} \cdots x_{b_1-1} [x_{b_1}, y_{a_1}] x_{b_1+1} \cdots x_{a_2-1} y_{a_1+1} \cdots y_{a_2-1} x_{a_2} \cdots x_{b_2-1}$$

$$\times [x_{b_2}, y_{b_2}] y_{a_2+1} \cdots y_{b_2-1} \cdots$$

This is not very illuminating. What might be better is to rewrite $y_1 \cdots dy_{a_1} \cdots dy_{a_2} \cdots$ in terms of $y \cdots d(y_{a_1} \cdots) d(y_{a_2} \cdots) \cdots$

Consider next $F = R * S * T$ and let J be the ideal generated by $[R,S]$, $[R,T]$, $[S,T]$ so that $F/J = R\otimes S\otimes T$. We have

$$J = F([R,S] + [R,T] + [S,T])F$$

It might be useful to study the square zero extension F/J^2 of $R\otimes S\otimes T$. We have the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & J/J^2 \arrow[r] & \Omega^1(F)/F^1_J(\Omega^1 F) \arrow[r] \arrow[d, equal] & (R\otimes S\otimes T)^2 \arrow[r] & R\otimes S\otimes T \\
& & \begin{array}{c} (R\otimes S\otimes T)\otimes_R \Omega^1 R\otimes_R (R\otimes S\otimes T) \\ \oplus \quad \text{———}\otimes_S \Omega^1 S\otimes_S \text{———} \\ \oplus \quad \text{———}\otimes_T \Omega^1 T\otimes_T \text{———} \end{array} & &
\end{tikzcd}$$

so it should be true that one has exact sequences

$$0 \longrightarrow J/J^2 \longrightarrow \begin{array}{c} \Omega^1 R\otimes(S\otimes S)\otimes(T\otimes T) \\ \oplus\, (R\otimes R)\otimes \Omega^1 S\otimes(T\otimes T) \\ \oplus\, (R\otimes R)\otimes(S\otimes S)\otimes\Omega^1 T \end{array} \longrightarrow (R\otimes R)\otimes(S\otimes S)\otimes(T\otimes T)$$

$$0 \longrightarrow \Omega^1 R\otimes\Omega^1 S\otimes\Omega^1 T \longrightarrow \begin{array}{c} \Omega^1 R\otimes\Omega^1 S\otimes(T\otimes T) \\ \oplus\, \Omega^1 R\otimes(S\otimes S)\otimes\Omega^1 T \\ \oplus\, (R\otimes R)\otimes\Omega^1 S\otimes\Omega^1 T \end{array} \longrightarrow J/J^2 \longrightarrow 0$$

Observe that J^2 is the sum of 9 ideals

$F[R,S]F[R,S]F$, $\quad F[R,T]F[R,S]F$, $\quad F[S,T]F[R,S]F$

$F[R,S]F[R,T]F$, $\quad F[R,T]F[R,T]F$, $\quad F[S,T]F[R,T]F$

$F[R,S]F[S,T]F$, $\quad F[R,T]F[S,T]F$, $\quad F[S,T]F[S,T]F$

We are concerned with the two maps

$$\begin{tikzcd}
X(R*S*T) \arrow[r, "\alpha_{R,S*T}"] \arrow[d, "\alpha_{R*S,T}"] & X(R)\otimes X(S*T) \arrow[d, "1\otimes\alpha_{S,T}"] \\
X(R*S)\otimes X(T) \arrow[r, "\alpha_{R,S}\otimes 1"] & X(R)\otimes X(S)\otimes X(T)
\end{tikzcd}$$

and whether they agree. ~~[illegible]~~ Let's try to compute $(\alpha_{R,S}\otimes 1)\alpha_{R*S,T}$ by finding a quotient complex of $X(R*S*T)$ to which this map descends. We know that $\alpha_{R,S}$ descends to $X'(R*S, I_{R,S})$. Thus $(\alpha_{R,S}\otimes 1)\,\alpha_{R*S,T}$ descends to the X-complex of

$$* \qquad \left(R*S/I_{R,S}^2\right)*T\Big/\left(I_{R*S/I^2_{R*S}},T\right)^2$$

Introduce the notation $R\# S = R*S/I_{R,S}^2$. This can be described as the quotient of $R*S$ by the relations $[R,S]^2=0$, and also as $R\otimes S \oplus \Omega^1R\otimes\Omega^1S$ with $\circ$ product. With this notation $*$ above is just $(R\#S)\#T$.

As far as the map $(\alpha_{R,S}\otimes 1)\circ\alpha_{R*S}$ is concerned we can actually get to a smaller quotient algebra. First let's examine the quotient $(R*S)\#T = R*S*T/I^2_{R*S,T}$. Now

$$\begin{aligned} I_{R*S,T} &= F[R*S,T]F \\ &= F[R,T]F + \cancel{R}[S,T]F \end{aligned}$$

$\therefore\ I^2_{R*S,T}$ consists of all linear combinations of expressions involving 2 T commutators.

Thus $(R\#S)\#T$ is the quotient of $R*S*T$ generated by the relations

$$[R,T]F[R,T] = [R,T]F[S,T] = [S,T]F[R,T] = [S,T]F[S,T] = 0$$

$$[R,S]^2 = 0.$$

Let us recall that instead of using $\Omega^1(R\#S)_\natural$ in $X(R)\otimes X(S) = X'(R\#S, I_{R,S})$, we actually use $\left(\Omega^1(R*S)/F^1_{I_{R,S}}\,\Omega^1(R*S)\right)_\natural = \left(\Omega^1R\otimes(S\otimes S) + (R\otimes R)\otimes\Omega^1S\right)_\natural$ This suggests looking at the following quotient

$$\begin{tikzcd}
 & R*S*T \arrow[d] \\
(R*S)\#T & (R*S)\otimes T \oplus \Omega^1(R*S)\otimes\Omega^1T \arrow[d] \\
(R\#S)\#T & (R\#S)\otimes T \oplus \Omega^1(R\#S)\otimes\Omega^1T \arrow[d] \\
 & \begin{pmatrix} R\otimes S \\ \oplus \\ \Omega^1R\otimes\Omega^1S \end{pmatrix}\otimes T \oplus \begin{pmatrix} \Omega^1R\otimes(S\otimes S) \\ (R\otimes R)\otimes\Omega^1S \end{pmatrix}\otimes\Omega^1T
\end{tikzcd}$$

It probably would be better to first write

$$\Omega^1(R*S)/F^1_{I_{R,S}}\,\Omega^1(R*S) = (R\otimes S)\otimes_R\Omega^1R\otimes_R(R\otimes S) \oplus (R\otimes S)\otimes_S\Omega^1S\otimes_S(R\otimes S)$$

$$= S\otimes\Omega^1R\otimes S \oplus R\otimes\Omega^1S\otimes R$$

Let's call this space $W(R,S)$. Then we have a canonical derivation

$$R\#S \longrightarrow W(R,S)$$

which is what one needs to form the extension

$$E = (R\# S)\otimes T \oplus W(R,S)\otimes \Omega' T$$

At some point we have to compute in this algebra which is a nilpotent extension of $R\otimes S\otimes T$ of order 2. The point is that $\Omega' R\otimes \Omega' S\otimes T$ has square $\neq 0$. NO see 345

Next let's discuss the relations in $R * S * T$ giving E. In passing from $\Omega'(R\# S)$ to $W(R,S)$ we kill $I_{R,S}\,\Omega'(R\# S) + \Omega'(R\# S) I_{R,S}$. I should be thinking of $\Omega'(R\# S)\otimes \Omega' T$ as ~~$((R\# S)\otimes T)[R\# S, T]$~~

$$(R\# S)\otimes T\,[R\# S, T]\,(R\# S)\otimes T$$

$$= (R\# S)\otimes T\,\big([R,T] + [S,T]\big)\,(R\# S)\otimes T$$

So now we propose to kill the kernel of $R\# S\to R\otimes S$, namely $\mathrm{Im}(I_{R,S}) = (R\# S)[R,S] = [R,S](R\# S)$. This means we have the additional relations

$$[R,S][R,T] = [R,S][S,T]$$

$$= [R,T][R,S] = [S,T][R,S] = 0$$

We should check that we have

$$[R,S]F[R,T] = [R,T]F[R,S] = 0$$

$$[R,S]F[S,T] = [S,T]F[R,S] = 0$$

But because of the 2 T-commutator relations, we have $[R,S]F[R,T] = [R,S](R*S)T[R,T]$

$$+ [R,S](R*S)T\underbrace{[R*S,T][R,T]}_{0}$$

$$= (R*S)\underbrace{[R,S][R,T]}_{0}T$$

and similarly for the others.

At this point we reach a puzzle. It would seem from the above discussion that the quotient E of $R*S*T$ is given by the relations of double [redacted] T commutator, unequal double R commutator (i.e. $[R,S]F[R,T] = [R,T]F[R,S] = 0$, unequal double S commutators, and finally

$$[R,S]^2 = 0$$

which is weaker NO than [redacted] $[R,S]F[R,S] = 0$. In fact we have assuming $[R,S]^2 = 0$

$$[R,S]F[R,S] = [R,S]\Big((R*S)T + \overset{F}{\text{[redacted]}}[R*S,T]\Big)[R,S]$$

$$\subset (R*S)[R,S]T[R,S] + \underbrace{[R,S]F\Big([R,T](R*S) + (R*S)[S,T]\Big)F[R,S]}_{=0}$$

Thus it seems that E should map onto [redacted] $R*S*T/J^2$ with kernel generated by the relation $[R,S]T[R,S] = 0$. This is a puzzle because the size of J/J^2 according to p 341 is the same as the kernel of $E \longrightarrow R\otimes S\otimes T$. The problem is solved because

$$[R,S]T[R,S] = \underbrace{[[R,S],T]}_{\subset R*S([R,T]+[S,T])R*S}[R,S] + \underbrace{T[R,S]^2}_{0}$$

is zero by the relations $[R,T][R,S] = [S,T][R,S] = 0$.

So the conclusion is that E is exactly F/J^2.

April 13, 1991

Let's check yesterday's conclusion that one has an isomorphism

$$\varphi : \begin{pmatrix} R\otimes S \\ \Omega^1R\otimes\Omega^1S \end{pmatrix}\otimes T \;\oplus\; \begin{pmatrix} \Omega^1R\otimes(S\otimes S) \\ \oplus \\ (R\otimes R)\otimes\Omega^1S \end{pmatrix}\otimes\Omega^1T \xrightarrow{\sim} \overbrace{R*S*T}^{F}/J^2$$

given by

$$\varphi(x\otimes y\otimes z) = xyz$$

$$\varphi(x_0dx_1\otimes y_0dy_1\otimes z_1) = x_0y_0[x_1,y_1]z_1$$

$$\varphi(x_0dx_1\otimes(y_0\otimes y_1)\otimes z_0dz_1) = x_0y_0z_0[x_1,z_1]y_1$$

$$\varphi((x_0\otimes x_1)\otimes y_0dy_1\otimes z_0dz_1) = x_0y_0z_0[y_1,z_1]x_1$$

Recall that $J = F[R,S]F + F[R,T]F + F[S,T]F$ and that J^2 contains $[?,?]F[?,?]$ where ? can be any of R,S,T. Thus J/J^2 is an $R\otimes S\otimes T$ bimodule generated by $[x,y]$, $[x,z]$, $[y,z]$, and these symbols are biderivations. This should imply immediately that φ is well-defined.

Let's now define an actions of R,S,T on the space on the left. For $x\in R$ we use the obvious left module structure over R. For $y\in S$ we use the obvious left S-module structure except for

$$\boxed{y(x_1\otimes y_1\otimes z_1) = x_1\otimes yy_1\otimes z_1 - dx_1\otimes \overbrace{dy\,y_1}^{d(yy_1)-y\,dy_1}\otimes z_1}$$

Observe that

$$\begin{aligned}\varphi(y(x_1\otimes y_1\otimes z_1)) &= x_1yy_1z_1 - [x_1,yy_1]z_1 + y[x_1,y_1]z_1 \\ &= x_1yy_1z_1 - [x_1,y]y_1z_1 \\ &= yx_1y_1z_1 = y\varphi(x_1\otimes y_1\otimes z_1)\end{aligned}$$

Next we define multiplication by $z \in T$ so as to be compatible with φ.

$$z(x_1 y_1 z_1) = \underbrace{x_1 z y_1 z_1}_{x_1 y_1 z z_1 - x_1[y_1,z]z_1} - [x_1,z]y_1 z_1$$

$$\therefore\ z\,\varphi(x_1\otimes y_1\otimes z_1) = x_1 y_1 z z_1 - [x_1,z]y_1 z_1 - x_1[y_1,z]z_1$$
$$= \varphi\left(x_1\otimes y_1\otimes z z_1 - dx_1\otimes(1\otimes y_1)\otimes dz\,z_1 - (x_1\otimes 1)\otimes dy_1\otimes dz\,z_1\right)$$

Thus define

$$\boxed{z(x_1\otimes y_1\otimes z_1) = x_1\otimes y_1\otimes z z_1 - dx_1\otimes(1\otimes y_1)\otimes dz\,z_1 - (x_1\otimes 1)\otimes dy_1\otimes dz\,z_1}$$

Verify assoc.

$$\begin{aligned} z'z(x_1\otimes y_1\otimes z_1) = {} & x_1\otimes y_1\otimes z'z z_1 && - dx_1\otimes(1\otimes y_1)\otimes z'dz\,z_1 \\ & - dx_1\otimes(1\otimes y_1)\otimes dz'\,z z_1 && - (x_1\otimes 1)\otimes dy_1\otimes z'dz\,z_1 \\ & - (x_1\otimes 1)\otimes dy_1\otimes dz'\,z z_1 \end{aligned}$$

and it's OK. Next

$$z\,\varphi(x_0 dx_1\otimes y_0 dy_1\otimes z_1) = z\,x_0 y_0[x_1,y_1]z_1$$
$$= x_0 y_0\, z[x_1,y_1]z_1$$
$$= x_0 y_0\left([x_1,y_1]z + [[z,x_1],y_1] + [x_1,[z,y_1]]\right)z_1$$
$$= x_0 y_0\left([x_1,y_1]z + [y_1,[x_1,z]] - [x_1,[y_1,z]]\right)z_1$$
$$= \varphi(x_0 dx_1\otimes y_0 dy_1\otimes z z_1)$$
$$+ \underbrace{x_0 y_0\left(y_1[x_1,z] - [x_1,z]y_1\right)z_1}_{\varphi(x_0 dx_1\otimes(y y_0\otimes 1 - y_0\otimes y_1)\otimes dz\,z_1}$$
$$- \underbrace{x_0 y_0\left(x_1[y_1,z] - [y_1,z]x_1\right)z_1}_{\varphi((x_0 x_1\otimes 1 - x_0\otimes x_1)\otimes y_0 dy_1\otimes dz\,z_1)}$$

Thus we define

$$\boxed{\begin{aligned} z(x_0dx_1\otimes y_0dy_1\otimes z_1) &= x_0dx_1\otimes y_0dy_1\otimes zz_1 \\ &+ x_0dx_1\otimes(y_0y_1\otimes 1-y_0\otimes y_1)\otimes dz z_1 \\ &-(x_0x_1\otimes 1-x_0\otimes x_1)\otimes y_0dy_1\otimes dz z_1 \\ &= x_0dx_1\otimes y_0dy_1\otimes zz_1 - \partial(x_0dx_1\otimes y_0dy_1)\otimes dz z_1 \end{aligned}}$$

Check assoc.

$$\begin{aligned} z'z\,(x_0dx_1\otimes y_0dy_1\otimes z_1) &= x_0dx_1\otimes y_0dy_1\otimes z'zz_1 \\ &-\partial(x_0dx_1\otimes y_0dy_1)\otimes dz'zz_1 \\ &-\partial(\qquad\qquad)\otimes z'dzz_1 \end{aligned}$$

so it's OK.

At this point we have an F-module structure on the source of φ, and so φ is necessarily surjective.

Next let's check J^2 kills the source of φ; denote this source by Σ, and let K be the kernel of the obvious projection $\Sigma \longrightarrow R\otimes S\otimes T$. K is clearly an F-submodule. We observe that $[x,y]=0$, $[x,z]=0$, and $[y,z]=0$ on K, so J kills K. On the other hand J kills $\Sigma/K \cong R\otimes S\otimes T$, so $J^2\Sigma \subset JK = 0$.

At this point we have a map $\psi: F/J^2 \longrightarrow \Sigma$ given by acting on 1, and we should check that $\psi\varphi = \mathrm{id}$. But this more or less clearly must be so, because of our description of J/J^2

April 16, 1991

Let $R \otimes S \otimes T = F/J$ where $F = R * S * T$ and $J = F([R,S] \cup [R,T] \cup [S,T])F$ as before. We want to identify $X(R) \otimes X(S) \otimes X(T)$ with a suitable quotient complex of $X(F)$.

Here are some ideas.

① Recall that for $A = R/I$, R quasi free one knows that the complex

$$Y'(R,I) : \quad R/I^2 \rightleftarrows \Omega^1 R/[R, \Omega^1 R] + I^2\Omega^1 R + IdI$$

computes $HP(A)$ if $\text{Hochdim}(A) \leq 3$. In general we have

$$H_0(Y'(R,I)) = \text{Ker}\{HC_2(A) \xrightarrow{B} HH_3(A)\}$$

$$H_1(Y'(R,I)) = HC_3(A).$$

and the Connes exact sequence:

$$0 \longrightarrow HC_5 \xrightarrow[\sim]{S} HC_3 \xrightarrow{B} \underset{0}{HH_4} \to HC_4 \xrightarrow{S} HC_2 \xrightarrow{B} HH_3$$

Thus we know in the case $R \otimes S \otimes T = F/J$ with R, S, T quasi-free that $X(R) \otimes X(S) \otimes X(T)$ should be quasi-isomorphic to $Y'(F,J)$. One can hope that $X(R) \otimes X(S) \otimes X(T)$ is a quotient of $Y'(F,J)$.

② In the case $R \otimes S = R * S/I$, $I = I_{R,S}$ we have isomorphisms of complexes

$$X(R) \otimes X(S) \xrightarrow[\psi(R,S)]{\sim} X'(R*S, I) \xleftarrow[\psi(S,R)]{\sim} X(S) \otimes X(R)$$

These are canonical it seems, but the composite is not the usual commutativity isomorphism σ for tensor product of complexes. In odd degree we have σ but in even degree we have

$$\begin{tikzcd}
x\otimes y \arrow[r, mapsto, "\psi_{R,S}"] & xy & y\otimes x \arrow[l, mapsto, "\psi_{S,R}"'] \arrow[d, phantom, "\shortparallel"] \\
-dx\otimes dy & -[x,y] & \sigma(x\otimes y)
\end{tikzcd}$$

$$\begin{tikzcd}
x_0dx_1\otimes y_0dy_1 \arrow[r, mapsto] \arrow[drr, bend right, "\sigma"'] & x_0y_0[x_1,y_1] \arrow[d, phantom, "\shortparallel"] & \\
 & -y_0x_0[y_1,x_1] & -y_0dy_1\otimes x_0dx_1 \arrow[l, mapsto]
\end{tikzcd}$$

Thus σ and $\psi_{R,S}\psi_{S,R}$ differ by the endomorphism $\psi_{R,S}\psi_{S,R}\sigma - 1$ which is zero on odd degree and on $\Omega^1R_\natural \otimes \Omega^1S_\natural$, and ~~as~~ $R\otimes S$ is $x\otimes y \longmapsto -dx\otimes dy$.

Notice that for $\mathbb{Z}/2$ graded complexes K, L we have the endomorphisms $d\otimes d$ on the tensor product $K\otimes L$ on both even and odd degree parts, which annihilate the total differential $d\otimes 1 + \varepsilon\otimes d$. Actually there are four $d\otimes d$ maps

$$\begin{tikzcd}
K_0\otimes L_0 \arrow[d, leftrightarrow, "d\otimes d"'] & & K_0\otimes L_1 \arrow[d, leftrightarrow, "d\otimes d"] \\
\oplus & \arrow[l, shift right] \arrow[r, shift right, "\text{total differential}"'] & \oplus \\
K_1\otimes L_1 & & K_1\otimes L_0
\end{tikzcd}$$

It seems these maps are null-homotopic, but that the homotopy is not constructible from the obvious operators. [redacted] The obvious operators in $\mathrm{End}(K)$ are $1, \varepsilon, d, \varepsilon d$, [redacted] and their linear combinations. One has

$$[d,1]_s = 0 \qquad [d,\varepsilon]_s = -2\varepsilon d$$
$$[d,d]_s = 0$$
$$[d,\varepsilon d]_s = 0$$

so the cohomology, i.e. maps of complexes modulo nullhomotopic maps, is 2-dimensional. There are

16 obvious operators in $\mathrm{End}(K)\otimes\mathrm{End}(L) = \mathrm{End}(K\otimes L)$ and the cohomology is four dimensional.

The point is that something like $X(R)\otimes X(S)$ has a lot of unipotent automorphisms, and that it might be wise to look for new tensor product structures, e.g. a new definition of the tensor product for $\mathbb{Z}/2$ graded complexes, or new associativity and commutativity isomorphisms.

Let's return to $A = R\otimes S\otimes T = F/J$. I recall that the ~~quotient~~ $(R\# S)\# T$ of F suggests the square zero extension of A

$$\begin{pmatrix} R\otimes S \\ \oplus \\ \Omega^1 R\otimes \Omega^1 S\end{pmatrix}\otimes T \quad\oplus\quad \begin{pmatrix}(R\otimes R)\otimes \Omega^1 S \\ \oplus \\ \Omega^1 R\otimes (S\otimes S)\end{pmatrix}\otimes \Omega^1 T \xrightarrow[\sim]{\psi} F/J^2$$

which turned out to be F/J^2. The obvious map $F/J^2 \to (X(R)\otimes X(S)\otimes X(T))_0$ is it seems.

~~$x\otimes y\otimes z \longmapsto xyz$~~

~~$x_0dx_1\otimes y_0dy_1\otimes z \longmapsto x_0dx_1\otimes y_0dy_1\otimes z$~~

$$xyz \longmapsto x\otimes y\otimes z$$

$$x_0y_0[x_1,y_1]z_1 \longmapsto x_0dx_1\otimes y_0dy_1\otimes z_1$$

$$x_0y_0z_0[y_1,z_1]x_1 \longmapsto (x_1x_0)\otimes y_0dy_1\otimes z_0dz_1$$

$$x_0y_0z_0[x_1,z_1]y_1 \longmapsto x_0dx_1\otimes(y_1y_0)\otimes z_0dz_1$$

What one does is to take the usual A-bimodule commutator ~~quotient~~ space of $(R\otimes R)\otimes\Omega^1 S\otimes\Omega^1 T \oplus \Omega^1 R\otimes(S\otimes S)\otimes\Omega^1 T$, but not for the other one. I think this means that the map $(\alpha_{R,S}\otimes 1)\,\alpha_{R*S,T}$ is not going to be same as $(1\otimes\alpha_{S,T})\,\alpha_{R,S*T}$; here $\alpha_{R,S}: X(R*S)\to X(R)\otimes X(S)$ is our canonical map. Thus our original goal of proving associativity is probably impossible. However the idea now is to see what's canonical, and hopefully adjust our concepts of tensor product for complexes so as to get the correct associativity and commutativity results. Ultimately I would like a description of the periodic theory of $R_1\otimes\cdots\otimes R_n$ which is consistent with R_A where $A=\mathbb{C}[t_1,\dots,t_n]$.

Let's consider the degree 1 part of $Y^1(F,J)$.

$$\Omega^1F/[F,\Omega^1F] + J^2\Omega^1F + JdJ$$

We have

$$\begin{aligned}\Omega^1F/[F,\Omega^1F]+J^2\Omega^1F = {} & (F/J^2)\otimes_R\Omega^1R\otimes_R\\ & \oplus (F/J^2)\otimes_S\Omega^1S\otimes_S\\ & \oplus (F/J^2)\otimes_T\Omega^1T\otimes_T\end{aligned}$$

An obvious quotient of this is where F/J^2 is replaced by $F/J = R\otimes S\otimes T$. We then get $\Omega^1R_\natural\otimes S\otimes T\oplus\cdots$ in $X(R)\otimes X(S)\otimes X(T)$ as desired. We want then to have left $\Omega^1R_\natural\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural$; so we ought to see if this occurs naturally as a quotient of

$$(J/J^2)\otimes_R\Omega^1R\otimes_R\oplus(J/J^2)\otimes_S\Omega^1S\otimes_S\oplus(J/J^2)\otimes_T\Omega^1T\otimes_T$$

Let's recall the basic presentation of J/J^2 from p. 341. This is an exact sequence

$$
* \qquad 0 \longrightarrow \Omega^1R \otimes \Omega^1S \otimes \Omega^1T \longrightarrow \begin{array}{c} (R\otimes R)\otimes \Omega^1S \otimes \Omega^1T \\ \oplus \\ \Omega^1R \otimes (S\otimes S) \otimes \Omega^1T \\ \oplus \\ \Omega^1R \otimes \Omega^1S \otimes (T\otimes T) \end{array} \xrightarrow{\pi} J/J^2 \longrightarrow 0
$$

Here the second map π is

$$\pi\left((x_0\otimes x_1)\otimes y_0dy_1 \otimes z_0dz_1\right) = x_0y_0z_0[y_1,z_1]x_1$$

$$\pi\left(x_0dx_1 \otimes (y_0\otimes y_1)\otimes z_0dz_1\right) = x_0y_0z_0[x_1,z_1]y_1$$

$$\pi\left(x_0dx_1 \otimes y_0dy_1 \otimes (z_0\otimes z_1)\right) = x_0y_0z_0[x_1,y_1]z_1$$

and the first map is

$$x_0dx_1 \otimes y_0dy_1 \otimes z_0dz_1 \longmapsto \begin{array}{c} (x_0(x_1\otimes 1 - 1\otimes x_1))\otimes y_0dy_1 \otimes z_0dz_1 \\ \oplus \\ - x_0dx_1 \otimes y_0(y_1\otimes 1 - 1\otimes y_1)\otimes z_0dz_1 \\ \oplus \\ x_0dx_1 \otimes y_0dy_1 \otimes z_0(z_1\otimes 1 - 1\otimes z_1) \end{array}$$

and this $\overset{\text{under } \pi}{\text{goes to}}$

$$\begin{aligned} &x_0y_0z_0\,[x_1,[y_1,z_1]] \\ -\,&x_0y_0z_0\,[y_1,[x_1,z_1]] \\ +\,&x_0y_0z_0\,[z_1,[x_1,y_1]] \end{aligned}$$

which is 0 by Jacobi's identity.

The presentation $*$ expresses J/J^2 as the sum of the images of $F[S,T]F$, $F[R,T]F$, $F[R,S]F$ with the relation given by the Jacobi identity. It seems natural to pass to the following quotient

$$J/J^2 \otimes_R \Omega^1R \otimes_R \longrightarrow \Big(\underbrace{J/J^2 + F[R,S]F + F[R,T]F}\Big)\otimes_R \Omega^1R \otimes_R$$

$$\uparrow \cong \qquad (\text{by } *)$$

$$R \otimes \Omega^1S \otimes \Omega^1T$$

$$\cong \Omega^1R_\natural \otimes \Omega^1S_\natural \otimes \Omega^1T$$

In this way we are killing $(F[R,S]F + F[R,T]F)\otimes dR$

in $\Omega^1F/[F,\Omega^1F]+J^2\Omega^1F$. If we want a quotient complex of $Y^1(F,J)$, then in F/J^2 we must kill the image under β which is the image of $[F[R,S]F+F[R,T]F\ ,\ R]$ in J/J^2. As J/J^2 is an $R\otimes S\otimes T$ bimodule, it seems that upon killing this in J/J^2 as well as $[F[R,S]F+F[S,T]F,\ S]$, $[F[R,T]F+F[S,T],T]$ we get the cokernel of

$$\Omega^1R\otimes\Omega^1S\otimes\Omega^1T \longrightarrow \begin{matrix}(R\otimes R)\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural\\ \Omega^1R_\natural\otimes(S\otimes S)\otimes\Omega^1T_\natural\\ \Omega^1R_\natural\otimes\Omega^1S_\natural\otimes(T\otimes T)\end{matrix}$$

So far we have introduced the relations $(F[R,S]F+F[R,T]F)dR$ sim. for $(\ \ )dS$ $(\ \ )dT$ so that we are left in odd degree with trying to collapse

(**) $ST[S,T]\otimes\Omega^1R_\natural \oplus RT[R,T]\otimes\Omega^1S_\natural \oplus RS[R,S]\otimes\Omega^1T_\natural$

to $\Omega^1R_\natural\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural$. Let's next divide by JdJ. Consider $Jd[R,S]$ and observe that

$$\big(F[R,S]F+F[R,T]F+F[S,T]F\big)\big([dR,S]+[R,dS]\big)$$

$$= RST[S,T][dR,S] + RST[R,T][R,dS]$$

since we have killed $(F[R,S]F+F[R,T]F)dR$ etc. These seems are independent; it would have been better to start with

$$F[S,T]F\,d[R,S] = RST[S,T][dR,S]$$

Note $x_0y_0z_0[y_1,z_1][dx,y] = [y,x_0y_0z_0[y_1,z_1]]dx$

so killing this is to kill $[S,\Omega^1S\otimes\Omega^1T\otimes\Omega^1R_\natural]$.

So it seems from JdJ we ~~[illegible]~~ reduces (**), which is isomorphic to

$$\Omega^1S\otimes\Omega^1T\otimes\Omega^1R_\natural \oplus \Omega^1R\otimes\Omega^1T\otimes\Omega^1S_\natural \oplus \cdots,$$

by the six versions of $RST[S,T][dR,S]$ to get three copies of $\Omega^1R_\natural\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural$.

But let's compute carefully the cokernel of

$$\Omega^1R\otimes\Omega^1S\otimes\Omega^1T \longrightarrow \begin{matrix}(R\otimes R)\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural\\ \oplus\\ \Omega^1R_\natural\otimes(S\otimes S)\otimes\Omega^1T_\natural\\ \oplus\\ \Omega^1R_\natural\otimes\Omega^1S_\natural\otimes(T\otimes T)\end{matrix}$$

Note that using the mult. map $R\otimes R\longrightarrow R$ with kernel Ω^1R and similarly for S,T we do get a map of the latter onto $(R\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural)\oplus(\Omega^1R_\natural\otimes S\otimes\Omega^1T_\natural)\oplus(\Omega^1R_\natural\otimes\Omega^1S_\natural\otimes T)$ and the kernel is $\Omega^1R\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural\oplus\cdots\oplus\cdots$. Next consider $[R,\Omega^1R]\otimes\Omega^1S\otimes\Omega^1T$ on the left. It maps to zero on the second two summands on the right, and the image in $\Omega^1R\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural$ is $\Omega^1R_\natural\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural$, similarly divide out by $\Omega^1R\otimes[S,\Omega^1S]\otimes\Omega^1T$ and then $\Omega^1R\otimes\Omega^1S\otimes[T,\Omega^1T]$ on the left, and one gets ~~[illegible]~~ the diagonal map

$$\Omega^1R_\natural\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural \longrightarrow \begin{matrix}\Omega^1R_\natural\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural\\ \oplus\\ \cdots\\ \oplus\\ \cdots\end{matrix}$$

Thus cokernel is the remaining part of $(X(R)\otimes X(S)\otimes X(T))_0$ plus as subspace two copies of $\Omega^1R_\natural\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural$. These might be killed by the "extra two copies

April 18, 1991.

$$F = R * S * T$$

$$J = F[R,S]F + F[R,T]F + F[S,T]F$$

$$F/J = R \otimes S \otimes T.$$

We have the following description of J/J^2:

$$0 \longrightarrow \Omega^1 R \otimes \Omega^1 S \otimes \Omega^1 T \xrightarrow{\partial} \begin{array}{c} (R\otimes R) \otimes \Omega^1 S \otimes \Omega^1 T \\ \oplus \\ \Omega^1 R \otimes (S \otimes S) \otimes \Omega^1 T \\ \oplus \\ \Omega^1 R \otimes \Omega^1 S \otimes (T \otimes T) \end{array} \xrightarrow{\pi} J/J^2 \longrightarrow 0$$

where ∂ is the differential in the complex

$$(\Omega^1 R \to R \otimes R) \otimes (\Omega^1 S \to S \otimes S) \otimes (\Omega^1 T \to T \otimes T)$$

and
$$\pi((x_0 \otimes x_1) \otimes y_0 dy_1 \otimes z_0 dz_1) = x_0 y_0 z_0 [y_1, z_1] x_1$$
$$\pi(x_0 dx_1 \otimes (y_0 \otimes y_1) \otimes z_0 dz_1) = x_0 y_0 z_0 [x_1, z_1] y_1$$
$$\pi(x_0 dx_1 \otimes y_0 dy_1 \otimes (z_0 \otimes z_1)) = x_0 y_0 z_0 [x_1, y_1] z_1$$

In other words π is the $A = R \otimes S \otimes T$ bimodule map such that $1 \otimes 1 \otimes dy \otimes dz \longmapsto [y,z]$ etc.

Observe that ∂ is the A-bimodule map such that

$$\begin{aligned} \partial(dx \otimes dy \otimes dz) = {} & (x \otimes 1 - 1 \otimes x) \otimes dy \otimes dz \\ & - dx \otimes (y \otimes 1 - 1 \otimes y) \otimes dz \\ & + dx \otimes dy \otimes (z \otimes 1 - 1 \otimes z) \end{aligned}$$

and that

$$\begin{aligned} \pi\partial(dx \otimes dy \otimes dz) &= [x,[y,z]] - [y,[x,z]] + [z,[x,y]] \\ &= 0 \end{aligned}$$

by the Jacobi identity.

Recall

$$\Omega^1F/[F,\Omega^1F]+J^2\Omega^1F = \left(F/J^2\right)\otimes_R \Omega^1R\otimes_R \oplus \left(F/J^2\right)\otimes_S \Omega^1S\otimes_S \oplus \left(F/J^2\right)\otimes_T \Omega^1T\otimes_T$$

When R is quasi-free we have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (J/J^2)\otimes_R \Omega^1R\otimes_R \arrow[r] & (F/J^2)\otimes_R \Omega^1R\otimes_R \arrow[r] & (F/J)\otimes_R \Omega^1R\otimes_R \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\ & & & \Omega^1R_\natural\otimes S\otimes T & \end{tikzcd}$$

Similarly for S, T, so this allows us to construct quotients of $\Omega^1F/[F,\Omega^1F]+J^2\Omega^1F$ by taking a suitable quotient of

$$J/J^2\otimes_R\Omega^1R\otimes_R \quad\oplus\quad \text{same for } S,T.$$

I propose first to use the surjection

$$\begin{tikzcd} J/J^2 \arrow[r, two heads] & J/J^2+F[R,S]F+F[R,T]F \arrow[d, "\beta"] \\ & R\otimes\Omega^1S\otimes\Omega^1T \end{tikzcd}$$

which means dividing by

$$F[R,S]F.\Omega^1R \quad + \quad F[R,T]F\,\Omega^1R$$

Now mod J^2 one has $F[R,S]F \equiv RST[R,S]T$, $F[R,T]F \equiv RST[R,T]S$, so mod $J^2\Omega^1R$ we have

$$F[R,S]F\Omega^1R + F[R,T]F\Omega^1R = ST[R,S]T\,\Omega^1R + ST[R,T]S\,\Omega^1R$$

If we wish to kill this on the $\Omega^1F_\natural$ side, then we must kill the image under β, which is $[RST[R,S]T,R] + [RST[R,T]S,R] \subset J/J^2$.

April 19, 1991

Recall what we found before about $F = R * S * T$, $F/J = R \otimes S \otimes T$, when R, S, T are quasi-free.

First, $Y^1(F,J) = \{ F/J^2 \rightleftarrows \Omega^1F/[F,\Omega^1F] + J^2\Omega^1F + JdJ \}$ computes the periodic homology of F/J.

2nd we construct a quotient Q of $Y^1(F,J)$ as follows: ~~[illegible]~~

$$Q_1 = \Omega^1F/[F,\Omega^1F] + J^2\Omega^1F + F[R,S]F\,dR + F[R,T]F\,dR + F[R,S]F\,dS + F[S,T]F\,dS + JdJ + F[R,T]F\,dT + F[S,T]F\,dT$$

$$Q_0 = F/J^2 + [F[R,S]F + F[R,T]F, R] + [F[R,S]F + F[S,T]F, S] + [F[R,T]F + F[S,T]F, T]$$

I conjecture that $Y^1(F,J) \longrightarrow Q$ is a quis, but it was not possible to see this directly from the construction.

We have exact sequences

~~[illegible]~~

$$0 \longrightarrow Q_0' \longrightarrow Q \longrightarrow Q_0'' \longrightarrow 0$$

$$0 \longrightarrow Q_1' \longrightarrow Q \longrightarrow Q_1'' \longrightarrow 0$$

where

$$Q_0'' = R \otimes S \otimes T \qquad x\,y\,z$$

$$Q_1'' = \begin{array}{ll} \Omega^1R_\natural \otimes S \otimes T & x_0dx_1\; y_0 z_0 \\ R \otimes \Omega^1S_\natural \otimes T & x_0\; y_0dy_1\; z_0 \\ R \otimes S \otimes \Omega^1T_\natural & x_0\; y_0\; z_0dz_1 \end{array}$$

$\uparrow$ representatives in F and $\Omega^1F_\natural$

and

$$\begin{tikzcd}
 & & Q_1'' = \begin{array}{c} \Omega^1R_\natural \otimes \Omega^1S_\natural \otimes \Omega^1T_\natural \\ \Omega^1R_\natural \otimes \Omega^1S_\natural \otimes \Omega^1T_\natural \\ \Omega^1R_\natural \otimes \Omega^1S_\natural \otimes \Omega^1T_\natural \end{array} \arrow[d, hook, "\beta"] & & \\
0 \arrow[r] & \Omega^1R_\natural \otimes \Omega^1S_\natural \otimes \Omega^1T_\natural \arrow[r, "\partial"] & \begin{array}{c} (R\otimes R/\partial[R,\Omega^1R]) \otimes \Omega^1S_\natural \otimes \Omega^1T_\natural \\ \Omega^1R_\natural \otimes (S\otimes S/\partial[S,\Omega^1S]) \otimes \Omega^1T_\natural \\ \Omega^1R_\natural \otimes \Omega^1S_\natural \otimes (T\otimes T/\partial[T,\Omega^1T]) \end{array} \arrow[r, "\pi"] & Q_0'' \arrow[r] & 0
\end{tikzcd}$$

$y_0z_0[y_1,z_1]x_0dx_1$

$x_0z_0[x_1,z_1]y_0dy_1$

$x_0y_0[x_1,y_1]z_0dz_1$

One has $b(y_0z_0[y_1,z_1]x_0dx_1)$

$$= [y_0z_0[y_1,z_1]x_0, x_1]$$

$$= \pi((1\otimes x_0x_1 - x_1\otimes x_0)\otimes y_0dy_1 \otimes z_0dz_1)$$

It would have been better to first give

$$\pi(x_0\otimes x_1 \otimes y_0dy_1 \otimes z_0dz_1) = x_0y_0z[y_1,z_1]x_1$$

$$\partial(x_0dx_1 \otimes y_0dy_1 \otimes z_0dz_1) = x_0(x_1\otimes 1 - 1\otimes x_1)\otimes y_0dy_1 \otimes z_0dz_1$$
$$- x_0dx_1 \otimes y_0(y_1\otimes 1 - 1\otimes y_1)\otimes z_0dz_1$$
$$+ \cdots\cdots\cdots$$

and to ~~state~~ that the differential $b: Q_1 \to Q_0$ induces a map $Q_1'' \xrightarrow{b} Q_0''$, which we can write

$$b = \pi\beta.$$

Now let us recall that

$$\partial : \Omega^1R_\natural \longrightarrow R\otimes R/\partial[R,\Omega^1R]$$

$$\partial(x_0dx_1) = x_0(x_1\otimes 1 - 1\otimes x_1)$$

is injective with cokernel R, the map to this cokernel being $x\otimes y \longrightarrow xy$. We have in addition

$$\beta : \Omega^1R_\natural \longrightarrow R\otimes R/\partial[R,\Omega^1R]$$

$$\beta(x_0dx_1) = 1\otimes x_0x_1 - x_1\otimes x_0 = \sigma(\partial(x_0dx_1))$$

One can check that $\partial[R,\Omega^1R]$ is stable under σ:

$$x\,x_0(x_1\otimes 1-1\otimes x_1) - x_0(x_1\otimes 1-1\otimes x_1)x$$

$$= xx_0x_1\otimes 1 - xx_0\otimes x_1 - x_0x_1\otimes x + x_0\otimes x_1x$$

$$\overset{\sigma}{\longmapsto} 1\otimes xx_0x_1 - x_1\otimes xx_0 - x\otimes x_0x_1 + x_1x\otimes x_0$$

$$= \left[1\otimes xx_0 - x\otimes x_0\,,\ x_1\right]$$

Thus β is well-defined, it's injective, and the cokernel is $\cong R$ via $x\otimes y \longmapsto yx$.

Thus ∂ is not a diagonal version of β except in the case where R, S, T are commutative.

Consider the following construction of quotient complexes of Q. Think of Q_1'' as $\mathbb{C}^3\otimes(\overbrace{\Omega^1R_\natural\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural}^{L})$, and choose a 2 diml subspace $W\subset\mathbb{C}^3$ such that $\beta(W\otimes L)$ is transverse to $\partial(L)$. For example take $W=\mathbb{C}e_1\oplus\mathbb{C}e_2$, then $Q_1''/W\otimes L = L$ and we have an exact sequence

$$0\longrightarrow \Omega^1R_\natural\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural \longrightarrow \begin{matrix} R\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural \\ \Omega^1R_\natural\otimes S\otimes\Omega^1T_\natural \\ \Omega^1R_\natural\otimes\Omega^1S_\natural\otimes(T\otimes T/\partial[T,\Omega^1T]) \end{matrix} \longrightarrow Q_0''/bW\longrightarrow 0$$

which gives an exact sequence

$$0\longrightarrow \begin{matrix} R\otimes\Omega^1S_\natural\otimes\Omega^1T_\natural \\ \Omega^1R_\natural\otimes S\otimes\Omega^1T_\natural \end{matrix} \longrightarrow Q_0''/bW \longrightarrow \Omega^1R_\natural\otimes\Omega^1S_\natural\otimes T\longrightarrow 0$$

It seems that this quotient complex of Q (which is qis to Q) is just the quotient of

$X(F)$ given by

$$X(F) \longrightarrow X(R*S)\otimes X(T) \longrightarrow X(R)\otimes X(S)\otimes X(T)$$

In effect we have maps

$$\begin{tikzcd}
 & F \arrow[r, shift right] \arrow[d] & \Omega^1F_\natural \arrow[l, shift right] \\
 & \begin{matrix}(R\#S)\otimes T\\ \Omega^1(R\#S)_\natural\otimes\Omega^1T_\natural\end{matrix} \arrow[r, shift right] \arrow[d] & \begin{matrix}(R\#S)\otimes\Omega^1T_\natural\\ \Omega^1(R\#S)_\natural\otimes T\end{matrix} \arrow[l, shift right] \\
* & \begin{matrix}\begin{pmatrix}R\otimes S\\ \Omega^1R_\natural\otimes\Omega^1S_\natural\end{pmatrix}\otimes T\\ \begin{pmatrix}R\otimes\Omega^1S_\natural\\ \Omega^1R_\natural\otimes S\end{pmatrix}\otimes\Omega^1T_\natural\end{matrix} \arrow[r, shift right] & \begin{matrix}\begin{pmatrix}R\otimes S\\ \Omega^1R_\natural\otimes\Omega^1S_\natural\end{pmatrix}\otimes\Omega^1T_\natural\\ \begin{pmatrix}\Omega^1R_\natural\otimes S\\ R\otimes\Omega^1S_\natural\end{pmatrix}\otimes T\end{matrix} \arrow[l, shift right]
\end{tikzcd}$$

One can check that the map from

$$(R\#S)\#T = \begin{pmatrix}R\otimes S\\ \Omega^1R\otimes\Omega^1S\end{pmatrix}\otimes T \oplus \underbrace{\begin{pmatrix}(R\otimes R)\otimes\Omega^1S\\ \Omega^1R\otimes(S\otimes S)\end{pmatrix}\otimes\Omega^1T}$$

to the even part of * takes the commutator quotient space of $\wedge$, but it treats $\Omega^1R\otimes\Omega^1S\otimes T$ strangely.

Similarly I think it should be the case that taking $W = \mathbb{C}e_2\oplus\mathbb{C}e_3$ yields the quotient complex

$$X(F)\longrightarrow X(R)\otimes X(S*T)\longrightarrow X(R)\otimes X(S)\otimes X(T).$$

April 30, 1991

Direct computation of $X(RA)$. Recall we have an isomorphism

$$\phi : \Omega^+ A \xrightarrow{\sim} RA$$

such that the product in RA corresponds to Fedosov product on even forms. Thus

$$\phi(x)\phi(y) = \phi(x \circ y) \qquad x\circ y = xy - dx\,dy$$

for $x, y \in \Omega^+ A$. Also we have

$$\Omega^- A = \Omega^+ A \otimes \bar{A} \xrightarrow{\sim} \Omega^1(RA),$$

$$\phi : x\,da \longmapsto \phi(x)\delta(\rho a)$$

We would now like to compute the basic pairing $x, y \longmapsto \phi^{-1}(\phi x\,\delta(\phi y))$ which will be a 1-coycle on $\Omega^+ A$ with values in $\Omega^- A$. Let's use the notation

$$\int x\,\delta y = \phi^{-1}(\phi x\,\delta(\phi y))$$

and recall the basic property

$$\int x\,\delta(y\circ z) = \int (x\circ y)\delta z + \int (z\circ x)\delta y$$

First formula:

$$\boxed{\int x\,\delta(da_1 da_2) = -b(x\,da_1 da_2) + (1+\kappa)d(x\,da_1 da_2)}$$

Proof: $da_1 da_2 = a_1 a_2 - a_1\circ a_2$, so

$$\phi x\,\delta(\phi(da_1 da_2)) = \phi x\,\delta(\phi(a_1a_2) - \phi(a_1)\phi(a_2))$$

$$= \phi x\,\delta(\phi(a_1a_2)) - \phi x\,\phi a_1\,\delta(\phi a_2) - \phi a_2\,\phi x\,\delta(\phi a_1)$$

$$= \phi\big(x\,d(a_1a_2) - (x\circ a_1)da_2 - (a_2\circ x)da_1\big)$$

Better would be

$$\int x\,\delta(da_1da_2) = \int x\,\delta(a_1a_2 - a_1\circ a_2)$$

$$= \int x\,\delta(a_1a_2) - \int (x\circ a_1)\delta a_2 - \int (a_2\circ x)\delta a_1$$

$$= x\,d(a_1a_2) - (x\circ a_1)da_2 - (a_2\circ x)da_1$$

where we have used $\boxed{\int x\,\delta a = x\,da}$, so

$$\int x\,\delta(da_1da_2) = x(da_1a_2 + \cancel{a_1da_2}) - \cancel{xa_1da_2} + dx\,da_1da_2 - a_2x\,da_1 + da_2dx\,da_1$$

$$= [x\,da_1, a_2] + dx\,da_1da_2 + da_2\,dx\,da_1$$

$$= -b(x\,da_1da_2) + (1+\kappa)\,d(x\,da_1da_2).$$

General formula: If $y \in \Omega^{2n}A$, then

$$\boxed{\int x\,\delta y = -\sum_{j=0}^{n-1}\kappa^{2j}b(x\circ y) + \sum_{j=0}^{2n-1}\kappa^j d(xy) + \kappa^{2n}(x\,dy)}$$

Proof. Use induction on n; true for $n=0$. Assume $n>0$; [illegible] we may assume $y = z\,da_1da_2$ with $z \in \Omega^{2n-2}A$. Then

$$\int x\,\delta y = \int x\,\delta(z\circ da_1da_2) = \int (x\circ z)\,\delta(da_1da_2) + \int (da_1da_2\circ x)\,\delta z$$

The first term is

$$-b((x\circ z)da_1da_2) + (1+\kappa)\,d((x\circ z)da_1da_2)$$

$$= -b(x\circ y) + (1+\kappa)\,d(x\circ y)$$

$$= -b(x\circ y) + (1+\kappa)\,d(xy)$$

where we have used that

$$x\circ y = xy - dx\,dy = xz\,da_1da_2 - dx\,dz\,da_1da_2 = (x\circ z)da_1da_2$$

The second term is

$$\int (da_1 da_2 x)\,\delta z = -\sum_{j=0}^{n-2} K^{2j} b(da_1 da_2 x\circ z)$$

$$+ \sum_{j=0}^{2n-3} K^j d(da_1 da_2 x\circ z) + K^{2n-2}(da_1 da_2 x\, dz)$$

$$= -\sum_{j=1}^{n-1} K^{2j} b(x\circ z\, da_1 da_2) + \sum_{j=2}^{2n-1} K^j d(xz\, da_1 da_2)$$

$$+ K^{2n}(x\, dz\, da_1 da_2)$$

$$= -\sum_{j=1}^{n-1} K^{2j} b(x\circ y) + \sum_{j=2}^{2n-1} K^j d(xy) + K^{2n}(x\,dy)$$

using induction hypothesis. This proves the formula. ☐

Consider the map $X(RA) \to X(R_A)$ when A is commutative. Actually I mean the map $\Omega A \longrightarrow \Omega_A$ which is compatible with d, b, K provided with define $b = 0$, $K = 1$ on Ω_A. Then

$$\int x\,\delta y \longrightarrow |y|\, d(xy) + x\,dy$$

$$= (1+|y|)\, x\,dy + |y|\, dx\, y$$

Another way of writing the formula is

$$\int x\,\delta y = \Big(-\sum_{j=0}^{\frac{1}{2}|y|-1} K^{2j}\Big) b(x\circ y) + \Big(\sum_{j=0}^{|y|-1} K^j\Big) dx\, y + \Big(\sum_{j=0}^{|y|} K^j\Big) x\,dy$$

May 5, 1991

Eventually we should work out relative theory as an exercise. We consider a homom. $S \to A$ of algebras and [redacted] relative to S. More precisely we consider the category of algebras under S.

Let's work out the relation between square zero extensions of S-algebras (algebras under S) and A-bimodules extensions of $\Omega^1(A;S)$. Given an extension of A in S-algebras

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] & R \arrow[r] & A \arrow[r] & 0 \\
 & & S \arrow[u, bend left] \arrow[ur] & &
\end{tikzcd}$$

we would like to know whether

$$0 \to I/I^2 \to A \otimes_R \Omega^1(R;S) \otimes_R A \to \Omega^1(A;S) \to 0$$

[redacted] is exact. Right exactness is formal; the issue is whether the injectivity at the left holds.

Use Tor calculation

$$0 \to \Omega^1(R;S) \to R \otimes_S R \to R \to 0$$

$$0 \to \Omega^1(R;S) \otimes_R A \to R \otimes_S A \to A \to 0$$

$$\begin{tikzcd}
\mathrm{Tor}_1^R(A, R\otimes_S A) \arrow[r] & \mathrm{Tor}_1^R(A,A) \arrow[r] \arrow[d, equal] & A \otimes_R \Omega^1(R;S) \otimes_R A \arrow[r] & A \otimes_S A \arrow[r] & A \arrow[r] & 0 \\
 & I/I^2 & & & &
\end{tikzcd}$$

To calculate $\mathrm{Tor}_1^R(A, R\otimes_S A)$ use

$$0 \to I \to R \to A \to 0$$

and tensor on the right with $R \otimes_S A$. This gives

$$\overset{\overset{0}{\shortparallel}}{\mathrm{Tor}_1^R(R, R\otimes_S A)} \to \mathrm{Tor}_1^R(A, R\otimes_S A) \to I\otimes_S A \to R\otimes_S A \to A\otimes_S A \to 0$$

$$\to \mathrm{Tor}_1^S(R, A) \to \mathrm{Tor}_1^S(A, A) \to I\otimes_S A \to R\otimes_S A \to A\otimes_S A \to 0$$

Thus we get

$$\mathrm{Tor}_1^S(R,A) \to \mathrm{Tor}_1^S(A,A) \to \mathrm{Tor}_1^R(A, R\otimes_S A) \to 0.$$

Thus we conclude

Lemma: If $\mathrm{Tor}_1^S(A,A) = 0$, then for any S-alg. extension

$$0 \to I \to R \to A \to 0$$

we have

$$\circledast \qquad 0 \to I/I^2 \to A\otimes_R \Omega^1(R;S)\otimes_R A \to \Omega^1(A;S) \to 0$$

From this it should follow formally that square zero S-alg extensions of A are equivalent to A-bimodule extensions of $\Omega^1(A;S)$ when $\mathrm{Tor}_1^S(A,A)=0$.

Next suppose injectivity in $\circledast$ holds for all S-alg. extensions $A = R/I$ of A. Choose R to be an S free algebra, mapping onto A. This means R is $T_S(M)$ with M a free S-bimodule. It should be true that $\Omega^1(R;S)$ is a free R-bimodule, whence in the exact sequence

$$0 \to \Omega^1(R;S)\otimes_R A \to R\otimes_S A \to A \to 0$$

$\Omega^1(R;S)\otimes_R A$ is a free R-module, so we have

$$0 \to \mathrm{Tor}_1^R(A, R\otimes_S A) \to I/I^2 \to A\otimes_R \Omega^1(R;S)\otimes_R A \to \cdots$$

so $\mathrm{Tor}_1^R(A, R\otimes_S A) = 0$ and then since

R is right S-flat, we have $\mathrm{Tor}_1^S(R,A) = 0$, and so $\mathrm{Tor}_1^S(A,A)=0$

Other point: Suppose A is quasi-free relative to S, i.e. has lifting wrt square zero extensions of S-algebras. Then choose $A = R/I$ with R S-free, and consider the square zero extension

$$0 \longrightarrow I/I^2 \longrightarrow R/I^2 \longrightarrow A \longrightarrow 0 .$$

This splits, which means we have the identity $R/I^2 \longrightarrow R/I^2$ and $R/I^2 \longrightarrow A \longrightarrow R/I^2$, whose difference is a derivation $R/I^2 \longrightarrow I/I^2$, whence an A-bimodule map

$$A \otimes_R \Omega^1(R;S) \otimes_R A \longrightarrow I/I^2$$

which shows

$$0 \longrightarrow I/I^2 \longrightarrow A \otimes_R \Omega^1(R;S) \otimes_R A \longrightarrow \Omega^1(A;S) \longrightarrow 0$$

is split exact. Then as above we conclude that we must have $\mathrm{Tor}_1^S(A,A) = 0$.

Thus it seems that [redacted] in so far as square zero extensions are concerned flatness is perhaps too strong - the natural condition is $\mathrm{Tor}_1^S(A,A) = 0$.

May 6, 1991

Fedosov algebra computations.
Let's compute the maps

$$K_i A \longrightarrow H_i(X(R_A)) \longrightarrow H_{i \bmod 2}(\Omega_A^{\pm})$$

If e is an idempotent matrix over A we know it lifts to the idempotent

$$\tilde{e} = \frac{1}{2} + \sum_{n \geq 0} \boxed{\frac{(2n\text{[illegible]})!}{(n!)^2}} (e - \tfrac{1}{2}) de^{2n} \qquad \frac{2^n(2n-1)!!}{n!}$$

over $RA = \Omega^+ A$ with Fedosov product. Thus in $H^+(\Omega_A^{\text{[illegible]}})$ we get the class represented by

$$tr(\tilde{e}) = tr\, e + \sum_{n \geq 1} \underbrace{\frac{(2n\text{[illegible]})!}{n!}}_{2^n(2n+1)!!} \underbrace{tr\left(e \frac{de^{2n}}{n!}\right)}_{\text{usual character form}}$$

If g is an invertible matrix of A, its inverse in R_A is

$$h = \sum_{n \geq 0} g^{-1}(dg\,dg^{-1})^n$$

Check: $$gh - dg\,dh = \sum_{n \geq 0} (dg\,dg^{-1})^n - \sum_{n \geq 0} dg\,dg^{-1}(dg\,dg^{-1})^n = 1$$

So in $H^-(\Omega_A)$ we get the class represented by the form

$$tr(h\,dg) = \sum_{n \geq 0} tr\, g^{-1}(dg\,dg^{-1})^n dg$$

$$= \sum_{n \geq 0} (-1)^n\, tr(g^{-1}dg)^{2n+1}$$

Recall usual Chern character form is $\cancel{(-1)}^n \frac{n!}{(2n+1)!} tr(g^{-1}dg)^{2n+1}$

$$2^n(2n+1)!! \qquad = \sum_{n \geq 0} \boxed{\frac{(2n+1)!}{n!}} \underbrace{(-1)^n \frac{n!}{(2n+1)!} tr(g^{-1}dg)^{2n+1}}_{\text{usual character form.}}$$

But we've not taken into account the differential on Ω_A which it inherits as a quotient of $X(RA)$. Recall this is

$$\begin{tikzcd} \Omega_A^+ \arrow[r, shift right, "Nd"'] & \Omega_A^- \arrow[l, shift right, "-2d"'] \end{tikzcd}$$

So to get rid of these constants we rescale

$$\begin{tikzcd} \Omega^0 \arrow[r, "d"] \arrow[d, "1"] & \Omega^1 \arrow[r, "-2d"] \arrow[d, "1"] & \Omega^2 \arrow[r, "3d"] \arrow[d, "-\frac{1}{2}"] & \Omega^3 \arrow[r, "-2d"] \arrow[d, "-\frac{1}{6}"] & \Omega^4 \arrow[r, "5d"] \arrow[d, "\frac{1}{12}"] & \Omega^5 \arrow[d, "\frac{1}{60}"] \\ \Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3 \arrow[r, "d"] & \Omega^4 \arrow[r, "d"] & \Omega^5 \end{tikzcd}$$

Thus

$$\begin{tikzcd} \Omega^{2n} \arrow[rr, "(2n+1)d"] \arrow[d, "\frac{(-1)^n}{2^n(2n-1)!!}"] & & \Omega^{2n+1} \arrow[d, "\frac{(-1)^n}{2^n(2n+1)!!}"] \\ \Omega^{2n} \arrow[rr, "d"'] & & \Omega^{2n+1} \end{tikzcd}$$

Under this rescaling

$$\begin{tikzcd} \operatorname{tr}(\tilde{e}) = \sum_{n\geq 0} 2^n(2n-1)!! \operatorname{tr}\left(\frac{e\,de^{2n}}{n!}\right) \arrow[d, mapsto] \\ \sum_{n\geq 0} (-1)^n \operatorname{tr}\left(\frac{e\,de^{2n}}{n!}\right) \quad \in \Omega_A^+ \end{tikzcd}$$

$$\begin{tikzcd} \operatorname{tr}(h\,dg) = \sum_{n\geq 0} (-1)^n \operatorname{tr}(g^{-1}dg)^{2n+1} \arrow[d, mapsto] \\ \sum_{n\geq 0} \frac{1}{2^n(2n+1)!!} \operatorname{tr}(g^{-1}dg)^{2n+1} \quad \in \Omega_A^- \end{tikzcd}$$

These are off by $(-1)^n$ which perhaps means we should change the sign of the b map in the X complex. Sort of thing like $i/2\pi$?

9

May 7, 1991

Given A, let $\tilde{A}$ be the algebra obtained by adjoining an identity to A. Because A has an identity we have obvious homomorphisms $\tilde{A} \twoheadrightarrow A$, $\tilde{A} \longrightarrow \mathbb{C}$ which combine to give a canonical isomorphism

$$(*) \qquad \tilde{A} \xrightarrow{\sim} A \times \mathbb{C}$$

of algebras. Standard notation for elements of $\tilde{A}$: $a + c1$, $a \in A$, $c \in \mathbb{C}$. The isomorphism $(*)$ is

$$a + c1 \longmapsto (a+c, c)$$

In terms of the standard notation

$$\tilde{A} = \underset{\text{ideal}}{A} \oplus \underset{\text{subalgebra}}{\mathbb{C}1}$$

and this corresponds to

$$A \times \mathbb{C} = A \times 0 \oplus \Delta\mathbb{C}$$

A linear map $\rho : \tilde{A} \longrightarrow R$ such that $\rho(1_{\tilde{A}}) = 1_R$ is the same as a linear map $\rho : A \longrightarrow R$ which can be arbitrary. Thus

$$R(\tilde{A}) \xleftarrow{\sim} T(A)$$

Canonical homomorphism

$$R(\tilde{A}) = R(A \times \mathbb{C}) \longrightarrow RA \times R\mathbb{C} = RA \times \mathbb{C}$$

is the homomorphism

$$T(A) \xrightarrow{\pi} RA \times \mathbb{C}$$
$$a \longmapsto (\hat{\rho}(a), 0)$$

Since $RA \times \mathbb{C}$ is quasi-free, we know there is a lifting homomorphism

$$RA \times \mathbb{C} \longrightarrow \widehat{T(A)}$$

where the completion is wrt the I-adic filtration, $I = \operatorname{Ker}\pi$.

Problem: Is there a canonical lifting homomorphism?

Notice that everything we have done so far makes sense for a vector space V equipped with a distinguished element $1_V \in V$ which is nonzero.

Idea: Examine the commutative analogue. Thus we consider the homomorphism

$$S(V) \longrightarrow S(V)/(1_{S(V)} - 1_V) \times \mathbb{C}$$

$$v \longmapsto (\hat{\rho} v , 0)$$

Let's denote $S(V)/(1_{S(V)} - 1_V)$ by $S_r(V)$, r standing for reduced. Geometrically we have

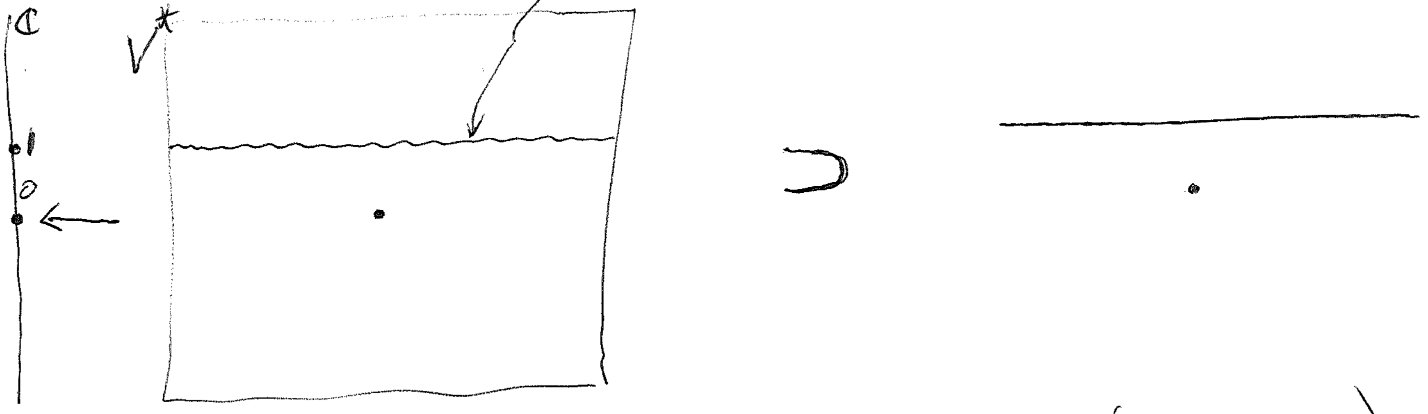
$\operatorname{Spec} S(V) \;=\;$ the vector space V^*

$\operatorname{Spec} S_r(V) \;=\;$ the affine space of splittings of the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow V \longrightarrow \bar{V} \longrightarrow 0$$

affine space of $\{\lambda \in V^* \mid \lambda(1_V) = 1\}. = \operatorname{Spec} S_r(V)$

Picture



$$\operatorname{Spec} S(\mathbb{C}1_V) \longleftarrow \operatorname{Spec} S(V) \xhookleftarrow[\text{closed immersion}]{} \operatorname{Spec}\big(S_r(V) \times \mathbb{C}\big)$$

$$\operatorname{Spec}\big(S_r(V) \times \mathbb{C}\big) = \operatorname{Spec}(S_r V) \cup pt$$

Now we ask whether there is a lifting homomorphism $S_\lambda(V)\times\mathbb{C} \longrightarrow \widehat{S(V)}$. Equivalently, is there a retraction of the formal neighborhood of $Z = \operatorname{Spec}(S_\lambda(V)\times\mathbb{C})$ in $\operatorname{Spec}(S(V)) = V^*$ onto Z? So Z is a formal neighborhood retract?

One way of producing such a retraction is to choose a splitting $V = \mathbb{C}\oplus\bar{V}$, but it's clear geometrically that there is a canonical way to proceed, namely to project from 0. This gives a <u>non-linear</u> retraction of the formal nbd. of the affine space it seems. In other words there seems to be an interesting lifting homomorphism $S_\lambda(V)\times\mathbb{C} \longrightarrow \widehat{S(V)}$ which is canonical.

Q: Given $\lambda\in V^*$ with $\lambda(1_V)=1$, can we construct an element of $\widehat{S(V)}$, actually a sequence in $S(V)$ which approximate to higher & higher order the projection onto the affine space from the origin and which vanish to high order at the origin.?

May 25, 1991

Section on the canonical maps

$$K_iA \longrightarrow HP_iA \qquad i=0,1$$

Prop. There are canonical additive maps

$$K_0A \longrightarrow \mathrm{Ker}\left(A_\natural \xrightarrow{d} \Omega^1A_\natural\right) = HD_0A \qquad [e] \mapsto tr(e)$$

$$K_1A \longrightarrow \mathrm{Ker}\left(\Omega^1A_\natural \xrightarrow{b} A\right) = HH_1A \qquad [g] \mapsto tr(g^{-1}dg)$$

Pf. K_0A = Groth grp of f.g. projective A-modules (right modules). Need

Lemma: Given idempotent matrices $e \in M_kA$, $e' \in M_{k'}A$. Then $Im(e) \simeq Im(e') \implies e \oplus 0_{k'}$ conjugate to $e' \oplus 0_k$ in $M_{k+k'}A$.

(NOTE: This lemma should be examined closely when we take up the Lundell business.)

Prop. Given $A = R/I$, there are canonical additive maps

$$K_0A \longrightarrow H_0(X(\hat{R}))$$

$$K_1A \longrightarrow \mathrm{Ker}\left(\varprojlim \Omega^1(R/I^{n+1})_\natural \longrightarrow \hat{R}\right)/d\hat{I}$$

hence canonical maps $K_iA \longrightarrow H_i(\hat{X}(R,I))$ $i=0,1$.

Need: Lemma: lifting of idempotents unique up to conjugation

Lemma: (1) 2) $p \in M_n\hat{R}$ is invertible $\iff$ its image in M_nA is invertible.

1) $1 + M_n\hat{I}$ is a group under multiplication

Pf of 1) is geometric series $(1-x)^{-1} = \sum_0^\infty x^n$

2) Let p lift $g \in GL_nA$, q lift g^{-1}, define x, y by $qp = 1-x$, $pq = 1-y$. Then $p^{-1} = (1-x)^{-1}q = q(1-y)^{-1}$.

$$GL_k A = GL_k \hat{R} \Big/ 1 + M_k \hat{I}$$

$$K_1 A = K_1 \hat{R} \Big/ \left\{ [1-x] \,\middle|\, x \in M_k \hat{I} \right\}$$

$$\left(\begin{array}{l} d\{\mathrm{tr}(\log(1-x))\} = \mathrm{tr}\left((1-x)^{-1} d(1-x)\right) \\ \text{holds in } \varprojlim \Omega^1(R_n)_{\natural} \end{array} \right)$$

Given idempotent $e \in M_k A$, let $x \in M_k \hat{R}$ lift e, then
$$\tilde{e} = \frac{1}{2} + \left(x - \frac{1}{2}\right) \sum_{n \geq 0} \frac{2^n (2n-1)!!}{n!} (x - x^2)^n$$
is an idempotent in $M_k \hat{R}$ lifting e.

Given $\tau : I^m/[R, I^m] \longrightarrow \mathbb{C}$, choose $\tilde{\tau} : R/[R, I^m] \longrightarrow \mathbb{C}$ extending τ, define a cyclic 1-cocycle on R/I^m by
$$f(r_0 + I^m, r_1 + I^m) = \tilde{\tau}([r_0, r_1])$$
Then $K_1 A \longrightarrow H_1(\hat{X}(R, I)) \longrightarrow H_1(X(R/I^m)) \xrightarrow{f} \mathbb{C}$ is given by

~~$[g] \longmapsto \mathrm{tr}(p^{-1}dp) \in H_1(X(R/I^m))$ p lift of g to R/I^m~~

~~$\xmapsto{f} \mathrm{tr} f(g_m, p)$ where g_m~~

$$[g] \longmapsto [\mathrm{tr}(p^{-1}dp)] \in H_1(\hat{X}(R, I))$$ p lift of g to R

$$\longmapsto \left[\mathrm{tr}\left((p+I^m)^{-1} d(p+I^m)\right)\right] \in H_1(X(R/I^m))$$

$$\longmapsto \mathrm{tr}\, f(g_m + I^m, p + I^m)$$ $g_m + I^m = (p+I^m)^{-1}$

$$= \mathrm{tr}\left(\tilde{\tau}([g_m, p])\right)$$

$$= \mathrm{tr}\left(\tau [g_m, p]\right)$$ since $[g_m, p] \in M_k(I^m)$

$$= \mathrm{tr}\left(\tau(1 - p g_m)\right) - \mathrm{tr}\left(\tau(1 - g_m p)\right)$$

Thus we have

$$[g] \longmapsto tr\left(\tau(1-pq_m)\right) - tr\left(\tau(1-q_m p)\right)$$

where p is a lift of g to R and $q_m + I^m = (p+I^m)^{-1}$.

Finally, if q is a lift of g^{-1} to R then can take $q_m = \sum_0^{m-1} x^n q = \sum_0^{m-1} p^n y$ and

$$1-pq_m = (1-pq)^m \qquad 1-q_m p = (1-qp)^m$$

so we have $\underline{[g] \longmapsto tr(1-pq)^m - tr(1-qp)^m}$

Prop. 1) e idempotent over A lifts to the idempotent

$$e + \sum_{n\geq 1} \frac{2^n(2n-1)!!}{n!}(e-\tfrac{1}{2})\,de^{2n}$$

over $\hat{R}A$. Further

$$tr(e) + \sum_{n \geq 1} (-1)^n 2^n(2n-1)!!\; tr\left((e-\tfrac{1}{2})\,de^{2n}\right) \in \hat{\Omega}^+A$$

is a K-invariant $b+B$ cocycle representing the image of $[e]$ in HP_0A.

2) g invertible over A lifts to g over $\hat{R}A$ which has inverse $g^{-1}\sum_{n\geq 0}(dg\,dg^{-1})^n$. Further

$$\sum_{n\geq 0} tr\left(g^{-1}dg\,(dg^{-1}dg)^n\right)$$

is a K^2-invariant $b+B$ cocycle representing the image of $[g]$ in HP_1A. Also

$$\frac{1}{2}\sum_{n\geq 0} tr\left(g^{-1}dg\,(dg^{-1}dg)^n\right) - tr\left(g\,dg^{-1}(dg\,dg^{-1})^n\right)$$

is a K-invariant $b+B$ cocycle rep. the image of $[g]$.

May 27, 1991

Universal property of R_A, A comm.

Define R_A to be a universal algebra equipped with a normalized linear map $\rho : A \to R_A$ satisfying

$$* \qquad \rho(a_1 a_2) = \frac{\rho a_1 \, \rho a_2 + \rho a_2 \, \rho a_1}{2}$$

We have a canonical homomorphism $\Phi : R_A \to \Omega_A^+$, where Ω_A^+ is equipped with the Fedosov product, such that $\rho a \mapsto a$.

Next we construct $\Psi : \Omega_A^+ \to R_A$. [illegible]

Set $x * y = \frac{1}{2}(xy + yx)$. We have the identity

$$4\big((x*y)*z - x*(y*z)\big) = [y,[x,z]].$$

Apply this to the elements $\rho a_0, \rho a_1, \rho a_2$ of R_A, use

$$(\rho a_0 * \rho a_1) * \rho a_2 = \rho(a_0 a_1) * \rho a_2 = \rho(a_0 a_1 a_2)$$

(by $*$ above), etc. to obtain

$$(1) \qquad [\rho a_0, [\rho a_1, \rho a_2]] = 0.$$

Since R_A is generated by the elements ρa_0, $a_0 \in A$, it follows that $[\rho a_1, \rho a_2] \in$ center of R_A.

Next

$$[\rho a_0, \rho a_1][\rho a_1, \rho a_2] = [\rho a_0, \rho a_1 [\rho a_1, \rho a_2]]$$

$$= [\rho a_0, [\rho a_1, \rho a_1 \rho a_2]]$$

$$= \underbrace{[\rho a_0, [\rho a_1, \rho(a_1 a_2)]]}_{0} + [\rho a_0, \underbrace{[\rho a_1, \tfrac{1}{2}[\rho a_1, \rho a_2]]}_{0}]$$

Here we have used

$$\rho a_1 \rho a_2 = \rho(a_1 a_2) + \frac{1}{2}[\rho a_1, \rho a_2].$$

$\therefore \; \omega(a_1, a_2) = -\frac{1}{2}[\rho a_1, \rho a_2]$

Now define

$$\Phi^n : \Omega_A^{2n} \longrightarrow R_A$$

$$a_0 da_1 \cdots da_{2n} \longmapsto \rho a_0\, \omega(a_1,a_2) \cdots \omega(a_{2n-1},a_{2n})$$

$$= \left(-\tfrac{1}{2}\right)^n \rho a_0 [\rho a_1, \rho a_2] \cdots [\rho a_{2n-1}, \rho a_{2n}]$$

The question is why is this well-defined. If so, then assembling Φ^n for different n, we obtain $\Phi : \Omega_A^+ \longrightarrow R_A$. As the homomorphism $\Psi : R_A \to \Omega_A^+$ satisfies $\Psi(\rho a) = a$, $\Psi\left(-\frac{1}{2}[\rho a_1, \rho a_2]\right) = da_1 da_2$, it is clear that $\Psi\Phi = 1$. As Ψ is surjective ([redacted] Ω_A^+ is a quotient of $\Omega^+ A = RA$ and R_A is a quotient of RA), we have Φ, Ψ are inverse isomorphisms.

To see that Φ^n is well-defined we have to understand the universal property of Ω_A^q as a vector space. As an A-module it is a universal A-module equipped with multilinear map $A^{\otimes q} \longrightarrow \Omega_A^q$, $(a_1, \ldots, a_q) \longmapsto da_1 \cdots da_q$, which is a derivation in each variable, and [redacted] which is alternating (vanishes if $a_i = a_{i+1}$ for some i, $1 \le i < q$).

But a linear map $\Omega_A^q \longrightarrow V$ is equivalent to an A-module map $\Omega_A^q \longrightarrow \mathrm{Hom}(A, V)$, hence a linear map $\Omega_A^q \to V$ is equivalent to a multilinear map [redacted] $f(a_0, \ldots, a_q)$ with values in V satisfying

1) Alternating condition $f(a_0, \ldots, a_q) = 0$ if $a_i = a_{i+1}$ for some $i = 1, \ldots, q-1$.

2) Derivation condition $f(a_0, a_1 a_1', a_2, \ldots)$
$= f(a_0 a_1, a_1', a_2, \ldots) + f(a_0 a_1', a_1, a_2, \ldots)$

Note that 1) implies $f(a_0, a_{\sigma 1}, \ldots, a_{\sigma q}) = \mathrm{sign}(\sigma) f(a_0, a_1, \ldots, a_q)$

for any permutation σ, and then 2) implies the derivation condition with respect to each variable a_i $i=1, \dots$ &c

Let's check now that these conditions hold for

$$f(a_0, \dots, a_{2n}) = \rho a_0 [\rho a_1, \rho a_2] \cdots [\rho a_{2n-1}, \rho a_{2n}] \in R_A$$

We have seen the alternating condition holds because of the identity $[\rho a_0, \rho a_1][\rho a_1, \rho a_2] = 0$. So we have to check the derivation condition. We will use that brackets $[\rho a_1, \rho a_2]$ lie in the center of R_A.

$$f(a_0, a_1 a_1', a_2, \dots) = \rho a_0 [\underbrace{\rho(a_1 a_1')}_{\rho a_1 \rho a_1' + \frac{1}{2}[\rho a_1, \rho a_1']}, \rho a_2] \cdots$$

$$= \rho a_0 [\rho a_1 \rho a_1', \rho a_2] \cdots$$

$$= \rho a_0 \big([\rho a_1, \rho a_2] \rho a_1' + \rho a_1 [\rho a_1', \rho a_2] \big) \cdots$$

$$= \big(\underbrace{\rho a_0 \rho a_1'}_{\rho(a_0 a_1') + \frac{1}{2}[\rho a_0, \rho a_1']} [\rho a_1, \rho a_2] + \underbrace{\rho a_0 \rho a_1}_{\rho(a_0 a_1) + \frac{1}{2}[\rho a_0, \rho a_1]} [\rho a_1', \rho a_2] \big) \cdots$$

$$= \big(\rho(a_0 a_1') [\rho a_1, \rho a_2] + \rho(a_0 a_1) [\rho a_1', \rho a_2] \big) \cdots$$

$$+ \frac{1}{2} \big(\underbrace{[\rho a_0, \rho a_1'][\rho a_1, \rho a_2] + [\rho a_0, \rho a_1][\rho a_1', \rho a_2]}_{} \big) \cdots$$

0 by polarizing $[\rho a_0, \rho a_1][\rho a_1, \rho a_2] = 0$ (better by alternating character of $[\rho a_1, \rho a_2] \cdots [\rho a_{2n+1}, \rho a_{2n+2}]$)

$$= f(a_0 a_1', a_1, a_2, a_n) + f(a_0 a_1, a_1', a_2, \dots, a_n).$$

□

7

Alternative: Apparently from Hochschild, Kostant, Rosenberg one knows that $f(a_0,\dots,a_n) \longmapsto \sum_{\sigma\in\Sigma_n} \operatorname{sign}(\sigma)\, f(a_0, a_{\sigma 1},\dots,a_{\sigma n})$

takes Hochschild cocycles into currents and it kills Hochschild coboundaries. This gives an explicit section

$$\begin{tikzcd}\Omega^n A_\natural \arrow[r, two heads, "\mu"'] & \Omega^n_A \arrow[l, dashed, bend right]\end{tikzcd}$$

and it allows one to describe n-currents (linear functions on Ω^n_A) as n-cocycles $f(a_0,\dots,a_n)$ which are alternating in $a_1,\dots,a_n$. Then the proof that $\Omega^n_A \xrightarrow{\rho\omega^n} R_A$ is well defined reduces to showing 1) $\rho\omega^n$ is a cocycle wrt b and 2) alternating with respect ~~all~~ the variables except the first. 2) we've done, ~~and~~ for 1) we use

$$(b(\rho\omega^n))(a_0,\dots,a_{2n+1})$$

$$= [(\rho\omega^n)(a_0,\dots,a_{2n}), \rho a_{2n+1}] + \omega^{n+1}(1+A)(a_0,\dots,a_{2n+1})$$

$$= [\rho a_0, \rho a_{2n+1}]\,\omega^n(a_1,\dots,a_{2n}) + \omega^{n+1}(a_0,\dots,a_{2n+1}) - \omega^{n+1}(a_{2n+1},a_0,\dots,a_{2n})$$

$$= \left(-\tfrac{1}{2}\right)^{n+1}\Big(-2[\rho a_0,\rho a_{2n+1}][\rho a_1,\rho a_2]\cdots[\rho a_{2n-1},\rho a_{2n}] + [\rho a_0,\rho a_1]\cdots[\rho a_{2n},\rho a_{2n+1}] - [\rho a_{2n+1},\rho a_0][\rho a_1,\rho a_2]\cdots[\rho a_{2n-1},\rho a_{2n}]\Big)$$

$= 0$ by the alternating character of $[\rho a_0,\rho a_1]\cdots[\rho a_{2n},\rho a_{2n+1}]$.

June 12, 1991

Given a homomorphism of algebras $S \longrightarrow A$, such that $A\otimes_S A$ is a projective bimodule over A, we would like to prove that the image of S in A is separable. So we can assume S is a subalgebra of A.

By hypothesis the surjection of bimodules

$$\pi : A\otimes A \longrightarrow A\otimes_S A \qquad 1\otimes 1 \longmapsto 1\otimes 1$$

has a section, which has to be given by an element $e\in A\otimes A$ satisfying $se = es$ $\forall s\in S$, and $\pi(e) = 1\otimes 1$. Let V, W be the smallest subspaces of A such that $e \in V\otimes W$. Then V, W are finite dimensional and they are canonically dual, e.g. there is a unique isom $W \simeq V^*$ such that e corresponds to the identity in $V\otimes V^* = \mathrm{Hom}(V,V)$. Write $e = \sum v_i\otimes w_i$, so that $\{v_i\}$ is a basis for V and $\{w_i\}$ is a basis for V^*. We can find linear functionals f_j on A such that $f_j(w_i) = \delta_{ij}$. Then

$$f_j(se) = f_j \sum_i s v_i \otimes w_i = \sum_i s v_i \underbrace{f_j(w_i)}_{\delta_{ji}}$$

$$\| \qquad = s v_j$$

$$f_j(es) = f_j \sum v_i \otimes w_i s = \sum_i v_i f_j(w_i s)$$

This shows $S\cdot V = V$. Similarly $WS = W$. So V is a left S-module and W is a right S-module, and because $se = es$, it follows that the dually mentioned above: $V \simeq W^*$ is an isomorphism of S-modules.

We now show the representation of S on V given by left multiplication is faithful. If $sv_j = 0$ for all j, then $se = 0$. But we have

$$\begin{tikzcd}
A\otimes A \arrow[r, "\pi"] & A\otimes_S A \arrow[r, "m"] & A \\
e \arrow[r, mapsto] & 1\otimes_S 1 \arrow[r] & 1
\end{tikzcd}$$

so $se = 0 \Rightarrow s = 0$ in $A \Rightarrow s = 0$ since we are assuming $S \subset A$. Thus $S \subset \mathrm{End}(V)$, ~~and~~ we see S is finite dimensional. We have

$$\begin{tikzcd}
V\otimes W \arrow[r, "?"] \arrow[d, phantom, "\cap"] & S \arrow[r, phantom, "="] \arrow[d, squiggly] & S \arrow[d, phantom, "\cap"] \\
A\otimes A \arrow[r, "\pi"] & A\otimes_S A \arrow[r] & A
\end{tikzcd}$$

Thus we reach the following situation: We have a subalgebra $S \subset \mathrm{End}(V)$, $\dim(V) < \infty$ and a projection $x \mapsto x^\natural$ from $\mathrm{End}(V)$ onto S which is an S-bimodule map:

$$(sx)^\natural = s x^\natural \qquad (xs)^\natural = x^\natural s$$

The question is whether this implies S is separable.

June 13, 1991

Review: Assume $S \subset A$ is a subalgebra and $A \otimes_S A$ is a projective A-bimodule. Choose bimodule section

$$\begin{tikzcd} A \otimes A \arrow[r] & A \otimes_S A \arrow[l, bend right, dashed] \end{tikzcd}$$

This is given by $e \in A \otimes A$ such that $\overbrace{e \longmapsto 1 \otimes_S 1}^{se = es\ \forall s \in S \text{ and}}$. We then have smallest subspaces $V, W \subset A$ such that $e \in V \otimes W \subset A \otimes A$, and we have $e = \sum v_i \otimes w_i$, where $V = \oplus \mathbb{C} v_i$, $W = \oplus \mathbb{C} w_i$.

Moreover V, W are in duality. From $se = es$, $\forall s \in S$ we derive $SV \subset V$, $WS \subset W$ and that $W = V^*$ as representations of S. From

$$\begin{tikzcd} e \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \arrow[rr, mapsto] & & 1 \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \\ V \otimes W \arrow[d, phantom, "\cap"] & & S \arrow[d] \arrow[rr, equal] & & S \arrow[d, phantom, "\cap"] \\ A \otimes A \arrow[rr] & & A \otimes_S A \arrow[rr, "m"] & & A \end{tikzcd}$$

we see that Se is finite dimensional and it maps isomorphically onto S. $\therefore$ S is finite dimensional.

Yesterday I made a mistake and assumed that there was a bimodule map $V \otimes W \longrightarrow S$ such that $e \longmapsto 1$. This doesn't seem to be true necessarily, however let us show it implies S is separable.

Lemma: Let V be a faithful finite dimensional representation of an algebra S, and assume that the obvious bimodule map $S \hookrightarrow V \otimes V^* = \operatorname{End} V$ is a direct injection. Then S is separable.

Proof (after what Benson told me)
Decompose V into indecomposables
$V = \bigoplus V_i$. Then we have a bimod
decomposition

$$\mathrm{Hom}_{\mathbb{C}}(V,V) = \bigoplus_{i,j} \underbrace{\mathrm{Hom}_{\mathbb{C}}(V_i,V_j)}_{V_i \otimes V_j^*}.$$

We claim that $V_i \otimes V_j^*$ is an indecomposable bimodule. It suffices to check its endomorphism ring is local. But

$$\mathrm{End}_{S\otimes S^{\circ}}(V_i \otimes V_j^*) = \mathrm{End}_S(V_i) \otimes \underbrace{\mathrm{End}_{S^{\circ}}(V_j^*)}_{\mathrm{End}_S(V_j)^{op}}$$

But the tensor product of f.d. local algebras is local:

$$R \otimes R' / \underbrace{m \otimes R' + R \otimes m'}_{\text{nilpotent}} = \underbrace{R/m}_{\mathbb{C}} \otimes \underbrace{R'/m'}_{\mathbb{C}} = \mathbb{C}$$

Now we have $S \hookrightarrow \mathrm{End}(V)$ is a direct summand as S-bimodule. As bimodule S decomposes uniquely into indecomposables — these are the blocks $e_k S$, where $1 = \sum e_k$ is the decomposition into central minimal idempotents. By Krull-Schmidt, for each k, $\exists\, i,j$ such that we have a bimodule iso.

$$\Phi : e_k S \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V_i, V_j)$$

From $\Phi(e_k s) = \Phi(e_k) s$ we see that $\Phi(e_k S) V_i \subset \Phi(e_k) V_i$, and so $\Phi(e_k) V_i = V_j$ otherwise there will be linear maps $V_i \to V_j$ not in the image of Φ.

Similarly from $\Phi(e_k s) = s\,\Phi(e_k)$ we have $\mathrm{Ker}(\Phi(e_k) \text{ on } V_i) \subset \mathrm{Ker}(\Phi(e_k)s \text{ on } V_i)$ and so Φ surjective $\Longrightarrow$ $\Phi(e_k): V_i \longrightarrow V_j$ is injective. Thus $\Phi(e_k): V_i \xrightarrow{\sim} V_j$ is an isomorphism of S-modules, and we have a comm. diag

$$\begin{tikzcd}
 & Se_k \arrow[dl, "\alpha"'] \arrow[dr, "\Phi\ \simeq"] & \\
\mathrm{Hom}(V_i, V_i) \arrow[rr, "\Phi(e_k)"] & & \mathrm{Hom}(V_i, V_j)
\end{tikzcd}$$

where α sends $s \in Se_k$ into mult by by s on V_i. Thus α is an algebra isomorphism of the block Se_k with the matrix algebra $\mathrm{Hom}(V_i, V_i)$. $\square$

Let's return to our mistake and try to construct an example in which $A \otimes_S A$ is a projective bimodule over A, but S is not separable. What have at the moment is the following. We know S is finite dimensional, that we have a finite-dimensional left S-module $V \subset A$ and right module $W \subset A$ such that $V^* \simeq W$, and such that

$$\begin{array}{ccccccccc}
S & \hookrightarrow & V \otimes V^* & \subset A \otimes A & \longrightarrow & A \otimes_S A & \longrightarrow & A \\
1 & \longmapsto & \sum v_i \otimes v_i^* & \longmapsto \sum v_i \otimes w_i & \longmapsto & 1 \otimes_S 1 & \longmapsto & 1.
\end{array}$$

The hard condition to analyze appears to be that $\sum v_i \otimes w_i \longmapsto 1 \otimes_S 1$.

Let us consider what we have inside A. We have the subalg S, the left S-module $V \subset A$ the right S-module $W \subset A$, the canonical element $c \in V \otimes W$ commuting with S. $c = \sum v_i \otimes w_i$

In A we have the image of the multiplication

$$V \otimes W \subset A \otimes A \xrightarrow{m} A$$

which gives the S-bimodules VW containing $\sum v_i w_i = 1$.

$$\begin{array}{ccc} & VW & \\ & \cup & \\ V & S & W \\ & \cup & \\ & \mathbb{C} & \end{array}$$

We can also consider WV which is the image of

$$\begin{tikzcd} V^* \otimes_S W \arrow[r] \arrow[d] & VW \arrow[d, phantom, "\cap"] \\ A \otimes_S A \arrow[r, "m"] & A \end{tikzcd}$$

It should be true that

$$\left(V^* \otimes_S V\right)^* = \left(V \otimes V^*\right)^S = \mathrm{Hom}(V,V)^S$$

Question: Are there interesting examples where $WV = \mathbb{C}$? In this case VW is a subalgebra.

Let's be more specific and assume the product $W \otimes_S V \longrightarrow WV = \mathbb{C}$ is the natural pairing of $W = V^*$ with V. Then it should be the case that the algebra VW is a quotient of $V \otimes V^* = \mathrm{End}(V)$, hence isomorphic to $\mathrm{End}(V)$ by simplicity.

Let's consider the algebra $A = \mathrm{End}(V)$ where V is a faithful finite dimensional S-module. We have lots of A-central elements $e \in A \otimes A$ lying over $1 \in A$. Can we find an e mapping to $1 \otimes_S 1$ in $A \otimes_S A$. The A-central elts. are

$$\sum_i v_i \otimes \alpha_i \otimes w_i^* \in V \otimes V^* \otimes V \otimes V^*$$

We have

$$A \otimes_S A = V \otimes (V^* \otimes_S V) \otimes V^*$$

So it looks like we have to analyze the map

$$\begin{tikzcd}
(V \otimes V^* \otimes V \otimes V^*)^S \arrow[r] & V \otimes (V^* \otimes_S V) \otimes V^* \arrow[d, dash, "\wr"] \\
(V^* \otimes V)^S \otimes V^* \otimes V \arrow[r] \arrow[u, "\wr"'] & V^* \otimes V \otimes (V^* \otimes_S V)
\end{tikzcd}$$

We need to be able to lift the element

$$\sum_{i,j} v_j^* \otimes v_i \otimes v_i^* \otimes v_j$$

on the right. The map has a cokernel which is $\left(V^* \otimes V / (V^* \otimes V)^S\right) \otimes (V^* \otimes_S V)$. Maybe it would be better to look at the image

$$(V^* \otimes V)^S \otimes (V^* \otimes_S V)$$

The two factors should be dual spaces.

Let's check carefully. We have

$$S \subset \underbrace{\mathrm{Hom}(V,V)}_{A} = V \otimes V^*.$$

We consider the obvious map

$$(A \otimes A)^S \longrightarrow A \otimes_S A$$

and we would like to see whether $1 \otimes_S 1$ lies in the image. (Observe that the two actions of S are different, that is, $A \otimes A$ is a bimodule over S in two ways which commute). Put in $A = V \otimes V^*$ and we are looking at

$$(V \otimes V^* \otimes V \otimes V^*)^S \longrightarrow (V \otimes V^* \otimes_S V \otimes V^*)$$

where S acts on the outside on the left. Now let's apply forward shift to get the obvious map

$$(V^* \otimes V)^S \otimes V^* \otimes V \longrightarrow V^* \otimes V \otimes (V^* \otimes_S V)$$

We have the element on the right which corresponds to $1 \otimes_S 1$ in $A \otimes_S A$. This is

$$\sum_{i,j} v_i \otimes v_i^* \otimes v_j \otimes v_j^* \in V \otimes V^* \otimes_S V \otimes V^*$$

and it becomes

$$\sum_{i,j} v_j^* \otimes v_i \otimes v_i^* \otimes v_j \in V^* \otimes V \otimes (V^* \otimes_S V)$$

after forward shift. Let us consider the element

$$\sum_{i,j} v_j^* \otimes v_i \otimes v_i^* \otimes v_j \in \underbrace{(V^* \otimes V)}_{} \otimes \underbrace{(V^* \otimes V)}_{}$$

and consider the two spaces [illegible] paired by coupling the insides and outsides:

~~[illegible]~~

$$\langle \lambda\otimes v \mid \lambda'\otimes v'\rangle = \langle \lambda | v'\rangle \langle \lambda' | v\rangle$$

Write Z for the first factor $V^*\otimes V$. It has the basis $z_{ji} = v_j^*\otimes v_i$

Let Z^* denote the second factor $V^*\otimes V$. It has the dual basis $z^*_{ji} = v_i^*\otimes v_j$. Then we are considering the canonical element

$$\sum z_{ji}\otimes z^*_{ji} \in Z\otimes Z^* .$$

Next consider the right S multiplication on $Z = V^*\otimes V$ which is $(\lambda\otimes v)s = \lambda s\otimes v$. Let's compute its transpose:

$$\langle (\lambda\otimes v)s \mid \lambda'\otimes v'\rangle = \langle \lambda s\otimes v \mid \lambda'\otimes v'\rangle$$

$$= \lambda(sv')\lambda'(v)$$

$$= \langle \lambda\otimes v \mid \lambda'\otimes sv'\rangle$$

We get left S multiplication on Z^*. Similarly the transpose of the left S mult on Z is the right S-mult on Z^*.

Thus we might think of Z as an $S\otimes S^\circ$ module, whence $Z\otimes Z^*$ is a bimodule over $S\otimes S^\circ$ and we are looking at the identity element

$$\begin{array}{ccc} Z\otimes Z^* & = & \mathrm{Hom}(Z,Z) \\ e & \longleftrightarrow & 1 \end{array}$$

Now we want to pass from $Z^* = V^*\otimes V$ on the right side to the quotient $V^*\otimes_S V$. A quotient of Z^* is of the form $Z^*/W^\perp$ where W is a subspace of Z. By definition of $V^*\otimes_S V$, the linear functionals on $Z^*/W^\perp = V^*\otimes_S V$ should be elements of Z equalized by left + right S-multiplication on Z. The quotient on Z^* should correspond to the subspace of Z. Thus we are looking at the surjection

$$\mathrm{Hom}(Z,Z) \longrightarrow \mathrm{Hom}(Z^S,Z) = Z \otimes \underbrace{(Z^S)^*}_{Z^*_S}$$

So what we are asking is whether in

$$\begin{tikzcd}
 & 1 \arrow[rr, leftrightarrow] & & 1\otimes 1 \\
 & \mathrm{Hom}(Z,Z) \arrow[d] & = & A\otimes A \\
\mathrm{Hom}(Z,Z^S) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(Z^S,Z) & = & A\otimes_S A \\
(A\otimes A)^S & & &
\end{tikzcd}$$

The image of $1 \in \mathrm{Hom}(Z,Z)$ $\overbrace{\text{(down in } \mathrm{Hom}(Z^S,Z)\text{)}}$ can be lifted to $\mathrm{Hom}(Z,Z^S)$. This is obvious.

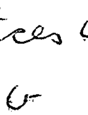
June 14, 1991

We know that finite [illegible]-dimensional quasi-free algebras are of the form $T_S(M)$ with S separable. Now $T_S(M)$ is [illegible] a graded algebra, so its periodic homology is the same as that of S, namely $S_\natural[0]$. This is also checked by the relative X-complex calculation

$$X(T_S(M);S): \quad S_\natural \oplus \bigoplus_{n\geq 1}[M\otimes_S]^n \underset{N}{\overset{1-\sigma}{\leftrightarrows}} \bigoplus_{n\geq 1}[M\otimes_S]^n$$

The question is whether a finite dimensional quasi-free algebra A can have nontrivial HH_1A, i.e. whether $HC_0A \neq HP_0A$. This apparently cannot happen and is probably in Benson's paper.

Up to Morita equivalence one can suppose $S = \underbrace{\mathbb{C}\times\cdots\times\mathbb{C}}$, whence $T_S(M)$ is the path algebra of a quiver with n-vertices. Then $[M\otimes_S]^{n-1}M$ has a basis consisting of all paths of length n, and finite-dimn [illegible] is equivalent to no oriented cycles in the quiver. Now $[M\otimes_S]^n$ should have as basis all (parametrized) loops of length n,* and there are none of these for $n\geq 1$ when there are no loops.
*(with n vertices)

Alternative method is to define a grading on the algebra such that the grading is specified by an inner automorphism in the spirit of triangular matrices . It suffices to assign integers n_v to [illegible] vertices v such that when there is an arrow from v to v' one has $n_v < n_{v'}$. One can take n_v to be the dimension of v, that is, the maximal length of a [illegible] path ending with v. Once we have the integers n_v we can attach to each path joining σ to σ' the degree $n_{\sigma'} - n_{\sigma}$. This gives a grading of the path algebra, and the automorphism $x \longmapsto t^{|x|} x$ is conjugation by the element of S which is $(t^{n_v})_{v \in \text{Vertices}}$.

The problem now is to get the relative case understood well enough to write. We consider $S \longrightarrow A$ (say $S \subset A$ to simplify) satisfying a separability condition, namely $A \otimes_S A$ is a projective A-bimodule (which is the case if S is separable). Then the surjective map

$$\Omega A \longrightarrow \Omega_S(A) \otimes_S$$

(which is compatible with d, b, etc.) is a quis with respect to b, and hence it gives an equivalence between the cyclic theory of A and the relative cyclic theory of A rel S.

June 15, 1991

Consider again $S \to A$ an algebras hom. such that $\exists\ e \in (A\otimes A)^S$ mapping to $1\otimes_S 1 \in A\otimes_S A$. First, notice that this condition uses only the bimodule structure of A over S and the map $S \to A$. Secondly, we saw that in the case of $S \to \mathrm{Hom}(V,V) = V\otimes V^*$, with V a finite-dimensional representation of S that such an e exists, but is apparently not unique. Here seems to be what happens. Given $e \in (A\otimes A)^S$ over $1\otimes_S 1$, we let $V, W \subset A$ be smallest subspaces such that $e \in V\otimes W$, whence we have $W = V^*$ and bimodule maps

$$\begin{tikzcd} S \arrow[r] & V\otimes V^* \arrow[r] & A \\ 1 \arrow[r, mapsto] & \sum v_i\otimes v_i^* \arrow[r, mapsto] & 1 \end{tikzcd}$$

Thus we have a homomorphism of algebras under S

$$R_S(V\otimes V^*) \longrightarrow A$$

On the other hand, given an algebra A with such a homomorphism, one obtains the desired element $e \in (A\otimes A)^S$ from the one we showed two days ago exists in $(V\otimes V^*\otimes V\otimes V^*)^S$. Therefore it seems we can prove

Prop. Given $S \to A$ a homomorphism of algebras, then $A\otimes_S A$ is a projective bimodule over A if and only if there exists a f.d. representation V of S and a factorization

$$S \longrightarrow R_S(V\otimes V^*) \longrightarrow A$$

of $S \to A$.

Let's check a few things.

If $A\otimes_S A$ is a projective A-bimodule, and $A \to A'$ is a homom., then

$$A'\otimes_A (A\otimes_S A)\otimes_A A' = A'\otimes_S A'$$

is a projective A'-bimodule.

Another point is that if $S\to M$ is a S-bimodule map such that $e\in (M\otimes M)^S$ exists over $1\otimes_S 1 \in M\otimes_S M$, then we don't seem to get a factorization $S\to V\otimes V^*\longrightarrow M$, we need the algebra structure of A for this.

Another interpretation of e is a bimodule lifting

$$\begin{tikzcd}
 & S \arrow[dl, dashed] \arrow[d] \\
M\otimes M \arrow[r] & M\otimes_S M
\end{tikzcd}$$

A natural question is what bimodules occur in the form $V\otimes V^*$.

June 17, 1991

Recall for $S \to A$ that one has $\mathrm{Tor}_1^S(A,A) = 0$, $\Omega^1_S A$ projective bimodule over A $\Longrightarrow$ A relatively quasi-free under S.

Proof uses the first condition to identify square-zero extensions of algebras under S of A with bimodule extensions of $\Omega^1_S A$, then second condition to see such extensions are trivial. I think the converse should also be true.

Check: S quasi-free, A rel quasi-free under S $\Longrightarrow$ A quasi-free. Because

$$0 \to \underset{\substack{\| \\ 0}}{\mathrm{Tor}_1^S(A,A)} \to \underbrace{A \otimes_S \Omega^1 S \otimes_S A}_{\text{proj.}} \longrightarrow \Omega^1 A \longrightarrow \underbrace{\Omega^1_S A}_{\text{proj.}} \longrightarrow 0$$

Suppose S quasi-free and A rel quasifree under S. Then consider

$$X(A) \twoheadrightarrow X_S(A)$$

The kernel we know is

$$\begin{tikzcd}
{[A,S]} \arrow[r, shift right] & A \otimes_S \Omega^1 S \otimes_S \arrow[l, shift right, "b"']
\end{tikzcd}$$

since $\mathrm{Tor}_1^S(A,A) = 0 \Longrightarrow A\,dS\,A = A \otimes_S \Omega^1 S \otimes_S A$. Thus we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & HP_0 A \arrow[r] & HP_0(A;S) \arrow[r] & H_1(S,A) \arrow[dll, rounded corners] & \\
 & HP_1 A \arrow[r] & HP_1(A;S) \arrow[r] & 0 &
\end{tikzcd}$$

Difficulties: The condition that $\Omega^1_S A$ be a projective bimodule over A is very strong. If S is smooth commutative of dim 1, then

$$\Omega^1_S(S\otimes B) = S\otimes\Omega^1 B$$

is not a projective $S\otimes B$ bimodule. One has to *distinguish* between *algebras* *under S* and *algebras over the comm. alg. S*, where S always has to *map* to the center.

Question: Suppose given $S \longrightarrow A$ with S separable. Then $R_S A$ is quasi-free. We can use both RA and $R_S A$ to compute the cyclic theory of A.

June 18, 1991

Let us consider $RA \twoheadrightarrow R_S A$. This is a map of extensions of A. When S is separable, the algebra $R_S A$ is quasi-free, in addition to RA being quasi-free. Then we know that both extensions can be used to compute the cyclic theory of A. Specifically, let $J = \operatorname{Ker}(RA \to R_S A)$; it is the ideal generated by $\omega(s,a)$, $\omega(a,s)$ for $s \in S$, $a \in A$. Then we can lift $R_S A$ into $\varprojlim_n RA/J^n$; moreover, $R_S A$ becomes an SDR of $\varprojlim RA/J^n$. In particular $\hat{R}_S A$ becomes an SDR of $\hat{R}A$, and so we know that we have an SDR situation for the towers $\mathcal{X}^q(R_S A, I_S A)$, $\mathcal{X}^q(RA, IA)$.

There are many points in this business which are unclear. ~~for what reason? The~~ First of all, the final result seems to be true under the weaker assumption that the bimodule $A \otimes_S A$ is projective. This we see from the "~~Hochschild~~" approach. More precisely, this means we have two projective resolutions

$$\cdots \xrightarrow{b'} \Omega^1 A \otimes A \xrightarrow{b'} A \otimes A \dashrightarrow A \dashrightarrow 0$$

$$\xrightarrow{b'} \Omega^1_S A \otimes_S A \xrightarrow{b'} A \otimes_S A \dashrightarrow A \dashrightarrow 0$$

and the lower is an SDR of the upper. This means $(\Omega_S A \otimes_S, b)$ is an SDR of $(\Omega A, b)$, and then using HPT one extends this to include the B operator and the whole cyclic theory.

The basic problem seems to be to find the link which should exist between the "Hochschild" approach and the universal extension approach.

Question: Assuming only that $A \otimes_S A$ is a projective bimodule is the inverse system of algebras $\{R_S A / I_S A^{n+1}\}$ a retract of the system $\{RA/IA^{n+1}\}$? Can you actually produce a lifting $\Omega_S A \longrightarrow \Omega A$ which is a homomorphism with respect to the Fedosov product? Let's observe this is not completely trivial since for separable S we are asking for a lifting homomorphism $S \longrightarrow \hat{R}S$. (The existence is clear, but the Hochschild approach gives a formula whose analogue in the universal extension approach is not yet known.

Here's a first step. Let's look at square zero extensions. We would like a lifting homomorphism

$$R_S A / I_S A^2 \longrightarrow RA/IA^2$$

which certainly implies that any square zero extension of A becomes trivial over S. This can be seen to be true as follows. A square zero extension of A ~~extension~~ is equivalent to a bimodule extension of $\Omega^1 A$. Now we have a bimod. splitting of

$$0 \longrightarrow A\,dS\,A \longrightarrow A \otimes A \longrightarrow A \otimes_S A \longrightarrow 0$$

which implies that $A\,dS\,A$ is a projective bimodule. Thus any bimodule extension of $\Omega^1 A$ becomes trivial when pulled back to $A\,dS\,A$, hence trivial as S-bimodule extension when pulled back to $\Omega^1 S$, which means that the corresponding square zero algebra extension of A becomes trivial when pulled back to S.

Let us observe further that the bimodule extension

$$0 \longrightarrow \Omega^2 A \longrightarrow \Omega^1 A \otimes A \longrightarrow \Omega^1 A \longrightarrow 0$$

corresp to RA/IA^2 decomposes as follows.

$$\begin{aligned}
\Omega^1 A \otimes A &= \Omega^1 A \otimes_A (A \otimes A) \\
&= (AdSA \oplus \Omega^1_S A) \otimes_A (AdSA \oplus A \otimes_S A) \\
&= (AdSA \otimes_A AdSA) \oplus (\Omega^1_S A \otimes_A AdSA) \\
&\quad \oplus (AdSA \otimes_S A) \oplus (\Omega^1_S A \otimes_S A)
\end{aligned}$$

~~Thus we have a map of bimod extensions~~

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2_S A \arrow[r] \arrow[d] & \Omega^1_S A \otimes_S A \oplus AdSA \arrow[r] \arrow[d] & \Omega^1_S A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^2 A \arrow[r] & \Omega^1 A \otimes A \arrow[r] & \Omega^1 A \arrow[r] & 0
\end{tikzcd}$$

One ought to be able to use this to construct a lifting $R_S A / I_S A^2 \longrightarrow RA/IA^2$.

June 19, 1991

There is a basic problem we seem to be running into, namely, homological algebra methods produce maps of cyclic theories that are not ~~apparently~~ obviously related to the homomorphisms on the level of R-algebras.

A simple example: suppose A separable, whence there is a canonical splitting of

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1 A \arrow[r] & A\otimes A \arrow[r, shift right] & A \arrow[l, dotted, shift right] \arrow[r] & 0 \end{tikzcd}$$

and thus a canonical SDR situation $\begin{tikzcd}(\Omega A, b) \arrow[r, shift right] & A_\natural \arrow[l, dotted, shift right]\end{tikzcd}$. This leads by HPT to an [redacted] SDR equivalence $(\hat{\Omega} A, b+B) \longrightarrow A_\natural[0]$. I'd like to show there is a canonical lifting $A \longrightarrow \hat{R}A$ that corresponds to this choice of splitting above.

It's possible that I am not trying to prove something strong enough. The fact is we should have a SDR situation on the level of algebras.

Suppose A quasi-free. Then we have a lifting homomorphism $A \longrightarrow \hat{R}A$. We then have a surjection of A-bimodules

$$\hat{I}A \longrightarrow \Omega^2 A$$

and we can choose a lifting $\Omega^2 A \longrightarrow \hat{I}A$, which is a map of A-bimodules, since $\Omega^2 A = \Omega^1 A \otimes_A \Omega^1 A$ is a projective bimodule. This extends to a homom.

$$\hat{T}_A(\Omega^2 A) \longrightarrow \hat{R}A$$

which has to be an isomorphism, because we know that $\operatorname{gr} \hat{R}A = \Omega^+ A = T_A(\Omega^2 A)$.

Thus we see $\hat{R}A$ is isomorphic to $\hat{T}_A(\Omega^2 A)$, which is a graded algebra, and this makes the deformation of the identity map $\hat{R}A \to \hat{R}A$ to $\hat{R}A \longrightarrow A \longrightarrow \hat{R}A$ transparent.

A further thing we obtain is that $\hat{Q}A$ is isomorphic as superalgebra to $\hat{\Omega}A$. In effect we have the lifting $A \longrightarrow \hat{R}A \subset \hat{Q}A$, so $\hat{Q}A$ becomes an A-bimodule, the bimodule structure being compatible with the $\mathbb{Z}/2$-grading. We consider the obvious surjection

$$(\hat{J}A)^- \longrightarrow \Omega^1 A$$

as a surjection of bimodules over A, and choose a lifting. This gives then a homom.

$$\hat{T}_A(\Omega^1 A) \longrightarrow \hat{Q}A$$

of superalgebras, which has to be an isomorphism.

All this works very nicely, but there is a puzzle in that from the b-complex viewpoint there is a canonical way to proceed after one makes the first few steps. A connection in the A-bimodule $\Omega^1 A$ determines an SDR equivalence

$$(\hat{\Omega}A, b) \longrightarrow (X(A), b)$$

in a fairly canonical way.

Suppose A quasi-free. Let's recall our idea for constructing a lifting homomorphism $A \longrightarrow \hat{R}A$. Such a homomorphism gives a natural way to project normalized linear maps $\rho : A \longrightarrow R$, which are homomorphisms modulo some nilpotent ideal, into homomorphisms $A \longrightarrow R$. The idea is to introduce a kind of Yang-Mills flow on the space of all ρ which decreases the curvature.

Let's review the derivation of formula for the flow. Given ρ we consider variation $\rho + \varepsilon\dot\rho$. The curvature is

$$b'(\rho+\varepsilon\dot\rho) - (\rho+\varepsilon\dot\rho)^2$$

$$= \underbrace{b'\rho - \rho^2}_{\omega} + \varepsilon(b'\dot\rho - \rho\dot\rho - \dot\rho\rho) + O(\varepsilon^2)$$

Thus to decrease the curvature we try to take

$$\rho\dot\rho - b'\dot\rho + \dot\rho\rho = +\omega$$

i.e.

$$\rho(a_1)\dot\rho(a_2) - \dot\rho(a_1a_2) + \dot\rho a_1\,\rho a_2 = \omega(a_1,a_2)$$

Now except for the fact that ρ is not a homomorphism, this says that the coboundary of $\dot\rho$ is ω, and we can solve $\delta\dot\rho = +\omega$ if ω were a cocycle, because A is quasi-free. A solution is $\dot\rho a = +(\rho\omega)\underbrace{\nabla(da)}_{\varphi a}$. Indeed

$$(a_1+\varphi a_1)\circ(a_2+\varphi a_2) \equiv a_1a_2 - da_1da_2 + \varphi a_1\,a_2 + a_1\varphi a_2$$

$$\equiv a_1a_2 + \varphi(a_1a_2)$$

(modulo IA^2) says that

$$(\delta\varphi)(a_1,a_2) = a_1(\varphi a_2) - \varphi(a_1a_2) + (\varphi a_1)a_2 = da_1da_2 = \omega(a_1,a_2)$$

So we take our flow to be

$$\boxed{\dot\rho = +(\rho\omega)\circ\varphi}$$

But because of the universal property of RA, such a natural vector field on normalized linear maps $\rho : A \to R$ for any R is equivalent to a derivation D on RA. Using the usual identification $RA = \Omega^+A$ we find D is the derivation such that

$$Da = +\varphi(a)$$

Notice that $D(RA) \subset IA$, hence

$$D(IA^n) \subset IA^n$$

for all n, and D induces a derivation on $\mathrm{gr}\, RA$, which is $\Omega^+ A$ with the usual multiplication. $D = 0$ on $\mathrm{gr}^0 = A$. Let's calculate D on $\mathrm{gr}^1 = \Omega^2 A$. We have

$$D(a_0 da_1 da_2) \equiv a_0 D(da_1 da_2) \pmod{IA^2}$$

$$D(da_1 da_2) = D(a_1 a_2 - a_1 \circ a_2)$$

$$= +\varphi(a_1 a_2) - Da_1 \circ a_2 - a_1 \circ Da_2$$

$$\equiv +\varphi(a_1 a_2) - \varphi a_1 \circ a_2 - a_1 \circ \varphi a_2$$

$$\equiv +\varphi(a_1 a_2) - \varphi a_1 a_2 - a_1 \varphi a_2$$

$$\equiv - da_1 da_2$$

Thus $D = -1$ on $\mathrm{gr}^1_{IA} RA$, and hence $D = -n$ on $\mathrm{gr}^n_{IA} RA$. Thus the eigenspaces of D on RA/IA^{n+1} will give a grading, i.e. an algebra isomorphism

$$RA/IA^{n+1} = \bigoplus_{g \le n} \Omega^{2g} A$$

and hence an alg. isomorphism $\hat{R}A \simeq \Omega^+ A$ for the usual product on forms.

The next question is whether we can extend the derivation D to QA, since we know that $\hat{Q}A \simeq \Omega A$. Recall that we have the homomorphism

$$\theta a = a + da$$

from A to QA:

$$(a_1 + da_1) \circ (a_2 + da_2) = (a_1 a_2 - \cancel{da_1 da_2}) + (a_1 da_2 + da_1 a_2) \cancel{+ da_1 da_2}$$

$$= a_1 a_2 + d(a_1 a_2)$$

We would like to define

$$\mathring{\theta} a = Da + D(da)$$

so that $\mathring{\theta}$ is a [redacted] derivation relative to θ. Of course we assume $D(da)$ is odd, [redacted] so the other homomorphism and derivation are

$$\theta^\gamma a = a - da$$

$$\mathring{\theta}^\gamma a = Da - D(da).$$

Let's try to find what $D(da)$ should be. We have

$$D(da_1 da_2) = D(a_1a_2) - Da_1 \circ a_2 - a_1 \circ Da_2$$

$$= \underbrace{\varphi(a_1a_2) - (\varphi a_1)a_2 - a_1(\varphi a_2)}_{-da_1da_2} + d(\varphi a_1)da_2 + da_1 d(\varphi a_2)$$

$$D(da_1 da_2) = D(da_1)da_2 + da_1 D(da_2)$$

Thus it appears that we want

$$\boxed{D(da) = -\tfrac{1}{2}da + d(\varphi a)}$$

Thus we [redacted] want to check that

$$\mathring{\theta} a = \varphi a - \tfrac{1}{2}da + d(\varphi a)$$

is a derivation relative to $\theta a = a + da$. Let's do this brutally

$$\mathring{\theta}(a_1a_2) = \varphi(a_1a_2) - \tfrac{1}{2}d(a_1a_2) + d(\varphi(a_1a_2))$$

$$= \overset{①}{a_1(\varphi a_2)} + \overset{②}{(\varphi a_1)a_2} \overset{③}{- da_1da_2} - \tfrac{1}{2}\overset{④}{da_1a_2} - \tfrac{1}{2}\overset{⑤}{a_1da_2}$$

$$+ \overset{⑥}{da_1\varphi a_2} + \overset{⑦}{a_1d(\varphi a_2)} + \overset{⑧}{d(\varphi a_1)a_2} + \overset{⑨}{\varphi a_1da_2}$$

$$\mathring{\theta} a_1 \circ \theta a_2 = (\varphi a_1 - \tfrac{1}{2}da_1 + d(\varphi a_1)) \circ (a_2 + da_2)$$

$= \overset{②}{\varphi a_1 a_2}$ [redacted] $\cancel{d(\varphi a)da_2} + \overset{⑨}{\varphi a_1 da_2} - \tfrac{1}{2}\overset{④}{da_1a_2} - \tfrac{1}{2}\overset{③}{da_1da_2}$

$+$ [redacted] $+ \cancel{d(\varphi a_1)da_2}$

⑧ $d(\varphi a_1)a_2$

$$\theta a_1 \circ \mathring{\theta} a_2 = (a_1 + da_1) \circ \left(\varphi a_2 - \tfrac{1}{2} da_2 + d(\varphi a_2)\right)$$

$$= \overset{①}{a_1} \varphi a_2 - \cancel{da_1 d(\varphi a_2)} + \overset{⑥}{da_1} \varphi a_2$$

$$- \tfrac{1}{2} \overset{⑤}{a_1} da_2 - \tfrac{1}{2} \overset{③}{da_1} da_2 + \overset{⑦}{a_1} d(\varphi a_2) + \cancel{da_1 d(\varphi a_2)}$$

Example: $A = \mathbb{C}[F]$, $D(F) = \varphi(F) = \frac{1}{2} F dF^2$

Then $D(F dF^{2n}) = \frac{1}{2} F dF^{2n+2} + F(2n)(dF^{2n-1})(-\frac{1}{2} dF + \frac{1}{2} dF^3)$

$$= -(n) F dF^{2n} + (n + \tfrac{1}{2}) F dF^{2n+2}$$

where we use $D(dF) = -\frac{1}{2} dF + \underbrace{\frac{1}{2} dF^3}_{d(\varphi F)}$.

So

$$D\left(\sum c_n F dF^{2n}\right) = \sum_{n \geq 0} c_n \left(-(n) F dF^{2n} + (n + \tfrac{1}{2}) dF^{2n+2}\right)$$

$$= \sum_{n \geq 0} \left(-(n) c_n + (n - \tfrac{1}{2}) c_{n-1}\right) F dF^{2n}$$

This is zero iff $n c_n = (n - \frac{1}{2}) c_{n-1}$ i.e.

$$c_n = \frac{(n - \frac{1}{2}) \cdots \frac{3}{2} \cdot \frac{1}{2}}{n!} = \frac{(2n-1)!!}{2^n n!}$$

So the involution in $\hat{R}A$ lifting F is

$$\sum_{n \geq 0} \frac{(2n-1)!!}{2^n n!} F dF^{2n} = z \sum_{n \geq 0} \frac{(-\frac{1}{2})(-\frac{3}{2}) \cdots (-\frac{2n-1}{2})}{n!} \left(-(1 - \underbrace{F \circ F}_{z^2})\right)^n$$

$$= z \left(1 - (1 - z^2)\right)^{-1/2}$$