March 2, 1991

If Ω is a DG algebra, define the Fedosov product to be

$$x \circ y = xy + c(-1)^{|x|} dx\, dy$$

Then
$$x \circ y \circ z = xyz + c\{(-1)^{|x|} dx\, dy\, z + (-1)^{|x|+|y|} dx\, y\, dz + (-1)^{|y|} x\, dy\, dz\}$$

$$x \circ y \circ z \circ w = xyzw +$$
$$c\{(-1)^{|x|} dx\, dy\, z\, w + (-1)^{|x|+|y|} dx\, y\, dz\, w$$
$$+ (-1)^{|x|+|y|+|z|} dx\, y\, z\, dw + (-1)^{|y|} x\, dy\, dz\, w$$
$$+ (-1)^{|y|+|z|} x\, dy\, z\, dw + (-1)^{|z|} x\, y\, dz\, dw\}$$
$$+ c^2\{(-1)^{|x|+|z|} dx\, dy\, dz\, dw\}$$

Now suppose Ω (super) commutative. We have

$$x \circ y = xy + c(-1)^{|x|} dx\, dy$$
$$y \circ x = yx + c(-1)^{|y|} dy\, dx$$
$$= (-1)^{|x||y|} xy + c(-1)^{|y|}(-1)^{|x||y|+|x|+|y|+1} dx\, dy$$
$$(-1)^{|x||y|} y \circ x = xy + c(-1)^{|x|+1} dx\, dy$$

$$\boxed{[x,y]^{\circ}_{s} = 2c(-1)^{|x|} dx\, dy}$$

Conclude that <u>any</u> <u>closed linear function on Ω is a supertrace relative to the Fedosov product.</u>

This is nice because it [redacted] allows one to associate to closed currents on A (currents are linear functions on Ω_A, A supposed commutative) higher traces in the sense of extensions. The Fedosov[redacted] algebra $(\Omega_A^+, \circ)$ is a nilpotent extension of A (assume A noetherian) and an even closed current gives a trace on this Fedosov algebra. On the other hand Ω_A^- is a bimodule over Ω_A^+ and an odd closed current gives an odd supertrace on $(\Omega_A, \circ)$, hence a trace on [redacted] Ω_A^- as bimodule over $(\Omega_A^+, \circ)$. We have that d is a superderivation on $(\Omega_A, \circ)$:

$$d(x \circ y) = d(xy) = dx\, y + (-1)^{|x|} x\, dy$$
$$= dx \circ y + (-1)^{|x|} x \circ dy.$$

Thus we have a derivation of $(\Omega_A^+, \circ)$ with values in a bimodule with trace, hence a Hochschild 1-cocycle, in fact, a cyclic 1-cocycle on $(\Omega_A^+, \circ)$ associated to an odd closed current. In down to earth terms, let $\int$ be an odd current, and consider

$$(x, y) \longmapsto \int x \circ dy = \int x\, dy$$

on Ω_A^+. Then

$$\int (x \circ y)\, dz - \int x\, d(y \circ z) + \int (z \circ x)\, dy$$

$$= \int (xy + c\, dx\, dy)\, dz - \int x (dy\, z + \text{[redacted]}\, y\, dz)$$
$$+ \int (zx + c\, dz\, dx)\, dy$$

$$= c \underbrace{\left(\int dx\, dy\, dz + dz\, dx\, dy \right)}_{\substack{=0 \\ \text{because } \int d\Omega = 0}} + \underbrace{\int zx\, dy - x\, dy\, z}_{\substack{=0 \text{ because} \\ \int [\Omega^+, \Omega^-] = 0}}$$

not sign cancellation

Note that we don't have to assume Ω_A bounded; we want the current to be bounded (e.g. to be homogeneous), then we can truncate Ω_A to get a nilpotent extension.

At this point we have described how to map cycles (i.e. closed linear functions on Ω_A) into cyclic cohomology represented by traces & cyclic 1-cocycles on nilpotent extensions. Now we want to improve things to a map of complex. We have the map

$$\begin{array}{cccc} X(RA): & RA & \underset{\partial}{\overset{\bar b}{\rightleftarrows}} & \Omega^1(RA)_\natural \\ \downarrow & \Phi^+ \downarrow & & \downarrow \Phi^- \\ \Omega_A: & \Omega_A^+ & \underset{Nd}{\overset{-2d}{\rightleftarrows}} & \Omega_A^- \end{array}$$

but we want the appropriate ~~quotient~~ of RA through which it factors. We have

$$\Phi^+ \circ (\rho\,\omega^n) = \theta d\theta^{2n} \qquad \Phi^-(\rho\omega^n d\rho) = \theta d\theta^{2n+1}$$

e.g.
$$\Phi^+(\rho a_0\, \omega^n(a_1,\dots,a_{2n})) = a_0 da_1 \dots da_{2n}$$
$$\Phi^-(\rho a_0\, \omega^n(a_1,\dots,a_{2n})\, d(\rho a_{2n+1})) = a_0 da_1 \dots da_{2n+1}$$

We know that Φ^+ is an algebra homomorphism if we equip Ω_A^+ with the Fedosov product with $c=-1$. Let's check this. It suffices to show $\Phi^+(\rho a x) = a \circ \Phi^+ x$ for all $x \in RA$, and we can suppose $x = \rho a_0 \omega^n(a_1,\dots,a_{2n})$. Then $\Phi^+(\rho a x) = \Phi^+(\rho(aa_0)\omega^n(a_1,\dots,a_{2n}) - \omega(a,a_0)\omega^n(a_1,\dots,a_{2n}))$

$$= a a_0 da_1 \dots da_{2n} - da\, da_0 \dots da_{2n} = a \circ \underbrace{(a_0 da_1 \dots da_{2n})}_{\Phi^+(x)}$$

We want to think of Φ^- as a trace

$$\begin{array}{c} \Phi^- : \Omega^1(RA) \longrightarrow \Omega_A^- \\ \qquad\quad \| \\ \qquad\quad RA \otimes d(\rho A) \otimes RA \end{array}$$

We have

$$\boxed{\Phi^-(x\, d(\rho a)) = \Phi^+ x\, da}$$

for all $x \in RA$; in effect one can assume $x = \rho a_0\, \omega^n(a_1, \ldots, a_{2n})$, whence it is the defn. of Φ^-. In general a trace on $\Omega^1(RA)$ is equivalent to its restriction to $RA\, d(\rho A) \subset \Omega^1(RA)$, and this restriction can be any linear functional. The above formula shows this ~~restriction~~ is a left module map rel. to Φ^+.

Let's calculate:

$$\Phi^-(x\, d(\rho a) y) = \Phi^-(yx\, d(\rho a))$$

$$= \Phi^+(yx)\, da = \Phi^+ y \circ \Phi^+ x\, da$$

$$= \underbrace{\Phi^+ x\, da \circ \Phi^+ y}_{\begin{array}{c}\Phi^+ x\, da\, \Phi^+ y \\ + c(-1)\, d\Phi^+ x\, da\, d\Phi^+ y\end{array}} + \underbrace{[\Phi^+ y, \Phi^+ x\, da]_s^{\circ}}_{\begin{array}{c}2c\, d\Phi^+ y\, d\Phi^+ x\, da \\ = +2c\, d\Phi^+ x\, da\, d\Phi^+ y\end{array}}$$

March 3, 1991

Let Ω be a DG algebra which is commutative. Let $R = \Omega^+$ equipped with Fedosov product $x \circ y = xy - dx\,dy$. Define a bilinear function on R with values in Ω^- by

$$\Phi(x,y) = (|y|+1)\,x\,dy + |y|\,dx\,y$$

I claim Φ is a Hochschild 1-cocycle on R

$$\Phi(x\circ y, z) = (|z|+1)(xy - dx\,dy)\,dz + |z|(dx\,y + x\,dy)\,z$$

$$\Phi(z\circ x, y) = (|y|+1)(zx - dz\,dx)\,dy + |y|(dz\,x + z\,dx)\,y$$

$$\Phi(x, y\circ z) = \Phi(x, yz) - \Phi(x, dy\,dz)$$

$$= (|y|+|z|+1)\,x\,(dy\,z + y\,dz) + (|y|+|z|)\,dx\,y\,z - \underbrace{(|dy| + |dz|)}_{|y|+1+|z|+1}\,dx\,dy\,dz$$

Check the following commutes

$$\begin{tikzcd} R \arrow[r, shift right, "\bar d"'] \arrow[d, "="'] & \Omega^1 R \arrow[l, shift right, "\bar b"'] \arrow[d, "\Phi"] & x\,\delta y \arrow[d, mapsto] \\ \Omega^+ \arrow[r, shift right, "Nd"'] & \Omega^- \arrow[l, shift right, "-2d"'] & \Phi(x,y) \end{tikzcd}$$

use δ for the diff in $X(R)$

$$\bar b(x\,\delta y) = [x,y]^\circ = -2\,dx\,dy$$

$$d\,\Phi(x,y) = d\{(|y|+1)\,x\,dy + |y|\,dx\,y\} = (|y|+1-|y|)\,dx\,dy = dx\,dy$$

Thus $\quad \bar{b} = -2d\,\Phi \quad$ on $\Omega^1 R_q$.

$$\Phi\,\bar{d}(x) = \Phi(1,x) = (|x|+1)\,dx = N\,dx.$$

Problem: How to handle Weyl algebras and more generally, generalized universal enveloping algebras. The latter are algebras A equipped with an increasing filtration F_pA such that $F_{-1}A = 0$, $\bigcup F_pA = A$, $\operatorname{gr} A \cong S(\operatorname{gr}_1 A)$.

Structure: Observe $F_1 A$ is a Lie algebra under bracket and $F_0 A = \mathbb{C}$ is in the center. So we have a central extension of Lie algebras

$$0 \longrightarrow \mathbb{C} \longrightarrow F_1A \longrightarrow F_1A/F_0A \longrightarrow 0$$

Notation: $F_1A = \tilde{\mathfrak{g}}$, $\quad F_1A/F_0A = \mathfrak{g}$

Then A is $U(\tilde{\mathfrak{g}})$ divided by the ideal generated by $1 - 1_{\tilde{\mathfrak{g}}}$. (Further, the graded alg.

$$\bigoplus_{p\geq 0} (F_pA)\,h^p \subset A[h]$$

is it isomorphic to $U(\tilde{\mathfrak{g}})$? Apparently not!)

Now Brylinski + Kassel analyze the cyclic homology of such an algebra by constructing a mixed complex, which I think has the following form. Firstly, if we ignore the differential b, it is the same as for the polynomial ring $S(\mathfrak{g})$. Recall that in this case the mixed complex is the DR complex $\Omega_{S(\mathfrak{g})} = S(\mathfrak{g}) \otimes \Lambda\mathfrak{g}$ with $b = 0$ and $B = d$. This is the De Rham complex of forms on $\mathfrak{g}^*$.

Actually we should be more careful in that from the viewpoint of filtered algebra, even in the commutative case we have an extension

$$0 \longrightarrow \mathbb{C} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

and so there is no natural way to embed $\mathfrak{g}$ in A. Thus $\mathfrak{g}^*$ is to be replaced by the splittings of the above sequence which is an affine space under $\mathfrak{g}^*$. The corresponding DR complex is the space of (polynomial coefficients) differential forms on this affine space.

Recall that the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ has a "moment" type structure, which maybe amounts to it being a Poisson manifold, i.e. a Poisson bracket structure on functions. The polynomial functions on $\mathfrak{g}^*$ form the algebra $S(\mathfrak{g}) = gr\, U(\mathfrak{g})$ which we know has a Poisson bracket induced by the bracket in $U(\mathfrak{g})$. Notice that the Poisson bracket of linear functions is linear. It seems clear that we are looking at <u>affine Poisson manifolds</u> in some sense.

Let's go over this. An affine space has its algebra of polynomial functions, which is a filtered algebra whose graded algebra is a polynomial rings. If we give a Poisson bracket such that the bracket of linear functions is linear (here linear means degree ≤ 1), then we get a Lie algebra out of the linear functions containing the scalars $\mathbb{C}$ in the center.

March 4, 1991

A Poisson structure on a manifold M is defined to be a bilinear map $f,g \longmapsto \{f,g\}$ on $C^\infty(M)$ satisfying

1) biderivation: ($\{f, ?\}$ derivation for f fixed, 2) $\{\ ,\ \}$ defines a Lie algebra structure on $C^\infty(M)$.

1) implies that $\{f,g\}(x)$ depends only on $df(x)$, $dg(x)$ so that at each $x \in M$ we have a bilinear form on $T(x)$, necessarily skew symmetric by 2). Thus a Poisson structure is specified by a section ω of $\Lambda^2 T$.

Let $J = \omega \lrcorner$ be the corresponding degree -2 operations on $\Omega(M)$. Locally if

$$J = \frac{1}{2}\omega_{jk}\, \iota_k \iota_j \qquad \iota_k = \iota(\partial_k)$$

then

$$Jdf\,dg = \frac{1}{2}\omega_{jk} \underbrace{\iota_k \iota_j (df\,dg)}_{\partial_j f \partial_k g - \partial_k f \partial_j g}$$

$$= \omega_{jk}\, \partial_j f\, \partial_k g$$

Thus $\boxed{\{f,g\} = Jdf\,dg = \omega_{jk}\, \partial_j f\, \partial_k g.}$

Now define b to be the degree -1 operator

$$\boxed{b = [J, d]}$$

so that $\boxed{[d,b] = 0}$ and

$$\boxed{b(f\,dg) = Jd(f\,dg) = Jdf\,dg = \{f,g\}}$$

Then

$$[b, f] = [[J, d], f] = [J, [d f]] = [J, df]$$

$$= \frac{1}{2}\omega_{jk}\left(\iota_k[\iota_j, df] - [\iota_k, df]\iota_j\right)$$

$$= \frac{1}{2}\omega_{jk}\left(\iota_k \partial_j f - \partial_k f \iota_j\right)$$

$$= \omega_{jk}\partial_j f \iota_k = \iota(X_f)$$

where $X_f = \omega_{jk}\partial_j f \partial_k$ is the vector field such that $X_f g = \{f, g\}$. Thus

$$\boxed{[b, f] = i(X_f)}$$

and $$\underset{\underset{-[b, df]}{\parallel}}{[d, [b, f]]} = [d, \iota(X_f)] = \mathcal{L}(X_f)$$

so $$\boxed{[b, df] = -\mathcal{L}(X_f)}$$

Now let's compute $b^2(f\,dg\,dh)$. We have

$$b(f\,dg\,dh) = i(X_f)dg\,dh + f\,b(dg\,dh)$$

$$= \{f, g\}dh - dg\{f, h\} - f\,d\,b(g\,dh)$$

$$= \{f, g\}dh - dg\{f, h\} - f\,d\{g, h\}$$

$$b^2(f\,dg\,dh) = \{\{f, g\}, h\} - \{\{f, h\}, g\} - \{f, \{g, h\}\}$$

so $b^2 = 0 \Longrightarrow$ Jacobi identity

March 6, 1991

$$A = \underbrace{T(V)}_{R} \Big/ \underbrace{([v_1, v_2] = \omega(v_1, v_2))}_{I}, \quad \text{a}$$

Heisenberg algebra. We propose to compute the square zero extension R/I^2 as the fibre product

$$\begin{tikzcd}
R/I^2 \arrow[r] \arrow[d] & A \arrow[d, "d"] & a \arrow[d, mapsto] \\
A \otimes V \otimes A \arrow[r] & A \otimes A & a\otimes 1 - 1 \otimes a \\
1 \otimes v \otimes 1 \arrow[r, mapsto] & v \otimes 1 - 1 \otimes v &
\end{tikzcd}$$

The whole construction is functorial in (V, ω). When $\dim V = 1$, one has $I = 0$ and $A \otimes V \otimes A \xrightarrow{\sim} \Omega^1 A \subset A \otimes A$. This means we have a canonical lifting into R/I^2 of $\mathbb{C}[v] \subset A$. Specifically we know how to lift elements v of V, hence powers v^n. Since we know this gives an isomorphism $S(V) \xrightarrow{\sim} A$ of vector spaces, we therefore have a canonical lifting lifting $\rho : A \to R$, in particular $\rho : A \to R/I^2$. We have

$$\rho(v) = (v, \ 1 \otimes v \otimes 1)$$

$$\rho(e^v) = \left(e^v, \ \int_0^1 dt\, e^{tv} \otimes v \otimes e^{(1-t)v}\right)$$

$$\rho(e^v)\rho(e^w) = \Big(e^v e^w, \ e^v \int_0^1 dt\, e^{tw} \otimes w \otimes e^{(1-t)w} +$$

$$\int_0^1 dt\, e^{tv} \otimes v \otimes e^{(1-t)v}\, e^w \Big)$$

$$\rho(e^v e^w) = e^{\frac{1}{2}[v,w]} \rho(e^{v+w})$$

$$= \left(e^{\frac{1}{2}[v,w]} e^{v+w}, \ \int_0^1 dt\, e^{\left(\frac{1}{2}[v,w]\right)} e^{t(v+w)} \otimes (v+w) \otimes e^{(1-t)(v+w)}\right)$$

$$\rho(e^v)\rho(e^w) - \rho(e^v e^w) = I + II$$

where $II \in A\otimes w\otimes A$, $I \in A\otimes v\otimes A$

Using the obvious isom $A\otimes w\otimes A = A\otimes A$ we have

$$II = \int_0^1 dt\; e^v e^{tw}\otimes e^{(1-t)w}$$

$$- \int_0^1 dt\; e^{\frac{1}{2}[v,w]}\, e^{t(v+w)}\otimes e^{(1-t)(v+w)}$$

We want to write this in the form of a linear combination of $a_1(v\otimes 1 - 1\otimes v)a_2$. (Why? Because $\rho(e^v)\rho(e^w) - \rho(e^v e^w)$ is in I/I^2 which is the image of

$$A\otimes \Lambda^2 V\otimes A \xrightarrow{\ \partial\ } A\otimes V\otimes A$$

$$a_1\otimes vw\otimes a_2 \longmapsto a_1 v\otimes w\otimes a_2 - a_1\otimes w\otimes v a_2$$
$$- a_1 w\otimes v\otimes a_2 + a_1\otimes v\otimes w a_2$$

Note this ∂ is injective when $\dim V = 2$.) So in the above expression for II we move the w's outward

$$II = \int_0^1 dt\; e^{t[v,w]}\, e^{tw}(e^v\otimes 1)e^{(1-t)w}$$

$$- \int_0^1 dt\; e^{\left\{\frac{1}{2}[v,w] + \frac{t^2}{2}[v,w] - \frac{(1-t)^2}{2}[v,w]\right\}}\, e^{tw}\left(e^{tv}\otimes e^{(1-t)v}\right)e^{(1-t)w}$$

$$\left\{\frac{1}{2}[v,w] + \frac{t^2}{2}[v,w] - \frac{(1-t)^2}{2}[v,w]\right\} = t[v,w]$$

$$= \int_0^1 dt\; e^{t[v,w]}\, e^{tw} e^{tv}\left(e^{(1-t)v}\otimes 1 - 1\otimes e^{(1-t)v}\right)e^{(1-t)w}$$

similarly $I = \int_0^1 dt\; e^{tv}\otimes e^{(1-t)v} e^w$

$$- \int_0^1 dt\; e^{\frac{1}{2}[v,w]}\, e^{t(v+w)}\otimes e^{(1-t)(v+w)}$$

$$= \int_0^1 dt \quad e^{tv}(1\otimes e^{w})\, e^{(1-t)v} \; e^{(1-t)[v,w]}$$

$$- \int_0^1 dt \quad e) \quad e^{tw}\left(e^{tw}\otimes e^{(1-t)w}\right) e^{(1-t)w}$$

$$\left\{\frac{1}{2}[v,w] - \frac{t^2}{2}[v,w] + \frac{(1-t)^2}{2}[v,w]\right\} = (1-t)[v,w]$$

$$I = \int_0^1 dt\, e^{(1-t)[v,w]} \; e^{tv} \underbrace{\left(1\otimes e^{tw} - e^{tw}\otimes 1\right)}_{-\int_0^t ds \; e^{sw}(w\otimes 1 - 1\otimes w)\, e^{(t-s)w}} e^{(1-t)w} \; e^{(1-t)v}$$

$$= -\int_0^1 dt \int_0^t ds \; e) \quad e^{tv+sw}(w\otimes 1 - 1\otimes w)\, e^{(1-t)v+(1-s)w}$$

$$\left\{(1-t)[v,w] + \frac{st}{2}[v,w] - \frac{(1-s)(1-t)}{2}[v,w]\right\}$$

$$= [v,w]\left\{1-t + \frac{\cancel{st}}{2} - \frac{1}{2} + \frac{s}{2} + \frac{t}{2} - \frac{\cancel{st}}{2}\right\} = [v,w]\left(\frac{1+s-t}{2}\right)$$

So we expect

$$\boxed{\rho(e^v)\rho(e^w) - \rho(e^v e^w) = \iint_{0\le s\le t\le 1} e^{\frac{1+s-t}{2}[v,w]} \; e^{tv+sw} \otimes vw \otimes e^{(1-t)v+(1-s)w}}$$

Now I want to check this using

$$II = \int_0^1 ds\, e^{s[v,w]} \; e^{sw} \; e^{sv} \underbrace{\left(e^{(1-s)v}\otimes 1 - 1\otimes e^{(1-s)v}\right)}_{\int_0^{1-s} du \; e^{(1-s-u)v}(v\otimes 1 - 1\otimes v)\, e^{uv}} e^{(1-s)w}$$

$0 \le u \le 1-s$ Put $u = 1-t$

$0 \le 1-t \le 1-s$

$s \le t \le 1$

$$II = \int_0^1 ds \int_s^1 dt \quad e^{s[v,w]} \; e^{sw} \; e^{tv} \left(v\otimes 1 - 1\otimes v\right) e^{(1-t)v} \; e^{(1-s)w}$$

$$\text{II} = \iint_{0\le s\le t\le 1} e^{\left\{ s[v,w] - \frac{st}{2}[v,w] + \frac{(1-t)(1-s)}{2}[v,w] \right\}} e^{tv+sw}(v\otimes 1 - 1\otimes v)e^{(1-t)v+(1-s)w}$$

$$= [v,w]\left(s - \frac{st}{2} + \frac{1-t-s+st}{2} \right) = [v,w]\left(\frac{1-t+s}{2} \right)$$

$$\therefore \text{II} = \iint_{0\le s\le t\le 1} e^{\frac{1+s-t}{2}} e^{tv+sw}(v\otimes 1 - 1\otimes v)e^{(1-t)v+(1-s)w}$$

which confirms the formula for $\omega(e^v, e^w)$.

Let us now try to represent the generator of $HC^2(A)$, when $\dim V = 2$, by a trace on R/I^2.

First consider a square zero extension with lifting.

$$0 \longrightarrow M \longrightarrow R \overset{\rho}{\underset{\longrightarrow}{\longleftarrow}} A \longrightarrow 0$$

and trace τ on R. We might as well look at $R = RA/IA^2$. The trace τ is then described by $\tau_0 = \tau(\rho)$, $\tau_2 = \tau(\rho\,\omega)$ satisfying

$$b\tau_0 = (1+K)s\tau_2 \qquad b\tau_2 = 0.$$

If $\tau_0(1) = 0$, we can write $\tau_0 = sT_1$ and remove $(Td)_0 = BT_1 = sT_1$ and $(Td)_2 = -bT_1$ from (τ_0, τ_2) to arrange $\tau_0 = 0$, $(1+K)s\tau_2 = 0$, $b\tau_2 = 0$. Then $(1+K)\tau_2$ is a reduced cyclic 2-cocycle.

But in the case of the Weyl algebra ~~[illegible]~~ we have $HC^2(A) \xrightarrow{\sim} HC^2(\mathbb{C}) = \mathbb{C}$ by restriction, so the cyclic class is definitely not reduced. Thus the trace we are after is non trivial on ~~both~~ $\rho(A)$ and I/I^2.

Now $I/I^2 = A\otimes \Lambda^2 V \otimes A$, so a trace on this bimodule is a linear function on $A\otimes\Lambda^2 V \otimes A$. Recall the Hochschild homology is computed by the

Koszul type complex

$$\cdots \longrightarrow A\otimes \Lambda^2 V \xrightarrow{b} A\otimes V \xrightarrow{b} A$$

whose homology is $\mathbb{C}[2] =$ the kernel of $b: A\otimes\Lambda^2 V = \Lambda^2 V$. The trace must restrict nontrivially to this line. [redacted] It's clear that a linear functional on $A\otimes\Lambda^2 V$ in the image of b^t extends to a null-cobordant trace on R/I^2. (Recall $(\Omega^1 R/I\Omega^1 R)_\natural = A\otimes V$). Thus we can suppose our τ on R/I^2 when restricted to I/I^2 corresponds to the linear function on $A\otimes\Lambda^2 V$, which is the given by the projection $e^v \mapsto 1$, $A\to\mathbb{C}$

Let's suppose τ is a trace on R/I^2. First we have to identify the element $1\otimes vw\otimes 1 \in A\otimes\Lambda^2 V\otimes A$ as an element in I/I^2.

$$\rho v = (v, 1\otimes v\otimes 1)$$
$$\rho w = (w, 1\otimes w\otimes 1)$$

$$\rho v\,\rho w = (vw, v\otimes w\otimes 1 + 1\otimes v\otimes w)$$
$$\rho w\,\rho v = (wv, w\otimes v\otimes 1 + 1\otimes w\otimes v)$$

$$[\rho v,\rho w] - \rho[v,w] = v\otimes w\otimes 1 - 1\otimes w\otimes v - w\otimes v\otimes 1 + 1\otimes v\otimes w$$
$$= \partial(1\otimes vw\otimes 1)$$

Let's take a trace τ on R/I^2. We have

$$\tau(\rho(e^v)\rho(e^w)) - e^{\frac{1}{2}[v,w]}\tau(\rho(e^{v+w}))$$

$$= \iint_{0\le s\le t\le 1} e^{\frac{1+s-t}{2}[v,w]}\,\tau\Big(\underbrace{e^{(1-t)v+(1-s)w}\,e^{tv+sw}}_{e^{v+w+\frac{1}{2}[v,w]((1-t)s-(1-s)t)}}\otimes vw\otimes 1\Big)$$

$$= \iint_{0\le s\le t\le 1} e^{\underbrace{\left\{\frac{1+s-t}{2} + \frac{1}{2}\left((1-t)s-(1-s)t\right)\right\}}[v,w]} \tau\left(e^{v+w}\otimes vw\otimes 1\right)$$

$$\left(\frac{1}{2}+s-t\right) = \frac{1}{2}\left\{1+s-t+\cancel{s}-\cancel{ts}-t+\cancel{st}\right\}$$

Thus

$$\boxed{\begin{aligned}\tau\left(\rho(e^{v})\rho(e^{w})\right) &= e^{\frac{1}{2}[v,w]}\tau\left(\rho(e^{v+w})\right)\\ &+ \iint_{0\le s\le t\le 1} e^{\left(\frac{1}{2}+s-t\right)[v,w]}\tau\left(e^{v+w}\otimes vw\otimes 1\right)\end{aligned}}$$

$$\iint_{0\le s\le t\le 1} e^{\left(\frac{1}{2}+s-t\right)a} = \int_0^1 dt\int_0^t ds\, e^{\frac{a}{2}-at}\, e^{sa} = \int_0^1 dt\, e^{\frac{a}{2}-at}\left[\frac{e^{sa}}{a}\right]_0^t$$

$$= \int_0^1 dt\; e^{\frac{a}{2}-at}\,\frac{e^{at}-1}{a} = \frac{e^{\frac{a}{2}}}{a}\int_0^1\left(1-e^{-at}\right)dt = \frac{e^{\frac{a}{2}}}{a}\left[t+\frac{e^{-at}}{a}\right]_0^1$$

$$= \frac{e^{\frac{a}{2}}}{a}\left(1+\frac{e^{-a}-1}{a}\right) = \frac{e^{\frac{a}{2}}}{a}+\frac{e^{-\frac{a}{2}}-e^{\frac{a}{2}}}{a^2}$$

$$\boxed{\therefore \iint_{0\le s\le t\le 1} e^{\left(\frac{1}{2}+s-t\right)a} = \frac{e^{\frac{a}{2}}}{a}+\frac{e^{-\frac{a}{2}}-e^{\frac{a}{2}}}{a^2}}$$

March 7, 1991

Let's go over what we learned yesterday. A is the Weyl algebra generated by V with relations $vw - wv = [v,w]$, where $[\ ,\ ]$ is a given skew-symm bilinear form on V. We have $A = R/I$ where $R = T(V)$ and I is generated by $(vw - wv - [v,w])$. There is a canonical linear lifting $\rho : A \longrightarrow R$ which is the canonical embedding $V \longrightarrow T(V)$ on V and which extends uniquely to A so that

$$\rho(v^n) = (\rho v)^n \qquad \forall n, v.$$

We prefer to write this in generating function form

$$\rho(e^v) = e^{\rho v}$$

Because of this lifting ρ, the square zero extension R/I^2 is equivalent to a 2-cocycle. We computed it and found

$$\boxed{\begin{aligned}&\rho(e^v)\rho(e^w) - \rho(e^v e^w) = \\ &\iint_{0\le s\le t\le 1} e^{\frac{1+s-t}{2}[v,w]}\, e^{tv+sw}\big([\rho v,\rho w] - [v,w]\big) \\ &\qquad\qquad \times e^{(1-t)v + (1-s)w}\end{aligned}}$$

Now I want to use this in the case $\dim V = 2$, where

$$A \otimes \Lambda^2 V \otimes A \xrightarrow{\ \sim\ } I/I^2$$
$$1 \otimes v \wedge w \otimes 1 \longmapsto [\rho v, \rho w] - [v,w]$$

in order to find the traces on R/I^2.

As a check on signs etc., let us look at both sides of the above formula to first order in v and w separately, so that the 2nd order effect vw is not lost. Precisely we should introduce indeterminates s,t say and replace v by sv, w by tw and then work over $\mathbb{C}[s,t]/(s^2,t^2)$.

Then on the left we have

$$(1+\rho v)(1+\rho w) - \left(1+\tfrac{1}{2}[v,w]\right)\left(1+\rho(v+w)+\tfrac{1}{2}\rho(v+w)^2+\cdots\right)$$

$$= 1+\rho v+\rho w+\rho v\rho w - \left(1+\rho v+\rho w+\tfrac{1}{2}(\rho v\rho w+\rho w\rho v)\right) - \tfrac{1}{2}[v,w]$$

$$= \tfrac{1}{2}\left([\rho v,\rho w]-\rho[v,w]\right)$$

(alternative calculation:

$$(1+\rho v)(1+\rho w) - \rho\left((1+v)(1+w)\right)$$

$$= \rho v\rho w - \rho(vw)$$

But it turns out this is skew-symmetric in v,w:

$$\rho v\rho w+\rho w\rho v - \rho(vw+wv)$$

$$= \begin{matrix}(\rho v+\rho w)^2 \\ -(\rho v)^2 \\ -(\rho w)^2\end{matrix} \quad - \quad \begin{matrix}\rho\{(v+w)^2\} \\ +\rho(v^2) \\ +\rho(w^2)\end{matrix} \quad = 0 \quad)$$

On the right we have

$$\iint_{0\le s\le t\le 1} \underbrace{\left([\rho v,\rho w]-[v,w]\right)}_{\text{already 2nd order}} = \tfrac{1}{2}\left([\rho v,\rho w]-[v,w]\right)$$

Lets describe the traces on R/I^2. Recall first of all that a trace τ on RA/IA^2 is a pair $\tau_0(a)=\tau(\rho a)$ $\quad \tau_2(a_0,a_1,a_2)=\tau(\rho\omega^2)(a_0,a_1,a_2)$

satisfying $b\tau_0 = (1+K)s\tau_2$, $b\tau_2 = 0$.

In the case of R/I^2 because we have a surjection $RA/IA^2 \longrightarrow R/I^2$ which is the identity on ρA we have the extra condition that τ_2 descends to a trace on I/I^2 (which is a quotient of $IA/IA^2 = \Omega^2 A$).

Suppose τ is a trace on R/I^2. Then

$$\tau_0(e^v) = \tau(\rho e^v), \quad \text{and}$$

$$(b\tau_0)(e^v, e^w) = \tau_0(e^v e^w - e^w e^v)$$

$$= \tau_0\left(\left(e^{\frac{1}{2}[v,w]} - e^{-\frac{1}{2}[v,w]}\right) e^{v+w}\right)$$

$$= \left(e^{\frac{1}{2}[v,w]} - e^{-\frac{1}{2}[v,w]}\right) \tau_0(e^{v+w})$$

Since $A \otimes \Lambda^2 V \otimes A \xrightarrow{\sim} I/I^2$ a trace on I/I^2 as A-bimodule is equivalent to a linear function on $A \otimes \Lambda^2 V \simeq A$; more precisely τ on I/I^2 is determined by $\tau(\rho e^v \cdot g)$ where g generates the line in I/I^2 which is the image of $1 \otimes \Lambda^2 V \otimes 1$.

Let's consider the condition $b\tau_0 = (1+K)s\tau_2$.

We have

$$((1+K)s\tau_2)(e^v, e^w) = \tau\left(\omega(e^v, e^w) - \omega(e^w, e^v)\right)$$

Now $-\tau(\omega(e^v, e^w)) = \tau(\rho e^v \rho e^w - \rho(e^v e^w))$

$$= \tau \iint_{0 \le s \le t \le 1} e^{\frac{1+s-t}{2}}\, \tau\Big(\underbrace{e^{(1-t)v+(1-s)w}\, e^{tv+sw}}_{e^{\{(1-t)s-(1-s)t\}\frac{1}{2}[v,w]}\, e^{v+w}}\big([\rho v, \rho w] - [v,w]\big)\Big)$$

$$= \iint_{0\le s\le t\le 1} e^{\left\{\frac{1+s-t}{2}+s-\cancel{ts}-t+\cancel{st}\right\}[v,w]}\;\tau\left(e^{v+w}\left([\rho v,\rho w]-[v,w]\right)\right)$$

$$-\tau(\omega(e^v,e^w)) = \underbrace{\left(\iint_{0\le s\le t\le 1} e^{\left(\frac{1}{2}+s-t\right)[v,w]}\right)}_{\substack{\text{call this}\\ F([v,w])}}\;\underbrace{\tau\left(e^{v+w}([\rho v,\rho w]-[v,w])\right)}_{\substack{\text{call this}\\ G(v,w)\\ \text{(anti-symmetric)}}}$$

We found $\boxed{F(x) = \frac{e^{\frac{x}{2}}}{x} + \frac{e^{-\frac{x}{2}}-e^{\frac{x}{2}}}{x^2}}$ odd in x

[redacted] so we have

$$\left((1+\kappa)s\,\tau_2\right)(e^v,e^w) = \tau\left(\omega(e^v,e^w)-\omega(e^w,e^v)\right)$$

$$= -F([v,w])\,G(v,w) + F(-[v,w])\,G(w,v)$$

$$= \left(-F([v,w]) - F(-[v,w])\right)\,G(v,w)$$

$$= -\left(\frac{e^{\frac{1}{2}[v,w]}-e^{-\frac{1}{2}[v,w]}}{[v,w]}\right)\tau\left(e^{v+w}([\rho v,\rho w]-[v,w])\right)$$

This has to equal

$$(b\tau_0)(e^v,e^w) = \left(e^{\frac{1}{2}[v,w]}-e^{-\frac{1}{2}[v,w]}\right)\tau_0(e^{v+w})$$

which yields the condition

$$\boxed{\tau\left(e^{v+w}([\rho v,\rho w]-[v,w])\right) = -[v,w]\,\tau\left(\rho(e^{v+w})\right)}$$

on the trace.

It seems therefore that a trace on R/I^2 is given by an arbitrary function [redacted] $\tau(\rho(e^v))$ on A.

The next to look at is how small a nilpotent extension of A is needed to represent the cyclic cohomology generator. We have a representative given by R/I^2 with any trace τ such that $\tau(1) = 1$. There is a lot of freedom in the choice of τ – any linear function on A extends uniquely to trace. The point is to make as small as possible the bimodule ~~quotient~~ of I/I^2 to which the trace descends. Put another way the trace τ on I/I^2 is equivalent to a bimodule map

$$A \otimes \Lambda^2 V \otimes A = I/I^2 \longrightarrow A^*$$

and we want to make the image as small as possible. It seems from dimensional considerations that for any τ this map ought to be surjective – this is vague but we are looking for a module over the Weyl algebra $A \otimes A^\circ$ generated by a single element, and these all have to half the dimension at least.

Example of a trace: Consider a linear fnl φ on A (i.e. ~~state~~) of the form $\varphi(a) = \langle 0 | a | 0 \rangle$ where $\langle 0|$ is a vector in a right A-module E', and $|0\rangle$ is a vector in a left module E, such that we have a pairing $E' \otimes_A E \longrightarrow \mathbb{C}$ such that $\langle 0 | 0 \rangle = 1$. Consider the corresponding trace on $A \otimes A$ considered as bimodule

$$\tau(a_1 \otimes a_2) = \varphi(a_2 a_1) = \langle 0 | a_2 a_1 | 0 \rangle$$

Look at

$$\begin{aligned} A \otimes A &\longrightarrow A^* \\ a_1 \otimes a_2 &\longmapsto (x \longmapsto \tau(a_2 x a_1)) \\ & \qquad\qquad\quad \shortparallel \\ & \qquad\qquad \langle 0 | a_2 x a_1 | 0 \rangle \end{aligned}$$

Then we observe that if
$\underline{l}$ is the left ideal annihilating $|0\rangle$
and $\underline{r}$ is the right ideal annihilating $\langle 0|$,
then this map factors through the bimodule

$$A/\underline{l} \otimes A/\underline{r}$$

To be more specific, let $E = \mathbb{C}[x]$, A acting on these polys as $A = \mathbb{C}[x, \partial_x]$, let E' be distributions supported at 0. Thus $|0\rangle = 1 \in E$ and $|0\rangle = \delta \in E'$. In this case

$$(A/\underline{l} \otimes A/\underline{r})_7 = A/\underline{l} + \underline{r} = A/A\partial_x + xA \cong \mathbb{C}$$

How should I think of the bimodule

$$A/\underline{l} \otimes A/\underline{r} = \mathbb{C}[x] \otimes \bigcup_n (\mathbb{C}[x]/(x^n))^* \; ?$$

These are operators on polynomials which are a distribution supported at 0 followed by multiplication by a polynomial.

Let us consider the linear functional $\varphi(e^v) = 1 \quad \forall v$ on A and the corresponding trace on $A \otimes A$: $\tau(a_1 \otimes a_2) = \varphi(a_2 a_1)$. Consider the bimodule map

$$A \otimes A \longrightarrow A^*$$
$$a_1 \otimes a_2 \longmapsto (a \mapsto \varphi(a_2 a a_1))$$

and try to find elements in the kernel. Claim $e^v \otimes e^v - 1 \otimes 1$ lies in the kernel, since

$$\varphi(e^v e^w e^v) = e^{\frac{1}{2}\{[v,w] + [v+w,v]\}} \varphi(e^{v+w+v})$$
$$= \varphi(e^{2v+w}) = 1$$

Thus working to first order in v the kernel should contain

$$(1+v)\otimes(1+v) - (1\otimes 1) = v\otimes 1 + 1\otimes v$$

Let l_v, r_v be left and right mult. by v on A. The bimodule structure on $A\otimes A$ we consider is given by the operators $l_v\otimes 1$, $1\otimes r_v$ for $v\in V$. Commuting with these are the operators $r_v\otimes 1$, $1\otimes l_v$. We wish to divide by the operators $r_v\otimes 1 + 1\otimes l_v$. Note that these operators commute

$$[r_v\otimes 1 + 1\otimes l_v, r_w\otimes 1 + 1\otimes l_w]$$

$$= [r_v, r_w]\otimes 1 + 1\otimes [l_v, l_w]$$

$$= -[v,w] + [v,w] = 0$$

so that dividing out by these shouldn't give zero. More precisely we have a Koszul complex of bimodules

$$0 \longrightarrow A\otimes \Lambda^2 V\otimes A \longrightarrow A\otimes V\otimes A \longrightarrow A\otimes A$$

$$a_1\otimes v\otimes a_2 \longmapsto a_1 v\otimes a_2 + a_1\otimes v a_2$$

At this point to see what we get we should use the isomorphism $A = S(V)$ and the standard forms for l_v, r_v.

March 8, 1991

Quivers: $X_1 \underset{t}{\overset{s}{\rightrightarrows}} X_0$, X_0 is a set of vertices, X_1 a set of arrows, s and t are source and target. A representation of the quiver is a family of vector space V_i for each $i \in X_0$ together with maps $V_{s(\alpha)} \longrightarrow V_{t(\alpha)}$ for each $\alpha \in X_1$.

Suppose X_0 finite, let $S = \mathbb{C}^{X_0} = \bigoplus_{i \in X_0} \mathbb{C} e_i$ where the e_i are annihilating idempotents. Let $M = \bigoplus_{\alpha \in X_1} \mathbb{C} e_\alpha$. M is an S-bimodule where the right structure comes from the source and the left from the target. Thus

$$e_i e_\alpha = \begin{cases} 0 & i \neq t(\alpha) \\ e_\alpha & i = t(\alpha) \end{cases}$$

$$e_\alpha e_i = \begin{cases} 0 & i \neq s(\alpha) \\ e_\alpha & i = s(\alpha) \end{cases}$$

A representation of the quiver is the same as an S-module V equipped with an S-module map $M \otimes_S V \longrightarrow V$. It is thus the same as a module over $T_S(M)$. $T_S(M)$ is called the path algebra of the quiver, because it has a basis indexed by paths in the quiver, i.e. sequences $(\alpha_k, \ldots, \alpha_2, \alpha_1)$ in X_1 such that $s(\alpha_j) = t(\alpha_{j-1})$, $j = k, \ldots, 2$. The path algebra of the quiver is finite-dimensional iff there are no loops.

Examples: 1)  gives free alg. $T(x,y)$

2) $\bullet \longrightarrow \bullet$ gives triangular matrices $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

More generally $\longrightarrow \bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet$ n vertices gives triangular $n \times n$ matrices.

Because S is separable, M is a projective S-bimodule and so $T_S(M)$ is quasi-free. Thus path algebras of quivers are quasi-free.

Note that if M is an arbitrary S bimodule, then it decomposes into $e_i M e_j$ for $i, j \in X_0$; so upon choosing a basis for $e_i M e_j$, call it $X_{1,ij}$ we get a quiver with $X_1 = \coprod_{i,j} X_{1,ij}$, which gives rise to M. Now if S is an arbitrary separable algebra and M is an S-bimodule, we know S is Morita equivalent to one of the form $\mathbb{C}^{X_0}$, so up to Morita equivalence $T_S(M)$ with S <u>separable</u> is equivalent to the path algebra of a quiver.

Let's now consider a finite dimensional quasi-free algebra A. We know it contains a largest nilpotent ideal I, and that $A/I = S$ is separable. We also can find a lifting $S \xrightarrow{l} A$. Also I/I^2 is a projective S-bimodule, so we can find a S-bimodule lifting $l : I/I^2 \longrightarrow I$. Putting these two liftings together gives an algebra map $\Phi : T_S(I/I^2) \longrightarrow A$ compatible with the augmentations to S. Put $N = I/I^2$. We have

$$\begin{tikzcd}
0 \arrow[r] & \overbrace{\bigoplus_{n>0} T_S^n(N)}^{J} \arrow[r] \arrow[d] & \overbrace{T_S(N)}^{R} \arrow[r] \arrow[d, "\Phi"] & S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] & A \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

Φ is a homomorphism of algebras carrying $J \to I$ inducing an isomorphism $J/J^2 \xrightarrow{\sim} I/I^2$, hence Φ must be surjective. Because I is nilpotent we have an induced maps

$$\begin{tikzcd}
0 \arrow[r] & J/J^r \arrow[r] \arrow[d] & R/J^r \arrow[r] \arrow[d, "\Phi"] & S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] & A \arrow[r] \arrow[u, dashed, bend left, "\sigma"] & S \arrow[r] & 0
\end{tikzcd}$$

for some $r \geq 2$. Then R/J^r is a nilpotent extension of [redacted] A, so we know Φ admits a section homomorphism $\sigma: A \longrightarrow R/J^r$. This σ is automatically compatible with the maps to S, hence it induces a map of filtered algebras $\sigma(I) \subset J/J^r$. Thus σ induces a section of $\Phi: J/J^2 \xrightarrow{\sim} I/I^2$, so σ is an isom. on gr^1, hence σ is surjective. Thus σ, Φ are inverse isomorphisms. [redacted] It follows that $J^r = J^{r+1} = \cdots$ which means $T_S^{r'}(N) = 0 \quad \forall r' \geq r$, hence $R = R/J^r$. Thus $R \xrightarrow{\sim} A$.

Conclude that a finite dimensional quasi-free algebra A is of the form $T_S(N)$, with S separable and N an S-bimodule such that $T_S^r(N) = 0$ [redacted] for some r. I guess this means that A is Morita equivalent to the path algebra of a quiver with [redacted] no loops.

Remark: If A is <u>quasi-free</u>, then for any (left) A-module we have the resolution of length 1

$$0 \longrightarrow \Omega^1 A \otimes_A M \longrightarrow A \otimes M \longrightarrow M \longrightarrow 0$$

by projective A-modules, so M has projective dim ≤ 1. Thus A is left (+ right) <u>hereditary</u> (submodules of projective modules are projective.)

Let's return to the task of representing the generator of $HC^2(A)$ by a trace on a square zero extension, where A is a Weyl algebra generated by V, $\dim(V)=2$. Recall $A = R/I$, $R = T(V)$ and one has a map of A-bimodules in general

$$A \otimes \Lambda^2 V \otimes A \longrightarrow I/I^2$$
$$v \wedge w \longmapsto [\rho v, \rho w] - [v,w]$$

which is an isomorphism when $\dim(V)=2$. We have seen that any trace on $A \otimes \Lambda^2 V \otimes A = I/I^2$, i.e. linear function on $A \otimes \Lambda^2 V \cong A$ extends uniquely to a trace on R/I^2. I now wish to look specifically at the simplest $\varphi \in A^*$, namely

$$\varphi(e^v) = 1 \qquad \forall v \in V.$$

The corresponding trace on $A \otimes A$ is

$$\tau(a_1 \otimes a_2) = \varphi(a_2 a_1)$$

and we want the smallest bimodule ~~quotient~~ of $A \otimes A$ to which τ descends; equivalently we want the coimage of the bimodule map

$$\begin{aligned} A \otimes A &\longrightarrow A^* \\ a_1 \otimes a_2 &\longmapsto (a \longmapsto \underbrace{\varphi(a_2 a a_1)}_{\begin{subarray}{c}\tau(a(a_1 \otimes a_2)) \\ = \tau((a_1 \otimes a_2)a)\end{subarray}}) \end{aligned}$$

We have seen that

$$\varphi(e^v e^w e^v) = e^{\frac{1}{2}([v,w] + [v+w, v])} \varphi(e^{2v+w})$$
$$= 1$$

for all v and w, hence to first order in v

$$\varphi(v e^w) + \varphi(e^w v) = 0$$

so

$$\boxed{\varphi(v a) + \varphi(a v) = 0}.$$

for all $v \in V$ and $a \in A$. Let's check this before going on.

Now φ is just the projection onto $\mathbb{C}1$ defined by the canonical linear isomorphism

$$A = S(V)$$

so we have $\varphi(v^{n+1}) = 0$ $\quad v \in V, \ n \geq 0$.

As $\quad \varphi\left((v+tw)^{n+1}\right) = \varphi(v^{n+1}) + t\sum_{j=0}^{n} \varphi(v^j w v^{n-j}) + O(t^2)$

we have

$$\sum_{j=0}^{n} \varphi(v^j w v^{n-j}) = 0$$

But

$$\varphi(v^j w v^{n-j}) = \varphi(v^{j-1} w v^{n-j+1}) + \underbrace{[v,w]\,\varphi(v^{n-1})}_{=0 \quad n \geq 2}$$

Thus

$$\varphi(v^j w v^{n-j}) = 0 \qquad 0 \leq j \leq n \quad \text{for } n \geq 2$$

$$\varphi(vw + wv) = 0$$

proving that $\varphi(v^n w + w v^n)$ for all $n \geq 0$, hence $\varphi(aw + wa) = 0$ for all $a \in A$, $w \in V$.

This means that the trace τ descends to the quotient bimodule $A\otimes A / \{a_1 v \otimes a_2 + a_1 \otimes v a_2\}$

Notice that $v \mapsto -v$ preserves $[v,w]$, hence gives rise to an automorphism σ of A of order 2. It's clear that

$$\begin{array}{ccc} A\otimes A / A\{(v\otimes 1 + 1\otimes v)\}A & \xrightarrow[\sim]{1\otimes\sigma} & A\otimes A / A\{(v\otimes 1 - 1\otimes v)\}A \\ & & \Big\downarrow{\scriptstyle m}\ \wr \\ a_1\otimes a_2 & \longmapsto \ a_1\sigma(a_2) & A \end{array}$$

Thus the trace τ on $A\otimes A$ descends to the bimodule A_σ (same left mult, but right multiplication twisted by σ). Moreover

$$A_\sigma/[A,A_\sigma] = A_\sigma/[V,A_\sigma]$$
$$\cong A/\{\underbrace{va+av}\} \xrightarrow[\varphi]{\sim} \mathbb{C}$$

includes $vv^n + v^n v = 2v^{n+1}$

so φ is the unique trace (up to scalars) on the bimodule A_σ.

Thus we have a square zero alg extension of A by the bimodule $A_\sigma \otimes \Lambda^2 V$ and a trace on it which realizes the generator of $HC^2(A)$; this is for $\dim V = 2$. (In the general case we ~~...~~ have only a surjection $A\otimes\Lambda^2V\otimes A \twoheadrightarrow I/I^2$)

Now I would like to understand this better. Recall the formula for the cocycle describing the extension R/I^2:

$$\boxed{\begin{aligned}&\rho(e^v)\rho(e^w) - \rho(e^v e^w) = \\ &\iint_{0\le s\le t\le 1} e^{\frac{1+s-t}{2}[v,w]}\, e^{tv+sw}\left([\rho v,\rho w]-[v,w]\right) e^{(1-t)v+(1-s)w}\end{aligned}}$$

Let us apply the ~~...~~ map

$$I/I^2 \xleftarrow{\sim} A\otimes\Lambda^2 V\otimes A \longrightarrow A_\sigma\otimes\Lambda^2 V$$
$$a_1([\rho v,\rho w]-[v,w])a_2 \leftarrow\!\shortmid a_1\otimes vw\otimes a_2 \longmapsto a_1\sigma(a_2)\otimes vw$$

and this becomes

$$\iint_{0\le s\le t\le 1} \underbrace{e^{\frac{1+s-t}{2}[v,w]}\, e^{tv+sw}\, e^{(t-1)v+(s-1)w}} \otimes vw$$

$$e^{(2t-1)v+(2s-1)w}$$
$$e^{\left\{\frac{1+s-t}{2}[v,w] + \frac{t(s-1)}{2}[v,w] + \frac{s(t-1)}{2}[w,v]\right\}}$$

$$1+s-t+\cancel{ts}-t-\cancel{st}+s \simeq 1+2s-2t.$$

whence for this extension of A by $A_\sigma \otimes \Lambda^2 V$ we have

$$\boxed{\rho(e^v)\rho(e^w) - \rho(e^v e^w) = \iint_{0\le s\le t\le 1} e^{\left(\frac{1}{2}+s-t\right)[v,w]}\, e^{(2t-1)v+(2s-1)w} \otimes (v\wedge w)}$$

Observe this is pretty complicated; even for the commutative case $[v,w]=0$ it is uninspiring:

$$\rho(e^v)\rho(e^w) - \rho(e^{v+w}) = \iint_{0\le s\le t\le 1} e^{(2t-1)v+(2s-1)w} \otimes (v\wedge w)$$

It seems therefore this idea of looking for a quotient <u>algebra</u> of R/I^2 carrying the trace is not a good approach.

Instead let us calculate $X^1(R,I)$.

Recall that we have

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r, "d"] & \Omega^1 R/F^1_I\Omega^1 R \arrow[d, "\cong"] \\
0 \arrow[r, "\dim 2"'] & A\otimes\Lambda^2 V\otimes A \arrow[r, "\partial"] & A\otimes V\otimes A
\end{tikzcd}$$

$$\partial(1\otimes vw\otimes 1) = v\otimes w\otimes 1 - 1\otimes w\otimes v - w\otimes v\otimes 1 + 1\otimes v\otimes w$$

$$d\big([\rho v,\rho w]-[v,w]\big) = [d(\rho v),\rho w] + [\rho v, d(\rho w)]$$

$$\longrightarrow 1\otimes v\otimes w - w\otimes v\otimes 1 + v\otimes w\otimes 1 - 1\otimes w\otimes v$$

Thus

$$\begin{array}{ccc} (I/I^2)\otimes A & \xrightarrow{\bar{d}} & (\Omega^1 R/I\Omega^1 R)_\natural \\ \wr\| & & \wr\| \\ A\otimes\Lambda^2 V & & A\otimes V \end{array}$$

is computed to be

$$a\otimes v\wedge w \longmapsto \natural\big(a v\otimes w\otimes 1 - a\otimes w\otimes v - aw\otimes v\otimes 1 + a\otimes v\otimes w\big)$$

$$= [a,v]\otimes w - [a,w]\otimes v$$

$$= b(a\otimes v\wedge w)$$

Now

$$\begin{array}{ccc} \rho(e^{v}) & \stackrel{d}{\longmapsto} & \int_0^1 e^{tv}\otimes v\otimes e^{(1-t)v} \\ \hat{R} & & A\otimes\hat{V}\otimes A \end{array}$$

so that $\bar{d}\rho(e^v) = \natural\left\{\int_0^1 e^{tv}\otimes v\otimes e^{(1-t)v}\right\} = e^v\otimes v$.

Thus we have the following description of $\bar{d}$

$$\begin{array}{ccc} R/I^2+[R,I] & \xrightarrow{\bar{d}} & (\Omega^1 R/I\Omega^1 R)_\natural \\ \| & & \| \\ \rho(A)\oplus(I/I^2+[R,I]) & & A\otimes V \\ \| & & \| \\ S(V)\oplus S(V)\otimes\Lambda^2 V & \xrightarrow[\begin{pmatrix} d \\ b\end{pmatrix}]{} & S(V)\otimes V \end{array}$$

where d, b refer the the mixed complex $(\Omega_{S(V)}, b, d)$ associated to the Weyl algebra A.

Next we compute $\bar{b}$. Given $e^v\otimes w\in A\otimes V$ lift to $\rho(e^v)\,d(\rho w)\in\Omega^1 R$ and apply $\bar{b}$ to get $[\rho e^v, \rho w]$. We can find this from our formula for $\rho(e^v)\rho(e^w)$ by taking the first order contribution of ω. Let

$$\chi(v \wedge \omega) = [\rho v, \rho\omega] - [v, \omega]$$

In $R/I^2 + [R,I]$ we have

$$\rho(e^v)\rho(e^\omega) = \rho(e^v e^\omega) + \iint_{s \le t} e^{\frac{1+s-t}{2}[v,\omega]} e^{tv+s\omega} \chi(v\wedge\omega) \times e^{(1-t)v+(1-s)\omega}$$

$$= e^{\frac{1}{2}[v,\omega]}\rho(e^{v+\omega}) + \underbrace{\left(\iint_{s\le t} e^{\left(\frac{1}{2}+s-t\right)[v,\omega]}\right)}_{\left(\frac{e^{\frac{1}{2}[v,\omega]}}{[v,\omega]} + \text{odd term in } [v,\omega]\right)} e^{v+\omega}\chi(v\wedge\omega)$$

Thus in $R/I^2+[R,I]$ we have

$$\boxed{[\rho(e^v), \rho(e^\omega)] = \left(e^{\frac{1}{2}[v,\omega]} - e^{-\frac{1}{2}[v,\omega]}\right)\rho(e^{v+\omega}) + \frac{e^{\frac{1}{2}[v,\omega]} - e^{-\frac{1}{2}[v,\omega]}}{[v,\omega]} e^{v+\omega}\chi(v\wedge\omega)}$$

The first order contribution of ω is

$$[\rho(e^v), \rho\omega] = [v,\omega](\rho e^v) + e^v\chi(v\wedge\omega)$$

which corresponds to the following element of $S(V) \oplus S(V)\otimes\Lambda^2 V$:

$$\underbrace{[v,\omega]e^v}_{\substack{[e^v,\omega] \\ \doteq\, b(e^v d\omega)}} + \underbrace{e^v\, dv\, d\omega}_{d(e^v d\omega)}$$

Thus

$$\begin{array}{ccc} R/I^2+[R,I] & \xleftarrow{\quad \bar{b} \quad} & (\Omega^1 R/I\Omega^1 R)_\natural \\ \| & & \| \\ S(V)\oplus S(V)\otimes\Lambda^2 V & \xleftarrow{\quad \binom{b}{d} \quad} & S(V)\otimes V \end{array}$$

Summarizing we have constructed an isomorphism of $\mathbb{Z}/2$-graded complexes

$$\begin{array}{ccc}
X'(R,I): \quad R/I^2+[R,I] & \underset{\bar{d}}{\overset{\bar{b}}{\rightleftarrows}} & (\Omega^1R/I\Omega^1R)_\natural \\
\| & & \| \\
\begin{matrix} S(V) \\ \oplus \\ S(V)\otimes\Lambda^2V \end{matrix} & \underset{\begin{pmatrix} d \\ b \end{pmatrix}}{\overset{\begin{pmatrix} b \\ d \end{pmatrix}}{\rightleftarrows}} & S(V)\otimes V
\end{array}$$

Now comes the interesting question which is how to understand this result. The interesting point is that we do not have an algebra homomorphism $R \twoheadrightarrow S(V) \oplus S(V)\otimes\Lambda^2V$; it is better to say that the kernel is not an ideal. So there is no obvious map to start with from R to $\Omega^+_{S(V)}$. In fact it seems unlikely that one can construct a Fedosov algebra out of $\Omega^+_{S(V)}$ as we did in the commutative case.

We might look at other 2 diml generalized enveloping algebras.

<u>March 14, 1991</u>

Let R_A be the Fedosov algebra associated to Ω_A^+, A supposed comm. We have constructed a map

$$\nu : X(R_A) \longrightarrow \Omega_A$$

of $\mathbb{Z}/2$ graded vector spaces, which is a map of complexes when rescaled properly. This map is bijective in even degree. Is it bijective in odd degree?

Recall $\qquad \nu : \Omega^1(R_A)_\natural \longrightarrow \Omega_A^-$

$$\nu : x \cdot \delta y \longmapsto x\,dy + |y|\,d(xy)$$

Thus $\quad \nu(x \cdot \delta a) = x\,da \quad$ for $\quad x \in R_A = \Omega_A^+$ and $a \in A$. This shows ν is surjective.

We have

$$x \cdot \delta(y \cdot z) = (z \cdot x) \cdot \delta y + (x \cdot y) \cdot \delta z$$

$$x \cdot \delta(z \cdot y) = (x \cdot z) \cdot \delta y + (y \cdot x) \cdot \delta z$$

Recall
$$\begin{aligned} x \cdot y &= xy - dx\,dy \\ y \cdot x &= yx - dy\,dx = xy + dx\,dy \end{aligned}$$

$$\tfrac{1}{2}(x \cdot y + y \cdot x) = xy$$

$$\tfrac{1}{2}(x \cdot y - y \cdot x) = -dx\,dy$$

Thus
$$\boxed{x \cdot \delta(yz) = (zx) \cdot \delta y + (xy) \cdot \delta z}$$

$$\begin{aligned} x \cdot \delta(dy\,dz) &= (dz\,dx) \cdot \delta y + (dx\,dy) \cdot \delta z \\ &= -(dx\,dz) \cdot \delta y + (dx\,dy) \cdot \delta z \end{aligned}$$

The first formula shows that

$$(x,y) \longmapsto x\cdot\delta y$$

is a 1-cocycle on Ω_A^+ with values in $\Omega(R_A)_\natural$. Thus we have maps

$$\Omega^1_{\Omega_A^+} \longrightarrow \Omega^1(R_A)_\natural \xrightarrow{\ \nu\ } \Omega_A^-$$

$$x\,\delta y \longmapsto x\cdot\delta y \longmapsto x\,dy + |y|\,d(xy)$$

Now Ω_A^+ is generated by the elements a, da_1da_2 and R_A is generated by the elements $a\in A$.

March 15, 1991

$R = R_A$ the Fedosov algebra constructed from Ω_A^+, A commutative. The problem is to describe $\Omega^1 R_\natural$. Let's denote Fedosov product with a $\circ$, and use $\delta : R \longrightarrow \Omega^1 R$ for the canonical derivation. Similarly we write $x \circ \delta y$ for the canonical pairing $R \times R \longrightarrow \Omega^1 R_\natural$. We have

$$* \quad \begin{cases} x \circ \delta(y \circ z) = (z \circ x) \circ \delta y + (x \circ y) \circ \delta z \\ x \circ \delta(z \circ y) = (x \circ z) \circ \delta y + (y \circ x) \circ \delta z \end{cases}$$

Since
$$x \circ y = xy - dx\,dy$$
$$y \circ x = yx - dy\,dx = xy + dx\,dy$$
we have
$$\tfrac{1}{2}(x \circ y + y \circ x) = xy$$
$$\tfrac{1}{2}(x \circ y - y \circ x) = -dx\,dy$$

Hence the equations $*$ are equivalent to

$$*' \quad \begin{cases} x \circ \delta(yz) = (zx) \circ \delta y + (xy) \circ \delta z \\ x \circ \delta(dy\,dz) = (dz\,dx) \circ \delta y + (dx\,dy) \circ \delta z \end{cases}$$

Put another way, a linear [redacted] function on $\Omega^1 R_\natural$, i.e. a 1-[redacted] current $f(x,y)$ on R, is equivalent to a bilinear function $f(x,y)$ on Ω_A^+ satisfying

$$* \quad \begin{cases} f(x, yz) = f(zx, y) + f(xy, z) \\ f(x, dy\,dz) = f(dz\,dx, y) + f(dx\,dy, z) \end{cases}$$

[redacted] Now we know that $\Omega^1 R_{\not\,}$ is spanned by elements of the form $x \cdot \delta a$, $a \in A$, $x \in \Omega_A^+$ which means that f is determined by the bilinear function $f(x, a)$ on $\Omega^+ \times A$. The first formula implies this $f(x,a)$ is equivalent to a linear function on

$$\Omega^+ \otimes_A \Omega^1 A \otimes_A \quad = \Omega^+ \otimes_A \Omega_A^1$$

(Here the point is that Ω^+ is a module over the commutative algebra A ([redacted] bimodule such that left & right multiplication coincide) so we have

$$[x \otimes_A a(a_0 da_1) \otimes_A] = [xa \otimes_A (a_0 da_1) \otimes_A] = [ax \otimes_A (a_0 da_1) \otimes_A]$$

$$= [x \otimes_A (a_0 da_1) \overset{a}{\underset{\wedge}{\otimes}}_A]$$

which means the image of $[x \otimes_A (a_0 da_1) \otimes_A]$ depends on the image of $a_0 da_1$ in $\Omega^1 A / [A, \Omega^1 A] \xrightarrow{\sim} \Omega_A^1$.)

Thus we have a map

$$\begin{array}{ccccc} \Omega^+ \otimes_A \Omega_A^1 & \longrightarrow & \Omega^1(R)_{\not\,} & \xrightarrow{\gamma} & \Omega_A^- \\ x \otimes da & \longmapsto & x \cdot \delta a & \longmapsto & x\,da \end{array}$$

which is surjective.

The thing to do now is to try to construct an inverse. In other words we know $x \cdot \delta y$ can be expressed as a sum of [redacted] $x_i \cdot \delta a_i$ and we would like to do this explicitly starting from $y = a_0 da_1 \cdots da_{2n}$.

We have

$$x \cdot \delta(ay) = (yx)\cdot\delta a + (xa)\cdot\delta y$$

$$x \cdot \delta(da_1\, da_2) = (da_2\, dx)\cdot\delta a_1 + (dx\, da_1)\cdot\delta a_2$$

$$x \cdot \delta(da_1\, da_2\, da_3\, da_4) =$$

$$(da_3\, da_4\, x)\cdot\delta(da_1\, da_2) + (x\, da_1\, da_2)\cdot\delta(da_3\, da_4)$$

$$= \begin{array}{ll} da_2\, da_3\, da_4\, dx \cdot \delta a_1 & + (da_4\, dx\, da_1\, da_2)\cdot\delta a_3 \\ da_3\, da_4\, dx\, da_1 \cdot \delta a_2 & + (dx\, da_1\, da_2\, da_3)\cdot\delta a_4 \end{array}$$

Thus

$$\boxed{\begin{aligned} & x\cdot\delta(a_0\, da_1\, da_2) \\ &= da_1\, da_2\, x\cdot\delta a_0 \\ &+ da_2\, d(xa_0)\cdot\delta a_1 \\ &+ d(xa_0)\, da_1\cdot\delta a_2 \end{aligned}}$$

$$\boxed{\begin{aligned} & x\cdot\delta(a_0\, da_1\, da_2\, da_3\, da_4) \\ &= da_1\, da_2\, da_3\, da_4\, x\cdot\delta a_0 \\ &+ da_2\, da_3\, da_4\, d(xa_0)\cdot\delta a_1 \\ &+ da_3\, da_4\, d(xa_0)\, da_1\cdot\delta a_2 \\ &+ da_4\, d(xa_0)\, da_1\, da_2\cdot\delta a_3 \\ &+ d(xa_0)\, da_1\, da_2\, da_3\cdot\delta a_4 \end{aligned}}$$

Next we would like to write this using operations defined on Ω_A. Recall that because

$$\Omega_A^q = \Lambda_A^q\, \Omega_A^1$$

There is a canonical interior product operation

$$\Omega^q_A \xrightarrow{\partial} \Omega^{q-1}_A \otimes_A \Omega^1_A$$

$$\xi_1 \cdots \xi_q \longmapsto \sum_{j=1}^{q} (-1)^{q-1} \xi_1 \cdots \hat{\xi}_j \cdots \xi_q \otimes \xi_j$$

Note that

$$\Omega^q_A \xrightarrow{\partial} \Omega^{q-1}_A \otimes_A \Omega^1_A \longrightarrow \Omega^q_A$$

is multiplication by $(-1)^{q-1} q$.

Formula we seem to have is

$$\boxed{x \cdot \delta y = \text{Image of } x\,\partial(dy) - dx\,\partial y \text{ in } \Omega^+_A \otimes_A \Omega^1_A.}$$

As a check we can compose with the multiplication map

$$x\,\partial(dy) - dx\,\partial y \stackrel{m}{\longmapsto}$$

$$x (-1)^{|dy|-1} |dy|\, dy - dx (-1)^{|y|-1} |y|\, y$$

$$= (1+|y|)\, x\, dy + |y|\, dx\, y$$

At this point it seems very likely that we have

$$\Omega^+_A \otimes_A \Omega^1_A \xrightarrow{\sim} \Omega^1(R_A)_H$$

$$x \otimes da \longmapsto x \cdot \delta a$$

Why? We have defined a pairing

$$\Omega^+_A \otimes \Omega^+_A \longrightarrow \Omega^+_A \otimes_A \Omega^1_A$$

$$x \otimes y \longmapsto x\,\partial(dy) - dx\,\partial y$$

All we have to do is verify this is a 1-cocycle with respect to the Fedosov product. One method of doing this is to consider the surjection $RA \longrightarrow R_A$ and use the fact that for any linear function on $RA \otimes d(\rho A)$, there is a unique 1-cocycle on $\Omega^1(RA)_\natural$. All one has to do then is check that

$$x \otimes y \longmapsto x\,\partial(dy) - dx\,\partial y$$

agrees with the [redacted] 1-cocycle on [redacted] RA such that $x\, d(\rho a) \longmapsto x \otimes da$. Thus one takes the image of $x\,d(\rho a_0\, \omega^n(a_1,\dots,a_{2n}))$ ~~calculates~~ the [redacted] $y = a_0\, da_1 \cdots da_{2n}$ and in $\Omega^+_A \otimes_A \Omega^1_A$, and checks if it is $x\,\partial(dy) - dx\,\partial y$.

<u>Let's do this for $n=1$.</u>

$$x\, d(\rho a_0\, \omega(a_1,a_2))$$
$$= x\rho a_0\, d(\omega(a_1,a_2)) + x\,\omega(a_1,a_2)\, d(\rho a_0)$$

$$d(\omega(a_1,a_2)) = d\{\rho(a_1a_2) - \rho(a_1)\rho(a_2)\}$$
$$= d(\rho(a_1a_2)) - \rho(a_2)\, d\rho(a_1) - \rho(a_1)\, d\rho(a_2)$$

$$\therefore\ x\, d(\rho a_0\, \omega(a_1,a_2)) = x\rho a_0\, d(\rho(a_1a_2))$$
$$- (x\rho a_0 \rho a_2\, d\rho a_1) - x\rho a_0 \rho a_1\, d(\rho a_2)$$
$$+ x\,\omega(a_1,a_2)\, d(\rho a_0).$$

This maps to

$$(x\cdot a_0)\otimes d(a_1a_2) - [redacted]\ (a_2\cdot x\cdot a_0)\otimes da_1$$
$$- (x\cdot a_0\cdot a_1)\otimes da_2 + x\,da_1da_2 \otimes da_0$$

$$= (\cancel{xa_0} - dx\,da_0)\otimes(\cancel{a_1\,da_2 + da_1\,a_2}) - (a_2\cancel{xa_0)\otimes da_1}$$
$$+ (da_2\,dx\,a_0 + da_2\,x\,da_0 + \cancel{a_2\,dx\,da_2})\otimes da_1$$
$$- \cancel{(xa_0a_1)\otimes}\, da_2 + (dx\,\cancel{da_0}\,a_1 + dx\,a_0\,da_1 + x\,da_0\,da_1)\otimes da_2$$
$$+ x\,da_1da_2 \otimes da_0$$

$$= \quad + x\, da_1 da_2 \otimes da_0$$

$$(da_2\, dx\, a_0 + da_2\, x\, da_0) \otimes da_1$$

$$(dx\, a_0\, da_1 + x\, da_0\, da_1) \otimes da_2$$

But $\quad x\, \partial(da_0\, da_1\, da_2) = x\, da_1 da_2 \otimes da_0$

$$- dx\, \partial(a_0\, da_1\, da_2) =$$

$$- dx\, a_0\, da_2 \otimes da_1 + dx\, a_0\, da_1 \otimes da_2$$

and it checks.

Let's try another method, namely let's assume A smooth, whence Ω^1_A is a projective A-module, and so we can understand ∂ via interior products with vector fields. In general given a derivation $\partial : A \to M$ with values in a module M, we should have interior product

$$i_\partial : \Omega^q_A \longrightarrow \Omega^{q-1}_A \otimes_A M.$$

Let's abbreviate i_∂ to ∂ and check that $x, y \longmapsto x\, \partial(dy) - dx\, \partial y$ is a 1-cocycle on Ω^+_A relative to the Fedosov product.

$$(x, y \cdot z) \longmapsto x\, \partial(d(y \cdot z)) - dx\, \partial(y \cdot z)$$

$$= x\, \partial(dy\, z + y\, dz) - dx\, \partial(y z - dy\, dz)$$

$$= x \left\{ \overset{1}{\partial(dy)\, z} - \overset{2}{dy\, \partial z} + \overset{3}{\partial y\, dz} + y\, \overset{4}{\partial(dz)} \right\}$$

$$+ dx \left\{ - \overset{5}{\partial y\, z} - \overset{6}{y\, \partial z} + \overset{7}{\partial(dy)\, dz} - dy\, \overset{8}{\partial(dz)} \right\}$$

$$(\cancel{z}\cdot x, y) \longmapsto (z\cdot x)\,\partial(dy) - d(z\cdot x)\,\partial y$$

$$= (zx - dz\,dx)\,\partial(dy) - (dz\,x + z\,dx)\,\partial y$$

$$(x\cdot y, z) \longmapsto (x\cdot y)\,\partial(dz) - d(x\cdot y)\,\partial z$$

$$+ \quad = (xy - dx\,dy)\,\partial(dz) - (dx\,y + x\,dy)\,\partial z$$

$$x\left\{ z\,\overset{1}{\partial(dy)} - dz\,\overset{3}{\partial y} + y\,\overset{4}{\partial(dz)} - dy\,\overset{2}{\partial z}\right\}$$

$$+dx\left\{ dz\,\overset{7}{\partial(dy)} - z\,\overset{5}{\partial y} - dy\,\overset{8}{\partial(dz)} - y\,\overset{6}{\partial z}\right\}$$

Note we use 3 $\quad \partial y\, dz = -dz\,\partial y \qquad$ both odd

7 $\quad \partial(dy)\,dz = dz\,\partial(dy) \qquad \partial(dy)$ even.

March 26, 1991

More on Poisson structures.

Recall that a Poisson structure on M is a Poisson bracket operation $f,g \mapsto \{f,g\}$ on functions which is $\mathbb{C}$ bilinear, skew-symmetric, a derivation in each variable when the other is held fixed, and finally such that the Jacobi identity holds. An operation satisfying all these conditions except the Jacobi identity is equivalent to a section σ of $\Lambda^2 T$ via the formula

$$\{f,g\} = \sigma \lrcorner (df\, dg)$$

We want to understand the Jacobi identity.

Let J be the operation $\omega \mapsto \sigma \lrcorner \omega$ on $\Omega(M)$ and set $b = [J,d]$, so that b is an operator on forms of degree -1 which anti commutes with d.

Prop: TFAE:

1) $\{f,g\} = J(df\, dg)$ satisfies the Jacobi identity

2) $bJ = Jb$

3) $b^2 = 0$.

Proof. 1) $\Rightarrow$ 2). J is locally the sum of operators of the form $\iota_X \iota_Y$ with X, Y vector fields. $bJ - Jb = [b,J] = [[J,d],J]$ is a sum of operators of the form

$$[[\iota_X \iota_Y, d], \iota_Z \iota_W] = [\iota_X L_Y - L_X \iota_Y, \iota_Z \iota_W]$$

$$= \iota_X [L_Y, \iota_Z] \iota_W + \iota_X \iota_Z [L_Y, \iota_W] - [L_X, \iota_Z] \iota_W \iota_Y - \iota_Z [L_X, \iota_W] \iota_Y$$

$$= \iota_X \iota_{[Y,Z]} \iota_W + \iota_X \iota_Z \iota_{[Y,W]} - \iota_{[X,Z]} \iota_W \iota_Y - \iota_Z \iota_{[X,W]} \iota_Y$$

Thus $bJ - Jb$ is interior product by a section of $\Lambda^3 T$ (Nijenhuis tensor?), and to see it vanishes it suffices to check it kills any 3 form of the type $df\,dg\,dh$.

Now

$$J(df\,dg\,dh) = \{f,g\}dh - \{f,h\}dg + \{g,h\}df$$

$$bJ(df\,dg\,dh) = \{\{f,g\},h\} - \{\{f,h\},g\} + \underbrace{\{\{g,h\},f\}}_{-\{f,\{g,h\}\}}$$

Here we have used

$$b(f\,dg) = (Jd - dJ)(f\,dg) = J(df\,dg) = \{f,g\}.$$

Next note that

$$bJ + Jb = (Jd - dJ)J + J(Jd - dJ) = [J^2, d]$$

kills $df\,dg\,dh$ as this is closed and J^2 has degree -4. Thus

$$[b,J](df\,dg\,dh) = 2bJ(df\,dg\,dh)$$

$$= 2\Big(\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}\Big)$$

[redacted] which completes the proof of 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 3): [redacted] We know b commutes with J, anti-commutes with d, as $b = Jd - dJ$ hence it anti commutes with itself.

3) $\Rightarrow$ 1) $\quad b^2(f\,dg\,dh) = b(Jd - dJ)(f\,dg\,dh)$

$$= bJ(df\,dg\,dh) + \underbrace{db J}_{\text{in degree } -3}(f\,dg\,dh)$$

$$= \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} \quad \text{by above.} \quad \square$$

Suppose from now on that we have a Poisson structure. Then we have a $\mathbb{Z}/2$ graded complex given by $\Omega(M)$ with the differential $b+d$.

~~Better~~: We have a mixed complex $(\Omega(M), b, d)$.

(Question: Is b homology the same as Poisson cohomology?)

Let's identify the $b+d$ homology with the even-odd de Rham cohomology. Consider the operator e^J. We have

$$[d, e^J] = \int_0^1 e^{(1-t)J} \underbrace{[d,J]}_{-b} e^{tJ}\, dt$$

$$= -b e^J \qquad \text{since } [b,J]=0.$$

i.e. $\boxed{(d+b)e^J = e^J d}$. Thus e^J is an invertible operator on $\Omega(M)$ preserving the $\mathbb{Z}/2$ grading, which intertwines d and $d+b$.

Next consider a symplectic manifold, which is the case of a Poisson manifold with nondegenerate Poisson structure. Let's first look at the linear algebra. Consider ΛV as the standard irreducible representation of $\mathrm{Cliff}(V \oplus V^*)$, and suppose we are given a nondegenerate skew form $\sigma \in \Lambda^2 V^*$. Let's use bases: $e^k = |k\rangle \wedge$? and $\iota_j = \langle k| \lrcorner$. Suppose $J = \frac{1}{2}\sigma^{jk}\iota_k\iota_j$ is interior product by σ, and let $\omega = \frac{1}{2}\sigma_{jk}e^j e^k$ be exterior multiplication by the corresponding elt of $\Lambda^2 V$ to σ. Here $(\sigma_{jk}) = (\sigma^{jk})^{-1}$.

We have

$$[J,\omega] = \left[\tfrac{1}{2}\sigma^{jk}\iota_k\iota_j,\ \tfrac{1}{2}\sigma_{lm}e^l e^m\right]$$

$$= \tfrac{1}{4}\sigma^{jk}\sigma_{lm}\left(\iota_k\,\delta^l_j\, e^m - \delta^l_k\,\iota_j e^m + e^l\,\delta^m_j\,\iota_k - e^l\,\delta^m_k\,\iota_j\right)$$

$$= \tfrac{1}{4}\left(\sigma^{jk}\sigma_{jm}\,\iota_k e^m - \sigma^{jk}\sigma_{km}\,\iota_j e^m + \sigma^{jk}\sigma_{lj}\,e^l\iota_k - \sigma^{jk}\sigma_{lk}\,e^l\iota_j\right)$$

$$= \tfrac{1}{4}\left(-\iota_k e^k - \iota_j e^j + e^k\iota_k + e^j\iota_j\right)$$

$$= \tfrac{1}{2}\left(e^j\iota_j - \iota_j e^j\right) = \underbrace{e^j\iota_j} - \tfrac{1}{2}\underbrace{\dim V}_{=2n}$$

gives mult by q on $\Lambda^q V$ call this N

Thus we have the commutation relations

$$[\omega,\omega] = 0 \qquad [J,J] = 0$$

$$[J,\omega] = N-n$$

$$[N-n, J] = -2J$$

$$[N-n,\omega] = 2\omega$$

which give a representation of sl_2

Let's consider now these operators on forms over a symplectic manifold. From

$$[J,\omega] = N-n$$

we have $[[J,d],\omega] = [[J,\omega],d] = [N-n,d] = d$

or $\quad [b,\omega] = d \quad$ to go along with $[J,d]=b$

Thus the [redacted] operators d, b span a 2-dim'l

representation of sl_2.

Let's review what we have. We have an action of the Lie algebra $sl_2(\mathbb{C})$ on $\Omega(M)$ given by the operators ω, J, $N-n$. (It might be useful to change the sign of J, b for then the relations would read.

$$[\omega, -J] = N-n \qquad [X^+, X^-] = H$$
$$[N-n, \omega] = 2\omega \qquad [H, X^+] = 2X^+$$
$$[N-n, (-J)] = -2(-J) \qquad [H, X^-] = -2X^-$$

$$[\omega, -b] = d \qquad [\omega, d] = 0$$
$$[-J, d] = -b \qquad [-J, -b] = 0$$

So that relative to $d \longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$ $\quad -b \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$

we have $\omega \longleftrightarrow \begin{pmatrix}0&1\\0&0\end{pmatrix}$ $\qquad (-J) \longleftrightarrow \begin{pmatrix}0&0\\1&0\end{pmatrix}$

Recall $\begin{pmatrix}1&u\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-u^{-1}&1\end{pmatrix}\begin{pmatrix}1&u\\0&1\end{pmatrix} = \begin{pmatrix}0&u\\-u^{-1}&0\end{pmatrix}$

Take $u=1$, whence

$$e^{\omega} e^{+J} e^{\omega} \longleftrightarrow \begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

Check: $d \longmapsto e^{\omega} d e^{-\omega} = d$

$\longmapsto e^{J} d e^{-J} = d + b$

$\longmapsto e^{\omega}(d+b)e^{-\omega} = d + b + \underbrace{e^{\omega}[b, e^{-\omega}]}_{-e^{-\omega}d} = b.$

$\therefore e^{\omega} b e^{-\omega} = b - d$

Similarly

$$\begin{aligned} b &\longmapsto e^{\omega} b e^{-\omega} = b - d \\ &\longmapsto e^{J}(b-d)e^{-J} = b-(d+b) = -d \\ &\longmapsto e^{\omega}(-d)e^{-\omega} = -d \end{aligned}$$

Thus the operator $e^{\omega} e^{J} e^{\omega}$ on $\Omega(M)$ should be of order 4, and conjugation by it should send d to b and b to $-d$. Conjugation by $e^{\omega} e^{J} e^{\omega}$ should [redacted] reverse the sign of $N-n$, so $e^{\omega} e^{J} e^{\omega}$ should give an isomorphism $\Omega^{i}(M) \xrightarrow{\sim} \Omega^{2n-i}(M)$.

March 28, 1991

Associated to a Poisson manifold is a sort of universal enveloping algebra $\mathcal{U}$ generated by functions f and elements X_f satisfying the following relations.

1) X_f $\mathbb{C}$-linear in f

2) $X_{fg} = fX_g + gX_f$

3) $X_f \cdot g = gX_f + \{f,g\}$

4) $[X_f, X_g] = X_{\{f,g\}}$

This enveloping algebra $\mathcal{U}$ operates naturally on the algebra of functions $R = \Omega^0(M)$, where a function f acts by multiplication, and where X_f acts as $\{f, \cdot\}$. In the non-degenerate case we get an isomorphism of $\mathcal{U}$ with the algebra $\mathcal{D}$ of differential operators on M (finite order if M not compact)

Let $F_n = F_n\mathcal{U}$ be the subspace spanned by elements of the form $f_0 X_{f_1} \cdots X_{f_g}$ for $g \leq n$. F_0 is the subalgebra of $\mathcal{U}$ of functions; it is a quotient of $R = \Omega^0(M)$. 1) & 2) give a map of left R-modules

$$R \oplus \Omega^1_R \longrightarrow F_1$$

$$(f\ ,\ g\,dh) \longmapsto f + gX_h$$

which is surjective. Using 3) we see that $F_1 \cdot R \subset F_1$, from which it follows that F_1 is an R-bimodule, and that the $\{F_n\}$ is an increasing algebra filtration. From 4) we see the associated graded algebra is a quotient of the symmetric algebra over R of Ω^1_R.

Here's how one might construct U concretely. Let us take the left R-module $R \oplus \Omega^1_R$ and define right multiplication by R by

$$f \cdot g = fg \qquad g \in R$$

$$f_0 df_1 \cdot g = f_0 g\, df + f_0\{f_1, g\}, \qquad f_0 df_1 \in \Omega^1_R$$

We have to see the second formula defines a left R-module map $\Omega^1 R \longrightarrow R \oplus \Omega^1 R$ for each $g \in R$. However

$$f \longmapsto g\,df + \{f, g\}$$

is a derivation $R \longrightarrow R \oplus \Omega^1_R$ (it would have been better to write $\Omega^1 R \oplus R$), so it extends to the desired left-R-module map $\Omega^1 R \longrightarrow \Omega^1 R \oplus R$. Next check compatible with right mult.

$$(df \cdot g_1) \cdot g_2 = (g_1\, df + \{f, g_1\}) \cdot g_2$$

$$= g_1(g_2\, df + \{f, g_2\}) + g_2\{f, g_1\}$$

$$df \cdot (g_1 g_2) = g_1 g_2\, df + \{f, g_1 g_2\}$$

These agree by the derivation property of the Poisson bracket.

Now we note that given an R-bimodule exact sequence

$$0 \longrightarrow R \longrightarrow M \longrightarrow N \longrightarrow 0$$

there is a obvious generalization of the R-algebra construction, i.e.

$$T_R(M)\Big/\left(1_{T_R(M)} - 1_M\right)$$

Now we can apply this to the R-bimodule made from $R \oplus \Omega^1 R$. This should give the universal algebra generated by R and X_f satisfying the relations 1) - 3). The associated graded algebra should be $T_R(\Omega^1_R)$.

When we further divide by the relations 4) it is not immediately clear what happens. However, it would appear that we have a (R, L) Lie algebra in Rinehart's sense. This should be the same as a formal groupoid, and perhaps my old game with jets and Spencer sequences can be applied.

March 30, 1991

Review KMS for Gaussian states on a Weyl algebra.

CCR: $\quad [\rho x, \rho y] = 2\omega(x,y)$

Here x, y are elements of a real vector space W, $\omega(x,y)$ is a skew-symmetric form on W with values in $i\mathbb{R}$, and ρx is a skew-adjoint operator depending linearly on x.

Weyl CCR: $\quad e^{\rho x} e^{\rho y} = e^{\rho(x+y)} e^{\omega(x,y)}$

Gaussian state: $\quad \varphi(e^{\rho x}) = e^{-\frac{1}{2}(x,x)}$

where (x,y) is an inner product on W

Example. $\rho(z) = za^* - \bar{z}a$, $\quad z \in \mathbb{C}$ considered as real v.s. $\quad \varphi(\alpha) = \langle 0|\alpha|0\rangle$, where $a|0\rangle = 0$.

We have
$$\langle 0|e^{\lambda a^* + \mu a}|0\rangle = \langle 0|e^{\lambda a^*} e^{\mu a} e^{\frac{1}{2}\lambda\mu[a,a^*]}|0\rangle$$
$$= e^{\frac{1}{2}\lambda\mu}$$

so
$$\langle 0|e^{za^* - \bar{z}a}|0\rangle = e^{-\frac{1}{2}|z|^2}$$

Suppose φ as above and write $\langle\alpha\rangle = \varphi(\alpha)$.
$$\langle e^{\rho x}\rangle = e^{-\frac{1}{2}(x,x)}$$

$$-\langle \rho x^2\rangle = (x,x)$$

$$-\langle \rho x \rho y + \rho y \rho x\rangle = 2(x,y)$$
$$-\langle \rho x \rho y - \rho y \rho x\rangle = -2\omega(x,y)$$
$$\boxed{-\langle \rho x \rho y\rangle = (x,y) - \omega(x,y)}$$

KMS in general. If φ is a suitable (faithful normal?) ~~state~~ on a von Neuman algebra, then there is a 1-parameter group α_t of automorphisms of the algebra such that

$$\varphi(xy) = \varphi(\alpha_i(y)x)$$

where the right side is defined by anal. cont.

<u>Example</u>. States on $M_n(\mathbb{C})$ (and more generally on $\mathcal{L}(H)$) are of the form $x \mapsto Tr(\rho x)$ with $\rho \geq 0$ (and of trace class). Faithful means $\rho > 0$, hence of the form e^{-H} with $H^* = H$. We have

$$Tr(e^{-H}xy) = Tr(e^{-H}(e^{H}ye^{-H})x)$$

and $\alpha_t(x) = e^{-itH}x\, e^{itH}$

Lets find α_t in the case of a Gaussian state on a Weyl algebra. We want

$$\langle e^{\rho x} e^{\rho y}\rangle = \langle \underbrace{\alpha_i(e^{\rho y})}_{e^{\rho \alpha_i y}} e^{\rho x}\rangle$$

where it seems reasonable to assume α_t "is" a 1-param. group of autom. of W preserving ω. Since these are Gaussian expressions, we want

$$\langle \rho x\, \rho y\rangle = \langle \rho\alpha_i y\, \rho x\rangle$$

$$(x,y) - \omega(x,y) = (\alpha_i y, x) - \omega(\alpha_i y, x)$$
$$= (x, \alpha_i y) + (x, \alpha_i y)$$

Suppose $\omega(x,y) = (x, iTy)$ where $T^t = -T$ on W; recall ω has values in $i\mathbb{R}$. Then

$$(1-iT)y = (1+iT)\alpha_i y \quad \text{so}$$

$$\boxed{\alpha_i = \frac{1-iT}{1+iT}}$$

Note that $\alpha_i^t = \alpha_i^{-1}$ so $\alpha_i \in$ complex orthogonal group assoc. to $(\,,\,)$. Also $\alpha_i^* = \alpha_i$ is hermitian.

If T has the pair of eigenvalues $\pm ia$, then α_i has the pair of eigenvalues $\frac{1+a}{1-a}$ and its inverse. Note

$$-1 < a < 1 \iff 0 < \frac{1+a}{1-a} < \infty$$

We remark that the positivity for the state $\langle\ \rangle$ is equivalent to

$$|\omega(x,y)| \leq |x|\,|y| \qquad |x| = (x,x)^{1/2}$$

so that necessarily $\|T\| \leq 1$, and so $|a| \leq 1$. But if $a = \pm 1$, the state φ is not faithful, e.g. for $\langle e^{za^* - \bar{z}a}\rangle = e^{-\frac{1}{2}|z|^2}$.

Suppose $\varphi(e^{\rho x}) = e^{-\frac{1}{2}(x,x)}$, $\quad \frac{1}{2}[\rho x, \rho y] = \omega(x,y)$ as above. Let's review the GNS representation associated to φ. This is a $*$ repn. on a Hilbert space with cyclic vector $|0\rangle$ such that $\varphi(\xi) = \langle 0|\xi|0\rangle$. Let V be the complex vector space of elements

$$(\rho x + i\rho y)|0\rangle$$

Thus we have $W \hookrightarrow V$, $x \mapsto (\rho x)|0\rangle$ and $W + iW = V$. In the case of a faithful state

$$W \oplus iW = V.$$

Observe that the hermitian inner product $\langle\,|\,\rangle$ on V is given by

$$\langle x|y\rangle = \langle 0|(\rho x)^*\rho y|0\rangle$$
$$= -\langle 0|\rho x \rho y|0\rangle$$

$$\boxed{\langle x|y\rangle = (x,y) - \omega(x,y)}$$

Put another way:

$$\boxed{|x+iy|^2 = (x,x)+(y,y)+\frac{2}{i}\omega(x,y)}$$

Recall a little about Tomita – Takesaki theory. Let A be a $*$ algebra and φ a state. We then have two hermitian inner products on A considered as complex vector space

$$H_\ell(x,y) = \varphi(x^*y)$$
$$H_r(x,y) = \varphi(y\,x^*)$$

The conjugate linear automorphism $x \longmapsto x^*$ of A can be used to induce one from the other. More precisely starting from H_ℓ we obtain a new hermitian inner product $(x,y) \longmapsto \overline{H_\ell(x^*,y^*)}$ $= H_\ell(y^*,x^*)$ and this is $\varphi((y^*)^*(x^*)) = \varphi(yx^*)$ $= H_r(x,y)$.

Let $\mathcal{H}_\ell$ be the Hilbert space obtained (assuming H_ℓ non-degenerate) by completing A with respect to H_ℓ. This means one has a dense isometric embedding(*) $\iota_\ell : A \longrightarrow \mathcal{H}_\ell$ and the pair $(\mathcal{H}_\ell, \iota_\ell)$ is unique up to canonical

isomorphism. Similarly define
$\iota_r : A \longrightarrow \mathcal{H}_r$. Then it's clear that

$$A \xrightarrow{*} \bar{A} \longrightarrow \overline{\mathcal{H}_\ell}$$

is an isometric embedding with dense image when A is equipped with H_r, so we have

$$\begin{tikzcd} A \arrow[r, "\iota_r"] \arrow[d, "*", "\simeq"'] & \mathcal{H}_r \arrow[d, "\simeq"] \\ \bar{A} \arrow[r, "\iota_\ell"] & \overline{\mathcal{H}_\ell} \end{tikzcd}$$

It would seem then that it suffices to consider one of these inner products and the corresponding completion.

To simplify suppose A finite-dimensional, whence $H_\ell(x,y) = \varphi(x^* y)$ gives a hermitian inner product on A. Notice that $*$ defines a real structure on A whose real subspace is the space of hermitian elements of A. Thus we have exactly the type of situation considered above, namely, a complex Hilbert space V with a real subspace W such that $V = W \oplus iW$. We've analyzed this in terms of the equation

$$\underbrace{\langle x|y\rangle}_{\substack{\text{herm.}\\ \text{inner product}}} = \underbrace{(x,y)}_{\substack{\text{real}\\ \text{part}}} - \underbrace{\omega(x,y)}_{\substack{\text{imag.}\\ \text{part}}}$$

Thus W is a real inner product space equipped with a skew-form $\omega(x,y)$ satisfying

$$|\omega(x,y)| < |x|\,|y| \qquad x \neq 0 \quad y \neq 0.$$

and having imaginary values

March 31, 1991

In dealing with Weyl algebras and Gaussian ~~states~~ it seems to be a good idea to consider a complex vector space W_c with symmetric form (x,y) and skew-symm form $\omega(x,y)$ and to write the basic relations in the form

$$e^x e^y = e^{x+y} e^{\omega(x,y)}$$

$$\langle e^x \rangle = e^{\frac{1}{2}(x,x)}$$

There is also a real structure on W_c given by $x \mapsto x^*$ such that

$$(x^*, y^*) = \overline{(x,y)}$$

$$\omega(x^*, y^*) = -\overline{\omega(x,y)} \qquad \therefore \ \omega \in i\mathbb{R} \text{ on } W_r$$

As usual let's assume $(\ ,\ )$ and ω non-degenerate and define β by $\omega(x,y) = (x, \beta y)$, whence $\beta^t = -\beta$ and the eigenvalues of β are stable under $\lambda \mapsto -\lambda$. The reality conditions imply $\beta = iT$ where T is real skew-symmetric on W_r, hence the eigenvalues of T are purely imaginary, and those of β are real.

Let W_ξ be the ξ-eigenspace of β. Then for $x \in W_\xi$, $y \in W_\eta$ we have

$$(x,y) = \xi^{-1}(\beta x, y) = -\xi^{-1}(x, \beta y) = -\xi^{-1}\eta(x,y)$$

hence $(W_\xi, W_\eta) \neq 0 \implies \xi = -\eta$. Also

$$\begin{aligned} \omega(x^*, y^*) &= (x^*, \beta y^*) \\ \parallel \\ -\overline{\omega(x,y)} &= -\overline{\omega(x^*, \beta y^*)} \\ &= -\omega(x^*, (\beta y)^*) \end{aligned}$$

Thus $(\beta y)^* = -\beta(y^*)$ so $(W_\xi)^* = W_{-\bar{\xi}}$ and since the eigenvalues are real, we have

$$\boxed{(W_\xi)^* = W_{-\xi}}$$

Different viewpoint. Suppose given W complex vector space with symmetric bilinear form (x,y) and skew-symmetric one $\omega(x,y)$. Consider an algebra with state satisfying

$$\omega(x,y) = \tfrac{1}{2}[x,y]$$

$$\langle x^2 \rangle = (x,x) \qquad \langle 1 \rangle = 1.$$

more generally $\left\langle \frac{xy+yx}{2} \right\rangle = (x,y)$. Observe we have

$$\langle xy \rangle = (x,y) + \omega(x,y).$$

Now suppose $\langle xy \rangle$ non degenerate. Then we can define α_i by

$$\langle xy \rangle = \langle \alpha_i y\, x \rangle$$

suppose $\omega(x,y)$ non-degenerate; then we can define S by $(x,y) = \omega(x, Sy)$. We have

$$\langle xy \rangle = (x,y) + \omega(x,y) = \omega(x,(1+S)y)$$

$$\begin{aligned}\langle \alpha_i y\, x \rangle &= (\alpha_i y, x) + \omega(\alpha_i y, x)\\ &= (x, \alpha_i y) - \omega(x, \alpha_i y)\\ &= \omega(x, (S-1)\alpha_i y).\end{aligned}$$

Since this holds for all x and ω is non-deg., we have

$$\alpha_i = \frac{S+1}{S-1}$$

Similarly if (x,y) is non-degenerate, we can define β by $\omega(x,y) = (x, \beta y)$ and we find $\alpha_i = \frac{1+\beta}{1-\beta}$. Of course $\beta = S^{-1}$ when both are defined.

The spectrum for this setup (assuming $(\,,\,)$ non degenerate is the set of λ such that the bilinear form $\underbrace{(x,y) - \lambda\,\omega(x,y)}_{\omega(x,(S-\lambda)y)}$ is degenerate.

If $(S-\lambda)y = 0$, then as

$$0 = \omega(x,(S-\lambda)y) = -\omega((S+\lambda)x, y)$$

for all x, we see that $S+\lambda$ is not surjective, thus $-\lambda$ also is in the spectrum.

Suppose we split the spectrum into two pieces carried into each other by $\lambda \longmapsto -\lambda$. Then we get a decomposition $W = W^+ \oplus W^-$ where W^+, W^- are isotropic for both $(\,,\,)$ and $\omega(\,,\,)$.

April 1, 1991

Dilating: Suppose we have a Gauss-Weyl structure on W given by (x,y), $\omega(x,y)$. Recall

$$\langle xy\rangle = (x,y) + \omega(x,y)$$

Let $V = W \oplus W$ and define

$$\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix}\right) = \frac{1}{2}\left(\langle x_2 y_1\rangle + \langle y_2 x_1\rangle\right)$$

$$\omega\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}y_1\\y_2\end{pmatrix}\right) = \frac{1}{2}\left(\langle x_2 y_1\rangle - \langle y_2 x_1\rangle\right)$$

so that $S\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}y_1\\-y_2\end{pmatrix}$. Define the

$\boxed{\text{embedding } W \hookrightarrow V \text{ by } x \longmapsto \begin{pmatrix}x\\x\end{pmatrix}}$. Then

$$\left(\begin{pmatrix}x\\x\end{pmatrix}, \begin{pmatrix}y\\y\end{pmatrix}\right) = \frac{1}{2}\left(\langle xy\rangle + \langle yx\rangle\right) = (x,y)$$

$$\omega\left(\begin{pmatrix}x\\x\end{pmatrix}, \begin{pmatrix}y\\y\end{pmatrix}\right) = \frac{1}{2}\left(\langle xy\rangle - \langle y,x\rangle\right) = \omega(x,y)$$

Suppose $x \longmapsto x^*$ is a real structure on W so that $(x,y)^* = (x^*,y^*)$, $\omega(x,y)^* = -\omega(x^*,y^*)$.

Then
$$\begin{aligned}\langle xy\rangle^* &= (x,y)^* + \omega(x,y)^* \\ &= (x^*,y^*) - \omega(x^*,y^*) \\ &= (y^*,x^*) + \omega(y^*,x^*) = \langle y^* x^*\rangle.\end{aligned}$$

$\boxed{\text{Define } \begin{pmatrix}x_1\\x_2\end{pmatrix}^* = \begin{pmatrix}x_2^*\\x_1^*\end{pmatrix}}$. Then $*$ on V is compatible with $*$ on W. Also

$$\left(S_V\begin{pmatrix}x_1\\x_2\end{pmatrix}\right)^* = \begin{pmatrix}x_1\\-x_2\end{pmatrix}^* = \begin{pmatrix}-x_2^*\\x_1^*\end{pmatrix} = -S_V\begin{pmatrix}x_2^*\\x_1^*\end{pmatrix} = -S_V\begin{pmatrix}x_1\\x_2\end{pmatrix}^*$$

which shows

$$\boxed{* \circ S_V = - S_V \circ *}$$

$$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)^* = \tfrac{1}{2}\left(\langle x_2 y_1 \rangle + \langle y_2 x_1 \rangle\right)^*$$

$$= \tfrac{1}{2}\left(\langle y_1^* x_2^* \rangle + \langle x_1^* y_2^* \rangle\right)$$

$$\left(\begin{pmatrix} x_2^* \\ x_1^* \end{pmatrix}, \begin{pmatrix} y_2^* \\ y_1^* \end{pmatrix}\right) = \tfrac{1}{2}\left(\langle x_1^* y_2^* \rangle + \langle y_1^* x_2^* \rangle\right)$$

$$\omega\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right)^* = \frac{1}{2}\left(\langle y_1^* x_2^* \rangle - \langle x_1^* y_2^* \rangle\right)$$

$$\omega\left(\begin{pmatrix} x_2^* \\ x_1^* \end{pmatrix}, \begin{pmatrix} y_2^* \\ y_1^* \end{pmatrix}\right) = \frac{1}{2}\left(\langle x_1^* y_2^* \rangle - \langle y_1^* x_2^* \rangle\right)$$

Thus the embedding $W \hookrightarrow V$ is compatible with real structures.

Example: $W = \mathbb{C}b \oplus \mathbb{C}b^*$ where $[b, b^*] = 1$ and $\langle b^* b \rangle = \mu^2, \mu > 0$. Also $\langle b^2 \rangle = \langle b^{*2} \rangle = 0$. Thus $\langle b b^* \rangle = \langle b^* b \rangle + 1 = \mu^2 + 1$. Set $\lambda^2 = \mu^2 + 1$ with $\mu, \lambda > 0$. Recall S defined by

$$(x, y) = \omega(x, Sy)$$

or

$$\langle xy \rangle = \omega(x, (S+1)y)$$

$$\langle yx \rangle = (y, x) + \omega(y, x)$$

$$= \omega(x, Sy) - \omega(x, y)$$

$$= \omega(x, (S-1)y)$$

$$\therefore \quad \langle \alpha(y) x \rangle = \langle xy \rangle \iff (S-1)\alpha(y) = (S+1)y \iff \boxed{\alpha = \frac{S+1}{S-1}}$$

$$\langle b b^* \rangle = \tfrac{1}{2}[b, (S+1) b^*]$$

$$\mu^2 + 1 \qquad \therefore \quad \boxed{S b^* = (2\mu^2+1) b^*}$$

$$\langle b^* b \rangle = \tfrac{1}{2}[b^*, (S+1) b] \qquad Sb = sb$$

$$\mu^2 = -\tfrac{1}{2}(s+1) \qquad s = -(2\mu^2+1).$$

$$\boxed{S b = -(2\mu^2+1) b}$$

$$\alpha(b^*) = \frac{2\mu^2+2}{2\mu^2} b^* \qquad \boxed{\alpha(b^*) = \frac{\mu^2+1}{\mu^2} b^*}$$

$$\alpha(b) = \frac{-2\mu^2}{-2\mu^2-2} b \qquad \boxed{\alpha(b) = \frac{\mu^2}{\mu^2+1} b}$$

Next look at $W \hookrightarrow V$.

$$b^* \longmapsto \begin{pmatrix} b^* \\ b^* \end{pmatrix} = \begin{pmatrix} b^* \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b^* \end{pmatrix}.$$

$$b \longmapsto \begin{pmatrix} b \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\omega\left(\begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} b^* \\ 0 \end{pmatrix}\right) = \frac{1}{2} \underbrace{\langle b b^* \rangle}_{\mu^2+1 = \lambda^2}$$

$$\tfrac{1}{2}\left[\begin{pmatrix} 0 \\ b \end{pmatrix}, \begin{pmatrix} b^* \\ 0 \end{pmatrix}\right].$$ Set $\begin{pmatrix} b^* \\ 0 \end{pmatrix} = \lambda a_1^*$, $\begin{pmatrix} 0 \\ b \end{pmatrix} = \lambda a_1$

$$\tfrac{1}{2}\left[\begin{pmatrix} b \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b^* \end{pmatrix}\right] = \tfrac{1}{2}(-\langle b^* b \rangle) = -\frac{\mu^2}{2}$$ Set $\begin{pmatrix} b \\ 0 \end{pmatrix} = \mu a_2^*$

$\begin{pmatrix} 0 \\ b^* \end{pmatrix} = \mu a_2$. Thus we have

$$\boxed{\begin{aligned} b^* &= \lambda a_1^* + \mu a_2 \\ b &= \mu a_2^* + \lambda a_1 \end{aligned}} \qquad \lambda^2 - \mu^2 = 1.$$

April 2, 1991

Gauss - Weyl. Consider W complex vector space with symm. (x,y) and skew symm. $\omega(x,y)$. Put

$$\langle xy \rangle = (x,y) + \omega(x,y).$$

Assume $\langle xy \rangle$ non degenerate. Then we can define the KMS operator α by

$$\langle xy \rangle = \langle \alpha(y)x \rangle$$

Suppose ω non-degenerate. Then we can define the operator S by

$$(x,y) = \omega(x,Sy).$$

Then we have

$$\omega(x,Sy) + \omega(Sx,y) = (x,y) - (y,x) = 0$$

$$(x,Sy) + (Sx,y) = \omega(x,S^2y) + \omega(Sx,Sy) = 0$$

showing that S is an infinitesimal symplectic transformation with respect to ω, and also that it is an infinitesimal orthogonal transformation with respect to (x,y). We have

$$\langle xy \rangle = (x,y) + \omega(x,y) = \omega(x,(S+1)y)$$

$$\langle \alpha(y)x \rangle = (\alpha(y),x) + \omega(\alpha(y),x)$$

$$= (x,\alpha(y)) - \omega(x,\alpha(y)) = \omega(x,(S-1)\alpha y)$$

Thus $(S-1)\alpha y = (S+1)y$ for all y, whence

$$\alpha = \frac{S+1}{S-1}$$

This is the Cayley transform of S, and it is simultaneously a symplectic transformation w.r.t. ω an orthogonal transf. w.r.t. (x,y).

Thus we learn that the KMS operator naturally lives in a group. This gives some justification for expecting the KMS automorphism to lie on a 1-parameter group of automorphisms of an algebra. (Symplectic autos of W are certain autos of the assoc. Weyl alg.)

The subtle point in all this seems to be the following calculation. ~~[illegible]~~ Suppose we consider a real symplectic vector space W_r ~~[illegible]~~ ($\omega = i \cdot$ symp. form) with pos definite symmetric form, which we take as Hamiltonian H. Then on the honest C^*-Weyl algebra, we have the thermal ~~state~~

$$\circledast \qquad \langle \xi \rangle = \frac{\mathrm{Tr}(e^{-H}\xi)}{\mathrm{Tr}(e^{-H})}$$

It turns out that this state is Gaussian. One calculates this and determines that the KMS operator is the operator on W_r given by $x \longmapsto e^{H}x$.

Old formulas. $\quad H = ta^*a$

$$\langle a^*a \rangle = \frac{\mathrm{Tr}(e^{-ta^*a}\, a^*a)}{\mathrm{Tr}(e^{-ta^*a})} = \underbrace{\frac{1}{e^t - 1}}_{\mu^2 \text{ as on } p\,319}$$

(Planck ~~[illegible]~~)

$$\alpha(a^*) = \frac{\mu^2+1}{\mu^2}\, a^* = e^t a^*$$

$$\alpha(a) = e^{-t} a$$

We know that $\alpha : \xi \longmapsto e^{H}\xi e^{-H}$ is the KMS autom assoc. to $\circledast$, so the only non-obvious point is why the thermal state $\circledast$ is Gaussian.

April 6, 1991

Universal description of R_A for A comm.

R_A comes equipped with a linear map $\rho : A \to R_A$ such that $\rho(1) = 1$, namely the inclusion $A = \Omega_A^0 \subset \Omega_A^+$. The corresp. curvature is

$$\omega(a_1, a_2) = \rho(a_1 a_2) - \rho a_1 \rho a_2$$
$$= a_1 a_2 - a_1 \circ a_2 = da_1 da_2$$

Note that $\omega(a_1, a_2)$ is in the center of R_A and that it is skew-symmetric:

$$(*) \qquad \begin{aligned} [\rho a_0, \omega(a_1, a_2)] &= 0 \\ \omega(a_1, a_2) + \omega(a_2, a_1) &= 0 \end{aligned}$$

Let's study the universal algebra for these relations, i.e. the quotient of RA by the ideal generated by these relations. Call this R. Observe

$$\rho a_1 \rho a_2 = \rho(a_1 a_2) - \omega(a_1, a_2)$$

is the decomposition of the former into symmetric and skew-symmetric forms, whence

$$\rho(a_1 a_2) = \tfrac{1}{2}(\rho a_1 \rho a_2 + \rho a_2 \rho a_1)$$
$$-2\omega(a_1, a_2) = [\rho a_1, \rho a_2]$$

We can eliminate ω using the latter and describe R, the universal normalized algebra with ρ whose curvature lies in the center and is skew-symmetric, by the relations

$$(*)' \qquad \begin{aligned} [\rho a_0, [\rho a_1, \rho a_2]] &= 0 \\ \rho(a_1 a_2) &= \tfrac{1}{2}(\rho a_1 \rho a_2 + \rho a_2 \rho a_1) \end{aligned}$$

We would like to show $R \xrightarrow{\sim} R_A$.

<u>Lemma</u>: The multilinear function of degree 4

$$[\rho a_1, \rho a_2][\rho a_3, \rho a_4]$$

on A is alternating.

Proof: Let's change notation temporarily and use x, y, z, w to denote elements of A. It will suffice to prove

$$[\rho x, \rho y]^2 = 0. \qquad (+)$$

In effect polarizing this, considered as quadratic function of x gives

$$0 = [\rho x + \rho z, \rho y]^2 = \overbrace{[\rho x, \rho y]^2}^{0} + \overbrace{[\rho z, \rho y]^2}^{0} + [\rho x, \rho y][\rho z, \rho y] + [\rho z, \rho y][\rho x, \rho y]$$

$$= 2\,[\rho x, \rho y][\rho z, \rho y]$$

since these brackets are in the center of R and therefore commute. Thus we conclude that

$$[\rho x, \rho y][\rho z, \rho w]$$

vanishes if any two arguments coincide, which means it is alternating.

To verify $(+)$ we use Weyl algebra type calculations, i.e. the formula

$$e^X e^Y = e^{X+Y} e^{\frac{1}{2}[X,Y]}$$

when $[X,Y]$ commutes with X, Y. Note that the identity $\rho(xy) = \frac{1}{2}(\rho x \rho y + \rho y \rho x)$ implies $\rho(xy) = \rho x \rho y$ when $\rho x, \rho y$ commute. In particular

$$\rho(x^n) = (\rho x)^n \qquad \rho(e^x) = e^{\rho x}$$

to consider

$$\begin{array}{ccc}
\rho(e^x e^y) & = & \frac{1}{2}\left(\rho(e^x)\rho(e^y) + \rho(e^y)\rho(e^x)\right) \\
\| & & \| \\
\rho(e^{x+y}) & & \frac{1}{2}\left(e^{\rho x} e^{\rho y} + e^{\rho y} e^{\rho x}\right) \\
\| & & \| \\
e^{\rho x+\rho y} & & \frac{1}{2} e^{\rho x+\rho y}\left(e^{\frac{1}{2}[\rho x,\rho y]} + e^{-\frac{1}{2}[\rho x,\rho y]}\right)
\end{array}$$

Since $e^{\rho x+\rho y}$ is invertible this implies

$$\frac{1}{2}\left(e^{\frac{1}{2}[\rho x,\rho y]} + e^{-\frac{1}{2}[\rho x,\rho y]}\right) = 1$$

$$= 1 + \frac{1}{2!}[\rho x,\rho y]^2 + \frac{1}{4!}[\rho x,\rho y]^4 + \cdots$$

whence $[\rho x,\rho y]^2 = 0$.

One can probably justify using these exponential calculations by generating function method - adjoin variable s, t and work with formal power series e^{sx}, $e^{s\rho x}$ etc. having coefficients in A, R. However, here's an elementary proof.

Let $X = \rho x$, $Y = \rho y$, $Z = [\rho x, \rho y]$ so that Z commutes with X, Y and $[X,Y] = Z$.

$$\rho(x^2y^2) = \frac{1}{2}\left(\rho(x^2)\rho(y^2) + \rho(y^2)\rho(x^2)\right)$$
$$= \frac{1}{2}\left(X^2Y^2 + Y^2X^2\right)$$

Use normal ordering with Y's on the left and X's to the right.

$$X^2Y^2 = X[X,Y^2] + XY^2X$$
$$= X(2YZ) + [X,Y^2]X + Y^2X^2$$
$$= 2XYZ + 2YXZ + Y^2X^2$$

Thus $\quad \rho(x^2y^2) = \frac{1}{2}(X^2Y^2 + Y^2X^2)$

$$= Y^2X^2 + XYZ + YXZ = Y^2X^2 + 2YXZ + Z^2$$

Next $\quad \rho(x^2y^2) = \rho(xyxy) = \rho(xy)\rho(xy)$

$$= \frac{1}{4}(\rho x \rho y + \rho y \rho x)^2 = \frac{1}{4}(XY + YX)^2$$

$$= \frac{1}{4}(XYXY + XY^2X + YX^2Y + YXYX)$$

$$XYXY = [X,Y]XY + YX^2Y$$

$$= Z([X,Y] + YX) + Y\underbrace{[X^2,Y]}_{2XZ} + Y^2X^2$$

$$= Y^2X^2 + 3YXZ + Z^2$$

$$XY^2X = [X,Y^2]X + Y^2X^2 = 2YXZ + Y^2X^2$$

$$YX^2Y = Y[X^2,Y] + Y^2X^2 = 2YXZ + Y^2X^2$$

$$YXYX = Y[X,Y]X + Y^2X^2 = YXZ + Y^2X^2$$

$$\therefore \rho(x^2y^2) = \frac{1}{4}(4Y^2X^2 + 8YXZ + Z^2)$$

$$= Y^2X^2 + 2YXZ + \frac{1}{4}Z^2$$

Comparing [redacted]

$$Y^2X^2 + 2YXZ + Z^2 = Y^2X^2 + 2YXZ + \frac{1}{4}Z^2$$

$$\therefore \quad Z^2 = 0.$$

Let's review. A commutative R is the universal algebra equipped with a linear map $\rho: A \to R$ satisfying $\rho(1) = 1$ such that

$$[\rho a_0, [\rho a_1, \rho a_2]] = 0 \qquad \rho(a_1 a_2) = \frac{\rho a_1 \rho a_2 + \rho a_2 \rho a_1}{2}$$

We want to show $R = R_A$.

First we show R is spanned by elements of the form

$$\rho a_0 [\rho a_1, \rho a_2] \cdots [\rho a_{2n-1}, \rho a_{2n}]$$

It suffices show that the subspace L spanned by these elements satisfies $\rho(a) L \subset L$. But we have

$$\rho a \, \rho a_0 = \tfrac{1}{2}(\rho a \rho a_0 + \rho a_0 \rho a) + \tfrac{1}{2}[\rho a, \rho a_0]$$
$$= \rho(a a_0) + \tfrac{1}{2}[\rho a, \rho a_0]$$

so this is clear.

Now we know that brackets $[\rho a_1, \rho a_2]$ belong to the center Z of R. It follows that

$$[\rho a, \rho a_0 [\rho a_1, \rho a_2] \cdots [\rho a_{2n-1}, \rho a_{2n}]]$$
$$= [\rho a, \rho a_0][\rho a_1, \rho a_2] \cdots [\rho a_{2n-1}, \rho a_{2n}]$$

Thus $[\rho a, R] \subset Z$ and hence

$$(*) \qquad [\rho a_1, [\rho a_2, R]] = 0 .$$

Next we will define an A-module structure on by the formula

$$\boxed{a \cdot x = \tfrac{1}{2}(\rho a \, x + x \, \rho a)}$$

We check that it's a module

$$a_1 \cdot (a_2 \cdot x) = \tfrac{1}{4}\left(\rho a_1 (\rho a_2 x + x \rho a_2) + (\rho a_2 x + x \rho a_2)\rho a_1\right)$$
$$(a_1 a_2) \cdot x = \tfrac{1}{4}\left((\rho a_1 \rho a_2 + \rho a_2 \rho a_1) x + x(\rho a_1 \rho a_2 + \rho a_2 \rho a_1)\right)$$

Subtracting and multiplying by 4 gives

$$\rho a_1 x \rho a_2 + \rho a_2 x \rho a_1 - \rho a_2 \rho a_1 x - x \rho a_1 \rho a_2$$
$$= [\rho a_1, x]\rho a_2 + \rho a_2 [x, \rho a_1] = [[\rho a_1, x], \rho a_2]$$

and we have seen this is zero by $(*)$

Another consequence of ⊛ is a simpler proof of the previous lemma. Thus observe

$$[\rho a_1, \rho a_2]^2 = [\rho a_1, \rho a_2][\rho a_1, \rho a_2]$$

$$= [\rho a_1, \rho a_2 [\rho a_1, \rho a_2]] \qquad \text{as } [\rho a_1, \rho a_2] \in Z$$

$$= [\rho a_1, [\rho a_2 \rho a_1, \rho a_2]]$$

$$= 0.$$

Observe that we can define a new product on R, not just an A-module structure. First show that $[R,R] \subset Z$. In effect we know that R is spanned by elements $\rho a\, z$ with $z \in Z$. Then

$$[\rho a_1 z_1, \rho a_2 z_2] = [\rho a_1, \rho a_2] z_1 z_2 \in Z.$$

Then define $\boxed{x \circ y = \tfrac{1}{2}(xy + yx)}$

and

$$4\, x \circ (y \circ z) = x(yz + zy) + (yz + zy)x$$

$$4\, (x \circ y) \circ z = (xy + yx)z + z(xy + yx)$$

$$\text{difference} = xzy + yzx - yxz - zxy$$

$$= [x,z]y + y[z,x] = [[x,z],y]$$

and this is zero because $[x,z] \in Z$.

Consider triples $\rho : A \longrightarrow R$ where A, R are algs. A commutative, ρ linear map $\rho(1) = 1$ and such that

$$\rho(a_1 a_2) = \tfrac{1}{2}(\rho a_1 \rho a_2 + \rho a_2 \rho a_1)$$

Then

$$\rho(a_1 a_2 a_3) = \tfrac{1}{2}\left(\rho(a_1 a_2)\rho a_3 + \rho a_3 \rho(a_1 a_2)\right)$$

$$= \tfrac{1}{4}\left((\rho a_1 \rho a_2 + \rho a_2 \rho a_1)\rho a_3 + \rho a_3(\rho a_1 \rho a_2 + \rho a_2 \rho a_1)\right)$$

$$\rho(a_1 a_2 a_3) = \tfrac{1}{2}\left(\rho a_1\, \rho(a_2 a_3) + \rho(a_2 a_3)\rho a_1\right)$$
$$= \tfrac{1}{4}\left(\rho a_1(\rho a_2 \rho a_3 + \rho a_3 \rho a_2) + (\rho a_2 \rho a_3 + \rho a_3 \rho a_2)\rho a_1\right)$$

4 × difference is

$$\begin{aligned} &\cancel{123} + 213 + 312 + \cancel{321} \\ -&\cancel{123} - 132 - 231 - \cancel{321} \end{aligned} \quad = 2[1,3] + [3,1]2$$
$$= 2[1,3] - [1,3]2 = [2,[1,3]] = [\rho a_2, [\rho a_1, \rho a_3]]$$

Thus the relation $[\rho a_1, [\rho a_2, \rho a_3]] = 0$ is a consequence of commutativity of A and $\rho(a_1 a_2) = \tfrac{1}{2}(\rho a_1 \rho a_2 + \rho a_2 \rho a_1)$.

Suppose R generated by the elements ρa. Then we have seen that R is spanned by elements of the form

$$\rho a_0 [\rho a_1, \rho a_2] \cdots [\rho a_{2n-1}, \rho a_{2n}]$$

where the brackets are central. It follows that $[R,R] \subset$ center, i.e. $[R,[R,R]] = 0$

<u>Lemma:</u> TFAE for an algebra

1) $[R,[R,R]] = 0$
2) $x \cdot y = \tfrac{1}{2}(xy + yx)$ is associative
3) $\exists\, \rho : A \longrightarrow R$ with A comm., $\rho(1) = 1$, $\rho(a_1 a_2) = \tfrac{1}{2}(\rho a_1 \rho a_2 + \rho a_2 \rho a_1)$ such that R is generated by the elements ρa.

<u>Pf.</u> 1) $\Leftrightarrow$ 2) by the identity

$$4\big((x \cdot y)\cdot z - x \cdot (y \cdot z)\big) = [y,[x,z]]$$

3) $\Rightarrow$ 1) we just did

2) $\Rightarrow$ 3) Take $A = R$ equipped with product $x \cdot y$ and ρ to be the identity map.

Call such an algebra nearly commutative. I'd like to identify these algebras with Poisson algebras (comm. algs. with Poisson bracket) such that $\{f,\{g,h\}\}=0$.

Let's start with such a Poisson algebra $A, \{\,,\}$ and define a new product on A by

$$f\circ g = fg + c\{f,g\}$$

Then

$$(f\circ g)\circ h = (fg + c\{f,g\})h + c\{fg + \cancel{c\{f,g\}}, h\}$$

$$= fgh + c\Big(\{f,g\}h + \underbrace{\{fg,h\}}_{\{f,h\}g + f\{g,h\}}\Big)$$

$$f\circ(g\circ h) = f(gh + c\{g,h\}) + c\{f, gh + \cancel{c\{g,h\}}\}$$

$$= fgh + c\Big(f\{g,h\} + \underbrace{\{f,gh\}}_{\{f,g\}h + g\{f,h\}}\Big)$$

so we have an associative product such that $\frac{1}{2}(f\circ g + g\circ f) = fg$. (Note the Jacobi identity is automatic since triple brackets are 0.). Also we have $[f,g]_\circ = 2c\{f,g\}$. so that $[f,[g,h]_\circ]_\circ = 0$.

Conversely given R with $[R,[R,R]]=0$ define $x\cdot y = \frac{1}{2}(xy+yx)$ which gives a commutative alg. structure. Let $\frac{1}{2c}[f,g]$ be the Poisson bracket. Then

$$[f\cdot g, h] = [fg - \tfrac{1}{2}[f,g], h] = [fg, h]$$

$$= [f,h]g + f[g,h] = [f,h]\cdot g + f\cdot[g,h]$$

so we have the <u>biderivation property</u>.

We should be able to show that the universal nearly commutative algebra generated by the commutative algebra A is just the Fedosov algebra R_A.

Consider $\rho : A \longrightarrow R$ such that $\rho(a_1 a_2) = \dots$ and such that R is generated by the $\rho(a)$. Then we know R is nearly commutative, so it's just a commutative algebra with Poisson bracket such that triple brackets are zero. Use $x \cdot y$, $[x,y]$ for these structures on R. ρ is a homom. of comm. algebras and $[\rho a_1, \rho a_2]$ is a biderivation on A with values in the A-module R; also its skew-symmetric, hence we get an induced A-module homomorphism $\Omega^2_A \longrightarrow R$ such that $da_1 da_2 \longmapsto [\rho a_1, \rho a_2]$. Then we get a induced homom. of comm. algs $\mathrm{Sym}^A(\Omega^2_A) \longrightarrow R$. Because of the identity $[\rho a_1, \rho a_2]^2 = 0$, we know that expressions $[\rho a_1, \rho a_2] \cdots [\rho a_{2n-1}, \rho a_{2n}]$ are alternating in the $a_1, \dots, a_{2n}$. So this means we get an induced homomorphism $\Omega^+_A \longrightarrow R$ of commutative algebras.