January 2, 1991

The Connes exact sequence via extensions

Let $A = R/I$ with R projective.

Consider the following ~~quotients~~ of $X(R)$:

$$X^n(R,I): \quad R/I^{n+1}+[R,I^n] \rightleftarrows \Omega^1R/I^n\Omega^1R+[R,\Omega^1R]$$

$$X^n(R,I)': \quad R/I^{n+1} \rightleftarrows \Omega^1R/I^{n+1}\Omega^1R+[R,\Omega^1R]+I^n dI$$

Consider the kernel of $X^n(R,I)' \twoheadrightarrow X^n(R,I)$:

$$\begin{tikzcd}
\dfrac{I^{n+1}+[R,I^n]}{I^{n+1}} \arrow[r, shift right, "d=0"'] \arrow[d, phantom, "\parallel" rotate=90] & \dfrac{I^n\Omega^1R+[R,\Omega^1R]}{I^{n+1}\Omega^1R+[R,\Omega^1R]+I^n dI} \arrow[l, shift right, "b"'] \arrow[d, phantom, "\parallel" rotate=90] \\
{[A, I^n/I^{n+1}]} & I^n/I^{n+1}\otimes_R \Omega^1R\otimes_R / \operatorname{Im}\{I^n dI\} \arrow[d, phantom, "\parallel" rotate=90] \\
 & I^n/I^{n+1}\otimes_A \Omega^1A\otimes_A \arrow[ul, "b"']
\end{tikzcd}$$

The homology of this complex is

$$0 \qquad\qquad H_1(A, I^n/I^{n+1}) = HH_{2n+1}A$$

Recall our calculations in the case of RA, IA:

$$\begin{tikzcd}
H^i(X^n(R,I)): & HC_{2n} \arrow[d, phantom, "\cup" rotate=90] & \operatorname{Ker}\{HC_{2n-1}A \xrightarrow{B} HH_{2n}A\} \\
\dot{H}(X^n(R,I)'): & \operatorname{Ker}\{HC_{2n}A \xrightarrow{B} HH_{2n+1}A\} & HC_{2n+1}A \arrow[u, "S"']
\end{tikzcd}$$

This is consistent with a six term exact sequence

$$\begin{tikzcd}
H_0X^n(R,I)' \arrow[r] & \overset{HC_{2n}A}{\overset{\parallel}{H_0X^n(R,I)}} \arrow[r, "(B)"] & HH_{2n+1}A \arrow[d, "(I)"] \\
0 \arrow[u] & H_1X^n(R,I) \arrow[l] & H_1X^n(R,I)' = HC_{2n+1}A \arrow[l]
\end{tikzcd}$$

Next consider the kernel of the surjection $X^n(R,I) \longrightarrow X^{n-1}(R,I)^{\sigma}$.

$$\begin{tikzcd}
⊛ \quad \dfrac{I^n}{I^{n+1}+[R,I^n]} \arrow[r, shift right, "d"'] \arrow[d, equal] & \dfrac{I^n\Omega^1R+[R,\Omega^1R]+I^{n-1}dI}{I^n\Omega^1R+[R,\Omega^1R]} \supset \mathcal{J} \arrow[l, shift right, "b"'] \arrow[d, equal] \\
(I^n/I^{n+1})\otimes_A \arrow[d, equal] & \text{Image of } I^{n-1}dI \text{ in } I^n/I^{n+1}\otimes_A(A\otimes_R\Omega^1R\otimes_R A)\otimes_A \\
{[I/I^2\otimes_A]^n} &
\end{tikzcd}$$

Let's introduce the notation $E = A\otimes_R\Omega^1R\otimes_R A$, $N = I/I^2$ and recall the exact sequence

$$0 \longrightarrow HH_{2n}A \longrightarrow [N\otimes_A]^n \xrightarrow{\ \partial\ } [N\otimes_A]^{n-1}E\otimes_A$$

where ∂ is induced by the embedding $N \hookrightarrow E$ (induced by d) applied to the last copy of N. Also $HH_nA \subset [N\otimes_A]^{n,\sigma}$ and d the cyclic norm of ∂.

But we have

$$0 \longrightarrow N \xrightarrow{\ d\ } E \longrightarrow \Omega^1A \longrightarrow 0$$

$$0 \longrightarrow [N\otimes_A]^{n-1}N \xrightarrow{1^{\otimes n-1}\otimes d} [N\otimes_A]^{n-1}E \longrightarrow [E\otimes_A]^{n-1}\Omega^1A \longrightarrow 0$$

this map has image $\mathcal{J} = \mathrm{Im}\,(I^{n-1}dI)$

$$0 \longrightarrow HH_{2n}A \longrightarrow [N\otimes_A]^n \xrightarrow{\partial = 1^{\otimes n-1}\otimes d} [N\otimes_A]^{n-1}E\otimes_A \longrightarrow [N\otimes_A]^{n-1}\Omega^1A \to 0$$

$$\therefore\ 0 \longrightarrow HH_{2n}A \longrightarrow [N\otimes_A]^n \xrightarrow{\ \partial\ } \mathcal{J} \longrightarrow 0$$

However we have $d =$ cyclic norm type sum of ∂. Thus in ⊛ above

$$\mathrm{Ker}(d) = HH_{2n}A \oplus (1-\sigma)[N\otimes_A]^n$$

But notice that the b maps $\mathcal{J}$ onto $(1-\sigma)[N\otimes_A]^n$

since $b(I^{n-1}dI) = [I^{n-1}, I]$.

So what seems to happen is that instead of the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & HH_{2n}A \arrow[r] & {[N\otimes_A]^n} \arrow[rr, "\partial"] \arrow[dr] & & {[N\otimes_A]^{n-1}E\otimes_A} \arrow[r] & {[N\otimes_A]^{n-1}\Omega^1A} \arrow[r] & 0 \\
\text{we have} & & (1-\sigma)[N\otimes_A]^n \arrow[d] & \overset{\partial}{\underset{b}{\xleftarrow{\ \sim\ }}} \arrow[ur] & I^{n-1}dI/d(I^n) & & \\
0 \arrow[r] & HH_{2n}A \arrow[r] & {[N\otimes_A]} \arrow[rr, "d=\partial\text{ norm}"] & & {[N\otimes_A]^{n-1}E\otimes_A} \arrow[r] & {[N\otimes_A]^{n-1}\Omega^1A} \arrow[r] & 0
\end{tikzcd}$$

Thus we should get the homology for $\operatorname{Ker}\{X^n \longrightarrow X^{n-1\prime}\}$

$HH_{2n}A$ $\qquad\qquad$ 0

which is also consistent with

$$\begin{tikzcd}
H^i(X^{n-1\prime}) : & \operatorname{Ker}\{HC_{2n-2} \xrightarrow{B} HH_{2n-1}\} & HC_{2n-1} \arrow[d, phantom, "\cup"] \\
H^i(X^n) : \arrow[u] & HC_{2n} \arrow[u, Rightarrow, "S"'] & \operatorname{Ker}\{HC_{2n-1} \xrightarrow{B} HH_{2n-1}\}
\end{tikzcd}$$

and a six term exact sequence

$$\begin{tikzcd}
HC_{2n}A \simeq H^0(X^n) \arrow[r] & H^0(X^{n-1\prime}) \arrow[r] & 0 \arrow[d] \\
HH_{2n}A \arrow[u, "(I)"'] & H^1(X^{n-1\prime}) \arrow[l, "(B)"'] \arrow[d, phantom, "\shortparallel"] & H^1(X^n) \arrow[l] \\
& HC_{2n-1}A &
\end{tikzcd}$$

Thus it seems to work quite nicely. The surjection $X^{n\prime} \longrightarrow X^n$ yields B, I maps through $HH_{2n+1}A$ and the surjection $X^n \longrightarrow X^{n-1\prime}$ yields the B, I map thru $HH_{2n}A$.

We have an interesting situation as follows. Instead of the complex

$$\longrightarrow N^{(n)} \xrightarrow{\partial} N^{(n-1)}\otimes_A E \otimes_A \xrightarrow{\beta} N^{(n-1)} \longrightarrow \cdots$$

which we know gives the Hochschild homology, we have a complex

$$\begin{tikzcd} \circledast & {} \arrow[r, "\beta"] & N^{(n)} \arrow[r, "\partial"] & N^{(n-1)}\otimes_A E \otimes_A \arrow[l, bend left, dashed] \arrow[r, "\beta"] & N^{(n-1)} \arrow[r] & {} \arrow[l, bend left, dashed] \end{tikzcd}$$

[redacted] together with a dotted arrow which is only defined on the image of ∂.

It's a puzzle what the appropriate viewpoint should be. See July 17, 1990 p.51ff for the exact sequence

$$\longrightarrow N^{(n)\sigma} \xrightarrow{\sigma} N^{(n-1)}\otimes_A E \otimes_A \Big/ \mathrm{Im}(1-\sigma)[N\otimes_A]^n \longrightarrow$$

which is a subquotient of $\circledast$ which gives the Hochschild homology.

This is reminiscent of what happens in Morse theory as one passes a critical point, or in flag manifold (Bruhat decomp.) as one crosses a wall.

January 5, 1991

Gauss-Manin. Let us consider a family of algebras $\{A_h, h \in \mathbb{C}\}$ of the form
$$A_h = \mathbb{C}_h \otimes_{\mathbb{C}[t]} \mathcal{A} \qquad \mathbb{C}_h = \mathbb{C}[t]/(t-h)$$
where $\mathcal{A}$ is an algebra over $\mathbb{C}[t]$. Example: The Weyl algebra family
$$\mathcal{A} = \underbrace{\mathbb{C}[t] \otimes \mathbb{C}\langle p, q\rangle}_{\mathcal{R}} / \mathcal{I}$$
where $\mathcal{I} = \mathcal{R}([p,q]-t)\mathcal{R}$.

Notice that for any commutative algebra S over $\mathbb{C}[t]$, i.e. equipped with a homomorphism $\mathbb{C}[t] \to S$ (or equivalently element of S) we have an S-algebra
$$A_S = S \otimes_{\mathbb{C}[t]} \mathcal{A}.$$
This allows us to treat the family $\{A_h\}$ infinitesimally.

~~Recall the idea behind the Gauss-Manin connection. Let's suppose for the sake of the discussion that we have a deformation of Banach algebras $\{A_h\}$~~

Let's discuss the intuitive picture of the Gauss-Manin connection. Consider the family $HP(A_h)$ of periodic homology groups. Then these should form a flat vector bundle over $\mathbb{C}$. Let's suppose for the sake of the discussion that $\{A_h\}$ is a nice family of Banach algebras and that a linear map $\rho: A \to A'$ of Banach algebras which is close to a homomorphism induces a map of periodic homology $HP(A) \longrightarrow HP(A')$ which doesn't change as ρ is deformed through linear maps close to homomorphisms.

Then in our family $\{A_h\}$ we should be find linear maps [redacted] $\rho_{h,h'} : A_h \to A_{h'}$ [redacted] which are close to homomorphisms for h, h' in some nbds of the diagonal. Then we have canonical [redacted]isomorphisms $HP(A_h) \simeq HP(A_{h'})$ for h, h' in some nbd of the diagonal. (These are ~~absolutely~~ canonical by the homotopy property. Also one has transitivity.)

(Side comments. 1) In reality one might only obtain a sheaf: An element of $HP(A_h)$ might determine an element of $HP(A_{h'})$ for h' near h, but the nbd might depend on the element.

2) Geometric picture of singular spaces: Any closed subset is a strong deformation retract of a nbd. This happens for simplicial complexes, where a subcomplex is an SDR of its open star.

3) Question: Is there an honest distinction between additive and multiplicative homotopy? Consider the two homotopies

$$\begin{tikzcd} R[x] \arrow[r, shift left, "x \mapsto x+t"] \arrow[r, shift right, "x \mapsto tx"'] & R[x] \end{tikzcd}$$

One feels the former is reversible and the latter is not.)

Let's return to our family $A_h = \mathbb{C}_h \otimes_{\mathbb{C}[t]} \mathcal{a}$. We need to be able to discuss the family $HP_i^{\bullet}(A_h)$ of periodic homology groups. ~~There is a [illegible] of HP cycles complex giving periodic homology as follows~~

What we would like is to have

$$HP_i(A_h) = \mathbb{C}_h \otimes_{\mathbb{C}[t]} M_i$$

for some $\mathbb{C}[t]$-module M_i. More generally for any comm. alg S over $\mathbb{C}[t]$ there should be periodic homology group $HP_i(A_S, S)$ and we would like to have

$$\textcircled{*} \qquad HP_i(A_S, S) = S \otimes_{\mathbb{C}[t]} M_i$$

whence $M_i = HP_i(\mathcal{A}, \mathbb{C}[t])$.

This hope is naive because there are $\varprojlim$ and Tor technicalities, which we will discuss later. For the moment let us assume $\textcircled{*}$ and consider what it means for there to be transitive isomorphisms

$$HP_i(A_h) \simeq HP_i(A_{h'})$$

for h, h' "close". Grothendieck has analyzed this and found the following.

Let S be a smooth commutative (finite type) algebra and M an S-module. Let $\mathcal{D} = \mathrm{Diff}(S)$ be the algebra of differential operators. The following data are equivalent.

1) $\mathcal{D}$-module structure on M (compat. with S-mod str.)

2) For any pair $S \overset{h}{\underset{h'}{\rightrightarrows}} T$ with $(h-h')(S)^N = 0$ one is given an isomorphism $T_{h'} \otimes_S M \simeq T_h \otimes_S M$, and these isomorphisms are transitive & natural.

3) An S-module map

$$M \longrightarrow \varprojlim \left\{ (S \otimes S / I_\Delta^n) \otimes_S M \right\} = J_\infty(M)$$

with comodule property relative to $\varprojlim (S \otimes S / I_\Delta^n) = J_\infty$ being a coalgebra in the category of S-bimodules

4) An integrable connection on M: $\quad M \xrightarrow{\nabla} \Omega^1_S \otimes_S M \xrightarrow{\nabla} \Omega^2_S \otimes_S M \to$

$\nabla^2 = 0$

(I once understood all of this in the [redacted] more general context of formal groupoids. It would be nice to [redacted] relate the basic inductive step in passing from order n to order $n+1$ and formal Poincaré lemma to the new ideas involving [redacted] polynomial homotopy. In particular, can one obtain the compatible isomorphisms $T_{h'} \otimes_S M \simeq T_h \otimes_S M$ using the "variety" of all homomorphisms $S \longrightarrow T$ which should be smooth? How does the inductive process

$$\begin{tikzcd}
 & & \dagger \\
J_{n+1} \arrow[r] \arrow[d] & \text{[redacted]} & T^* \otimes J_n \arrow[d] \\
J_n \arrow[r] & {} & T^* \otimes J_{n-1}
\end{tikzcd}$$

(not cart)

fit with polynomial homotopies of degree $\leq n$?)

So the family of transitive isomorphisms

$$HP_i(A_h) \simeq HP_i(A_{h'})$$

for h, h' "close" (infinitesimal interpretation) should be equivalent to an operator ∂_t on $HP_i(\mathcal{A}, \mathbb{C}[t]) = M_i$ consistent with ∂_t on $\mathbb{C}[t]$.

[redacted] Actually there's something stronger one can ask for than just the $\mathcal{D}$-module stronger on M_i. Namely we can ask [redacted] for enough flat elements of M_i at least formally.

Let's now recall the way periodic homology can be computed. Consider the Weyl algebra example, where $\mathcal{A} = \mathcal{R}/\mathcal{I}$, $\mathcal{R} = \mathbb{C}[t] \otimes R$ with R free. Here $A_h = R/I_h$ and [redacted] we know that $HP(A_h)$ is the homology of the complex

$$\varprojlim_{n} \left\{ R/I_h^{n+1} + [R, I_h^n] \rightleftarrows \Omega^1 R/I_h^n \Omega^1 R + [R, \Omega^1 R] \right\}$$

January 9, 1991

Recall the following from the proof of the Krull-Schmidt theorem. Suppose L is an operator on V (a module) such that $\mathrm{Ker}(L^n) = \mathrm{Ker}(L^{n+1})$ and $\mathrm{Im}(L^n) = \mathrm{Im}(L^{n+1})$. Then one has a direct sum decomposition

$$V = \mathrm{Ker}(L^n) \oplus \mathrm{Im}(L^n)$$

and L is the direct sum of L on $\mathrm{Ker}(L^n)$ which is nilpotent, and L on $\mathrm{Im}(L^n)$ which is bijective. (This is used to show when V is Artinean that V indecomposable $\Longrightarrow$ $\mathrm{End}(V)$ local, which in turn leads to Krull-Schmidt.) Thus we get B, G defined for such an L.

New viewpoint about Karoubi operator. Recall

$$\begin{tikzcd}
0 \arrow[r] & \bar{A}^{\otimes n} \arrow[r, "s"] \arrow[loop below, "\lambda_n"] & A \otimes \bar{A}^{\otimes n} \arrow[r] \arrow[loop below, "\kappa"] & \bar{A}^{\otimes n+1} \arrow[r] \arrow[loop below, "\lambda_{n+1}"] & 0
\end{tikzcd}$$

It follows that $\kappa^{n+1} - 1$ carries $A \otimes \bar{A}^{\otimes n}$ into $s\bar{A}^{\otimes n}$ which is killed by $\kappa^n - 1$. Thus

$$(\kappa^n - 1)(\kappa^{n+1} - 1) = 0$$

But the roots of the polynomial $(X^n - 1)(X^{n+1} - 1)$ are all simple except for $X = 1$ which is a double root. The point is that n and $n+1$ are relatively prime, so the n-th roots of unity ζ which are $\neq 1$ are not $(n+1)$th roots of unity. Thus ΩA is the direct sum of the eigenspaces of κ corresponding to roots of unity $\zeta \neq 1$ and the kernel of $(1-\kappa)^2$

January 11, 1991

I've run into writing difficulties with the new approach to the Karoubi operator.

Here's what I've done. 1) Consider K on Ω^n; it leaves the subspace $d\Omega^{n-1}$ invariant and is of order n on this subspace and of order $n+1$ on $\Omega^n/d\Omega^{n-1}$. Thus $(K^n-1)(K^{n+1}-1)=0$.

2) The roots of $(x^n-1)(x^{n+1}-1)$ are n-th + $(n+1)$-th roots of 1, those $\neq 1$ are simple and 1 is a double root. This gives generalized eigenspace decomposition

$$\Omega = \Omega_1 \oplus \Big(\bigoplus_{\zeta} \Omega_\zeta\Big) \qquad \begin{array}{l} \Omega_1 = \operatorname{Ker}(1-K)^2 \\ \Omega_\zeta = \operatorname{Ker}(\zeta - K) \end{array}$$

where ζ runs over roots of unity $\neq 1$. 3) Define P to be projection on Ω_1 wrt this decomposition. For $\zeta \neq 1$ $1-K = 1-\zeta$ on Ω_ζ is invertible. Thus define $G = 0$ on Ω_1, $(1-K)^{-1}$ on $\bigoplus_\zeta \Omega_\zeta = P^\perp\Omega$. 4) Ω_1, Ω_ζ stable under b,d so P, G commute with b,d.

5) On $P^\perp\Omega$ we have $1 = G(1-K) = bGd + dGb$ hence a "Hodge" decomposition: $\operatorname{Im} b = \operatorname{Ker} b$, $\operatorname{Im} d = \operatorname{Ker} d$,

$$P^\perp\Omega = \operatorname{Im} b \oplus \operatorname{Im} d$$

$$b : \operatorname{Im} d \xrightarrow{\sim} \operatorname{Im} b \qquad d : \operatorname{Im} b \xrightarrow{\sim} \operatorname{Im} d$$

bGd projects onto $\operatorname{Im} b$ with kernel $\operatorname{Im} d$
dGb ——— $\operatorname{Im} d$ ——— $\operatorname{Im} b$

The thing I need still is what P, G look like when restricted to $d\Omega^{n-1}$ where $K^n = 1$. What you have is two spaces with K operating and a map

$$\begin{tikzcd} \Omega^{n-1} \arrow[r, "d"] \arrow[loop below, "K"] & \Omega^n \arrow[loop below, "K"] \end{tikzcd}$$

And you have defined P, G on both Ω^{n-1}, Ω^n. You

Now consider the induced operator K on the image $d\Omega^{n-1}$. You would like to define P, G for it and to show compatible with P, G defined already on Ω^{n-1}, Ω^n. This should be clear from eigenvalue considerations.

But I think I want the general picture: P, G can be defined for any operator L such that $\operatorname{Ker} L^n = \operatorname{Ker} L^{n+1}$, $\operatorname{Im} L^n = \operatorname{Im} L^{n+1}$ for some n. It's not true that given such an L on V and a ~~subspace~~ $W \subset V$ stable under L that W, L_W is again of this type: Take L invertible on V and W such that $LW \subsetneq W$. But consider L_V on V and L_W on W of this type and a map $f: V \to W$ $\ni$ $L_W f = f L_V$. Then we have

$$\begin{tikzcd} V \arrow[d, "f"] & = & \operatorname{Ker} L_V^n \arrow[d, dashed, "f"] & \oplus & \operatorname{Im} L_V^n \arrow[d, dashed, "f"] \\ W & = & \operatorname{Ker} L_W^n & \oplus & \operatorname{Im} L_W^n \end{tikzcd}$$

Thus
$$fV = \underbrace{f(\operatorname{Ker} L_V^n)}_{L_W \text{ nilp}} \oplus \underbrace{f \operatorname{Im}(L_V^n)}_{L_W \text{ invertible since}}$$

$$\begin{tikzcd} \operatorname{Im} L_V^n \arrow[r, "L_V", "\simeq"'] \arrow[d, "f"] & \operatorname{Im} L_V^n \arrow[d, "f"] \\ \operatorname{Im} L_W^n \arrow[r, "L_W", "\simeq"'] & \operatorname{Im} L_W^n \end{tikzcd}$$

So L has the correct property on $\operatorname{Ker} f$, $\operatorname{Im} f$, $\operatorname{Coker} f$.

We are looking at $k[T]$-modules which are the direct sum of a $k[T, T^{-1}]$ module and a $k[T]/(T^n)$ module for some n. One has cut out 0 from the line and made it an isolated point. (You might review *essential points* in the theory of topoi.)

But for the purpose of writing and understanding, it should be possible to show that the condition

$$V = \mathrm{Ker}\{(1-K)^2 \text{ on } V\} \oplus (1-K)^2 V$$

for $V = \Omega$ implies the same thing for $d\Omega$, $b\Omega$.

January 18, 1991

Connes tangent groupoid. Given a manifold M form $\mathbb{R}\times M\times M$ and blow up $0\times\Delta M$.

In order to understand this let's first understand what we get when we blow up $(0,0)$ in $\mathbb{R}\times M$ where M is a vector space. The blow up $\widetilde{\mathbb{R}\times M}$ is the set of triples (h,x,ℓ) where $(h,x)\in\mathbb{R}\times M$ and ℓ is a line through the origin in $\mathbb{R}\times M$ containing (h,x). For $h\neq 0$, the line ℓ is determined by (h,x). For $h=0$ we have pairs (x,ℓ) with $(0,x)\in\ell$, ℓ a line in $\mathbb{R}\times M$. Now lines in $\mathbb{R}\times M$ project non-trivially onto $\mathbb{R}$ or not. This gives

$$\mathbb{P}(\mathbb{R}\times M) = M\cup\mathbb{P}M$$

where we identify a line ℓ projecting non-trivially onto $\mathbb{R}$ with the unique point $x\in M \ni (1,x)\in\ell$.

Another way to think of $\widetilde{\mathbb{R}\times M}$ is that we remove $(0,0)$ and put in the limiting lines as $(h,x)\longrightarrow(0,0)$, i.e. we put in $\mathbb{P}(\mathbb{R}\times M)$.

We want to consider smooth functions on $\widetilde{\mathbb{R}\times M}$ which vanish on $\mathbb{P}M$. These will be functions on M for $h\neq 0$ and functions on the tangent space to M at 0 for $h=0$.

Examples of the functions to consider. Take $M=\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$. Let $\varphi\in\mathcal{S}(\mathbb{R})$. Consider

$$f(h,x) = \langle x|\varphi(p)|0\rangle \qquad \text{where } p=\frac{h}{i}\partial_x$$

$$= \sum_{k\in 2\pi\mathbb{Z}}\langle x|\varphi(p)e^{ik} = \sum_{k\in 2\pi\mathbb{Z}}\varphi(hk)e^{ikx}$$

[redacted]

$$= \sum_{p\in 2\pi h\mathbb{Z}} \varphi(p)\, e^{-ip\frac{x}{h}} \sim \frac{1}{h}\hat{\varphi}\left(\frac{x}{h}\right)$$

(This has to multiplied by h). Thus a function of the form

$$\sum_{n\in\mathbb{Z}} \hat{\varphi}\left(\frac{x}{h}+\frac{n}{h}\right) \qquad x\in\mathbb{R}/\mathbb{Z}$$

[redacted] where $\hat{\varphi}(p)\in\mathcal{S}(\mathbb{R})$ is contemplated.

Let us look at the blowup process algebraically. Ultimately we are interested in doing this deformation to the normal bundle in the case of $\Delta M\subset M\times M$, in which case we get a noncommutative algebra deformation of the Schwartz functions on T^*M, or equivalently the [redacted] Schwartz functions on TM under convolution. Here **we** would like to link up with differential operators $\mathrm{Diff}(M)$ and jets J^∞ and the **theory** of formal groupoids.

First we want to understand what happens at a point, say $0\in M$. Then we have the [redacted] "jets" at the point

$$\varprojlim \mathcal{O}/\mathfrak{m}^n \qquad\qquad \mathcal{O}=\mathcal{O}_{M,0}$$

$\mathfrak{m}$ = maximal ideal

and the differential operators

$$\varinjlim (\mathcal{O}/\mathfrak{m}^n)^*$$

The problem is to link these with what arises above when we blowup, i.e. take the deformation to the tangent space. Also I should be able to link this with the idea of ΨDO's as sums of homogeneous functions approximately.

What exactly is the blowup of $(0,0) \subset \mathbb{R}\times M$? Algebraic we have $\mathcal{O}[T]$ and the ideal $T\mathcal{O}[T] + m[T]$. Blowing up means we take

$$\mathrm{Proj}\left\{ \bigoplus_{n\geq 0} m^n + m^{n-1}T + \cdots + \mathcal{O}T^n + \mathcal{O}T^{n+1} + \cdots \right\}$$

Where $T \neq 0$ is the localization wrt T, which is just $\mathcal{O}[T,T^{-1}]$. What do we have after setting $T=0$?

$$\mathrm{Proj}\left\{ \bigoplus_{n\geq 0} m^n + (m^{n-1}/m^n)T + (m^{n-2}/m^{n-1})T^2 + \cdots \right\}$$

If we localize wrt T we get it seems $\mathrm{Spec}\left\{ \bigoplus_n m^{n-1}/m^n \right\}$ = tangent space to $0 \in M$.
And if we look at the rest

$$\mathrm{Proj} \bigoplus_n m^n$$

we get the blow up $\tilde{M}$ of $0 \in M$. There is some amazing geometry here!

In the blowup $\widetilde{\mathbb{R}\times M}$ the point $(0,0)$ is replaced by the space of tangent lines through this point. Lines tangent to $0\times M$ form a closed submanifold which is the singular fibre of $\tilde{M}$. We look at smooth functions on $\widetilde{\mathbb{R}\times M}$ which vanish to all orders on $\tilde{M}$.

January 20, 1991

Morita aspect of cyclic theory.

To an algebra A associate the additive category $\mathcal{P}(A)$ of finitely generated projective right A-modules. It's a Karoubienne additive category with a distinguished generator.

Given two algebras A, B consider the additive functors from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. Such a functor is canonically isomorphic to

$$P \longmapsto P \otimes_A E$$

where E is **the** representation of A in $\mathcal{P}(B)$ which is the value of the functor on A. The category of additive functors $\mathcal{P}(A) \rightarrow \mathcal{P}(B)$ is equivalent to the category of representations of A in $\mathcal{P}(B)$.

Suppose given such a representation P. We can choose a direct embedding $P \underset{i}{\overset{j}{\rightleftarrows}} B^n$ and thus obtain a nonunital homomorphism

$$A \longrightarrow \mathrm{End}_B(P) = e(M_nB)e \subset M_nB \qquad \begin{matrix} ji=1 \\ ij=e \end{matrix}$$

since HC and HH are functors on nonunital algebras equipped with trace maps for matrices, we get an induced map $HC(A) \longrightarrow HC(B)$ and also for HH. I think these maps are independent of the choice of direct embedding.

In any case there should be a sense in which the choice of direct embedding is unique up to "inner derivation type" homotopy.

Let's next consider the complexes $X(A)$ and $\Omega(A)$. I believe $X(A)$ is functorial for non unital homomorphisms, in fact, it can be defined on the category of nonunital algebras. ~~But there are~~ Also there are problems with $\Omega(A)$ perhaps, which is why Connes uses

$$\operatorname{Ker}\{\Omega(\tilde{A}) \longrightarrow \Omega(\mathbb{C})\}.$$

Recall

$$H_n^{DR}(A) = \operatorname{Ker}\{\overline{HC}_n(A) \xrightarrow{B} HH_{n+1}A\} \qquad n \geq 1$$

$$H_n^{DR}(\tilde{A}) = \operatorname{Ker}\{HC_n(A) \xrightarrow{B} HH_{n+1}A\} \qquad n \geq 1$$

The former is not a functor for nonunital homoms.

$u: A \longrightarrow B$

$$\begin{tikzcd}
HC_{2n}(\mathbb{C}) \arrow[d, "\text{induced by } 1 \mapsto 1"'] \arrow[dr, bend left, "\text{induced by } 1 \mapsto e"] & \\
HC_{2n}(A) \arrow[r] \arrow[d] & HC_{2n}(B) \arrow[d] \\
\overline{HC}_{2n}(A) \arrow[d] & \overline{HC}_{2n}(B) \arrow[d] \\
0 & 0
\end{tikzcd}$$

Check: that $\Omega^1(\tilde{A})_\natural \xrightarrow{\sim} \Omega^1 A_\natural$. Obviously surjective. A linear functional f on $\Omega^1(\tilde{A})_\natural$ is pair (ψ, φ) where $\psi \in (A^{\otimes 2})^*$, $\varphi \in A^*$ such that

$$b\psi = 0 \qquad \text{and} \qquad (1-\lambda)\psi = b'\varphi$$

Note ψ is normalized: $\psi(x, e) = 0$, $e = 1_A$. Then $\varphi(xy) = \psi(x,y) + \psi(y,x) \implies \varphi(x) = \varphi(xe) = \psi(e, x)$, so f comes from ψ which is a linear fn. on $\Omega^1 A_\natural$.

Thus we have for A unital the formula

$$X(A) = \operatorname{Ker}\{X(\tilde{A}) \longrightarrow X(\mathbb{C})\}$$

or better

$$X(\tilde{A}) \xrightarrow{\sim} X(A) \oplus X(\mathbb{C})$$

More generally we have

$$\boxed{X(A \times B) \xrightarrow{\sim} X(A) \oplus X(B)}$$

Proof. Let T be a trace on $\Omega^1(A \oplus B)$. $\Omega^1(A\oplus B)$ is spanned by elements of the form $a_1 da_2$, $b_1 db_2$, adb, bda. To show $T(adb) = 0$ and similarly $T(bda) = 0$. But

$$T(adb) = T(a 1_A db) = T(a\{d(1_A b) - d1_A b\})$$

$$= - T(\underset{\substack{\| \\ 0}}{\underline{ba}}\, d1_A) = 0$$

It now follows that we can define for a nonunital algebra $\mathfrak{a}$

$$X(\mathfrak{a}) = \operatorname{Ker}\{X(\tilde{\mathfrak{a}}) \longrightarrow X(\mathbb{C})\}$$

An important problem seems to be to relate ΩA with $\operatorname{Ker}\{\Omega(\tilde{A}) \to \Omega\mathbb{C}\}$. As mixed complexes they are equivalent. We have a map

$$\Omega(\tilde{A}) \longrightarrow \Omega(A) \times \Omega\mathbb{C}$$

which is a quis with respect to b. It's also compatible with b, d, κ, etc. However the induced map on the d homology of the supercommutator quotients spaces is not an isomorphism.

Let $S = \mathbb{C}\times\mathbb{C} \subset A\times B$. Then we have

$$\Omega(A\times B ; \mathbb{C}\times\mathbb{C}) = \Omega A \times \Omega B$$

In general if $S \longrightarrow R$ is a homomorphism with S commutative and the image of S central in R, then $\Omega(R;S)$ should be the universal DGA generated by R in the category of algebras over the commutative ring S.

To be more concrete note that if $e = (1,0) \in S = \mathbb{C}\times\mathbb{C}$, then in $\Omega^1(A\times B; S)$ we have

$$a\,db = ae\,db = ad(eb) = 0$$

etc.

Conclusion: We have the obvious map

$$\Omega(A\times B) \longrightarrow \Omega(A\times B ; \mathbb{C}\times\mathbb{C})\otimes_{\mathbb{C}\times\mathbb{C}} = \Omega A \times \Omega B$$

compatible with all the operators d, b, etc. We also know it is a qis wrt b (Kassel or Loday: Loday's proof most interesting: it uses inner derivations acting trivially).

However it would be nice to understand why

$$\Omega(R) \longrightarrow \Omega(R;S)\otimes_S$$

is a qis wrt b when S is separable. This should be true because we know $R\otimes_S R$ is a direct factor of $R \otimes R$ and so one can get at Hochschild using the b' resolution made of $T^n_S(R)$. Also I think Lars Kadison has a cyclic object proof along these lines.

Proposition: If A, B are quasi-free, so is $A \times B$.

Proof. Consider a square zero extension

$$0 \longrightarrow M \longrightarrow C \longrightarrow A \times B \longrightarrow 0$$

Lift the idempotent $(1,0)$ to an idempotent $e \in C$. Consider conjugation by $F = 2e - 1$; this is an action of $\mathbb{Z}/2$, so taking fixpts. is exact. This shows we have an extension

$$0 \longrightarrow M^e \longrightarrow C^e \longrightarrow A \times B \longrightarrow 0$$

where C^e denotes the centralizer of e. Thus we can suppose e central in C, whence we have a direct sum of square zero extensions

$$0 \longrightarrow eMe \longrightarrow eCe \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow e^\perp M e^\perp \longrightarrow e^\perp C e^\perp \longrightarrow B \longrightarrow 0$$

Each of these has lifting homos. so one wins.

Better: You lift $(1,0)$ to $e \in C$, then

eCe is a square zero extension of A
$e^\perp C e^\perp$ ________________________ B

so there exist lifting homos. $A \to eCe$, $B \to e^\perp C e^\perp$, whence $A \oplus B \longrightarrow eCe \oplus e^\perp C e^\perp \subset C$.

Alternative proof based on

$$\Omega^1(A \oplus B) = \begin{pmatrix} \Omega^1 A & A \otimes B \\ B \otimes A & \Omega^1 B \end{pmatrix}$$

where $A \otimes B$ maps to $A\, de\, B$ and $B \otimes A$ to $B\, de^{\perp} A$. More precisely given $m \in M$, where M is a bimodule, where $m = em = me^{\perp}$ we have a derivation $D : A \oplus B \longrightarrow M$ given by $D(a) = am \quad D(b) = -mb$.

(Check: Suppose $M = eMe^{\perp}$ [redacted], then

$$D(a) = D(ae) = aDe + \cancel{(Da)e}$$

$$D(b) = D(e^{\perp}b) = D(e^{\perp})b + \cancel{e^{\perp}Db}$$

$$= -(De)b .)$$

[redacted] Now $A \otimes B$, $B \otimes A$ are projective bimodules over $A \oplus B$ in general, so if $\Omega^1 A$ and $\Omega^1 B$ are projective bimodules (over A, B respectively, whence also over $A \oplus B$) then so is $\Omega^1(A \oplus B)$.

January 21, 1990

Morita aspect of cyclic theory. Recall

1. An additive functor $P(A) \longrightarrow P(B)$, where $P(A)$ is the category of finite projective right A-modules, is equivalent to a repn. of A in $P(B)$, in other words an object E of $P(B)$ equipped with a homom. $A \to \mathrm{End}_B(E)$.

[redacted] 2. [redacted] [redacted] Given such an E one can construct a non unital homomorphism

$$A \longrightarrow \mathrm{End}_B(E) \subset M_n(B) \qquad E \underset{i}{\overset{s}{\leftrightarrows}} B^n$$

by choosing a direct embedding. The choice of such a direct embedding can be regarded as innocuous. For example taking $A = B = \mathbb{C}$, [redacted] then the space of direct embeddings $E \leftrightarrows \mathbb{C}^n$ [redacted] is heq the space of embeddings $E \hookrightarrow \mathbb{C}^n$, say isometric wrt an inner product on E, which is a Stiffel manifold, and so its connectivity increases with n.

3. The space of [redacted] non-unital homomorphisms $\mathbb{C} \longrightarrow M_n\mathbb{C}$ is the space of direct summands of $\mathbb{C}^n$, in other words, the direct sum Grassmannian.

[redacted] Thus I can think of a fd vector space V as giving a non unital homomorphism

$$\mathbb{C} \longrightarrow M_n\mathbb{C}$$

which is unique up to homotopy, however [redacted] when we look at families of vector spaces then we [redacted] get interesting topology.

January 23, 1991

I want to understand properly why the mixed complex $(\Omega A, b, B)$ gives the cyclic theory of A. The cyclic theory is defined more generally for nonunital algebras $\mathfrak{a}$ using the mixed complex $(\tilde{\Omega}\mathfrak{a}, b, B)$, where $\tilde{\Omega}\mathfrak{a}$ is Connes DG algebra of forms, that is, the nonunital DG algebra generated by $\mathfrak{a}$. We have

$$\tilde{\Omega}\mathfrak{a} = \operatorname{Ker}\{\Omega(\tilde{\mathfrak{a}}) \longrightarrow \overbrace{\Omega\mathbb{C}}^{\mathbb{C}}\}$$

1. We have canonical maps for A unital:

$$\Omega(\tilde{A}) = \Omega(A\times\mathbb{C}) \longrightarrow \Omega A \times \Omega\mathbb{C}$$

$$\tilde{\Omega}(A) \longrightarrow \Omega A$$

and the result we wish to understand says these maps are quasi-isomorphisms of complexes with the differential b, and also that the induced maps

$$\mathcal{B}_{\geq 0}(\tilde{\Omega}A) \longrightarrow \mathcal{B}_{\geq 0}(\Omega A)$$

$$\hat{\mathcal{B}}(\tilde{\Omega}A) \longrightarrow \hat{\mathcal{B}}(\Omega A)$$

$$(\widehat{\tilde{\Omega}A}, b+B) \longrightarrow (\widehat{\Omega A}, b+B)$$

are qis.

2. There seems to be a principle that if F is a functor on unital algebras such that

$$F(A\times\mathbb{C}) \xrightarrow{\sim} F(A)\times F(\mathbb{C})$$

then we can extend F to nonunital algebras by setting $F(\mathfrak{a}) = \operatorname{Ker}\{F(\tilde{\mathfrak{a}}) \longrightarrow F(\mathbb{C})\}$.

Notice that $F(\tilde{a}) \to F(\mathbb{C})$ is onto, in fact has a ~~canonical~~ section so that canonically

$$F(\tilde{a}) = F(a) \times F(\mathbb{C}).$$

3. *Example*: $A = \mathbb{C}$. $\tilde{C} = \mathbb{C}[e]$, $e^2 = e$

$$b(de^{2n}) = -[de^{2n-1}, e] = (-(1-e)+e)de^{2n-1} = (2e-1)de^{2n-1}$$

$$b((2e-1)de^{2n}) = -[(2e-1)de^{2n-1}, e] = \Big(+(2e+1)(1-e)+e(2e-1)\Big)de^{2n-1}$$
$$\qquad\qquad 1-e+2e-e$$
$$= [illegible]\, de^{2n-1}$$

Thus $\tilde{\Omega}\mathbb{C} = \operatorname{Ker}\{\Omega\mathbb{C}[e] \xrightarrow{e \mapsto 0} \Omega\mathbb{C}\}$ has basis

$$\begin{tikzcd}
 & de & de^2 \arrow[dl] & de^3 & de^4 \arrow[dl] \\
e & & & & \\
 & (2e-1)de & (2e-1)de^2 \arrow[ul] & (2e-1)de^3 & (2e-1)de^4 \arrow[ul]
\end{tikzcd}$$

where the arrows are b. Note also that

$P\tilde{\Omega}\mathbb{C}$: e $\quad de \leftarrow (2e-1)de^2 \quad de^3 \leftarrow (2e-1)de^4$

$P^{\perp}\tilde{\Omega}\mathbb{C}$: 0 $\quad (2e-1)de \leftarrow de^2 \quad (2e-1)de^3 \leftarrow de^4$

and that $\tilde{\Omega}\mathbb{C}$ is the sum of the $K = \pm 1$ eigenspaces. This reflects the fact that $(\bar{A}^{\otimes n})^{\lambda}$ is zero for n even and the fact that $(1-\lambda^2)\bar{A}^{\otimes 2m} = 0$ $m \geq 1$, so that $\tilde{\Omega}\mathbb{C} = P_2\tilde{\Omega}\mathbb{C}$.

Here it is obvious that

$$\tilde{\Omega}\mathbb{C} \longrightarrow \Omega\mathbb{C} = \mathbb{C}[0]$$

is a quis with respect to b. Notice however that this is not true for $b+B$. In fact because b is isomorphism $\tilde{\Omega}^{2n}\mathbb{C} \longrightarrow \tilde{\Omega}^{2n-1}\mathbb{C}$, $n \geq 1$ there is a unique way to construct a $b+B$ cycle starting with e and this gives an element of $\hat{\tilde{\Omega}}\mathbb{C}$ as follows: $(b+B)\left(e+\sum_{n\geq 1} c_{2n}(2e-1)de^{2n}\right)=0$

$$c_{2n-2}\underbrace{B\left((2e-1)de^{2n-2}\right)}_{2(2n-1)\,de^{2n-1}} + c_{2n}\,de^{2n-1} = 0$$

$$\therefore \quad c_{2n} = -2(2n-1)c_{2n-2}$$

$$c_{2n} = (-1)^n 2^n (2n-1)!!$$

Note the necessity of using $\hat{\tilde{\Omega}}\mathbb{C}$. In general we know that $\tilde{\Omega}\mathcal{A}$ is acyclic for $b+B$.

4. Proof from LQ that $(\tilde{\Omega}A,b)\rightarrow(\Omega A,b)$ is a quis. We identify $\tilde{\Omega}\mathcal{A}$ in general with the mapping cone of $1-\lambda$ going from the b-complex to the b' complex of $\mathcal{A}$. In the case of a unital algebra A, the b' complex is contractible, and the b-complex is homotopy equivalent to the normalized b-complex ΩA via the simplicial normalization thm. ~~Thus $(\tilde{\Omega}A,b)$ is the direct sum of $(\Omega A,b)$ and the mapping cone of a map between contractible complexes. More precisely the kernel of $\tilde{\Omega}A\rightarrow\Omega A$ is the cone on $1-\lambda$~~

The normalization thm. writes the unnormalized Hochschild complex of A as the direct sum of the normalized one and the degenerate subcomplex which is contractible. We thus get a map $i:\Omega A\rightarrow\tilde{\Omega}A$ compatible with b which is section of the canonical map the other way. The cokernel of i is the mapping cone on $1-\lambda$ followed by projection onto the degenerate subcomplex.

5. <u>Question:</u> Are there explicit homotopy inverses for the maps

$$(\tilde\Omega A, b) \longrightarrow (\Omega A, b)$$

$$\left.\begin{array}{l} \mathcal{B}_{\geq 0}(\tilde\Omega A) \longrightarrow \mathcal{B}_{\geq 0}(\Omega A) \\ \widehat{\tilde\Omega A} \longrightarrow \widehat{\Omega A} \end{array}\right\} \text{ wrt } b+B.$$

Two approaches:

i) Homological perturbation theory ~~starting~~ from an explicit fibre contraction of $(\tilde\Omega A, b)$ over $(\Omega A, b)$.

ii) Using $R\tilde A \longrightarrow RA \times \overbrace{R\mathbb{C}}^{\mathbb{C}}$ and the fact that $RA \times \mathbb{C}$ is quasi-free, hence there is a lifting are suitably completing $R\tilde A$.

The second seems the most promising.

6. Recall that for $A = \mathbb{C}$ we have $\kappa^2 = 1$ on $\Omega\tilde A = \Omega(\mathbb{C}[e])$. Thus we have

$$X(R\mathbb{C}[e]) \cong \Omega(\mathbb{C}[e]) \quad \text{with diff'l } b+B.$$

$$\hat X(R\mathbb{C}[e], I\mathbb{C}[e]) \cong \hat\Omega(\mathbb{C}[e]) \quad \underline{\hspace{4cm}}\,.$$

7. Here is a proof that <u>periodic homology is additive with respect to direct sums.</u> Given A, B we have the quasi-free cover $RA \times RB$ of $A \times B$, so using it to compute periodic homology. Thus $HP_\bullet(A\times B)$ is the homology of

$$\varprojlim_n X(RA \times RB / IA^n \times IB^n) = \varprojlim_n X(RA/IA^n) \times X(RB/IB^n)$$

$$= \hat X(RA) \times \hat X(RB)$$

8. Addition to 5: iii) Kassel's notion of strongly homotopic maps of mixed complexes allows one to deal ~~[illegible]~~ to some extent with maps of complexes computing HH which are compatible with B only up to homotopy. One might hope to apply this to ~~[illegible]~~ construct a suitable homotopy inverse for $\tilde{\Omega}A \longrightarrow \Omega A$.

January 24, 1991 (David is 27)

Consider $R(A \times B) \longrightarrow RA \times RB$. Observe

$$\mathrm{Hom}_{alg}\big(R(A\times B), \Lambda\big) = \Big\{(\rho', \rho'') \in \mathrm{Hom}_{\mathbb{C}}(A,\Lambda)\times \mathrm{Hom}_{\mathbb{C}}(B,\Lambda) \ \Big|\ \rho'(1_A) + \rho''(1_B) = 1_\Lambda\Big\}$$

$$\mathrm{Hom}_{alg}\big(R(\mathbb{C}\times\mathbb{C}), \Lambda\big) = \{x \in \Lambda\}$$

The embedding $R(\mathbb{C}\times\mathbb{C}) \hookrightarrow R(A\times B)$ corresponds to $(\rho', \rho'') \longmapsto x = \rho'(1_A) = 1_\Lambda - \rho''(1_B)$

In the quotient $RA \times RB$ we have

$$\circledast \qquad \rho'(a)\rho''(b) = 0 = \rho''(b)\rho'(a)$$

Conversely given $R(A\times B) \longrightarrow \Lambda$ described by $\rho' : A \to \Lambda$, $\rho'' : B \longrightarrow \Lambda$ $\ni$ $\rho'(1) + \rho''(1) = 1$, if $\circledast$ holds we have

$$\rho'(a) = \rho'(a)\big(\rho'(1) + \rho''(1)\big) = \rho'(a)\rho'(1)$$

$$\rho'(a) = \big(\rho'(1) + \rho''(1)\big)\rho'(a) = \rho'(1)\rho''(a)$$

showing that $e = \rho'(1)$ is an idempotent and $\rho'(A) \subset e\Lambda e$. Similarly $\rho''(B) \subset e^\perp \Lambda e^\perp$, so the homomorphism $R(A\times B) \longrightarrow \Lambda$ induces a homom. $RA \times RB \longrightarrow \Lambda$.

Let us now take $B = \mathbb{C}$ in which case we are looking at $R(\tilde{A}) \longrightarrow \widetilde{RA}$. A homom. $u : R(\tilde{A}) \longrightarrow \Lambda$ is the same as a linear map $\rho' : A \longrightarrow \Lambda$. A homom. $v : \widetilde{RA} \longrightarrow \Lambda$ is the same as an idempotent $e \in \Lambda$ and a linear map $\sigma : A \longrightarrow e\Lambda e$ such that $\sigma(1) = e$. Given u we would like to construct a v under suitable conditions. Let $x = \rho'(1) \in \Lambda$. We will assume the spectrum of x is $\{0, 1\}$ in the sense that $(x(1-x))^n = 0$ for some n. In this case there is is a unique idempotent e which is a polynomial in x such that $\mathrm{Im}(e)$ is the generalized null space of x and $\mathrm{Im}(e^\perp)$ is the generalized eigenspace for x and the eigenvalue 1, for any $\mathbb{C}[x]$-module. Thus for any Λ-module M we have $M = eM \oplus e^\perp M$ with $x^n = 0$ on eM and $(1-x)^n = 0$ on $e^\perp M$.

Next we want $\sigma : A \longrightarrow e\Lambda e$ such that $\sigma(1) = e$. All we have apparently is $\rho' : A \to \Lambda$. which is arbitrary subject to the fact that $\rho'(1) = x$.

In order to obtain σ, the only way I can see to proceed is to split A: $A = \mathbb{C} \oplus \bar{A}$ & define $\sigma(1) = e$ and $\sigma(\alpha) = e\rho'(\alpha)e$ for $\alpha \in \bar{A}$.

January 26, 1991

Recall the exact sequence with splitting associated to $R = T(V)$:

$$\begin{tikzcd} 0 \arrow[r] & \Omega^2 R_\natural \arrow[r, shift right, "-b"'] \arrow[d, phantom, "\parallel"] & \Omega^1 R \arrow[l, shift right, "-k"'] \arrow[r, shift right, "\natural"'] \arrow[d, phantom, "\parallel"] & \Omega^1 R_\natural \arrow[l, shift right, "h"'] \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\ & R \otimes V \otimes R \otimes V & R \otimes V \otimes R & R \otimes V & \end{tikzcd}$$

$$\begin{tikzcd} & a_1 dv a_2 \arrow[dl, mapsto] \arrow[r, mapsto] & a_2 a_1 dv \arrow[dl, mapsto] \\ a_1 dv da_2 \arrow[dr, mapsto] & a_2 a_1 dv & \\ & a_1 dv a_2 - a_2 a_1 dv & \end{tikzcd}$$

This leads to the following ~~[illegible]~~ null homotopy of L_D on $X(R)$ with $Dv = v$ $\forall v \in V$:

$$\begin{tikzcd} R \arrow[r, shift right, "d"'] & \Omega^1 R_\natural \arrow[l, shift right, "I_D = h_1"'] \arrow[r, shift right, "b"'] & R \arrow[l, shift right, "I_D = h_0"'] \end{tikzcd}$$

$$h_1 : \quad \Omega^1 R_\natural \xrightarrow{h} \Omega^1 R \xrightarrow{\iota_D} R$$

$$a\,dv \longmapsto a\,dv \longmapsto a\,Dv = av$$

is the identity from $V^{\otimes n} = V^{\otimes(n-1)} dV \longrightarrow V^{\otimes n}$

$$h_0 : \quad R \xrightarrow{d} \Omega^1 R \xrightarrow{-k} \Omega^2 R_\natural \xrightarrow{\iota_D} \Omega^1 R_\natural$$

$$v_1 \cdots v_n \overset{d}{\longmapsto} \sum_{i=1}^{n} v_1 \cdots v_{i-1}\, dv_i\, v_{i+1} \cdots v_n$$

$$\overset{-k}{\longmapsto} \sum v_1 \cdots v_{i-1}\, dv_i\, d(v_{i+1} \cdots v_n)$$

$$\overset{\iota_D}{\longmapsto} \sum_{i=1}^{n} v_1 \cdots v_i\, d(v_{i+1} \cdots v_n)$$

$$= \sum_{j=2}^{n} (j-1) v_1 \cdots dv_j \cdots v_n$$

$$= \sum_{j=2}^{n} (j-1) v_{j+1} \cdots v_n v_1 \cdots v_{j-1}\, dv_j$$

This is the map

$$\sum_{j=2}^{n} (j-1)\sigma^{-j} = \sum_{i=0}^{n-2} (n-1-i)\sigma^{i}$$

from $V^{\otimes n}$ to $V^{\otimes n-1} dV = V^{\otimes n}$

Thus we have

$$\begin{tikzcd} V^{\otimes n} \arrow[r, shift right, "d = N"'] & V^{\otimes n} \arrow[l, shift right, "h_1 = 1"'] \arrow[r, shift right, "1-\sigma"'] & V^{\otimes n} \arrow[l, shift right, "h_0 = \sum_{i=0}^{n-1}(n-1-i)\sigma^i"'] \end{tikzcd}$$

and indeed $[d, h] = n$. But of course we don't have $I_D^2 = 0$.

The natural question is whether one can arrange things to define I_D satisfying $I_D^2 = 0$. Return to the first order variation map

$$X(A) \xrightarrow{\delta} X(A \oplus \Omega^1 A)_{(1)}$$ = degree 1 part relative to $\mathbb{N}$-grading on $A \oplus \Omega^1 A$.

$$\begin{tikzcd} \arrow[r] & A \arrow[r, "d"] \arrow[d, "\delta"] & \Omega^1 A_\natural \arrow[r, "b"] \arrow[d, "\delta = \binom{1}{B}"] \arrow[dl, dashed, "h_1"'] & A \arrow[r] \arrow[d, "\delta"] \arrow[dl, dashed, "(h_0', h_0'')"'] & {} \\ \arrow[r] & \Omega^1 A \arrow[r, "\binom{\natural}{0}"'] & \Omega^1 A_\natural \oplus \Omega^2 A_\natural \arrow[r, "(0 \; -b)"'] & \Omega^1 A \arrow[r] & {} \end{tikzcd}$$

Here we describe $\Gamma = \Omega^1(A \oplus \Omega^1 A)_{\natural (1)}$ by

$$\Omega^1 A_\natural \oplus \Omega^2 A_\natural \xrightarrow{\sim} \Gamma$$

$$(a_0, a_1) \longmapsto d(a_0 \delta a_1)$$

$$(a_0, a_1, a_2) \longmapsto a_0 \delta a_1 \, da_2$$

We have

$$\delta(a_0 da_1) = \delta a_0 da_1 + a_0 d\delta a_1$$
$$d(a_0 \delta a_1) = da_0 \delta a_1 + a_0 d\delta a_1$$

$$\therefore \delta(a_0 da_1) = d(a_0\delta a_1) + \underbrace{\delta a_0 da_1 - \delta a_1 da_0}$$

corresponds to $B(a_0 da_1)$ under
$(a_0, a_1, a_2) \longrightarrow a_0 da_1 da_2$

Remark to be recorded is that in the case $HH_2(A) = 0$ (whence $b: \Omega^2 A_\natural \hookrightarrow \Omega^1 A$) we obtain a null-homotopy for δ from maps h_1, h_0' such that

$$\begin{tikzcd}
& \Omega^1 A_\natural \arrow[r, "b"] \arrow[dl, "h_1"'] & A \arrow[dl, "h_0'"] \\
\Omega^1 A \arrow[r, "\natural"'] & \Omega A^1_\natural &
\end{tikzcd}$$

$$1 = \natural h + h_0' b$$

Then $h_0'' : A \longrightarrow \Omega^2 A_\natural$ can be filled in uniquely. This is more general than what we have been looking at, which is the case where $\natural h_1 = 1$ on $\Omega^1 A_\natural$.

Let's return to $R = T(V)$

$$\begin{tikzcd}
R\otimes V\otimes R\otimes V \arrow[r, shift right, "-b"'] & R\otimes V\otimes R \arrow[l, shift right, "-k"'] \arrow[r, shift right, "\natural"'] & R\otimes V \arrow[l, shift right, "h"']
\end{tikzcd}$$

and lets modify h, k to h', k'. The idea is to have

$$h'(v_1 \dots v_{n-1} dv_n) = \frac{1}{n}\sum_{j=1}^{n} v_j \cdots v_{n-1} dv_n v_1 \cdots v_{j-1}$$

What is the corresponding $-k'$? Take

$$\alpha = v_i \cdots v_{n-1} dv_n v_1 \cdots v_{i-1} \quad \in V^{\otimes n-i} \otimes dV \otimes V^{\otimes i-1}$$

$(-k')\alpha = (-k)(\alpha - h' \natural \alpha)$ defines $(-k')$.

$$(-k)h'\natural\alpha = (-k)\frac{1}{n}\sum_{j=1}^{n} v_j\cdots v_{n-1}\,dv_n v_1\cdots v_{j-1}$$

$$= \frac{1}{n}\sum_{j=2}^{n} v_j\cdots v_{n-1}\,dv_n\, d(v_1\cdots v_{j-1}).$$

Thus

$$(-k')(v_i\cdots v_{n-1}\,dv_n v_1\cdots v_{i-1})$$

$$= v_i\cdots v_{n-1}\,dv_n\, d(v_1\cdots v_{i-1})$$
$$-\frac{1}{n}\sum_{j=2}^{n} v_j\cdots v_{n-1}\,dv_n\, d(v_1\cdots v_{j-1})$$

Now let's calculate I_D using h', k'.

$$\Omega^1 R_\natural \xrightarrow{h'} \Omega^1 R \xrightarrow{\iota_D} R$$

$$v_1\cdots v_{n-1}dv_n \longmapsto \frac{1}{n}\sum_{j=1}^{n} v_j\cdots v_{n-1}\,dv_n\, v_1\cdots v_{j-1}$$
$$\longmapsto \frac{1}{n}\sum_{j=1}^{n} v_j\cdots v_{n-1} v_n v_1\cdots v_{j-1}$$

which is obviously the N map $V^{\otimes n} \longrightarrow V^{\otimes n}$

Next

$$R \xrightarrow{d} \Omega^1 R \xrightarrow{-k'} \Omega^2 R_\natural \xrightarrow{\iota_D} \Omega^1 R_\natural$$

$$v_1\cdots v_n \longmapsto \sum_{\ell=1}^{n} v_1\cdots v_{\ell-1}\,dv_\ell\, v_{\ell+1}\cdots v_n$$

$$\longmapsto \sum_{\ell=1}^{n}\Big\{ v_1\cdots v_{\ell-1}\,dv_\ell\, d(v_{\ell+1}\cdots v_n)$$
$$-\frac{1}{n}\sum_{j=\ell+2}^{\ell+n} v_j\cdots v_{\ell-1}\,dv_\ell\, d(v_{\ell+1}\cdots v_{j-1})\Big\}$$

$$\xmapsto{\iota_D} \sum_{\ell=1}^{n} v_1\cdots v_\ell\, d(v_{\ell+1}\cdots v_n)$$
$$-\frac{1}{n}\sum_{\ell=1}^{n}\sum_{j=\ell+1}^{\ell+n-1} v_{j+1}\cdots v_\ell\, d(v_{\ell+1}\cdots v_j)$$

Let us compute

$$\frac{1}{n}\sum_{l=1}^{n}\sum_{j=l+1}^{l+n-1} v_{j+1}\cdots v_{l}\, d(v_{l+1}\cdots v_{j})$$

Think of $v_1,\dots,v_n$ arranged circularly and choosing an interval starting at $l+1$ and going to j. We want both the intervals $[l+1,j]$ and $[j+1,l]$ to be non-empty The answer will be

$$\frac{1}{n}\sum_{i=1}^{n} \int v_{i+1}\cdots v_{i-1}\, dv_i$$

where $\int$ is the number of the intervals $[l+1,j]$ containing i. There is one interval of length 1, two intervals of length 2, up to $n-1$ intervals of length $n-1$. $\therefore \int = \frac{1}{2}n(n-1)$

Similarly the first term

$$\sum_{l=1}^{n} v_1\cdots v_l\, d(v_{l+1}\cdots v_n)$$

is $\sum c_j \, v_{j+1}\cdots v_{j-1}\, dv_j$ where is the number of intervals $[l+1,n]$ containing j when $1\leq l\leq n$. The intervals in question are $[2,\dots,n],\dots,[j,\dots,n]$ and the number is $j-1$. Thus we get the

$$\sum_{j=1}^{n}\left(j-1-\frac{n-1}{2}\right) v_{j+1}\cdots v_{j-1}\, dv_j$$

which gives the Green's function.

January 28, 1991

Connections. Let E be a right R-module. Define (following Connes) a connection in E to be a map

$$\nabla : E \longrightarrow E \otimes_R \Omega^1 R$$

satisfying $\nabla(\xi r) = (\nabla\xi)r + \xi dr$. I claim this is equivalent to a section of the right R-module map $E \otimes R \xrightarrow{m_r} E$ given by right multiplication. Indeed, we have

$$\begin{tikzcd} & & & E \arrow[dll, dashed, "\nabla"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "\otimes 1_R"] & \\ 0 \arrow[r] & E \otimes_R \Omega^1 R \arrow[rr] & & E \otimes R \arrow[r, "m_r"] & E \arrow[r] & 0 \end{tikzcd}$$

giving a 1-1 corresp. between s such that $m_r s = 1$ and ∇ given by

$$\nabla(\xi) = \xi \otimes 1_R - s(\xi).$$

Then

$$\nabla(\xi r) = \xi r \otimes 1 - s(\xi r)$$
$$= \xi(r \otimes 1 - 1 \otimes r) + \xi \otimes r - s(\xi)r + s(\xi)r - s(\xi r)$$

$$\boxed{\nabla(\xi r) = \xi dr + (\nabla\xi)r + s(\xi)r - s(\xi r)}$$

showing ∇ is a connection iff s is a right R-module homomorphism.

In particular ~~the~~ R-module E has a connection iff it is projective.

Next consider connections in the "tangent" bundle, that is, a connection

$$\nabla : \Omega^1 R \longrightarrow \Omega^1 R \otimes_R \Omega^1 R = \Omega^2 R$$

compatible with left R-multiplication:

$$\nabla(r\omega) = r\nabla\omega \qquad \nabla(\omega r) = \omega dr + (\nabla\omega)r$$

Such a thing is equivalent to a section of the bimodule map

$$\Omega^1R\otimes R \xrightarrow{m_r} \Omega^1R$$

hence exists only for R quasi-free.

Let's compute $\natural\nabla(-b)$:

$$\begin{tikzcd}
\Omega^2R_\natural \arrow[r, shift right, "-b"'] & \Omega^1R \arrow[l, shift right, "\natural\nabla"']
\end{tikzcd}$$

We have $\natural\nabla(-b)(r_0dr_1dr_2) = \natural\nabla(r_0dr_1r_2 - r_2r_0dr_1)$

$$= \natural\{r_0\underbrace{\nabla(dr_1r_2)}_{dr_1dr_2+\nabla(dr_1)r_2} - r_2r_0\nabla(dr_1)\}$$

$$= \natural(r_0dr_1dr_2).$$

Thus the connection ∇ determines a splitting

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2R_\natural \arrow[r, shift right, "-b"'] & \Omega^1R \arrow[l, shift right, "-k"'] \arrow[r, shift right, "\natural"'] & R \arrow[l, shift right, "h"'] \arrow[r] & 0
\end{tikzcd}$$

with $-k = \natural\nabla$.

Now I want to check that the interior product operators on the X complexes associated to this splitting behave nicely with respect to adic filtrations.

Recall h^0 is the composition

$$R \xrightarrow{d} \Omega^1R \xrightarrow{\natural\nabla} \Omega^2R_\natural \xrightarrow{\iota} \Omega^1R'_\natural$$

where $\iota = \iota(u,\ddot{u})$, $uI\subset I'$, $\ddot{u}R\subset I'$. We have $h^0 = \natural\iota\nabla d$.

$$I^{n+1} \xrightarrow{d} \sum_{j=0}^{n} I^{n-j}\,dI\,I^j$$

$$\xrightarrow{\nabla} \sum_{j=0}^{n}\Big\{I^{n-j}\,dI\,\underbrace{d(I^j)}_{\sum_{k=1}^{j} I^{j-k}dI\,I^{k-1}} + I^{n-j}\,\nabla(dI)\,I^j\Big\}$$

$$\subset \sum_{j=\text{[illegible]}}^{n} \sum_{k=1}^{j} I^{n-j} dI\, I^{j-k} dI\, I^{k-1} + \sum_{j=1}^{n} I^{n-j} \nabla dI\, I^{j}$$

$$\xrightarrow{\iota} \sum_{1 \le k \le j \le n} I'^{\,n-j} \overbrace{(\mathring{u} I)}^{I'} I'^{\,j-k} dI' I'^{\,k-1}$$

$$+ \sum_{j=1}^{n} I'^{\,n-j} (\iota \nabla dI)\, I'^{\,j}$$

Use $\mathring{u} I \subset I'$, $\iota : \Omega^2 R \longrightarrow I' \Omega^1 R'$

$$r_0\, dr_1\, dr_2 \longmapsto u r_0 \underbrace{\mathring{u} r_1}_{I'} d(u r_2)$$

$$\therefore \quad (\iota \nabla d)(I^{n+1}) \subset \sum_{1 \le k \le n} I'^{\,n-k+1} dI' I'^{\,k-1}$$

$$+ \sum_{1 \le j \le n} I'^{\,n-j+1} \Omega^1 R' I'^{\,j}$$

$$\therefore \boxed{(\underbrace{\natural\, \iota\, \text{[redacted]} \nabla d}_{h_0})(I^{n+1}) \subset I'^{\,n} dI' + I'^{\,n+1} \Omega^1 R'}$$

Next consider $h^0([R, I^n]) = h^0(b(I^n dR))$.

$$b(I^n dR) \xrightarrow{d} db(I^n dR) = bB(I^n dR)$$

$$\xrightarrow{\natural \nabla} B(I^n dR) \subset \Omega^1 R_\natural \qquad -\natural \nabla b = 1.$$

$$B(x\, dr) = dx\, dr - dr\, dx$$

$$B(I^n dR) \subset d(I^n) dR + dR\, d(I^n)$$

$$\subset \sum_{1 \le j \le n} I^{n-j} dI\, I^{j-1} dR + dR\, I^{n-j} dI\, I^{j-1}$$

$$\xrightarrow{\iota} \sum_{1 \le j \le n} I'^{\,n-j} I' I'^{\,j-1} dR' + I' I'^{\,n-j} dI' I'^{\,j-1}$$

$$\subset I'^{\,n} dR' + I'^{\,n} dI' = I'^{\,n} dR'$$

$$\therefore \boxed{h^0([R, I^n]) \subset I'^{\,n} dR'}$$

Next we consider

$$h' : \Omega^1 R_\natural \xrightarrow{h} \Omega^1 R \xrightarrow{\iota} R'$$

and use that $h_\natural = 1 + b(\natural \nabla)$. We want to find $h'(\underline{I^n dR})$. First $I^n \Omega^1 R$

$$\iota(\underline{I^n dR}) \subset I'^n \mathring{u} R \cdot R' \subset I'^{n+1}$$

Then

$$b\natural\nabla(I^n \overset{\Omega^1 R}{\cancel{dR}}) = b(I^n \underbrace{\nabla \overset{\Omega^1 R}{\cancel{dR}}}_{\subset \Omega^2 R}) \subseteq [I^n \Omega^1 R, R]$$

$$\xrightarrow{\iota} [I'^{n+1}, R'] \subset I'^{n+1}$$

$$\therefore \boxed{h'(I^n \overset{\Omega^1 R}{\cancel{dR}}) \subset I'^{n+1}}$$

color red version only looks nicer

Next consider the relevant complexes

$$X^n(R,I) : \quad R/I^{n+1} + [R, I^n] \rightleftarrows \Omega^1 R/[R, \Omega^1 R] + I^n \Omega^1 R$$

$$X^n(R,I)' : \quad R/I^{n+1} \rightleftarrows \Omega^1 R/[R, \Omega^1 R] + I^{n+1} \Omega^1 R + I^n dI$$

February 1, 1991

First order variation of the X-complex.

$$\Omega^1(R\oplus M) \cong (R\oplus M)\otimes(\bar{R}\oplus M)$$

$$\therefore\ \Omega^1(R\oplus M)_{(1)} \xleftarrow{\sim} R\otimes M \oplus M\otimes\bar{R}$$

$$x\,dm \longleftarrow (x,m)$$

$$m\,dx \longleftarrow (m,x)$$

Thus a linear fn. T on $\Omega^1(R\oplus M)$ of degree 1 is equivalent to the pair of bilinear fns.

$$T(x\,dm) \qquad T(m\,dx)$$

which can be arbitrary such that the second vanishes for $x=1$.

When is T a trace? Set

$$\varphi(m) = T(dm) \qquad \psi(m,x) = T(m\,dx)$$

Prop. These formulas give an equivalence between traces on $\Omega^1(R\oplus M)$ of degree 1 and pairs (φ,ψ), where $\varphi(m)$ is a trace on M and $\psi(m,x)$ is a 1-cocycle [redacted] i.e. a bilinear fn on $M\times R$ such that $b\psi=0$.

Proof. If T is a trace

$$(b\varphi)(m,x) = \varphi[m,x] = Td[m,x]$$
$$= T([dm,x]+[m,dx]) = 0$$
$$(b\psi)(m,x,y) = T(mx\,dy - m\,d(xy) + ym\,dx)$$
$$= T([m\,dx,y]) = 0.$$

Further

$$T(x\,dm) = T(d(xm) - dx\,m) = Td(xm) - T(m\,dx)$$
$$= \varphi(xm) - \psi(m,x)$$

which shows T is determined by (φ, ψ).

Conversely given $\varphi(m)$, $\psi(m,x)$ satisfying $b\varphi = b\psi = 0$, we define a linear fn T of degree 1 on $\Omega^1(R \oplus M)$ by

$$T(m\,dx) = \psi(m, x)$$

$$T(x\,dm) = \varphi(xm) - \psi(m,x)$$

To check T is a check it suffices to verify

1) $T([m\,dx, y]) = 0$ (follows from $b\psi = 0$ see above)

2) $T([x\,dm, y]) = 0$

3) $T([x\,dy, m]) = 0$

2): $T(x\,dm\,y) = T(x\,d(my)) - T(xm\,dy)$

$$= \varphi(\underline{x\check{m}}y) - \psi(my, x) - \psi(xm, y)$$

$$T(yx\,dm) = \varphi(y\underline{\check{x}m}) - \psi(m, yx)$$

These agree as $b\varphi = b\psi = 0$.

3): $T(x\,dy\,m) = T(x\,d(ym)) - T(xy\,dm)$

$$= \cancel{\varphi(xym)} - \psi(ym, x)$$

$$- \cancel{\varphi(xym)} + \psi(m, xy)$$

$$T(mx\,dy) = \psi(mx, y)$$

These agree as $b\psi = 0$.

Corollary:

$$\begin{tikzcd} M_\natural \oplus (M \otimes_R \Omega^1 R \otimes_R) \arrow[r, "\sim"] & \Omega^1(R \oplus M)_\natural \ (1) \\ m \arrow[r, mapsto] & dm \\ m\,dx \arrow[r, mapsto] & m\,dx \end{tikzcd}$$

We have the following description of $X(R\oplus M)_{(1)}$:

$$\begin{tikzcd} M \arrow[r, shift right, "\begin{pmatrix} f \\ 0 \end{pmatrix}"'] & (M\otimes_R)\oplus(M\otimes_R\Omega^1R\otimes_R) \arrow[l, shift right, "(0\ b)"'] \end{tikzcd}$$

Because of the exact sequence

$$0\longrightarrow H_1(R,M)\longrightarrow M\otimes_R\Omega^1R\otimes_R\xrightarrow{b}M\longrightarrow M\otimes_R\longrightarrow 0$$

this implies

$$H_i\big(X(R\oplus M)_{(1)}\big)=\begin{cases}0 & i=0\\ H_1(R,M) & i=1.\end{cases}$$

In the above we [redacted] found the linearization (or derivative) of the functors $R\longmapsto(\Omega^1R)_\natural$ and $R\longmapsto X(R)$. We now discuss something closely related in some way which needs to be better understood, namely homotopy. Consider a pair $(\theta,\mathring{\theta}):R\longrightarrow R'$ where θ is a homomorphism and $\mathring{\theta}$ is a derivation relative to θ. There is an induced map

$$L(\theta,\mathring{\theta}):X(R)\longrightarrow X(R') \qquad \begin{aligned} x&\longmapsto\mathring{\theta}x\\ xdy&\longmapsto\mathring{\theta}x\,d(\theta y)+\theta x\,d(\mathring{\theta}y)\end{aligned}$$

called Lie derivative. It is a map of complexes. Example if $\theta_t:R\to R'$ is a differentiable 1-parameter family of homorphisms and $X(\theta_t)$ is the family of induced maps on X-complexes, we have

$$L(\theta_t,\mathring{\theta}_t)=\partial_t X(\theta_t).$$

We wish to find when $L(\theta,\dot{\theta})$ is homotopic to zero. Note that $\dot{\theta}$ induces an R-bimodule map $\Omega^1 R \longrightarrow R'$ which in turn induces an algebra homomorphism

$$\Omega R = T_R(\Omega^1 R) \longrightarrow R'.$$

Thus for R fixed, there is a universal algebra R' equipped with $(\theta,\dot{\theta}) : R \to R'$, and it is given by $R' = \Omega R$ with θ the obvious inclusion and $\dot{\theta}$ the derivation

$$R \xrightarrow{d} \Omega^1 R \subset \Omega R$$

~~Let us calculate $L(\theta,\dot{\theta})$ in this universal case. Write δ for $\dot{\theta}$ in order to avoid confusion with the differential d in the X-complex.~~

We now describe the Lie derivative L in this universal case. Note that $X(\Omega R)$ has a grading derived from the grading on ΩR and that the image of L is contained in the subcomplex $X(\Omega R)_{(1)}$ of degree 1 for this grading. Further one has

$$X(\Omega R)_{(1)} \xrightarrow{\sim} X(R \oplus \Omega^1 R)_{(1)}$$

where $R \oplus \Omega^1 R$ is regarded as the quotient algebra $\Omega R / \Omega R^{\geq 2}$.

Let us now apply our calculation for semi-direct products. We need to distinguish the derivation $\dot{\theta} : R \longrightarrow R \oplus \Omega^1 R$ from the differential d in the X-complex. Write δ for $\dot{\theta}$, so that $\Omega^1 R$ is spanned by elements $x \delta y$ modulo the relations of bilinearity and vanishing for $y = 1$.

We then have $\overset{\text{an}}{\wedge}$ isomorphisms

$$\circledast \qquad \begin{aligned} \Omega^1 R_\natural \oplus \Omega^2 R_\natural &\xrightarrow{\sim} \Omega^1(R\oplus\Omega^1 R)_{\natural(1)} \\ (x\,dy \quad , \quad x\,dy\,dz) &\longmapsto d(x\delta y) + x\delta y\,dz \end{aligned}$$

by our calculation above of $\Omega^1(R\oplus M)_{\natural(1)}$ in the case $M = \Omega^1 R$. We also have the following calculation of the Lie derivative map L:

$$\begin{tikzcd} X(R) \arrow[d, "L"] : & R \arrow[r, "d"] \arrow[d, "d"'] & \Omega^1 R_\natural \arrow[r, "b"] \arrow[d, "\binom{1}{B}"] & R \arrow[d, "d"] \\ X(\Omega R)_{(1)} : & \Omega^1 R \arrow[r, "\binom{\natural}{0}"'] & \Omega^1 R_\natural \oplus \Omega^2 R_\natural \arrow[r, "(0\ -b)"'] & \Omega^1 R \end{tikzcd}$$

Let's verify the formula $\binom{1}{B}$ for L in degree 1, i.e. on $X_1(R)$. Let $x\,dy \in \Omega^1 R_\natural = X_1(R)$; then

$$\begin{aligned} L(x\,dy) &= \delta x\,dy + x\,d\delta y \\ &= d(x\delta y) + \delta x\,dy - \delta y\,dx \end{aligned}$$

which under the isomorphism ⊛ above corresponds to

$$\begin{pmatrix} x\,dy \\ dx\,dy - dy\,dx \end{pmatrix} = \begin{pmatrix} 1 \\ B \end{pmatrix} x\,dy$$

[redacted] The sign in the arrow $(0\ -b)$ is explained as follows: [redacted]

$$b(x\delta y\,dz) = [x\delta y, z]$$

$$b(x\,dy\,dz) = -[x\,dy, z]$$

[redacted] We can now calculate the map on homology induced by L. We have

$$H_i\{X(R\oplus\Omega^1 R)_{(1)}\} = \begin{cases} 0 & i=0 \\ H_1(R,\Omega^1 R) = H_2(R,R) = HH_2(R) & i=1 \end{cases}$$

Thus $L_n H_1$ can be identified with

$$B: HC_1(R) \longrightarrow HH_2(R)$$

February 3, 1991

Let us review extensions, but this time for (non-commutative) algebras A over a commutative ground ring S.

Let $\Omega^1(A;S)$ be the relative differentials - here we suppose only that $S \to A$ is a map of algebras, and we define $\Omega^1(A;S)$ by

$$0 \longrightarrow \Omega^1(A;S) \longrightarrow A \otimes_S A \longrightarrow A \longrightarrow 0$$

This sequence splits as left A-modules because of the section $a \longmapsto a \otimes 1$. As

$$S \longrightarrow A \longrightarrow A/S \longrightarrow 0$$

$$A \xrightarrow{\otimes 1} A \otimes_S A \longrightarrow A \otimes_S (A/S) \longrightarrow 0$$

we obtain

$$A \otimes_S (A/S) \xrightarrow{\sim} \Omega^1(A;S)$$

$$a_0 \otimes a_1 \longmapsto a_0 \otimes a_1 - a_0 a_1 \otimes 1$$

$$= a_0 \underbrace{(1 \otimes a_1 - a_1 \otimes 1)}_{-da_1}$$

From the exactness of

$$A \otimes_S A \otimes_S A \otimes_S A \xrightarrow{b'} A \otimes_S A \otimes_S A \xrightarrow{b'} A \otimes_S A \xrightarrow{b'} A$$

we obtain the universal property of $\Omega^1(A;S)$ with respect to derivations vanishing on S. (I've reviewed this to check that one does not need S commutative + central in A, or any other assumptions)

Let us now consider $S \to A$ as above an an extension of A-bimodules:

$$0 \to M \to E \to \Omega^1(A;S) \to 0$$

Then we define a square zero extension R of A by pull-back

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & R \arrow[r, "p"] \arrow[d, "{(p,D)}"] & A \arrow[r] \arrow[d, "1+d"] & 0 \\
0 \arrow[r] & M \arrow[r] & A\oplus E \arrow[r] & A\oplus\Omega^1(A;S) \arrow[r] & 0
\end{tikzcd}$$

Note D is a derivation relative to the homom. p. Also we have a unique homom. $S \to R$ lifting $S \to A$ and such that $D(S) = 0$.

Next suppose we have an extension

$$0 \to I \to R \to A \to 0$$

of algebras under S.

$$0 \to \Omega^1(R;S) \to R\otimes_S R \to R \to 0$$

splits as left or right R-modules.

$$0 \to \Omega^1(R;S)\otimes_R A \to R\otimes_S A \to A \to 0$$

$$\begin{tikzcd}
\mathrm{Tor}_1^R(A, R\otimes_S A) \arrow[r] & \mathrm{Tor}_1^R(A,A) \arrow[r] \arrow[d, equal] & A\otimes_R\Omega^1(R;S)\otimes_R A \arrow[r] & A\otimes_S A \arrow[r] & A \arrow[r] & 0 \\
& I/I^2 & & & &
\end{tikzcd}$$

If we <u>assume</u> A <u>is</u> <u>left</u> S-<u>flat</u>, then $R\otimes_S A$ is left R flat and $\mathrm{Tor}_1^R(A, R\otimes_S A) = 0$ yielding an exact sequence of bimodules over A

$$0 \longrightarrow I/I^2 \longrightarrow A\otimes_R \Omega^1(R;S)\otimes_R A \longrightarrow \Omega^1(A;S) \longrightarrow 0$$

supposing that

$$0 \longrightarrow M \longrightarrow R \longrightarrow A \longrightarrow 0$$

is a square zero extension of rings under S, we therefore obtain an extension of A-bimodules

$$0 \longrightarrow M \longrightarrow \underbrace{A\otimes_R \Omega^1(R;S)\otimes_A A}_{E} \longrightarrow \Omega^1(A;S) \longrightarrow 0$$

together with an S-derivation $D: R \longrightarrow E$ such that

$$\begin{tikzcd}
 & R \arrow[d, "D"] \\
M \arrow[ur] \arrow[r] & E
\end{tikzcd}$$

commutes. Thus we have

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & R \arrow[r, "p"] \arrow[d, "{(p,D)}"] & A \arrow[r] \arrow[d, "1+d"] & 0 \\
0 \arrow[r] & M \arrow[r] & A\oplus E \arrow[r] & A\oplus\Omega^1(A;S) \arrow[r] & 0
\end{tikzcd}$$

and the second square is cartesian.

We conclude that if A is either left or right S-flat, then there is an equivalence between square zero extensions of A in the category of rings under S, and A-bimodule extensions of $\Omega^1(A;S)$.

Let us now assume S is a commutative ring and that A is a possibly noncommutative alg over S. We consider $S = \mathbb{C}[h]$ to fix the ideas.

Let $S' = S \otimes S / I_\Delta^{n+1} = \mathbb{C}[h, \varepsilon]/(\varepsilon^{n+1})$

We want the infinitesimal analogue of an isomorphism $A_{h_1} \xrightarrow{\sim} A_{h_2}$ for all pairs of sufficiently close points. Let $\iota_1, \iota_2 : S \rightrightarrows S'$ be the two canonical embeddings: $\iota_2(h) = h$, $\iota_1(h) = h + \varepsilon$. We want an isomorphism of S' algebras (over A)

$$S'_{\iota_1} \otimes_S A = (\iota_1)_* A \xrightarrow{\sim} (\iota_2)_* A = S'_{\iota_2} \otimes_S A$$

Such an isomorphism is given by a homomorphism

$$A \xrightarrow{\varphi} S'_{\iota_2} \otimes_S A \qquad \text{(over } A\text{)}$$

of S-algebras where the latter is regarded as an S-alg via $S \xrightarrow{\iota_1} S' \longrightarrow S'_{\iota_2} \otimes_S A$.

[redacted] Now

$$S'_{\iota_2} \otimes_S A = \mathbb{C}[\varepsilon]/(\varepsilon^{n+1}) \otimes A$$

so we are trying to lift in the following:

$$\begin{tikzcd}
 & \mathbb{C}[\varepsilon]/(\varepsilon^{n+1}) \otimes A \arrow[d] \\
S \arrow[ur, "h \mapsto h+\varepsilon"] \arrow[r, "h \mapsto h"'] & A \arrow[u, dashed, bend left, "\varphi"]
\end{tikzcd}$$

For example if $n = 1$, we have this picture

$$\begin{tikzcd}
 & A \oplus \varepsilon A \arrow[d, shift left] \\
\mathbb{C}[h] \arrow[ur, "h \mapsto h+\varepsilon"] \arrow[r] & A \arrow[u, dashed, shift left, "\varphi"]
\end{tikzcd}$$

The homomorphism φ is equivalent to a derivation of A compatible with the derivation ∂_h on $\mathbb{C}[h]$.

February 5, 1991

Let S be a commutative $\mathbb{C}$ algebra such that $\mathbb{C} \xrightarrow{\sim} S/\sqrt{0}$ and $(\sqrt{0})^{N+1} = 0$.

Let A be an algebra (noncomm. in general) over S. More precisely we mean A is an algebra equipped with a homomorphism $S \to A$ whose image lies in the center of A. Better terminology: let A be an S-algebra.

Put $m = \sqrt{0} \subset S$. We have the m-adic filtration m^n of S and the mA-adic filtration $(mA)^n = m^n A$ of A. We have a canonical homomorphism

$$gr_m S \otimes A_0 \longrightarrow gr_{mA} A$$

which is surjective. When A is flat over S, this map is an isomorphism, since one has

$$m^n \otimes_S A \xrightarrow{\sim} m^n A$$

$$m^n/m^{n+1} \otimes A_0 = (m^n/m^{n+1}) \otimes_{S/m} A/mA$$

$$= (m^n/m^{n+1}) \otimes_S A \xrightarrow{\sim} m^n A/m^{n+1}A.$$

I think the converse is also true (I recall the result that for $I \subset S$ and an S module M, one has M/IM flat over S/I and $gr_I S \otimes_{S/M} M/IM \xrightarrow{\sim} gr_I M \implies M$ flat over S provided some condition of "ideally separated" is satisfied)

Now suppose A flat over S and that A_0 is quasi-free. Now A is a nilpotent extension of A_0, hence there exists a lifting homom. $A_0 \to A$. This induces (since S, A_0 commute in A) a homom. of S-algebras

$$S \otimes A_0 \longrightarrow A.$$

This homomorphism yields a map of associated graded algebras

$$\begin{tikzcd} gr_m(S\otimes A_0) \arrow[r] & gr_m(A) \\ gr_m(S)\otimes A_0 \arrow[u, "\cong"] \arrow[r, equal] & gr_m S \otimes A_0 \arrow[u, "\cong \text{ because } A \text{ flat.}"'] \end{tikzcd}$$

so we conclude that $S\otimes A_0 \xrightarrow{\sim} A$ for any lifting of the algebra A_0 into A.

In fact we have an equivalence between liftings of A_0 into A and S-algebra isomorphisms $S\otimes A_0 \xrightarrow{\sim} A$ reducing to the identity of A_0 modulo m.

Notice that this gives ~~some~~ the following structure on the space of liftings. Let G be the group of automorphisms of the S-algebra $S\otimes A_0$ reducing to identity modulo m. Then G acts simply transitively on the space of liftings of A_0 into A. Now G is a nilpotent group, so the exponential map from its Lie algebra gives global coordinates. This shows that the space of liftings in this situation has global coordinates. In general I know that, by writing A_0 as a quotient of a free algebra, one obtains the space of liftings as a retract of a vector space.

So far we have checked some assertions needed to put the following discussion on firm ground. We wish to understand the Gauss-Manin connection in the quasi-free case. Let S, A, A_0, G be as above. We have a cartesian square

$$\begin{tikzcd} A \arrow[r] & A_0 \\ S \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

and G acts nontrivially on A but trivially on the others. Let's consider the noncommutative analogue of the DR complex for the S-algebra A, namely $X(A;S)$. The GM idea says there is a flat connection in some sense on the complex $X(A;S)$ of S-modules. In particular when we pass to homology we have a canonical isomorphism

$$* \qquad S\otimes H_i(X(A_0)) \xrightarrow{\sim} H_i(X(A;S))$$

We can produce such an isomorphism as follows. Choose an isom. $S\otimes A_0 \xrightarrow{\sim} A$, equivalently a lifting of A_0 into A. Then we have isomorphisms

$$X(A;S) \simeq X(S\otimes A_0;S) \underset{\text{canon isom.}}{=} S\otimes X(A_0)$$

and so passing to homology the isomorphism $*$.

Notice that we can describe the isomorphism thus obtained as follows. The lifting $A_0 \to A$ induces $X(A_0) \to X(A) \to X(A;S)$ which induces

$$H_i(X(A_0)) \longrightarrow H_i(X(A;S))$$

which extends to the desired isomorphism

$$S\otimes H_i(X(A_0)) \xrightarrow{\sim} H_i(X(A;S))$$

~~[illegible]~~

The point of Gauss-Manin is that this isomorphism is canonical, i.e. independent of the choice of the lifting. And this follows from the fact that the map $X(A_0) \to X(A)$ up to homotopy is independent of the lifting because any two liftings are homotopic.

February 6, 1991

Consider $A = R/I$, A quasifree, I nilpotent & let M be the space of liftings $\theta : A \to R$. For each θ we have a map of complexes $X(\theta) : X(A) \longrightarrow X(R)$. But more is true: the function $\theta \longmapsto X(\theta)$ from M to $E = \text{Hom}_{\mathbb{C}}(X(A), X(R))$ is smooth, so we have an element

$$S_0 \in \Omega^0(M, E^0) \qquad d_E S_0 = 0.$$

Thus each distribution on M gives a cycle in E^0.

Here there's an important idea. Instead of the normal homotopy setup, where points and paths and higher homotopies one has a situation with the extra features of linearity and differentiability. By linearity I mean we are dealing with homology instead of homotopy, so that complexes suffices to describe the information. By differentiability I mean we can use differential forms with their nice commutative product instead of singular cochains. Moreover we can do our constructions infinitesimally and then integrate.

We wish to express the idea that there is a map $X(A) \longrightarrow X(R)$ unique up to homotopy and higher homotopy. Unique up to homotopy we have already handled by producing a contracting homotopy for $L(\theta, \dot{\theta})$ and then integrating. This amounts to a 1-form

$$s_1 \in \Omega^1(M, E^{-1})$$

satisfying $d_M s_0 = d_E s_1$.

Generalizing we want to produce a family $s_n \in \Omega^n(M, E^{-n})$ such that $s = \{s_n\}$ is a cocycle in the double complex $\Omega(M, E^{\cdot})$. This should be equivalent to a maps of complexes

$$\begin{tikzcd}
X(A) \arrow[r, "s"] & \Omega(M, X(R)) \\
& X(R) \arrow[u, hook]
\end{tikzcd}$$

*

Note the the vertical arrow is a quis by the contractibility of M. (Here use retract of an affine space if required to be explicit).

The above diagram I think is the good way to express the idea that we have a ~~well defined~~ map $X(A) \longrightarrow X(R)$ unique up to higher homotopy.

It remains to construct s, which should be an infinitesimal affair around each point of M. It might be possible to do the construction with R universally constructed from A, following the analysis of $L(O, \mathring{O})$ (cf. 175-180).

February 7, 1991

Free products. Let's consider two algebras R, S. One has a map

$$X(R) \underset{X(\mathbb{C})}{\amalg} X(S) \longrightarrow X(R*S)$$

and one would like to prove it is a quasi-isomorphism.

Let's consider the case where R, S are augmented algebras $R = \mathbb{C} \oplus \bar{R}$ where $\bar{R}$ is an ideal in R. $R*S$ has a direct sum decomposition indexed by words in the letters r, s without repeated letters:

words		summands	
1		$\mathbb{C}$	
r	s	$\bar{R}$	$\bar{S}$
rs	sr	$\bar{R}\otimes\bar{S}$	$\bar{S}\otimes\bar{R}$
rsr	srs	$\bar{R}\otimes\bar{S}\otimes\bar{R}$	$\bar{S}\otimes\bar{R}\otimes\bar{S}$
rsrs	srsr		
rsrsr			

Consider $R*S$ as an R-bimodule. One sees it has a decomposition

$$R \oplus (R\otimes\bar{S}\otimes R) \oplus (R\otimes\bar{S}\otimes\bar{R}\otimes\bar{S}\otimes R) \oplus \cdots$$

where all the summands except the first are free R-bimodules. Recall that we have a decomposition

$$\Omega^1(R*S) = (R*S)\otimes_R \Omega^1 R \otimes_R (R*S) \oplus (R*S)\otimes_S \Omega^1 S \otimes_S (R*S)$$

$$\Omega^1(R*S)_\natural = (R*S)\otimes_R \Omega^1 R \otimes_R \quad \oplus \quad (R*S)\otimes_S \Omega^1 S \otimes_S$$

Let us consider the complex

$$X(R*S;R)/R_\natural : \quad (R*S/R)^{\otimes}_R \rightleftarrows (R*S)\otimes_S \Omega^1 S\otimes_S$$

We claim that

$$X(R*S)/\mathbb{C} \xrightarrow[\text{NO}]{\sim} X(R*S;R)/R_\natural \oplus X(R*S;S)/S_\natural$$

This is clear in degree 1, and so we only have to check that

$$\circledast \qquad R*S/\mathbb{C} \longrightarrow (R*S/R)^{\otimes}_R \oplus (R*S/S)^{\otimes}_S$$

is an isomorphism. NO Now we have

$$(R*S/R)^{\otimes}_R \simeq R\otimes\bar S \oplus R\otimes\bar S\otimes\bar R\otimes\bar S \oplus R\otimes\bar S\otimes(\bar R\otimes\bar S)^{\otimes 2} \oplus \cdots$$

So ~~this~~ has a decomposition indexed by the words s, rs, srs, rsrs, .. i.e. all words ending with s. Similarly $(R*S/S)^{\otimes}_S$ will decompose according to words ending with r. So $\circledast$ ought to be an isomorphism because the different summands on the left are accounted for on the right. To be more careful, let us consider the projection $(R*S)/\mathbb{C} \longrightarrow (R*S/R)^{\otimes}_R$. This is onto because the summands $\bar S$, $\bar R\otimes\bar S$, $\bar S\otimes\bar R\otimes\bar S$, .. of the former are mapped isomorphically onto the corresponding summands of the latter. The kernel of this projection admits a decomposition into pieces isomorphic to $\bar R$, $\bar S\otimes\bar R$, $\bar R\otimes\bar S\otimes\bar R$ which are embedded into $R*S/\mathbb{C}$ in the following way. Take for example $\bar S\otimes\bar R$. This is embedded

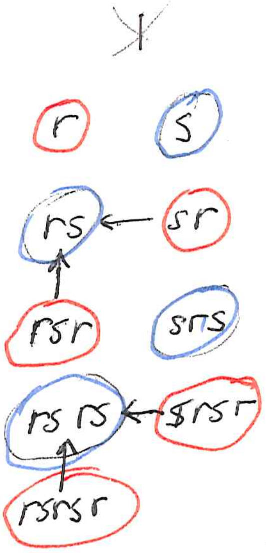
$$\begin{aligned} \bar S\otimes\bar R &\longrightarrow \bar R\otimes\bar S + \bar S\otimes\bar R \\ y\otimes x &\longmapsto -x\otimes y + y\otimes x \end{aligned}$$

Similarly

$$\bar R \otimes \bar S \otimes R \longrightarrow \bar R \otimes \bar S \oplus \bar R \otimes \bar S \otimes \bar R$$

$$x_1 \otimes y \otimes x_2 \longmapsto -x_2 x_1 \otimes y + x_1 \otimes y \otimes x_2$$

One can see what is happening from the picture



blue circles indicate part mapped isomorphically onto $(R*S/R)\otimes_R$. Red circles with the arrows to show how they are embedded give the pieces of the kernel of the projection

Now we can see there are problems, for the elements

$$xy - yx \in [R,S] \subset [R*S,S] \cap [R*S,R]$$

$$xyxy - yxyx = \underbrace{[x,yxy]}_{\in[R,R*S]} = \underbrace{[xyx,y]}_{[R*S,S]}$$

are in the kernel of $\circledast$.

Nevertheless let's try to understand the homology of $X(R*S;R)/R_\natural$.

$(R*S/R)\otimes_R$ has decomposition corresponding to the words s, rs, srs, … ending in s.

$(R*S)\otimes_S \Omega^1 S \otimes_S$ has a decomposition corresponding to words ds, rds, rsds, … . More precisely As S-bimodule we have

$$R*S = S \oplus R*S/S$$

where $R*S/S$ is a free S-bimodule with generating subspaces r, rsr, $rsrsr$, $\ldots$ so

$$(R*S)\otimes_S \Omega^1 S \otimes_S = \Omega^1 S \oplus (\bar{R}\otimes \Omega^1 S) \oplus (\bar{R}\otimes\bar{S}\otimes\bar{R}\otimes \Omega^1 S) \oplus \cdots$$

Picture of the differentials

$$\begin{tikzcd}
(rs)^{n+1}, n\geqslant 0 \arrow[r, shift left] & (rs)^n r\,ds \quad n\geqslant 0 \arrow[l, shift left] & \text{] subcomplex} \\
s(rs)^{n+1} n\geqslant 0 \arrow[r, shift left] \arrow[ur] & (rs)^{n+1}ds \quad n\geqslant 0 \arrow[l, shift left] \arrow[ul] & \text{] quotient complex} \\
s & ds, \text{[illegible]}\ sds &
\end{tikzcd}$$

Let's look at the differentials in the subcomplex:

$$x_1 y_1 \cdots x_n y_n\, x\,dy \xmapsto{b} x_1 y_1 \cdots x_n y_n\, xy$$
$$- xy\, x_1 y_1 \cdots x_n y_n$$

(recall we can move $x \in \bar{R}$ around)

$$x_0 y_0 \cdots x_n y_n \xmapsto{d} \sum x_{j+1} y_{j+1} \cdots x_{j-1} y_{j-1} x_j\, dy_j$$

Thus [illegible] it appears we have [illegible] $(\bar{R}\otimes\bar{S})^{\otimes n+1}$, $n\geqslant 0$ on both sides of the subcomplex, and that $b = 1-\sigma$ and $d = N$. Thus the subcomplex should be acyclic.

Let us consider the quotient complex. We have

$$y\, x_1 y_1 \cdots x_{n+1} y_{n+1} \xmapsto{d} dy\, x_1 y_1 \cdots x_{n+1} y_{n+1}$$
$$+ y x_1 \cdots x_{n+1}\, dy_{n+1}$$

$$= x_1 y_1 \cdots x_{n+1}\, d(y_{n+1} y) \quad = 0 \quad \text{in quotient complex}$$

$$x\, y \cdots x_{n+1} y_{n+1}\, dy \xmapsto{b} -y\, x_1 y_1 \cdots x_{n+1} y_{n+1}$$

(other term is in subcomplex)

It seems therefore that the part of the quotient complex complementary to $\bar{X}(S)$ has b an isomorphism and $d = 0$. Thus we obtain a quis

$$\bar{X}(S) \longrightarrow X(R*S;R)/R_\natural .$$

This is not inconsistent with what we have ~~done~~ found: Observe we have maps

$$\begin{tikzcd}
 & \bar{X}(R) \arrow[d] \arrow[dr, dashed, "quis"] & \\
\bar{X}(S) \arrow[r] \arrow[dr, dashed, "quis"'] & \bar{X}(R*S) \arrow[r] \arrow[d] & X(R*S;S)/S_\natural \\
 & X(R*S;R)/R_\natural &
\end{tikzcd}$$

where the compositions $\dashrightarrow$ and $\downarrow\downarrow$ are zero. This is consistent with a direct sum decomposition on the homology level, the point being that the map

$$\bar{X}(R*S) \longrightarrow X(R*S;R)/R_\natural \oplus X(R*S;S)/S_\natural$$

can have both a kernel & cokernel.

Here's how to finish the calculation. Let us consider the kernel K of the projection

$$\bar{X}(R*S) \longrightarrow X(R*S;R)/R^\natural$$

In degree 1 we have $(R*S) \otimes_R \Omega^1 R \otimes_R$ which we have seen has the decomposition corresponding to the words

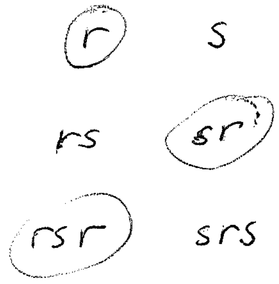
$$(sr)^n s\, dr \quad n \geq 0 \quad ; \quad (sr)^{n+1} dr \quad n \geq 0$$

$$dr \quad r\,dr \qquad \text{corresp. to} \quad \Omega^1 R_\natural$$

Now in degree 0 we have the projection

$$(R*S)/\mathbb{C} \longrightarrow (R*S/R)\otimes_R$$

So K^0 will contain $\bar{R}$ and then $K^0/\bar{R} = [R*S/R, R]$. But commutators are killed by the d map. Hence the homology will be determined by the kernel + cokernel of the b map. Picture:



We know K^0 has a decomposition described by the circled words. We need to describe the pieces of K^0 explicitly

r stands for $\bar{R} \subset \overline{R*S}$.

sr stands for $[S,R] \subset \overline{R*S}$, i.e. to the subspace spanned by $\{y_1x_1 - x_1y_1 = [y, x_1]\}$

rsr Here my first choice was the space spanned by $x_0 y x_1 - x_1 x_0 y = [x_0 y, x_1]$, but a better choice is the space spanned by

$$x_0 y_1 x_1 - y_1 x_1 x_0 = -[y_1 x_1, x_0]$$

Observe these spaces are congruent modulo $[S,R]$.

srsr stands for the space spanned by

$$\{y_1 x_1 y_2 x_2 - x_2 y_1 x_1 y_1 = [y_1 x_1 y_2, x_2]\}$$

Now $b : (sr)^n s dr \longrightarrow (sr)^{n+1}$ is an isom.

$$b(y_1 x_1 \cdots y_n x_n y_{n+1} dx_{n+1}) = [y_1 x_1 \cdots x_n y_{n+1}, x_{n+1}]$$

and $b : (sr)^{n+1}dr \longrightarrow$ [illegible] $r(sr)^{n+1}$

$$b(y_1x_1 \cdots y_{n+1}x_{n+1}dx_{n+2}) = [y_1x_1 \cdots y_{n+1}x_{n+1}, x_{n+2}].$$

is an isomorphism. Therefore we should have that the cokernel of

$$\overline{X}(R) \hookrightarrow \mathrm{Ker}\left\{\overline{X}(R*S) \longrightarrow X(R*S;R)/R_\natural\right\}$$

is a complex having its b map an isomorphism, & hence this map is a quis.

February 8, 1991

Notes for future calculations with $R*S$. The first problem is to handle the case where R,S are not augmented. The second problem is to handle the I-adic filtration with $R\otimes S = R*S/I$. This should be needed in order to understand the Kunneth formulas.

It appears one can proceed by analogy with the case of RA. One has the decreasing I-adic filtration and a funny sort of increasing filtration

$$\begin{tikzcd}
 & \mathbb{C} \arrow[dl, dash] \arrow[dr, dash] & \\
R \arrow[d, dash] \arrow[drr, dash] & & S \arrow[d, dash] \arrow[dll, dash] \\
RS \arrow[d, dash] \arrow[drr, dash] & & SR \arrow[d, dash] \arrow[dll, dash] \\
RSR \arrow[d, dash] \arrow[drr, dash] & & SRS \arrow[d, dash] \arrow[dll, dash] \\
 {} & & {}
\end{tikzcd}$$

There ~~are~~ two natural complements to I in $R*S$ namely RS and SR, and one apparently has to make a choice unlike the case of RA where ρA is the unique complement. Then one gets a direct sum decomposition

$$\bigoplus_{n\geq 0} R\otimes S\otimes(\bar{R}\otimes\bar{S})^{\otimes n} \xrightarrow{\sim} R*S$$

$$(x_0,y_0,\dots,x_n,y_n) \longmapsto x_0y_0[x_1,y_1]\cdots[x_n,y_n]$$

and an isomorphism

$$gr_I(R*S) = T_{R\otimes S}(\Omega^1R\otimes\Omega^1S).$$

I forgot to mention that the increasing filtration

$$RS\subset(RS)^2\subset(RS)^3\subset\cdots\cdots$$

is complementary to the I-adic filtration.

The symbol $[x,y]$ will play the role of $\omega(a_1, a_2)$. We have

$$[x_0, y]x_1 + x_0[x_1, y] = [x_0 x_1, y]$$

hence

& similarly

$$\boxed{\begin{aligned} [R,S]R &= R[R,S] \\ [R,S]S &= S[R,S] \end{aligned}}$$

The difficulty here is with the two complements RS SR to I. The situation seems to be analogous with the two complements $\theta(A) = A$ and $\theta^\gamma(A) = A^\gamma$ ~~to~~ JA in QA. There we used $\rho(A)$ as a nicer complement.

We have to calculate with the following maps with values in $R * S$

$$\alpha : \quad x \in R \longmapsto x \in R * S$$

$$\beta : \quad y \in S \longmapsto y \in R * S$$

$$\gamma : \quad (x,y) \in \bar{R} \otimes \bar{S} \longmapsto [x,y] \in R * S$$

The decomposition of $R * S$ is given in some sense by the cochains $\alpha \beta \gamma^n$.

Let's try to construct a DG algebra of cochains. It's seems natural to consider the identity

$$[x_0, y]x_1 + x_0[x_1, y] = [x_0 x_1, y_1]$$

as something like

$$\gamma \alpha + \alpha \gamma = b'_R \gamma$$

If so, then we have to shuffle the x's and y's as they are feed to the cochains.

For example

$$(\gamma\alpha)(x_0, x_1, y) = \gamma(x_0, y)\,\alpha(x_1)$$

What this means I think is that we want to consider the bigraded differential algebra

$$\mathrm{Hom}_{\mathbb{C}}\big(B(R)\otimes B(S)\,,\; R * S\big)$$

The two differentials will be

$$d\varphi_{pq} = -(-1)^{p+q}\varphi_{pq}(b'_R\otimes 1)$$

$$\delta\varphi_{pq} = -(-1)^{p+q}\varphi_{pq}(\underbrace{1\otimes b'_S}_{\text{sign in here.}})$$

$\alpha(x) = x$ has degree $(1,0)$

$\beta(y) = y$ has degree $(0,1)$

$\gamma(x,y) = [x,y]$ has degree $(1,1)$

$$(d\gamma)(x_1, x_2, y) = -\gamma(x_1x_2, y) = -[x_1x_2, y]$$
$$= -x_1[x_2, y] - [x_1, y]x_2$$

$$\Delta(x_1, x_2, y) = \big((x_1, x_2)\otimes 1 + x_1\otimes x_2 + 1\otimes(x_1, x_2)\big)\cdot(y\otimes 1 + 1\otimes y)$$

$$= (x_1, x_2, y)\otimes 1 + (x_1, x_2)\otimes y$$
$$- (x_1, y)\otimes x_2 + x_1\otimes(x_2, y)$$
$$+ y\otimes(x_1, x_2) + 1\otimes(x_1, x_2, y)$$

Thus $(\alpha\gamma)(x_1, x_2, y) = m(\alpha\otimes\gamma)\Delta(x_1, x_2, y)$

$$= m\big(\alpha(x_1)\otimes\gamma(x_2, y)\big) = \alpha(x_1)\gamma(x_2, y)$$

$$= x_1[x_2, y]$$

$$(\gamma\alpha)(x_1,x_2,y) = m(\gamma\otimes\alpha)\underbrace{\Delta(x_1,x_2,y)}_{-(x_1,y)\otimes x_2 \text{ counts}}$$

$$= m\left(-\gamma(x_1,y)\otimes\alpha(x_2)\right)$$

$$= -\gamma(x_1,y)\alpha(x_2) = -[x_1,y]x_2$$

$$\therefore \quad \boxed{\begin{aligned} d\gamma &= -\alpha\gamma + \gamma\alpha \\ [d+\alpha,\gamma] &= 0 \end{aligned}}$$

$$(\delta\gamma)(x,y_1,y_2) = -\gamma(1\otimes b'_S)(x,y_1,y_2)$$

$$= \gamma(x,y_1y_2) = [x,y_1y_2]$$

$$= y_1[x,y_2] + [x,y_1]y_2$$

$$\Delta(x,y_1,y_2) = (x\otimes 1 + 1\otimes x)\left((y_1,y_2)\otimes 1 + y_1\otimes y_2 + 1\otimes(y_1,y_2)\right)$$

$$= (x,y_1,y_2)\otimes 1 + (x,y_1)\otimes y_2 + x\otimes(y_1,y_2)$$

$$+ (y_1,y_2)\otimes x - y_1\otimes(x,y_2) + 1\otimes(x,y_1,y_2)$$

$$(\beta\gamma)(x,y_1,y_2) = m(\beta\otimes\gamma)(-y_1\otimes(x,y_2))$$

$$= -\beta(y_1)\gamma(x,y_2) = -y_1[x,y_2]$$

$$(\gamma\beta)(x,y_1,y_2) = m(\gamma\otimes\beta)((x,y_1)\otimes y_2)$$

$$= \gamma(x,y_1)\beta(y_2) = [x,y_1]y_2$$

$$\therefore \quad \boxed{\begin{aligned} \delta\gamma &= -\beta\gamma + \gamma\beta \\ [\delta+\beta,\gamma] &= 0 \end{aligned}}$$

Since $d\alpha + \alpha^2 = 0$ and $\delta\beta + \beta^2 = 0$ and $\delta\alpha = d\beta = 0$, it's clear that we should have

$$(d+\delta+\alpha+\beta)^2 =$$

$$\underbrace{(d+\alpha)^2}_{0} + \underbrace{(\delta+\beta)^2}_{0} + [d+\alpha, \delta+\beta] = [\alpha,\beta]$$

equal to γ.

$$\Delta(x,y) = (x\otimes 1 + 1\otimes x)(y\otimes 1 + 1\otimes y)$$
$$= (x,y)\otimes 1 + x\otimes y - y\otimes x + 1\otimes(x,y)$$

$$(\alpha\beta)(x,y) = m(\alpha\otimes\beta)(x\otimes y) = -\alpha(x)\beta(y) = -xy$$
$$(\beta\alpha)(x,y) = m(\beta\otimes\alpha)(-y\otimes x) = \beta(y)\alpha(x) = yx$$

Thus $\boxed{\alpha\beta + \beta\alpha = -\gamma}$

The geometric picture is that of a connection on a product manifold which is flat in both directions, whence it's curvature is of type (1,1).

Let us now try to obtain the traces on $R*S$. We know a linear linear function τ on $R*S$ is equivalent to the family of cochains

$$\tau(\alpha\beta[\alpha,\beta]^n) \qquad n\geq 0$$

which can be arbitrary vanishing whenever an α or β value other than at the beginning is 1. Let us fix the β values and calculate b for the α values; call this b_α:

$$b_\alpha \tau(\alpha\beta[\alpha,\beta]^n) = \tau\{\alpha^2\beta[\alpha,\beta]^n - \alpha\beta(\alpha[\alpha,\beta]^n - (-1)^n[\alpha,\beta]^n\alpha)\}$$
$$+ \lambda_\alpha \tau(\alpha^2\beta[\alpha,\beta]^n)$$
$$= \tau(\alpha[\alpha,\beta]^{n+1}) + (-1)^n\tau(\alpha\beta[\alpha,\beta]^n\alpha) + \lambda_\alpha\tau(\alpha^2\beta[\alpha,\beta]^n)$$

Now

$$\tau b(\alpha\beta[\alpha,\beta]^n d\alpha) = \tau(\alpha\beta[\alpha,\beta]^n\alpha) + (-1)^n\lambda_\alpha \tau(\alpha^2\beta[\alpha,\beta]^n)$$

Thus we get

$$\boxed{b_\alpha \tau(\alpha\beta[\alpha,\beta]^n) - \tau(\alpha[\alpha,\beta]^{n+1}) = (-1)^n(\tau b)(\alpha\beta[\alpha,\beta]^n d\alpha)}$$

This allows us to analyze linear functions on $R*S$ which are traces with respect to the R-bimodule structure. They are families of cochains $\tau_n = \tau(\alpha\beta[\alpha,\beta]^n)$ satisfying

$$b_\alpha \tau_n = s_\beta \tau_{n+1}$$

Let's analyze this. It seems that there are many solutions. $\tau_0 = \tau(\alpha\beta)$ can be chosen arbitrarily on $R\otimes S$, NO then $s_\beta \tau_1 = \tau(\alpha[\alpha,\beta])$ is determined. This determines τ_1 on [redacted]

$$\begin{tikzcd} R[R,S] \arrow[r, phantom, "\subset"] & RS[R,S] \\ R\otimes\bar{R}\otimes\bar{S} \arrow[u, phantom, "\wr"] \arrow[r, hook] & R\otimes S\otimes\bar{R}\otimes\bar{S} \arrow[u, phantom, "\wr"] \end{tikzcd}$$

The arbitrariness in τ_1 is an element of $(R\otimes\bar{S}\otimes\bar{R}\otimes\bar{S})^*$. Similarly the arbitrariness in τ_2 is a linear function on $R\otimes\bar{S}\otimes(\bar{R}\otimes\bar{S})^2$. This agrees with what we want on p. 192 in the augmented case except at the beginning.

We've made a mistake. It's necessary to check that $b_\alpha\tau_0$ satisfies the normalization conditions in order that it is in the form $s_\beta\tau_1$

for some τ_1. We need to have

$$b_\alpha \tau(\alpha\beta) = \tau(\alpha[\alpha,\beta])$$

Thus $(b_\alpha \tau_0)(x_1, x_2 y)$ must vanish if x_2 or $y = 1$. This is automatic for x_2, but holds for $y=1$ if $b_\alpha \tau(\alpha) = 0$, i.e. τ restricted to R is a trace.

Now the rest seems to be OKAY. Assuming τ_0 chosen such that $s_\beta b_\alpha \tau_0 = b_\alpha s_\beta \tau_0 = 0$, we can choose τ_1 such that $b_\alpha \tau_0 = s_\beta \tau_1$. Then $s_\beta(b_\alpha \tau_1) = b_\alpha s_\beta \tau_1 = b_\alpha^2 \tau_0 = 0$, so we can choose τ_2 with $b_\alpha \tau_1 = s_\beta \tau_2$, etc.

Next let us consider the b_β

$$b_\beta \tau(\alpha\beta[\alpha,\beta]^n) = \tau\{\alpha\beta^2[\alpha,\beta]^n - \alpha\beta(\beta[\alpha,\beta]^n - (-1)^n[\alpha,\beta]^n\beta)\} + \lambda_\beta \tau(\alpha\beta^2[\alpha,\beta]^n)$$

$$= (-1)^n \tau(\alpha\beta[\alpha\beta]^n\beta) + \lambda_\beta \tau(\alpha\beta^2[\alpha,\beta]^n)$$

$$(-1)^n(\tau b)(\alpha\beta[\alpha,\beta]^n d\beta) = (-1)^n \tau(\alpha\beta[\alpha,\beta]^n\beta) + \cancel{(-1)^n}\lambda_\beta \tau(\beta\alpha\beta[\alpha,\beta]^n).$$

$$\boxed{\begin{aligned} b_\beta \tau(\alpha\beta[\alpha,\beta]^n) - \lambda_\beta \tau([\alpha,\beta]\beta[\alpha,\beta]^n) \\ = (-1)^n(\tau b)(\alpha\beta[\alpha,\beta]^n d\beta) \end{aligned}}$$

But we have

$$K_\beta \tau(\beta[\alpha,\beta]^{n+1}) = \lambda_\beta(1 - b'_\beta s_\beta)\tau(\beta[\alpha,\beta]^{n+1})$$

$$= \lambda_\beta \tau\Big(\big(\underbrace{\beta[\alpha,\beta] - b'_\beta[\alpha,\beta]}_{\beta[\alpha,\beta] - [\alpha,\beta^2] = -[\alpha,\beta]\beta}\big)[\alpha,\beta]^n\Big)$$

$$= -\lambda_\beta \tau([\alpha,\beta]\beta[\alpha,\beta]^n)$$

Thus we have

$$\boxed{\begin{aligned} & b_\beta \tau(\alpha\beta[\alpha,\beta]^n) + K_\beta \tau(\beta[\alpha,\beta]^{n+1}) \\ & = (-1)^n (\tau b)(\alpha\beta[\alpha,\beta]^n d\beta) \end{aligned}}$$

Let's now use the formulas above to describe traces on $R * S$. Traces are linear functions τ on $R * S$ such that the cochains $\tau_n = \tau(\alpha\beta[\alpha,\beta]^n)$ $n \geq 0$ satisfy

$$\begin{cases} b_\alpha \tau_n = s_\beta \tau_{n+1} \\ b_\beta \tau_n = -K_\beta s_\alpha \tau_{n+1} \end{cases}$$

or equivalently

$$\circledast \quad \begin{cases} s_\alpha \tau_{n+1} = -K_\beta^{-1} b_\beta \tau_n \\ s_\beta \tau_{n+1} = b_\alpha \tau_n \end{cases}$$

Suppose $\tau_0, \ldots, \tau_m$ have been found verifying $\circledast$ where applicable, i.e. $n < m$. We would like to show τ_{m+1} can be found satisfying the [illegible] equations for $n = m$.

We first want to be able to solve

$$s_\alpha \tau_{n+1} = f \qquad \text{~~(degree n+1 in α, n+2 in β)~~}$$

$$s_\beta \tau_{n+1} = g \qquad \text{~~(degree n+2 in α, n+1 in β)~~}$$

~~Obvious necessary conditions are~~

~~$s_\alpha f = 0$ $\quad s_\beta g = 0$~~

~~$s_\beta f = s_\alpha g$~~

~~In fact these are sufficient because suppose we~~

Clearly f has to be a linear function on $S\otimes(\bar R\otimes\bar S)^{\otimes n+1}$ and g has to be a function on $R\otimes(\bar R\otimes\bar S)^{\otimes n+1}$.

If $s_\beta f = s_\alpha g$, on $(\bar R\otimes\bar S)^{\otimes n+1}$, then f and g fit together to give a linear function on $(1_R\otimes S + R\otimes 1_S)\otimes(\bar R\otimes\bar S)^{\otimes n+1} \subset R\otimes S\otimes(\bar R\otimes\bar S)^{\otimes n+1}$, which then extends to a linear function τ_{n+1} on $R\otimes S\otimes(\bar R\otimes\bar S)^{\otimes n+1}$ which is unique up to a linear fn. on $(\bar R\otimes\bar S)^{\otimes n+2}$.

So now suppose we have found $\tau_0,\dots,\tau_n$ and we want to solve

$$s_\alpha\tau_{n+1} = -\kappa_\beta^{-1}b_\beta\tau_n \quad \text{(call this } f)$$

$$s_\beta\tau_{n+1} = b_\alpha\tau_n \quad \text{(call this } g)$$

We first check whether $f\in(S\otimes(\bar R\otimes\bar S)^{\otimes n+1})^*$. Now $\tau_n\in(R\otimes S\otimes(\bar R\otimes\bar S)^{\otimes n})^*$, better say $\tau_n\in(\Omega^nR\otimes\Omega^nS)^*$ so $\kappa_\beta^{-1}b_\beta\tau_n\in(\Omega^nR\otimes\Omega^{n+1}S)^*$. We have to check that it belongs to $(\bar R^{\otimes n+1}\otimes S\otimes\bar S^{\otimes n+1})^*$, which means we want $s_\alpha\kappa_\beta^{-1}b_\beta\tau_n \overset{?}{=} 0$. But

$$s_\alpha\kappa_\beta^{-1}b_\beta\tau_n = \kappa_\beta^{-1}b_\beta\underbrace{s_\alpha\tau_n}_{-\kappa_\beta^{-1}b_\beta\tau_{n-1}} = 0$$

Similarly $g\in(R\otimes(\bar R\otimes\bar S)^{\otimes n+1})^*$. Next we have to check that $s_\beta f = s_\alpha g$

$$s_\beta f = -\kappa_\beta^{-1}s_\beta b_\beta\tau_n = -\kappa_\beta^{-1}(1-\kappa_\beta - b_\beta s_\beta)\tau_n$$

$$= (1-\kappa_\beta^{-1})\tau_n + \kappa_\beta^{-1}b_\beta b_\alpha\tau_{n-1}$$

$$s_\alpha g = s_\alpha b_\alpha \tau_n = (1 - K_\alpha - b_\alpha s_\alpha)\tau_n$$

$$= (1 - K_\alpha)\tau_n + \underbrace{b_\alpha K_\beta^{-1} b_\beta \tau_{n-1}}_{= K_\beta^{-1} b_\beta b_\alpha \tau_{n-1}}$$

Thus $s_\beta f = s_\alpha g$ is equivalent to $K_\alpha \tau_n = K_\beta^{-1} \tau_n$ or that

$$\boxed{K_\alpha K_\beta \tau_n = \tau_n}$$

At this point we see that the arbitrariness in the choice of τ_{n+1}, assuming we can find it invariant under $K_\alpha K_\beta$, is a linear function on $(\bar{R} \otimes \bar{S})^{\otimes (n+2)}$ invariant under the cyclic shift by 2 steps.

February 9, 1991

Recall the formulas

$$\boxed{\begin{aligned} & b_\alpha \tau(\alpha\beta[\alpha,\beta]^n) - s_\beta \tau(\alpha\beta[\alpha,\beta]^{n+1}) \\ & \quad = (-1)^n(\tau b)(\alpha\beta[\alpha,\beta]^n d\alpha) \\ \hline & b_\beta \tau(\alpha\beta[\alpha,\beta]^n) + \kappa_\beta s_\beta \tau(\alpha\beta[\alpha,\beta]^{n+1}) \\ & \quad = (-1)^n(\tau b)(\alpha\beta[\alpha,\beta]^n d\beta) \end{aligned}}$$

Let's describe linear functions on

$$\Omega^1(R*S)_\natural = (R*S)\otimes_R \Omega^1 R \otimes_R \quad \oplus \quad (R*S)\otimes_S \Omega^1 S \otimes_S$$

Note $(R*S)\otimes_R \Omega^1 R = (R*S)\otimes dR$ where $dR \simeq \bar{R}$ so the cochains $\alpha\beta[\alpha,\beta]^n d\alpha$ span this space. We have to describe the relation resulting from applying $\otimes_R$. Let T be a linear function on $(R*S)\otimes_R \Omega^1 R$; its equivalent to the cochains $T(\alpha\beta[\alpha,\beta]^n d\alpha)$, $n \geq 0$ which can be arbitrary subject to the evident normalization conditions.

T is a trace for the R-bimodule structure if

$$T(\alpha\beta[\alpha,\beta]^n d\alpha\,\alpha) \overset{?}{=} (-1)^n \lambda_\alpha T(\alpha\,\alpha\beta[\alpha,\beta]^n d\alpha)$$

But

$$\begin{aligned} b_\alpha T(\alpha\beta[\alpha,\beta]^n d\alpha) = T\{ & \alpha^2\beta[\alpha,\beta]^n d\alpha \\ & -\alpha\beta(\alpha[\alpha,\beta]^n - \cancel{(-1)^n[\alpha,\beta]^n}\alpha)d\alpha \\ & +(-1)^{n+1}\alpha\beta[\alpha,\beta]^n(\cancel{\alpha d\alpha} + d\alpha\,\alpha)\} \\ & + \lambda_\alpha T(\alpha^2\beta[\alpha,\beta]^n d\alpha) \end{aligned}$$

$$\begin{aligned} = T(\alpha[\alpha,\beta]^{n+1} d\alpha) & + (-1)^{n+1} T(\alpha\beta[\alpha,\beta]^n d\alpha\,\alpha) \\ & + \lambda_\alpha T(\alpha^2\beta[\alpha,\beta]^n d\alpha) \end{aligned}$$

Thus T is a linear function on $(R*S)\otimes_R \Omega^1 R \otimes_R$ iff

$$\boxed{b_\alpha T(\alpha\beta[\alpha,\beta]^n d\alpha) = s_\beta T(\alpha\beta[\alpha,\beta]^{n+1} d\alpha)}$$

Next

$$T(\alpha\beta[\alpha,\beta]^n d\beta\,\beta) \overset{?}{=} (-1)^n \lambda_\beta T(\beta\alpha\beta[\alpha,\beta]^n d\beta)$$

$$b_\beta T(\alpha\beta[\alpha,\beta]^n d\beta) = T\{\alpha(\beta^2\cancel{[\alpha,\beta]^n} - \beta(\cancel{\beta[\alpha,\beta]^n} - (-1)^n \cancel{[\alpha,\beta]^n\beta}))d\beta$$

$$+ (-1)^{n+1}\alpha\beta[\alpha,\beta]^n(\cancel{\beta d\beta} + d\beta\,\beta)\}$$

$$+ \lambda_\beta T(\alpha\beta^2[\alpha,\beta]^n d\beta)$$

$$= (-1)^{n+1} T(\alpha\beta[\alpha,\beta]^n d\beta\,\beta) + \lambda_\beta T([\alpha,\beta]\beta[\alpha,\beta]^n d\beta)$$

$$+ \lambda_\beta T(\beta\alpha\beta[\alpha,\beta]^n d\beta)$$

Thus T is a trace on $(R*S)\otimes_S \Omega^1 S \otimes_S$ iff

$$b_\beta T(\alpha\beta[\alpha,\beta]^n d\beta) = \lambda_\beta T([\alpha,\beta]\beta[\alpha,\beta]^n d\beta)$$

$$[\alpha,\beta]\beta = \beta[\alpha,\beta] - [\alpha,\beta^2] = (1 - b'_{\beta\beta} s)(\beta[\alpha,\beta])$$

Thus T is a trace on $(R*S)\otimes_S \Omega^1 S \otimes_S$ iff

$$\boxed{b_\beta T(\alpha\beta[\alpha,\beta]^n d\beta) = -K_\beta s_\alpha T(\alpha\beta[\alpha,\beta]^{n+1} d\beta)}$$

Let us now [redacted] discuss the homotopy formula. Suppose $T \in ((R*S)\otimes_R \Omega^1 R \otimes_R)^*$. We have

$$(Td)(\alpha\beta[\alpha,\beta]^{n+1}) = T(d\alpha\,\beta[\alpha,\beta]^{n+1})$$

$$+ \sum_{j=0}^{n} T(\alpha\beta[\alpha,\beta]^{n-j}[d\alpha,\beta][\alpha,\beta]^j)$$

Let's see if the terms in this sum are relate by $K_\alpha K_\beta$ -

$$(\kappa_\alpha \kappa_\beta)^j \; T(\alpha\beta[\alpha,\beta]^n[d\alpha,\beta])$$

$$= (\lambda_\alpha \lambda_\beta)^j (1 - b_{\alpha j} s_\alpha)(1 - b_{\beta j} s_\beta) \; T(\alpha\beta[\alpha,\beta]^n[d\alpha,\beta])$$

Now $(1 - b_{\beta j} s_\beta)\; \alpha\beta[\alpha,\beta]^j = \alpha\beta[\alpha,\beta]^j - b_\beta \alpha[\alpha,\beta]^j$

$$= \alpha\left(\beta[\alpha,\beta]^j - \beta[\alpha,\beta]^j + (-1)^j[\alpha,\beta]^j\beta\right)$$

$$= (-1)^j \alpha[\alpha,\beta]^j \beta$$

so $(1 - b_{\alpha j} s_\alpha)(1 - b_{\beta j} s_\beta)(\alpha\beta[\alpha,\beta]^j) = (-1)^j(1 - b_{\alpha j} s_\alpha)\; \alpha[\alpha,\beta]^j\beta$

$$= [\alpha,\beta]^j \alpha \beta.$$

Thus

$$\boxed{\begin{aligned} &(\kappa_\alpha \kappa_j)^j T(\alpha\beta[\alpha,\beta]^n[d\alpha,\beta]) \quad \text{[redacted]} \\ &= (\lambda_\alpha\lambda_\beta)^j T([\alpha,\beta]^j \alpha\beta[\alpha,\beta]^{n-j}[d\alpha,\beta]) \\ &= T(\alpha\beta[\alpha,\beta]^{n-j}[d\alpha,\beta][\alpha,\beta]^j) \end{aligned}}$$

Next

$$b_\beta T(\alpha\beta[\alpha,\beta]^n d\alpha) = T\{\alpha\cancel{\beta^2[\alpha,\beta]^n} d\alpha$$

$$-\alpha\beta(\beta\cancel{[\alpha,\beta]^n} - (-1)^n[\alpha,\beta]^n\beta)\, d\alpha\}$$

$$+ \lambda_\beta T(\alpha\beta^2[\alpha,\beta]^n d\alpha)$$

$$= (-1)^n T(\alpha\beta[\alpha,\beta]^n \beta d\alpha)$$

$$+ \underbrace{\lambda_\beta T(\beta\alpha\beta[\alpha,\beta]^n d\alpha)}_{(-1)^{n+1}T(\alpha\beta[\alpha,\beta]^n d\alpha\beta)} + \underbrace{\lambda_\beta T([\alpha,\beta]\beta[\alpha,\beta]^n d\alpha)}_{-\kappa_\beta T(\beta[\alpha,\beta]^{n+1} d\alpha)}$$

$$\boxed{\begin{aligned} &(-1)^{n+1} T(\alpha\beta[\alpha,\beta]^n[d\alpha,\beta]) \\ &= b_\beta T(\alpha\beta[\alpha,\beta]^n d\alpha) + \kappa_\beta s_\alpha T(\alpha\beta[\alpha,\beta]^{n+1} d\alpha) \end{aligned}}$$

So

$$(-1)^{n+1}(Td)(\alpha\beta[\alpha,\beta]^{n+1}) =$$ [redacted]

$$(-1)^{n+1}\sum_{j=0}^{n}(K_\alpha K_\beta)^j\, T(\alpha\beta[\alpha,\beta]^n[d\alpha,\beta]) + (-1)^{n+1}T(d\alpha\beta[\alpha,\beta]^{n+1})$$

$$= \sum_{j=0}^{n}(K_\alpha K_\beta)^j\left(b_\beta T(\alpha\beta[\alpha,\beta]^n d\alpha) + K_\beta s_\alpha T(\alpha\beta[\alpha,\beta]^{n+1}d\alpha)\right)$$

$$+ K_\alpha^{-1}s_\alpha T(\alpha\beta[\alpha,\beta]^{n+1}d\alpha)$$

Multiply by K_β^{-1} to obtain

$$\boxed{\begin{aligned}&(-1)^{n+1}K_\beta^{-1}(Td_\alpha)(\alpha\beta[\alpha,\beta]^{n+1})\\ &= \sum_{j=0}^{n}(K_\alpha K_\beta)^j(K_\beta^{-1}b_\beta)T(\alpha\beta[\alpha,\beta]^n d\alpha)\\ &\quad + \left(\sum_{j=0}^{n}(K_\alpha K_\beta)^j + (K_\alpha K_\beta)^{-1}\right)s_\alpha T(\alpha\beta[\alpha,\beta]^{n+1}d\alpha)\end{aligned}}$$

Next

$$(Td_\beta)(\alpha\beta[\alpha,\beta]^{n+1}) = \sum_{j=0}^{n}T(\alpha\beta[\alpha,\beta]^{n-j}[\alpha,d\beta][\alpha,\beta]^j)$$

$$+ T(\alpha d\beta[\alpha,\beta]^{n+1})$$

$$b_\alpha T(\alpha\beta[\alpha,\beta]^n d\beta) = T\{\alpha^2\beta[\alpha,\beta]^n d\beta$$

$$- \alpha\beta(\alpha[\alpha,\beta]^n - (-1)^n[\alpha,\beta]^n\alpha)d\beta\}$$

$$+ \lambda_\alpha T(\alpha^2\beta[\alpha,\beta]^n d\beta)$$

$$= T(\alpha[\alpha,\beta]^{n+1}d\beta)$$

$$+ (-1)^n T(\alpha\beta[\alpha,\beta]^n\alpha d\beta) + (-1)^{n+1}T(\alpha\beta[\alpha,\beta]^n d\beta\alpha)$$

$\therefore$

$$\boxed{\begin{aligned}&(-1)^n T(\alpha\beta[\alpha,\beta]^n[\alpha, d\beta])\\ &= b_\alpha T(\alpha\beta[\alpha,\beta]^n d\beta) - s_\beta T(\alpha\beta[\alpha,\beta]^{n+1} d\beta)\end{aligned}}$$

$$\begin{aligned}&(-1)^n (T d_\beta)(\alpha\beta[\alpha,\beta]^{n+1})\\ &= \sum_{j=0}^{n} (\kappa_\alpha \kappa_\beta)^j \, b_\alpha T(\alpha\beta[\alpha,\beta]^n d\beta)\\ &\quad - \sum_{j=0}^{n} (\kappa_\alpha \kappa_\beta)^j \, s_\beta T(\alpha\beta[\alpha,\beta]^{n+1} d\beta)\\ &\quad + (-1)^n T(\alpha\, d\beta[\alpha,\beta]^{n+1})\end{aligned}$$

February 10, 1991

Problem. $A = R/I$ quasifree, I nilpotent M = space of liftings $\theta : A \to R$. To produce a complex of complexes

$$(*) \qquad X(A) \longrightarrow \Omega(M, X(R))$$

extending the obvious map $X(A) \to \Omega^0(M, X(R))$, which associates to [redacted] $\theta \in M$ the induced map $X(\theta)$ on X-complexes. (p 188-189)

Since M is "nonsingular" and "contractible" (retract of affine space) the map $(*)$ expresses precisely the idea that there is a map $X(A) \to X(R)$ unique up to higher homotopy.

Unlike general higher homotopy situations where one has points, paths, etc, here one has two special features: 1) Linearity: Points are replaced by distributions, paths by 1-currents, etc. One is dealing with an abelian (or homology) situation. 2) Differentiability: It suffices to construct $(*)$ infinitesimally at each point. We can work with forms (commutative cochains)

(This last point might not be important in the noncommutative setting.)

Note that $A \longrightarrow \Omega^0(M, R)$ is roughly of the form

$$A \longrightarrow S \otimes R \qquad\qquad S = \Omega^0(M)$$

which leads to

$$\Gamma A \longrightarrow \Gamma(S \otimes R) \;\overset{\sim}{[redacted]}\; \Gamma(S) \boxtimes \Gamma(R)$$

where Γ denotes cyclic bicomplex and $\boxtimes$ is some [redacted] tensor product over $\mathbb{C}[u]$ and the homotopy equivalence is the Kunneth formula of Jones - Kassel.

Point: Close connection between Kunneth and bivariant constructions

Special case of the effect of $A \longrightarrow S \otimes R$ is

$$\Omega A \longrightarrow \Omega S \otimes R_\natural \longrightarrow \Omega(M) \otimes R_\natural$$

which we have mentioned before.

Strategy. You have $X(A) \longrightarrow \Omega^{\leq 1}(M, X(R))$ and want to get to Ω^2 which requires looking at a 2-parameter family $\theta_{st} : A \longrightarrow R$. I think it suffices to work to first order in s, t and the take R to be the universal algebra for $(\theta, \dot{\theta}, \theta')$ where $\dot{\theta}$ and θ' are two variations. This we have described already at least the degree 1,1 part. We now have to compute the relevant part of the X complex for this algebra.

It seems we will end up computing multiple first order variation algebras and their traces. Is there a link with with your Fedosov proof?

February 15, 1991

If $A = R \otimes S$ there is an important square zero extension $R * S / I^2$, where $I = \operatorname{Ker}\{R*S \to R\otimes S\}$. This extension comes with a linear lifting $x\otimes y \overset{\rho}{\mapsto} xy$, hence we have a map

$$X^1(RA, IA) \longrightarrow X^1(R*S/I^2, I/I^2) \simeq X(R)\otimes X(S)$$

$$\begin{pmatrix} \begin{matrix} A \\ \oplus \\ \Omega^2 A_\natural \end{matrix} & \underset{(B\ \ -\text{Б})}{\overset{\binom{b}{-\natural B}}{\rightleftarrows}} & \Omega^1 A \end{pmatrix} \longrightarrow \begin{pmatrix} \begin{matrix} R\otimes S \\ \oplus \\ \Omega^1 R_\natural \otimes \Omega^1 S_\natural \end{matrix} & \rightleftarrows & \begin{matrix} [redacted]\ \Omega^1 R_\natural \otimes S \\ \oplus \\ R\otimes \Omega^1 S_\natural \end{matrix} \end{pmatrix}$$

The problem is that the map $\Omega^1 A \longrightarrow \Omega^1 R_\natural \otimes S$ is ugly: $(x_0\otimes y_0)\, d(x_1\otimes y_1) \xmapsto{\rho d\rho} x_0 y_0\, d(x_1 y_1) \longmapsto x_0 dx_1 \otimes \underline{y_1 y_0}$ (note: reversal)

Let's begin by looking at the homomorphism

$$RA/IA^2 \longrightarrow R*S/I^2$$

Both of these are square zero extensions with linear lifting, hence they are equivalent to 2-cocycles with values in bimodules. These 2-cocycles in both cases are cup products of 1-cocycles, i.e. derivations. We have derivations

$$R\otimes S \longrightarrow \Omega^1(R\otimes S; S) \simeq \Omega^1 R\otimes S$$

$$d\otimes 1 : \quad x\otimes y \longmapsto dx\otimes y$$

$$R\otimes S \longrightarrow \Omega^1(R\otimes S; R) \simeq R\otimes \Omega^1 S$$

$$1\otimes d : \quad x\otimes y \longmapsto x\otimes dy$$

The cup product is

$$(d\otimes 1 \cup 1\otimes d)(x_1\otimes y_1, x_2\otimes y_2)$$

$$= (d\otimes 1)(x_1\otimes y_1)\cdot(1\otimes d)(x_2\otimes y_2)$$

$$= (dx_1\otimes y_1)\cdot(x_2\otimes dy_2) \in (\Omega^1R\otimes S)\otimes_{(R\otimes S)}(R\otimes\Omega^1S)$$

$$= dx_1x_2\otimes y_1dy_2 \in \Omega^1R\otimes\Omega^1S$$

similarly

$$(1\otimes d \cup d\otimes 1)(x_1\otimes y_1, x_2\otimes y_2)$$

$$= (x_1\otimes dy_1)\cdot(dx_2\otimes y_2)$$

$$= x_1dx_2\otimes dy_1y_2 \in \Omega^1R\otimes\Omega^1S$$

These two cup products are cohomologous up to sign: If $f(x\otimes y) = dx\otimes dy$, then

$$-(\delta f)(x_1\otimes y_1, x_2\otimes y_2) =$$

$$d(x_1x_2)\otimes d(y_1y_2) - x_1dx_2\otimes y_1dy_2 - dx_1x_2\otimes dy_1y_2$$

$$= dx_1x_2\otimes y_1dy_2 + x_1dx_2\otimes dy_1y_2$$

These 2-cocycles correspond to the two liftings $R\otimes S \rightrightarrows R*S/I$, $x\otimes y \longmapsto xy$ or yx.
The curvatures are

$$x_1x_2y_1y_2 - x_1y_1x_2y_2 = x_1[x_2,y_1]y_2$$

$$y_1y_2x_1x_2 - y_1x_1y_2x_2 = y_1[y_2,x_1]x_2$$

One has the isomorphism

$$\boxed{\begin{array}{lcl} R\otimes S \oplus \Omega^1R\otimes\Omega^1S & \xrightarrow{\sim} & R*S/I^2 \\ x\otimes y & \longmapsto & xy \\ x_1dx_2\otimes dy_1y_2 & \longmapsto & x_1[x_2,y_1]y_2 \end{array}}$$

and the product in $R*S/I^2$ is given by

$$(x_1\otimes y_1)*(x_2\otimes y_2) = x_1x_2\otimes y_1y_2 - x_1dx_2\otimes dy_1y_2$$

Before leaving $R*S/I^2$ let's describe the isomorphism

$$X^1(R*S,I) \cong X(R)\otimes X(S)$$

(see also)

$$\boxed{\begin{array}{ccc} R\otimes S \;\oplus\; \Omega^1R_\natural\otimes\Omega^1S_\natural & \xrightarrow{\sim} & R*S/I^2+[R*S,I] \\ x\otimes y\,,\; x_1dx_2\otimes dy_1y_2 & \longmapsto & xy\,,\; x_1[x_2,y_1]x_2 \end{array}}$$

$$\boxed{\begin{array}{ccc} \Omega^1R_\natural\otimes S \;\oplus\; R\otimes\Omega^1S_\natural & \xrightarrow{\sim} & \Omega^1(R*S)/[R*S,\Omega^1(R*S)]+I\Omega^1(R*S) \\ x_1dx_2\otimes y\,,\; x\otimes dy_1y_2 & \longmapsto & x_1dx_2y\,,\; xdy_1y_2 \end{array}}$$

$$\begin{aligned} b(x_1dx_2\otimes y) &= b(x_1d(x_2y) - x_1x_2\,dy) \\ &= [x_1,x_2y] - [x_1x_2,y] \\ &= [x_1,x_2]y + \underbrace{x_2[x_1,y] - [x_1,y]x_2}_{\equiv 0 \text{ mod } [R*S,I]} - x_1[x_2,y] \end{aligned}$$

$$\boxed{b(x_1dx_2\otimes y) = b(x_1dx_2)\otimes y - (x_1dx_2)\otimes dy}$$

similarly

$$\boxed{b(x\otimes dy_1y_2) = dx\otimes dy_1y_2 + x\otimes b(dy_1y_2)}$$

$$\boxed{\begin{aligned} d(x\otimes y) &= dx\otimes y + x\otimes dy \\ d(x_1dx_2\otimes dy_1y_2) &= b(x_1dx_2)\otimes dy_1y_2 - (x_1dx_2)\otimes b(dy_1y_2) \end{aligned}}$$

Another point about $R\otimes S$ is that we have a canonical homom. of DG algebras

$$\Omega(R\otimes S) \longrightarrow \Omega(R)\otimes\Omega(S)$$

which sends

$$\Omega^1(R\otimes S) \longrightarrow (\Omega^1R\otimes S) \oplus (R\otimes\Omega^1S)$$

$$(x_1\otimes y_1)\,d(x_2\otimes y_2) \longmapsto x_1dx_2\otimes y_1y_2 \oplus x_1x_2\otimes y_1dy_2$$

and $\Omega^2(R\otimes S) \to (\Omega^2R\otimes S)\oplus(\Omega^1R\otimes\Omega^1S)\oplus(R\otimes\Omega^2S)$

$$(x_0\otimes y_0)\,d(x_1\otimes y_1)\,d(x_2\otimes y_2) \longmapsto (x_0\otimes y_0)(dx_1\otimes y_1+x_1\otimes dy_1)(dx_2\otimes y_2+x_2\otimes dy_2)$$

$$= (x_0dx_1dx_2\otimes y_0y_1y_2)$$

$$+(x_0dx_1x_2\otimes y_0y_1dy_2) - (x_0x_1dx_2\otimes y_0dy_1y_2)$$

$+$ [illegible] $x_0x_1x_2\otimes y_0dy_1dy_2$

Let us now consider the extension $RA/IA^2 \simeq A\oplus\Omega^2A$ with $a_1 * a_2 = a_1a_2 - da_1da_2$. We have the following description of $X^1(RA,IA)$:

$$\begin{tikzcd} A\oplus\Omega^2A_\natural \arrow[r, shift right, "(B\ \ -\bar{b})"'] & \Omega^1A \arrow[l, shift right, "\begin{pmatrix} b \\ -4B \end{pmatrix}"'] \end{tikzcd}$$

Here the funny part is perhaps the isomorphism

$$\rho d\rho \ :\ \Omega^1A \xrightarrow{\sim} \Omega^1RA\big/[RA,\Omega^1RA]+IA\,\Omega^1RA$$

which leads in the case $A = R\otimes S$ to the funny map

$$\Omega^1(R\otimes S)\longrightarrow\Omega^1R\otimes S \qquad x_1\otimes y_1\,d(x_2\otimes y_2)\longmapsto x_1dx_2\otimes y_2y_1$$

February 16, 1991

Let's review the equivalence between square zero algebra extensions ~~of~~ A and bimodule extensions of $\Omega^1 A$.

Suppose given an algebra extension $A = R/I$ and a bimodule extension $E \xrightarrow{f} \Omega^1 A$. Let us associate to the pair $R \overset{\pi}{\twoheadrightarrow} A$, $E \xrightarrow{f} \Omega^1 A$ the space of derivations $D: R \longrightarrow E$, where E is regarded as A-bimodule via π, such that

$$* \qquad \begin{tikzcd} R \arrow[r, "\pi"] \arrow[d, "D"] & A \arrow[d, "d"] \\ E \arrow[r, "f"] & \Omega^1 A \end{tikzcd}$$

commutes. Call this space $H(R,E)$. I claim we have $\boxed{\text{adjoint functors}}$

$$H(R,E) = \mathrm{Hom}(R, G(E)) = \mathrm{Hom}(F(R), E)$$

where $G(E) = A \times_{\Omega^1 A} E = \{(a,\xi) \mid D\xi = da\}$

and $F(R) = A \otimes_R \Omega^1 R \otimes_R A = \Omega^1 R / F_I^1(\Omega^1 R)$

Note that $*$ is equivalent to a commutative diagram of homomorphisms

$$\begin{tikzcd} R \arrow[r, "\pi"] \arrow[d, "\pi+D"] & A \arrow[d, "1+d"] \\ A \oplus E \arrow[r, "1+f"] & A \oplus \Omega^1 A \end{tikzcd}$$

i.e. to a homom.

$R \longrightarrow (A$ [illegible] $) \times_{(A\oplus\Omega^1 A)} (A \oplus E) = A \times_{\Omega^1 A} E$

If D is as in $*$, then one has

$$\begin{tikzcd} A\otimes_R \Omega^1 R\otimes_R A \arrow[d, "\tilde{D}"] \arrow[dr, bend left, "\tilde{d\pi}"] & \\ E \arrow[r, "p"] & \Omega^1 A \end{tikzcd}$$

which is a map of bimodule extensions $F(R) \longrightarrow E$ of $\Omega^1 A$.

So we haven't used the interesting point which is the injectivity part of the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow A\otimes_R \Omega^1 R \otimes_R A \longrightarrow \Omega^1 A \longrightarrow 0$$

This tells us that $GF(R)$ is the extension R/I^2. ~~It should be obvious that $FG(E) \xrightarrow{\sim} E$; actually this follows from the right exactness~~ Also from

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & FG(E) \arrow[r] \arrow[d] & \Omega^1 A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r] & E \arrow[r] & \Omega^1 A \arrow[r] & 0 \end{tikzcd}$$

we conclude $FG(E) \xrightarrow{\sim} E$. Thus we obtain an equivalence between the categories of square zero extensions of A and bimodule extensions of $\Omega^1 A$.

Let's now identify the bimodule extension associated to the universal extension RA. We have $\Omega^1 RA = RA \otimes \bar{A} \otimes RA$, $x\,d(\rho a)y \leftrightarrow x\otimes a\otimes y$

so $A \otimes_{RA} \Omega^1 RA \otimes_R A = A\otimes \bar{A} \otimes A \simeq \Omega^1 A$

Consider

$$\omega(a_1,a_2) \longmapsto \rho(a_1a_2) - \rho a_1 \rho a_2$$

$$\begin{tikzcd}
0 \arrow[r] & IA/IA^2 \arrow[r] \arrow[d, "\cong"] & RA/IA^2 \arrow[r] \arrow[d, "D"] & A \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \Omega^2 A \arrow[r] & \Omega^1 A \otimes A \arrow[r, "m"] & \Omega^1 A \arrow[r] & 0
\end{tikzcd}$$

$$\rho a_0\, \omega(a_1,a_2) \longmapsto a_0 da_1 da_2$$

$$a_0 da_1 da_2 \longmapsto a_0 da_1 (a_2 \otimes 1 - 1 \otimes a_2)$$

where D is the derivation such that $D\rho a = da \otimes 1$. Then

$$D\omega(a_1,a_2) = d(a_1a_2)\otimes 1 - (da_1 \otimes 1)a_2 - a_1(da_2 \otimes 1)$$

[redacted]

$$= da_1(a_2 \otimes 1 - 1 \otimes a_2)$$

We know that $\tilde{D} : \Omega^1 RA / F^1_{IA}\Omega^1 RA \xrightarrow{\sim} \Omega A \otimes A$, so we can conclude $IA/IA^2 \xrightarrow{\sim} \Omega^2 A$ (which we already know).

We have learned that RA/IA^2 is the square zero extension associated to the bimodule extension $A \otimes \bar{A} \otimes A \twoheadrightarrow \Omega^1 A$. Thus homomorphisms of square zero extensions $RA/IA^2 \longrightarrow$ [redacted] S $\qquad S/J = A$ [redacted], $J^2 = 0$

(which are equivalent to linear liftings in $S \to A$) are equivalent to linear liftings of $dA \subset \Omega^1 A$ in the corresponding bimodule extension. This is also clear from the formula:

$$S = A \times_{\Omega^1 A} E$$

Next let us consider $R * S \xrightarrow{\pi} R \otimes S$, so we have a canonical square zero extension $R \otimes S = R * S / I^2$, where $I = \operatorname{Ker} \pi$. We have

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, equal] & R*S/I^2 \arrow[r] \arrow[d] & R\otimes S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I/I^2 \arrow[r, "d"] \arrow[d, "s"] & (R\otimes S)\otimes_{R*S}\Omega^1(R*S)\otimes_{R*S}(R\otimes S) \arrow[r] \arrow[d, equal] & \Omega^1(R\otimes S) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1 R\otimes\Omega^1 S \arrow[r] & \Omega^1 R\otimes(S\otimes S) \oplus (R\otimes R)\otimes\Omega^1 S \arrow[r] & \Omega^1(R\otimes S) \arrow[r] & 0
\end{tikzcd}$$

Take $[x,y] \in I/I^2$, then $\quad [x,y] \longleftrightarrow dx\otimes dy$

$$d[x,y] = [dx,y] + [x,dy]$$

$$\longrightarrow \begin{matrix} dx\otimes(1\otimes y) \\ -dx\otimes(y\otimes 1)\end{matrix} + \begin{matrix}(x\otimes 1)\otimes dy \\ -(1\otimes x)\otimes dy\end{matrix}$$

$$= -dx\otimes(y\otimes 1-1\otimes y) + (x\otimes 1-1\otimes x)\otimes dy$$

Thus we can identify

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] & \Omega^1(R*S)/F^1_I(R*S) \arrow[r] & \Omega^1(R\otimes S) & & \\
& & & \cap & & \\
& & & R\otimes R\otimes S\otimes S \arrow[r] & R\otimes S \arrow[r] & 0
\end{tikzcd}$$

with the tensor product of complexes

$$(\Omega^1 R \longrightarrow R\otimes R)\otimes(\Omega^1 S \longrightarrow S\otimes S)$$

So far I am discussing a square zero extension canonically associated to the tensor product $R\otimes S$. I have described the corresponding bimodule extension of $\Omega^1(R\otimes S)$.

Next I can consider $X^1(R*S, I) = X^1(R*S/I^2, I/I^2)$. This is a $\mathbb{Z}/2$ graded complex canonically associated to the square zero extension $R*S/I^2$ of $R\otimes S$. Assuming R, S are quasi free, we know this complex gives the periodic homology of $R\otimes S$.

Actually there seems to be the general question of how best to represent the periodic homology of an algebra of projective dimension 2. What you seem to be doing is to take a versal square zero extension and the [illegible] associated X' complex. Let's continue with $R \otimes S$ and return to this idea later.

In this case we can identify $X'(R * S, I)$ with $X(R) \otimes X(S)$. This seems to depend upon a linear lifting $R \otimes S \rightarrow R * S$, however one should check carefully.

February 17, 1991

Problem: How to handle the periodic homology of an algebra of projective dim ≤ 2?

We know that the X complex gives the periodic homology for dim $\leqslant 1$. However there is more than this statement - there should be something one can say about functorality, namely if A is quasi-free and R is a nilpotent extension of A, then we obtain a map unique up to homotopy from $X(A)$ to the standard $b+B$ complex giving the periodic homology of R, in fact to any complex giving the periodic homology. [redacted]

I guess the thing I would like to understand is the sort of data that might [redacted] link an algebra and a $\mathbb{Z}/2$ graded complex together in such a way that one could say the latter gives the cyclic homology of the former. For example in the case of a smooth commutative algebra one ~~knows~~ [redacted] the even-odd de Rham complex gives the periodic homology. In this example there is a map [redacted]

$$(\hat{\Omega} A, b+B) \xrightarrow{\mu} (\Omega_A^{\pm}, d)$$

which is a quis. I guess I would like to be able to construct an inverse map in this case.

[redacted] Connections in some generalized sense seem to be the sort of things one is after.

Example. Recall that a connection $\nabla : M \longrightarrow M \otimes_R \Omega^1 R$ in a bimodule M over R extends to an operator $\nabla : M \otimes_R \Omega R \longrightarrow M \otimes_R \Omega R$ of degree $+1$ such that

$$\nabla(\xi \omega) = (\nabla \xi)\, \omega + (-1)^{|\xi|} \xi\, d\omega \qquad \xi \in M \otimes_R \Omega^1 R, \quad \omega \in \Omega R$$

and

$$\nabla(x\xi) = x\nabla\xi \qquad \begin{array}{l} x \in R \\ \xi \in M\otimes_R \Omega^1 R \end{array}$$

Thus [redacted] $(-1)^k\nabla : M\otimes_R \Omega^k R \longrightarrow M\otimes_R \Omega^{k+1} R$ gives a connection in $M\otimes_R \Omega^k R$.

Also recall that one has

$$b\nabla + \nabla b = 1 \quad \text{on } M\otimes_R \Omega^{k+1} R$$

and there is a unique [redacted] map $l : M_\natural \longrightarrow M$ given by $l\natural = 1 - b\nabla$ on M, and $\natural l = 1$ on $M_\natural$.

Thus a connection [redacted] in $\Omega^n R$ gives a SDR of the b-complex

$$\begin{tikzcd}
{} \arrow[r, "b"] & \Omega^{n+1}R \arrow[l, bend right, "\nabla"'] \arrow[r, "b"] & \Omega^n R \arrow[l, bend right, "\nabla"'] \arrow[r, "b"] \arrow[d, shift right, "\natural"'] & \Omega^{n-1}R \arrow[r] \arrow[d, equal] & {} \\
 & & \Omega^n R_\natural \arrow[u, shift right, "l"'] \arrow[r, "\bar b"] & \Omega^{n-1}R \arrow[r] & {}
\end{tikzcd}$$

Then by HPT one obtains a [redacted] [redacted] lifting of the $\mathbb{Z}/2$ graded complex associated to the truncated $b + B$ complex into the whole one.

Thus a connection in the bimodule $\Omega^n R$ (which exists [redacted] iff this bimodule is projective) leads to a lifting of the truncated $b+B$ complex into the big one, which is in fact an SDR situation.

Example: Suppose A is separable, i.e. projective dim 0. Here we [redacted] know the periodic homology is given by $A_\natural[0]$. What is a connection in A? It is an operator $\nabla : A \longrightarrow \Omega^1 A$ satisfying

$$\nabla(a_1 a_2) = a_1 \nabla a_2$$

$$\nabla(a_1, a_2) = (\nabla a_1) a_2 + \text{[redacted]}\ a_1 da_2$$

Thus if $Y = \nabla(1)$ we have

$$\nabla a = \nabla(a1) = aY$$

$$= \nabla(1a) = Ya + da$$

so [redacted] $da = [a, Y]$ $\forall a \in A$.

In general we have

$$\boxed{\nabla \omega = (d + Y)\omega \qquad \omega \in \Omega A}$$

Notice this implies

$$\omega = (b\nabla + \nabla b)\omega = (bd + db)\omega + b(Y\omega) + Y(b\omega)$$

$$= \omega - K\omega + b(Y\omega) - Y(b\omega).$$

$$\therefore \boxed{K\omega = b(Y\omega) - Y(b\omega) \qquad |\omega| \geq 1}$$

Subexample: $A = \mathbb{C}[F]$, $Y = \frac{1}{2} F dF$

Notice that the curvature of the connection in this case is

$$dY + Y^2 = \frac{1}{2} dF^2 + \frac{1}{4} FdFFdF$$

$$\boxed{dY + Y^2 = \frac{1}{4} dF^2}$$

Let's review the stuff on $R\otimes S$.

The first point is that we have the canonical extension $R\otimes S = R*S/I$.

This gives a canonical square zero extension

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, equal] & R*S/I^2 \arrow[r] \arrow[d] & R\otimes S \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & I/I^2 \arrow[r] & \Omega^1(R*S)/F^1_I\Omega^1(R*S) \arrow[r] & \Omega^1(R\otimes S) \arrow[r] & 0
\end{tikzcd}$$

Next we have a canonical isom.

$$\Omega^1(R*S)/F^1_I\Omega^1(R*S) = (R\otimes S)\otimes_{(R*S)}\Omega^1(R*S)\otimes_{(R*S)}(R\otimes S)$$

$$* \qquad = (R\otimes S)\otimes_R\Omega^1R\otimes_R(R\otimes S)\oplus(R\otimes S)\otimes_S\Omega^1S\otimes_S(R\otimes S)$$

$$= \Omega^1R\otimes(S\otimes S)\quad\oplus\quad(R\otimes R)\otimes\Omega^1S$$

which induces a canonical isomorphism

$$\begin{tikzcd}
I/I^2 \arrow[r] \arrow[d, "\simeq"'] & R*S/I^2 \arrow[d] \\
\Omega^1R\otimes\Omega^1S \arrow[r, "-1\otimes\partial+\partial\otimes 1"] & \Omega^1R\otimes(S\otimes S)\oplus(R\otimes R)\otimes\Omega^1S
\end{tikzcd}$$

$$\begin{tikzcd}
{[x,y]} \arrow[r, mapsto, "d"] \arrow[d, leftrightarrow] & {[dx,y]+[x,dy]\in\Omega^1(R*S)/F^1} \arrow[d, leftrightarrow] \\
dx\otimes dy \arrow[r, mapsto] & -dx\otimes(y\otimes 1-1\otimes y)+(x\otimes 1-1\otimes x)\otimes dy
\end{tikzcd}$$

The canonical isom $*$ induces one

$$*_\natural : (\Omega^1(R*S)/F^1)_\natural \xrightarrow{\sim} \Omega^1R_\natural\otimes S\oplus R\otimes\Omega^1S_\natural$$

Next we claim there is an isomorphism of complexes

$$X'(R*S, I) \xrightarrow{\ \sim\ } X(R)\otimes X(S)$$

$$R*S/I^2+[\overset{R*S}{\vphantom{x}},I] \xrightarrow{\ \sim\ } R\otimes S \ \oplus\ \Omega^1 R_\natural \otimes \Omega^1 S_\natural$$

$$\left(\Omega^1(R*S)/F^1\right)_\natural \underset{(*_\natural)}{\xrightarrow{\ \sim\ }} \Omega^1 R_\natural \otimes S \ \oplus\ R\otimes \Omega^1 S_\natural$$

Here the isomorphism is slightly less canonical, because it depends on choosing one of the liftings $x\otimes y \longmapsto xy,\ yx$ of $R\otimes S$ into $R*S/I^2$. Let's check this carefully. Let's first compute ~~the~~ ambiguity. The problem is we have to split

$$0 \longrightarrow I/I^2+[R*S,I] \longrightarrow R*S/I^2+[R*S,I] \longrightarrow R\otimes S \to 0$$

$$\begin{matrix} I/I^2+[R*S,I] \\ \wr\Vert \\ \Omega^1 R_\natural \otimes \Omega^1 S_\natural \end{matrix}$$

The ~~two~~ difference of the liftings is $x\otimes y \longrightarrow xy-yx \Longleftrightarrow dx\otimes dy$

Next let's compute differentials. First b

$x_1 dx_2 \otimes y$ lifts to $x_1 dx_2\, y \in \left(\Omega^1(R*S)/F^1\right)_\natural$

$$\xmapsto{b} b(yx_1 dx_2) = [yx_1, x_2] = y[x_1,x_2] + [y,x_2]x_1$$

$$= [x_1,x_2]\,y - x_1[x_2,y]$$

$$\Longleftrightarrow b(x_1dx_2)\otimes y - x_1dx_2\otimes dy$$

$x\otimes dy_1 y_2$ lifts to $x\,dy_1 y_2 = x\,d(y_1y_2) - xy_1dy_2$

$$\xrightarrow{b} [x,y_1y_2]-[xy_1,y_2] = [x,y_1]y_2 + \cancel{y_1[x,y_2]} - x[y_1,y_2] - \cancel{[x,y_2]y_1}$$

$$\Longleftrightarrow +x\otimes b(dy_1y_2) + dx\otimes dy_1y_2$$

Notice that $d \otimes d$ kills

$$b(x_1 dx_2) \otimes y \quad \text{and} \quad x \otimes b(dy_1 y_2)$$

so these differentials are not affected by the ambiguity. Next note that

$$x_1 dx_2 \otimes dy_1 y_2 \xrightarrow{\text{lifts}} x_1 [x_2, y_1] y_2 \xrightarrow{d}$$

$$x_1 [dx_2, y_1] y_2 + x_1 [x_2, dy_1] y_2$$

$$= x_1 dx_2 \otimes [y_1, y_2] + [x_1, x_2] \otimes dy_1 y_2$$

$$= -(x_1 dx_2) \otimes b(dy_1 y_2) + b(x_1 dx_2) \otimes dy_1 y_2$$

so the ambiguity $x \otimes y \longmapsto [x,y] \leftrightarrow dx\,dy$ is killed by this differential.

I guess the point maybe is that the two liftings $R \otimes S \rightrightarrows R * S / I^2$ give us isomorphisms

$$X(R) \otimes X(S) \simeq X^1(R * S, I) \simeq X(R) \otimes X(S)$$

so we have an automorphism of $X(R) \otimes X(S)$ which is the identity on all the parts except $R \otimes S$ where it is $x \otimes y \longmapsto x \otimes y - dx \otimes dy$

This leads to the question of whether the endomorphism of $X(R) \otimes X(S)$ given by $x \otimes y \mapsto dx \otimes dy$ on $R \otimes S$ and 0 on the other three pieces is null-homotopic. Here's how to see this is true

$$\begin{tikzcd} R \otimes S \arrow[r, "d \otimes 1"] & \Omega^1 R_\natural \otimes S \arrow[dl, "-1 \otimes d"'] \\ \Omega^1 R_\natural \otimes \Omega^1 S_\natural & \end{tikzcd}$$

Define $h : R \otimes S \to \Omega^1 R_\natural \otimes S$ to be $d \otimes (S \xrightarrow{\natural} S_\natural \xrightarrow{l} S)$ where l is a lifting of $S_\natural$ into S, i.e. $\natural l = \text{id}$ on $S_\natural$. Extend h to be 0 on the other three pieces.

Then $h\underline{d} = 0$ where $\underline{d}$ is the differential in $X(R)\otimes X(S)$. In effect we only have to consider the maps in $\underline{d}$ coming into $R\otimes S$, and these are $b\otimes 1 : \Omega^1 R_\natural \otimes S \longrightarrow R\otimes S$ which is killed by $d\otimes l\natural$ as $db=0$, and also $1\otimes b : R\otimes \Omega^1 S_\natural$ which is killed by $d\otimes l\natural$ as $\natural b = 0$.

Next $\underline{d}h = -d\otimes d : R\otimes S \longrightarrow \Omega^1 R_\natural \otimes \Omega^1 S_\natural$. In effect the image of h is contained in $\Omega^1 R_\natural \otimes S$ and there are two maps $b\otimes 1$ and $-1\otimes d$ issuing from this spot. Clearly $(b\otimes 1)(d\otimes l\natural) = 0$ and also $(+1\otimes d)(d\otimes l\natural) = d\otimes d$ since the d in $dl\natural$ is real $\natural d$; thus $\natural d l \natural = d^\natural \natural l \natural = d^\natural \natural = \natural d$. Clear.

Summarize: We have a canonical extension $R\otimes S = R*S/I$ and an almost canonical isomorphism

$$* \qquad X^1(R*S, I) = X(R)\otimes X(S).$$

More precisely corresponding to the two liftings $R\otimes S \rightrightarrows R*S$ we have two isomorphisms $*$ which agree up to the square zero endomorphism $x\otimes y \longmapsto dx\otimes dy$. This endomorphism is homotopic to zero.

Now let's return to our original project of trying to derive the homotopy formula for X on quasi-free algebras. The idea is that given $A \longrightarrow R\otimes S$ we get a canonical square zero extension of A by pulling back $R*S/I^2$. If

A is quasi-free then we have a lifting in this extension, hence a lifting $A \longrightarrow R*S/I^2$ and so a map

$$X(A) \longrightarrow X^1(R*S/I^2, I/I^2) \simeq X(R) \otimes X(S)$$

We now want to understand the choices. The main choice is lifting A into $R*S/I^2$, which amounts to writing a 2-cocycle as a coboundary. The other choice is the choice of linear lifting of $R \otimes S$ into $R*S/I^2$ that we have discussed. Actually one picks a linear lifting of $R \otimes S$ in order to obtain a cocycle, so one should compare doing everything with $x \otimes y \to xy$ or alternating xy with yx.

February 19, 1991

Consider a homomorphism $\theta : A \to R \otimes S$ where A is quasi-free. Then we can lift θ to a homomorphism $A \to R*S/I^2$ whence we have a map

$$X(A) \longrightarrow X'(R*S, I) \cong X(R) \otimes X(S)$$

Suppose we try to carry this out explicitly. The square zero extension $R*S/I^2 \to R\otimes S$ corresponds to the $R\otimes S$ bimodule extension:

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, "\wr"] & \Omega^1(R*S)/F^1_I\Omega^1(R*S) \arrow[r] \arrow[d, "\wr"] & \Omega^1(R\otimes S) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1R\otimes\Omega^1S \arrow[r, "{-1\otimes\partial,\ \partial\otimes 1}"'] & \Omega^1R\otimes(S\otimes S) \oplus (R\otimes R)\otimes\Omega^1S \arrow[r] & \Omega^1(R\otimes S) \arrow[r] & 0
\end{tikzcd}$$

To lift A into $R*S/I^2$ is equivalent to lifting A into the induced square zero extension of A by I/I^2, and this is equivalent to splitting the assoc. bimodule extension of $\Omega^1 A$ by I/I^2, which in turn amounts to a bimodule lifting:

$$\begin{tikzcd}
& \Omega^1(R*S)/F^1_I \arrow[d] \\
\Omega^1 A \arrow[r] \arrow[ur, dashed, "①"] & \Omega^1(R\otimes S)
\end{tikzcd}$$

Suppose we choose a connection on $\Omega^1 A$ i.e. a lifting $A\otimes\bar{A}\otimes A \overset{\xleftarrow{s}}{\longrightarrow} \Omega^1 A$. Then to obtain ① all we need do is to lift $dA \subset \Omega^1 A$, extend to a bimodule map $A\otimes dA\otimes A \longrightarrow \Omega^1(R*S)/F^1_I$, then compose with s.

Thus we need to consider how to lift $d(x\otimes y) \in d(R\otimes S) \subset \Omega^1(R\otimes S)$.

Now [redacted] we have two lifts of $x\otimes y \in R\otimes S$ to $R*S/I^2$, namely xy and yx, so we obtain two lifts of $d(x\otimes y)$ into

$\Omega^1(R*S)/F^1\Omega^1(R*S) =$ [redacted]
$\Omega^1 R\otimes(S\otimes S) + (R\otimes R)\otimes\Omega^1 S$

namely

$$dx\,y + x\,dy \longmapsto dx\otimes(1\otimes y) + (x\otimes 1)\otimes dy$$

$$y\,dx + dy\,x \longmapsto dx\otimes(y\otimes 1) + (1\otimes x)\otimes dy$$

February 20, 1991

1. For any A we have the following description of $X(RA)$:

$$\begin{tikzcd}
RA \arrow[r, shift right, "\bar{d}"'] & \Omega^1(RA)_\natural \arrow[l, shift right, "\bar{b}"'] \\
\bigoplus_{n\geq 0} A\otimes\bar{A}^{\otimes 2n} \arrow[u, "\cong"', "(\rho\omega^n)"] & \bigoplus_{n\geq 0} A\otimes\bar{A}^{\otimes 2n+1} \arrow[u, "\cong"', "(\rho\omega^n d\rho)"]
\end{tikzcd}$$

$$\bar{b}(\rho\omega^n d\rho) = b(\rho\omega^n) - (1+\kappa)s(\rho\omega^{n+1})$$

$$\bar{d}(\rho\omega^n) = -nP_2 b(\rho\omega^{n-1}d\rho) + B(\rho\omega^n d\rho)$$

In particular we have the following description of $X'(RA, IA)$:

$$\begin{tikzcd}
A\oplus\Omega^2A_\natural \arrow[r, shift right, "\begin{pmatrix} B & -\bar{b}\end{pmatrix}"'] & \Omega^1A \arrow[l, shift right, "\begin{pmatrix} b \\ -\natural B\end{pmatrix}"']
\end{tikzcd}$$

2. Suppose A quasi-free and let $\nabla : \Omega^1A \to \Omega^2A$ be a connection in the bimodule Ω^1A. This means ~~$\nabla(a_1a_2) = a_1\nabla a_2$~~ ~~$\nabla(a_1a_2) = \nabla a_1 a_2 + a_1 da_2$~~

$$\nabla(a\omega) = a\nabla\omega$$
$$\nabla(\omega a) = \nabla\omega\, a + \omega\, da$$

The first equation shows $\nabla(a_0da_1) = a_0(\nabla da_1)$, so that ∇ is determined by $\varphi(a) = \nabla d(a)$. One has

$$\begin{aligned}\varphi(a_1a_2) &= \nabla(da_1\, a_2 + a_1 da_2)\\ &= \varphi(a_1)a_2 + da_1da_2 + a_1\varphi(a_2)\end{aligned}$$

so that φ is a 1-cochain such that

$$(\delta\varphi)(a_1, a_2) + da_1da_2 = 0.$$

Next observe this identity for a [redacted] map $\varphi : A \longrightarrow \Omega^2 A$ implies

$$\varphi(1) = \varphi(1) + \varphi(1) \implies \varphi(1) = 0$$

so that we can then define $\nabla : \Omega^1 A \to \Omega^2 A$ by $\nabla(a_0 da_1) = a_0 \varphi(a_1)$. It is clear that ∇ is connection. Thus connections in $\Omega^1 A$ are equivalent to 1-cochains [redacted] whose coboundary is the universal 2-cocycle $da_1 da_2$.

Such a 1-cochain is equivalent to a [redacted] splitting of the square zero extension

$$\begin{tikzcd} 0 \arrow[r] & IA/IA^2 \arrow[r] & RA/IA^2 \arrow[r] \arrow[d, phantom, "\wr"] & A \arrow[r] & 0 \\ & & A \oplus \Omega^2 A & & \end{tikzcd}$$

with Fedosov product

In effect a lifting $A \longrightarrow RA/IA^2$ is of the form $a \mapsto \rho a$ [redacted] $- \varphi a$, $\varphi : A \longrightarrow IA/IA^2 \cong \Omega^2 A$

and

$$\rho(a_1 a_2) - \varphi(a_1 a_2) \overset{?}{=} (\rho(a_1) - \varphi(a_1))(\rho(a_2) - \varphi(a_2))$$

$$= \rho(a_1 a_2) - \omega(a_1, a_2) - a_1 \varphi(a_2) - \varphi(a_1) a_2$$

iff $$\varphi(a_1 a_2) = a_1 \varphi(a_2) + \varphi(a_1) a_2 + \frac{\omega(a_1, a_2)}{da_1 da_2}$$

In other words a lifting $A \longrightarrow RA/IA^2$ is [redacted] of the form $a \longmapsto \rho a - \nabla da$ for a unique connection ∇.

3. Let us consider the induced map of X-complexes

$$X(A) \longrightarrow X'(RA/IA^2 \text{[redacted]}, IA/IA^2) = X'(RA, IA)$$

Recall the lifting $\ell : \Omega^1 A_\natural \longrightarrow \Omega^1 A$ is defined by $\ell \natural = 1 + b\nabla$. Then this map of X-complexes is

$$\begin{tikzcd}
A \arrow[r, shift right, "\natural d = \natural B"'] \arrow[d, "\begin{pmatrix} 1 \\ -\natural\nabla d \end{pmatrix}"'] & \Omega^1 A_\natural \arrow[l, shift right, "\bar{b}"'] \arrow[d, "\ell"] \\
A \oplus \Omega^2 A_\natural \arrow[r, shift right, "\begin{pmatrix} d & -\bar{b} \end{pmatrix}"'] & \Omega^1 A \arrow[l, shift right, "\begin{pmatrix} b \\ -\natural B \end{pmatrix}"']
\end{tikzcd}$$

In effect the homomorphism from A to $RA/IA^2 = A \oplus \Omega^2 A$ is $a \longmapsto a - \nabla da = a - \varphi a$, which gives $\begin{pmatrix} 1 \\ -\natural\nabla d \end{pmatrix}$ on the left. The induced map in degree 1 is

$$\natural(a_0 da_1) \longmapsto (a_0 - \varphi a_0)\, d(a_1 - \varphi a_1) \in (\Omega^1 RA / F^1_I)_\natural$$

here use $a_0 d(\varphi a_1) = d(a_0 \varphi a_1)$

$$(\Omega^1 RA / F^1_I)_\natural \cong \Omega^1 A$$

$$a_0 da_1 - d(a_0 \varphi a_1)$$

(as $d^\times$ on $\Omega^2 A_\natural$ is $-\bar{b}$)

$$= a_0 da_1 + b(a_0 \varphi a_1)$$

$$= (1 + b\nabla)(a_0 da_1)$$

$$= \ell \natural (a_0 da_1).$$

4. Consider the extension $R \otimes S = R * S / I$ and the linear lifting $\rho : R \otimes S \longrightarrow R * S$, $x \otimes y \longmapsto xy$. This induces a map of complexes

$$X^1(R(R\otimes S), I(R\otimes S)) \longrightarrow X^1(R*S, I) \simeq X(R) \otimes X(S)$$

$$\begin{pmatrix} \begin{matrix} R\otimes S \\ \oplus \\ \Omega^2(R\otimes S)_\natural \end{matrix} & \Omega^1(R\otimes S) \end{pmatrix} \longrightarrow \begin{pmatrix} \begin{matrix} R\otimes S \\ \oplus \\ \Omega^1 R_\natural \otimes \Omega^1 S_\natural \end{matrix} & \begin{matrix} \Omega^1 R_\natural \otimes S \\ \oplus \\ R \otimes \Omega^1 S_\natural \end{matrix} \end{pmatrix}$$

which can be described as follows. Firstly the complex $X'(R(R\otimes S), I(R\otimes S))$ is described via the cochains $\rho, \rho\omega, \rho d\rho$ (for the universal ρ), so this map can be described by the corresponding cochains associated to $x\otimes y \longmapsto xy \in R*S/I^2$. [redacted]

[redacted] Using

$$\omega(x_1\otimes y_1, x_2\otimes y_2) = x_1x_2y_1y_2 - x_1y_1x_2y_2$$

$$= x_1[x_2,y_1]y_2 \Longleftrightarrow x_1dx_2\otimes dy_1y_2 \in \Omega^1R\otimes\Omega^1S$$

We see the maps are

ρ $\qquad R\otimes S \xrightarrow{id} R\otimes S$

$\rho\omega$ $\qquad \Omega^2(R\otimes S)_\natural \longrightarrow \Omega^1R_\natural\otimes\Omega^1S_\natural$

$$(x_0\otimes y_0)d(x_1\otimes y_1)d(x_2\otimes y_2) \longmapsto x_0x_1dx_2\otimes y_0dy_1y_2$$

$\rho d\rho$ $\qquad \Omega^1(R\otimes S) \longrightarrow \Omega^1R_\natural\otimes S \oplus R\otimes\Omega^1S_\natural$

$$(x_0\otimes y_0)\,d(x_1\otimes y_1) \longrightarrow x_0dx_1\otimes y_1y_0 + x_0x_1\otimes y_0dy_1$$

(For the last $\rho(x_0\otimes y_0)\,d\rho(x_1\otimes y_1) =$ [redacted] $x_0y_0\,d(x_1y_1)$ [redacted]

$= x_0y_0(dx_1\,y_1 + x_1dy_1) \in \Omega^1(R*S)/F^1_I\Omega^1(R*S)$

$$\begin{tikzcd}
x_0y_0(dx_1\,y_1 + x_1dy_1) \arrow[d, leftrightarrow] & \Omega^1(R*S)/F^1_I\Omega^1(R*S) \arrow[d, phantom, "\parallel"] \\
x_0dx_1\otimes(y_0\otimes y_1) + (x_0x_1\otimes 1)\otimes y_0dy_1 \arrow[d, mapsto] & \in \Omega^1R\otimes(S\otimes S)\oplus(R\otimes R)\otimes\Omega^1S \arrow[d] \\
x_0dx_1\otimes y_1y_0 + x_0x_1\otimes y_0dy_1 & \in \Omega^1R_\natural\otimes S + R\otimes\Omega^1S_\natural
\end{tikzcd}$$
)

5. Let us now take a ~~[redacted]~~ homom. $A \longrightarrow R\otimes S$, where $S = \mathbb{C}[\varepsilon]/(\varepsilon^2) = \mathbb{C}\oplus\mathbb{C}\varepsilon$, $\varepsilon^2 = 0$. Such a homom. has the form $\theta + \dot\theta\varepsilon$, where $\theta : A \longrightarrow R$ is a homom. and $\dot\theta$ is a derivation rel θ.

What is $X(\mathbb{C}[\varepsilon]/(\varepsilon^2))$? Note that $\Omega^1(\mathbb{C}[\varepsilon]/(\varepsilon^2))$

has the basis $d\varepsilon$, $\varepsilon d\varepsilon$ and that $d\varepsilon\,\varepsilon + \varepsilon d\varepsilon = d(\varepsilon^2) = 0$, so that $\natural(\varepsilon d\varepsilon) = 0$. Thus $X(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ is

$$\begin{tikzcd} \mathbb{C} \oplus \mathbb{C}\varepsilon \arrow[r, shift right, "d"'] & \mathbb{C}d\varepsilon \arrow[l, shift right, "b=0"'] \end{tikzcd}$$

The maps are found as follows

$$\rho: \quad A \xrightarrow{\theta + \mathring{\theta}\varepsilon} R + R\varepsilon$$

$$\rho\omega: \quad \Omega^2 A_\natural \longrightarrow \Omega^1 R_\natural \otimes d\varepsilon$$

Here we take $a_0 da_1 da_2$ into

$$(\theta a_0 \otimes 1 + \mathring{\theta} a_0 \otimes \varepsilon)\, d(\theta a_1 \otimes 1 + \mathring{\theta} a_1 \otimes \varepsilon)\, d(\theta a_2 \otimes 1 + \mathring{\theta} a_2 \otimes \varepsilon)$$

which breaks into 8 terms of the form

$$x_0 \otimes y_0\, d(x_1 \otimes y_1)\, d(x_2 \otimes y_2)$$

which maps to $x_0 x_1 dx_2 \otimes y_0 dy_1 y_2$. Only $y_0 = y_2 = 1$ and $y_1 = \varepsilon$ counts and we get $x_0 = \theta a_0$, $x_1 = \mathring{\theta} a_1$, $x_2 = \theta a_2$. Thus we have

$$\begin{tikzcd} \rho\omega: \quad \Omega^2 A_\natural \arrow[r] & \Omega^1 R_\natural \otimes d\varepsilon \\ a_0 da_1 da_2 \arrow[r, mapsto] & \underbrace{\theta a_0\, \mathring{\theta} a_1\, d(\theta a_2)}_{i(\theta, \mathring{\theta})(a_0 da_1 da_2)} \otimes d\varepsilon \end{tikzcd}$$

$$\rho d\rho: \quad \Omega^1 A \longrightarrow \Omega^1 R_\natural + \Omega^1 R_\natural\, \varepsilon + R \otimes d\varepsilon$$

Here we take $a_0 da_1$ into

$$(\theta a_0 \otimes 1 + \mathring{\theta} a_0 \otimes \varepsilon)\, d(\theta a_1 \otimes 1 + \mathring{\theta} a_1 \otimes \varepsilon) \quad \in \Omega^1(R \otimes S)$$

which breaks into 4 terms of the form

$$(x_0 \otimes y_0)\, d(x_1 \otimes y_1)$$

which maps to

$$x_0 dx_1 \otimes y_1 y_0 + x_0 x_1 \otimes y_0 dy_1$$

We then have for $y_0 = y_1 = 1$

$$\theta a_0\, d(\theta a_1) \otimes 1$$

for $y_0 = 1, y_1 = \varepsilon$ and $y_0 = \varepsilon, y_1 = 1$ we have

$$\big(\underbrace{\theta a_0\, d(\mathring{\theta} a_1) + \mathring{\theta} a_0\, d(\theta a_1)}_{L(\theta,\mathring{\theta})(a_0 da_1)}\big) \otimes \varepsilon$$

and for $y_0 = 1, y_1 = \varepsilon$ we have

$$\theta a_0\, \mathring{\theta} a_1 \otimes d\varepsilon.$$

Thus we have

$$\rho d\rho : \quad \Omega^1 A \longrightarrow \Omega^1 R_\natural + \Omega^1 R_\natural \otimes \varepsilon + R \otimes d\varepsilon$$

$$a_0 da_1 \longmapsto \underbrace{\theta a_0\, d(\theta a_1)}_{\theta(a_0 da_1)} + \big(\underbrace{\theta a_0\, d(\mathring{\theta} a_1) + \mathring{\theta} a_0\, d(\theta a_1)}_{L(\theta,\mathring{\theta})(a_0 da_1)}\big) \otimes \varepsilon$$

$$+ \underbrace{\theta a_0\, \mathring{\theta} a_1}_{\iota(\theta,\mathring{\theta})(a_0 da_1)} \otimes d\varepsilon$$

Summary: Given $\theta + \mathring{\theta}\varepsilon : A \longrightarrow R + R\varepsilon$ the induced map

$$\overline{\Phi} : X^1(RA, IA) \longrightarrow X(R) \otimes X(\mathbb{C}[\varepsilon]/\varepsilon^2)$$

has the components as follows relative to the basis $1, \varepsilon, d\varepsilon$ of $X(\mathbb{C}[\varepsilon]/(\varepsilon^2))$:

Coeff of 1 : $X^1(RA, IA) \twoheadrightarrow X(A) \xrightarrow{X(\theta)} X(R)$

Coeff of ε : $X^1(RA, IA) \twoheadrightarrow X(A) \xrightarrow{L(\theta,\mathring{\theta})} X(R)$

Coeff of $d\varepsilon$: $X^1(RA, IA) \longrightarrow X(R)$

$$\begin{tikzcd}
\boxed{A} \arrow[rr] & & 0 & \\
\oplus & \boxed{\Omega^1 A} \arrow[r, "\iota(\theta,\mathring{\theta})"] \arrow[r, shift right] & R & \Omega^1 R_\natural \\
\boxed{\Omega^2 A_\natural} \arrow[rrru, "i(\theta,\mathring{\theta})"'] & & &
\end{tikzcd}$$

Let's write

$$\underline{\Phi} = u_0 \otimes 1 + u_1 \otimes \varepsilon + v \otimes d\varepsilon$$

Then as $\underline{\Phi}$ is a map of complexes we have

$$0 = [\underline{d}, \underline{\Phi}] = [\underline{d}, u_0] \otimes 1 + [\underline{d}, u_1] \otimes \varepsilon + u_1 \otimes d\varepsilon + [\underline{d}, v] \otimes d\varepsilon$$

so u_0, u_1 are maps of complexes, while $[\underline{d}, v] + u_1 = 0$, so that $-v$ is a contracting homotopy for u_1.

Note (April 26) $v \otimes d\varepsilon$ has to be interpreted with the usual signs: $(v \otimes d\varepsilon)(\xi) = (-1)^{|\xi|} v(\xi) \otimes d\varepsilon$. Thus v differs from $\iota(\theta, \dot{\theta})$ by a sign.

Further once a connection is chosen one has liftings for any square zero extension equipped with linear lifting. Since the surjections

$$\longrightarrow RA/IA^{n+1} \longrightarrow RA/IA^{n} \longrightarrow \cdots$$

come with obvious linear liftings (via the isom $RA \simeq \Omega^{ev}A$), one therefore obtains a lifting $A \to \widehat{RA}$. We have seen that this may not be unique since one can go also from RA/IA^n to RA/IA^{2n}.

Here's an attempt to do things canonically. What we want to do is to associate to a linear map $\rho : A \longrightarrow R$ $(\rho 1 = 1)$ which is close to a homomorphism (say $IA^N \longrightarrow 0$) a homomorphism $A \longrightarrow R$. Let us try to construct a flow which tends to decrease the curvature.

Let $\varphi = \nabla d : \bar{A} \longrightarrow \Omega^2 A$ satisfy

$$\varphi(a_1a_2) = a_1\varphi a_2 + (\varphi a_1)a_2 + da_1da_2$$

let $\omega = b'\rho + \rho^2$ and $\rho\omega : \Omega^2 A \longrightarrow R$. Consider the differential equation

$$* \qquad \mathring{\rho} = -(\rho\omega)\varphi$$

In the case of a square zero extension $\rho\omega$ is compatible with left + right multiplication by $\rho(a)$. Thus

$$(\rho+s\mathring{\rho})(a_1)\ (\rho+s\mathring{\rho})(a_2) = (\rho a_1 - s(\rho\omega\cdot\varphi)a_1)(\rho a_2 - s(\rho\omega\cdot\varphi)a_2)$$

$$= \rho(a_1a_2) - \omega(a_1,a_2) - s(\rho\omega)(\varphi a_1)\rho a_2 - s\rho a_1\,(\rho\omega)(\varphi a_2)$$

$$= \rho(a_1a_2) - \underset{(\rho\omega)(da_1da_2)}{} - s(\rho\omega)((\varphi a_1)a_2 + a_1\varphi(a_2))$$

$$= \underset{-(1-s)\ \omega(a_1,a_2)}{\rho(a_1a_2) - s(\rho\omega)(\varphi(a_1a_2))} = (\rho+s\mathring{\rho})(a_1a_2) - (1-s)\omega(a_1,a_2)$$

February 21, 1991

Let's make a list of ideas for future references

1. Problem: How to handle periodic homology for an algebra A of projective dimension 2. Let $A = E/J$ be a square zero extension which is "versal" i.e. $\Omega^1(E)/F^1_J\Omega^1(E)$ is a projective bimod over A. Then the periodic homology of A is given by the complex $X^1(E,J)$. The problem is now with the functorality: Given a nilpotent extension $A = R/I$, how do we construct a map $X^1(E,J) \longrightarrow X(R)$ unique up to homotopy.

More specifically, given $A = R/I$ quasi-free how do we obtain a homotopy inverse for

$$\hat{X}(R,I) \longrightarrow X^1(R/I^2, I/I^2) = X^1(R,I)?$$

(Examples to consider: functions on a 2 manifold, Heisenberg algebra.)

2. Assume $\Omega^n A$ projective. Choosing a connection in it we obtain a strong deformation retraction of $(\Omega A, b)$ onto a subcomplex of length n. We can then use HPT to construct a corresponding SDR for $(\hat{\Omega}A, b+B)$. The question is whether there is something analogous in the extension picture.

For example suppose A separable (resp. quasi free) Then a connection in the A-bimodule A (this is equivalent to a $Y \in \Omega^1 A$ such that $da = [a, Y]$ $\forall a$) (resp. a connection in $\Omega^1 A$, i.e. $\varphi : A \longrightarrow \Omega^1 A$ [redacted] $\ni$ $(\delta\varphi)(a_1,a_2) + da_1 da_2 = 0$) gives rise to a SDR of $(\hat{\Omega}A, b+B)$ onto $A_\natural[0]$ (resp. $X(A)$).

3. A natural question is to relate connections in $\Omega^1 A$ to liftings $A \longrightarrow \widehat{RA}$. We know that a connection is equivalent to a lifting in the square zero extension $A \longrightarrow RA/IA^2$.

In this case the flow should be

$$\rho_t = \rho + (1-e^{-t})\dot{\rho}$$

$$\omega_t = e^{-t}\omega$$

Example: Take $A = \mathbb{C}[F]$. Then $\varphi(F) = -\frac{1}{2}F(dF)^2$.

Check $\varphi(F^2) = F\varphi(F) + \varphi(F)F +$ [redacted] dF^2

$$= -\frac{1}{2}dF^2 - \frac{1}{2}FdF^2F + dF^2 = 0.$$

Let $\rho(F) = z$, then $\omega(F,F) = \rho(1) - \rho(F)^2 = 1 - z^2$ and the DE is

$\dot{z} =$ [redacted] $(-\rho\,\omega)\left(-\frac{1}{2}FdF^2\right)$

$$= \frac{1}{2}z(1-z^2)$$

Note that this is not a linear DE, but 3rd degree.

$$\frac{dz}{z(1-z^2)} = \frac{1}{2}dt$$

$$\frac{1}{z(1-z^2)} = \frac{1}{z} + \frac{1}{2}\frac{1}{1-z} - \frac{1}{2}\frac{1}{1+z}$$

$d \log \frac{z}{\sqrt{1-z^2}} = \frac{t}{2}$ [redacted] $+$ const,

$$\therefore \quad \frac{z_t}{\sqrt{1-z_t^2}} = \frac{z_0}{\sqrt{1-z_0^2}}e^{t/2}$$

$$\frac{z_t^2}{1-z_t^2} = \frac{z_0^2 e^t}{1-z_0^2} \qquad \frac{1}{z_t^2} - 1 = \left(\frac{1}{z_0^2} - 1\right)e^{-t}$$

$$z_t^2 = \frac{1}{1+\left(\frac{1}{z_0^2}-1\right)e^{-t}} = \frac{z_0^2}{e^{-t} + (1-e^{-t})(1-(1-z_0^2))}$$

$$\boxed{z_t = \frac{z_0}{\sqrt{1-(1-e^{-t})(1-z_0^2)}}}$$

[redacted]

At first sight this deformation seems strange, but it has an interpretation in terms of Cayley transform ideas. We should be thinking of z_0 as a self-adjoint operator $-1 \leq z_0 \leq 1$. Working on the set where z_0 doesn't have the eigenvalues ± 1, we can write

$$z_0 = \frac{Y_0}{\sqrt{1+Y_0^2}}$$

with Y_0 self-adjoint. Then a natural deformation of z_0 to an involution is

$$z_t = \frac{Y_0}{\sqrt{e^{-t}+Y_0^2}} = \frac{Y_0/\sqrt{1+Y_0^2}}{\sqrt{\frac{e^{-t}+Y_0^2}{1+Y_0^2}}}$$

$$= \frac{z_0}{\sqrt{e^{-t}(1-z_0^2)+z_0^2}} = \frac{z_0}{\sqrt{e^{-t}(1-z_0^2)+1-(1-z_0^2)}}$$

$$= \frac{z_0}{\sqrt{1-(1-e^{-t})(1-z_0^2)}}$$

So the deformation is natural.

Let's return to the DE

$$\dot{\rho} = -(\rho\omega)\varphi$$

and look for fixpts. Notice that the image of $\varphi : \bar{A} \longrightarrow \Omega^2 A$ generates $\Omega^2 A$ as an A-bimodule since $A \otimes \bar{A} \otimes A \longrightarrow \Omega^2 A$ is the surjection corresponding to the splitting of $0 \to \Omega^2 A \to A \otimes \bar{A} \otimes A \longrightarrow \Omega^1 A \to 0$.

We would like to show that $(\rho\omega)\varphi = 0$ $\Longrightarrow \omega = 0$. We assume $\rho : A \longrightarrow R$ such that $\rho(1) = 1$ and such that the induced homomorphism $RA \longrightarrow R$ carries IA^N to 0 for some N.

We can suppose $R = RA/J$ with $\rho : A \longrightarrow R$ the image of the canonical linear map. Write $I = IA$ and consider the square zero extension RA/I^2+J and the linear map which is the image of the universal linear map. Since this is a square zero extension we know that $\rho\omega : \Omega^2 A \to RA/I^2+J$ is a bimodule morphism. Since $(\rho\omega)\varphi = 0$, and the $\varphi(a)$ generate $\Omega^2 A$ as bimodule we conclude that $\rho\omega : \Omega^2 A \longrightarrow RA/I^2+J$ is zero. This implies that $I \subset I^2 + J$. Then $I^n \subset I^{n+1} + J$ and $I^n + J \subset I^{n+1} + J$ for all $n \geq 1$, showing that $I \subset I^N + J = J$. Thus $\rho\omega : \Omega^2 A \longrightarrow RA/J$ is zero as claimed.

1990-91

February 22, 1991

Remark: If A is quasi-free, then we have a lifting $A \longrightarrow \widehat{RA}$, and since $\Omega^2 A$ is a projective bimodule over A we can find an A-bimodule section of

$$\widehat{IA}/\widehat{IA^2} \longrightarrow\!\!\!\!\rightarrow \Omega^2 A$$

and obtain a homomorphism

$$T_A(\Omega^2 A) \longrightarrow \widehat{RA}.$$

Thus $\widehat{RA} \cong \widehat{\Omega A^+}$ as algebras. Similarly as $\Omega^1 A$ is projective we can find an A-bimodule lifting for the surjection

$$(\widehat{JA})^- \longrightarrow (\widehat{JA}/\widehat{JA^2})^- = \Omega^1 A;$$

here A acts via $A \to \widehat{RA} = (\widehat{QA})^+$. Thus we get $\widehat{QA} \simeq \widehat{\Omega A}$ as superalgebras.

It might be easier to [redacted] integrate the flow $\dot{\rho} = -(\rho\omega)\varphi$ provided one [redacted] [redacted] incorporates liftings of $\Omega^1 A$ or $\Omega^2 A$. Thus one should simultaneously lift A and $\Omega^1 A$ into $\widehat{QA}$.

Example. Take $A = T(V)$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 A \arrow[r, "-b"'] & \Omega^1 A \otimes A \arrow[r] & \Omega^1 A \arrow[r] \arrow[d, equal] & 0 \\
 & & A\otimes V\otimes A\otimes A & A\otimes V\otimes A & \\
 & & dv\otimes 1 & dv \arrow[l, mapsto] & \\
 & & dva\otimes 1 \arrow[r, mapsto] \arrow[dll, "\nabla"] & dva \arrow[dl, mapsto] & \\
dvda \arrow[drr, mapsto] & & dv\otimes a & & \\
 & & dv(a\otimes 1-1\otimes a) & &
\end{tikzcd}$$

Thus $\nabla(a_1 dv a_2) = a_1 dv da_2$ is the connection on $\Omega^1 A = A \otimes dV \otimes A$ such that $\nabla(dv) = 0$ for all $v \in V$. Also

$$\varphi(v_1 \cdots v_n) = \nabla d(v_1 \cdots v_n)$$

$$= \nabla \sum_{j=1}^{n} v_1 \cdots v_{j-1} dv_j v_{j+1} \cdots v_n$$

$$= \sum_{j=1}^{n} v_1 \cdots v_{j-1} dv_j d(v_{j+1} \cdots v_n)$$

Let's consider the DE $\dot{\rho} = -(\rho\omega)\varphi$. Consider the map $\rho \longmapsto \rho|V$. One has

$$\dot{\rho}(v) = -(\rho\omega)\varphi(v) = 0$$

which means that the flow leaves the elements $\rho(v)$ fixed. Thus if the flow as $t \to +\infty$ takes ρ to a fixpoint ρ_∞, then as we know ρ_∞ is a homomorphism, it follows that ρ_∞ is the unique homomorphism with $\rho_\infty(v) = \rho(v)$ for all $v \in V$.

Observe that the curvature is zero for the standard connection in a free algebra. In effect the curvature ∇^2 is an A-bimodule map which vanishes on the generators dv of $\Omega^1 A$. More generally if $E = W \otimes A$ is a free right A-module and we use the standard connection $\nabla(wa) = w\,da$, then $\nabla^2(wa) = \nabla(w\,da) = \nabla w\,da + w\,d^2 a = 0$.

What is the *curvature* for *separable algebras*? Look at $A = \mathbb{C}[F]$. The elements of $A \otimes A$ centralized by A are those in $\mathbb{C}(1 \otimes 1 + F \otimes F) + \mathbb{C}(F \otimes 1 + 1 \otimes F)$, and the multiplication $A \otimes A \longrightarrow A$ maps this subspace isomorphically to A. Thus there is a unique connection in A given by

$$Z = \frac{1}{2}(1 \otimes 1 + F \otimes F)$$

which is the unique central elt in $A\otimes A$ of augmentation 1. The corresponding elt $Y = \nabla 1 \in \Omega^1 A$ is determined by $\tilde{\partial}(Y) = 1\otimes 1 - Z$ in

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1 A \arrow[r, "\tilde{\partial}"'] & A\otimes A \arrow[r, "m"'] \arrow[l, dashed, bend right, "-id\otimes d"'] & A \arrow[r] \arrow[l, dashed, bend right, "\otimes 1_A"'] & 0 \end{tikzcd}$$

[redacted] where $(-id\otimes d)(a_1\otimes a_2) = -a_1 da_1$ is a splitting because

$$a_1\otimes a_2 - a_1a_2\otimes 1 = a_1(1\otimes a_2 - a_2\otimes 1)$$
$$= -a_1\,\partial a_2 = -\tilde{\partial}(a_1 da_2).$$

Thus $\quad Y = (-id\otimes d)(1\otimes 1 - Z) = (id\otimes d)Z = \frac{1}{2}FdF.$

For this unique connection in $A = \mathbb{C}[F]$ the curvature is non zero:

$$\nabla^2 = (d+Y)^2 = dY + Y^2 = \frac{1}{4}dF^2$$

Consider next a matrix algebra $A = \mathrm{End}\,V = V\otimes V^*$. Then $A\otimes A = V\otimes V^*\otimes V\otimes V^*$ with left and right multiplication acting on the outside factors. Note that $V\otimes V^*$ is an A-bimodule because it's the tensor product of the left A-module V and the right A-module V^*. It is isomorphic as bimodule to A, the [redacted] central element being the identity of $V\otimes V^*$ corresponding to $1\in A$

$$\sum_i |i\rangle\otimes\langle i| \in V\otimes V^*$$

The central elements of the bimodule $A\otimes A$ are of the form

$$\sum_i |i\rangle\otimes\alpha\otimes\langle i| \in V\otimes(V^*\otimes V)\otimes V^*$$

where $\alpha\in V^*\otimes V$. We want a central element augmenting to the identity in A, which means $\alpha\mapsto 1$ under $V^*\otimes V\longrightarrow\mathbb{C}$, $\lambda\otimes\sigma\longmapsto\lambda(\sigma)$.

Notice that the map $\begin{array}{ccc} A\otimes A & \longrightarrow & A \\ a_1\otimes a_2 & \longmapsto & a_2a_1 \end{array}$

gives an isomorphism of the space of central elements in $A\otimes A$ with A. On the other hand we have the multiplication or augmentation from the central elements in $A\otimes A$ to the central elements in A which are multiples of the identity. Thus we have a canonical linear functional on A, which clearly has to be the trace up to some scalar.

So we learn that there will be many connections in the bimodule A, but that there is a unique one invariant under automorphisms of A.

For $A = M_n(\mathbb{C})$ we have

$$Z = \frac{1}{n}\sum_{i,j}\big(|i\rangle\langle j|\big)\otimes\big(|j\rangle\langle i|\big) \in A\otimes A$$

This is central, ~~augments~~ and maps to the identity under both maps $A\otimes A \Longrightarrow A$
$a_1\otimes a_2 \longrightarrow a_1a_2,\ a_2a_1$. Thus

$$y = (1\otimes d)Z = \frac{1}{n}\sum_{i,j} e_{ij}\,de_{ji}$$

where we write $e_{ij} = |i\rangle\langle j|$. Then

$$y^2 = \frac{1}{n^2}\sum_{ijkl} \underbrace{e_{ij}\,de_{ji}\,e_{kl}\,de_{lk}}$$

$$e_{ij}\,d(e_{ji}e_{kl})\,de_{lk} - e_{ij}e_{ji}\,de_{kl}\,de_{lk}$$

$$= e_{ij}\,de_{jl}\,de_{lk}\,\delta_{ik} - e_{ii}\,de_{kl}\,de_{lk}$$

$$y^2 = \frac{1}{n^2}\sum_{ijl} e_{ij}\,de_{jl}\,de_{li} - \frac{1}{n}\underbrace{\sum_{i,k,l} e_{ii}\,de_{kl}\,de_{lk}}_{\sum_{k,l} de_{kl}\,de_{lk}}$$

$$dY = \frac{1}{n}\sum_{ij} de_{ij}\,de_{ji}$$

Thus the curvature is

$$\boxed{dY + Y^2 = \frac{1}{n^2} \sum_{ijk} e_{ij}\, de_{jk}\, de_{ki}}$$

Suppose we use another connection

$$Z = \sum_i (|i\rangle\langle 1|) \otimes (|1\rangle\langle i|)$$

$$= \sum_i e_{i1} \otimes e_{1i}$$

$$Y = \sum_i e_{i1}\, de_{1i}$$

$$Y^2 = \sum_{ij} \underbrace{e_{i1}\, de_{1i}\, e_{j1}\, de_{1j}}_{e_{i1}\, d(\underbrace{e_{1i} e_{j1}}_{\delta_{ij} e_{11}}) de_{1j} \;-\; \underbrace{e_{i1} e_{1i}}_{e_{ii}}\, de_{j1}\, de_{1j}}$$

$$= \sum_j e_{j1}\, de_{11}\, de_{1j} - \sum_j de_{j1}\, de_{1j}$$

$$dY = \sum_j de_{j1}\, de_{1j}$$

$$\therefore \boxed{dY + Y^2 = \sum_j e_{j1}\, de_{11}\, de_{1j}}$$

In both cases the curvature is non zero it seems. In the second case

$$e_{11}(dY + Y^2)e_{11} = e_{11}\, de_{11}\, de_{11} \neq 0$$

February 24, 1991

Separable algebras. A is separable when it is projective as A-bimodule, i.e. when there exists a bimodule map $l: A \longrightarrow A \otimes A$ which is a section of the multiplication $m: A \otimes A \longrightarrow A$.

The first remark is that for any left A-module E we have that $E = A \otimes_A E$ is a direct summand of $(A \otimes A) \otimes_A E = A \otimes E$, hence is a projective A-module. Thus Wedderburn theory says A is a product of matrix algebras over skew-fields (Recall the steps: Exact sequences split, so left ideals are generated by idempotents, in particular A is left noetherian. Also since in general any $\neq 0$ finitely generated module has a simple quotient module, one has any nonzero module has a simple submodule, and then by Zorn is semi-simple. Then any finitely generated A-module is a finite sum of simple submodules, etc.)

Next the bimodule lifting $l: A \longrightarrow A \otimes A$ is given by an elt $e \in A \otimes A$ such that $ae = ea$ $\forall a \in A$ and $m(e) = 1$. Choose a representation

$$e = \sum_{i=1}^{n} x_i \otimes y_i \qquad\qquad x_i, y_i \in A$$

with n least. Then the y_i are linearly independent over $\mathbb{C}$, since if $y_j = \sum_{i \neq j} c_i y_i$, one has

$$e = \sum_{i \neq j} x_i \otimes y_i + x_j \otimes \sum_{i \neq j} c_i y_i$$

$$= \sum_{i \neq j} (x_i + c_i x_j) \otimes y_i$$

Since the y_i are independent there exist $\varphi_i \in A^*$ such that $\varphi_j(y_i) = \delta_{ij}$, so $ae = ea$:

$$\sum a x_i \otimes y_i = \sum x_i \otimes y_i a$$

yields

$$a x_j = \sum_i a x_i \varphi_j(y_i) = (1 \otimes \varphi_j)(ae)$$
$$= (1 \otimes \varphi_j)(ea) = \sum_i x_i \varphi_j(y_i a)$$

showing $A x_j \subset \sum_i \mathbb{C} x_i$.

Thus $\sum \mathbb{C} x_i$ is a left ideal in A. On the other hand the condition $\sum x_i y_i = 1$ shows A acts faithfully on $\sum_i \mathbb{C} x_i$ by left mult. Thus $A \subset \operatorname{End}_{\mathbb{C}}(\sum_i \mathbb{C} x_i)$, showing A is finite dimensional over $\mathbb{C}$.

Combining the above, we see A is a product of matrix algebras over $\mathbb{C}$.

Let's consider connections, that is, the possible e as above.

Suppose A is commutative. In the exact sequence

$$0 \longrightarrow \Omega^1 A \longrightarrow A \otimes A \xrightarrow{m} A \longrightarrow 0$$

$A \otimes A$ is an algebra (commutative), m is a homom. and $\Omega^1 A$ is an ideal; call it I. We have $Ie = 0$ and $1 - e \subset I$, whence $e^2 = e$

Put $R = A \otimes A$

$$R = Re \oplus R(1-e)$$

$$I = \cancel{Ie} \oplus I(1-e) \implies I = R(1-e).$$

[redacted]

Now if $e' \in R$ also satisfies $(1-e') \in I$, $Ie' = 0$ then we have $(1-e')e = (1-e)e' = 0$, so $e = e'e$, $e' = ee'$. Since R is commutative this implies $e = e'$.

Thus ~~there~~ is a unique connection when A is commutative separable.

Next consider $A = M_n(\mathbb{C}) = V \otimes V^*$. Then we have seen that the central elements of the bimodule $A \otimes A$ are

$$(A\otimes A)^\natural = \left\{ \sum_{i=1}^{n} |i\rangle \alpha \langle i| \;\middle|\; \alpha \in V^* \otimes V \right\}$$

Consider the map

$$(A\otimes A)^\natural \hookrightarrow A \otimes A \xrightarrow{\natural} (A\otimes A)_\natural \cong A$$

If $\alpha = \sum c_{jk} (\langle j|) \otimes (|k\rangle)$, then this map sends:

$$\sum |i\rangle \alpha \langle i| = \sum_{ijk} c_{jk} \, |i\rangle \otimes \langle j| \otimes |k\rangle \otimes \langle i|$$

$$\longmapsto \sum_{ijk} \text{[redacted]} \; c_{jk} \overset{\langle i|i\rangle = 1}{{}_\wedge} |k\rangle \otimes \langle j| = n\alpha^t$$

Thus we have an isomorphism $(A\otimes A)^\natural \xrightarrow{\sim} (A\otimes A)_\natural$ because we are in characteristic zero.

So one ought to be able to conclude that for any bimodule M over a separable algebra A (in characteristic 0) that one has an isom.

$$M^\natural \xrightarrow{\sim} M_\natural$$

(In effect M [redacted] is a direct summand of $(A\otimes A) \otimes_A M \otimes_A (A \otimes A) = A \otimes M \otimes A$.)

Thus in a matrix algebra we have a canonical connection, where c corresponds under

$$(A\otimes A)^\natural \xrightarrow{\sim} (A\otimes A)_\natural \cong A$$

to a multiple of the identity. For a product of matrix algebras, it should be the case that c can be chosen uniquely so that it maps under the above isomorphism into the center $A^\natural$ of A.

Let's try to understand $M^\natural \longrightarrow M_\natural$ on the category of A-bimodules. Notice that $R = A\otimes A^\circ$ is separable. (We have an identification of A-bimodules and A°-bimodules such that A as A-bimodule corresponds to A° as A°-bimodules. Thus A separable $\Leftrightarrow$ A° is. Clearly separable algebras are closed under tensor products.) The simple R-modules are the simple A-bimodules.

Thus to understand $M^\natural \longrightarrow M_\natural$ it suffices to consider M a simple A-bimodule. Now because $\mathbb{C}$ is alg. closed, Schur's lemma tells us that the endo. ring of a simple A-module is $\mathbb{C}$. Simple R-modules must be of the form $V\otimes W^*$ where V, W are simple A-modules. It's clear that $M^\natural$ and $M_\natural$ will be zero if V is not isomorphic to W; take a central elt of A acting as 0 on V and 1 on W. If $V\cong W$, then $M^\natural = \mathbb{C}\cdot id \subset End(V)$, $M_\natural \cong End(V)_\natural \cong \mathbb{C}$ via trace.

Then $M^7 \longrightarrow M_7$ will be given by the trace of the identity map which is $\neq 0$ in characteristic zero.

February 28, 1991

Suppose A commutative. Its de Rham complex Ω_A is the commutative DG alg generated by A in degree 0. One therefore has a canonical surjection of DG algs

$$\pi : \Omega A \longrightarrow \Omega_A$$

The ideal should be generated by the relations

$$[a_0 da_1, a_2] = 0$$

$$[da_1, da_2] = 0$$

Let us define b to be zero on Ω_A. Since $b(\omega da) = (-1)^{|\omega|}[\omega, a]$ defines b in ΩA and this goes to zero in Ω_A, we see π is compatible with b operators. Simpler to say $\pi b = 0$. It follows that

$$\pi(1-\kappa) = \pi(bd+db) = d\pi b = 0$$

simpler to use that $\kappa(\omega da) = (-1)^{|\omega|} da\, \omega$ becomes ωda in Ω_A.

Thus we see the map π kills $(1-\kappa)\Omega A$ which contains $(1-\kappa)^2 \Omega A = P^1 \Omega A$. Hence we get an induced map

$$P\Omega(A) \cong \Omega A/(1-\kappa)^2 \Omega A \longrightarrow \Omega_A$$

which is compatible with d, b. If we put in factorials it becomes compatible with $b+B$ on the former and d on the latter.

March 1, 1991

Question. Let $f(x,y)$ be a Hoch 1-cocycle, i.e. f trace on $\Omega^1 R$. Is there a largest ideal I in R such that f descends to $\Omega^1(R/I)$?

Let $J = \{ z \in R \mid f(z\Omega^1 R) = 0 \}$. Then J is an ideal: $f(zx\Omega^1 R) \subset f(z\Omega^1 R) = 0$ and $f(xz\Omega^1 R) = f(z\Omega^1 R\, x) \subset f(z\Omega^1 R) = 0$. Clearly J is the largest ideal so that f descends to $\Omega^1 R / J\Omega^1 R + \Omega^1 R J$.

Let $I = \{ z \in J \mid f(R\,dz) = 0 \}$. Then for $x \in R$

$$f(Rd(xz)) = f(Rdx\, z) + f(Rx\,dz)$$
$$\subset f(\underset{I}{zR}\, dx) + \cancel{f(Rdz)}$$
$$\subset f(I\Omega^1 R) = 0$$
$$f(Rd(zx)) = f(\underbrace{Rdz\,x} + Rz\,dx)$$
$$\subset f(Rdz) + f(I\Omega^1 R) = 0.$$

(Check: Given a trace τ on a bimodule M over R, we have a map of bimodules $R \longrightarrow M^*$, $1 \longmapsto \tau$ and the kernel J of this map is and ideal in R. $J = \{ z \in R \mid \tau(zM) = 0 \}$ and τ descends to $M/F^1_J M = M/JM + MJ$.)

This shows I is an ideal such that $f(I\Omega^1 R) = 0$ and $f(RdI) = 0$, whence because of

$$0 \longrightarrow I/I^2 \longrightarrow \Omega^1 R / F^1_I \Omega^1 R \longrightarrow \Omega^1(R/I) \longrightarrow 0$$

f descends to $\Omega^1(R/I)$. Clearly I is the largest such ideal.

Let's recall now the description of $X(RA)$:

$$X(RA): \quad \begin{tikzcd} RA \arrow[r, shift right, "\bar{d}"'] & \Omega^1(RA)_\natural \arrow[l, shift right, "\bar{b}"'] \\ \bigoplus\limits_{n\geq 0}\Omega^{2n}A \arrow[u, "\rho\omega^n", "\cong"'] & \bigoplus\Omega^{2n+1}A \arrow[u, "\cong", "\rho\omega^n d\rho"'] \end{tikzcd}$$

Recall

$$\bar{b}(\rho\omega^n d\rho) = (\rho\omega^n)b - (\rho\omega^{n+1})(1+K)d$$

$$\bar{d}(\rho\omega^n) = \text{[redacted]} - (\rho\omega^{n-1}d\rho)\sum_{j=0}^{n-1}K^{2j}b + (\rho\omega^n d\rho)\sum_{j=0}^{2n}K^j d$$

Thus we have the nice formulas:

$$\bar{b} = b - (1+K)d$$

$$\bar{d} = -N_{K^2}b + N_K d \quad \overset{\text{wrong}}{\boxed{\cancel{= -N_{K^2}(b-(1+K)d)}}}$$

Suppose now that A is commutative. Then we have

$$\begin{tikzcd} RA \arrow[r, shift right, "\bar{d}"'] \arrow[d, dash, "\cong"'] & \Omega^1(RA)_\natural \arrow[l, shift right, "\bar{b}"'] \arrow[d, dash, "\cong"'] \\ \Omega^+A \arrow[r, shift right, "\cancel{-N_{K^2}(b-(1+K)d)}"', "-N_{K^2}b+N_K d"' near end] \arrow[d] & \Omega^-A \arrow[l, shift right, "b-(1+K)d"'] \arrow[d] \\ \Omega^+_A \arrow[r, shift right, "\text{[redacted]}\ Nd"'] & \Omega^-_A \arrow[l, shift right, "-2d"'] \end{tikzcd}$$

$\Leftarrow$ obvious DG map $\Omega A \twoheadrightarrow \Omega_A$

$N = q$ on $\Omega^q A$

This shows that after rescaling the $\mathbb{Z}/2$-graded complex Ω_A appears as a quotient of $X(RA)$, in fact of $\hat{X}(RA)$.(*) Now I would like to find a nilpotent extension of A whose X-complex maps naturally to Ω_A. It's clear now that there is a rather nice choice as quotient algebras of RA. (*) I'm assuming here that A is finitely generated, so that $\rho\omega^n = 0$ for $n >> 0$. So RA/IA^N works for large N.