November 16, 1990

I am trying to decide whether the result about $X(R)^\wedge \simeq \varprojlim X(R/I^n)$ being unique up to homotopy is really interesting. (Here $A = R/I$ and $\Omega^1 R$ is proj.) The point is that I haven't really explored possible consequences of the main result

$$(1) \qquad P\hat{\Omega}(A) \sim X(RA)^\wedge = X_{top}(\hat{R}A)$$

Recall we have an explicit homotopy equivalence. I think it should be possible to prove that we have a heq given by the canonical quotient map

$$(2) \qquad P\hat{\Omega}_{top}(\hat{R}A) \longrightarrow X_{top}(\hat{R}A)$$

(both sides being understood in the topological algebra sense) This should be true for any adic algebra R with $\Omega^1_{top}(R)$ projective. On the other hand, there is ~~[illegible]~~ Goodwillie's theorem that if E is a nilpotent extension of A, then

$$(3) \qquad P\hat{\Omega}(E) \longrightarrow P\hat{\Omega}(A)$$

is a homotopy equivalence. Now (2) + (3) seem to imply (1), namely one has heqs.

$$P\hat{\Omega}(A) \longleftarrow P\hat{\Omega}_{top}(\hat{R}A) \longrightarrow X_{top}(\hat{R}A)$$

In fact for any adic algebra R with $\Omega^1 R$ proj. and $A = R/I$, I an ideal of definition for the topology one should have heq's

$$P\hat{\Omega}(A) \xleftarrow{(3)} P\hat{\Omega}_{top}(R) \xrightarrow{(2)} X_{top}(R)$$

which gives the result that $X_{top}(R)$ up to homotopy depends only on A.

So the issue is whether there is an independent proof of Goodwillie's thm (3).

In particular we want a map backwards: $P\hat{\Omega}(A) \longrightarrow P\hat{\Omega}(E)$, when E is a nilpotent extension of A. One way to do this is to use our result (1). This means we want a map

$$X_{top}(\hat{R}A) \longrightarrow X_{top}(\hat{R}E)$$

and the obvious way to get this is from a homomorphism $\hat{R}A \longrightarrow \hat{R}E$.

More generally consider

$$\begin{tikzcd} \hat{R}A \arrow[r, dashed] \arrow[d] & \hat{R}E \arrow[d] \\ & E \arrow[d] \\ A \arrow[r] & E/J \end{tikzcd} \qquad J^N = 0$$

It's clear that we have many choices for the dotted arrow, so the fact that they induce homotopic maps on X_{top} becomes significant.

I originally thought that a linear map $\rho : A \longrightarrow B$, whose curvature is nilpotent, should induce a map $P\hat{\Omega}(A) \longrightarrow P\hat{\Omega}(B)$. It looks now like other data is involved. We have to find

$$\begin{tikzcd} & \hat{R}B \arrow[d] \\ \hat{R}A \arrow[ur, dashed] \arrow[r] & B \end{tikzcd}$$

which means lifting $\begin{tikzcd} & \hat{R}B \arrow[d] \\ A \arrow[ur, dashed] \arrow[r, "\rho"'] & B \end{tikzcd}$

Of course we have the obvious candidate $\rho B \rho$.

Let's see if this choice is functorial.

$$RA \xrightarrow{u_1} RB \xrightarrow{u_2} RC$$

$$A \xrightarrow[\rho_1]{} B \xrightarrow[\rho_2]{} C$$

u_1 is the unique homomorphism such that $u_1 \rho_A = \rho_B \rho_1$. Similarly u_2 is characterized by $u_2 \rho_B = \rho_C \rho_2$. Then

$$u_2 u_1 \rho_A = u_2 \rho_B \rho_1 = \rho_C \rho_2 \rho_1 .$$

Of course this functorality is obvious if you recall that RA is a functor of the underlying vector space of A.

At this point we ~~understand~~ appreciate Connes' statement that asymptotic morphisms induce maps on entire theory. The issue in the formal setting is whether the map $RA \to RB$ associated to $\rho: A \to B$ is continuous for the topologies, and this will be the case iff the curvature is nilpotent.

Recall the old problem of associating to $\rho: A \to R$ a map from X of the bar construction of A to $X(R)$. This map was to be based on cochains ρe^{ω} in some way, and it was supposed to refine what I was able to do if I looked at cyclic cochains with values in $X(R)$ or Hochschild & bar cochains with values in $R_{\natural}$. I have now a solution to this problem given by a map

$$P\Omega(A) \longrightarrow X(R)$$

which is an embedding in the case where $R = RA$. However the question remains as to whether there might be a nice link with the bar construction. In other words, <u>is there some nice algebra of cochains with values in</u> R <u>in which to work?</u> Recall that in the augmented algebra case $A = \mathbb{C} \oplus \bar{A}$ we worked with the DG algebra $\mathrm{Hom}(B(\bar{A}), R)$ which is roughly a tensor product.

The point is that we've learned a lot ~~that~~ since looking at this question before and we should try to use our new knowledge.

November 17, 1990

I have studied the effect of a derivation D of an algebra A on $\Omega(A)$. I now wish to carry the results over to the more general situation of a homomorphism $u : A \to B$ together with a derivation $\dot{u} : A \to B$ relative to u. (First order variation of a homomorphism).

Let's consider homotopy for the DR complex of a manifold. One is given a 1-parameter family of maps $f_t : M \to N$, whence we have induced homomorphisms $f_t^* : \Omega(N) \to \Omega(M)$ and we can take the derivative $\partial_t f_t^*$ which is a derivation with respect to f_t^*. The key fact is that $L = \partial_t f_t^*$ is homotopic to zero, that is, there is a derivation $i = i(f_t^*, \partial_t f_t^*) : \Omega(N) \to \Omega(M)$ of degree -1 and rel to f_t^* such that $L = [d, i]$. This is derived from Cartan's homotopy formula for derivations as follows. One considers the maps

$$f : \mathbb{R} \times M \to N \qquad j_t : \begin{array}{ccc} M & \longrightarrow & \mathbb{R} \times M \\ x & \longmapsto & (t, x) \end{array}$$

such that $f_t = f j_t$. This reduced one to the case of the family $j_t : M \to \mathbb{R} \times M$. One has

$$\Omega(\mathbb{R} \times M) = \underbrace{\Omega(\mathbb{R})}_{\Omega^0(\mathbb{R}) + dt\,\Omega^0(\mathbb{R})} \otimes \Omega(M)$$

so any form ω on $\mathbb{R} \times M$ has a unique expression $\omega = \alpha + dt\,\beta$ with $\alpha, \beta \in \Omega^0(\mathbb{R}, \Omega(M))$. On $\mathbb{R} \times M$ we have the vector field $X = \partial_t$, hence Lie derivative and interior product operators L_X, ι_X which satisfy the Cartan homotopy fmla $L_X = d\iota_X + \iota_X d$.

If $\omega = \alpha + dt\,\beta$ as above we have

$$j_t^* \omega = j_t^* \alpha = \alpha_t$$

hence $\partial_t j_t^* \omega = \partial_t \alpha_t = (L_X \alpha)_t = j_t^* L_X \omega$

Thus $\partial_t j_t^* = j_t^* L_X$ so if we put $\iota_t = j_t^* i_X$ we have

$$L_t = \partial_t j_t^* = j_t^*(d\iota_X + \iota_X d) = d\iota_t + \iota_t d$$

~~Thus in the general case $f : R \times M \to N$ we have~~

$$\cancel{\partial_t f_t^* = \partial_t j_t^* f^* = [d, \iota_t] f^* = [d, \iota_t f^*]}$$

$$\Omega(\mathbb{R}\times M) \underset{i_X}{\overset{L_X}{\rightrightarrows}} \Omega(\mathbb{R}\times M) \xrightarrow{j_t^*} \Omega(M)$$

So now when we have a map $f : \mathbb{R}\times M \to N$ we get map

$$\Omega(N) \xrightarrow{f^*} \Omega(\mathbb{R}\times M) \underset{\iota_X}{\overset{L_X}{\rightrightarrows}} \Omega(\mathbb{R}\times M) \xrightarrow{j_t^*} \Omega(M)$$

$$\partial_t f_t^* = \partial_t j_t^* f^* = j_t^* L_X f^* = [d, j_t^* \iota_X f^*]$$

Thus we establish the infinitesimal homotopy property for a general family f_t using the Cartan homotopy fmla for derivations.

I'm interested in the case of a first order variation $(f, \dot f)$, where f is a map $: M \to N$ and $\dot f$ is a section of $f^* TN$, rather than a whole 1-parameter family. (Why? Because ~~[illegible]~~ it is the basic miracle of cyclic theory that homotopy for traces reduces to this first order situation.)

A map ~~plus~~ first order variation (also called a 1-jet) $(f,\mathring{f}) : M \to N$ is the same as a map $M \to TN$. Recall that we discussed the "variation map" for Lie algebra cohomology in these terms. Recall we have three maps

$$\Omega(N) \underset{i}{\overset{\pi^*}{\underset{L}{\rightrightarrows}}} \Omega(TN) \qquad L = di + id$$

π^* is a DG homomorphism ($\pi : TN \to N$ is the canonical projection), L is a degree 0 derivation (Lie derivative) and i is a degree -1 derivation, both of these derivations being relative to π^*.

Recall that if $x = (x_i)$ are coordinates for N we have (locally)

$$\Omega^0 N = \mathbb{C}[x] \qquad \Omega^0(TN) = \mathbb{C}[x,\mathring{x}]$$

$$\Omega N = \mathbb{C}[x, dx] \qquad \Omega(TN) = \mathbb{C}[x, dx, \mathring{x}, d\mathring{x})$$

and $\pi^* x = x$, $\pi^* dx = dx$, $Lx = \mathring{x}$, $L dx = d\mathring{x}$, $i(x) = 0$, $i(dx) = \mathring{x}$. Check:

$$(di + id)x = i\,dx = \mathring{x} = Lx$$

$$(di + id)(dx) = d\mathring{x} = L(dx)$$

If $N = G$ a Lie group, $TG = G \ltimes \mathfrak{g}$

$$\begin{array}{cc}
\Omega(G) & \Omega(TG) \\
\cup & \cup \\
\Lambda \mathfrak{g}^* & \Lambda(\mathfrak{g} \oplus \mathfrak{g})^* \\
\| & \| \\
\Lambda[\theta] & \Lambda[\theta, \mathring{\theta}] \\
\delta\theta + \frac{1}{2}[\theta,\theta] = 0 & \delta\theta + \frac{1}{2}[\theta,\theta] = 0 \\
 & \delta\mathring{\theta} + [\theta, \mathring{\theta}] = 0
\end{array}$$

$$\pi^*\theta = \theta, \quad L\theta = \mathring{\theta}$$

Note that i doesn't preserve left-invariant forms. Thus $i\theta$ is the function on TG which sends a tangent vector to its pairing with θ. This is not constant on TG.

Comment: Algebraically a homomorphism $(u, \dot{u}) : A \longrightarrow B$ with first order variation is the same as an element of

$$\mathrm{Hom}_{alg}(A, \mathbb{C}[\varepsilon]\otimes R) = \mathrm{Hom}_{alg}(\underbrace{T_A(\Omega^1 A)}_{\Omega A}, R)$$

Thus ΩA appears as the non-commutative tangent bundle. Geometrically the analogue of $\mathbb{C}[\varepsilon]\otimes R$ is the first order inf. nbd of $\{0\}\times M$ in $\mathbb{R}\times M$.

Let's now try to work with $R[\varepsilon] = \mathbb{C}[\varepsilon]\otimes R$ instead of $R[t] = \mathbb{C}[t]\otimes R$. Our aim is to derive the infinitesimal homotopy formula for a general first order variation of a homomorphism from the case we've already handled for derivations.

The idea is that we have three basic maps

$$\Omega(\mathbb{C}[\varepsilon]\otimes R) \overset{\delta_0}{\underset{\iota^*}{\overset{L}{\rightrightarrows}}} \Omega(R)$$

(In the noncommutative setting i should become ι^*.) We want the homotopy formula $L = [d, \iota^*]$ to follow from the case of derivations.

Let us consider then the possible derivations of $\mathbb{C}[\varepsilon]$. A derivation $D : \mathbb{C}[\varepsilon] \longrightarrow M$, where M is a bimodule is the same as the element $D\varepsilon$ of M which can be arbitrary subject to the condition $\varepsilon D\varepsilon + D\varepsilon\, \varepsilon = 0$. If left + right multiplication on M coincides, this means $\varepsilon D\varepsilon = 0$. Thus

$$\Omega^1_{\mathbb{C}[\varepsilon]} = \left(\mathbb{C}[\varepsilon]/(\varepsilon)\right) d\varepsilon$$

and the (commutative) DR complex of $\mathbb{C}[\varepsilon]$ is

$$\underset{0}{\mathbb{C}[\varepsilon]} \xrightarrow{d} \underset{1}{\left(\mathbb{C}[\varepsilon]/(\varepsilon)\right)d\varepsilon} \longrightarrow 0 \longrightarrow 0 \longrightarrow$$

so it's 3-dimensional with basis $1, \varepsilon, d\varepsilon$.
All derivations $D: \mathbb{C}[\varepsilon] \to \mathbb{C}[\varepsilon]$ are scalar multiples of $\varepsilon\partial_\varepsilon$.

Recall that for commutative algebras we have

$$\Omega_{A\otimes B} = \Omega_A \otimes \Omega_B$$

There are various ways to see this. For example, $A \mapsto \Omega_A$ is left adjoint to the forgetful functor for comm. DGAs to comm. algebras $R \mapsto R^0$. And left adjoint functors commute with direct sums. Also [redacted] it suffices to check that

$$\Omega^1_{A\otimes B} = (\Omega^1_A \otimes B) \oplus (A \otimes \Omega^1_B)$$

and then apply Λ over $A\otimes B$ to both sides. But one has

$$\Omega^1_{A\otimes B} = HH_1(A\otimes B) = \left(HH(A)\otimes HH(B)\right)_{(1)}$$
$$= (A \otimes \Omega^1_B) \oplus (\Omega^1_A \otimes B)$$

[redacted]

Thus in the commutative case we have

$$\Omega_{\mathbb{C}[\varepsilon]\otimes R} = \Omega_{\mathbb{C}[\varepsilon]} \otimes \Omega_R$$

and since $\Omega_{\mathbb{C}[\varepsilon]}$ has the basis $1, \varepsilon, d\varepsilon$, we obtain three operators $\Omega_{\mathbb{C}[\varepsilon]\otimes R} \to \Omega_R$. A similar method works in the non-comm. case. We have canonical maps of DG algs

$$\Omega(\mathbb{C}[\varepsilon]\otimes R) \longrightarrow \Omega(\mathbb{C}[\varepsilon]) \otimes \Omega(R) \longrightarrow \Omega_{\mathbb{C}[\varepsilon]} \otimes \Omega(R)$$

hence operators $\Omega(\mathbb{C}[\varepsilon]\otimes R) \longrightarrow \Omega R$
given by the coefficients of $1, \varepsilon, d\varepsilon$.
Let's compute these operators more generally where $\mathbb{C}[\varepsilon]\otimes R$ is replaced by an algebra A with a hom.

$$u+\varepsilon\hat{u} : A \longrightarrow \mathbb{C}[\varepsilon]\otimes R$$

One then gets DGA morphisms

$$\Omega A \longrightarrow \Omega(\mathbb{C}[\varepsilon]\otimes R) \longrightarrow \Omega_{\mathbb{C}[\varepsilon]} \otimes \Omega A$$

which we can write $\quad \omega \longmapsto u_*(\omega) + \varepsilon L\omega + d\varepsilon\, \iota^*\omega$

One has

$$a_0\, da_1 \cdots da_n \longmapsto (ua_0 + \varepsilon\hat{u}a_0)\big(d(ua_1) + \varepsilon d(\hat{u}a_1) + d\varepsilon(\hat{u}a_1)\big)\cdots$$

$$\longmapsto ua_0\, d(ua_1)\cdots d(ua_n)$$
$$+\varepsilon\Big\{\hat{u}a_0\, d(ua_1)\cdots d(ua_n) + \sum_{j=1}^{n} ua_0 \cdots du(a_{j-1})\, d(\hat{u}a_j)\, d(ua_{j+1})\cdots\Big\}$$
$$+d\varepsilon\Big\{\sum_{j=1}^{n}(-1)^{j-1} ua_0 \cdots d(ua_{j-1})(\hat{u}a_j)\, d(ua_{j+1})\cdots\Big\}$$

so that L is the Lie derivative and ι^* the interior product associated to $u, \hat{u}$.

Recall that because $\omega \longmapsto u_*\omega + \varepsilon L\omega + d\varepsilon\,\iota^*\omega$ is a DGA morphism we have that L, ι^* are derivations relative to u_* and also

$$0 = [d, u_* + \varepsilon L + d\varepsilon\,\iota^*] = d\varepsilon\, L + \varepsilon[d, L] - d\varepsilon[d, \iota^*]$$

so that $[d, L] = 0$, $[d, \iota^*] = L$.

Now at this point we understand the three operators associated to $(u, \hat{u})$ in the framework of operators

$$\Omega(\mathbb{C}[\varepsilon]\otimes R) \Rrightarrow \Omega(R).$$

This is the analogue of the maps $\Omega(N) \Rrightarrow \Omega(TN)$

in the 1-jet situation.

What I want to do next is to improve the understanding from the (Ω, d) level to the $(\Omega, d, b, \kappa, \dots)$ level. Having done this already for derivations I want to show there's an obvious extension to the case of $A \xrightarrow{u+\varepsilon\dot{u}} \mathbb{C}[\varepsilon]\otimes R$.

First suppose we have a derivation $D: A \to A$. Then the operators L_D, ι_D^* arise from the general construction above:

$$\begin{tikzcd} \Omega(A) \arrow[r, "(1+\eta D)_*"] \arrow[rr, bend right, "1+\eta L_D + d\eta\, \iota_D^*"'] & \Omega(\mathbb{C}[\eta]\otimes A) \arrow[r] & \Omega_{\mathbb{C}[\eta]} \otimes \Omega(A) \end{tikzcd}$$

Now take $A = \mathbb{C}[\varepsilon]\otimes R$ and $D = \varepsilon\partial_\varepsilon$. Let's calculate

$$\begin{tikzcd} \Omega(\mathbb{C}[\varepsilon]\otimes R) \arrow[r, "1+\eta L_D + d\eta\, \iota_D^*"] \arrow[dr] & \Omega_{\mathbb{C}[\eta]} \otimes \Omega(\mathbb{C}[\varepsilon]\otimes R) \arrow[d] \\ & \Omega_{\mathbb{C}[\eta]} \otimes \mathbb{C}[\varepsilon] \otimes \Omega R \end{tikzcd}$$

This is a DGA map such that

$$x+\varepsilon y \longmapsto (1+\eta L_D + d\eta\, \iota_D^*)(x+\varepsilon y)$$

$$\longmapsto x + \eta\varepsilon y$$

However observe that we have a DG homom.

$$\Omega_{\mathbb{C}[\varepsilon]} \longrightarrow \Omega_{\mathbb{C}[\eta]} \otimes \mathbb{C}[\varepsilon]$$

$$\varepsilon \longmapsto \eta\varepsilon$$

$$d\varepsilon \longmapsto d(\eta\varepsilon) = (d\eta)\varepsilon$$

Thus we have a commutative diagram

$$\begin{tikzcd}
\Omega(\mathbb{C}[\varepsilon]\otimes R) \arrow[r, "1+\eta L_D + d\eta\, i_D^*"] \arrow[d, "j_0 + \varepsilon L + d\varepsilon\, \iota^*"] & \Omega_{\mathbb{C}[\eta]} \otimes \Omega(\mathbb{C}[\varepsilon]\otimes R) \arrow[d] \\
\Omega_{\mathbb{C}[\varepsilon]} \otimes \Omega R \arrow[r] & \Omega_{\mathbb{C}[\eta]} \otimes \mathbb{C}[\varepsilon] \otimes \Omega R \\
\varepsilon, d\varepsilon \arrow[r, mapsto] & \eta\varepsilon, d\eta\,\varepsilon
\end{tikzcd}$$

This shows that one can get L, ι^* from L_D, ι_D by taking the coefficient of ε.

Another way to say this is that the canonical L, ι^* maps are given by

$$\begin{tikzcd}
\Omega(\mathbb{C}[\varepsilon]\otimes R) \arrow[r, shift left, "L_D"] \arrow[r, shift right, "\iota_D^*"'] \arrow[rdd, bend right, "L"] \arrow[rdd, bend right=40, "\iota^*"'] & \Omega(\mathbb{C}[\varepsilon]\otimes R) \arrow[d] \\
 & \mathbb{C}[\varepsilon]\otimes \Omega R \arrow[d, "\text{coeff of } \varepsilon"] \\
 & \Omega R
\end{tikzcd}$$

Now we know the vertical arrows are compatible with d, b. Thus I_D will give rise to an interior product I compatible with b such that $[d, I] = L$, etc.

Further comments. You have in effect defined a map $\mathfrak{g} \longrightarrow \mathrm{Der}(\Omega A)$ where $\mathfrak{g}$ is the DG Lie algebra with basis L_D, ι_D^*. So far one has a map of complexes, but one still should give a nice proof that it is a Lie homomorphism, in particular that $(\iota_D^*)^2 = 0$. Maybe there is a nice way to get the BRS DGA $\Lambda\mathfrak{g}^* \otimes \Omega A$.

Observe that the map

$$\Omega(A\otimes B) \longrightarrow \Omega A \otimes \Omega B$$

is not compatible with b, not even in degree one.

<u>Idea</u>: To what extent can the above homotopy discussion based on $\mathbb{C}[\varepsilon]$ be generalized to an S with a null homotopic trace. (Note that $\mathbb{C}[\varepsilon] \xrightarrow{d} (\mathbb{C}[\varepsilon]/(\varepsilon))\,d\varepsilon$ shows that the trace on $\mathbb{C}[\varepsilon]$ giving the coefficient of ε is null-homotopic) Thus I would like to know that given $A \longrightarrow B\otimes S$ together with a null-homotopic trace τ, then the map $P\Omega(A) \longrightarrow P\Omega(B\otimes S) \longrightarrow P\Omega(B)\otimes S_\natural \xrightarrow{1\otimes\tau} P\Omega(B)$ is nullhomotopic.

November 18, 1990

Suppose $\Omega^1 R$ is projective, i.e. there is a splitting

$$(*) \qquad 0 \longrightarrow \Omega^2 R \longrightarrow \Omega^1 R \otimes R \underset{\overset{h}{\curvearrowleft}}{\longrightarrow} \Omega^1 R \longrightarrow 0$$

where $\boxed{h(\xi) = \sum \varphi_i(\xi) \otimes x_i}$ is a bimodule map such that $\sum \varphi_i(\xi) x_i = \xi$. (Here we take x_i to be a basis of R, whence the φ_i are well-defined left R-module maps $\Omega^1 R \to \Omega^1 R$. This is somehow unnecessary, but I don't have a better notation for a map into a tensor product. Should I think of such a map as a "diagonal approximation"?)

The corresponding projection $\Omega^1 R \otimes R \xrightarrow{k} \Omega^2 R$ is found as follows

$$\xi \otimes 1 - h(\xi) = \xi \otimes 1 - \sum' \varphi_i(\xi) \otimes x_i$$

$$= \sum \varphi_i(\xi)(x_i \otimes 1 - 1 \otimes x_i) = b'\left\{-\sum' \varphi_i(\xi)\, dx_i\right\}$$

so the projection is $\boxed{k(\xi \otimes 1) = -\sum \varphi_i(\xi)\, dx_i}$.

Example: $R = T(V)$

$$\begin{array}{ccc}
\Omega^1 R \otimes R & & \Omega^1 R \\
(R \otimes V \otimes R) \otimes R & & R \otimes V \otimes R \\
dv \otimes 1 & \longleftarrow\!\shortmid & dv \\
 & & d(v_1 \cdots v_n) \\
 & & \| \\
\sum v_1 \cdots v_{j-1}\, dv_j \otimes v_{j+1} \cdots v_n & \longleftarrow\!\shortmid & \sum v_1 \cdots v_{j-1}\, dv_j\, v_{j+1} \cdots v_n
\end{array}$$

$$\therefore \quad \boxed{h(d(v_1 \cdots v_n)) = \sum v_1 \cdots v_{j-1}\, dv_j \otimes v_{j+1} \cdots v_n}$$

$$\boxed{\begin{aligned} k(d(v_1 \cdots v_n) \otimes 1) &= -\sum_j v_1 \cdots v_{j-1}\, dv_j\, d(v_{j+1} \cdots v_n) \\ &= -\sum_{j<k} v_1 \cdots dv_j \cdots dv_k \cdots v_n \end{aligned}}$$

By tensoring (*) on the left with $\Omega^{n-1}R$ over R we obtain splittings

$$\begin{tikzcd} 0 \arrow[r] & \Omega^{n+1}R \arrow[r] & \Omega^n R\otimes R \arrow[r] \arrow[l, bend right, dashed, "k"'] & \Omega^n R \arrow[r] \arrow[l, bend right, dashed, "h"'] & 0 \end{tikzcd}$$

$$h(\omega\xi) = \omega\sum\varphi_i(\xi)\otimes X_i \qquad \xi\in\Omega^1R$$

$$k(\omega\xi\otimes 1) = (-1)^{|\omega|+1}\,\omega\sum\varphi_i(\xi)\,dX_i$$

Splicing the splittings together we obtain a contracting homotopy for the bimodule exact sequence

$$\begin{tikzcd} {} \arrow[r] & \Omega^2R\otimes R \arrow[r] \arrow[l, bend right, dashed] & \Omega^1R\otimes R \arrow[r] \arrow[l, bend right, dashed, "H"'] & \Omega^1 R \arrow[r] \arrow[l, bend right, dashed, "h"'] & 0 \end{tikzcd}$$

where
$$H(\omega\xi\otimes 1) = hk(\omega\xi\otimes 1)$$
$$= h\Big\{(-1)^{|\omega|+1}\sum_i \omega\varphi_i(\xi)\,dX_i\Big\}$$

$$\boxed{H(\omega\xi\otimes 1) = (-1)^{|\omega|+1}\sum_i \omega\,\varphi_i(\xi)\sum_j\varphi_j(dX_i)\otimes X_j}$$

Let's check this works i.e. $b'H + Hb' = 1$ on $\Omega^nR\otimes R$ for $n\geq 2$. (a special argument needs to be done for $n=1$.) We have

$$(b'H)(\omega da\otimes 1) = b'\Big\{(-1)^{|\omega|+1}\,\omega\varphi_i(da)\,\varphi_j(dX_i)\otimes X_j\Big\} \qquad \text{summ. conv.}$$

$$= b'\Big\{(-1)^{|\omega|+1}\,\omega\varphi_i(da)\underbrace{\varphi_j(dX_i)X_j}_{dX_i}\Big\}$$

$$= \omega\varphi_i(da)(X_i\otimes 1 - 1\otimes X_i) = \omega da\otimes 1 - \omega\varphi_i(da)\otimes X_i$$

$$Hb'(\omega da\otimes 1) = H\big\{(-1)^{|\omega|}(\omega a\otimes 1 - \omega\otimes a)\big\} \qquad \text{Suppose } \omega = \omega_1 da_1$$

$$= (-1)^{|\omega|}H(\omega_1 da_1 a\otimes 1 - \omega_1 da_1\otimes a)$$

$$= \omega_1\Big\{\underbrace{\varphi_i(da_1 a)\,\varphi_j(dX_i)}_{\varphi_i(da_1)\varphi_j(d(X_i a))}\otimes X_j - \varphi_i(da_1)\underbrace{\varphi_j(dX_i)\otimes X_j a}_{\varphi_j(dX_i a)\otimes X_j}\Big\}$$

where we use the identity

$$\sum \varphi_i(\xi a) \otimes X_i = \sum \varphi_i(\xi) \otimes X_i a$$

in $\Omega^1 R \otimes R$ and ~~the various~~ take its image under various bilinear maps.

Continuing we have $\quad (\ d(X_i a) = dX_i a + X_i da$

$$Hb'(\omega da \otimes 1) = \omega_1 \varphi_i(da_1)\, \varphi_j(X_i da) \otimes X_j$$

$$= \omega_1 \varphi_i(da_1) X_i\, \varphi_j(da) \otimes X_j$$

$$= \omega_1 da_1\, \varphi_j(da) \otimes X_j = \omega\, \varphi_j(da) \otimes X_j$$

Therefore $(b'H + Hb')(\omega da \otimes 1) = \omega da \otimes 1.$

November 20, 1990

Let us consider an inverse system of algebras $\{R_n, n \geq 1\}$ such that if $I_n = \mathrm{Ker}\{R_n \to R_1\}$, then we have

$$R_n / I_n^q \xrightarrow{\sim} R_q$$

for $1 \leq q \leq n$. Then if we put $\hat{R} = \varprojlim R_n$, $\hat{I} = \varprojlim I_n$, $\hat{R}$ is a topological algebra with nbd basis given by $\hat{I}^q$, $q \geq 0$. Also $\hat{I}$ is an open ideal. I think the converse is true and discussed by Grothendieck under the term adique topological algebra (topological algebra with nbd nbd given by ideals for which $\exists$ an open ideal whose closed powers form a nbd basis.)

Example is the I-adic tower $R_n = R/I^n$ associative to an algebra with an ideal.

Given $\{R_n, n \geq 1\}$ an "adic" inverse system, let F be an algebra, and let $F \twoheadrightarrow R_n$ be a compatible family of surjective homomorphisms. Put

$$J_n = \mathrm{Ker}\{F \to R_n\}$$

so that $I_n = J_1/J_n$ and $I_n^q = (J_1/J_n)^q = J_1^q + J_n/J_n$ in $F/J_n = R_n$. Thus we have

$$J_q = J_1^q + J_n \qquad \text{for} \quad n \geq q \geq 1$$

by the adic property of $\{R_n\}$. Let's compare $\hat{F} = \varprojlim F/J_1^n$ with $\hat{R}$. We have

$$0 \to J_n/J_1^n \longrightarrow F/J_1^n \longrightarrow R_n \longrightarrow 0$$

and the inverse system J_n/J_1^n has surjective maps because

$$J_n/J_1^n \longrightarrow J_{n-1}/J_1^{n-1} \quad \text{has image}$$
$$J_n + J_1^{n-1}/J_1^{n-1} \quad \text{and} \quad J_n + J_1^{n-1} = J_{n-1}.$$

Thus $R^1 \varprojlim_n (J_n/J_1^n) = 0$ and we have a surjection

$$\hat{F} = \varprojlim F/J_1^n \longrightarrow\!\!\!\!\to \hat{R}$$

I would like now to construct a lifting $\hat{R} \longrightarrow \hat{F}$ assuming that

$$R_n \otimes_{R_{2n}} \Omega^1 R_{2n} \otimes_{R_{2n}} R_n \quad \text{is a projective } R_n\text{-bimod.}$$

Recall that in the case $R_n = R/I^n$, where $\Omega^1 R$ is a projective R-bimodule, we have that

$$R_n \otimes_{R} \Omega^1 R \otimes_R R_n \xrightarrow{\sim} R_n \otimes_{R_{2n}} \Omega^1 R_{2n} \otimes_{R_{2n}} R_n$$

is a projective R_n-bimodule.

November 21, 1990

An "adic algebra" is a topological algebra R such that there exists an open ideal I whose closed powers I^n form a nbd basis of 0. Then

$$R = \varprojlim R/I^n$$

is an inverse limit of algebras R_n such that if $R_n/I_n \xrightarrow{\sim} R_1$ we have $R_n/I_n^{\,q} \xrightarrow{\sim} R_q$.

Let's choose an algebra F together with a compatible family of surjections $F \longrightarrow\!\!\!\!\to R_n$, and put $J_n = \operatorname{Ker}(F \to R_n)$, $J = J_1$. Then we have seen that $\hat{F} = \lim F/J^n$ maps onto $R = \varprojlim R_n$. Now what I want to do is to produce a section homomorphism for $\hat{F} \longrightarrow R$, i.e. a lifting $R \to \hat{F}$. We assume $R_n \otimes_{R_{2n}} \Omega^1(R_{2n}) \otimes_{R_{2n}} R_n$ is projective as an R_n bimodule. We proceed inductively

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
R_4 \arrow[r, dashed] \arrow[d] & F/J^4 \arrow[r] \arrow[d] & R_4 \arrow[d] \\
R_2 \arrow[r, dashed] \arrow[d] & F/J^2 \arrow[r] \arrow[d] & R_2 \arrow[d] \\
R_1 \arrow[r, equal] & F/J \arrow[r, equal] & R_1
\end{tikzcd}$$

The first step is to lift R_2 into the (square zero) extension F/J^2 of R_1. We can do this because $R_1 \otimes_{R_2} \Omega^1(R_2) \otimes_{R_2} R_1$ is projective, which means R_2 is a versal square zero extension of R_1. Having chosen $R_2 \to F/J^2$ we now have a lifting problem

$$\begin{tikzcd}
 & F/J^4 \arrow[d, "\pi"] \\
R_4 \arrow[r] \arrow[ur, dotted] & F/J^2 \times_{R_2} R_4
\end{tikzcd}$$

The key point is to see that π is surjective, I think. Let's check this.

$$\begin{tikzcd}
0 \arrow[r] & J_4/J^4 \arrow[r] \arrow[d] & F/J^4 \arrow[r] \arrow[d] & R_4 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J_2/J^2 \arrow[r] & F/J^2 \arrow[r] & R_2 \arrow[r] & 0
\end{tikzcd}$$

It suffices to show is surjective, which we verified earlier in proving $\hat{F} \twoheadrightarrow R$. i.e.

$$J^2 + J_4 = J_2 \qquad \text{(in general } J^q + J_n = J_q \text{ for } q \le n.\text{)}$$

Thus π is surjective so when pulled back over R_4 we have an extension of R_4. It is a square zero extension F/J^4 is a square zero extension of F/J^2.

$$\begin{tikzcd}
 & F/J^4 \arrow[d, "\pi"] \\
R_4 \arrow[r] \arrow[d] & F/J^2 \times_{R_2} R_4 \arrow[d] \\
R_2 \arrow[r] & F/J^2
\end{tikzcd}$$

All I have to show is that the kernel of π has square zero, and that the result R_4-bimodule structure on $\mathrm{Ker}(\pi)$ comes by restriction of scalars from an R_2-bimodule. But this is clear since $\mathrm{Ker}\,\pi \subset J^2/J^4$ and the R_4 multiplication on J^2/J^4 comes from the F/J^2 multiplication and $R_4 \to F/J^2$ factors through R_2.

Conclusion: Suppose we are given an algebra A and an adic inverse system $\{R_n\}$ with $R_1 = A$ such that the condition $R_n \otimes_{R_{2n}} \Omega^1(R_{2n}) \otimes_{R_{2n}} R_n$ projective holds. Then we can find $A = F/J$ with F free such that $\{R_n\}$ is a retract of $\{F/J^n\}$.

For example suppose $A = R/I$ with $\Omega^1 R$ projective. Then we know $R_n = R/I^n$ satisfies the projectivity condition. On the other hand suppose we choose $F/K = R$ with F

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free. We can then successively lift

$$\begin{tikzcd} & F/K^4 \arrow[d] \\ & F/K^2 \arrow[d] \\ R \arrow[ur, dashed] \arrow[uur, dashed] \arrow[r, equal] & F/K \end{tikzcd}$$

But if $I = J/K$, ~~then~~ we get maps

$$\begin{tikzcd} \arrow[d] & \arrow[d] & \arrow[d] \\ R/I^2 \arrow[r] \arrow[d] & F/J^2 \arrow[r] \arrow[d] & R/I^2 \arrow[d] \\ R/I \arrow[r] & F/J \arrow[r] & R/I \end{tikzcd}$$

giving a retraction situation. (Note that

$$\begin{tikzcd} & \arrow[d] & \arrow[d] & \\ & F/K^2 \arrow[r] \arrow[d] & F/J^2 \arrow[r] \arrow[d] & R/I^2 \arrow[d] \\ R \arrow[ur] \arrow[r, equal] & F/K \arrow[r] & F/J \arrow[r] & R/I \end{tikzcd}$$

so we get $R \to F/K^{2^n} \to F/J^{2^n} \to R/I^{2^n}$ and the point is that $R \to F/J^{2^n}$ carries I into J/J^{2^n}, hence descends to $R/I^{2^k} \to F/J^{2^n}$)

November 21, 1990

Recall that I had a way to prove the injectivity of

$$I/I^2 \longrightarrow R/I \otimes_R \Omega^1 R \otimes_R R/I$$

via a Tor_1 calculation. Here's a simpler version of this idea. Put $A = R/I$

$$0 \to \Omega^1 R \to R \otimes R \to R \to 0$$

$$* \left\{ \begin{array}{l} 0 \to \Omega^1 R \otimes_R A \to R \otimes A \to A \to 0 \\ \qquad\qquad\qquad\qquad\qquad\quad \| \\ 0 \to I \longrightarrow R \longrightarrow A \to 0 \end{array} \right.$$

The last two sequences are exact sequences of left R-modules and since $R \otimes A$, R are free there are maps both ways over the identity. In fact either sequence can be used to compute $\mathrm{Tor}_1^R(A,A)$. We get

$$\begin{array}{l} 0 \to \mathrm{Tor}_1^R(A,A) \to A \otimes_R \Omega^1 R \otimes_R A \longrightarrow A \otimes A \to A \to 0 \\ \qquad\quad\ \| \\ 0 \to \mathrm{Tor}_1^R(A,A) \to I/I^2 \longrightarrow A \xrightarrow{1} A \to 0 \end{array}$$

whence the desired injectivity. Explicitly we can use the maps

$$\begin{array}{ccc} x & \longmapsto & x \otimes 1 \\ R & \rightleftarrows & R \otimes A \\ x\rho(a) & & x \otimes a \end{array}$$

which induce maps

$$\begin{array}{ccc} y & \longmapsto & dy \otimes 1 \Rightarrow dy \\ I & \rightleftarrows & \Omega^1 R \otimes_R A = \Omega^1 R / (\Omega^1 R) I \\ x_0(x_1 - \rho\pi x_1) & \longleftarrow & x_0 dx_1 \end{array}$$

Observe these make the second sequence in $*$ a retract of the first. We obtain maps

$$y \longmapsto dy$$
$$I/I^2 \rightleftarrows A \otimes_R \Omega^1 R \otimes_R A$$
$$x_0(x_1 - \rho\pi x_1) \longleftarrow\!\!\shortmid\; x_0 dx_1$$

making I/I^2 a retract of $A \otimes_R \Omega^1 R \otimes_R A$. This establishes the injectivity.

November 23, 1990

An important problem is to handle deformations, in particular, to prove Block's theorem. For example, consider the Weyl algebra deformation $A_h = k\langle x,y\rangle/(xy-yx-h)$ of the polynomial algebra $A_0 = k[x,y]$. We know that the periodic cyclic theory of A_h is independent of h. On A_0 we have a class in $HP^0(A_0)$ represented by the trace given by evaluating at any point, say $x=y=0$. How do we realize this class on A_h for $h \neq 0$? From what we know about nilpotent extensions, we can do the following.

Let $R = k\langle x,y\rangle[h]/(xy-yx-h)$ be the algebra over $k[h]$ whose specializations give the family A_h, $h \in k$. We have the tower of nilpotent extensions

$$(*) \quad A_0 = R/hR \longleftarrow R/h^2R \longleftarrow R/h^3R \longleftarrow \cdots$$

So these all have the same periodic cyclic cohomology. The problem is to be able to reach $h \neq 0$.

I think the way we want to proceed is to work over $k[h]$ in some sense. One sense is to work with algebras over the commutative ground ring $k[h]$, but we have seen that in the noncommutative setup there is an interesting relative theory.

So I want to consider traces on R/h^nR with values in $k[h]/(h^n)$. Also I want to consider traces on R with values in $k[h]$. These traces should be linear over $k[h]$.

Now the periodic cyclic cohomology of R as algebra over $k[h]$ should have a connection, a Gauss–Manin connection. This connection should express the idea that the periodic cyclic cohomology of A_h is independent of h.

I think what this all means is that we have to make ∂_h act on the cyclic cohomology. Perhaps the best framework is to develop the relative theory $X(R;S)$, or maybe $\Omega(R;S)$, in the case where Ω^1S is projective, for example $S = k[h]$. This would perhaps also handle Künneth.

Returning to the tower $(*)$, we ought to take the trace on A_0 with values in k and "extend" it to a higher trace on R/h^2R with values in $k[h]/(h^2)$. I would expect this to be possible using a square zero extension of R/h^2R. For the next stage – a h-linear trace on R/h^3R with values in $k[h]/(h^3)$ I would expect in general to have to use a 2nd order extension, but because the Hochschild homology for A_0 vanishes in degrees ≥ 2, one might get away with a square zero extension. If this is true for higher orders one should have the Gauss–Manin connection.

One expects Hochschild homology to appear in this game as obstructions, and Block's hypothesis that Hochschild dimension is finite should give the uniformity required to integrate the connection.

November 28, 1990

Here's a technical point requiring clarification. Consider

$$\varprojlim X(\underbrace{R_n}_{R/I^n}) : \qquad \hat{R} \overset{b}{\underset{}{\rightleftarrows}} \varprojlim \Omega^1(R_n)_\natural$$

What is the relation between

$$\varprojlim_n [R_n, R_n] \subset \hat{R}$$

$$(*) \qquad \mathrm{Im}\left\{ \varprojlim \Omega^1(R_n)_\natural \xrightarrow{b} \hat{R} \right\}$$

$$\mathrm{Im}\left\{ \varprojlim \Omega^1 R_n \xrightarrow{b} \hat{R} \right\} \qquad ?$$

Note that the exact sequence of inverse systems

$$0 \longrightarrow [R_n, R_n] \longrightarrow R_n \longrightarrow (R_n)_\natural \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \varprojlim [R_n, R_n] \longrightarrow \hat{R} \longrightarrow \varprojlim (R_n)_\natural \longrightarrow 0$$

because the sub inverse system has surjective maps and hence $R^1 \varprojlim [R_n, R_n] = 0$. We have

$$[R_n, R_n] = [R,R] + I^n / I^n = [R,R] + \hat{I^n}/\hat{I^n} = [\hat{R},\hat{R}] + \hat{I^n}/\hat{I^n}$$

and $\varprojlim [R_n, R_n] =$ closed commutator subspace of $\hat{R}$.

To simplify the discuss let's consider the interesting case where $\Omega^1 R$ is projective, in which case we know that the inverse system of complexes

$$\longrightarrow \Omega^2 R_n \xrightarrow{b} \Omega^1 R_n \xrightarrow{\natural} \Omega^1(R_n)_\natural \longrightarrow 0$$

has a contracting homotopy from level $2n$ to level n.

This means it gives a split exact after taking the inverse limit. Thus in this case

$$\varprojlim (\Omega^1 R_n) \twoheadrightarrow \varprojlim \Omega^1(R_n)_\natural$$

So the two bottom spaces in $(*)$ are the same. In general these are not the same as $\overline{[\hat{R},\hat{R}]}$, and the difference is $R^1\varprojlim HC_1(R_n)$

To see this use

$$\longrightarrow R_n \xrightarrow{d} \Omega^1(R_n)_\natural \xrightarrow{b} R_n \longrightarrow$$

$$0 \longrightarrow \underset{\substack{\| \\ HH_1(R_n)}}{Z_n} \overset{i}{\hookrightarrow} \Omega^1(R_n)_\natural \xrightarrow{b} [R_n, R_n] \longrightarrow 0$$

This shows that the cokernel of b in the limit is $R^1\varprojlim HH_1(R_n)$. But we have

$$\begin{tikzcd}
R_n \arrow[rr, "d"] \arrow[dr] & & Z_n \arrow[r, two heads] & HC_1(R_n) \arrow[r] & 0 \\
& \mathrm{Im}\, d \arrow[ur, hook] & & &
\end{tikzcd}$$

R_n surj $\Rightarrow$ $\mathrm{Im}\,d$ surj $\Longrightarrow$ $R^1\lim Z_n = R^1\lim HC_1(R_n)$.

Thus we get

$$\boxed{\frac{\varprojlim [R_n, R_n]}{b\{\varprojlim \Omega^1(R_n)_\natural\}} = R^1\varprojlim HH_1(R_n) = R^1\varprojlim HC_1(R_n)}$$

Observe that it should be true that

$$HH_q(R_{2n}) \longrightarrow HH_q(R_n) \quad \text{is zero}$$

for $q \geq 2$, since we have a contracting homotopy from level $2n$ to n:

$$\begin{tikzcd}
{} \arrow[r] & \Omega^2(R_{2n}) \arrow[r] \arrow[d] \arrow[dl, dashed] & \Omega^1(R_{2n}) \arrow[r] \arrow[d] \arrow[dl, dashed] & \Omega^1(R_{2n})_\natural \arrow[r] \arrow[d] \arrow[dl, dashed] & 0 \\
{} \arrow[r] & \Omega^2(R_n) \arrow[r] & \Omega^1(R_n) \arrow[r] & \Omega^1(R_n)_\natural \arrow[r] & 0
\end{tikzcd}$$

Let's return to the question of defining in a suitable derived category framework the Hochschild complex $A \overset{\mathbb{L}}{\otimes}_A$ with all its structure. Here's a situation of which we have several examples. Consider a chain complex E of length m of A-bimodules which is a resolution of A:

$$0 \to E_m \to \cdots \to E_1 \longrightarrow E_0 \longrightarrow A \longrightarrow 0$$

and is such that $E_0, \dots, E_{m-1}$ are projective bimodules.

Examples:

1) $$0 \to \Omega^1 A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0$$

2) $$0 \longrightarrow I/I^2 \longrightarrow A \otimes_R \Omega^1 R \otimes_R A \longrightarrow A \otimes A \longrightarrow A \to 0$$

where $A = R/I$ and $\Omega^1 R$ is projective

3) $$0 \to \Omega^1(A;S) \longrightarrow A \otimes_S A \longrightarrow A \longrightarrow 0$$

where S is "relatively separable" in A in the sense that $A \otimes_S A$ is a projective A-bimodule. (Note that separable means S is a projective S-bimodule, hence $A \otimes_S S \otimes_S A = A \otimes_S A$ is a projective A-bimodule.)

Now we can construct a projective bimodule resolution of A by concatenation. We also can construct finite resolutions which are projective except at the top by tensor product $E \otimes_A \cdots \otimes_A E$. ~~Must compare~~. To see this we need

<u>Lemma</u>: If M, E_0 are bimodules such that E_0 is projective and M is projective as a left bimodule, then $M \otimes_A E_0$ is projective.

Must compare tensor powers $E \otimes_A \cdots \otimes_A E$ and concatenated resolutions.

November 29, 1990

Observation: In discussing adic filtrations [redacted] I encountered homotopies $u_t: R \longrightarrow$ [redacted] $\bigoplus_{n\geq 0} t^n I'^n \subset R'[t]$. On the other hand I-adic traces on I^n are just degree n traces on the algebra $\bigoplus_{n\geq 0} I^n$. The observation is just that

$$\bigoplus_{n\geq 0} I^n \cong \bigoplus_{n\geq 0} t^n I^n \subset R[t].$$

The question is whether there is anything here [redacted] capable of being exploited.

Given an I'-adic trace τ' on I'^m, this is the same as a trace of degree m on $\bigoplus_{n\geq 0} t^n I'^n$, we can pull it back by u_t to get a [redacted] 1-parameter family of traces on R. Actually we get $t^m \tau'(u_{(m)})$, which maybe isn't interesting.

A better version arises if we remember that we want to evaluate u_t at $t=1$ and at other values of t.

$$\begin{tikzcd}
R \arrow[r] & \bigoplus_{n\geq 0} t^n I'^n \arrow[r, "ev_{t_0}"] & R' \\
I^m \arrow[u, phantom, "\cup"] \arrow[r] & \bigoplus_{n<m} t^n I'^m \oplus \bigoplus_{n\geq m} t^n I'^n \arrow[u, phantom, "\cup"] \arrow[r] & I'^m \arrow[u, phantom, "\cup"] & ?
\end{tikzcd}$$

Question: Is it possible that one can prove homotopy invariance for

$$\frac{\{I\text{-adic traces on } I^n\}}{\{\text{traces on } R\}}$$

using some variant of the above?

The problem is the following. If one pulls back an I'-adic trace τ' on I'^m via u_t and differentiates we know the derivatives $\tau' \dot{u}_t$ comes from an I-adic trace $\tau' \tilde{\dot{u}}_t$ on $F_I^{m-1}\Omega^1 R$. It would be nice if in fact $\tau' \dot{u}_t$ extends to I^{m-1}. In the case where R we know that I-adic traces on $F_I^{m-1}\Omega^1 R$ extend to traces on $\Omega^1 R$.

So what I would like to prove is that $\tau' \tilde{\dot{u}}_t$ on $F_I^{m-1}\Omega^1 R$ comes from an I-adic trace on $F_I^{m-2}\Omega^1 R$, then from one on $F_I^{m-3}\Omega^1 R$, etc.

Review. If $I \subset R$ is an ideal let $\tilde{R} = \bigoplus_{n \geq 0} I^n$; it's a graded alg wrt $\mathbb{N}$. A trace on $\tilde{R}$ of degree m is the same as an I-adic trace on I^m. Since $\tilde{R}$ is generated by R, I this is a linear fn on I^m vanishing on $[R, I^m] + [I, I^{m-1}]$ (say $m \geq 1$). One has $[R, I^m] \subset [I^{m-1}, I]$ by the circular bracket identity, so I-adic trace on I^m means vanishing on $[I, I^{m-1}]$.

If M is an R-bimodule, we have the ideal $I \oplus M \subset R \oplus M$ and

$$\widetilde{R \oplus M} = \bigoplus_n (I \oplus M)^n$$

$$= \underbrace{\bigoplus_{n \geq 0} I^n}_{\tilde{R}} \oplus \underbrace{\bigoplus_{n \geq 1} F_I^{n-1} M}_{\tilde{M}}$$

An $I \oplus M$ adic trace on $(I \oplus M)^n$ $n \geq 2$ vanishes on

$$[I \oplus M, I^{n-1} \oplus F^{n-2}M] = [I, I^{n-1}] \oplus \left([I, F^{n-2}M] + \underbrace{[M, I^{n-1}]}_{\subset [F^{n-2}M, I]} \right).$$

If D is a derivation: $D : R \to M$, then

$$1 + D : R \longrightarrow R \oplus M \qquad I \longrightarrow I \oplus M$$

so $1 + D : I^n \longrightarrow I^n \oplus F^{n-1}M$ and an I-adic trace τ on $F^{n-1}M$ yields an I-adic trace τD on I^n.

Impression: Nothing so far seems to be much easier using $\tilde{R}$.

Let's now identify

$$\tilde{R} = \oplus I^n = \oplus t^n I^n \subset R[t].$$

Suppose we have a homomorphism

$$A \longrightarrow \tilde{R}$$

$$a \longmapsto \sum t^n u_n(a) \qquad u_n : A \longrightarrow I^n$$

and recall that traces on $\tilde{R}$ are the same as families $\{\tau_n, n \geq 0\}$, where τ_n is an I-adic trace on I^n.

Let τ be an I-adic trace on I^m and take $\tau_n = \begin{cases} \tau \text{ restricted to } I^n & \text{for } n \geq m \\ 0 & n < m \end{cases}$

Then we get a family of traces on A

$$a \longmapsto \sum_{n \geq m} t^n \tau u_n(a)$$

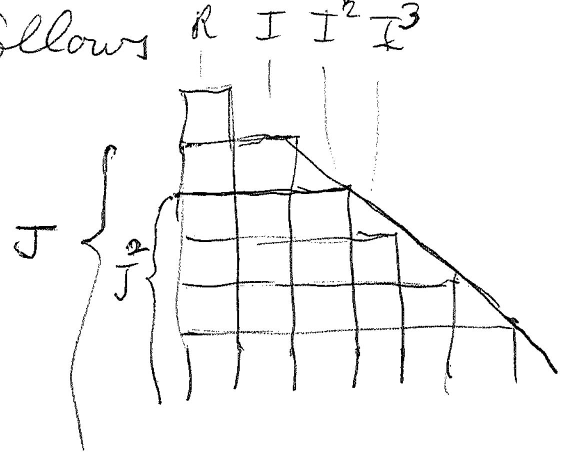
Moreover if we restrict to J^m, where J is the inverse image of $I \oplus \bigoplus_{n \geq 1} tI^n$, then this agrees up to a polynomial in t with

$$\sum_{n \geq 0} t^n \tau(u_n a) = \tau(u_t a).$$

Something strange is going on.

On $\tilde{R} = \bigoplus t^n I^n$ we have the derivation ∂_t: $\partial_t (t^n I^n) = n t^{n-1} I^n \subset t^{n-1} I^{n-1}$. If τ_m is the degree m trace on $\tilde{R}$ given by an I-adic trace τ on I^m, then $\tau_m \partial_t$ up to a scalar is the degree $m+1$ trace on $\tilde{R}$ associated to the restriction of τ to I^{m+1}.

Note that the universal A is $\tilde{R}$ itself and the powers J^n can be pictured as follows



Given an I-adic trace τ on I^2 we get a family τ_t on J^2 given by $t^n \tau$ on the n-th column. In other words we pull τ back by evaluation at time t to yield τ_t. The derivative of this family is $n t^{n-1} \tau$ on the nth column. For $n \geq 2$ this component is a trace on $\tilde{R}$, so modulo traces defined on $\tilde{R}$, we have just the trace τ on the first column which is defined only on I^2.

Conclude: The hope that the derivative $\tau \dot{u}_t$ should come from a global trace is not true.

November 30, 1990

Consider nilpotent extensions of order n $A = R/I$ where $I^n = 0$. There is a "path space" object

$$\tilde{R} = R \oplus I \oplus I^2 \oplus \cdots \oplus I^{n-1} \oplus \cdots$$

such that a homomorphism $R' \longrightarrow \tilde{R}$ over A is the same as a family $u_t : R' \to \tilde{R}$ such that $R' \longrightarrow R \longrightarrow R/I^k$ is a polynomial of degree $< k$. Notice that we have maps

$$R \hookrightarrow \tilde{R} \xrightarrow{ev_t} R$$

corresponding to constant paths and evaluation at time t. The basic homotopy for traces we use should be given already for $R' = \tilde{R}$. Thus if τ is a trace on R we have the family of traces $\tau\, ev_t$ on $\tilde{R}$ and we should have $\tau(ev_1) - \tau(ev_0) = Td$ for some trace T on $\Omega^1\tilde{R}$. But this is clear since we we have $\tilde{R} \subset R[t]$ and the derivation ∂_t preserves $\tilde{R}$. So you pull back your argument

$$\begin{tikzcd}
\tilde{R} \arrow[r] \arrow[d, "\partial_t"] & R[t] \arrow[r, "\tau"] \arrow[d, "\partial_t"] & \mathbb{C}[t] \arrow[d, "\partial_t"] \arrow[dr, "x_1 - x_0"] & \\
\tilde{R} \arrow[r] & R[t] \arrow[r, "\tau"] & \mathbb{C}[t] \arrow[r, "\int_0^1"'] & \mathbb{C}
\end{tikzcd}$$

December 2, 1990

Reduced cyclic homology again. The problem arose with defining the map $K_1 A \longrightarrow HC_1(A)$. Now we know the image is contained in

$$\mathrm{Ker}\left\{ HC_1(A) \xrightarrow{B} HH_2 A \right\}$$

which should be $H_1^{DR}(\tilde{A})$ by Connes-Karoubi. The situation is confusing and should be straightened out. An <u>important point</u> I think is the fact that functors like K_0, K_1, HC_n, HH_n are functors on the category of nonunital algebras. Moreover there is some sort of extended functoriality because these functors are Morita invariant. Now $H_\bullet^{DR}(A)$ is not such a functor*, nor is $\overline{HC}_\bullet(A)$. Thus Connes' idea of using $H_\bullet^{DR}(\tilde{A})$ makes sense.

Diagram

$$\begin{tikzcd}
 & & 0 & & \\
0 \arrow[r] & \bar{H}_0^{DR} A \arrow[r] & \overline{HC}_0 A \arrow[r, "B"] \arrow[u] & HH_1 A & \\
0 \arrow[r] & H_0^{DR} A \arrow[r] \arrow[u] & HC_0 A \arrow[r, "B"] \arrow[u] & HH_1 A \arrow[u, equal] & \\
 & & \mathbb{C} \arrow[u] & & \\
0 \arrow[r] & H_1^{DR} A \arrow[r] & \overline{HC}_1 A \arrow[r, "B"] \arrow[u] & HH_2 A \arrow[r, "I"] & \overline{HC}_2 A \\
0 \arrow[r] & H_1^{DR} \tilde{A} \arrow[r] \arrow[u, hook] & HC_1 A \arrow[r, "B"] \arrow[u] & HH_2 A \arrow[r, "I"] \arrow[u, equal] & HC_2 A \arrow[u] \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

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* at least not in an obvious way (see p. 98)

Actually I probably should have drawn this as some sort of braid diagram. To keep things simple note that one has exact sequences

$$\mathbb{C} \longrightarrow H_0^{DR}(A) \longrightarrow \bar{H}_0^{DR} A \longrightarrow 0$$

$$0 \longrightarrow H_1^{DR}(\tilde{A}) \longrightarrow H_1^{DR} A \longrightarrow \mathbb{C}$$

Look at <u>example</u> of the Weyl algebras A_n. For $n=1$.

$$H_0^{DR} = \bar{H}_0^{DR} = 0 \qquad H_1^{DR}(\tilde{A}) = HC_1 A = 0$$

$$\bar{HC}_1 A \xrightarrow[B]{\sim} HH_2 A \xrightarrow[I]{\sim} HC_2 A = \mathbb{C} \qquad H_1^{DR}(A) = 0$$

For $n \geq 2$. $\quad H_0^{DR} = \bar{H}_0^{DR} = 0 \qquad H_1^{DR}(\tilde{A}) = HC_1(A) = HH_2(A) = 0$

but $\quad H_1^{DR} A = \bar{HC}_1 A = \mathbb{C}$

Notice that $\overbrace{\text{one}}^{\text{when } A \text{ is unital}}$ has a map $\tilde{A} \longrightarrow A$ in the category of unital algebras, but not necessarily a map in the other direction unless A is augmented. This is why there are maps

$$HC(A) = \bar{HC}(\tilde{A}) \longrightarrow \bar{HC}(A)$$
$$H^{DR}(\tilde{A}) \longrightarrow H^{DR}(A)$$

but not <u>apparently</u> maps in the other direction.

Let's check that

$$X(\tilde{A}) \xrightarrow{\sim} X(A) \oplus X(\mathbb{C})$$

hence that
$$\boxed{\begin{aligned} H_0^{DR}(\tilde{A}) &= H_0^{DR}(A) \oplus \mathbb{C} \\ HC_1(\tilde{A}) &= HC_1(A). \end{aligned}}$$

The easy way to see this is to check that $\Omega^1\tilde{A}_{\natural} \xrightarrow{\sim} \Omega^1 A_{\natural}$. Let f be a Hochschild 1-cocycle on $\tilde{A}$; denote the unit of $\tilde{A}$ by 1, the unit of A

by e. We know f is equivalent to cochains

$$\psi(x,y) = f(x,y) \qquad x \in A$$

$$\varphi(x) = f(1,x) \qquad x \in A$$

such that $\quad b\psi = 0, \quad b'\varphi = (1-\lambda)\psi$

Check this.

$$f(1x,y) - f(1,xy) + f(y1,x) = 0$$

or $\quad f(1,xy) = f(x,y) + f(y,x)$

Now because $b\psi = 0$ we also have

$$\psi(e,xy) = \psi(x,y) + \psi(y,x)$$

Thus $\quad \psi(e,xy) = f(1,xy) \implies \varphi(x) = \psi(e,x)$ which means that f is determined by ψ. This shows $\Omega^1\tilde{A}_{\natural} \longrightarrow \Omega^1 A_{\natural}$ is injective, etc.

Another way to proceed is to consider the separable subalgebra $S = \mathbb{C}[e] \subset \tilde{A}$ which is central. We know $X(\tilde{A}) \simeq X(\tilde{A};S) \oplus K$, where K is a contractible complex which is both degrees is $[\tilde{A},S] = 0$. Thus $X(A) \xrightarrow{\sim} X(\tilde{A};S)$, which with some more work should imply $\Omega^1 A_{\natural} = \Omega^1 A_{\natural}$. Thus one has $X(\tilde{A}) = \mathbb{C}[0] \oplus X(A)$.

Now $HC_1(\tilde{A}) = HC_1(A)$ should be no surprise because HC_1 is a functor on nonunital algebras.

December 3, 1990

Why $\overline{HC}_n$, H^{DR}_n are not Morita invariant: When is a functor on unital algebras Morita invariant? First of all it must be functorial for non-unital homomorphisms. Secondly it must give an isomorphism when applied to any of the standard "rank 1" embeddings $A \longrightarrow M_n A$. (I definitely need a good characterization of "Morita functors".)

Consider a non unital homomorphism $A \longrightarrow B$, and suppose $1_A \longmapsto e$. Then look at what happens to $\overline{HC}_0$:

$$\begin{tikzcd} 0 \arrow[d] & & 0 \arrow[d] \\ \mathbb{C} \arrow[d] & & \mathbb{C} \arrow[d] \\ A_\natural \arrow[rr] \arrow[d] & & B_\natural \arrow[d] \\ \bar{A}_\natural \arrow[d] & & \bar{B}_\natural \arrow[d] \\ 0 & & 0 \end{tikzcd}$$

Now

$$\begin{tikzcd} 1 \arrow[d] & \\ {[1]} \arrow[r, mapsto] & {[e]} \end{tikzcd}$$

and $[e]$ doesn't have to be 1.

So you have a problem whenever

$\mathbb{C}_\natural \longrightarrow B_\natural$ is not surjective.

<u>Question.</u> The cyclic cohomology classes associated to p-summable Fredholm modules are reduced, but what about pre-Fredholm modules? (Important not to forget the condition that $[F, a]$ and $(F^2-1)a$ compact.)

December 9, 1990

Exposition problems: Consider the problem of defining the opposite algebra R^o and the enveloping algebra R^e of [redacted] (or associated to) an algebra R. To define an algebra you must specify a vector space and a product, i.e. a bilinear function from the vector space to itself. You need notations for the vector space and the product. An ideal notation might be V_R for the vector space and $m_R: V_R \times V_R \longrightarrow V_R$ for the product, and to write $R = (V_R, m_R)$ for the algebra.* However one normally uses a standard notation for the product: $(x,y) \longmapsto xy$ and one uses the same letter [redacted] for the algebra and underlying vector space.

* More precisely, [redacted] an algebra R is defined to be a pair (V, m), where .... V is called the underlying vector space and m is called the product of the algebra R. Write V_R, m_R for the underlying vector space and product of the algebra R.

In terms of this ideal notation the definition of R^o is easy

$$V_{R^o} = V_R \qquad\qquad m_{R^o}(x,y) = m_R(y,x)$$

Similarly for the enveloping algebra

$$V_{R^e} = R \otimes R \qquad\qquad m_{R^e}(x_1 \otimes y_1, x_2 \otimes y_2) = m_R(x_1, x_2) \otimes m_R(y_2, y_1)$$

Notice that trying to define R^o as the vector space R with the product $(x,y) \mapsto yx$ is inconsistent with the standard juxtaposition notation for product. To remedy this we can say that R° is an algebra with the same underlying vector space as R. Let x° denote the element of R° corresponding to (or given by) the element x of R. The product in the algebra R° is defined by $(xy)^\circ = y^\circ x^\circ$. (Better: We define R° as follows: the underlying vector space of R° is the underlying vector space of V. Let x° ....).

An alternative is to define R° as an algebra equipped with an isomorphism of underlying vector spaces $R \xrightarrow{\sim} R^\circ$, $x \longmapsto x^\circ$ such that $(xy)^\circ = y^\circ x^\circ$. (Better: The opposite algebra R° is defined (up to canonical isomorphism) to be the algebra equipped with ......)

Next consider the enveloping algebra. Here we can define R^e as the vector space $R \otimes R$ with the product

$$(x \otimes y)(x_1 \otimes y_1) = x x_1 \otimes y_1 y$$

or we can define it as the vector space $R \otimes R^\circ$ with the product $(x \otimes y^\circ)(x_1 \otimes y_1^\circ) = x x_1 \otimes y^\circ y_1^\circ$, i.e. R^e is the tensor product of the algebras R and R°.

However I like to suppress tensor signs in the same spirit as replacing $a \times b$ by ab. Thus I like to use the maps $a \longmapsto a \otimes 1$, $b \longmapsto 1 \otimes b$ to identify A, B with subalgebras of $A \otimes B$ and then write

$$a \otimes b = (a \otimes 1)(1 \otimes b) = ab$$

also $a \otimes b = (1 \otimes b)(a \otimes 1) = ba$

The tensor product of the algebras A and B is the algebra C (unique up to canonical isomorphism) equipped with homomorphisms $A \to C \leftarrow B$ whose images commute such that the map of vector spaces $A \otimes B \to C$, $a \otimes b \mapsto ab$ is an isomorphism. (Alternative: equipped with a pair of homs. $A \to C$, $B \to C$ which is an universal pair of homs. whose images commute.)

A virtue of the standard notation $A \otimes B$ is that it provides notations for the canonical homoms. $a \mapsto a \otimes 1$, $b \mapsto 1 \otimes b$.

On $C^n = (\Omega^n)^*$ we have

$$\kappa^n = 1 + sK^{-1}b \qquad \kappa^{n+1} = 1 - bs$$

On $\operatorname{Ker} b \cap C^n$ we have $\kappa^n = 1$

$\operatorname{Ker} s \cap C^n$ $\qquad \kappa^{n+1} = 1$

Clear because $\Omega^n / b\Omega^{n+1} = [\Omega^1 \otimes_A]^n$

$\Omega^n / d\Omega^{n-1} = \bar{A}^{\otimes n+1}$

$$1 - P = b\,G(\lambda)s + G(\lambda)s\,b$$

implies a) f is $\tilde{\kappa}$ invariant $\iff$ sf, sbf are λ-inv.

b) f is $\tilde{\kappa}^d$ inv. $\iff$ sf, sbf are λ^i-inv

(Apply a) to $g = f + \tilde{\kappa}f + \cdots + \tilde{\kappa}^{d-1}f$)

c) $bf = Pf = 0 \iff f = bG(\lambda)sf$

Get 1-1 corresp.

$$\{f \in C^n \mid bf = Pf = 0\} \xrightarrow{\sim} \{\varphi \in (\bar{A}^{\otimes n})^* \mid P(\lambda)\varphi = 0\}$$

$$f \longmapsto Gsf$$
$$b\varphi \longleftarrow \varphi$$

Identity: $1 - P(\tilde{\kappa}^2) = b\left((1+\lambda)G(\lambda^2)s\right) + \left((1+\lambda)G(\lambda^2)s\right)b$

implies

d) $bf = P(\tilde{\kappa}^2)f = 0 \iff f = b\,(1+\lambda)G(\lambda^2)s\,f$

$$\{f \in C^n \mid bf = P(\tilde{\kappa}^2)f = 0\} \xrightarrow{\sim} \{\varphi \in (\bar{A}^{\otimes n})^* \mid P(\lambda^2)\varphi = 0\}$$

$$f \longmapsto (1+\lambda)G(\lambda^2)sf$$
$$b\varphi \longleftarrow \varphi$$

(only n even matters since $\tilde{\kappa}^n = 1$ on $\operatorname{Ker} b \cap C^n$)

Identities for an operator of finite order

$$\frac{1+T}{2}P(T^2) = P(T) \qquad \frac{1-T}{2}P(T^2) = P(-T)$$

$$G(T) = (1+T)G(T^2) + \tfrac{1}{2}P(-T)$$

$$G(-T) = (1-T)G(T^2) + \tfrac{1}{2}P(T)$$

($\therefore$ $(1+\lambda)G(\lambda^2) = G(\lambda) - \frac{1}{2}P(-\lambda)$ above)

December 12, 1990

Lemma: If T is a transformation of finite order then

$$\frac{1+T}{2}P(T^2) = P(T) \qquad \frac{1-T}{2}P(T^2) = P(-T)$$

$$G(T) = (1+T)\,P(T^2) + \tfrac{1}{2}P(-T)$$
$$G(-T) = (1-T)\,P(T^2) + \tfrac{1}{2}P(T)$$

Proof: $\displaystyle \frac{1+T}{2}P(T^2) = \frac{1+T}{2}\,\frac{1}{m}\sum_{j=0}^{m-1}T^{2j} = \frac{1}{2m}\sum_{k=0}^{2m-1}T^k = P(T)$

$$(1-T)\left\{(1+T)\,G(T^2) + \tfrac{1}{2}P(-T)\right\}$$
$$= (1-T^2)\,G(T^2) + P(-T) \qquad \text{since } (-T)P(-T) = P(-T)$$
$$= 1 - P(T^2) + P(-T)$$
$$= 1 - P(T) \qquad \text{since } P(T^2) = P(T) + P(-T) \text{ orthog. idems.}$$

$$P(T)\left\{(1+T)\,G(T^2) + \tfrac{1}{2}P(-T)\right\}$$
$$= P(T^2)\left\{(1+T)^2\,G(T^2) + \frac{1+T}{2}P(-T)\right\} = 0.$$

December 18, 1990

Towards understanding homotopy and restricted homotopy (when ideals are present). First ignore ideals. We have the following notions

1) homotopy $\quad u_t : R \longrightarrow E[t]$

2) first order infinitesimal homotopy

$$u + \dot{u}\varepsilon : R \longrightarrow E \oplus E\varepsilon \qquad \varepsilon^2 = 0$$

3) More generally $\quad u + \dot{u} : R \longrightarrow E \oplus M$

4) More generally

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & E' \arrow[d, "\text{square zero extension.}"] \\ & E \end{tikzcd}$$

In the latter three cases we have universal gadgets as follows.

2): The universal pair for pairs $(u, \dot{u})$ from R to another algebra is $(1, d) : R \longrightarrow \Omega R = T_R(\Omega^1 R)$.

$$\mathrm{Hom}_{alg}(R, E \oplus E\varepsilon) = \mathrm{Hom}_{alg}(T_R(\Omega^1 R), E)$$

3): Here let (E, M) range over algs + bimods. Then

$$\mathrm{Hom}_{alg}(R, E \oplus M) = \mathrm{Hom}_{alg + bim}(R \oplus \Omega^1 R, E \oplus M)$$

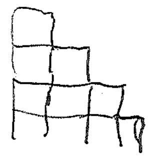
4):

$$\left\{ \begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & E' \arrow[d] \\ & E \end{tikzcd} \right\} = \mathrm{Hom}_{\substack{alg + \\ sq\ 0 \\ extns}} \left(\begin{matrix} R \oplus \Omega^1 R \\ \downarrow \\ R \end{matrix} , \begin{matrix} E' \\ \downarrow \\ E \end{matrix} \right)$$

Next suppose we consider pairs consisting of an algebra together with an ideal. Then we have encountered a special kind of homotopy $u_t : (R, I) \longrightarrow (E, J)$, namely a homotopy $u_t : R \to E$

such that $u_t(I) \subset J$ and such that

$$u_t : R \longrightarrow \bigoplus_{n \geq 0} J^n t^n \subset E[t]$$

Picture 

I don't know if this concept is useful. The corresponding first order infinitesimal homotopy is

$$(*) \quad \begin{array}{ccc} R & \xrightarrow{u+\dot{u}\varepsilon} & E \oplus J\varepsilon \\ \cup & & \cup \\ J & \longrightarrow & J \oplus J\varepsilon \end{array}$$

We can generalize this to

$$\begin{array}{ccc} R & \xrightarrow{u+\dot{u}} & E \oplus M \\ \cup & & \cup \\ I & \longrightarrow & J \oplus M \end{array}$$

but an interesting point ~~about~~ about $(*)$ is that the nilpotence order is preserved. Thus

$$J^{m+1} = 0 \implies (J \oplus J\varepsilon)^{m+1} = 0$$

but

$$\not\Longrightarrow \quad (J+M)^{m+1} = \underbrace{J^{m+1}}_{0} + \underbrace{J^m M}_{\neq 0} \quad \text{in general.}$$

Question: Does it make sense to speak of restricted homotopy classes of traces on an extension $A = R/I$?

Apparently not. Consider the extension $R' = R \oplus \Omega^1 R$, $I' = I \oplus \Omega^1 R$. Then

$$u_t = 1 + td : \begin{array}{ccc} R & \longrightarrow & R' \\ I & & I' \end{array}$$

is a restricted homotopy, so any trace Td with T on $\Omega^1 R$ is restricted homotopic to zero.

It only becomes meaningful when you put in the order of nilpotence. Thus if $\tau' = 0 + T$ on $R' = R \oplus \Omega^1 R$ is required to vanish on

$$I'^{m+1} = I^{m+1} \oplus F_I^m \Omega^1 R$$

we get fewer traces of the form Td.

December 20, 1990

Let us consider a ~~covariant~~ functor $F(R)$ on algebras. Suppose given a polynomial family of homomorphisms $u_t : R \longrightarrow R'$, $t \in \mathbb{C}$. Then we can ask when the family $F(u_t) : F(R) \longrightarrow F(R')$ is a polynomial family i.e. obtained from a map $F(R) \longrightarrow F(R')[t]$ which is unique if it exists. Note that taking $R = R'[t]$ then gives a ! map

$$F(R'[t]) \longrightarrow F(R')[t]$$

such that evaluate at $\xi \in \mathbb{C}$ on the right is consistent with $F(ev_\xi)$ on the left.

Let's assume F has this property. Let us consider a semi-direct product algebra $R \oplus M$. Then we have the polynomial family associated to the grading $u_t : x + m \longmapsto x + tm$. Consider $F(u_t)$ on $F(R \oplus M)$. Given $\xi \in F(R \oplus M)$ we have that $F(u_t)\xi$ is a polynomial in t. Thus we can write

$$F(u_t)\xi = \sum_{n \geq 0} t^n f_n \xi$$

where f_n, $n \geq 0$ is a family of operators on $F(R \oplus M)$ such that for each ξ only finitely many $f_n(\xi)$ are $\neq 0$. Since $u_s u_t = u_{st}$ we have $F(u_{st}) = F(u_s) F(u_t)$ so

$$\sum_n s^n t^n f_n = \sum_p s^p f_p \sum_q t^q f_q$$

i.e. $\quad f_p f_q = \delta_{pq} f_q \quad$ i.e. the f_n, $n \geq 0$ are mutually annihilating idempotents. Moreover, since $F(u_1) = \mathrm{id}$ we have $\sum f_n = 1$. This means that we have decomposed $F(R\oplus M)$ into functors of the R-bimodule M which are homogeneous:

$$F(R\oplus M) = \bigoplus_{n\geq 0} \underbrace{F^{(n)}(R;M)}_{\text{homogeneous of degree } n \text{ in } M}$$

Example: Not all interesting functors have the property of preserving polynomial families. For example take $F(R) = \bar{R}$ the conjugate complex vector space. Other automorphisms or even field homomorphisms $\mathbb{C} \to \mathbb{C}$ can be used.

Generalization. In place of $\mathbb{C}[t]$ take a finitely generated $\mathbb{C}$ algebra S without nilpotent elements $\neq 0$. Then we can ask for a map

$$(*) \qquad F(R\otimes S) \longrightarrow F(R)\otimes S$$

consistent with evaluation at each point of the variety of S. This map is necessarily unique if it exists, since an element of $F(R)\otimes S = \bigoplus_I S$ is determined by its values at all points.

Let's consider derivatives. If $u_t : R \to R'$ is a poly family, then $F(u_t): F(R) \to F(R')$ is a poly family so we can differentiate to obtain

$$\partial_t F(u_t)|_{t=0} : F(R) \to F(R')$$

It would be nice to know if this depends only on $u_t, \partial_t u_t$ at $t=0$. This is true if we assume the existence of the maps $(*)$ for algebras S such as the dual numbers $\mathbb{C}[\varepsilon]$. In effect we have a commutative diagram

$$\begin{tikzcd}
F(R) \arrow[r] \arrow[dr, "F(u_0+\dot{u}_0\varepsilon)"'] \arrow[rr, bend left, "F(u_0)+\partial_t F(u_t)\big|_{t=0}\varepsilon"] & F(R'[T]) \arrow[r] \arrow[d] & F(R')[T] \arrow[d] \\
 & F(R'[\varepsilon]) \arrow[r] & F(R')[\varepsilon]
\end{tikzcd}$$

Let's us discuss an application of these ideas. Let us fix the algebra R and consider all polynomial families $u_t : R \to R'$. For each one we can look at the equalizer of the maps $F(u_t) : F(R) \to F(R')$, then look at the intersection taken over all polynomial families. This gives a subspace $E \subset F(R)$. We want to show ~~that~~

$$E = \overset{\text{equalizer over all } t}{\mathrm{Ker}} \left\{ F(R) \xrightarrow[F(1+td)]{} F(R \oplus \Omega^1 R) \right\}$$

In fact we want to show

$$E = \mathrm{Ker} \left\{ \partial_t \left(F(1+td) \right) \Big|_{t=0} : F(R) \to F(R \oplus \Omega^1 R) \right\}$$

It should be clear that E is contained in this kernel. It should also be clear that $\xi \in F(R)$ belongs to the equalizer of $F(u_t) : F(R) \to F(R')$ iff $\partial_t \{F(u_t)\xi\} = 0$ for any t. (The point is that $F(u_t)\xi$ is a polynomial in t, hence it is constant iff it's derivative is 0 at each point.)

We can suppose $t=0$. Then we have

$$\begin{tikzcd}
F(R) \arrow[r, "F(u_0+\dot{u}_0\varepsilon)"] \arrow[d] \arrow[rr, bend left, "F(u_0)+\partial_t F(u_t)\big|_{t=0}\varepsilon"] & F(R' \oplus R'\varepsilon) \arrow[r] & F(R')[\varepsilon] \\
F(R \oplus \Omega^1 R) \arrow[ur] \arrow[r] & F(R) \oplus F(R; \Omega^1 R) \arrow[ur] &
\end{tikzcd}$$

?

December 22, 1990

Consider $A = R/I$ with $\Omega^1 R$ projective. We have exact sequences of A-bimodules

$$0 \to I/I^2 \to A \otimes_R \Omega^1 R \otimes_R A \to A \otimes A \to A \to 0$$

$$0 \to I/I^2 \otimes_A I/I^2 \to I/I^2 \otimes_R \Omega^1 R \otimes_R A \to I/I^2 \otimes A \to I/I^2 \to 0$$

etc, which can be spliced together to yield a projective bimodule resolution of A:

$$\to N^{(2)} \otimes A \to N \otimes_A E \to N \otimes A \to E \to A \otimes A \to A \to 0$$

where $N = I/I^2$, $N^{(n)} = T^n_A(N)$, $E = A \otimes_R \Omega^1 R \otimes_R A$

This in turn yield a complex giving the Hochschild homology

$$\to N^{(2)} \to \underset{\substack{\parallel \\ N \otimes_R \Omega^1 R \otimes_R}}{N \otimes_A E \otimes_A} \to N \to \underset{\substack{\parallel \\ A \otimes_R \Omega^1 R \otimes_R}}{E \otimes_A} \to A \to 0 \to \cdots$$

The problem is to find the B operator on this complex. The B operator is an endomorphism of this complex of degree $+1$.

If A has homological dimension ≤ 2, then we expect there to be a B-operator on the truncated complex

$$\to 0 \to N \otimes_A \to A \otimes_R \Omega^1 R \otimes_R \to A \to 0 \to \cdots$$

In fact this is so without the the assumption. In effect we have the $\mathbb{Z}/2$ graded complex

$$R/I^2 + [R, I] \rightleftarrows (\Omega^1 R / I \Omega^1 R)_\natural$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$A \oplus (I/I^2 + [R,I]) \qquad A \otimes_R \Omega^1 R \otimes_R$$

provided we use a lifting $\rho : A \to R$ to split the exact sequence $0 \to I/I^2 \to R/I^2 \to A \to 0$.

The arrows in the $\mathbb{Z}/2$-graded complex then furnish all the expected maps. The d map gives rise to

$$\begin{array}{ccc} A \longrightarrow A\otimes_R \Omega^1 R\otimes_R & & a \longmapsto d\rho(a) \\ \qquad\quad \| & & \\ I/I^2\otimes_A \longrightarrow \left(\Omega^1 R/I\Omega^1 R\right)_\natural & & z \longmapsto dz \end{array}$$

The b map gives rise to

$$\left(\Omega^1 R/I\Omega^1 R\right)_\natural \longrightarrow R/I^2+[R,I] \longrightarrow A\oplus\left(I/I^2\otimes_A\right)$$

$$\rho(a)dx \longmapsto [\rho(a),x] \longrightarrow [a,\bar{x}],\quad [\rho(a),x]-\rho[a,\bar{x}]$$

$$(\bar{x} = x \bmod I)$$

Thus we have

$$I/I^2\otimes_A \longrightarrow A\otimes_R\Omega^1 R\otimes_R \longrightarrow A$$

$$[\rho a,x]-\rho[a,\bar{x}] \longleftarrow\!\!\mapsfrom\; a\,dx \longmapsto [a,\bar{x}]$$

$$d\rho[a,\bar{x}] \longleftarrow [a,\bar{x}]$$

$$[\rho a,x]-\rho[a,\bar{x}] \longmapsto d\left(\underbrace{[\rho a,x]}-\rho[a,\bar{x}]\right) = 0 \quad \text{since } db=0.$$

The next question is how to continue this B-operator to the rest of the complex

$$\longrightarrow N\otimes_R\Omega^1 R\otimes_R \longrightarrow N \longrightarrow A\otimes_R\Omega^1 R\otimes_R \longrightarrow A$$

I attempted to lift the map $a\,dx \longmapsto [\rho a,x]-\rho[a,\bar{x}]$ from $A\otimes_R\Omega^1 R\otimes_R \longrightarrow N\otimes_A$ to a map into N. It seems necessary to use the fact that $E = A\otimes_R \Omega^1 R\otimes_R A$ is a projective A-bimodule explicitly. Thus if P is our proj bimodule resolution of A, we want a map $P \to P\otimes_A P$ which in dimension 1 amounts to

$$E \longrightarrow E\otimes A \oplus A\otimes E$$

December 24, 1990

More on B from the derived category viewpoint. Suppose we try to define $B : HH_0 \longrightarrow HH_1$ starting from a bimod resolution

$$0 \longrightarrow X_1 \xrightarrow{i} X_0 \xrightarrow{p} A \longrightarrow 0$$

with X_0 projective. One idea I had was to compare both a left + right module splitting. Thus let's pick $\xi, \eta \in X_0$ such that $p\xi = p\eta = 1$. Then we have the splittings associated to $a \mapsto a\xi, \eta a$. The difference of these splittings gives a map $X_0 \to X_1$:

$$x \longmapsto -i^{-1}(x - p(x)\xi) + i^{-1}(x - \eta p(x)) = i^{-1}\{-\eta p(x) + p(x)\xi\}$$

which descends to A. Thus we have the map

$$a \overset{\varphi}{\longmapsto} i^{-1}(a\xi - \eta a) \qquad \varphi : A \longrightarrow X_1$$

and we can ask if it induces a map from $A_\natural$ to $X_{1\natural}$.
This means it carries $[A,A]$ into $[A,X_1]$. Now this is true if $\xi = \eta$ because then we have a map from $0 \longrightarrow \Omega^1 A \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0$ into the above sequence and for this sequence $i^{-1}(a\xi - \xi a) = i^{-1}(a\otimes 1 - 1\otimes a) = da$. Check

$$[a_1,a_2]\eta - \eta[a_1,a_2] = \underbrace{[[a_1,\eta],a_2] + [a_1,[a_2,\eta]]}\in [A,K_1]$$

so

$$[a_1,a_2]\xi - \eta[a_1,a_2] = [a_1,a_2]\underbrace{(\xi-\eta)}_{\text{can be arbitrary elt of } K_1} + \ldots$$

so is $[A,A]K_1 \subset [K_1,A]$, e.g. is $[A,A]\Omega^1 A \subset [\Omega^1 A, A]$?
Now $[\Omega^1 A, A] = b\Omega^2 A$ is killed by b, so is $b([A,A]\Omega^1 A) = 0$? False for Weyl algs.

Conclude that we have to pick $\xi = \eta$, and in working with longer length resolutions it seems hard to find an analogue of this condition. So it's unlikely that this idea of comparing left + right splittings leads anywhere.

Let us next consider $K \otimes_A K \overset{\varepsilon \otimes 1}{\underset{1 \otimes \varepsilon}{\rightrightarrows}} K$

$$\begin{tikzcd}
0 \arrow[r] & K_1 \otimes_A K_1 \arrow[r] & K_0 \otimes_A K_1 \oplus K_1 \otimes_A K_0 \arrow[r] & K_0 \otimes_A K_0 \arrow[dll, "\psi"'] \arrow[d, shift left] \arrow[d, shift right] & x \otimes y \arrow[d, mapsto, shift right] \arrow[d, mapsto, shift left] \\
& 0 \arrow[r] & K_1 \arrow[r, "i"] & K_0 & p(x)y \quad x p(y)
\end{tikzcd}$$

Thus we get the homotopy given by
$$\psi(x \otimes y) = i^{-1}(p(x)y - x p(y))$$
which joins $\varepsilon \otimes 1$ and $1 \otimes \varepsilon$. ψ is a bimodule map so it descends to commutator quotient spaces. We thus have
$$K \otimes_A K \otimes_A \overset{\varepsilon \otimes 1}{\underset{1 \otimes \varepsilon}{\underset{\psi}{\rightrightarrows}}} K \otimes_A$$
so $\psi + \sigma\psi$ (σ = flip on $[K \otimes_A]^2$) is a homotopy from $\varepsilon \otimes 1$ to itself, i.e. a map $\Sigma^{-1}[K \otimes_A]^2 \longrightarrow K \otimes_A$

$$\begin{aligned}
(\psi + \sigma\psi)(x \otimes y) &= \psi(x \otimes y) + \psi(y \otimes x) \\
&= i^{-1}\{p(x)y - x p(y) + p(y)x - y p(x)\} \\
&= i^{-1}\{[p(x), y] + [p(y), x]\}
\end{aligned}$$

Thus ψ should induce a map $A_\natural = H_0([K \otimes_A]^2) \longrightarrow H_1(K \otimes_A)$

Given $a \in A$ lift it to $a\xi \otimes \eta$ where $p(\xi) = p(\eta) = 1$. Then

$$(\varphi + \sigma\psi)(a\xi \otimes \eta) = i'\{[a,\eta] - [1, a\xi]\}$$
$$= i'([a,\eta])$$

which agrees with the map we got before.

Let's discuss the situation more generally. Let P be a projective bimodule resolution of A. Consider the operator $X \longmapsto P \otimes_A X$ on complexes of bimodules - maybe modules would be better. This replaces a complex by a projective complex, and it is like a projection operator, i.e. it is idempotent (up to homotopy).

Review the basics about

$$f \otimes g = (f \otimes 1)(1 \otimes g) = (-1)^{|f||g|}(1 \otimes g)(1 \otimes f)$$

$$\begin{tikzcd} X \otimes Y \arrow[r, "1 \otimes g"] \arrow[d, "f \otimes 1"'] \arrow[dr, "f \otimes g"] & X \otimes Y' \arrow[d, "f \otimes 1"] \\ X' \otimes Y \arrow[r, "1 \otimes g"'] & X' \otimes Y' \end{tikzcd}$$

One uses this to show for maps $f : A \longrightarrow \Sigma^p M$, $g : A \longrightarrow \Sigma^q N$ in the derived category that the Yoneda (or composition) product:

$$A \xrightarrow{g} \Sigma^q N = A \otimes_A \Sigma^q N \xrightarrow{f \otimes 1} \Sigma^p M \otimes_A \Sigma^q N$$

coincides with $f \otimes g$.

Suppose f realized by

$$
\begin{tikzcd}
\{\underset{\overset{\parallel}{M}}{X_p} \arrow[r] & \cdots \arrow[r] & X_0\} \arrow[r, "\varepsilon", "\text{quis}"'] & A \\
\Sigma^p M \arrow[from=1-1, "\text{projection onto } X_p = M. \text{ Call this } \hat{f}"] & & &
\end{tikzcd}
$$

and g by

$$
\begin{tikzcd}
\{\overset{N}{\overset{\parallel}{Y_q}} \arrow[r] \arrow[d, "\hat{g}"] & \cdots \arrow[r] & Y_0\} \arrow[r, "\varepsilon"] & A \\
\Sigma^q N & & &
\end{tikzcd}
$$

then we have

$$
\begin{tikzcd}
X \otimes_A Y \arrow[r, "\varepsilon \otimes 1"] \arrow[d, "1 \otimes \hat{g}"] & A \otimes_A Y \arrow[r, "1 \otimes \varepsilon"] \arrow[d, "1 \otimes \hat{g}"] & A \otimes_A A \\
X \otimes_A \Sigma^q N \arrow[r, "\varepsilon \otimes 1"] \arrow[d, "\hat{f} \otimes 1"] & A \otimes_A \Sigma^q N & \\
\Sigma^p M \otimes_A \Sigma^q N & &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
X \otimes_A Y \arrow[r, "1 \otimes \varepsilon"] \arrow[d, "\hat{f} \otimes 1"] & X \otimes_A A \arrow[r, "\varepsilon \otimes 1"] \arrow[d, "\hat{f} \otimes 1"] & A \otimes_A A \\
\Sigma^p M \otimes_A Y \arrow[r, "1 \otimes \varepsilon"] \arrow[d, "1 \otimes \hat{g}"] & \Sigma^p M \otimes_A A & \\
\Sigma^p M \otimes_A \Sigma^q N & &
\end{tikzcd}
$$

Notice that the two ways of concatenating the X and Y complexes are dominated by the complex $X \otimes_A Y$.

December 25, 1990

The problem is to understand the Hochschild complex and all its structure which leads to cyclic homology. For the moment we would like to understand the B operator on the Hochschild complex

Here are some patterns or analogies which might be fruitful. In order to compute Hochschild homology we need a chain complex P of projective A-bimodules which is a resolution of A, i.e. equipped with a quis $\varepsilon : P \longrightarrow A$. Another way of viewing P is via the interpretation of bimodules as operators on (left) modules. Then $K \longmapsto P \otimes_A K$ is an operation on complexes of modules which replaces K by a quasi-isomorphic projective complex. This means $P\otimes_A$ is a sort of smooth approximation to the identity functor, in analogy to the way a Thom form in the normal bundle of $\Delta X \subset X \times X$ gives a smooth approximation to the identity operator on forms. There might be links with diagonal approximation (Steenrod), Lefschetz fixpoint formalism, Grothendieck's analysis of constructibility and calculating cohomology as the image of restriction between two coverings.

Another analogy is with heat kernels. We know if it is possible to construct such a P starting from a quis $\underbrace{\{N \rightarrow E\}}_{X} \xrightarrow{\varepsilon} A$ and concatenating. This assumes E is projective as a bimodule. This seems similar to the idea of constructing a heat kernel by starting with an approximate one and iterating. Except here the parameter is not continuous but discrete.

Idea now: Let's fix $X = \{N \to E\}$ $X \xrightarrow{\varepsilon} A$ quis, E projective. We have $X \otimes_A X$ and the two concatenated complexes which are resolutions of length 2 of A projective in degrees < 2. Any of these ~~—~~ calculates the Hochschild homology. We can ask if we can define B on any of them. ~~—~~

~~—~~

December 26, 1990

Let us consider an exact sequence of A-bimodules

$$0 \longrightarrow \underbrace{N \longrightarrow E}_{X} \longrightarrow A \longrightarrow 0$$

[redacted]

Assuming E is projective as a bimodule, we then know that $X \otimes_A \overset{n}{\cdots} \otimes_A X$ is a complex supported in $[0,n]$, which is projective in degrees $< n$, and which resolves A. We know such a complex is unique up to homotopy and that it can be used to compute $HH_i A$ for $i \leq n$.

Next let us recall standard facts about Postnikov towers. Given a complex K we have the "Postnikov" quotient complex, which is the quotient complex supported in degrees $\leq n$ [redacted] [redacted] having the same homology in degrees $\leq n$.

$$\begin{tikzcd}
\arrow[r] & K_{n+1} \arrow[r] \arrow[d] & K_n \arrow[r] \arrow[d] & K_{n-1} \arrow[r] \arrow[d, equal] & {} \\
\arrow[r] & 0 \arrow[r] & K_n/bK_{n+1} \arrow[r] & K_{n-1} \arrow[r] & {}
\end{tikzcd}$$

Notation: $\mathrm{Post}_{\leq n}(K)$.

[redacted] It should be true that if K = Hochschild complex $\mathcal{B} \otimes_A$, $\mathcal{B}$ standard resolution of A, then we have homotopy equivalences

$$\mathrm{Post}_{\leq n}(\mathcal{B} \otimes_A) \sim [X \otimes_A]^n$$

For all n. Also the B map $\Sigma\, \mathcal{B} \otimes_A \longrightarrow \mathcal{B} \otimes_A$ should induce a map

$$\Sigma\, Post_{\leq n}(B\otimes_A) \longrightarrow Post_{\leq n+1}(B\otimes_A)$$

and hence a map

$$\Sigma\, [X\otimes_A]^n \longrightarrow [X\otimes_A]^{n+1}.$$

The problem is to construct these maps explicitly. Especially we want to understand the choices involved in their construction; the choices include those where the fact that E is projective is used to lift a map.

Consider $X\otimes_A X \overset{\varepsilon\otimes 1}{\underset{1\otimes\varepsilon}{\rightrightarrows}} X$. There is a unique homotopy between these

$$\begin{tikzcd}
0 \arrow[r] & N\otimes_A N \arrow[r] & E\otimes_A N \oplus N\otimes_A E \arrow[r] & E\otimes_A E \arrow[r] \arrow[dll, dashed, "h"'] \arrow[d, shift right, "\varepsilon\otimes 1"'] \arrow[d, shift left, "1\otimes\varepsilon"] & A \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[rr] & & E \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Thus we get the following maps

$$\begin{tikzcd}
X\otimes_A X\otimes_A \arrow[d, shift right, "\varepsilon_0"'] \arrow[d, shift left, "\varepsilon_1"] \\
X\otimes_A
\end{tikzcd}
\qquad
\varepsilon_0 \overset{h}{\Longrightarrow} \varepsilon_1 \overset{h\sigma}{\Longrightarrow} \varepsilon_0$$

Thus we obtain a degree 1 map

$$h + h\sigma : [X\otimes_A]^2 \longrightarrow X\otimes_A$$

which we have seen does yield $B : HH_0 A \longrightarrow HH_1 A$.

Now lets generalize this

$$\begin{tikzcd}
X\otimes_A X\otimes_A X \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
X\otimes_A X
\end{tikzcd}
\qquad
\begin{matrix} \varepsilon\otimes 1\otimes 1 \\ 1\otimes\varepsilon\otimes 1 \\ 1\otimes 1\otimes\varepsilon \end{matrix}
\quad
\begin{matrix} \Downarrow h\otimes 1 \\ \Downarrow 1\otimes h \end{matrix}$$

This gives us:

$$\begin{array}{ccl} [X\otimes_A]^3 & & \varepsilon_0\sigma = \varepsilon_2 : \quad x,y,z \mapsto z,x,y \mapsto xy \\ \varepsilon_0 \downarrow \varepsilon_1 \downarrow \varepsilon_2 & & \varepsilon_1\sigma = \sigma\varepsilon_0 \quad\quad z,x,y \mapsto z,y \\ [X\otimes_A]^2 & & \varepsilon_2\sigma = \sigma\varepsilon_1 \quad\quad \mapsto z,x \end{array}$$

Homotopies $\quad h_0 : \varepsilon_0 \Longrightarrow \varepsilon_1$

Note that $1\otimes h = \sigma(h\otimes 1)\sigma^{-1} \quad \therefore \quad h_1 = \sigma h_0 \sigma^{-1}$

Check $\quad h_0 : \varepsilon_0 \longrightarrow \varepsilon_1 \Longrightarrow h_1 = \sigma h_0 \sigma^{-1} : \underbrace{\sigma\varepsilon_0\sigma^{-1}}_{\varepsilon_1} \longrightarrow \underbrace{\sigma\varepsilon_1\sigma^{-1}}_{\varepsilon_2}$

Then we have

$$\varepsilon_0 \xRightarrow{h_0} \underset{\sigma\varepsilon_0\sigma^2}{\varepsilon_1} \xRightarrow{\sigma h_0 \sigma^{-1}} \underset{\varepsilon_0\sigma}{\varepsilon_2} \xRightarrow{h_0\sigma} \underset{\sigma\varepsilon_0}{\varepsilon_1\sigma} \xRightarrow{\sigma h_0} \underset{\varepsilon_0\sigma^2}{\varepsilon_2\sigma} \xRightarrow{h_0\sigma^{-1}} \underset{\sigma\varepsilon_0\sigma}{\varepsilon_1\sigma^{-1}} \xRightarrow{\sigma h_0 \sigma} \underset{\varepsilon_0}{\varepsilon_0}$$

So $\quad h_0 + \sigma h_0 \sigma^2 + h_0\sigma + \sigma h_0 + h_0\sigma^2 + \sigma h_0 \sigma$ is a degree 1 map from $[X\otimes_A]^3$ to $[X\otimes_A]^2$.

It seems this construction ought to generalize. However there seem to be two many terms for it to represent B. Notice however the form of this map namely

$$N h_0 N = (1+\sigma) h_0 (1+\sigma+\sigma^2)$$

which makes it clear that it's a map of complexes of degree 1:

$$[d, Nh_0N] = N[d,h_0]N = N(\varepsilon_0 - \sigma\varepsilon_0\sigma^{-1})N$$
$$= N\varepsilon_0 N - N\varepsilon_0 N = 0.$$

Let's now use the fact that E is projective. Diagram with short exact rows + columns.

$$\begin{tikzcd}
N\otimes_A N \arrow[r, "1\otimes i"] \arrow[d, "i\otimes 1"'] & N\otimes_A E \arrow[r, "1\otimes\varepsilon"] \arrow[d, "i\otimes 1"] \arrow[dl, dashed, "\varphi\ \cong"'] \arrow[dr, dotted, "i\otimes\varepsilon"] & N \arrow[d, "i"] \\
E\otimes_A N \arrow[r, "1\otimes i"] \arrow[d, "\varepsilon\otimes 1"'] \arrow[dr, dotted, "\varepsilon\otimes i"] & E\otimes_A E \arrow[r, "1\otimes\varepsilon"] \arrow[d, "\varepsilon\otimes 1"] & E \arrow[d, "\varepsilon"] \\
N \arrow[r, "i"] & E \arrow[r, "\varepsilon"] & A
\end{tikzcd}$$

From this we get

$$\begin{tikzcd}
 & E\otimes_A E \arrow[d, "(\varepsilon\otimes 1,\ 1\otimes\varepsilon)"] \\
E \arrow[ur, dashed, "\Delta"] \arrow[r, "\mathrm{diag}"] & E\times_A E
\end{tikzcd}$$

i.e. [redacted] $\boxed{(\varepsilon\otimes 1)\Delta = (1\otimes\varepsilon)\Delta = id_E}$. This means ΔE is complementary to both [redacted] $E\otimes_A N$ and $N\otimes_A E$ in $E\otimes_A E$

$$E\otimes_A E = (1\otimes i)\,E\otimes_A N \oplus (\Delta)E$$

$$= (i\otimes 1)\,N\otimes_A N \oplus (\Delta)E$$

Define $N\otimes_A E \underset{\varphi}{\overset{\psi}{\rightleftarrows}} E\otimes_A N$ by

$$(i\otimes 1)\xi = (1\otimes i)\varphi(\xi) + \Delta(i\otimes\varepsilon)\xi$$

$$(1\otimes i)\eta = (i\otimes 1)\psi(\eta) + \Delta(\varepsilon\otimes i)\eta$$

Then φ, ψ are inverses and

$$\boxed{(\varepsilon\otimes i)\varphi = -(i\otimes\varepsilon)}$$

Also for $\xi \in N\otimes_A N$ we have

$$(i\otimes 1)\underbrace{(1\otimes i)\xi}_{\in N\otimes_A E} = (1\otimes i)\underbrace{(i\otimes 1)\xi}_{E\otimes_A N}$$

so that $\boxed{\varphi(1\otimes i) = (i\otimes 1)}$

We conclude that we have an isomorphism of resolutions of A:

$$\begin{tikzcd}
0 \arrow[r] & N\otimes_A N \arrow[r, "-(1\otimes i)"] \arrow[d, equal] & N\otimes_A E \arrow[r, "i\otimes\varepsilon"] \arrow[d, "-\varphi"] & E \arrow[r, "\varepsilon"] \arrow[d, equal] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N\otimes_A N \arrow[r, "i\otimes 1"] & E\otimes_A N \arrow[r, "\varepsilon\otimes i"] & E \arrow[r, "\varepsilon"] & A \arrow[r] & 0
\end{tikzcd}$$

Let's try to summarize ~~this~~ ~~situation~~. We have three ~~[illegible]~~ resolutions of length 2 of A $X\otimes_A X$ and the two concatenated sequences; denote these by Y (with $E\otimes_A N$) and Z (with $N\otimes_A E$). ~~[illegible]~~ These concatenated sequences are quotient complexes of $X\otimes_A X$ so we canonical surjections

$$\begin{tikzcd}
 & X\otimes_A X \arrow[dl, "p"] \arrow[dr, "q"] & \\
Y \arrow[ur, dotted, bend left, "s"] & & Z \arrow[ul, dotted, bend right, "t"']
\end{tikzcd}$$

Choosing $\Delta : E \to E\otimes_A E$ with $(\varepsilon\otimes 1)\Delta = (1\otimes\varepsilon)\Delta$ gives rise to sections of p, q. I forgot to say the kernels of p, q are contractible (canonically because they have the form $\{M \xrightarrow{1} M\}$). Thus the choice of Δ makes p, q strong deformation retractions.

Picture of q:

$$\begin{tikzcd}
N\otimes_A N \arrow[r, "(i\otimes 1,\, -1\otimes i)"] \arrow[d, equal] & E\otimes_A N \oplus N\otimes_A E \arrow[r, "(1\otimes i,\, i\otimes 1)"] \arrow[d, "(0,1)"] & E\otimes_A E \arrow[r] \arrow[d, "1\otimes\varepsilon"] & A \arrow[d, equal] \\
N\otimes_A N \arrow[r, "-(1\otimes i)"] & N\otimes_A E \arrow[r, "i\otimes\varepsilon"] \arrow[u, dashed, bend left, "(-\varphi,1)"] & E \arrow[r] \arrow[u, dashed, bend left, "\Delta"] & A
\end{tikzcd}$$

Picture of p:

$$\begin{tikzcd}
N\otimes_A N \arrow[r, "(i\otimes 1,\, -1\otimes i)"] \arrow[d, equal] & E\otimes_A N \oplus N\otimes_A E \arrow[r, "(1\otimes i,\, i\otimes 1)"] \arrow[d, "(1,0)"] & E\otimes_A E \arrow[r] \arrow[d, "\varepsilon\otimes 1"] & A \arrow[d, equal] \\
N\otimes_A N \arrow[r, "i\otimes 1"] & E\otimes_A N \arrow[r, "\varepsilon\otimes i"] \arrow[u, dashed, bend left, "(1,-\varphi)"] & E \arrow[r] \arrow[u, dashed, bend left, "\Delta"] & A
\end{tikzcd}$$

The interesting point is that the maps

$$Y \overset{s}{\dashrightarrow} X\otimes_A X \xrightarrow{q} Z$$
$$Z \overset{t}{\dashrightarrow} X\otimes_A X \xrightarrow{p} Y$$

are inverse isomorphisms. In fact the images of s and t are the same. Thus we have

$$X\otimes_A X = W \oplus \mathrm{Ker}(p) = W \oplus \mathrm{Ker}(q)$$

where $W = \mathrm{Im}(s) = \mathrm{Im}(t)$ depends on the choice of Δ. Incidently

$$\mathrm{Ker}(p) = N\otimes_A E \oplus (i\otimes 1) N\otimes_A E$$
$$\mathrm{Ker}(q) = E\otimes_A N \oplus (1\otimes i) E\otimes_A N$$

Example: $0 \to \Omega^1 A \xrightarrow{\partial} A\otimes A \xrightarrow{m} A \to 0$

$da \mapsto a\otimes 1 - 1\otimes a$

$$\begin{tikzcd}
\Omega^2 A \arrow[r, "-1\otimes\partial"] \arrow[d, "\partial\otimes 1"'] & \Omega^1 A\otimes A \arrow[r, "m_r"] \arrow[d, "\partial\otimes 1"] & \Omega^1 A \arrow[d, "\partial"] \\
A\otimes\Omega^1 A \arrow[r, "1\otimes\partial"] \arrow[d, "m_l"] & A\otimes A\otimes A \arrow[r, "1\otimes m"] \arrow[d, "m\otimes 1"] & A\otimes A \arrow[d, "m"] \\
\Omega^1 A \arrow[r, "\partial"] & A\otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

$\Delta : A\otimes A \longrightarrow A\otimes A\otimes A$, $\boxed{\Delta(a_1\otimes a_2) = a_1\otimes 1\otimes a_2}$

$$(1\otimes\partial)\varphi(da\otimes 1) = (\partial\otimes 1)(da\otimes 1) - \Delta\underbrace{\partial m_r(da\otimes 1)}_{a\otimes 1 - 1\otimes a}$$

$$= a\otimes\cancel{1\otimes} 1 - 1\otimes a\otimes 1 - \cancel{a\otimes 1}\otimes 1 + 1\otimes 1\otimes a$$

$$= -1\otimes(a\otimes 1 - 1\otimes a) = -(1\otimes\partial)(1\otimes da).$$

$$\therefore \boxed{-\varphi(da\otimes 1) = 1\otimes da}$$

Observe then that $-\varphi(a_0 da_1\otimes a_2) = a_0(1\otimes da_1)a_2 = a_0\otimes da_1 a_2$

In other words if we make the idents

$$A\otimes\Omega^1 A = A\otimes\bar{A}\otimes A = \Omega^1 A\otimes A$$

$$a_0\otimes da_1 a_2 \longleftarrow (a_0, a_1, a_2) \longmapsto a_0 da_1\otimes a_2$$

then $-\varphi$ is the identity.

Next we want to apply $\natural$. It turns out best to use the isom.

$$(A\otimes\Omega^1 A)_\natural \simeq \Omega^1 A$$

$$a_0\otimes a_1 da_2 \longmapsto a_1 da_2 a_0$$

in addition to the standard ident.

$$(M\otimes A)_\natural \xrightarrow{\sim} M \qquad m\otimes a \mapsto am$$

Let's compute $(-\varphi)_\natural$.

$$(-\varphi)(\underbrace{a_1\,da_2 \otimes a}_{a a_1\,da_2}) = a_1 \otimes da_2\, a \overset{\natural}{\mapsto} da_2\, a a_1$$

And we recognize $-\varphi = \kappa : a_1 da_2 \longmapsto da_2 a_1$

$$(1\otimes\partial)(1 \otimes a_1 da_2) = 1 \otimes a_1 a_2 \otimes 1 - 1 \otimes a_1 \otimes a_2$$

$$\overset{\natural}{\longmapsto} 1 \otimes a_1 a_2 - a_2 \otimes a_1$$

$$\sigma\partial(a_1 da_2) = \sigma(a_1 a_2 \otimes 1 - a_1 \otimes a_2) = 1 \otimes a_1 a_2 - a_2 \otimes a_1$$

$\therefore\ (1\otimes\partial)_\natural = \sigma\partial : \Omega^1 A \longrightarrow A \otimes A.$ Answer:

$$\begin{tikzcd}
\Omega^2 A_\natural \arrow[r, "b"] \arrow[d, "\kappa b"'] & \Omega^1 A \arrow[r, "\natural"] \arrow[d, "\partial"] \arrow[dl, dashed, "-\varphi=\kappa"'] \arrow[drr, dotted, "b"] & \Omega^1 A_\natural \arrow[d, "b"] \\
\Omega^1 A \arrow[r, "\sigma\partial"] \arrow[d, "\natural"'] \arrow[dr, dotted, "b"] \arrow[ur, phantom, "\text{not comm.}"] & A \otimes A \arrow[r, "\sigma m"] \arrow[d, "m"] & A \arrow[d, "\natural"] \\
\Omega^1 A_\natural \arrow[r, "b"] & A \arrow[r, "\natural"] & A_\natural
\end{tikzcd}$$

It's important to notice that the triangle labelled ~~not~~-commutative is not-commutative:

$$\begin{array}{ccccc}
a_1 da_2 & \overset{-\kappa}{\longmapsto} & da_2 a_1 & \overset{\partial}{\longmapsto} & -a_2 \otimes a_1 + 1 \otimes a_2 a_1 \\
\downarrow \partial & & & & \downarrow \sigma \\
a_1 a_2 \otimes 1 - a_1 \otimes a_2 & & & & -a_1 \otimes a_2 + a_2 a_1 \otimes 1.
\end{array}$$

even when φ is used. Notice the defect is $(a_1 a_2 - a_2 a_1) \otimes 1 = \Delta_\natural\, b(a_1 da_2)$. Notice

$$\boxed{\Delta_\natural\, a = a \otimes 1}$$

December 28, 1990

Recall $0 \to N \to E \xrightarrow{\varepsilon} A \to 0$

$$X = \{N \to E\}$$

and E is projective. We then have quasi-isomorphisms

$$Y \xleftarrow{\ p\ } X \otimes_A X \xrightarrow{\ q\ } Z$$

where Y, Z are the concatenated resolutions:

$$\begin{tikzcd}
Z & 0 \arrow[r] & N\otimes_A N \arrow[r] & N\otimes_A E \arrow[r] & E \arrow[r] & A \arrow[r] & 0 \\
X\otimes_A X \arrow[u, "q"] \arrow[d, "p"] & 0 \arrow[r] & N\otimes_A N \arrow[r] \arrow[u, "1"] \arrow[d, equal] & E\otimes_A N \oplus N\otimes_A E \arrow[r] \arrow[u, "(0,1)"] \arrow[d, "(1,0)"] & E\otimes_A E \arrow[r] \arrow[u, "1\otimes\varepsilon"] \arrow[d, "\varepsilon\otimes 1"] & A \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
Y & 0 \arrow[r] & N\otimes_A N \arrow[r] & E\otimes_A N \arrow[r] & E \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

We have seen how choosing $\Delta : E \to E \otimes_A E \ni$ $(\varepsilon\otimes 1)\Delta = (1 \otimes \varepsilon)\Delta = id_E$ leads to a subcomplex $W \subset X \otimes_A X$ mapped isomorphically onto Y and Z. Thus Y and Z are isomorphic.

Here's an idea for constructing B. We know how to construct a B map

$$\Sigma [X\otimes_A]^2 \longrightarrow X \otimes_A$$

using $X\otimes_A X \underset{\varepsilon_1}{\overset{\varepsilon_0}{\rightrightarrows}} X$ and $h : \varepsilon_0 \Rightarrow \varepsilon_1$. Try to carry this over to $X \otimes_A X \underset{p}{\overset{q}{\rightrightarrows}} \begin{matrix} Z \\ \parallel \\ Y \end{matrix}$. In order to do this we need the analogue of

$$[X\otimes_A]^2 \underset{\varepsilon_1}{\overset{\varepsilon_0}{\rightrightarrows}} X\otimes_A \qquad \varepsilon_1 = \varepsilon_0 \sigma$$

i.e. commutativity of

$$\begin{tikzcd}
{[X\otimes_A]^2} \arrow[r, leftrightarrow, "\sigma"] \arrow[d, "p"] & X\otimes_A \arrow[d, "q"] \\
X\otimes_A \arrow[r, equal] & Z\otimes_A
\end{tikzcd}$$

But this ~~…~~ is false because the isomorphism $Y\otimes_A = Z\otimes_A$ is the identity on $N\otimes_A N\otimes_A$, whereas σ is the flip. Example:

$$\begin{tikzcd}
Z\otimes_A & \Omega^2A_\natural \arrow[r, "b"] & \Omega^1A \arrow[r, "b"] & A \\
{[X\otimes_A]^2} \arrow[u, "q"] \arrow[d, "p"'] & \Omega^2A_\natural \arrow[u, shift left] \arrow[u, shift right] \arrow[r, "{(\kappa b,\, b)}"] \arrow[d, equal] & \Omega^1A\oplus\Omega^1A \arrow[u, "{(0,1)}"'] \arrow[r, "{(\sigma\partial,\, \partial)}"] \arrow[d, "{(1,0)}"] & A\otimes A \arrow[u, "m\sigma"'] \arrow[d, "m"] \\
Y\otimes_A & \Omega^2A_\natural \arrow[r, "\kappa b"] & \Omega^1A \arrow[r, "b"] & A
\end{tikzcd}$$

Now σ on $[X\otimes_A]^2$ is given by the flip σ in degrees zero and one and by κ on $\Omega^2A_\natural = [\Omega^1A\otimes_A]^2$. Here is the isomorphism obtained from σ

$$\begin{tikzcd}
Z\otimes_A \arrow[d, leftrightarrow, "\sigma", "\cong"'] & \Omega^2A_\natural \arrow[r, "b"] \arrow[d, leftrightarrow, "\kappa"] & \Omega^1A \arrow[r, "b"] \arrow[d, equal] & A \arrow[d, equal] \\
Y\otimes_A & \Omega^2A_\natural \arrow[r, "b\kappa"] & \Omega^1A \arrow[r, "b"] & A
\end{tikzcd}$$

Here is the isomorphism obtained from $(-\varphi)$

$$\begin{tikzcd}
Z\otimes_A \arrow[d, "\cong"] & \Omega^2A_\natural \arrow[r, "b"] \arrow[d, "i"] & \Omega^1A \arrow[r, "b"] \arrow[d, "\kappa"] & A \arrow[d, equal] \\
Y\otimes_A & \Omega^2A_\natural \arrow[r, "b\kappa"] & \Omega^1A \arrow[r, "b"] & A
\end{tikzcd}$$

So this idea doesn't work in any obvious way.

Another idea for constructing B on $[X\otimes_A]^2$ is the following. We have seen how to construct a canonical map

$$\Sigma\,[X\otimes_A]^3 \longrightarrow [X\otimes_A]^2$$

out of $X\otimes_A X \underset{\varepsilon_1}{\overset{\varepsilon_0}{\rightrightarrows}} X \qquad h : \varepsilon_0 \Rightarrow \varepsilon_1$. Look at the induced map

$$\begin{tikzcd} \mathrm{Post}_{\leq 2}\left\{\Sigma\left[X\otimes_A\right]^3\right\} \arrow[r] \arrow[d, equal] & \left[X\otimes_A\right]^2 \\ \Sigma\ \mathrm{Post}_{\leq 1}\left\{\left[X\otimes_A\right]^3\right\} & \end{tikzcd}$$

Finally we should have a homotopy equivalence

$$\mathrm{Post}_{\leq 1}\left\{\left[X\otimes_A\right]^3\right\} \longrightarrow X\otimes_A$$

Thus we should get a map

$$\Sigma\, X\otimes_A \longrightarrow \left[X\otimes_A\right]^2$$

In fact there should be a canonical diagram

$$\begin{tikzcd} ? \arrow[r] \arrow[d, "\text{quis}"] & \left[X\otimes_A\right]^2 \\ \Sigma X\otimes_A & \end{tikzcd}$$

Something else we should be able to do is to prove simply that σ on $\left[X\otimes_A\right]^2$ is homotopic to the identity. Here's a proof I think:

$$\begin{tikzcd} \left[X\otimes_A\right]^3 \arrow[d, shift right=2, "\varepsilon_0"'] \arrow[d] \arrow[d, shift left=2] \\ \left[X\otimes_A\right]^2 \end{tikzcd} \qquad \begin{aligned} \varepsilon_0 &= \varepsilon\otimes 1\otimes 1 \\ \varepsilon_1 &= \sigma\varepsilon_0\sigma^{-1} \end{aligned} \qquad h: \varepsilon_0 \Rightarrow \varepsilon_1$$

Then we have $\varepsilon_0 \overset{h}{\Longrightarrow} \sigma\varepsilon_0\sigma^{-1} \Longrightarrow \varepsilon_0\sigma \Longrightarrow \sigma\varepsilon_0$

So we conclude that id and σ become homotopic after composing with ε_0. The homotopy should descend to

$$\mathrm{Post}_{\leq 2}\left[X\otimes_A\right]^3 \xrightarrow[\text{ind. by } \varepsilon_0]{} \left[X\otimes_A\right]^2$$

But this <u>map is a</u> homotopy equivalence.

Let's list things which we don't understand.

Consider $[X \otimes_A]^2$:

$$\begin{tikzcd} N\otimes_A N\otimes_A \arrow[r] \arrow[d] & N\otimes_A E\otimes_A \arrow[d, hook] \\ E\otimes_A N\otimes_A \arrow[r, hook] & E\otimes_A E\otimes_A \end{tikzcd}$$

injections assuming E projective

I think the basic mystery is why the σ action is trivial on the homology.

Here is an explicit proof in the case of H_0. Let $x, y \in E$. Then we have

$$x \otimes y - \underbrace{\Delta(x\,\varepsilon(y))}_{(\Delta x)\,\varepsilon(y)} \in \mathrm{Im}(E \otimes_A N)$$

$$y \otimes x - \underbrace{\Delta(\varepsilon(y)\,x)}_{\varepsilon(y)\,\Delta x} \in \mathrm{Im}(N \otimes_A E)$$

Thus

$$x\otimes y - y \otimes x - [\Delta x, \varepsilon(y)] \in \mathrm{Im}(E\otimes_A N \oplus N \otimes_A E)$$

Here we use the projections

$$E \otimes_A E \longrightarrow E\otimes_A N,\ N \otimes_A E$$

obtained from Δ. But the question is how to construct a map of any nontrivial sort from $E\otimes_A N \otimes_A$ to $N \otimes_A N \otimes_A$.

IDEA: Start with $E/N = A$. Then we have exact sequences of A-bimodules

$$0 \longrightarrow N \longrightarrow E \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow N^{(n+1)} \longrightarrow E \otimes_A N^{(n)} \longrightarrow N^{(n)} \longrightarrow 0$$

which we can splice together to obtain a resolution

$$\circledast \qquad \to E\otimes_A N\otimes_A N \longrightarrow E\otimes_A N \longrightarrow E \to A \to 0 \qquad \circledast$$

But this looks like the normalized complex associated to the (semi-)simplicial bimodule

$$\cdots\; E\otimes_A E\otimes_A E \;\rightrightarrows\; E\otimes_A E \;\overset{\Delta}{\underset{}{\leftleftarrows\!\!\rightrightarrows}}\; E$$

Assume the simplicial bimodule is well defined, and further that

$$\cdots\; [E\otimes_A]^3 \rightrightarrows [E\otimes_A]^2 \rightrightarrows E\otimes_A$$

is a cyclic vector space. Then it should follow that there is a B operator on $\circledast_A$. Of course it depends on the choice of Δ.

Check: $\Delta - \sigma\Delta : E\otimes_A \longrightarrow [E\otimes_A]^2$. We have

$$\varepsilon_0(\Delta-\sigma\Delta) = \varepsilon_0\Delta - \varepsilon_1\Delta = 1-1 = 0,$$ and similarly $\varepsilon_1(\Delta-\sigma\Delta)$. Thus $B = \Delta - \sigma\Delta$ maps $E\otimes_A$ into $\operatorname{Ker}\varepsilon_1 \simeq E\otimes_A N\otimes_A$ and in fact into $\operatorname{Ker}\varepsilon_0 \cap \operatorname{Ker}\varepsilon_1$, which gives $bB = 0$.

At this point there's a lot to be checked. An interesting point is that we started this business without Δ and a lot of things we have done are independent of Δ. In particular $\circledast$ and the relation of concatenated resolutions to the tensor products $X\otimes_A \cdots \otimes_A X$. So I have the feeling that a lot can be done, should be done, without getting lost in simplicial identities.

Example. Consider $T_A(E) = \bigoplus_{n\geq 0} E^{(n)}$, the tensor algebra of the bimodule E. We have $\varepsilon : E \to A$ a bimodule map, an it should extends to a degree -1 (anti) derivation which should be a differential as the square is a derivation vanishing on the generators. Consider the relative X-complex

$$\bar{T}_A(E)\otimes_A \underset{d}{\overset{b}{\rightleftarrows}} T_A(E)\otimes_A E\otimes_A$$

where $\bar{\ }$ means removes A in degree zero, and E is to be viewed as differentials $d(E)$. Thus it is killed by ε, and so $T_A(E)\otimes_A E\otimes_A$ should be a direct summand of $T_A(E)\otimes$ (some vector space); in particular it should be acyclic. On the other hand $\bar{T}_A(E)\otimes_A$ ought to compute $HH(A)$, so then we get a double complex

$$\bar{T}_A(E)\otimes_A \xleftarrow{\ b\ } T_A(E)\otimes_A E\otimes_A \xleftarrow{\ d\ } \bar{T}(E)\otimes_A \xleftarrow{\ b\ }$$

which horizontally resolves the "cyclic complex."

$$\bigoplus_{n\geq 1}\left[E\otimes_A\right]^n_\lambda$$

Notice that if all this is correct, then in the case $E = A\otimes A$, we really recover the usual cyclic double complex as a type of relative X-complex

Further observation is that what distinguishes $A\otimes A$ is that it is an E with $\varepsilon : E\longrightarrow A$ and $\Delta : E\longrightarrow E\otimes_A E$ such that there is an elt $\xi\in E$ satisfying $\varepsilon(\xi)=1$ and $\Delta(\xi) = \xi\otimes\xi$.

December 31, 1990

Problem. Let R be an algebra such that $\Omega^1 R$ is projective, let I be an ideal in R. To show the truncated complex $X^n(R,I)$ is invariant under restricted homotopy, specifically, given $u_t : (R,I) \longrightarrow (R',I')$ a restricted homotopy, then $u_{t*} : X^n(R,I) \longrightarrow X^n(R',I')$ modulo chain homotopy is constant in t.

I think I can prove this for R free. In the general case, we ~~write~~ $R = F/K$ with F free, and let $J \subset F$ be the inverse image of I so that $F/J = R/I$. Then we have a lifting $R \xrightarrow{s} \varprojlim F/K^n$ whence the inverse system $\{R/I^{n+1}\}$ is a retract of $\{F/J^{n+1}\}$. Since

$$X^m(R,I) = X^m(R/I^{n+1}, I/I^{n+1})$$

for $n \geq m$ we see that $X^m(R,I)$ is a retract of $X^m(F,J)$. The maps are given by the canonical surjection $r : F \to R$ and by $s : R \to \{F/K^{n+1}\}$

Now suppose we have a restricted homotopy $(R,I) \xrightarrow{u_t} (R',I')$; then $u_t r : (F,J) \longrightarrow (R',I')$ is one too, so we know $u_{t*} r_* : X^m(F,J) \longrightarrow X^m(R',I')$ mod homotopy is independent of t. $\therefore$ The same is true for $u_{t*} = u_{t*} r_* s_* : X^m(R,I) \longrightarrow X^m(R',I')$

First order analysis of homotopy for the X complex. (I remain confused about the proper functorial setting for this, but the "answer" is as follows.)

We wish to describe the map

$$(*) \qquad X(A) \longrightarrow X(A \oplus \Omega^1 A)_{(1)} \qquad (= X(\Omega A)_{(1)})$$

where (1) subscript means the degree 1 part for the obvious $\mathbb{N}$ grading on $A \oplus \Omega^1 A$ or on $\Omega A = T_A(\Omega^1 A)$. Let

$$\Gamma = \Omega^1(A \oplus \Omega^1 A)_{(1)}$$

and use $1 + t\delta : A \longrightarrow A \oplus \Omega^1 A$ for the canonical family, and d for the differential in the X complex.

We claim that we have an isomorphism

$$\Omega^1 A_\natural \oplus \Omega^2 A_\natural \xrightarrow{\sim} \Gamma$$

$$a_0 da_1 \longmapsto \delta(a_0 da_1)$$

$$a_0 da_1 da_2 \longmapsto a_0 \delta a_1 da_2$$

In other words, a trace T of degree 1 on $\Omega^1(A \oplus \Omega^1 A)$ (or on $\Omega^1(\Omega A)$) is equivalent to the Hochschild 1, 2 cocycles φ, ψ given by

$$\varphi(a_0, a_1) = T\delta(a_0 da_1) = T(\delta a_0 da_1) + T(a_0 d\delta a_1)$$

$$\psi(a_0, a_1, a_2) = T(a_0 \delta a_1 da_2)$$

Using this isom. we have the ~~[illegible]~~ formulas for $(*)$

$$\begin{tikzcd}
{} \arrow[r, "b"] & A \arrow[r, "d"] \arrow[d, "\delta"] & \Omega^1 A_\natural \arrow[r, "b"] \arrow[d, "\binom{1}{0}"] & A \arrow[r, "d"] \arrow[d, "\delta"] & {} \\
{} \arrow[r] & \Omega^1 A \arrow[r, "\binom{\natural}{-B}"'] & \Omega^1 A_\natural \oplus \Omega^2 A_\natural \arrow[r, "{(Bb, -b)}"'] & \Omega^1 A \arrow[r] & {}
\end{tikzcd}$$

Homology of $X(A \oplus \Omega^1 A)_{(1)}$: Have canonical surjection

$$\begin{tikzcd}
\arrow[r] & \Omega^1 A \arrow[r, "\binom{\natural}{-B}"] \arrow[d, "\natural"] & \Omega^1 A_\natural \oplus \Omega^2 A_\natural \arrow[r, "{\overset{-bB}{\overset{\parallel}{(Bb,\, -b)}}}"] \arrow[d, "(1\ \ 0)"] & \Omega^1 A \arrow[r] \arrow[d] & {} \\
\arrow[r, "0"] & \Omega^1 A_\natural \arrow[r, "1"] & \Omega^1 A_\natural \arrow[r, "0"] & \Omega^1 A_\natural \arrow[r] & {}
\end{tikzcd}$$

whose kernel is

$$\longrightarrow [A, \Omega^1 A] \xrightarrow{\ 0\ } \Omega^2 A_\natural \xrightarrow{\ -b\ } [A, \Omega^1 A] \longrightarrow$$

There's a canonical surjection of this to

$$\xrightarrow{\ 1\ } [A, \Omega^1 A] \xrightarrow{\ 0\ } [A, \Omega^1 A] \xrightarrow{\ 1\ } [A, \Omega^1 A] \xrightarrow{\ 0\ }$$

whose kernel is

$$\longrightarrow 0 \longrightarrow HH_2 A \longrightarrow 0 \longrightarrow$$

We have ~~have~~ described a filtration of $X(A \oplus \Omega^1 A)_{(1)}$ with three quotients, two contractible and the third being $HH_2 A\,[1]$. One can split this filtration by choosing 1) a section of $\Omega^1 A \xrightarrow{\natural} \Omega^1 A_\natural$
2) a section of $\Omega^2 A_\natural \xrightarrow{b} [A, \Omega^1 A] \subset \Omega^1 A$

The homotopy class of the map $X(A) \longrightarrow X(A \oplus \Omega^1 A)_{(1)}$ is therefore described by the induced map on H_1 which is $B : HC_1 A \longrightarrow HH_2 A$ up to sign.

Thus it seems that we have the homotopy property for $X(A)$ precisely when this B-map is zero, i.e. when $HC_1 A$ has the homotopy property.

The goal: I want a stronger property than homotopy for $X(A)$ rather I want invariance under deformation. Deformation should involve nilpotent extensions of higher orders and so it should require vanishing of higher Hochschild homology not just $B: HC_1 A \to HH_2 A$.

Observation: $HH_2 A = 0 \iff$ the map $X'(RA, IA) \longrightarrow X(A)$ is a qis.

In effect the homology is

$$HC_2 A \xrightarrow{S} \mathrm{Ker}\{HC_0 A \to HH_1 A\}$$

$$\mathrm{Ker}\{HC_1 A \xrightarrow{B} HH_2 A\} \subset HC_1 A$$

and one has the Connes exact sequence

$$HC_1 \xrightarrow{B} HH_2 \xrightarrow{I} HC_2 \xrightarrow{S} HC_0 \xrightarrow{B} HH_1$$

so it's clear.