April 11 (cont) - August 11, 1990 | 377-525
p. 377 - 525

p521, 528. Complex $R/I^{m+1}+[R,I^m] \rightleftarrows (\Omega^1 R/I^m\Omega^1 R)_\natural$

517. Connes exact sequence using $A=R/I$

501. Square zero alg ext. (Injectivity of I/I^2)

500. $A\otimes B$ 475. $A*B \to A\otimes B$ and decomp.

479-94. Coalgebras in cat of A-bimodules. Attempt to do bimodule version of bar construction

453-474 X-complex on nilpotent extensions

449. $e^{\tau X + tX^2}$

440 Comparing $\bar{\Omega}(\tilde{A})$ with ΩA

432 Continuity of homotopy wrt I-adic filtration

398-424 Homotopy for $X(A)$.

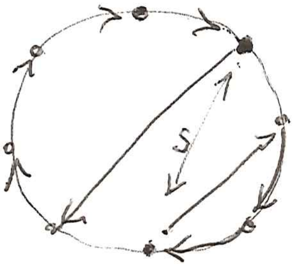
398 Defining L_D, ι_D^* on ΩA

April 11 (cont.)
Feit conference: April 8-11

Call a sequence of conjugacy classes $C_1, \dots, C_k$ in a finite group G rigid if G acts simply-transitively by conjugation on the set

$$A = \left\{ (x_1, \dots, x_k) \in C_1 \times \dots \times C_k \;\middle|\; \begin{array}{l} x_1 \cdots x_k = 1 \\ \langle x_1, \dots, x_k \rangle = G \end{array} \right\}$$

Example. $G = \Sigma_n$, $C_1 =$ n-cycles, $C_2 =$ transpositions, $C_3 = (n-1)$-cycles. Note that if γ is an n-cycle and s is a transposition then γs will be an $(n-1)$ cycle iff s transposes [illegible] adjacent elements wrt γ. Picture of γs:



~~It follows that the [illegible] pairs (γ, s) with γs an $(n-1)$ cycles is [illegible] the same as the set of pairs (γ, p).~~ Thus given an n-cycle γ, to give a transposition s such that γs is an n-1 cycle is the same as picking a point, that is, a linear ordering compatible with the cyclic ordering defined by γ. Thus G acts simply-transitively on A in this case.

Thompson showed that if G has a rigid sequence $C_1, \dots, C_k$ then one can construct a Galois extension M with group G of $K[X]$, where is the field over $\mathbb{Q}$ generated by the values of the irreducible characters on the classes C_i. Moreover K is algebraically closed in the extension M (regular). Then Hilbert's irreducibility theorem gives G as Galois group of an extension of K.

Further comments on rigidity (learned from Serre)

Given a finite group G there's a variety whose rational points over $\mathbb{Q}$ give the ways of realizing G as a Galois group over $\mathbb{Q}$. But the problem is to show this variety has rational points.

Riemann's treatment of the hypergeometric function gives an example of rigidity in a Lie group situation.

Rigidity is used in descending a Galois extension of $\mathbb{C}(t)$ to $\mathbb{Q}(t)$. Ordinary descent (Weil or Grothendieck) is used.* It's necessary to have the classes C_i rational in order to get down to $\mathbb{Q}(t)$. C_i rational means the characters have rational values or equivalently that if $x \in C_i$ and y is another generator of the cyclic group $\langle x \rangle$, then $y \in C_i$. *(so it should be an exercise)

Action of the braid group on A:

$$(x_1, \dots, x_k) \longmapsto (x_1, \dots, x_i x_{i+1} x_i^{-1}, x_i, \dots, x_k)$$

Outline of Serre's lectures on Galois Groups and cohomology.

The first two lectures were concerned with the problem of realizing a central extension $\tilde{G}$ of G as a Galois group, assuming G already realized.

$$\begin{tikzcd}
 & {} \arrow[d] \\
 & C \arrow[d] \\
 & \tilde{G} \arrow[d] \\
\mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[ur, dashed] & G \arrow[d] \\
 & 1
\end{tikzcd}$$

The dotted arrow will be onto assuming $G = (G, G)$, so one has a cohomological obstruction in $H^2(k, C)$.

Interesting case $k = \mathbb{Q}(t)$, $C = \mathbb{Z}/2\mathbb{Z}$. $H^2(k, \mathbb{Z}/2\mathbb{Z}) = Br_2(k)$ which is generated by quaternion algebra symbols (thm. of Mercuriev). Serre defines poles + residues for such classes. Probably he is making explicit an exact sequence in Galois cohomology

$$\longrightarrow H^2(\mathbb{Q}(t), \mathbb{Z}/2) \longrightarrow \bigoplus_{m} k_m^*/k_m^{*2} \longrightarrow$$

Second lecture concerns a central extension $\tilde{G}$ of G by $\mathbb{Z}/2\mathbb{Z}$, odd elements $s_1, .., s_k \in G$. These have unique odd order lifts $\tilde{s}_1, .., \hat{s}_k$. Assuming $s_1 .. s_k = 1$, then $\tilde{s}_1 -- \tilde{s}_k = \pm 1$ and Serre gives a formula for it. He supposes G realized over a genus zero field, otherwise the formula needs the genus.

Third lecture exposes stuff of Lenstra where one looks at a Galois extension L/k with group G as a quadratic G-space for the trace form (hence a free rank 1 $*$ rep of $k[G]$). Here Galois cohomology of the unitary group of $k[G]$ enters.

April 12, 1990

Given D a derivation on A we have on ΩA derivations ι_D^*, L_D satisfying the usual identities.

$$[L_D, L_{D'}] = L_{[D,D']} \qquad [d, \iota_D^*] = L_D$$

$$[L_D, \iota_{D'}^*] = \iota_{[D,D']}^*$$

$$[\iota_D^*, \iota_{D'}^*] = 0$$

We would like to obtain similar formulas on $(\Omega A)_{inv}$. As L_D is an infinitesimal automorphism, it commutes with $b, d, \kappa, \tilde{\kappa}, P$, and so it induces an operator on the invariant complex. The problem is with ι_D^*, which we recall is given by

$$\iota_D^* = \sum_{j=0}^{n-1} \kappa^j \iota_D \kappa^{-j} \qquad \text{on } \Omega^n$$

Recall from yesterday

$$P b \iota_D^* = P b \sum_{j=0}^{n-1} \tilde{\kappa}^j \iota_D \tilde{\kappa}^{-j} = n P b \iota_D P$$

$$\kappa^n = 1 + b\lambda^{-1} d$$

$\therefore$ [illegible]

$$\boxed{\begin{aligned} P b \iota_D^* &= n\, b \iota_D^\# \qquad \text{on } \Omega^n \\ \iota_D^* b P &= (n-1)\, \iota_D^\# b \end{aligned}}$$

From this we see that $\boxed{I_D = \frac{1}{n} P \iota_D^* P \quad \text{on } \Omega^n}$ commutes with b. It satisfies

$$\boxed{[b, I_D] = 0 \qquad [B, I_D] = PL_D}$$

Also because L_D comes from an infinitesimal automorphism it is clear that we have

$$[L_{D'}, I_D] = I_{[D', D]}$$

so the remaining point is whether $I_D{}^2 = 0$? By polarization this gives $[I_{D'}, I_D] = 0$.

Since $(\iota_D^*)^2 = 0$ we have

$$I_D{}^2 = \frac{1}{(n-1)n} P\iota_D^* P \iota_D^* P = \frac{-1}{(n-1)n} \underbrace{P\iota_D^*(1-P)\iota_D^* P}_{P\iota_D^*(Gdb+bGd)\iota_D^* P}$$

~~$P\iota_D^* Gdb\iota_D^* P = P\iota_D^* dGb\iota_D^* P$~~

$$P\iota_D^* Gd = P\iota_D^* dG = P(L_D - d\iota_D^*)G = -Pd\iota_D^* G$$

$$= -Pd\sum_{j=0}^{n-1} \cancel{K^j}\iota_D K^{-j} G$$

$$P\iota_D^* Gdb = -Pd\sum_{j=0}^{n-2} \iota_D K^{-j} bG$$

~~$K^{n-1} = (1+db)$ on Ω^{n-1}~~

~~$= -Pd\, n\, \iota_D P bG = 0$~~

Now $K^n = 1 - db$ on Ω^{n-1} so

$K^n = 1$ on $bG\Omega^{n-1}$ $\therefore$

$$\sum_{j=0}^{n-2} K^{-j} bG = \underbrace{\sum_{j=0}^{n-1} K^{-j} Gb}_{nPG = 0} - \underbrace{K^{-(n-1)}}_{K} Gb$$

So we have

$$P\iota_D^* G d b = -P d \iota_D \sum_{j=0}^{n-2} \kappa^{-j} b G$$

$$= P d \iota_D \kappa G b$$

On the other hand

$$b \iota_D^* P = b \sum_{j=0}^{n-1} \kappa^{j} \underbrace{\iota_D \cancel{\kappa^{-j}} P}_{\text{on } \Omega^{n-1}} \qquad \kappa^{n-1} = (1 + b\lambda^{-1} d)$$

$$= b((n-1)P+1) \iota_D P$$

$$\therefore G b \iota_D^* P = G b \iota_D P$$

$$P\iota_D^* G d b \iota_D^* P = (P d \iota_D \kappa G b) \iota_D^* P$$

$$= P d \iota_D \kappa G b \iota_D P$$

$$= d(P \iota \kappa G b \iota_D P)$$

Next compute $P\iota_D^* b G d \iota_D^* P = -P\iota_D^* \underbrace{b G \iota_D^* P}_{b G \iota_D P} d$

$$P\iota_D^* b = P \sum_{j=0}^{n-2} \cancel{\kappa^{j}} \iota_D \underbrace{\kappa^{-j} b}$$

$$\kappa^{n} = 1 - db \quad \text{on } \Omega^{n-1}$$

$$= P\iota_D (nP - \kappa) b = P\iota_D ((n-1)P - (1-P)\kappa) b$$

$$P\iota_D^* b G = -P\iota_D \kappa b G$$

$$\therefore \quad P\iota_D^* b G d \iota_D^* P = -\underbrace{P\iota_D^* b G}_{P\iota_D \kappa b G} \iota_D P d$$

$$P\iota_D^* G bd \iota_D^* P = (P\iota_D \kappa b G \iota_D P) d$$

Let's check this calculation

We have

$$\boxed{\begin{aligned} P b \iota_D^* &= n b \iota_D^\# \\ b \iota_D^* P &= n b \iota_D^\# + b(1-P)\iota_D P \\ \iota_D^* b P &= (n-1)\iota_D^\# b \\ P\iota_D^* b &= (n-1)\,\iota_D^\# b - P\iota_D(1-P)Kb \end{aligned}}$$

Then $I_D = \frac{1}{n} P\iota_D^* P$ on Ω^n

$$I_D^2 = \frac{1}{(n-1)n} P\iota_D^* P \iota_D^* P$$

$$-(n-1)n I_D^2 = P\iota_D^*(1-P)\iota_D^* P \qquad \text{using } \iota_D^{*2} = 0$$

$$= P\iota_D^*(Gdb + bGd)\iota_D^* P$$

$$= P(\cancel{\iota_D} - d\iota_D^*)Gb\iota_D^* P + P\iota_D^* bG(\cancel{\iota_D} - \iota_D^* d)P$$

$$(n-1)n I_D^2 = +d(P\iota_D^* Gb\iota_D^* P) + (P\iota_D^* Gb\iota_D^* P)d$$

But $P\iota_D^* Gb\iota_D^* P = P\iota_D^* G\big(\overset{n b \iota_D^\#}{\text{[illegible]}} + b(1-P)\iota_D P\big)$

$$= P\iota_D^* b(G\iota_D P)$$

$$= \{\cancel{(n-2)\,\iota_D^\#} b - P\iota_D(1-P)Kb\}G\iota_D P$$

$$= -P\iota_D GKb\iota_D P$$

$$= b(P\iota_D GK\iota_D P) = (P\iota_D GK\iota_D P)b$$

$$\boxed{I_D^2 = B\left(\frac{P\iota_D^* G b \iota_D^* P}{(n-2)(n-1)n}\right) + \left(\frac{P\iota_D^* G b \iota_D^* P}{(n-1)n(n+1)}\right)B}$$

Thus $I_D^2 = [B, Q]$ which checks with

$$[B, I_D^2] = [L_D, I_D] = 0.$$

Here

$$Q = \frac{P\iota_D^* GKb\, \iota_D^* P}{(n-2)(n-1)n} = + \frac{P\iota_D GK\iota_D P}{(n-2)(n-1)n} b = \frac{b(P\iota_D GK\iota_D P)}{(n-2)(n-1)n}$$

Set $A = P\iota_D GK\iota_D P$ so that $bA = Ab$. Let's try to produce an [redacted] $H =$ [redacted] $c_n A$ such that $bH - Hb = Q$. Then we want

$$b c_n A - c_{n-1} A b \overset{?}{=} \frac{bA}{(n-2)(n-1)n}$$

i.e. $$c_n - c_{n-1} = \frac{1}{(n-2)(n-1)n} = \frac{1}{n-1}\left(\frac{1}{n-2} - \frac{1}{n}\right)\frac{1}{2}$$

It thus appears the solution is

$$c_n = -\frac{1}{2n(n-1)} + c$$

which is OKAY since A has degree -2 and so $n \geq 2$.

This shows that I_D^2 has the form

$$\boxed{I_D^2 = [B, [b, H]]}$$

with

$$\boxed{H = \left(-\frac{1}{2n(n-1)} + c\right) P\iota_D GK\iota_D P}$$

April 13, 1990

The problem is to decides whether on $(\Omega A)_{\mathrm{inv}}$ it is possible to associate an operator I_D to a derivation such that $[B, I_D] = L_D$, $[b, I_D] = 0$, $I_D^2 = 0$. More precisely I want to represents $\mathfrak{g}[\varepsilon]$, where $\mathfrak{g} = \mathrm{Der}(A)$ and $\mathfrak{g}[\varepsilon] = \mathfrak{g} + \varepsilon\mathfrak{g}$ is the DGL with $d(\varepsilon) = 1$ occurring in Weil algebra theory, on $(\Omega A)_{\mathrm{inv}}$. It is the last condition $I_D^2 = 0$ which gives trouble. There may be a real obstruction to doing this.

Let's consider a simpler problem. Suppose we start with a complex E having a splitting $E = E' \oplus E''$ with E'' contractible and an endomorphism L preserving the splitting. Suppose given I on E of degree -1 such that $[d, I] = L$ and $I^2 = 0$ (whence $0 = [d, I^2] = [L, I]$). We do not assume I is compatible with the splitting. Then we want to obtain from I an I' on E' satisfying the same properties. If $E' \xrightarrow{\iota} E \xrightarrow{\iota^*} E'$ are the inclusion + projection, then $I' = \iota^* I \iota$ satisfies $[d, I'] = \iota^* L \iota = L'$, where $L' = L|E'$, and $[L', I'] = 0$. But I'^2 needn't be zero.

A simpler version of this situation occurs if we suppose $L = 0$. Then I gives a mixed complex structure on E, and we want to understand a SDR (strong deformation retract) of a mixed complex structure. This is similar to asking what happens to a DG algebra structure on E when one passes to a SDR E' of E. Here one knows that E' has the structure of A_∞-algebra, that is, a certain kind of differential on the tensor coalgebra of ΣE. One expects in the mixed complex case to obtain a mixed-complex-up-to-higher-homotopy structure on E', which is to be defined precisely as some sort of differential on $E'[u] = k[u] \otimes E'$.

Shift to mixed complex notation (E, b, B). One has on $E[u]$ the differential $b + uB$; $E[u]$ with $b+uB$ is bigraded with horizontal differential B and vertical differential b. Now let $E' \xrightarrow{\iota} E \xrightarrow{\iota^*} E'$, $\iota^*\iota = 1$, $\iota\iota^* = 1 + [b,h]$, $h\iota = \iota^* h = h^2 = 0$ be a SDR. Put $B' = \iota^* B \iota$. Then $[b, B'] = 0$, and

$$B'^2 = \iota^* B \iota \iota^* B i = \iota^* B(\iota^*\iota - 1) B i$$
$$= \iota^* B[b,h] B \iota = -[b, \iota^* B h B i]$$

is homotopic to zero.

Let's suppose we have a differential $d = b + uB' + u^2 d_2 + \dots$ on $E'[u]$. Then

$$0 = d^2 = b^2 + u([b,B']) + u^2(B'^2 + [b,d_2])$$
$$+ u^3([b,d_3] + [B',d_2]) + \cdots$$

giving the equations $b^2 = 0$, $[b, B'] = 0$, $B'^2 = -[b, d_2]$, $[b,d_3] + [B',d_2] = 0$.

Consider what one learns from HPT (homological perturbation theory)

$$\begin{tikzcd} E[u] \arrow[d, shift right, "i^*"'] \\ E'[u] \arrow[u, shift right, "i"'] \end{tikzcd}$$

$$b + \underbrace{uB}_{\theta}$$

$$b + \theta'$$

$$\theta' = i^* \frac{1}{1-\theta h} \theta i$$

$$= i^* \frac{1}{1-uBh} uBi$$

Thus the differential on $E'[u]$ is

$$d = b + u(i^*Bi) + u^2(i^*BhBi) + u^3(i^*(Bh)^2Bi) + \cdots$$

The moral is that this is the best one can expect on E'. It seems that $E'[u]$ will not have a <u>double complex structure</u>.

Remarks on HPT. Recall the basic setup, where one has a splitting of complexes $E = E' \oplus E''$ with E'' contractible, a perturbation $d + \theta$ of the differential of E, and one constructs (assuming convergence) a perturbed differential on E' and perturbed splitting.

I think that much of what is happening is independent of $d^2 = 0$. The rough idea is that one has a splitting invariant under d and a perturbation of d, so if things are nice there should be a perturbed splitting. The normal behavior is expected when the spectrum of d on E' is disjoint from the spectrum on E'' and if the perturbation is small. This doesn't happen for d, but the splitting is also invariant under $d+h$, which is invertible on E'' since $(d+h)^2 = \cancel{d^2} + [d,h] + \cancel{h^2} = 1$.

In general suppose we have a perturbation $\mathring{A}$ of an operator A and a splitting represented by an involution F which is invariant under A: $[A,F]=0$. We seek a perturbation $\mathring{F}$ of F so that $F+\mathring{F}$ is an involution invariant under $A+\mathring{A}$:

$$[A+\mathring{A}, F+\mathring{F}] = 0 \qquad (F+\mathring{F})^2=1$$

To first order this is

$$[A,\mathring{F}] + [\mathring{A},F] = 0 \qquad F\mathring{F}+\mathring{F}F=0.$$

Thus we have to solve

$$[A,\mathring{F}] = [F,\mathring{A}]$$

with $\mathring{F}$ anticommuting with F.

As $[F,\mathring{A}]$ anticommutes with F, it suffices to assume $ad(A)$ invertible on the operators anticommuting with F. This is like asking the spectrum of A on the two pieces of the splitting to be disjoint.

In our case $A=d$ and this doesn't hold, however $\mathring{A}=\theta$ which satisfies $[d,\theta]=0$ ~~to~~ first order. Thus we have to solve

$$[d,\mathring{F}] = [F,\theta]$$

which is possible on the space of operators anti-commuting with F: $\mathrm{Hom}(E'',E')\oplus\mathrm{Hom}(E',E'')$, since E'' is contractible.

Formulas:
$$\begin{tikzcd} E' \arrow[r, shift left, "\iota"'] & E \arrow[l, shift left, "\iota^*"'] \arrow[r, shift left, "j^*"] & E'' \arrow[l, shift left, "j"] \end{tikzcd}$$

$$d = (\iota \ \ j)\begin{pmatrix} d' & 0 \\ 0 & d'' \end{pmatrix}\begin{pmatrix} \iota^* \\ j^* \end{pmatrix}$$

write this $d \leftrightarrow \begin{pmatrix} d' & 0 \\ 0 & d'' \end{pmatrix}$. Then $\overbrace{e}^{=\iota\iota^*} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$\overset{\circ}{c} = (\iota\iota^*)^\circ = \iota^\circ\iota^* + \iota\iota^{*\circ}$$

$$\overset{\circ}{e} \longleftrightarrow \begin{pmatrix} 0 & \iota^{*\circ} j \\ j^*\iota^\circ & 0 \end{pmatrix}$$

$$[d,\overset{\circ}{e}] \longleftrightarrow \begin{pmatrix} 0 & d'\iota^{*\circ}j - \iota^{*\circ}jd'' \\ \underbrace{d''(j^*\iota^\circ) - (j^*\iota^\circ)d'}_{[d,\, j^*\iota^\circ]} & 0 \end{pmatrix}$$

write :

Want $[d,\overset{\circ}{e}] = [e,\theta] = \begin{pmatrix} 0 & \iota^*\theta j \\ -j^*\theta\iota & 0 \end{pmatrix}$

To solve $[d, j^*\iota^\circ] = -j^*\theta\iota$. Try $j^*\iota^\circ = h''j^*\theta\iota$.

$$\begin{aligned} [d, h''j^*\theta] &= d''h''j^*\theta\iota - h''j^*\theta\iota d' \\ &= (d''h'' + h''d'')j^*\theta\iota & \text{since } [d,\theta]=0 \\ &= -j^*\theta\iota & \text{as } 1+[d,h] = \iota\iota^* \end{aligned}$$

Thus $\iota^\circ = jh''j^*\theta\iota = h\theta\iota$, so

$$\iota + \iota^\circ = (1+h\theta)\iota = \frac{1}{1-h\theta}\iota \quad \text{to first order.}$$

I think this explains what happens at least in the case when θ can be reached by a path of θ_t satisfying $d\theta_t + \theta_t^2 = 0$, but it's not clear why the general formulas hold.

Problem: Consider the action of $Der(A,A) = \mathfrak{g}$ on $(\Omega A)_{inv}$ via $D \mapsto L_D$. Does this extend to an action of the DG Lie algebra $\mathfrak{g}[\varepsilon]$? In other words can one define I_D satisfying the usual formulas? So far I haven't been able to get the condition $I_D^2 = 0$. It seems likely that I_D^2 is ∞-ly homotopic to zero. It is a problem to make this precise, but presumably one might do it using the "cochains" on $\mathfrak{g}[\varepsilon]$ with values in $(\Omega A)_{inv}$. The problem is then to produce an appropriate BRS differential.

A related question concerns whether the natural $\mathfrak{g} = \mathrm{Der}(A,A)$ action on the little complex $\longrightarrow A \longrightarrow \Omega^1 A_\natural \longrightarrow A \longrightarrow$ extends to $\mathfrak{g}[\varepsilon]$, assuming of course that $\Omega^1 A$ is a projective A-bimodule. In this case we know L_D is homotopic to zero, but the resulting homotopy operator $\tilde{I}_D$ doesn't satisfy $\tilde{I}_D^2 = 0$, and probably there is also difficulty with $[L_D, \tilde{I}_D] = 0$.

April 14, 1990

Rinehart's formulas

$$L_D = d\sum_{j=0}^{n-1} K^j \iota_D K^{-j} + \sum_{j=0}^{n} K^j \iota_D K^{-j} d$$

$$B\iota_D + \iota_D B = \sum_{j=0}^{n-1} K^j d\iota_D + \sum_{j=0}^{n} \iota_D d K^{-j}$$

$$L - [B, \iota_D] = \sum_{i=0}^{n-1} K^i d\iota_D (K^{-i} - 1) + \sum_{j=0}^{n} (K^j - 1)\iota_D d K^{-j}$$

$$K^j - 1 = (K-1)(1 + K + \cdots + K^{j-1})$$

$$K^{-i} - 1 = (1-K)(K^{-i} + K^{-i+1} + \cdots + K^{-1})$$

$$L - [B, \iota_D] = \Big(\sum_{0<j\le i<n} K^i d\iota_D K^{-j}\Big)(bd + db) - (bd + db)\Big(\sum_{0\le i<j\le n} K^i \iota_D d K^{-j}\Big)$$

$$= \Big(\sum_{0<j\le i<n} K^i d\iota_D d K^{-j}\Big) b - b\Big(\sum_{0\le i<j\le n} K^i d\iota_D d K^{-j}\Big)$$

$$+ \Big(\sum_{0<j\le i<n} K^i (d\iota_D b d) K^{-j}\Big) + \Big(\sum_{0\le i<j\le n} K^i (d\iota_D b d) K^{-j}\Big)$$

$0 \le i < n$
$1 \le j \le n$
$j \le i$

$0 \le i < n$
$1 \le j \le n$
$i < j$

So the last two terms combine to yield

$$\sum_{0\le i<n} K^i \overbrace{(d\iota_D b d)}^{\Omega^{n-1}} \sum_{j=1}^{n} K^{-j} = B\iota_D b d \sum_{j=1}^{n} K^{-j}$$

$$= b\, B\iota_D d \sum_{j=1}^{n} K^{-j} = b \sum_{i=0}^{n} \sum_{j=1}^{n} K^i \underbrace{d\iota_D d K^{-j}}_{\Omega^n}$$

Combine with the second term $-b \sum_{0\le i<j\le n} K^i d\iota_D d K^{-j}$.

In the set $\begin{matrix} 0\le i\le n \\ 1\le j\le n\end{matrix}$ remove $i<j$ to get $i\ge j$ which gives $0<j\le i\le n$ yielding

$$L-[B,\iota_D] = \Big(\sum_{0<j\le i<n} K^i d\iota_D d K^{-j}\Big) b$$

$$\boxed{L=[B,\iota_D]+[b,J_D]} \qquad + b\Big(\sum_{0<j\le i\le n} K^i d\iota_D d K^{-j}\Big) = [b,J_D]$$

with $\boxed{J_D = \sum_{0<j\le i\le n} K^i d\iota_D d K^{-j} \quad \text{on } \Omega^n}$

which is Rinehart's formula since

$$K^j d\iota_D d K^{-j} = K^j d L_D^{(0)} K^{-j} = d L_D^{(j)}$$

Here's a sort of check on a previous calculation (p384) which ends up expressing I_D^2 in terms of $b(P\iota_D G K \iota_D P)$. Set $A_j = bP\iota_D K^j \iota_D P$ on Ω^{n+1}

$= bP\iota_D K^j \iota_D K^{-j} P$ (because b present)

$= bP\iota_D \iota_D^{(j+1)} P$ $\qquad j=1,\dots,n$

$= -bP\iota_D^{(j)}\iota_D P = -bP\iota_D K^{-j+1}\iota_D P.$

Now $A_j = bP\iota_D K^j \underbrace{\iota_D P}_{\Omega^n}$, $K^n = 1 + b\lambda^{-1}d$ on Ω^n, depends on j modulo n. Thus we have

$$A_j = -A_{n-j+1}.$$

so

$$\begin{aligned} A_1 &= -A_n \\ A_2 &= -A_{n-1} \\ &\vdots \\ A_n &= -A_1 \end{aligned} \qquad \Rightarrow \qquad bP\iota_D P\iota_D P = 0$$

Now $G = \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{n-1}{2}-j\right)K^j$

$$GK = \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{n+1}{2}-j-1\right)K^{j+1} = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{n+1}{2}-j\right)K^j$$

Thus in forming $bP\iota_D GK\iota_D P$ we combine A_j and A_{n+1-j} with opposite sign, which means we have no cancellation. This checks the occurrence of GK in a sense, and suggests strongly that $I_D^2 \neq 0$.

April 16, 1990

Let D be a derivation of A. Let us [redacted] try to produce L_D, ι_D^* simultaneously using the universal property of ΩA. We are after a representation of the DG Lie algebra with basis L, ι of degrees $0, -1$ satisfying $[L,L] = [L,\iota] = [\iota,\iota] = 0$, [redacted] $d(\iota) = L$. Introduce the cochains maybe. The DG algebra of cochains is the commutative algebra generated by [redacted] a shifted dual basis χ of degree 1 and φ of degree 2.

Here's how to proceed. Consider the DGA $k + (k\varepsilon + k\sigma)$, where $\varepsilon^2 = \varepsilon\sigma = \sigma\varepsilon = \sigma^2 = 0$ and $d(\varepsilon) = \sigma$. Here ε has degree 0 and σ of degree 1. This is the commutative DR algebra of the dual numbers $k + k\varepsilon$, where $d\varepsilon = \sigma$. Then we consider the tensor product DG algebra

$$k[\varepsilon, d\varepsilon]/(\varepsilon^2, \varepsilon d\varepsilon, d\varepsilon\,\varepsilon, d\varepsilon^2) \otimes \Omega A$$

and the map

$$a \longmapsto a + \varepsilon Da$$

which is a homomorphism from A into this DGA. Let the extension to ΩA be denoted

$$\omega \longmapsto \omega + \varepsilon L_D \omega + \sigma \iota_D^* \omega$$

since this is compatible with d we have

$$d(\omega + \varepsilon L_D \omega + \sigma \iota_D^* \omega) = d\omega + \varepsilon L_D d\omega + \sigma \iota_D^* d\omega$$

$$d\omega + \sigma L_D \omega - \sigma d\iota_D^* \omega$$
$$+ \varepsilon d L_D \omega$$

whence

$$[L_D, d] = 0$$
$$L_D = [d, \iota_D^*]$$

April 25, 1990

Elementary facts about mixed complexes. By definition, a mixed complex (C, b, B) is a graded vector space C equipped with operators b, B of degrees 1 and -1 respectively (for the upper indexing C^n) such that $b^2 = B^2 = [b, B] = 0$. Define the associated ordinary and 'cyclic' cohomology

$$H^{\cdot}_{ord} = H^{\cdot}(C, b), \quad H^{\cdot}_{cyc} = H(C[u], b + uB)$$

From the short exact sequence of complexes

$$0 \longrightarrow uC[u] \longrightarrow C[u] \longrightarrow C \longrightarrow 0$$

one obtains the Connes exact sequence

$$\longrightarrow H^n_{cyc} \longrightarrow H^n_{ord} \longrightarrow H^{n-1}_{cyc} \xrightarrow{u} H^{n+1}_{cyc} \longrightarrow$$

From the row filtration $C^{\geq m}[u]$ which exhausts $C[u]$ as $m \downarrow -\infty$ one obtains the spectral sequence

$$E_2 = H_{ver} H_{hor} \Longrightarrow H_{cyc}$$

In the special case $H(\mathrm{Ker}(B)/\mathrm{Im}(B), b) = 0$ this yields

$$H_{cyc} = H(\mathrm{Im}(B), b)$$

and one can obtain the Connes exact sequence directly from the short exact sequence + qis

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(B) \arrow[r] & C \arrow[r, "B"] & \Sigma^{-1}\mathrm{Im}(B) \arrow[r] & 0 \\ & \mathrm{Im}(B) \arrow[u, "qis"'] & & & \end{tikzcd}$$

($\Sigma K = \mathbb{Z}[\sigma] \otimes K$ where σ has upper degree -1)

Thus in the special case, which is what one uses to get started in cyclic theory, mixed complexes are unnecessary.

May 8, 1990

I-adic filtrations.

If I is an ideal in R, then following standard practice in commutative algebra, we introduce the graded algebra

$$R_I = \bigoplus_{m \geq 0} I^m$$

This is the image of the map

$$T_R(I) \longrightarrow T_R(R) = k[x] \otimes R$$

and it coincides with the tensor algebra $T_R(I)$ if I is flat as either a left or right R-module. To begin with, let's consider this case; better the case where I is flat both as a left and as a right R-module.

Given a left module M over R we then have

$$I^m \otimes_R M \xrightarrow{\sim} I^m M$$

for all m, hence

$$R_I \otimes_R M \xrightarrow{\sim} M_I \overset{\text{def}}{=} \bigoplus_{m \geq 0} I^m M$$

similarly for M a right R-module.

If M is an R-bimodule then

$$R_I \otimes_R M \otimes_R R_I \simeq \bigoplus_{m,n} I^m M I^n$$

and [illegible] the commutator quotient space is

$$R_I \otimes_R M \otimes_R R_I / [R_I, \quad] \cong R_I \otimes_R M \otimes_R = \bigoplus_m I^m M \otimes_R$$

I want, instead of $\bigoplus_{m,n} I^m M I^n$, the R_I-bimodule $\bigoplus_m F_I^m M$ which is a quotient of the former. Recall that

$$F_I^m M = \sum_{i+j=m} I^i M I^j$$

May 21, 1990

Here's a puzzle:

Consider constructing L_D and ι_D on $\Omega = \Omega A$ where $D: A \longrightarrow A$ is a derivation. The best method (see p. 394) is the following: Let K be the complex with basis h, k such that $|h| = 0$, $|k| = 1$, and $dh = k$. Form $K \otimes \Omega$ which is naturally a DG-module over Ω. Form the semi-direct product DG algebra

$$\Omega \oplus (K \otimes \Omega) = \Omega \oplus (h\Omega \oplus k\Omega)$$

One has a homomorphism $A \longrightarrow (\Omega \oplus K \otimes \Omega)^0$ given by $a \longmapsto a + hDa$. If $u : \Omega \longrightarrow \Omega \oplus (K \otimes \Omega)$ is the extension to a DGA map, then u has the form $u = 1 + hL_D + k\iota_D$ where L_D, ι_D are derivations of Ω of degree $0, -1$ respectively. One has

$$0 = [d, 1 + hL_D + k\iota_D]$$

$$= \underbrace{[d,h]}_{k} L_D + h[d, L_D] + \underbrace{[d,k]}_{0} \iota_D - k[d, \iota_D]$$

$$\Longrightarrow \quad [d, L_D] = 0 \qquad [d, \iota_D] = L_D$$

[redacted] Let us now weaken [redacted] the restriction of "square zero". Let us consider instead of a semi-direct product the following. Let R be the DG algebra over $\mathbb{C}$ generated by h, k where $|h| = 0$, $|k| = 1$ [redacted], $h^2 = 0$, $dh = k$ [redacted].

Let's take up this line later and for now just state the puzzle.

We can summarize the first approach to L_D, ι_D as consisting in forming the semi-direct product DG algebra + homom. from A into its degree zero subalgebra

$$A \longrightarrow \Omega \oplus (h\Omega \oplus k\Omega) \qquad dh=k$$

$$a \longmapsto a + hDa$$

then the extension to Ω is $1 + hL_D + k\iota_D$.

An older approach consisted of defining L_D, ι_D separately. First L_D is defined by

$$A \longrightarrow \Omega \oplus h\Omega$$

$$a \longmapsto a + hDa$$

extends to $\Omega \longrightarrow \Omega \oplus h\Omega$, $\quad 1 + hL_D$.

Secondly one forms $\Omega \oplus k\Omega$ with $d' = d + kL_D$, and takes $a \longmapsto a$. The extension: $\Omega \longrightarrow \Omega \oplus k\Omega$ has the form $1 + k\iota_D$.

Check:

$$(d+kL_D)(1+k\iota_D) = d + k(L_D - d\iota_D)$$

$$(1+k\iota_D)d = d + k(\iota_D d)$$

The puzzle is to link these two approaches. Can be second be interpreted as a quotient of the first?

Observe $d(h^2) = dh\,h + h\,dh$
$= kh + hk$ has to vanish. Thus R is not commutative. But it seems that the elements $hk = -kh$, k^2, hk^2, .. are non-zero. This should be formally analogous to the DGA $\Omega(\mathbb{C}\oplus\mathbb{C}e)$. In fact we see that $R = \Omega(\mathbb{C}\oplus\mathbb{C}h)$ where $h^2 = 0$.

Now form the DGA

$$R\otimes\Omega = \Omega(\mathbb{C}\oplus\mathbb{C}h)\otimes\Omega(A)$$

and consider the homomorphism

$$A \longrightarrow R\otimes\Omega \qquad a \longmapsto a + hDa$$

This induces a DG algebra map

$$u : \Omega \longrightarrow R\otimes\Omega$$

which we want to understand.

Potentially u will be of the form

$$u(\omega) = \sum_{n\geq 0}\left((dh)^n u'_n(\omega) + h(dh)^n u''_n(\omega)\right)$$

$$u(a) = a + hDa$$

$$u(da) = d(a + hDa)$$

$$= da + (dh)Da + h\,d(Da)$$

$$u(a_0da_1) = (a_0 + hDa_0)(da_1 + dh\,Da_1 + h\,dDa_1)$$

$$= a_0da_1 + h(Da_0\,da_1 + a_0\,dDa_1) + hdh(Da_0\,Da_1) + dh(a_0\,Da_1)$$

May 27, 1990

Let $S = T_A(M)$ where M is a bimodule over A. Then we have a right exact sequence

$$\circledast \quad S\otimes_A \Omega^1 A \otimes_A S \longrightarrow \Omega^1 S \longrightarrow S \otimes_A M \otimes_A S \longrightarrow 0$$

In effect given a bimodule N over S, a lifting $S \xrightarrow{1+D} S \oplus N$ is determined by its restriction to A and M. The former is a lifting $A \longrightarrow A \oplus N$, i.e. derivation on A with values in N, or equivalently an S-bimodule map $S\otimes_A \Omega^1 A \otimes_A S \longrightarrow N$. If this vanishes the restriction to M is equivalent to an A-bimodule map $M \longrightarrow N$. ∴ We have

$$0 \longrightarrow \mathrm{Hom}_{S\otimes S^\circ}(S\otimes_A M \otimes_A S, N) \longrightarrow \mathrm{Der}(S,N) \longrightarrow \mathrm{Hom}_{S\otimes S^\circ}(S\otimes_A \Omega^1 A \otimes_A S, N)$$

for all N proving exactness of $\circledast$.

Next we exact whether the first map in $\circledast$ is injective. This is equivalent to being able to extend any derivation on A with values in an S-bimodule N to a derivation on S after possibly enlarging N. Then if we enlarge N further we can even arrange the derivation on S be inner. Once one has the idea of looking at inner derivations the following occurs naturally:

$$\begin{tikzcd}
0 \arrow[d] & & \\
\mathrm{Tor}_1^{A\otimes A^{\circ}}(S\otimes S, A) \arrow[d] & & \\
S\otimes_A \Omega^1 A \otimes_A S \arrow[r] \arrow[d] & \Omega^1 S \arrow[r] \arrow[d, phantom, "\cap"] & S\otimes_A M \otimes_A S \arrow[r] & 0 \\
S\otimes_A (A\otimes A)\otimes_A S \arrow[r, phantom, "="] \arrow[d] & S\otimes S & & \\
S\otimes_A S \arrow[d] & & & \\
0 & & &
\end{tikzcd}$$

$$\bigoplus_{m,n} \mathrm{Tor}_1^A(M^{(m)}, M^{(n)})$$

So we conclude that we have an exact sequence

$$\boxed{0 \longrightarrow \mathrm{Tor}_1^{A\otimes A^{\circ}}(S\otimes S, A) \longrightarrow S\otimes_A \Omega^1 A \otimes_A S \longrightarrow \Omega^1 S \longrightarrow S\otimes_A M \otimes_A S \longrightarrow 0}$$

This Tor_1 should vanish when M is flat as a left and as a right A-module.

Let's assume this case holds so we have a bimodule exact sequence

$$0 \longrightarrow S\otimes_A \Omega^1 A \otimes_A S \longrightarrow \Omega^1 S \longrightarrow S\otimes_A M \otimes_A S \longrightarrow 0$$

over S. This looks almost as if $\Omega^1 S$ comes from $\Omega^1 A \oplus M$, which is an A-bimodule, via "base change". Inside $\Omega^1 S$ we have the sub bimodule over A $\Omega^1 A$ and the subspace dM.

Consider $A\,dM\,A \subset \Omega^1 S$. From

$$(da)m = d(am) - a\,dm$$
$$m(da) = d(ma) - dm\,a$$

we see it contains $M\Omega^1A + \Omega^1AM$. Moreover modulo this subspace one has $a\,dm \equiv d(am)$, $dm\,a \equiv d(ma)$. Thus we have an exact sequence of A-bimodules

$$\boxed{0 \longrightarrow \Omega^1A\otimes_A M \oplus M\otimes_A\Omega^1A \longrightarrow AdMA \longrightarrow M \longrightarrow 0}$$

which generates Ω^1S in some sense.

Let us now consider $S = \Omega A = T_A(\Omega^1 A)$. Thus M is Ω^1A. One way to understand Ω^1S is to note that it is canonically a DG bimodule over S characterized by the same universal property relative to liftings but for DG algebras. Let us therefore analyze liftings into a semi-direct product

$$S \xrightarrow{1+D} S \oplus N$$

where N is a DG bimodule. By the universal prop. of S these are the same as alg. liftings $A \longrightarrow A \oplus N^0$, which are the same as A-bimodule maps $\Omega^1A \longrightarrow N^0$. So now we are led to the question of describing the DG bimodule over S generated by the A-bimodule Ω^1A. It should contain the S bimodule generated by the A-bimodule Ω^1A, which is $S\otimes_A\Omega^1A\otimes_A S$, but then it must also have more. A typical generating element in $S\otimes_A\Omega^1A\otimes_A S$ can be written $\omega' Da\,\omega''$, but then we must

add elements $d(Da) = D(da)$
to have something closed under d.

It might be useful to take a bigraded differential algebra viewpoint. Thus $\Omega S = \Omega(\Omega A)$ should be the bigraded differential algebra generated by A. Initially one would get a DG algebra in the category of DG algebras, but then one ought to be able introduce the signs so that the differentials anti-commute. So the D becomes δ anti-commuting with d. Picture of $\Omega S = \Omega(\Omega A)$:

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "d"] & \Omega^1 A \\ \Omega^1 A & \bigcirc \end{tikzcd}$$

Here we can expect two copies of $\Omega^1 A \otimes_A \Omega^1 A = \Omega^2 A$ extended by elements $a_0 d\delta a$, which is exactly the sort of picture we have.

May 28, 1990

Let $L(R): \longrightarrow R \longrightarrow \Omega^1 R_\natural \longrightarrow R \longrightarrow$ denote the little periodic complex of R. Given a homomorphism $u: R \longrightarrow R'$ it induces a map of complex $u_*: L(R) \longrightarrow L(R')$. A family of homs. $u_t: R \longrightarrow R'[t]$ induces a family $u_{t*}: L(R) \to L(R')[t]$ of maps of complexes. We are interested in the derivative at time 0:

$$\partial_t (u_{t*})\Big|_{t=0} : L(R) \longrightarrow L(R')$$

When R is free we would like to show it is homotopic to zero.

To calculate $\partial_t (u_{t*})|_{t=0}$ we need only the homomorphism $u = u_t|_{t=0} : R \to R'$ and the derivation $\mathring{u} = \partial_t u_t|_{t=0}$ relative to u. In effect

$$u_{t*}(x\,dy) = u_t x\, d(u_t y)$$

$$\partial_t u_{t*}(x\,dy)\Big|_{t=0} = \mathring{u} x\, d(uy) + ux\, d(\mathring{u} y)$$

Such a pair $(u, \mathring{u})$ is equivalent to a homom. $R \longrightarrow R' \oplus hR'$, $h^2 = 0$. So we have a natural map which we want to lift:

$$\begin{tikzcd}
\mathrm{Hom}_{alg}(R, R'+hR') \arrow[r] \arrow[d, equal] \arrow[rd, dashed] & \mathrm{Hom}^0_{cxs}(L(R), L(R')) \\
\mathrm{Hom}_{algs}(\Omega R, R') & \mathrm{Hom}^{-1}(L(R), L(R')) \arrow[u, "{[d,\ ]}"']
\end{tikzcd}$$

Now one has $\mathrm{Hom}_{algs}(\Omega R, R')$. Thus to find the desired homotopy in a fashion natural wrt R' we must do it in the case $R' = \Omega R$.

It is often [redacted] easier to prove stronger results. Instead of homom. $R \longrightarrow R' \oplus hR'$ let us consider more generally homomorphisms $R \longrightarrow R' \oplus M$, [redacted], where M is a bimodule over R'.

(Digression: There is no point in writing $R' \oplus hM$ since $R' \oplus M$ is not an algebra over $k \oplus kh$ the way $R' \oplus hR'$ is. Is this somehow related to the presence of the anti-derivation d on ΩR? One has

$$\mathrm{Hom}_{alg}(R, R' \oplus hR') = \mathrm{Hom}_{algs}(\Omega R, R').$$

This explains the grading on ΩR, but where does d come from?)

If we consider homomorphisms $R \longrightarrow R' \oplus N$, then we need some sort of analogue of the little periodic complex to assign to $R' \oplus N$. The obvious candidate is the degree 1 part in N in $L(R' \oplus N)$; denote this $L'(R';N)$ where the prime in L' now [redacted] stands for derivative. Then we have

$$\begin{tikzcd}
\mathrm{Hom}_{alg}(R, R' \oplus N) \arrow[rr] \arrow[d, equal] \arrow[rrd, dashed] & & \mathrm{Hom}^0(L(R), L'(R';N)) \\
\mathrm{Hom}((R, \Omega^1 R), (R', N)) & & \mathrm{Hom}^{-1}(L(R), L'(R';N)) \arrow[u]
\end{tikzcd}$$

category of algebras + bimodules = graded algebras length ≤ 1.

So it suffices to consider $R' \oplus N = R \oplus \Omega^1 R = \Omega R / (\Omega^{>0} R)^2$.

There is also a problem of relating $L'(R';R')$ to $L(R')$. Hopefully there is a canonical map

$$L'(R';R') \longrightarrow L(R')$$

so that we have a commutative diagram

$$\begin{tikzcd}
L(R) \arrow[rr, "\text{assoc to}", "R\to R'\oplus hR'"'] \arrow[dr, "\text{assoc. to } R\to R'\oplus R'"'] & & L(R') \\
& L'(R';R') \arrow[ur, "\text{canon.}"'] &
\end{tikzcd}$$

for then if we show the map $\searrow$ is homotopic to zero, we see the map $\longrightarrow$ is homotopic to zero.

Let us consider then the little complex for a semi direct product $A\oplus M$. The problem is to understand $\Omega^1(A\oplus M)_\natural$.

First method.

$$\begin{array}{lccc}
\Omega^1(A\oplus M) & \dashrightarrow (A\oplus M)^{\otimes 2} & \longrightarrow & A\oplus M \\
 & \parallel & & \\
\Omega^1 A & A\otimes A & \longrightarrow & A \\
 & A\otimes M \oplus M\otimes A & \longrightarrow & M \\
M\otimes M & M\otimes M & \longrightarrow & 0
\end{array}$$

It's clear we have an exact sequence

$$\boxed{0 \longrightarrow \left(\Omega^1 A\otimes_A M\right)\oplus\left(M\otimes_A \Omega^1 A\right) \longrightarrow \Omega^1(A\oplus M)_{(1)} \longrightarrow M \longrightarrow 0}$$

where we use:

$$0 \longrightarrow \Omega^1 A \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow \Omega^1 A\otimes_A M \longrightarrow A\otimes M \longrightarrow M \longrightarrow 0$$

We have seen this exact sequence above in

Connection with $\Omega^1(T_A(M))$.

Next we want to compute the commutator quotient space. First take commutators wrt A. In degree 1

$$H_1(A,M) \longrightarrow \left(\Omega^1 A \otimes_A M \otimes_A\right) \oplus \left(M \otimes_A \Omega^1 A \otimes_A\right) \longrightarrow \Omega^1(A\oplus M)_{(1)} \otimes_A \longrightarrow M \otimes_A \to 0$$

In degree 2 we get $(M\otimes M)\otimes_A$. Next divide by commutators wrt M. This should gives an exact sequence in degree one

$$\Omega^1 A \otimes_A M \otimes_A \longrightarrow \Omega^1(A\oplus M)_{\natural (1)} \longrightarrow M\otimes_A \longrightarrow 0$$

since we identify dam and mda. As for degree 2 we have

$$dm_1\, m_2 = (m_1\otimes 1 - 1\otimes m_1)\, m_2 = m_1 \otimes m_2$$
$$m_1\, dm_2 = m_1 (m_2 \otimes 1 - 1 \otimes m_2) = - m_1 \otimes m_2$$
$$\therefore \quad dm_1\, m_2 + m_1\, dm_2 = d(m_1 m_2) = 0.$$

in $M\otimes M = \Omega^1(A\oplus M)(2)$. When we pass to commutator quotient space we have also $m_1\, dm_2 = dm_2\, m_1$. Thus $dm_1\, m_2 = - m_1\, dm_2 = - dm_2\, m_1$ which means we should have

$$\Omega^1(A\oplus M)_{\natural(2)} = (M\otimes_A)^2_\lambda$$

May 29, 1990

Let's review our earlier discussion of homotopy for traces, especially two-parameter homotopy. The point of departure is the idea that a family of traces on A of the form τu_t, where $u_t : A \rightarrow R[t]$ is a 1-parameter family of homomorphisms, should serve as model for a homotopy of traces. For reasons which aren't yet clearly understood, it is enough to consider first order families

$$u_t = u_0 + t u_1 \ : \ A \longrightarrow R[t]/(t^2)$$

and moreover it seems to be a good idea to consider more generally families

$$u_t = u_0 + t u_1 \ : \ A \longrightarrow R^0 \oplus R^1 \oplus 0 \oplus \cdots$$

To get to the point when looking at 2-parameter families we consider

$$u_{st} : A \longrightarrow \begin{array}{ccc} R^{00} & \oplus & R^{10} \\ & \oplus & \\ R^{01} & \oplus & R^{11} \end{array}$$

$$u_{st} = u_{00} + s u_{10} + t u_{01} + st\, u_{11}$$

and to simplify we suppose $R^{00} = A$, $u_{00} = id$. Then we write

$$u_{st} = 1 + sX + tY + st\,W$$

The condition that u_{st} be a homomorphism is that

$$\delta X = \delta Y = 0$$

$$\delta W + X \cup Y + Y \cup X = 0$$

Here X, Y, W are 1-cochains on A with values in the A-bimodules R^{10}, R^{01}, R^{11} respectively.

In the case of [redacted] one-parameter families $u_t = u_0 + tu : A \longrightarrow R^0 \oplus R^1$ the universal choice for $R^0 \oplus R^1$ is $A \oplus \Omega^1 A$. [redacted] If we consider only $R = R^0 = R^1$, then the universal choice for R is ΩA. Thus

$$\mathrm{Hom}_{alg}(A, R \otimes k[t]/t^2) = \mathrm{Hom}_{alg}(\Omega A, R)$$

$$\mathrm{Hom}_{alg}(A, R \otimes (k[s]/s^2) \otimes (k[t]/t^2))$$

$$= \mathrm{Hom}_{alg}(\Omega A, R \otimes (k[s]/s^2))$$

$$= \mathrm{Hom}_{alg}(\Omega(\Omega A), R).$$

May 30, 1990

Let's continue the study of homotopy for the little complex. We want to understand its deformations to first order. This means looking at homomorphisms $A \longrightarrow B + hB$, $h^2 = 0$, and taking the map

$$L(A) \longrightarrow L(B+hB) \xrightarrow{\text{projection}} \begin{array}{l} h\text{-degree 1 part} \\ \text{of } L(B+hB) \end{array}$$

Fixing A the universal B is ΩA as we have seen. For reasons I don't understand, except for the principle that it is nicer to prove stronger results, it is better to consider homomorphism $A \longrightarrow B \oplus M$, where M is a B-bimodule. In this case $A \oplus \Omega^1 A$ is the universal algebra.

So let's consider

$$L(A) \xrightarrow{(1+\delta)_*} L(A \oplus \Omega^1 A) \xrightarrow{\text{projection}} \text{degree 1 part}$$

Here we use $1+\delta : A \longrightarrow A \oplus \Omega^1 A$ for the canonical homomorphism to distinguish from d used in $L(A)$. Draw this map:

$$\begin{tikzcd}
{} \arrow[r] & A \arrow[r, "d"] \arrow[d, "1+\delta"] & \Omega^1 A_\natural \arrow[r, "c"] \arrow[d, "(1+\delta)_*"] & A \arrow[r] \arrow[d, "1+\delta"] & {} \\
{} \arrow[r] & A \oplus \Omega^1 A \arrow[r, "d"] \arrow[d] & \Omega^1(A \oplus \Omega^1 A)_\natural \arrow[r, "c"] \arrow[d] & A \oplus \Omega^1 A \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \Omega^1 A \arrow[r, "d"] & \Gamma \arrow[r, "c"] & \Omega^1 A \arrow[r] & {}
\end{tikzcd}$$

Let's recall what we think is true about Γ the degree 1 part of $\Omega^1(A\oplus\Omega^1A)_\natural$ relative to the grading inherited from the natural $\mathbb{N}$-grading on $A\oplus\Omega^1A$. The degree one part of $\Omega^1(A\oplus\Omega^1A)$ is

$$\Omega^1(A\oplus\Omega^1A) \xleftarrow{\sim} A\otimes\bar{A}\otimes A \oplus \Omega^2A$$

$$a_0(d\delta a_1)a_2 \longleftarrow\!\shortmid (a_0,a_1,a_2)$$

$$a_0da_1\delta a_2 \longleftarrow\!\shortmid a_0da_1da_2$$

Basic identity.

$$\delta a_1 da_2 + da_1\delta a_2 + d\delta a_1\, a_2 \text{ [illegible] } d\delta(a_1a_2) + a_1d\delta a_2 = 0$$

Actually it looks like we ought to use instead the map

$$a_0\delta a_1da_2 \longleftarrow\!\shortmid a_0da_1da_2$$

because this should fit better with c. We then have an isomorphism

$$A\otimes\bar{A} \oplus \Omega^2A_\natural \xrightarrow{\sim} \Omega^1(A+\Omega^1A)_{(1)}\otimes_A$$

$$(a_0,a_1) \longmapsto a_0\delta da_1$$

[illegible] $(a_0,a_1,a_2) \longmapsto a_0\delta a_1da_2$

We have

$$d(a_0\delta a_1) = a_0d\delta a_1 + da_0\delta a_1$$

$$= a_0d\delta a_1 + \delta a_1da_0$$

$$\delta(a_0da_1) = a_0d\delta a_1 + \delta a_0da_1$$

$$c(a_0d\delta a_1) = [a_0,\delta a_1] \qquad c(a_0\delta a_1da_2) = [a_0\delta a_1, a_2]$$

We propose now to calculate linear functionals on

$$\begin{tikzcd}
\arrow[r] & A \arrow[r, "d"] \arrow[d, "\delta"] & \Omega^1 A_\natural \arrow[r, "c"] \arrow[d, "\delta"] & A \arrow[r, "d"] \arrow[d, "\delta"] & {} \\
\arrow[r] & \Omega^1 A \arrow[r, "d"] & \Gamma \arrow[r, "C"] & \Omega^1 A \arrow[r, "d"] & {}
\end{tikzcd}$$

Let's write $\quad \tau \in A^* \qquad T \in (\Omega^1 A_\natural)^*$

$\qquad \phi \in (\Omega^1 A)^* \qquad \psi \in (\Gamma)^*$

Note that ϕ is equivalent to the 1-cochain $\phi_1(a_0,a_1) = \phi(a_0\delta a_1)$, T to $T_1(a_0,a_1) = T(a_0 da_1)$ which is 1-cocycle, and ψ is equivalent to the pair $\binom{\psi_1}{\psi_2}$ given by

$$\psi_1(a_0,a_1) = \psi(a_0\, d\delta a_1)$$

$$\psi_2(a_0,a_1,a_2) = \psi(a_0\,\delta a_1\, da_2)$$

which satisfies $\boxed{b\psi_2 = 0}$ - this means trace as A-bimodule - and $\boxed{b\psi_1 + (1-K)\psi_2 = 0}$:

$$\psi(\delta a_2\, a_0\, da_1) \overset{?}{=} \psi(a_0\, da_1\, \delta a_2)$$

$$\begin{array}{ll}
\quad\| & \quad\| \\
+\psi(\delta(a_2a_0)\, da_1) & -\psi(a_0\,\delta a_1\, da_2) \\
-\psi(a_2\,\delta a_0\, da_1) & -\psi(a_0\, d\delta a_1\, a_2) \\
\quad\| & +\psi(a_0\, d\delta(a_1a_2)) \\
-\psi_2(a_2,a_0,a_1) & -\psi(a_0 a_1\, d\delta a_2) \\
+\psi_2(1,a_2a_0,a_1) & \quad\| \\
\quad\| & (-\psi_2 - b\psi_1)(a_0,a_1,a_2) \\
-(K\psi_2)(a_0,a_1,a_2) &
\end{array}$$

Let's compute the induced maps.

$$(\psi d)_1(a_0,a_1) = \psi(d(a_0\delta a_1))$$

$$= \psi(a_0 d\delta a_1) + \underbrace{\psi(da_0\,\delta a_1)}_{\psi(\delta a_1\, da_0)}$$

$$= \psi_1(a_0,a_1) + (s\psi_2)(a_1,a_0)$$

$$\boxed{(\psi d)_1 = \psi_1 - K s\psi_2}$$

$$(\phi c)_1(a_0,a_1) = \phi(c(a_0\, d\delta a_1)) = \phi([a_0,\delta a_1])$$

$$= \phi(a_0\delta a_1) - \phi(\delta a_1\, a_0)$$

$$= \phi(a_0\delta a_1) - \phi(\delta(a_1a_0)) + \phi(a_1\delta a_0)$$

$$= \phi_1(a_0,a_1) - \phi_1(1,a_1a_0) + \phi_1(a_1,a_0)$$

$$\boxed{(\phi c)_1 = (1-K)\phi_1}$$

$$(\phi c)_2(a_0,a_1,a_2) = \phi(c(a_0\delta a_1\, da_2))$$

$$= \phi([a_0\delta a_1, a_2]) = \phi(a_0\delta a_1 a_2 - a_2a_0\delta a_1)$$

$$= \phi(a_0\delta(a_1a_2)) - \phi(a_0a_1\delta a_2) - \phi(a_2a_0\delta a_1)$$

$$\boxed{(\phi c)_2 = -b\phi_1}$$

Check $(\phi c d)_1 = (\phi c)_1 - Ks(\phi c)_2$

$$= (1-K)\phi_1 - Ks(-b\phi_1)$$

$$= bs\phi_1 + sb\phi_1 + Ksb\phi_1 = (bB+Bb)\phi_1 = 0$$

$$(\psi dc)_1 = (1-\kappa)(\psi d)_1$$
$$= (1-\kappa)(\psi_1 - \kappa s \psi_2)$$
$$= (1-\kappa)\psi_1 - \kappa s (1-\kappa)\psi_2$$
$$= (1-\kappa)\psi_1 + \kappa s b \psi_1$$
$$= bs\psi_1 + sb\psi_1 + \kappa s b\psi_1 = (bB + Bb)\psi_1 = 0$$

$$(\psi dc)_2 = -b(\psi d)_1 = -b(\psi_1 - \kappa s \psi_2)$$
$$= (1-\kappa)\psi_2 + \kappa \underbrace{bs\psi_2}_{(1-\kappa)\psi_2}$$
$$= (1-\kappa+\kappa-\kappa^2)\psi_2 = 0$$

$\kappa^2 = 1 + \kappa^{-1} s b$

Next $(\psi\delta)_1(a_0,a_1) = \psi(\delta(a_0 da_1))$
$$= \psi(\delta a_0 da_1) + \psi(a_0 d\delta a_1)$$

$$\boxed{(\psi\delta)_1 = \psi_1 + s\psi_2}$$

Check: $(\phi c \delta)_1 = (\phi c)_1 + s(\phi c)_2$
$$= (1-\kappa)\phi_1 + s(-b\phi_1)$$
$$= (1-\kappa-sb)\phi_1 = bs\phi_1$$

which is OKAY because

$$(\phi\delta)_0(a_0) = \phi(\delta a_0) = \phi_1(1, a_0)$$
$$(\tau c)_1(a_0,a_1) = \tau c(a_0 da_1) = \tau([a_0,a_1]) = (b\tau_0)(a_0,a_1)$$

$$\therefore \boxed{(\phi\delta)_0 = s\phi_1} \qquad \boxed{(\tau c)_1 = b\tau_0}$$

$\Rightarrow (\phi\delta c)_1$
$= b(\phi\delta)_0$
$= bs\phi_1$

also $\boxed{(Td)_0 = sT_1}$ similarly

Check $(\psi d\delta)_0 = s(\psi d)_1 = s(\psi_1 - Ks\psi_2) = s\psi_1$

$(\psi\delta d)_0 = s(\psi\delta)_1 = s(\psi_1 + s\psi_2) = s\psi_1$

Final picture for the induced map on cochains:

$$\begin{tikzcd}
{} & \tau_0 \arrow[l, mapsto] & T_1 \arrow[l, mapsto, "s"'] & \tau_0 \arrow[l, mapsto, "b"'] & {} \arrow[l, mapsto, "s"'] \\
{} & \phi_1 \arrow[u, mapsto, "s"] \arrow[l, mapsto] & \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \arrow[u, mapsto, "(1 \ \ s)"'] \arrow[l, mapsto, "(1 \ \ -Ks)"] & \phi_1 \arrow[u, mapsto, "s"'] \arrow[l, mapsto, "\begin{pmatrix}1-K\\ -b\end{pmatrix}"] & {} \arrow[l, mapsto]
\end{tikzcd}$$

The next stage is to look at invariant cochains. Note $P(1-K) = P(bs+sb) = bB + \frac{1}{2}Bb = \frac{1}{2}bB$ on 1-cochains. Thus we get

$$\begin{tikzcd}
{} & \tau_0 \arrow[l] & T_1 \arrow[l, "B"'] & \tau_0 \arrow[l, "b"'] & {} \arrow[l] \\
{} & \phi_1 \arrow[u, "B"] \arrow[l] & \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \arrow[u, "(1 \ \ \frac{1}{2}B)"'] \arrow[l, "(1 \ \ -\frac{1}{2}B)"] & \phi_1 \arrow[u, "B"'] \arrow[l, "\begin{pmatrix}\frac{1}{2}bB\\ -b\end{pmatrix}"] & {} \arrow[l]
\end{tikzcd}$$

$b\psi_1 = 0 \ , \ b\psi_2 = 0$

May 31, 1990

What is the homology of the degree 1 part of $L(A \oplus \Omega^1 A)$? Look at invariant cochains:

$$\begin{tikzcd} \cdots & \phi_1 \arrow[l] & \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \arrow[l, "d^t"', "(1 \ \ -\frac{1}{2}B)" below] & \phi_1 \arrow[l, "c^t"', "\begin{pmatrix} \frac{1}{2}bB \\ -b \end{pmatrix}" below] & \cdots \arrow[l] \end{tikzcd}$$

where $b\psi_1 + (1-K)\psi_2 = 0$, $b\psi_2 = 0$ become

$$\boxed{b\psi_1 = b\psi_2 = 0}$$

for invariant cochains. We have

$$\mathrm{Ker}(c^t) = \{\phi_1 \mid b\phi_1 = 0\} = \mathrm{Im}(d^t)$$

$$\mathrm{Ker}(d^t) = \left\{ \begin{pmatrix} \frac{1}{2}B\psi_2 \\ \psi_2 \end{pmatrix} \middle| \ b\psi_2 = 0 \right\}$$

$$\mathrm{Im}(c^t) = \left\{ \begin{pmatrix} \frac{1}{2}Bb\phi_1 \\ b\phi_1 \end{pmatrix} \middle| \ \phi_1 \text{ inv} \right\}$$

Thus $\boxed{\begin{array}{ll} \text{even cohomology} = 0 \\ \text{odd cohomology} = HH^2(A) \end{array}}$

On the ~~anti~~-invariant cochains we have $\tilde{K} = -1$ so the arrows are

$$\begin{tikzcd} \cdots & \phi_1 \arrow[l] & \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \arrow[l, "d^t"', "(1 \quad s)" below] & \phi_1 \arrow[l, "c^t"', "\begin{pmatrix} 2 \\ -b \end{pmatrix}" below] & \cdots \arrow[l] \end{tikzcd}$$

and $b\psi_1 + 2\psi_2 = 0$. Thus c^t is an isom.

Check: $$(1 \quad s)\begin{pmatrix} \psi_1 \\ -\frac{1}{2}b\psi_1 \end{pmatrix} = \underbrace{\left(1 - \frac{1}{2}sb\right)}_{1-\frac{1}{2}(1-K) = 0}\psi_1$$

Note $s\psi_1$ is invariant (degree 0). $\therefore s\psi_1 = 0$

Thus there is no homology on the anti-invariant part.

What's the map on odd cohomology induced by δ [redacted] $: L(A) \longrightarrow L(A\oplus\Omega^1A)_{(1)}$?

It should be [redacted] the map $HH^2(A) \xrightarrow{B} HC^1(A)$.

Check: Using invariant cochains something in $\operatorname{Ker}(d^t)$ is of the form $\begin{pmatrix}\frac{1}{2}B\psi_2\\ \psi_2\end{pmatrix}$ and δ^t carries it to $\begin{pmatrix}1 & \frac{1}{2}B\end{pmatrix}\begin{pmatrix}\frac{1}{2}B\psi_2\\ \psi_2\end{pmatrix} = B\psi_2$.

Finally we want to prove that when $HH^2(A)=0$ the map $L(A) \longrightarrow L(A\oplus\Omega^1A)_{(1)}$ is null homotopic. First proceed on the cochain level.

[redacted]

Recall our previous analysis. One has

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & A^* \arrow[r] \arrow[d, equal] & (\Omega^1A_\natural)^* \arrow[r] \arrow[d, "i"] & 0 \arrow[r] & {} \\
0 \arrow[r] & A^* \arrow[r, "b"] \arrow[d] & (\Omega^1A)^* \arrow[r, "b"] \arrow[d] & (\Omega^2A)^* \arrow[r] \arrow[d, equal] & {} \\
 & 0 \arrow[r] & X \arrow[r, hook] \arrow[d] & (\Omega^2A)^* \arrow[r] \arrow[d] & {} \\
 & & 0 & 0 &
\end{tikzcd}$$

an exact sequence of complexes ↑. [redacted] Assuming $HH^i(A)$ for $i \geq 2$ the quotient complex is acyclic, so we can choose a contracting homotopy and also split off the quotient complex so as to obtain a SDR situation.

Concretely, [redacted] let

us choose a complement for

$(\Omega^1 A_\natural)^* = \mathrm{Ker}(b) \subset (\Omega^1 A)^*$

This then gives a projection operator $j : (\Omega^1 A)^* \longrightarrow (\Omega^1 A_\natural)^*$ and a map h which together split the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & (\Omega^1 A_\natural)^* \arrow[r, "i"'] & (\Omega^1 A)^* \arrow[r, "b"] \arrow[l, dashed, bend right, "j"'] & \mathrm{Im}\, b \arrow[r] \arrow[l, dashed, bend right, "h"'] & 0
\end{tikzcd}$$

If we assume $HH^2 A = 0$, then $\mathrm{Im}\, b = \mathrm{Ker}\, b \subset (\Omega^2 A)^*$.
I claim we then have a null homotopy:

$$\begin{tikzcd}
{} & T_0 \arrow[l] & T_1 \arrow[l, "s"'] & \dot{T}_0 \arrow[l, "b"'] & {} \arrow[l] \\
{} & \phi_1 \arrow[l] \arrow[u, "s"] \arrow[ur, dashed, "ij"'] & \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \arrow[l, "(1\ \ -Ks)"] \arrow[u, "(1\ \ s)"'] \arrow[ur, dashed, "(0\ \ -sh)"'] & \phi_1 \arrow[l, "\begin{pmatrix}1-K\\ -b\end{pmatrix}"] \arrow[u, "s"'] & {} \arrow[l]
\end{tikzcd}$$

Check:

$$ij\,(1\ \ -Ks)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} + b(0\ \ -sh)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} =$$

$$\underbrace{ij\,(\psi_1 - Ks\psi_2)}_{\substack{\text{already a}\\ \text{cocycle as } c^t d^t = 0}} - \underbrace{bs}_{bB} h\psi_2 = \psi_1 - Ks\psi_2 + \underbrace{B\,\overbrace{bh}^{1}\psi_2}_{(s+Ks)\,\psi_2}$$

$$= \psi_1 + s\psi_2 \qquad ✓$$

$$(0\ \ -sh)\begin{pmatrix}1-K\\ -b\end{pmatrix}\phi_1 + sij\phi_1 = s(hb+ij)\phi_1 = s\phi_1 \qquad ✓$$

Thus what we need for a contracting homotopy of $L(A) \to L(A \oplus \Omega^1 A)_{(1)}$ is a splitting:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 A \arrow[r, "b"] & \Omega^1 A \arrow[r, "\pi"] \arrow[l, bend right, dashed, "h"'] & \Omega^1 A_\natural \arrow[r] \arrow[l, bend right, dashed, "j"'] & 0
\end{tikzcd}$$

[redacted]

exactness here iff $HH_2 = 0$.

Need
$hb = 1$
$\pi j = 1$
$j\pi + bh = 1$.

and apparently one doesn't need $hj = 0$.

(It follows $b(hj) = (1 - j\pi)j = j - j = 0$ & b is injective)

June 1, 1990

Deformation theory of $L(R)$: We study

$$L(R) \xrightarrow{(1+\delta)_*} L(R\oplus\Omega^1 R) \xrightarrow{proj} L(R\oplus\Omega^1 R)_{(1)}$$

Picture

$$\circledast \qquad \begin{tikzcd}
{} \arrow[r] & R \arrow[r, "d"] \arrow[d, "\delta"] & \Omega^1 R_\natural \arrow[r, "c"] \arrow[d, "\delta"] & R \arrow[r] \arrow[d, "\delta"] & {} \\
{} \arrow[r] & \Omega^1 R \arrow[r, "d"] & \Gamma \arrow[r, "c"] & \Omega^1 R \arrow[r] & {}
\end{tikzcd}$$

Description of Γ:

$$\Omega^1(R\oplus\Omega^1 R)_{(1)} = \underset{a_0\, d\delta a_1\, a_2}{R\otimes\bar R\otimes R} \oplus \underset{a_0\,\delta a_1\, da_2}{\Omega^2 R}$$

$$\Gamma = \Big(\underset{\substack{\rotatebox{90}{$\in$} \\ a_0\, d\delta a_1}}{R\otimes\bar R} \oplus \underset{\substack{\rotatebox{90}{$\in$} \\ a_0\,\delta a_1\, da_2}}{\Omega^2 R_\natural}\Big) \Big/ \text{relations}$$

where the relations are

$$a_0\,\delta a_1\, da_2 + \delta a_2\, a_0\, da_1 + a_0\, d\delta a_1\, a_2 - a_0\, d\delta(a_1 a_2) + a_0 a_1\, d\delta a_2 = 0$$

A linear fnl ψ on Γ is a pair of cochains

$$\psi_1(a_0,a_1) = \psi(a_0\, d\delta a_1)$$
$$\psi_2(a_0,a_1,a_2) = \psi(a_0\,\delta a_1\, da_2)$$

such that $\quad b\psi_2 = 0 \quad , \quad b\psi_1 + (1-\kappa)\psi_2 = 0$

Contracting homotopy for $\circledast$ when $HH_2(R) = 0$

Pick a splitting

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 R_\natural \arrow[r, "b"'] & \Omega^1 R \arrow[r, "\pi"'] \arrow[l, bend right, "h"'] & \Omega^1 R_\natural \arrow[r] \arrow[l, bend right, "j"'] & 0
\end{tikzcd}$$

Then we have the homotopy

$$\begin{tikzcd}
 & \Omega^1 R_\natural \arrow[r, "c"] \arrow[dl, "\iota_\delta j"'] \arrow[d, "\delta^\natural"] & R \arrow[r, "d"] \arrow[dl, "H"] \arrow[d, "\delta"] & \Omega^1 R_\natural \arrow[dl, "j"] \\
\Omega^1 R \arrow[r, "d"'] & \Gamma \arrow[r, "c"'] & \Omega^1 R &
\end{tikzcd}$$

where H is the composition

$$R \xrightarrow{d} \Omega^1 R \xrightarrow{-h} \Omega^2 R_\natural \xrightarrow{(\iota_\delta)} \Gamma$$

$$a_0 da_1 da_2 \longmapsto a_0 \delta a_1 da_2$$

Proof: $a_0 da_1 \overset{c}{\longmapsto} [a_0, a_1] \overset{d}{\longmapsto} [da_0, a_1] + [a_0, da_1]$

$$= b\{-da_0 da_1 + da_1 da_0\}$$

$$\overset{-h}{\longmapsto} da_0 da_1 - da_1 da_0 \longmapsto \delta a_0 da_1 - \underbrace{\delta a_1 da_0}_{da_0 \delta a_1} \in \Gamma$$

in Γ

$$d\overset{\iota_\delta}{j}(a_0 da_1) = d(a_0 \delta a_1)$$

$$= da_0 \delta a_1 + a_0 d\delta a_1$$

$$\therefore \left(d\overset{\iota_\delta}{j} + Hc\right)(a_0 \delta a_1) = \delta a_0 da_1 + a_0 d\delta a_1 = \delta(a_0 da_1)$$

[redacted] Note $\Omega^2 R_\natural \longrightarrow \Gamma \xrightarrow{c} \Omega^1 R$

$$a_0 da_1 da_2 \longmapsto a_0 \delta a_1 da_2 \longmapsto [a_0 \delta a_1, a_2]$$

is $-b$: [redacted] $\Omega^2 R_\natural \longrightarrow \Omega^1 R$ followed by the substitution $da \mapsto \delta a$. Thus cHa is

[redacted] $(-b)(-h)\,da = da - j\pi' da$

followed by this substitution (which is denoted ι_δ above.) Thus

$$cH + \overset{(\iota_\delta)}{\,} jd = \delta$$

□

Next we suppose $R = T(V)$ where we have the canonical splitting

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2R_\natural \arrow[r, shift right, "b"'] \arrow[d, equal] & \Omega^1R \arrow[l, shift right, "h"'] \arrow[r, shift right, "\pi"'] \arrow[d, equal] & \Omega^1R_\natural \arrow[l, shift right, "\gamma"'] \arrow[r] \arrow[d, equal] & 0 \\
& R\otimes V\otimes R\otimes V & R\otimes V\otimes R & R\otimes V & \\
& & adv & adv \arrow[l, mapsto, "\gamma"'] & \\
& -a_0dv\,da_1 & a_0dva_1 \arrow[l, mapsto, "h"'] \arrow[r, mapsto, "\pi"] & a_1a_0dv & \\
& a_0da_1da_2 \arrow[r, mapsto, "b"] & -[a_0da_1, a_2] & &
\end{tikzcd}$$

Check: $hb(a_1dv_1a_2dv_2) = -h[a_1dv_1a_2, v_2]$

$$= a_1dv_1\,d(a_2v_2) - \underbrace{v_2a_1dv_1da_2}_{} = a_1dv_1da_2dv_2$$

See p 368 for a more general situation

Then our homotopy becomes

$$\begin{tikzcd}
& v_1\cdots v_n dv_0 \arrow[rr, mapsto, "c"] \arrow[dl, mapsto] & & v_1\cdots v_nv_0 - v_0v_1\cdots v_n \arrow[ddl, mapsto, "(-)"'] \\
v_1\cdots v_n\delta v_0 \arrow[r, mapsto] & \boxed{\begin{array}{l} v_1\cdots v_n d\delta v_0 \\ +d(v_1\cdots v_n)\delta v_0 \end{array}} & & \\
& & \boxed{\begin{array}{l} -\sum\limits_{0\le i<j\le n} v_0\cdots\delta v_i\cdots dv_j\cdots v_n \\ +\sum\limits_{1\le i<j\le n+1} v_1\cdots\delta v_i\cdots dv_j\cdots v_{n+1} \end{array}} \arrow[d, equal] & v_{n+1} = v_0 \\
& & -\delta v_0\,d(v_1\cdots v_n) + \delta(v_1\cdots v_n)\,dv_0 &
\end{tikzcd}$$

Note it checks as the sum $\bigcirc + \bigcirc$

$$= v_1\cdots v_n\delta dv_0 + \delta(v_1\cdots v_n)dv_0 = \delta\{v_1\cdots v_n dv_0\}$$

I have left out the formula for H which is derived as follows:

$$v_1\cdots v_n \xmapsto{d} \sum_i^n v_1\cdots v_{i-1}\,dv_i\,v_{i+1}\cdots v_n$$

$$\xmapsto{-h} \sum_1^n v_1\cdots v_{i-1}\,dv_i\,d(v_{i+1}\cdots v_n)$$

$$\xmapsto{\iota\delta} \sum_1^n v_1\cdots v_{i-1}\,\delta v_i\,d(v_{i+1}\cdots v_n)$$

$$\boxed{H(v_1\cdots v_n) = \sum_{1\le i<j\le n} v_1\cdots\delta v_i\cdots dv_j\cdots v_n}$$

Here's how I would like to apply this formula. Suppose we have a homomorphism and derivation $u+\dot{u} : R \longrightarrow R'\oplus R'$ and ideals $I\subset R$, $I'\subset R'$ such that $uI\subset I'$ and $\dot{u}I\subset I'$. First just consider $u,\dot{u}$. Let's treat u as an inclusion: $u=1$ and let $D=\dot{u} : R\longrightarrow R'$ be a derivation, in order to simplify the notation. Then we have

$$\begin{tikzcd}
R \arrow[r, "d"] \arrow[d, "D"] & \Omega^1 R \arrow[r, "\frown"] \arrow[d, "D"] \arrow[dl, "\tilde{D}"'] & R \arrow[d, "D"] \arrow[dl, "H"'] \\
R' \arrow[r, "d"] & \Omega^1 R'_\natural \arrow[r, "d"] & R'
\end{tikzcd}$$

$$H(v_1\cdots v_n) = \sum_{1\le i<j\le n} v_1\cdots Dv_i\cdots dv_j\cdots v_n$$

$$= \sum_1^n v_1\cdots v_{i-1}, Dv_i\,d(v_{i+1}\cdots v_n)$$

$$= \sum_1^n D(v_1\cdots v_{j-1})\,dv_j\,v_{j+1}\cdots v_n$$

The point of interest is whether the homotopy $(\tilde{D}, H)$ is compatible with the adic filtrations. The first hope would be that if $DI \subset I'$ (which means $D = 0$ from $R/I \to R'/I'$), then

$$\tilde{D}\left(I^m \Omega^1 R\right)_\natural \subset \left(I'\right)^{m+1}$$

$$H\left(I^m\right) \subset \left(I'^m \Omega^1 R'\right)_\natural$$

so that D would be homotopic to zero on the complex

$$\begin{tikzcd}
{} \arrow[r] & I^{m+1} \arrow[r] \arrow[d] & \left(I^m \Omega^1 R\right)_\natural \arrow[r] \arrow[d] \arrow[dl, dashed] & I^m \arrow[r] \arrow[d] \arrow[dl, dashed] & {} \\
{} \arrow[r] & (I')^{m+1} \arrow[r] & \left(I'^m \Omega^1 R'\right)_\natural \arrow[r] & I'^m \arrow[r] & {}
\end{tikzcd}$$

$\tilde{D}$ is OK because $\tilde{D}\,\Omega^1 R \subset I'$. H is the composition

$$\begin{tikzcd}
R \arrow[r, "d"] & \Omega^1 R \arrow[r, "-h"] & \Omega^2 R_\natural \arrow[r, "\iota_D"] & \Omega^1 R'_\natural \\
\cup & & & \\
I^m & F_I^{m-1}\Omega^1 R & & \\
 & \cup & & \\
 & I^{i-1} dv (I^j) \arrow[r, mapsto, "-h"] & \subset I^{i-1} dv\, d(I^j) \arrow[r, mapsto] & \underbrace{I^{i-1} D v}_{\in I'^i}\, d(I^j)
\end{tikzcd}$$

So one has a problem because the end lies in $\left(I'^i F_{I'}^{j-1}(\Omega^1 R')\right)_\natural \subset (I')^{m-1}\Omega^1 R_\natural$.

June 5, 1990

Summary of things about $F_I^n M$ which might not end up in the paper.

Good case: Assume $I^p M \cap M I^q = I^p M I^q$ and $I^p \otimes_R M \otimes_R I^q \xrightarrow{\sim} I^p M I^q$. Then

$$I^n M \otimes_R \xrightarrow{\sim} F^n M / [I, F^{n-1} M] \qquad n \geq 1$$

[redacted]

$$F^n / F^{n+1} = \bigoplus_{j=0}^{n} \left(I^{n-j} / I^{n-j+1} \right) \otimes_{R/I} \left(M / F^1 M \right) \otimes_{R/I} \left[I^j / I^{j+1} \right)$$

i.e.

$$gr(M) = \underline{gr(R) \otimes_{R/I}} \left(M / IM + MI \right) \otimes_{R/I} gr(R).$$

Recall two chains in a modular lattice generate a distributive lattice, Zassenhaus lemma or $gr_F\, gr_G = gr_G\, gr_F$ gives following picture:

Given filtrations F^p, G^q decreasing say with $p, q \geq 0$, then the lattice generated by them is noetherian and distributive. Everything in the lattice is a union of irreducibles, the irreducibles being the $F^p \cap G^q$ such that

$$\frac{F^p \cap G^q}{F^{p+1} \cap G^q + F^p \cap G^{q+1}} \neq 0.$$

If $A = R/I$ [redacted] then

$$\textcircled{*} \qquad 0 \to I/I^2 \to A \otimes_R \Omega^1 R \otimes_R A \to \Omega^1 A \to 0$$

is exact. The right exact follows by considering derivations; the injectivity is special and [redacted] is proved using the fact that the square zero extension $0 \to I/I^2 \to R/I^2 \to A \to 0$ can be split by a map $I/I^2 \to E$ of A-bimodules.

It seems useful to view $\circledast$ as the <u>noncommutative cotangent complex</u>. When R is free one has

$$(**) \qquad 0 \longrightarrow I/I^2 \longrightarrow A\otimes_R \Omega^1 R\otimes_R A \longrightarrow \boxed{A\otimes A} \longrightarrow A \longrightarrow 0$$

a resolution of A-bimodules where things are free in degrees $0, 1$.

Idea — one can <u>tensor</u> over A or <u>concatenate</u> ~~this~~ resolution with itself to get interesting complexes computing $HH_\bullet(A)$. It seems necessary to understand this in order to be able to derive the Connes exact sequence ~~starting~~ from an arbitrary $R/I = A$ with R free.

June 6, 1990

There's a puzzle about homotopy, or rather a difficulty in writing down a satisfactory account, which suggests some idea is lacking. Consider a functor on algebras such as $L(R)$. Given a polynomial homotopy $A \xrightarrow{u_t} B[t]$ that is a 1-parameter polynomial family of homomorphisms one gets a 1-param. family of induced maps from $L(A)$ to $L(B)$:

$$L(A) \longrightarrow L(B[t]) \longrightarrow L(B)[t]$$

One has

$$L(u_1) - L(u_0) = \int_0^1 L(u_t)^\circ \, dt$$

so if $L(u_t)^\circ$ is homotopic to zero, say $L(u_t)^\circ = [d, v_t]$, then we have

$$L(u_1) - L(u_0) = \left[d, \int_0^1 v_t \, dt\right]$$

R This leads us to concentrate ~~on~~ the derivative $L(u_t)^\circ : L(A) \longrightarrow L(B)[t]$. This derivative is a multilinear expression in u_t and $\mathring{u}_t$ and it is linear in $\mathring{u}_t$; e.g.

$$L(u_t)^\circ (a_0 da_1) = \mathring{u}_t a_0 d(u_t a_1) + u_t a_0 d(\mathring{u}_t a_1)$$

We know the derivative depends on the homomorphism $A \xrightarrow{u_t} B[t]$ and the derivation $A \xrightarrow{\mathring{u}_t} B[t]$ relative to this ~~[illegible]~~ homomorphism. Consider then maps $A \xrightarrow{u+\mathring{u}} B \oplus B$ = semi direct product. These are "infinitesimal homotopies."

TFAE

1) homomorphisms $A \xrightarrow{u+\dot{u}} B \oplus B$

2) homomorphisms of semi direct products

$$A \oplus \Omega^1 A \longrightarrow B \oplus B$$

3) homoms: $\Omega A \longrightarrow B$.

Thus the universal "infinitesimal homotopy" from A to another algebra is $1+d : A \longrightarrow \Omega A$.

The puzzle is that ΩA is so big. To understand $L(u_t)^{\cdot}$ we only need u_t to first order.

One idea is that

$$L(1+td)^{\cdot}\Big|_{t=0} : L(A) \longrightarrow L(\Omega A)$$

lands in the degree 1 part $L(\Omega A)_{(1)}$ for the $\mathbb{N}$-grading on $L(\Omega A)$ induced by the natural $\mathbb{N}$-grading on ΩA. Thus one can recover the derivative using the map

$$L(\Omega A) \longrightarrow L(\Omega A / \Omega^{\geq 2} A) = L(A \oplus \Omega^1 A)$$

which induces an isom. on the degree 1 part.

Other ideas along this line are to introduce the derivative in the sense of nonadditive functors

$$L(A; M) = L(A \oplus M)_{(1)}$$

+ maybe ideas [illegible] could be obtained from Goodwillie's calculus

But I feel that I don't yet have the correct setting. The issue is to understand when different maps $A \rightrightarrows B$ induce the same map on L up to homotopy. This leads one to look at all homomorphisms $A \longrightarrow B$ and to organize them into families $A \longrightarrow B \otimes S$.

Now usually S is commutative, but we should be ~~[illegible]~~ careful and not make this assumption unnecessarily. For example if $S = M_n\mathbb{C}$ do we get an induced map $L(A) \to L(B)$? YES Is $L(M_nB) = M_nL(B)$? NO

Recall that when $S = M_n\mathbb{C}$ the Hochschild homology is trivial ~~[illegible]~~ and one has an exact sequence

$$0 \longrightarrow \Omega^1S_\natural \xrightarrow{\ b\ } S \xrightarrow{\ tr\ } \mathbb{C} \longrightarrow 0$$

Also one has exact sequences (in general)

$$0 \longrightarrow \Omega^1R \otimes \Omega^1S \longrightarrow (R\otimes R)\otimes\Omega^1_S \oplus \Omega^1_R\otimes(S\otimes S) \longrightarrow \Omega(R\otimes S) \longrightarrow 0$$

$$\Omega^1R_\natural \otimes \Omega^1_\natural S \longrightarrow R\otimes\Omega^1S_\natural \oplus \Omega^1R_\natural \otimes S \longrightarrow \Omega^1(R\otimes S)_\natural \longrightarrow 0$$

When $S = M_n$ the arrow at the left is injective and one gets an exact sequence

$$0 \longrightarrow R\otimes\Omega^1S_\natural \longrightarrow \Omega^1(R\otimes S)_\natural \longrightarrow \Omega^1R_\natural \otimes S_\natural \longrightarrow 0$$

$$r\otimes s_1ds_2 \longmapsto rs_1ds_2$$

Thus one has

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & R\otimes\Omega^1 S_\natural \arrow[r, equal] \arrow[d] & R\otimes\Omega^1 S_\natural \arrow[d] & & = \\
{} \arrow[r, "c"] & R\otimes S \arrow[r, "d"] \arrow[d] & \Omega^1(R\otimes S)_\natural \arrow[r, "c"] \arrow[d] & R\otimes S \arrow[r] \arrow[d] & {} \\
{} \arrow[r, "c"] & R\otimes S_\natural \arrow[r, "d"] \arrow[d] & \Omega^1 R_\natural\otimes S_\natural \arrow[r, "c"] \arrow[d] & R\otimes S_\natural \arrow[r] & {} \\
 & 0 & 0 & &
\end{tikzcd}$$

showing that the trace $tr: S \longrightarrow \mathbb{C}$ induces a homotopy equivalence

$$L(R\otimes S) \longrightarrow L(R)$$

A map in the inverse direction is given by $r \longmapsto r\otimes 1$. Since $tr(1) = n$ it is a homotopy inverse after dividing by n.

~~[illegible] it seems to be natural to~~

~~[illegible]~~ In general one has a map of complexes

$$L(R\otimes S) \longrightarrow L(R)\otimes S_\natural$$

as in the above diagram. Thus a trace on S induces a map $L(R\otimes S) \to L(R)$. Moreover given ~~[illegible]~~ a homomorphism $A \to B\otimes S$ and trace on S we get a map $L(A) \longrightarrow L(B)$.

The geometric picture of this is as follows. Let $X = Spec(A)$ $Y = Spec(B)$ $T = Spec(S)$

We have then maps

$$\begin{tikzcd} Y \times T \arrow[r] \arrow[d, "pr_1"] & X \\ Y & \end{tikzcd}$$

and the trace on T can be viewed as giving an integration-over-the-fibre map pr_{1*}.

June 8, 1990

Continuity of the homotopy with respect to the I-adic filtration. Consider an A such that $\Omega^1 A$ is a projective A-bimodule, & let $I \subset A$ be an ideal. Then we can split the sequence of A-bimodules

$$0 \longrightarrow \Omega^2 A \longrightarrow \Omega^1 A \otimes A \longrightarrow \Omega^1 A \longrightarrow 0$$

and the issue is whether the splitting extends to the inverse limit of the sequences

$$0 \to \Omega^2(A/I^n) \longrightarrow \Omega^1(A/I^n) \otimes A/I^n \longrightarrow \Omega^1(A/I^n) \to 0$$

It might be better to say ~~that~~ the exact sequence of proobjects splits.

Choose a lifting $\sigma : \Omega^1 A \to \Omega^1 A \otimes A$. Suppose given n. Then we want to find an m such that the dotted arrow exists in:

$$\begin{tikzcd}
\Omega^1 A \otimes A \arrow[dd] & \Omega^1 A \arrow[l, "\sigma"'] \arrow[d] \\
 & \Omega^1(A/I^m) \arrow[dl, dashed] \\
\Omega^1(A/I^n) \otimes (A/I^n) &
\end{tikzcd}$$

Recall that we have an exact sequence of A/I^k bimodules

$$0 \longrightarrow I^k/I^{2k} \longrightarrow (A/I^k) \otimes_A \Omega^1 A \otimes_A (A/I^k) \longrightarrow \Omega^1(A/I^k) \longrightarrow 0$$

So we have $(A_k = A/I^k)$

$$\begin{tikzcd}
I^{2k}/I^{4k} \arrow[r] \arrow[d, "0"] & A_{2k} \otimes_A \Omega^1 A \otimes_A A_{2k} \arrow[r] \arrow[d] & \Omega^1 A_{2k} \arrow[r] \arrow[d] \arrow[dl, dashed] & 0 \\
I^k/I^{2k} \arrow[r] & A_k \otimes_A \Omega^1 A \otimes_A A_k \arrow[r] & \Omega^1 A_k \arrow[r] & 0
\end{tikzcd}$$

In other words the inverse systems

$$A_k \otimes_A \Omega^1 A \otimes_A A_k = \Omega^1 A / I^k \Omega^1 A + \Omega^1 A I^k$$

$$\Omega^1(A_k) = \Omega^1 A / I^k \Omega^1 A + \Omega^1 A I^k + d(I^k)$$

are equivalent essentially because

$$d(I^{2k}) \subset \sum_{i+j=2k-1} I^i \, \Omega^1 A \, I^j = F_I^{2k-1} \Omega^1 A$$

and one has $F_I^{2k-1} \Omega^1 A \subset I^k \Omega^1 A + \Omega^1 A \, I^k$ because at least one of i, j must be $\geq k$. Thus we can also add the inverse system

$$\Omega^1 A / F_I^k \Omega^1 A$$

to the above list.

Now because $\sigma : \Omega^1 A \longrightarrow \Omega^1 A \otimes A$ is a bimodule morphism we have

$$\sigma(I^k \Omega^1 A + \Omega^1 A \, I^k) \subset I^k \Omega^1 A \otimes A + \Omega^1 A \otimes I^k$$

The last bimodule goes to zero in $\Omega^1(A/I^n) \otimes A/I^n$ for $k \geq n$. Thus we have proved the desired continuity.

To be more precise, we have the following picture

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 A \arrow[r, shift right] & \Omega^1 A \otimes A \arrow[l, shift right] \arrow[r, shift right] & \Omega^1 A \arrow[l, shift right, "\sigma"'] \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 A_{4k} \arrow[r] \arrow[d] & \Omega^1 A_{4k} \otimes A_{4k} \arrow[r] \arrow[d, "\text{not comm.}"'] \arrow[dl, dashed] & \Omega^1 A_{4k} \arrow[r] \arrow[d] \arrow[dl, dashed, "\text{not comm.}"'] & 0 \\
0 \arrow[r] & \Omega^2 A_{2k} \arrow[r] \arrow[d] & \Omega^1 A_{2k} \otimes A_{2k} \arrow[r] \arrow[d, "\text{not comm.}"'] \arrow[dl, dashed] & \Omega^1 A_{2k} \arrow[r] \arrow[d] \arrow[dl, dashed, "\text{not comm.}"'] & 0 \\
0 \arrow[r] & \Omega^2 A_k \arrow[r] & \Omega^1 A_k \otimes A_k \arrow[r] & \Omega^1 A_k \arrow[r] & 0
\end{tikzcd}$$

The point is that in passing from level 2k to k the map is nullhomotopic as a map of complexes.

Now it's clear that we have the desired continuity. It seems desirable to write up the homotopy property of the little periodic complex starting from the lifting σ.

June 10, 1990

Some ideas. First an observation

Consider $$A\otimes\bar{A}\otimes A \twoheadrightarrow \Omega^1 A$$
$$(a_0, a_1, a_2) \longmapsto a_0 da_1 a_2$$

There are two sections which are respectively left and right A-module morphisms

$$a_1 da_2 \longmapsto (a_1, a_2, 1)$$

$$a_1 da_2 = d(a_1a_2) - da_1 a_2 \longmapsto (1, a_1a_2, 1) - (1, a_1, a_2)$$

The difference of this two liftings gives a map $\Omega^1 A \longrightarrow \Omega^2 A$ which is just d. In effect

$$b'd(a_1da_2) = b'(da_1da_2) =$$ [redacted]

$$-da_1(a_2\otimes 1 - 1\otimes a_2) = a_1da_2\otimes 1 - d(a_1a_2)\otimes 1 + da_1\otimes a_2$$

$$= (a_1, a_2, 1) - (1, a_1a_2, 1) + (1, a_1, a_2)$$

Idea: $\Omega^1 R$ projective is the non commutative analogue of smooth, since it implies lifting with respect to nilpotent extensions. Can one show that when $A = R/I$ and $\Omega^1 R$ is projective, then the [redacted] inverse system of complexes $L(R/I^n)$ gives the periodic cyclic homology of A?

June 15, 1990

New idea. The observation is that the mixed complex (Ω, b, B) gives rise to the double complex

$$\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
& \Omega^2 \arrow[d] & \Omega^1 \arrow[l] \arrow[d] & \Omega^0 \arrow[l] \\
\mathcal{B}_{norm}(A) & \Omega^1 \arrow[d] & \Omega^0 \arrow[l] & \\
& \Omega^0 & &
\end{tikzcd}$$

which gives the cyclic $\overbrace{\text{homology of}}^{HC(A)}$ A, and the mixed complex $(\bar{\Omega}, b, B)$ gives rise to the double complex

$$\begin{tikzcd}
& \Omega^2 \arrow[d] & \Omega^1 \arrow[l] \arrow[d] & \bar{\Omega}^0 \arrow[l] \\
\mathcal{B}_{red}(A) & \Omega^1 \arrow[d] & \bar{\Omega}^0 \arrow[l] & \\
& \bar{\Omega}^0 & &
\end{tikzcd}$$

whose homology is defined to be $\bar{HC}(A)$ in [LQ] With this definition one gets the exact seq

$$\longrightarrow HC_i(\mathbb{C}) \longrightarrow HC_i(A) \longrightarrow \bar{HC}_i(A) \longrightarrow HC_{i-1}(\mathbb{C}) \longrightarrow$$

from the short exact sequence of complexes

$$0 \longrightarrow \underbrace{\mathcal{B}_{norm}(\mathbb{C})}_{\substack{\mathbb{C}\text{'s on the} \\ \text{bottom diagonal}}} \longrightarrow \mathcal{B}_{norm}(A) \longrightarrow \mathcal{B}_{red}(A) \longrightarrow 0$$

But now apply P everywhere but mainly to $\mathcal{B}red(A)$. One obtains a double complex whose rows resolve the reduced cyclic complex $\overline{CC}(A)$

$$\begin{tikzcd}
\overline{CC}^1 & P\Omega^1 \arrow[l, dashed] \arrow[d, "b"] & \bar\Omega^0 \arrow[l, "B"'] \\
\overline{CC}^0 & \bar\Omega^0 \arrow[l, dashed] &
\end{tikzcd}$$

so we obtain a quis $PB_{red}(A) \longrightarrow \overline{CC}(A)$. This shows the two definitions of reduced cyclic homology are equivalent.

It would be nice to understand what is happening here. We have somehow managed to establish that

$$CC(k) \hookrightarrow CC(A) \twoheadrightarrow \overline{CC}(A)$$ [redacted]

is a triangle without a filtration type argument. Recall that there are two of these arguments which [redacted] we have used in the past. First the filtration of $B_{red}(A)$ used in [LQ], then the filtration on $CC(A)$ resulting from the alg. filtration $k = F_0A \subset F_1A = A$.

Let's analyze the [LQ] argument that $B_{norm}(A)$ is quis to $CC(A)$. It's based on the resolution of $CC(A)$ by alternating b and b' complexes, then eliminating the b' complex via s, to obtain Connes mixed complex (unnormalized Hochschild for A, b, B), where B has two sums; then we map the unnormalized Hochschild to the normalized one Ω - this is a quis by the <u>semisimplicial normalization thms</u>

(<u>PROBLEM</u>: K-theory for cyclic modules, cyclic in the sense of cyclic objects)

Let's proceed as follows. Let's identify $CC(A)$ with $\bar{C}C(\tilde{A})$ and note that $(\bar{\Omega}(\tilde{A}), b)$ is the two column double complex with columns the b and b′ complexes. Let's replace $\bar{\Omega}(\tilde{A})$ with $P\bar{\Omega}(\tilde{A})$ and use the resolution

$$\begin{tikzcd} 0 & CC(A) \arrow[l] & P\bar{\Omega}(\tilde{A}) \arrow[l] & P\bar{\Omega}(\tilde{A}) \arrow[l, "B"'] & \cdots \arrow[l, "B"'] \end{tikzcd}$$

This is the double cx of $(P\bar{\Omega}(\tilde{A}), b, B)$. Then as A is assumed to be unital we have a map of resolutions

$$\begin{tikzcd} 0 & CC(A) \arrow[l] \arrow[d] & P\bar{\Omega}(\tilde{A}) \arrow[l] \arrow[d] & \arrow[l] \\ 0 & \bar{C}C(A) \arrow[l] & P\bar{\Omega}(A) \arrow[l] & \arrow[l] \end{tikzcd}$$

I guess the point is that we are looking at the map of mixed complexes

$$P\bar{\Omega}(\tilde{A}) \longrightarrow P\bar{\Omega}(A)$$

and the induced map of associated double complexes. Now the b-homology of $P\bar{\Omega}(\tilde{A})$, $P\bar{\Omega}(A)$ is $HH(\tilde{A}) = HH(A) \oplus \mathbb{C}[0]$, $HH(A)$ resp.

Here is what seems to be going on.

From the K-theory one learns that there are quis

$$\begin{tikzcd}
\bar{C}C(A) & D.C.(P\bar\Omega A) \arrow[l] \arrow[r] & D.C.(\bar\Omega A) \\
\bar{C}C(\tilde A) \arrow[d, phantom, "\cong" rotate=90] & D.C.(P\bar\Omega\tilde A) \arrow[l] \arrow[r] & D.C.(\bar\Omega\tilde A) \arrow[d, phantom, "="] \\
CC(A) & & b, b' \text{ colum double complex.}
\end{tikzcd}$$

On the other hand we have maps upward induced by $\tilde A \to A$. Look at $\bar\Omega\tilde A \to \bar\Omega A$.

~~One has a map $\Omega A \to \bar\Omega\tilde A$ which should be a quis because~~ We have $\bar\Omega^0\tilde A = A$, really eA, and we have a map $\mathbb{C}[0] \to \bar\Omega\tilde A$ sending $1 \in \mathbb{C}$ to $e \in A$. We want to see that $\mathbb{C}[0] \longrightarrow \bar\Omega\tilde A \longrightarrow \bar\Omega A$ is a triangle.

Use that $\bar\Omega\tilde A$ is the mapping cone of $1-\lambda$ from the b' complex to the b complex.

$$\begin{tikzcd}
 \arrow[d] & \arrow[d] & & \arrow[d, "b"] \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-\lambda"'] \arrow[d, "-b'"] \arrow[rr] & & A \otimes \bar A \arrow[d, "b"] \\
A & A \arrow[l, "1-\lambda"'] & & \bar A
\end{tikzcd}$$

This is a slightly subtle map. Hopefully it is compatible with the map from the b complex into the former.

There are lots of things to analyze here. One would like a homotopy equivalence between $\bar\Omega(\tilde A)$ and ΩA compatible with B.

What we have are heq's

$$\begin{tikzcd}\bar{\Omega}(\tilde{A}) & \left(\begin{smallmatrix} b \text{ cx} \\ \text{of } \tilde{A} \end{smallmatrix}\right) \arrow[l, hook', "\substack{\uparrow \\ \text{contractibility} \\ \text{of } b' \text{ cx}}"'] \arrow[r, "\substack{\text{by the} \\ \text{norm. thm.}}"'] & \Omega A\end{tikzcd}$$

and in LQ we did show how to make the first map on the level of the double complex. One has to look at the inverse given by the normalization thm. and see how it behaves relative to B.

There might be something interesting going on because one has a Morita map $A \longrightarrow \tilde{A}$ (in this case a non unital homomorphism). Such a map does not yield obviously a map $\Omega A \longrightarrow \Omega \tilde{A}$.

June 21, 1990

Consider a mixed complex $M = \oplus M_n$, $M_n = 0$ $n < 0$, b of deg -1, B of deg $+1$. Let $\mathcal{B}$ be the total complex of

$$\begin{tikzcd} {} \arrow[d, "b"] & {} \arrow[d, "b"] \\ M_1 \arrow[d, "b"] & M_0 \arrow[l, "B"'] \\ M_0 & \end{tikzcd}$$

and suppose B exact on M. Then we have a quis $\varepsilon : \mathcal{B} \longrightarrow M/BM$. We have exact sequences

$$0 \longrightarrow M \longrightarrow \mathcal{B} \longrightarrow \Sigma^2 \mathcal{B} \longrightarrow 0$$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(B) \arrow[r] & M \arrow[r] & M/BM \arrow[r] & 0 \\ & \Sigma(M/BM) \arrow[u, "\text{induced by } B"'] & & & \end{tikzcd}$$

and we want to compare the associated long exact sequences via the isomorphism $\varepsilon : H(\mathcal{B}) \xrightarrow{\sim} H(M/BM)$. Thus we ask whether

$$\begin{tikzcd} H_n M \arrow[r] \arrow[d, equal] & H_n \mathcal{B} \arrow[r, "①"] \arrow[d, "\varepsilon"] & H_{n-2} \mathcal{B} \arrow[r, "②"] \arrow[d, "\varepsilon"] & H_{n-1} M \arrow[d, equal] \\ H_n M \arrow[r] & H_n(M/BM) \arrow[r, "③"] & H_{n-2}(M/BM) \arrow[r, "④"] & H_{n-1} M \end{tikzcd}$$

commutes. The first square obviously commutes. Let's consider the second. Take a class in $H_n \mathcal{B}$ represent it by a $(b+B)$-cycle $x = (x_n, x_{n-2}, x_{n-4}, \dots)$ in $\mathcal{B}_n = M_n \oplus M_{n-2} \oplus \cdots$

Then ① sends it to $(x_{n-2}, x_{n-4}, \ldots)$ in $\mathcal{B}_{n-2}$, and ε sends the image to the cycle x_{n-2} mod BM. On the other hand ③ is a connecting homom. so ③$\varepsilon[x]$ = ③$(x_n \bmod BM)$ is found by lifting x_n mod BM to M applying b to get an element in $\mathrm{Ker}(B)$, ~~and~~ then writing the result as Bu with $u + BM \in (M/BM)_{n-2}$. The lift of x_n mod BM we take to be x_n, then since $bx_n + Bx_{n-2} = 0$, we have u mod BM $= -x_{n-2}$ mod BM. Thus the second square anti commutes.

For the third square start with the class of a cycle $(x_{n-2}, x_{n-4}, \ldots)$ in $\mathcal{B}_{n-2}$. ② is a connecting homom., so we lift via $S : \mathcal{B} \longrightarrow \Sigma^2 \mathcal{B} \simeq \mathcal{B}$ to get $(0, x_{n-2}, x_{n-4}, \ldots) \in \mathcal{B}_n$ apply the boundary to get

$$(b+B)(0, x_{n-2}, x_{n-4}, \ldots) = (Bx_{n-2}, 0, \ldots, 0)$$

which comes from the cycle $Bx_{n-2} \in \mathcal{B}_{n-1}$. On the other hand ④$\varepsilon(x_{n-2}, \ldots)$ = ④$(x_{n-2} + BM) = Bx_{n-2} \in \mathcal{B}_{n-1}$. Thus the third square commutes, and we have

$$\begin{tikzcd}
H_n M \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "+"] & H_n \mathcal{B} \arrow[r] \arrow[d, "\cong"'] \arrow[dr, phantom, "-"] & H_{n-2} \mathcal{B} \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "+"] & H_{n-1} M \arrow[d, equal] \\
H_n M \arrow[r] & H_n(M/BM) \arrow[r] & H_{n-2}(M/BM) \arrow[r] & H_{n-1} M
\end{tikzcd}$$

If we change the sign so as to have the differential $b - B$ in $\mathcal{B}$, then this makes the second square commute but the third square anti-commutes.

Examine case of circle actions. Let E be a principal S^1-bundle with connection A and base B. The equivariant cohomology of E is computed from

$$\{W(\mathfrak{g}) \otimes \Omega(E)\}_{bas} \simeq \mathbb{C}[u] \otimes \Omega(E)^{S^1}$$

with differential $d - u\iota_X$

Let's review this. $W(\mathfrak{g}) = \mathbb{C}[\theta, d\theta]$ where $\iota_X\theta = 1$, $L_X\theta = 0$. We have a map

$$\mathbb{C}[u] \otimes \Omega(E) \longrightarrow \{W(\mathfrak{g}) \otimes \Omega(E)\}_{hor}$$

$$\alpha \longmapsto (1-\theta\iota_X)\alpha$$

$$u \longmapsto d\theta$$

which turns out to be an isomorphism. It induces an isomorphism on invariants

$$\mathbb{C}[u] \otimes \Omega(E)^{S^1} \xrightarrow{\sim} \{W(\mathfrak{g}) \otimes \Omega(E)\}_{bas}$$

so we have the differential. But

[redacted]

$$\begin{tikzcd}
\alpha \arrow[rr, mapsto] \arrow[d, mapsto] & & (1-\theta\iota_X)\alpha \arrow[d, mapsto] \\
d\alpha - u\iota_X\alpha \arrow[r, mapsto] & (1-\theta\iota_X)(d\alpha) - d\theta(1-\cancel{\theta\iota_X})\iota_X\alpha & d(1-\theta\iota_X)\alpha
\end{tikzcd}$$

$$d(1-\theta\iota_X)\alpha = (1-\theta\iota_X)d\alpha - [d,\theta]\iota_X\alpha + \theta\underbrace{[d,\iota_X]\alpha}_{L_X\alpha = 0}$$

$$= (1-\theta\iota_X)d\alpha - d\theta(\iota_X\alpha)$$

$$(1-\theta\iota_X)(d\alpha) - d\theta(1-\cancel{\theta\iota_X})\iota_X\alpha = (1-\theta\iota_X)d\alpha - d\theta\iota_X\alpha$$

$\therefore$ agree

So now use the fact we have a principal bundle with connection. Then we have

$$\Omega(B)\oplus\Omega(B)\xrightarrow[\sim]{(1,A)}\Omega(E)^{S^1}$$

splitting the exact sequence

$$0\longrightarrow\Omega(B)\xrightarrow{\pi^*}\Omega(E)^{S^1}\xrightarrow{\pi_*}\Omega(B)\longrightarrow 0$$

We also have

$$\begin{array}{ccc}\Omega(B) & \longrightarrow & \mathbb{C}[u]\otimes\Omega(E)^{S^1}\\ \alpha & \longmapsto & \pi^*\alpha\end{array}$$

which is compatible with differentials. We have

$$\circledast\quad \Omega(B)\longrightarrow\mathbb{C}[u]\otimes\Omega(E)^{S^1}=\mathbb{C}[u]\otimes\mathbb{C}[A]\otimes\Omega(B)$$

where on the latter $(d-u\iota_X)(A)=dA-u$. But we have

$$\mathbb{C}[u]\otimes\overbrace{\mathbb{C}[A]}^{\mathbb{C}\oplus\mathbb{C}A}\otimes\Omega(B)\simeq\underbrace{\mathbb{C}[\overset{\in\Omega^2(B)}{\boxed{dA}}-u]\otimes\mathbb{C}[A]}\otimes\Omega(B)$$

showing that $\circledast$ is a quis because the DGA is contractible.

Now let's compare long exact sequences resulting from

$$0\longrightarrow\mathbb{C}[u]\otimes\Omega(E)^{S^1}\xrightarrow{u}\mathbb{C}[u]\otimes\Omega(E)^{S^1}\longrightarrow\Omega(E)^{S^1}\longrightarrow 0$$

$$0\longrightarrow\Omega(B)\xrightarrow{\pi^*}\Omega(E)^{S^1}\xrightarrow{\pi_*}\Omega(B)\longrightarrow 0$$

We have the maps

$$\begin{tikzcd}
H^i(\Omega B) \arrow[r, "\pi^*"] \arrow[d, "\pi^*"] \arrow[dr, phantom, "\text{commutes obviously}"] & H^i(\Omega E^{S^1}) \arrow[r, "\pi_*"] \arrow[d, "="] & H^{i-1}(\Omega B) \arrow[r] \arrow[d] & H^{i+1}(\Omega B) \arrow[d, "\pi^*"] \\
H^i(\mathbb{C}[u]\otimes\Omega E^{S^1}) \arrow[r] & H^i(\Omega E^{S^1}) \arrow[r, "\delta"] & H^{i-1}(\mathbb{C}[u]\otimes\Omega E^{S^1}) \arrow[r] & H^{i+1}(\mathbb{C}[u]\otimes\Omega E^{S^1})
\end{tikzcd}$$

Take $x \in \Omega^i E^{S^1}$, $dx = 0$. To compute δ of its class one lifts it to $\mathbb{C}[u]\otimes\Omega E^{S^1}$, say to $X = u^0 \otimes x$, one applies the differential: $(d - u\iota_X)x = -u\,\iota_X x$, then one pulls back via multiplecation by u to get $-\iota_X x$. However $\pi^*\pi_* x = \iota_X x$ for invariants elements in ΩE. $\therefore$ The second square anticommutes.

Next given $z \in \Omega^{i-1}B$ we multiply by $dA = F$ then apply π^* to get $\pi^*(\overset{F.}{}\,z) = dA\,\pi^*(z)$. The other way gives [redacted] $u\,\pi^*(z)$. But

$$(d - u\iota_X)(A\,\pi^* z) = dA\,\pi^* z - u\pi^* z$$

so those represent the same class. Thus the third square commutes.

I have used the following computation from the connecting hom. assoc. to the π^*, π^* exact sequence. Given $z \in \Omega(B)$, $dz = 0$ lift to $A\pi^* z \in \Omega(E)^{S^1}$; apply d to get $dA \cdot \pi^* z$, which comes from $Fz \in \Omega(B)$.

It is necessary to get a coherent set of conventions. Look at the principal circle bundle case $S^1 \to E \to B$. The equivariant form complex is $\mathbb{C}[u] \otimes \Omega(E)^{S^1}$ with diffl $d - u\iota_X$. Here u is the universal curvature. You ~~therefore~~ get a degree 2 map on $H^\cdot(B)$:

$$\begin{tikzcd}
H^i(\mathbb{C}[u]\otimes\Omega E^{S^1}) \arrow[r, "u\cdot"] & H^{i+2}(\mathbb{C}[u]\otimes\Omega E^{S^1}) \\
H^i(B) \arrow[u, "\pi^*", "\cong"'] \arrow[r] & H^{i+2}(B) \arrow[u, "\pi^*", "\cong"']
\end{tikzcd}$$

which you want to be cup product with the curvature class of E/B. But if A is a connection, we have

$$(d - u\iota_X)(A\pi^*(z)) = dA\cdot\pi^*z - u\pi^*z$$

so multiplication by u corresponds to multiplying by $[dA]$.

This seems to show that one must use the differential $d - u\iota_X$ on $\mathbb{C}[u]\otimes\Omega E^{S^1}$ if one wants u to be multiplication by the universal curvature.

Next we look at the square before

$$\begin{tikzcd}
H^{i+1}(\Omega E^{S^1}) \arrow[r, "\delta"] \arrow[d, equal] & H^i(\mathbb{C}[u]\otimes\Omega E^{S^1}) \\
H^{i+1}(E) & H^i(B) \arrow[u, "\cong"']
\end{tikzcd}$$

δ is a connection homomorphism, so that given a closed invariant form ω on E, one lifts ω to itself viewed as an elt of $\mathbb{C}[u]\otimes\Omega E^{S^1}$, then applies the diffl

$$(d-u\iota_X)(\omega) = -u(\iota_X\omega)$$

then divides by u to get $-\iota_X\omega$.

Thus the map $H^{i+1}(E)\longrightarrow H^i(B)$ we must use is induced by averaging a form with respect to the circle action followed by $-\iota_X$. This is $-\pi_*$ assuming that π_* is such that $\pi_*(A)=1$.

Next let's go back to a mixed complex (M,b,B) (homology setting) such that B is exact. We consider the double complex but take the total differential $b-B$. The reason for this is so that, when we have a cycle, say $(x_n, x_{n-2},\dots)\in\mathcal{B}_n$, it is true that the class of $x_n+BM\in M/BM$ is mapped by S into the class of $x_{n-2}+BM\in M/BM$. This is not clear, so consider cohomology.

Here we have $f_n\in \mathrm{Ker}(B)$ which is a cocycle: $bf_n=0$, we choose g_{n+1}, with $Bg_{n+1}=f_n$ and then take $f_{n+2}=bg_{n+1}$. We want $S[f_n]=[f_{n+2}]$, which means that we want the elements $f_n\in u^0M$ and $uf_n\in uM$ which are killed by both b, B to be cohomologous. But

$$(b-uB)g_{n+1} = f_{n+2}-uf_n$$

Really I want to compute $\delta : H_n(M/BM) \longrightarrow H_{n-2}(M/BM)$ as follows. Given $x_n \in M_n$ with $bx_n = Bx_{n-2}$ so that $[x_n + BM] \in H_n(M/BM)$ is defined, then $\delta[x_n + BM] = [x_{n-2} + BM]$.

Anyway consider

$$\begin{tikzcd}
H_n M \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\text{commutes}"] & H_n \mathcal{B} \arrow[r, "S"] \arrow[d, "\simeq"] & H_{n-2} \mathcal{B} \arrow[r] \arrow[d, "\simeq"] & H_{n-1} M \arrow[d, "1"] \\
H_n M \arrow[r] & H_n(M/BM) \arrow[r] & H_{n-2}(M/BM) \arrow[r] & H_{n-1} M
\end{tikzcd}$$

where the top row is defined via

$$0 \longrightarrow M \longrightarrow \mathcal{B} \xrightarrow{S} \Sigma^2 \mathcal{B} \longrightarrow 0$$

where $S(x_n, x_{n-2}, \dots) = (x_{n-2}, x_{n-4}, \dots)$. The second square commutes

$$\begin{tikzcd}
(x_n, x_{n-2}, \dots) \arrow[rrr, mapsto] \arrow[dd] & & & (x_{n-2}, x_{n-4}, \dots) \arrow[d, mapsto] \\
& & & {[x_{n-2} + BM]} \\
{[x_n + BM]} & x_n \arrow[l, mapsto, "\text{lift}"'] \arrow[r, mapsto, "b"] & bx_n = Bx_{n-2} & \\
& & x_{n-2} \arrow[r, mapsto] & {[x_{n-2} + BM]}
\end{tikzcd}$$

so we use the exact sequence

$$0 \longrightarrow \Sigma(M/BM) \xrightarrow{B} M \longrightarrow M/BM \longrightarrow 0$$

However the last square ~~[illegible]~~ anticommutes

$$\begin{tikzcd}
(x_{n-2}, \dots) \arrow[dd, mapsto] & (0, x_{n-2}, \dots) \arrow[l, mapsto, "\text{lift}"'] \arrow[r, mapsto, "b-B"] & (-Bx_{n-2}, 0, 0, \dots) \\
& & -Bx_{n-2} \arrow[u, mapsto] \\
x_{n-2} + BM \arrow[rr, mapsto, "-B"] & & Bx_{n-2}
\end{tikzcd}$$