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More on Karoubi's operator K.

Recall the formula $K^{n+1}f_n = (1-bs)f_n$.

This gives

$$K^{n+1}f_n = (K+sb)f_n$$

$$K(K^n-1)f_n = sbf_n$$

$$\boxed{(K^n-1)f_n = \lambda^{-1}sbf_n = \lambda_{n+1}^{n}\, sbf_n}$$

since K is an autom $= \lambda$ on $\mathrm{Im}\, s$. This gives directly the result

$$sbf_n = 0 \implies K^n f_n = f_n$$

and it's $\iff$.

Check: $K^n f_n = \lambda^n(1-b's)f_n = \lambda^n sb'f_n$

$$= \lambda^n sbf_n - \lambda^n s\,\mathrm{cross}\, f_n$$

where $\mathrm{cross}\, f_n$ is the crossover term in bf_n.

$$\left(\lambda^n s\,\mathrm{cross}\,f_n\right)(a_0,\dots,a_n) = (-1)^n\left(s\,\mathrm{cross}f_n\right)(a_1,\dots,a_n,a_0)$$

$$= (-1)^n(\mathrm{cross}\,f_n)(1,a_1,\dots,a_n,a_0)$$

$$= (-1)^n(-1)^{n+1} f_n(a_0,\dots,a_n) = -f_n(a_0,\dots,a_n)$$

giving $K^n f_n = \lambda^n sbf_n + f_n$.

Next

$$\left(1-K^{n(n+1)}\right)f_n = \sum_{j=0}^{n-1} (K^{n+1})^j(1-K^{n+1})f_n$$

$$= \sum_{j=0}^{n-1}(K^{n+1})^j\, bs f_n = b\sum_0^{n-1} K^j sf_n = bBf_n$$

Check:

$$\left(1 - \text{[redacted]}\, K^{n(n+1)}\right) f_n = \sum_{j=0}^{n} (K^n)^j (1-K^n) f_n$$

$$= -\sum_{j=0}^{n} (K^n)^j \lambda_{n+1}^{n} s \underbrace{b f_n}_{n+1\text{ coch}} = -\sum_{j=0}^{n} \lambda_{n+1}^{-j-1} s b f_n = -Bbf_n$$

This also gives a different proof of $bB + Bb = 0$

Summarizing we have

$$\boxed{\begin{aligned} (1-K^{n+1}) f_n &= bs f_n \\ (1-K^n) f_n &= -\lambda^{-1} sbf_n \\ (1-K^{n(n+1)}) f_n &= bBf_n = -Bbf_n \\ \hline K^{n(n+1)} f_n &= (1+Bb) f_n \end{aligned}}$$

Thus $K^{n(n+1)}$ ~~is~~ unipotent, and we have seen this [redacted] means there are not enough K-invariant cochains.

We propose now to modify the operators s, K to remedy this defect.

Let $\tilde{s} = s + c_n B : C^n \longrightarrow C^{n-1}$, $n \geq 1$, where the scalars c_n are to be determined. $\tilde{s} = s = 0$ on C^0. Let $\tilde{K}$ be defined by

$$1 - \tilde{K} = b\tilde{s} + \tilde{s}b.$$

Then on C^n one has

$$b\tilde{s} + \tilde{s}b = b(s + c_n B) + (s + c_{n+1} B) b$$

$$= bs + sb + c_n bB + c_{n+1} Bb$$

$$1 - \tilde{K} = 1 - K + (c_{n+1} - c_n) Bb$$

$$\tilde{K} = K - (c_{n+1} - c_n) Bb$$

Since $KB = BK = B$ and $bK = Kb$ and $(Bb)^2 = 0$, we have

$$\tilde{K}^j = K^j - j(c_{n+1} - c_n)Bb$$

In particular

$$\tilde{K}^{n(n+1)} = 1 + Bb - n(n+1)(c_{n+1} - c_n)Bb$$

so we can arrange $\tilde{K}^{n(n+1)} = 1$ by taking

$$c_{n+1} - c_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \qquad n \geq 1$$

Thus $\quad c_n = c - \frac{1}{n} \quad$ c constant.

It seems most natural to take $c = 1$ so that $c_1 = 0$ and so $\tilde{s} = s : C^1 \longrightarrow C^0$. In any case we want $\operatorname{Ker} \tilde{s} = \operatorname{Im} \tilde{s}$. Now

$$\tilde{s} = (1 + c_n N_n)s \quad \text{on } C^n$$

and the eigenvalues of N_n are $0, n$. Thus $\operatorname{Ker} \tilde{s} = \operatorname{Ker} s$ provided $c_n \neq -\frac{1}{n}$. Also $\operatorname{Im} \tilde{s} = \operatorname{Im} s$ as $1 + c_n N$ is invertible on sC^n in this case.

$$\boxed{\begin{array}{ll} \tilde{K}^j = K^j - \dfrac{j}{n(n+1)} Bb & \text{on } C^n \\ \tilde{K}^{n(n+1)} = 1 & \text{on } C^n \end{array}}$$

Because $\tilde{K}$ generates an action of $\mathbb{Z}/n(n+1)\mathbb{Z}$ (which is a finite gp) on C^n, one has an exact sequence

$$0 \longrightarrow (sC^{n+1})^{\lambda} \longrightarrow (C^n)^{\tilde{K}} \longrightarrow (sC^n)^{\lambda} \longrightarrow 0$$

(Observe that as $sBb = Bbs = 0$ one has $s\tilde{K} = sK = Ks = \lambda s$ and $\tilde{K}s = Ks = \lambda s$)

[redacted] Also $b\tilde{K} = bK = Kb = \tilde{K}b$

so the $\tilde{K}$ fixpts form a subcomplex. Thus we have an exact sequence of complexes

$$0 \to \operatorname{Im} B \longrightarrow \mathcal{C}^{\tilde{K}} \xrightarrow{B} \operatorname{Im} B \to 0$$

which is included in

$$0 \to \operatorname{Ker} B \longrightarrow \mathcal{C} \xrightarrow{B} \operatorname{Im} B \to 0$$

Since $\operatorname{Im} B \hookrightarrow \operatorname{Ker} B$ is a quis, it follows that $\mathcal{C}^{\tilde{K}} \hookrightarrow \mathcal{C}$ is a quis.

Next we would like to find [redacted] an analogue of averaging over S^1. Actually before we go further we should check that $(b, \tilde{s}, \tilde{K}, B)$ have the properties analogous to (d, ι_X, L_X, π_*). In particular we should check whether

$$B = \sum_{j=0}^{n-1} \tilde{K}^j \tilde{s} \qquad \text{on } \mathcal{C}^n \,?$$

Now we have seen that $\tilde{K}^j \tilde{s} =$

$$\left(K^j - \frac{j}{n(n+1)} Bb\right)\left(s + \left(1 - \tfrac{1}{n}\right)B\right) = K^j\left(s + \left(1-\tfrac{1}{n}\right)B\right) = K^j s + \left(1 - \tfrac{1}{n}\right)B$$

Thus

$$\sum_{j=0}^{n-1} \tilde{K}^j \tilde{s} = Ns + n\left(1 - \tfrac{1}{n}\right)B = B + (n-1)B = nB$$

and so

$$\boxed{\frac{1}{n}\sum_{j=0}^{n-1} \tilde{K}^j \tilde{s} = B \quad \text{on } \mathcal{C}^n}$$

$$\tilde{K}B = \left(K - \frac{1}{n(n+1)} Bb\right)B = KB = B$$

Also $B\tilde{K} = B$ similarly.

better is that $\tilde{K} = \lambda$ on $\operatorname{Im} s$

[redacted] Let us consider the projection operator P on C with image $C^{\tilde K}$ given by

$$P = \frac{1}{n(n+1)}\sum_0^{n(n+1)-1} \tilde K^j \quad \text{on } C^n$$

Consider $b: C^n \longrightarrow C^{n+1}$. One has

$1-K^{n+1} = bs$ on C^n, hence

$b - bK^{n+1} = b^2 s = 0$ on C^n. Also

$b\tilde K^j = b\left(K^j - \frac{j}{n(n+1)}Bb\right) = bK^j$. Thus

$$bP = \frac{1}{n(n+1)}\sum_0^{n(n+1)-1} bK^j = \frac{1}{n(n+1)}\sum_{q=0}^{n-1}\sum_{r=0}^{n} \underbrace{bK^{(n+1)q+r}}_{bK^r}$$

$$= \frac{1}{n+1}\sum_{r=0}^{n} bK^r = \frac{1}{n+1}\sum_{r=0}^{n} K^r b$$

Also on C^n we have $b = bK^{n+1} = K^{n+1}b$ and

$$Pb = \frac{1}{(n+1)(n+2)}\sum_0^{(n+1)(n+2)-1} K^j b$$

$$= \frac{1}{(n+1)(n+2)}\sum_{q=0}^{n+1}\sum_{r=0}^{n} \underbrace{K^{(n+1)q+r}b}_{K^r b}$$

$$= \frac{1}{n+1}\sum_{r=0}^{n} K^r b.$$

Thus $\boxed{Pb = bP}$ and

shows we want $c_n = 1-\frac{1}{n}$ since otherwise $P\tilde s = \frac{1}{n}(1+nc_n)B$

$$P\tilde s = \frac{1}{\text{[redacted]}(n-1)n}\sum_0^{(n-1)n-1} \tilde K^j \tilde s = \frac{1}{(n-1)n}\sum_0^{(n-1)n-1} \lambda_n^j \tilde s$$

$$= \frac{1}{(n-1)n}\sum_{q=0}^{n-1}\sum_{r=0}^{n-1} \lambda_n^{nq+r}\tilde s = \frac{1}{n}\sum_0^{n-1}\lambda^r \tilde s$$

$$= \frac{1}{n}N\tilde s = \frac{1}{n}N\left(s + \text{[redacted]}\left(1-\frac{1}{n}\right)B\right) = \frac{1}{n}\left(B+(n-1)B\right) = B$$

Similarly

$$\tilde{s}P = \frac{1}{n(n+1)}\sum_{j=0}^{n(n+1)-1}\tilde{s}\tilde{K}^j = \frac{1}{n(n+1)}\sum_{j=0}^{n(n+1)-1}\tilde{s}K^j$$

$$= \frac{1}{n(n+1)}\sum_{j}\left(s+(1-\tfrac{1}{n})B\right)K^j$$

$$= \frac{1}{n(n+1)}\sum_{j}\left(\lambda_n^j s + (1-\tfrac{1}{n})B\right)$$

$$= \frac{1}{n(n+1)}\sum_{q=0}^{n}\sum_{r=0}^{n-1}\lambda_n^{nq+r}s + (1-\tfrac{1}{n})B$$

$$= \frac{1}{n}\sum_{0}^{n-1}\lambda_n^r s + (1-\tfrac{1}{n})B = \tfrac{1}{n}B + (1-\tfrac{1}{n})B$$

$$= B.$$

Thus $\boxed{P\tilde{s} = \tilde{s}P = B}$.

Finally we want to show that [redacted] $C \xrightarrow{P} C^{\tilde{K}} \hookrightarrow C$ is homotopic to the identity. We want to write $I-P$ in the form $bh+hb$. On C^n one has

$$1 - P_n = \frac{1}{n(n+1)}\sum_{j=0}^{n(n+1)-1}(1-\tilde{K}^j)$$

$$= \underbrace{\left(\frac{1}{n(n+1)}\sum_{j=0}^{n(n+1)-1}\sum_{0\le i<j}\tilde{K}^i\right)}_{Q_n}\underbrace{(1-\tilde{K})}_{b\tilde{s}+\tilde{s}b}$$

$$= b(\underbrace{Q_n\tilde{s}}_{C^n\to C^{n-1}}) + (Q_n\tilde{s})b$$

The problem with this is that if we compare it with

$$1 - P_{n-1} = b(Q_{n-1}\tilde{s}) + (\underbrace{Q_{n-1}\tilde{s}}_{C^n\to C^{n-1}})b \quad \text{on } C^{n-1}$$

then we have two candidates for $h_n : C^n \longrightarrow C^{n-1}$ namely $Q_{n-1}\tilde{S}_n$ and $Q_n \tilde{S}_n$. Let's calculate what these are using the fact that $\tilde{K} = \lambda_n$ on $\tilde{S}_n C^n = sC^n$.

$$Q_n \tilde{s} = \frac{1}{n(n+1)} \sum_{j=0}^{n(n+1)-1} \sum_{i=0}^{j-1} \lambda_n^i \tilde{s}$$

$$= \frac{1}{n(n+1)} \sum_{q=0}^{n} \sum_{r=0}^{n-1} \underbrace{\sum_{i=0}^{qn+r-1} \lambda_n^i \tilde{s}}_{\left(qN_n + \sum_{i=0}^{r-1} \lambda_n^i\right)\tilde{s}}$$

$$= \frac{1}{n(n+1)} \sum_{q=0}^{n} \left(nqN_n + \sum_{r=0}^{n-1} \sum_{i=0}^{r-1} \lambda_n^i \right) \tilde{s}$$

$$= \frac{1}{n(n+1)} \left(n \frac{n(n+1)}{2} N_n + (n+1) \sum_{r=0}^{n-1} \sum_{i=0}^{r-1} \lambda_n^i \right) \tilde{s}$$

$$= \left(\frac{n}{2} N_n + \frac{1}{n} \sum_{r=0}^{n-1} \sum_{i=0}^{r-1} \lambda_n^i \right) \tilde{s}$$

$$Q_{n-1} \tilde{s} = \frac{1}{(n-1)n} \sum_{j=0}^{(n-1)n-1} \sum_{i=0}^{j-1} \lambda_n^i \tilde{s}$$

$$= \frac{1}{(n-1)n} \sum_{q=0}^{n-2} \sum_{r=0}^{n-1} \underbrace{\sum_{i=0}^{qn+r-1} \lambda_n^i \tilde{s}}_{\left(qN_n + \sum_{i=0}^{r-1} \lambda_n^i\right)\tilde{s}}$$

$$= \frac{1}{(n-1)n} \sum_{q=0}^{n-2} \left(nqN_n + \sum_{r=0}^{n-1} \sum_{i=0}^{r-1} \lambda_n^i \right) \tilde{s}$$

$$= \frac{1}{(n-1)n} \left(n \tfrac{1}{2}(n-2)(n-1) N_n + (n-1) \sum_{r=0}^{n-1} \sum_{i=0}^{r-1} \lambda_n^i \right) \tilde{s}$$

$$= \left(\frac{n-2}{2} N_n + \frac{1}{n} \sum_{r=0}^{n-1} \sum_{i=0}^{r-1} \lambda_n^i \right) \tilde{s}$$

However note that Q_n has to satisfy

$$1 - P_n = Q_n (1 - \tilde{K})$$

and can be altered by adding a multiple of P_n. So therefore it should be possible to use $Q_n + c_n P_n$ for suitable constants c_n.

We have proved

$$Q_n \tilde{s}_n = \underbrace{N_n \tilde{s}_n}_{= nB = nP_{n-1}\tilde{s}_n} + \tilde{Q}_{n-1} \tilde{s}_n$$

mistake

$N_n \tilde{s}_n = nB$

$P_{n-1}\tilde{s}_n = P_n \tilde{s}_n$

$$\therefore \boxed{Q_n \tilde{s}_n = (Q_{n-1} + nP_{n-1}) \tilde{s}_n}$$

Propose:

$$1 - P_n = (Q_n + c_n P_n)(1 - \tilde{K})$$

$b\tilde{s}_n + \tilde{s}_{n+1} b$

$$= b\big(\underbrace{(Q_n + c_n P_n)\tilde{s}_n}_{h_n}\big) + \big(\underbrace{(Q_n + c_n P_n)\tilde{s}_{n+1}}_{h_{n+1}}\big) b$$

$$1 - P_{n-1} = b\big(\underbrace{(Q_{n-1} + c_{n-1}P_{n-1})\tilde{s}_{n-1}}_{h_{n-1}}\big) + \big(\underbrace{(Q_{n-1} + c_{n-1}P_{n-1})\tilde{s}_n}_{h_n}\big) b$$

Thus we want $(Q_n + c_n P_n)\tilde{s}_n = (Q_{n-1} + c_{n-1} P_{n-1}) \tilde{s}_n$

$$\| $$

$$(Q_{n-1} + (n + c_n) P_n) \tilde{s}_n$$

Thus $n + c_n = c_{n-1}$ $\qquad c_n = -\tfrac{1}{2}n(n+1) + \text{constant}$

$$\boxed{\begin{aligned} h_n &= Q_n \tilde{s}_n - nB_n = Q_{n-1}\tilde{s}_n - (n-1)B_n \\ &= \Big(-\frac{n}{2} N_n + \frac{1}{n}\sum_{r=0}^{n-1}\sum_{i=0}^{r-1} \lambda_n^i\Big)\tilde{s}_n \quad + \text{possibly } cB_n \text{ where } c \text{ ind of } n \end{aligned}}$$

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Circle action: On $\Omega(M)$ we have the operators $d, \iota_X, L_X, P = \int_0^1 e^{tL_X} dt$, $\pi_* = P\iota_X$

$\boxed{d^2 = \iota_X^2 = 0}$ $\boxed{[d, \iota_X] = L_X}$

Then $L_X \iota_X = (d\iota_X + \iota_X d)\iota_X = \iota_X d \iota_X = \iota_X L_X$ gives

$$\boxed{[L_X, d] = [L_X, \iota_X] = 0}$$

Integrating gives

$$\boxed{[P, d] = [P, \iota_X] = 0}$$

Next $L_X P = P L_X = \int_0^1 e^{tL_X} L_X dt = \int_0^1 \frac{d}{dt}(e^{tL_X}) dt$

$= \left[e^{tL_X}\right]_0^1 = 0.$

observe we get $\operatorname{Im} P = \operatorname{Ker} L_X$ as the inclusion $\supset$ is obvious

Thus $\boxed{L_X P = P L_X = 0}$ $\Rightarrow$ $\operatorname{Im} P \subset \operatorname{Ker} L_X$, $\operatorname{Im} L_X \subset \operatorname{Ker} P$

$e^{tL_X} P = P$ (as $\frac{d}{dt} e^{tL_X} P = e^{tL_X} L_X P = 0$)

so $\boxed{P^2 = P}$ $\Rightarrow$ $\operatorname{Ker} P = \operatorname{Im}(1-P)$, $\operatorname{Ker} 1-P = \operatorname{Im} P$

$$1 - P = \int_0^1 (1 - e^{tL_X}) dt = \int_0^1 (1 - e^{tL_X}) \frac{d}{dt}(t) dt$$

$$= \cancel{\left[(1-e^{tL_X})t\right]_0^1} - \int_0^1 (-e^{tL_X} L_X) t\, dt$$

Thus $\boxed{1 - P = Q L_X = L_X Q \qquad Q = \int_0^1 e^{tL_X} t\, dt}$

Note that Q is unique up to a multiple of P

$$\boxed{\begin{aligned} \operatorname{Ker} L_X &= \operatorname{Im} P \\ \operatorname{Ker} P &= \operatorname{Im} L_X \end{aligned}}$$

~~as $\operatorname{Ker} L_X \subseteq \operatorname{Ker}(1-P) = \operatorname{Im} P$~~

as $\operatorname{Ker} P = \operatorname{Im}(1-P) \subseteq \operatorname{Im} L_X$

~~from $1-P = QL_X$~~ from $1 - P = L_X Q$

Next

$$1-P = QL_X = Q(d\iota_X + \iota_X d)$$

$$\boxed{1-P = d(Q\iota_X) + (Q\iota_X)d}$$

shows that P <u>is homotopic to the identity</u>, and hence that the inclusion of invariant forms $\underset{\substack{\shortparallel \\ \operatorname{Ker} L_X}}{\Omega^{L_X}} \subset \Omega$ is a quis.

So far we have not used freeness of the action which would give $\operatorname{Ker}\iota_X = \operatorname{Im}\iota_X$. This suggests it is possible to work with normalized Hochschild cochains, treating the extra k in degree zero as fixpoint cohomology.

Let's next consider the discrete analogue arising in cyclic theory.

In general if we have a representation of $\mathbb{Z}/n$, then

$$P = \frac{1}{n}N = \frac{1}{n}\sum_{i=0}^{n-1}\lambda^i$$

is the projector onto the invariants. ($\operatorname{Ker}(1-\lambda) \subset \operatorname{Im} P$ clear and $(1-\lambda)P = 0$ (which uses $\lambda^n = 1$) gives $\operatorname{Im} P \subset \operatorname{Ker}(1-\lambda)$. Thus $\operatorname{Ker}(1-\lambda) = \operatorname{Im} P$ is easy.)

For the other: $\operatorname{Ker} P = \operatorname{Im}(1-\lambda)$ we write

$$1-P = \frac{1}{n}\sum_{i=0}^{n-1}(1-\lambda^i) = \underbrace{\left\{\frac{1}{n}\sum_{i=0}^{n-1}(1+\lambda+\cdots+\lambda^{i-1})\right\}}_{g}(1-\lambda)$$

This g can be changed by adding a multiple of N. The best thing is to take the operator which is 0 on $\operatorname{Im} P$ and the inverse of $1-\lambda$ on $\operatorname{Im}(1-\lambda) = \operatorname{Im}(1-P)$. This is

$$Q = (1-P)\, q$$

Now

$$q = \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\lambda^j = \frac{1}{n}\sum_{j=0}^{n-2}\lambda^j \sum_{j<i<n} 1$$

$$= \frac{1}{n}\sum_{j=0}^{n-2}\lambda^j (n-1-j)$$

or

$$q = \frac{1}{n}\sum_{i=0}^{n-1}(n-1-i)\lambda^i$$

Also

$$Pq = \frac{1}{n} N q = \frac{1}{n}\sum_{i=0}^{n-1}(n-1-i) N$$

$$= \frac{1}{n}\, n\, \frac{n-1}{2} N = \frac{n-1}{2} N$$

$$\therefore \quad \boxed{Q = \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{n-1}{2} - i\right)\lambda^i}$$

is a canonical choice for an operator satisfying $\boxed{1-P = Q(1-\lambda) = (1-\lambda)Q}$

Now let us return to the reduced cyclic formalism. In degree n we have

$$K^{n+1} = 1 - bs \qquad 1 - K^{n+1} = bs$$

$$1 - K^{n(n+1)} = \sum_{j=0}^{n-1}(K^{n+1})^j\, bs = b\sum_{j=0}^{n-1}\lambda_n^j s = bB$$

$$K^{n(n+1)} = 1 - bB$$

$$\tilde{K} \overset{\text{def}}{=} K\left(1 + \frac{1}{n(n+1)} bB\right)$$

$$= K + \frac{1}{n(n+1)} bB$$

$$\tilde{s} \overset{\text{defn}}{=} s + \left(1 - \frac{1}{n}\right) B$$

$$P = \frac{1}{n(n+1)} \sum_{j=0}^{n(n+1)-1} \tilde{K}^j$$

$$P\tilde{s} = \frac{1}{n(n+1)} \sum_{j=0}^{n(n+1)-1} \lambda_n^j \tilde{s} = \frac{1}{n} N \tilde{s}$$

$$= \frac{1}{n} Ns + \left(1 - \frac{1}{n}\right) B = B$$

Now I want to look at proof that $1-P$ is homotopic to zero, and to see if it simplifies by using the canonical Q.

Let's recall previous formulas:

$$P_n = \frac{1}{n(n+1)} \sum_{j=0}^{n(n+1)-1} \tilde{K}^j$$

$$Q_n = \frac{1}{n(n+1)} \sum_{j=0}^{n(n+1)-1} \sum_{i=0}^{j-1} \tilde{K}^i$$

These formulas apply operators in any degree, but P_n is a projector only on C^n.

$$1 - P_n = Q_n(1 - \tilde{K}) = Q_n(b\tilde{s} + \tilde{s}b)$$

$$= b\left(Q_n \tilde{s}\right) + \left(Q_n \tilde{s}\right) b$$

On C^n $\quad 1 - P_n = b_{n-1}\left(Q_n \tilde{s}_n\right) + \left(Q_n \tilde{s}_{n+1}\right) b_n$

" C^{n-1} $\quad 1 - P_{n-1} = b_{n-2}\left(Q_{n-1} \tilde{s}_{n-1}\right) + \left(Q_{n-1} \tilde{s}_n\right) b_{n-1}$

and to get a well-defined homotopy operator we need to get $Q_n \tilde{s}_n$ the same as $Q_{n-1} \tilde{s}_n$.

Recall also that we computed

$$Q_n \tilde{s}_n = \left(\frac{n}{2} N_n + \frac{1}{n}\sum_{r=0}^{n-1}\sum_{i=0}^{r-1} \lambda_n^i\right) \tilde{s}_n$$

$$Q_{n-1} \tilde{s}_n = \left(\frac{n-2}{2} N_n + \text{——}\right) \tilde{s}_n$$

Suppose $m \mid n$: $n = km$

Let P_n, Q_n denote the operators on any representation of $\mathbb{Z}/n\mathbb{Z}$ given by

$$Q_n = \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{n-1}{2} - i\right)\lambda^i$$

$$P_n = \frac{1}{n}\sum_{i=0}^{n-1}\lambda^i$$

Thus $(1-\lambda)Q_n = 1 - P_n$ and $P_n Q_n = 0$.

Suppose we look at a representation of $\mathbb{Z}/m\mathbb{Z}$ but view it as a representation of $\mathbb{Z}/n\mathbb{Z}$ via the canonical surjection. Then we can compare P_n, P_m and Q_n, Q_m. One has

$$P_n = \frac{1}{n}\sum_{q=0}^{k-1}\sum_{r=0}^{m-1}\lambda^{qm+r} = \underbrace{\frac{1}{n}k}_{1/m}\sum_{r=0}^{m-1}\lambda^r = P_m$$

$$\cancel{Q_n = \frac{n-1}{2}P_n - \frac{1}{n}\sum_{q=0}^{k-1}\sum_{r=0}^{m-1}(qm+r)\lambda^{qm+r}}$$

$$Q_n = \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{n-1}{2} - j\right)\lambda^j$$

$$= \frac{1}{km}\sum_{g=0}^{k-1}\sum_{r=0}^{m-1}\left(\underbrace{\frac{km-1}{2} - gm - r}\right)\lambda^{gm+r}$$

$$\frac{(k-1)m}{2} + \frac{m-1}{2} - gm - r$$

$$= \left(\frac{(k-1)}{2} - g\right)m + \left(\frac{m-1}{2} - r\right)$$

$$\therefore\ Q_n = \frac{1}{km}\sum_{g=0}^{k-1}\sum_{r=0}^{m-1}\left\{\left(\frac{k-1}{2} - g\right)m + \left(\frac{m-1}{2} - r\right)\right\}\lambda^r$$

$$= \frac{1}{k}\sum_{g=0}^{k-1}\left\{\left(\frac{k-1}{2} - g\right)mP_m + \underbrace{\frac{1}{m}\sum_{r=0}^{m-1}\left(\frac{m-1}{2} - r\right)\lambda^r}_{Q_m}\right\}$$

$$= \left\{\underbrace{\frac{1}{k}\sum_{g=0}^{k-1}\left(\frac{k-1}{2} - g\right)}_{0}\right\}mP_m + Q_m$$

$$\therefore\ Q_n = Q_m$$

This makes we feels stupid because we have characterized Q as the unique operator $= 0$ on $\operatorname{Im} P$ and $=$ the inverse of $1-\lambda$ on $\operatorname{Im}(1-P)$.

The same considerations apply to Q in the circle action case. It is unique if we require $QP = 0$ and to be an inverse for L_X on $\operatorname{Im}(1-P)$. Thus

$$\boxed{Q = \int_0^1 e^{tL_X}\left(t - \tfrac{1}{2}\right)dt}$$

is the good choice.

March 18, 1990

It seems that [redacted] a good viewpoint to [redacted] adopt is that we have an action of $\hat{\mathbb{Z}}$ on our complex of reduced cochains with 1 acting as the operator $\tilde{K}$. The operators P and Q are then intrinsically associated and given by

$$P = \frac{1}{n}\sum_{i=0}^{n-1} \tilde{K}^i \qquad Q = \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{n-1}{2}-i\right)\tilde{K}^i$$

on any invariant subspace on which $\tilde{K}^n = 1$.

Observation: Consider $\tilde{K}$ acting on

$$0 \longrightarrow sC^{n+1} \longrightarrow C^n \xrightarrow{s} sC^n \longrightarrow 0$$

$$\lambda_{n+1} \qquad\qquad \tilde{K} \qquad\qquad \lambda_n$$

~~[illegible]~~

This gives an exact sequence

$$0 \longrightarrow (sC^{n+1})^{\lambda_{n+1}} \longrightarrow (C^n)^{\tilde{K}} \longrightarrow (sC^n)^{\lambda_n} \longrightarrow 0$$

$$(sC^{n+1})^{\lambda_{n+1}} = (\operatorname{Im} B)^{n+1} \qquad (sC^n)^{\lambda_n} = (\operatorname{Im} B)^n$$

On $(C^n)^{\tilde{K}}$ one has $B = ns$ and $\tilde{s} = B$. Check:

~~[illegible]~~

$$\tilde{s} = s + \left(1-\frac{1}{n}\right)B = s + \left(1-\frac{1}{n}\right)ns = ns = B$$

By analogy with circle actions we want to use $B = \tilde{s}$ in the exact sequence. Thus we use

$$0 \longrightarrow \mathrm{Im}\, B \xrightarrow{I} C^{\tilde{K}} \xrightarrow{B = \tilde{s}} \mathrm{Im}\, B \longrightarrow 0$$

Now consider the analogue of a connection in a principal circle bundle, which gives a splitting of the corresponding sequence.

Choose $\rho : A \to k$, $\rho(1) = 1$. If $g_{n-1} \in sC^n$ is a completely reduced cochain, let

$$(\rho g_{n-1})(a_0, a_1, \dots, a_n) = \rho(a_0)\, g_{n-1}(a_1, \dots, a_n)$$

so that

$$\boxed{\begin{aligned} s(\rho g_{n-1}) &= g_{n-1} \\ B(\rho g_{n-1}) &= N g_{n-1} \end{aligned}}$$

Thus $P(\rho g_{n-1})$ is an invariant cochain such that $B P(\rho g_{n-1}) = B(\rho g_{n-1}) = N g_{n-1}$. Thus if $g_{n-1} \in \mathrm{Im}\, B$ is a cyclic cochain we have

$$B\left\{\frac{1}{n} P(\rho g_{n-1})\right\} = \frac{1}{n} N g_{n-1} = g_{n-1}.$$

<u>Thus</u> $g_{n-1} \longmapsto \frac{1}{n} P(\rho g_{n-1})$ <u>is a section of</u> $B : C^{\tilde{K}} \to \mathrm{Im}\, B$, <u>and so the</u> S-<u>operation is realized by the map</u>

$$g_{n-1} \longmapsto b \frac{1}{n} P(\rho g_{n-1}) - \frac{1}{n+1} P(\rho\, b g_{n-1})$$

Now let's work this out when $A = k \oplus \mathfrak{a}$ is augmented and $\rho : A \to k$ is the augmentation.

Review formulas in the augmented case. Given $f_n(\tilde{a}_0, a_1, \cdots, a_n) \in (A \otimes \bar{\mathcal{A}}^{\otimes n})^*$ let

$$\varphi_n(a_1, \cdots, a_n) = f_n(1, a_1, \cdots, a_n) \qquad a_i \in \bar{\mathcal{A}}$$

$$\psi_{n+1}(a_0, \cdots, a_n) = f_n(a_0, \cdots, a_n)$$

and write $f_n = (\psi_{n+1}, \varphi_n)$. Then

$$\boxed{\begin{aligned} b(\psi_{n+1}, \varphi_n) &= (b\psi_{n+1}, (1-\lambda)\psi_{n+1} - b'\varphi_n) \\ s(\psi_{n+1}, \varphi_n) &= (\varphi_n, 0) \\ B(\psi_{n+1}, \varphi_n) &= (N\varphi_n, 0) \end{aligned}}$$

$$bs(\psi_{n+1}, \varphi_n) = (b\varphi_n, (1-\lambda)\varphi_n)$$

$$sb(\psi_{n+1}, \varphi_n) = ((1-\lambda)\psi_{n+1} - b'\varphi_n, 0)$$

$$(bs+sb)(\psi_{n+1}, \varphi_n) = ((1-\lambda)\psi_{n+1} + (b-b')\varphi_n, (1-\lambda)\varphi_n)$$

$$\boxed{\kappa(\psi_{n+1}, \varphi_n) = (\lambda\psi_{n+1} - (b-b')\varphi_n, \lambda\varphi_n)}$$

Set $-c = b - b'$ = cross over term:

$$(-c)\varphi_n(a_0, \cdots, a_n) = (-1)^n \varphi_n(a_n a_0, a_1, \cdots, a_{n-1})$$

Putting $f_n = (\psi_{n+1}, \varphi_n)$ we have

$$\kappa f_n = (\lambda\psi_{n+1} + c\varphi_n, \lambda\varphi_n)$$

$$\kappa^2 f_n = (\lambda^2\psi_{n+1} + (\lambda c + c\lambda)\varphi_n, \lambda^2\varphi_n)$$

$$\kappa^3 f_n = (\lambda^3\psi_{n+1} + (\lambda^2 c + \lambda c\lambda + c\lambda^2)\varphi_n, \lambda^3\varphi_n)$$

and in general

$$\boxed{K^j f_n = \left(\lambda^j \varphi_{n+1} + (\lambda^{j-1} c + \lambda^{j-2} c\lambda + \dots + c\lambda^{j-1})\varphi_n, \lambda^j \varphi_n\right)}$$

Suppose now that φ_n is a cyclic $(n-1)$-cocycle: $\lambda\varphi_n = \varphi_n$. One has

$$\rho\varphi_n = (0, \varphi_n)$$

for any $\varphi_n \in (\mathcal{A}^{\otimes n})^*$. We want to compute $P(\rho\varphi_n)$. From $\lambda\varphi_n = \varphi_n$ one has (with $f_n = \rho\varphi_n$)

$$K^j f_n = \left((\lambda^{j-1} + \lambda^{j-2} + \dots + 1)c\varphi_n, \cancel{\lambda^j}\varphi_n\right)$$

$$K^{n+1} f_n = (Nc\varphi_n, \varphi_n) = (-b\varphi_n, \varphi_n)$$

$$\tilde{K}^{n+1} f_n = \left(K^{n+1} + \frac{1}{n} bB\right) f_n \qquad \cancel{Bf_n = N\varphi_n = n\varphi_n}$$

$$\frac{1}{n} bBf_n = \frac{1}{n} bB(0, \varphi_n) = \frac{1}{n} b(N\varphi_n, 0)$$
$$= \frac{1}{n}(bN\varphi_n, \cancel{(1-\lambda)}N\varphi_n) = (b\varphi_n, 0)$$

$\therefore \tilde{K}^{n+1} f_n = f_n$, so $Pf_n = \frac{1}{n+1}\sum_0^n \tilde{K}^j f_n$

$$= \left(\frac{1}{n+1}\sum_{j=0}^{n}\sum_{i=0}^{j-1}\lambda^i c\varphi_n, \varphi_n\right)$$
$$+ \frac{1}{n+1}\sum_{j=0}^{n}\frac{j}{n(n+1)} bB f_n$$

Better. $$Pf_n = \frac{1}{n+1}\sum_{j=0}^{n}\tilde{K}^j f_n = \frac{1}{n+1}\sum_{j=0}^{n}\left(K^j + \frac{j}{n(n+1)} bB\right) f_n$$

$$= \frac{1}{n+1}\sum_{j=0}^{n}\left(\left(\sum_{i=0}^{j-1}\lambda^i c\varphi_n, \varphi_n\right) + \frac{j}{n+1}(b\varphi_n, 0)\right)$$

$$= \left(\frac{1}{n+1}\sum_{j=0}^{n}(n-j)\lambda^j c\varphi_n, \varphi_n\right)$$

$$+ \left(\frac{1}{n+1}\frac{n}{2}\underbrace{b\varphi_n}_{-\sum_{j=0}^{n}\lambda^j c\varphi_n}, 0\right)$$

$$= \left(\frac{1}{n+1}\sum_{j=0}^{n}\left(\frac{n}{2}-j\right)\lambda^j c\varphi_n, \varphi_n\right)$$

$$= (Qc\varphi_n, \varphi_n)$$

Conclusion is that $\boxed{\text{if } \lambda\varphi_n = \varphi_n \text{ then}}$

$$\boxed{P(\rho\varphi_n) = (Qc\varphi_n, \varphi_n) \qquad c = -(b-b')}$$

Note: Q here is $Q(\lambda)$

Check that $(Qc\varphi_n, \varphi_n)$ is $\tilde{K}$-invariant

$$\tilde{K}(Qc\varphi_n, \varphi_n) = K(Qc\varphi_n, \varphi_n) + \frac{1}{n(n+1)}bB(Qc\varphi_n, \varphi_n)$$

$$= (\lambda Qc\varphi_n + c\varphi_n, \underbrace{\lambda\varphi_n}_{\varphi_n}) + \frac{1}{n(n+1)}b(\underbrace{N\varphi_n}_{n\varphi_n}, 0)$$

$$= (Qc\varphi_n, \varphi_n) + (\underbrace{(\lambda-1)Qc\varphi_n}_{(P-1)c\varphi_n} + c\varphi_n, 0) + \frac{1}{n+1}(b\varphi_n, 0)$$

$$= (Qc\varphi_n, \varphi_n) + (\underbrace{Pc\varphi_n}_{\frac{1}{n+1}\underbrace{Nc\varphi_n}_{-b\varphi_n}} + \frac{1}{n+1}b\varphi_n, 0)$$

OKAY.

Improvement: Use

$$Qc\varphi = Q(b'-b)\varphi = Qb'\varphi$$

when φ cyclic, since $b\varphi$ is cyclic. Thus the lift of a cyclic cochain φ_n to a $\tilde{K}$-invariant normalized cochain is

$$\varphi_n \mapsto \frac{1}{n}\varphi_n \longmapsto \frac{1}{n}P(\rho\varphi_n) = \frac{1}{n}(Qb'\varphi_n, \varphi_n).$$

Now it is clear what one is doing is the following. One has the part of the cyclic double complex

$$\begin{tikzcd}
 & {} & {} & {} & {} & \\
0 \arrow[r] & \bar{C}_\lambda \arrow[u] \arrow[r, "I"] & \cdot \arrow[u] \arrow[l, bend right, "P"'] \arrow[r, "1-\lambda"] & \cdot & {} & \\
0 \arrow[r] & \bar{C}_\lambda \arrow[u] \arrow[r, "I"] & \cdot \arrow[u, "b"] \arrow[r, "1-\lambda"] & \cdot \arrow[u, "b'"] \arrow[l, bend right, "{[\frac{1}{1-\lambda}]}"'] \arrow[r, two heads, "N"] & \bar{C}_\lambda \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & C_\lambda \arrow[u] \arrow[r, "I"] & \cdot \arrow[u] \arrow[r, "1-\lambda"] & \cdot \arrow[u, "b'"] \arrow[r, two heads, "N"] & C_\lambda \arrow[u] \arrow[l, bend right, "{[\frac{1}{N}]}"'] \arrow[r] & 0 \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

To obtain an explicit S-operation one chooses horizontal homotopies h, better one splits the horizontal sequences and gets a horizontal homotopy h such that $h^2=0$. This horizontal homotopy which is natural to use consists of P, $Q=\left[\frac{1}{1-\lambda}\right]$, $\left[\frac{1}{N}\right]$.

Here $\left[\frac{1}{N}\right]\varphi_n = \frac{1}{n}\varphi_n$ if φ_n is cyclic.

The S operator is realized by the following process. First to get things clear look at a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "d"] & Y \arrow[r, "d"] \arrow[l, bend right, "h"'] & Z \arrow[r] \arrow[l, bend right, "h"'] & 0 \end{tikzcd}$$

and let δ denote the vertical differentials. Then $\delta h - h\delta : Z \to Y$ satisfies

$$d(\delta h - h\delta) = \delta(dh) - (dh)\delta = \delta - \delta = 0$$

and so $\delta h - h\delta$ lands in $\mathrm{Ker}\{d : Y \to Z\} = \mathrm{Im}\{X \xrightarrow{d} Y\}$. Since h projects onto this image the map

$$h(\delta h - h\delta) = h\delta h : Z \to X$$

is a map of complex realizing the connecting map $H^i(Z) \to H^{i+1}(X)$. More generally given

$$0 \to X_0 \to \cdots\cdots \to X_n \to 0$$

the map of complexes we are after is

$$h[\delta, h]^n = h(\delta h - \cancel{h\delta})[\delta, h]^{n-1} = \cdots = h(\delta h)^n$$

assuming $h^2 = 0$.

Applying this in the case of the cyclic situation gives the map

$$Pb\left[\frac{1}{1-\lambda}\right] b'\left[\frac{1}{N}\right]$$

realizing the S-operation on the cyclic cochain complex.

Let's continue with our checking. We have found the natural lift of a cyclic cochain φ to a $\tilde{K}$-invariant cochain to be

$$P\left(\rho \tfrac{1}{n}\varphi\right) = \tfrac{1}{n}(Qb'\varphi, \varphi)$$

if $\varphi = \varphi_n$ is an $(n-1)$-cyclic cochain.

Idea: We know $C^{\tilde{K}}$ is an extension of cyclic complexes and we have the above ~~splitting~~, so $C^{\tilde{K}}$ is the mapping cone ~~for~~ the map $[\delta, h]$ from the quotient complex to the subcomplex. We compute $[\delta, h]$.

$$\frac{1}{n} b(Qb'\varphi, \varphi) + \frac{1}{n+1}(Qb'b\varphi, b\varphi)$$

(This sign is due to B anti commuting with b, so the quotient complex has differential $-b$.)

$$= \frac{1}{n}\Big(bQb'\varphi, \underbrace{(1-\lambda)Qb'\varphi}_{(1-P)b'\varphi} - b'\varphi\Big) + \frac{1}{n+1}(Qb'b\varphi, b\varphi)$$

$$= \left(\frac{1}{n} bQb'\varphi + \frac{1}{n+1} Qb'b\varphi, \; -\frac{1}{n(n+1)} \overset{bN\varphi = bn\varphi}{Nb'\varphi} + \frac{1}{n+1} b\varphi\right)$$

$$= \left(\frac{1}{n} bQb'\varphi + \frac{1}{n+1} Qb'b\varphi, \; 0\right)$$

$$= \left(b\left(Qb'\tfrac{1}{n}\right)\varphi + \left(Qb'\tfrac{1}{n+1}\right)b\varphi, \; 0\right)$$

We want $h[\delta, h] = h\delta h$ which means apply P. This gives the map realizing the

$$PbQb'\frac{1}{n}\varphi = \frac{1}{n+2}NbQb'\frac{1}{n}\varphi$$

which agrees with Kassel's formula if one notes that [redacted]

$$Q = \frac{(1-\lambda)D^2}{(n+1)^2} \qquad \text{for } \mathbb{Z}/n+1$$

for his D. His $\frac{-D}{n+1}$ satisfies

$$(1-\lambda)\left(\frac{-D}{n+1}\right) = 1-P$$

and I know Q can be obtained by applying $1-P$ to anything X satisfy $(1-\lambda)X = 1-P$. Thus

$$Q = (1-P)\left(\frac{-D}{n+1}\right) = \frac{(1-\lambda)D^2}{(n+1)^2}$$

Notation? $\quad Q = (1-P)(1-\lambda)^{-1}(1-P)$

March 20, 1990

Consider the periodic complex

$$\longrightarrow \bar{Q} \xrightarrow{\text{[redacted]}\, d} (\Omega^1 Q)_\natural \xrightarrow{\beta} \bar{Q} \longrightarrow$$

appropriate to the superalgebra structure on $Q = QA$. This means $\beta(x\,dy) = +[x,y]$ where $[x,y]$ is the superbracket. Let's review this

$\partial y = y\otimes 1 - 1\otimes y$ (notation conflicts with cochain paper.) [redacted]

$$\begin{tikzcd}
Q^{\otimes 3} \arrow[r] \arrow[d] \arrow[rr, bend left, "b'"] & \Omega^1 Q \arrow[r, hook, "\tilde{\partial}"] \arrow[d] & Q^{\otimes 2} \arrow[d] \\
Q^{\otimes 2} \arrow[r] & (\Omega^1 Q)_\natural \arrow[r, "\beta"] & Q
\end{tikzcd}$$

$$\begin{tikzcd}
x\otimes y\otimes 1 \arrow[r, mapsto] \arrow[d, mapsto] & x\,dy \arrow[r, mapsto] \arrow[d, mapsto] & xy\otimes 1 - x\otimes y \arrow[d, mapsto] \\
x\otimes y \arrow[r, mapsto] & x\,dy \ \text{[redacted]} & xy - (-1)^{|x||y|} yx
\end{tikzcd}$$

Let's see the effect of the maps on the periodic complex [redacted] on the invariant cochains associated to supertrace. Consider even supertraces first. Actually I should have said linear fuls on $\bar{Q}$ and $(\Omega^1 Q)_\natural$.

Let τ be an even linear ful on $\bar{Q}$ and T an even linear ful on $(\Omega^1 Q)_\natural$. Then we have

$$Td(\rho q^{2n}) = -\tfrac{1}{2}\,\text{[redacted]} \sum_{j=0}^{2n-1} \kappa^j b T(\theta q^{2n-2} d\bar{\theta}) + B\,T(\theta q^{2n} d\bar{\theta})$$

$$\tau\tilde{\partial}(\theta q^{2n} d\bar{\theta}) = b\tau(\rho q^{2n}) - 2s\,\tau(\rho q^{2n+2})$$

Applying P gives

$$Td(\rho q^{2n}) = -\frac{2n}{2}\, b\, PT(\theta q^{2n-2}d\bar\theta) + B\, PT(\theta q^{2n}d\bar\theta)$$

or

$$\boxed{Td\left(\frac{\rho q^{2n}}{n!}\right) = -b\left\{PT\left(\frac{\theta q^{2n-2}d\bar\theta}{(n-1)!}\right)\right\} + B\left\{PT\left(\frac{\theta q^{2n}d\bar\theta}{n!}\right)\right\}}$$

$$P\left(\tau\tilde\partial(\theta q^{2n}d\bar\theta)\right) = b\{P\tau(\rho q^{2n})\} - 2\underbrace{Ps}_{\frac{1}{2n+2}B = \frac{1}{2n+2}BP}\tau(\rho q^{2n+2})$$

$$= b\{P\tau(\rho q^{2n})\} - \frac{1}{n+2}B\{P\tau(\rho q^{2n+2})\}$$

or

$$\boxed{P\left(\tau\tilde\partial\left(\frac{\theta q^{2n}d\bar\theta}{n!}\right)\right) = b\left\{P\tau\left(\frac{\rho q^{2n}}{n!}\right)\right\} - B\left\{P\tau\left(\rho\frac{q^{2n+2}}{(n+1)!}\right)\right\}}$$

Consider next odd linear functionals τ, T. We have

$$Td(\rho q^{2n+1}) = \frac{1}{2}\sum_{j=0}^{2n} K^j\, bT(\theta q^{2n-1}d\theta) - BT(\theta q^{2n+1}d\theta)$$

$$\tau\tilde\partial(\theta q^{2n-1}d\theta) = -b\tau(\rho q^{2n-1}) + 2s\,\tau(\rho q^{2n+1})$$

Applying P gives

$$Td(\rho q^{2n+1}) = (n+\tfrac{1}{2})\, b\, PT(\theta q^{2n-1}d\theta) - B\, PT(\theta q^{2n+1}d\theta)$$

$$P\left(\tau\tilde\partial(\theta q^{2n-1}d\theta)\right) = b\{P\tau(\rho q^{2n-1})\} + \frac{2}{2n+1}B\{P\tau(\rho q^{2n+1})\}$$

or

$$\boxed{\begin{aligned} Td\left(\rho\frac{q^{2n+1}}{(n+\frac12)!}\right) &= b\left\{PT\left(\theta\frac{q^{2n-1}}{(n-\frac12)!}d\theta\right)\right\} - B\left\{PT\left(\theta\frac{q^{2n+1}}{(n+\frac12)!}d\theta\right)\right\} \\ P\left\{\tau\tilde\partial\left(\theta\frac{q^{2n-1}}{(n-\frac12)!}d\theta\right)\right\} &= -b\left\{P\tau\left(\rho\frac{q^{2n-1}}{(n-\frac12)!}\right)\right\} + B\left\{P\tau\left(\rho\frac{q^{2n+1}}{(n+\frac12)!}\right)\right\}\end{aligned}}$$

where $(n+\frac{1}{2})! = \Gamma(n+\frac{3}{2})$

Problem: Consider the periodic complexes

$$\longrightarrow \bar{R} \xrightarrow{d} (\Omega^1 R)_\natural \xrightarrow{\beta} \bar{R} \longrightarrow$$

$$\longrightarrow \bar{Q} \xrightarrow{d} (\Omega^1 Q)_\natural \xrightarrow{\beta = (\tilde{\partial})_\natural} \bar{Q} \longrightarrow$$

and take "continuous" linear functionals, where continuous means vanishing on F_I^m, F_J^m for large m. These $\mathbb{Z}/2$-graded complexes should give the periodic reduced cyclic cohomogy of A.

We know the top sequence is exact, so another problem is to find the homology of the bottom sequence.

The idea is to use the explicit formulas to calculate what happens to the linear functionals. In the R-case we have the formulas

$$Td\left(\rho\frac{\omega^n}{n!}\right) = \underbrace{-\frac{1}{n}\sum_{j=0}^{n-1}\tilde{\kappa}^{2j}}_{P(\tilde{\kappa}^2)}\, bT\left(\rho\frac{\omega^{n-1}}{(n-1)!}d\rho\right) + BT\left(\rho\frac{\omega^n}{n!}d\rho\right)$$

$$\tau\beta\left(\rho\frac{\omega^n}{n!}d\rho\right) = b\tau\left(\rho\frac{\omega^n}{n!}\right) - (n+1)(1+\lambda)s\tau\left(\rho\frac{\omega^{n+1}}{(n+1)!}\right)$$

so [redacted] we have

$$\boxed{\begin{aligned} (d^t g)_{2n} &= -P(\tilde{\kappa}^2)\, b g_{2n-1} + B g_{2n+1} \\ (\beta^t f)_{2n+1} &= b f_{2n} - (n+1)(1+\lambda) s f_{2n+2} \\ f_{2n} = \tau\left(\rho\frac{\omega^n}{n!}\right) &\qquad g_{2n+1} = T\left(\rho\frac{\omega^n}{n!}d\rho\right) \end{aligned}}$$

In the even Q-case we have

$$Td\left(\rho\frac{g^{2n}}{n!}\right) = -\frac{1}{2n}\sum_{j=0}^{2n-1}\tilde{\kappa}^j\, bT\left(\theta\frac{g^{2n-2}}{(n-1)!}d\bar\theta\right) + BT\left(\theta\frac{g^{2n}}{n!}d\bar\theta\right)$$

$$\tau\tilde\partial\left(\theta\frac{g^{2n}}{n!}d\theta\right) = b\tau\left(\rho\frac{g^{2n}}{n!}\right) - 2(n+1)s\,\tau\left(\rho\frac{g^{2n+2}}{(n+1)!}\right)$$

so we have

$$\boxed{\begin{aligned} &(d^t g)_{2n} = -Pbg_{2n-1} + Bg_{2n+1} \\ &(\tilde\partial^t f)_{2n+1} = bf_{2n} - 2(n+1)s\,f_{2n+2} \\ &f_{2n} = \tau\left(\rho\frac{g^{2n}}{n!}\right) \qquad g_{2n+1} = T\left(\theta\frac{g^{2n}}{n!}d\bar\theta\right)\end{aligned}}$$

The idea is as follows. P is a projection operator on these complexes, and we have seen that the image of P coincides with the periodic complex of invariant cochains. So our problem is to show $\operatorname{Ker} P$ has zero cohomology, say by showing $1-P$ is nullhomotopic.

Let us return to an S^1-manifold M and review Bismut's construction. E vector bundle over M with ∇ but not equivariant

$$(-\nabla + u\iota_X)^2 = -u\nabla_X + \nabla^2$$

$$(-d + u\iota_X)\,\mathrm{tr}\left(e^{-\nabla_X + u^{-1}\nabla^2}\right) = 0$$

So if

$$\mathrm{tr}\left(e^{-\nabla_X + u^{-1}\nabla^2}\right) = \omega_0 + u^{-1}\omega_2 + u^{-2}\omega_4 + \cdots$$

one has $\boxed{d\omega_{2n} = \iota_X \omega_{2n+2}}$. Note that since $(-d+u\iota_X)^2 = -uL_X$, it follows that the ω_{2n} are invariant forms.

Now $\iota_X \omega_{2n+2} = \pi_* \omega_{2n+2}$ in the case of a free action, so we obtain odd closed forms on the base.

Let's consider the DR class of $\iota_X \omega_{2n+2}$. Since ω_0 is an invariant function it is basic, so $d\omega_0 = \iota_X \omega_2$ shows $[\iota_X\omega_2] = 0$. Fix a connection A in the principal S^1-bundle $M \to M/S^1$. This splits the exact sequence

$$0 \longrightarrow \Omega_{bas} \xrightarrow{\ I\ } \Omega_{inv} \xrightarrow{\pi_* = \iota_X} \Omega_{bas} \longrightarrow 0$$

and it should allow us to write any ω killed by $-d+\iota_X$ as $(-d+\iota_X)\eta$ in a canonical way. Let's take this in stages.

First, if $\omega_{2n+2} = -d\eta_{2n+1} + \iota_X \eta_{2n+3}$, then $\iota_X\omega_{2n+2} = -\iota_X d\eta_{2n+1} = d\iota_X\eta_{2n+1}$, and $\iota_X\eta_{2n+1}$ is basic, showing that $[\iota_X\omega_{2n+2}] = 0$.

Secondly, if A is a connection, one has

$$(-d+\iota_X)(A\omega) + A(-d+\iota_X)\omega$$

$$= (1-dA)\omega$$

so one gets the

contracting homotopy

$$h\omega = \frac{1}{1-dA} A\omega$$

Let's review the explicit calculation of the space of invariant cochains in the augmented case.

Let's proceed generally

Claim: If $f \in C^n$ and $\lambda s f = sf$, then $\tilde{K}^{n+1} f = f$.

Abstract proof. Decompose C^n into irreducibles for the action of $\tilde{K}$. Nontrivial irreducibles occurring embed either in sC^{n+1}, where $\tilde{K}^{n+1} = 1$, or sC^n, where $\tilde{K}^n = 1$. Since $n, n+1$ are relative prime both can't happen. The subspace of f such that $\lambda s f = sf$ is the direct sum of the invariant subspace and the irreducibles embedding in sC^{n+1}. Thus $\tilde{K}^{n+1} = 1$ on this subspace.

Formula proof.

$$\tilde{K}^{n+1} f = K^{n+1} f + \frac{\cancel{n+1}}{n(\cancel{n+1})}\, b\overbrace{Bf}^{Nsf = nsf}$$

$$= (1-bs)f + bsf = f.$$

Suppose $\lambda s f = sf$. Then

$$Pf = f - (1-P)f$$

$$= f - Q(1-\tilde{K})f$$

$$= f - Q(1-K)f \qquad \text{because } \tilde{K} - K = \tfrac{1}{n(n+1)} bB \text{ is invariant } \therefore Q\tilde{K} = QK$$

$$= f - Q(sb+bs)f$$

$$= f - \underset{Q(A)}{\underset{\shortparallel}{Q}}\, sbf \qquad \text{because } sf \text{ cyclic} \Rightarrow bsf \text{ cyclic} \therefore \text{inv.}$$

In particular if we take $f=(0,\varphi)$ with $\lambda\varphi=\varphi$, we have

$$\begin{aligned} Pf &= (0,\varphi) - Q(\lambda)s(0,-b'\varphi) \\ &= (0,\varphi) + (Q(\lambda)b'\varphi, 0) \\ &= (+Q(\lambda)b'\varphi, \varphi) \end{aligned}$$

Another general ~~point~~ is that $\tilde{K} = K$ on any subspace where $\tilde{K}$ has finite order. Besides $\boxed{Q\tilde{K} = QK}$ we have $\boxed{Q\tilde{s} = Qs}$ since $\operatorname{Im}(\tilde{s} - s = (1-\frac{1}{n})B)$ is invariant.

Other facts:

$$\forall n \qquad bf_{2n} = (1+\lambda)sf_{2n+2} \implies (1-K^2)f_{2n} = (bs+sb)(1+K)f_{2n}$$
$$= b^2 f_{2n-2} + s(1+K)(1+K)sf_{2n+2}$$
$$= 0$$

Thus $bf_{2n} = (1+\lambda)sf_{2n+2} \iff \begin{cases} (1-K^2)f_{2n} = 0 \\ bf_{2n} = \frac{1}{n}Bf_{2n+2} \end{cases}$

Similarly
$$bf_n = 2sf_{n+2} \iff \begin{cases} (1-K)f_n = 0 \\ bf_n = \frac{2}{n+2}Bf_{n+2} \end{cases}$$

Use notation G instead of Q:

$$1 = P + (1-\tilde{K})G = P + G(1-\tilde{K})$$

March 21, 1990

Questions + Ideas:

1) $R = RA$ has many symmetries. Any derivation of R operates on the periodic complex $\to \bar{R} \to (\Omega^1 R)_\natural \to \cdots$ and hence on cochains in a fashion compatible with the funny differential on p300. Perhaps the most important symmetries come from the affine group of transformations of A which fix 1 and induce scalar transformations on $\bar{A}$. Splitting $0 \to k \to A \to \bar{A} \to 0$ gives an action of G_m. This is related to Chern-Simons forms.

Is there any analogy with Virasoro?

2) Equivariant cohomology for the circle action on $L(BU)$ and Bismut's forms. A concrete question: View the cohomology of $L(BU)$ as giving characteristic classes for nonequivariant vector bundles over manifolds with circle action. Choosing a connection we get a monodromy transformation of the bundle, thus we have a bundle with automorphism and there are even + odd character classes. When the circle action is free we can produce odd + even classes on the quotient space by integrating the even, odd character classes over the fibre. How are these all related?

Construction of big cocycles analogous to Bismut forms.

Isomorphisms of periodic cxs. — Artin-Rees property.

March 23, 1990

Let us consider the exact sequence

$$0 \to sC^{n+1} \to C^n \xrightarrow{\tilde{s}} sC^{n} \to 0$$

analogous to

$$0 \to \Omega_{hor} \to \Omega \xrightarrow{\iota_X} \Omega_{hor} \to 0$$

associated to a principal circle bundle. Given a connection in the latter - this is a connection form A: $\iota_X A = 1$, $L_X A = 0$ - one obtain an invariant splitting of the second exact sequence given by $\omega \mapsto A\omega$. Hence one obtains an operator ∇ on Ω_{hor} by lifting, applying d, and then projecting:

$$\begin{aligned}\omega \mapsto \quad \iota_X d(A\omega) &= \iota_X(dA\omega - A\,d\omega)\\ &= \iota_X(dA)\,\omega + dA(\iota_X\omega)\\ &\quad - \iota_X(A)\,d\omega + A\,\iota_X d\omega\\ &= -d\omega + A\,\iota_X d\omega\end{aligned}$$

We've got the wrong sign because of the degree shift caused by ι_X, A. Thus ∇ is really

$$\nabla\omega = d\omega - A\iota_X d\omega = (d - AL_X)\omega$$

In our cochain situation let us choose $\rho: A \to k$ such that $\rho(1) = 1$. This gives a splitting of the exact sequence

$$\to C^{n+1} \xrightarrow{s} C^n \xrightarrow{s} C^{n-1} \xrightarrow{s}$$

as follows. First we split

$$0 \to sC \to C \xrightarrow{s} sC \to 0$$

by the lifting which takes a completely reduced φ to

$$(\rho\varphi)(a_0, \ldots, a_n) = \rho(a_0)\varphi(a_1, \ldots, a_n)$$

Then given $f \in \mathcal{C}$, one has

$$s(f - \rho s f) = sf - sf = 0 \quad \text{so} \quad f - \rho s f \in \mathcal{C}$$

and we can define

$$h(f) = \rho(f - \rho s f)$$

Then
$$shf = f - \rho s f$$
$$hsf = \rho(sf - \rho s s f) = \rho s f$$

so
$$sh + hs = id$$

Next to the exact sequence

$$\circledast \quad \xrightarrow{\tilde{s}} \mathcal{C}^{n+1} \xrightarrow{\tilde{s}} \mathcal{C}^n \xrightarrow{\tilde{s}} \mathcal{C}^{n-1} \xrightarrow{\tilde{s}}$$

where
$$\tilde{S} = s + (1 - \tfrac{1}{n})B = \left(1 + (1 - \tfrac{1}{n})nP\right)s$$

$$\boxed{\tilde{s} = (1 - P + nP)s} \quad \text{on } \mathcal{C}^n$$

we use $\boxed{\tilde{\rho} = \rho(1 - P + \tfrac{1}{n}P)}$ instead

and let $\tilde{h}f = \tilde{\rho}(f - \tilde{\rho}\tilde{s}f)$. Note $\tilde{\rho}\tilde{s} = \rho s$,

so $\boxed{\tilde{h}f = \tilde{\rho}(f - \rho s f)}$

Now that we have a splitting of $\circledast$ we can make it invariant by averaging

$$\frac{1}{M} \sum_{j=0}^{M} \tilde{K}^j \tilde{h} \tilde{K}^{-j}$$

The problem is then to find ∇.

~~Another way to proceed is this~~

Actually this h business is not essential. We just need the lifting $\tilde{\rho}$ averaged to find ∇.

Thus we look at

$$\frac{1}{n(n+1)}\sum_{j=0}^{n(n+1)} \tilde{K}^j \tilde{\rho}(\lambda_n^{-j}\varphi) = \frac{1}{n(n+1)}\sum_{r=0}^{n-1}\sum_{q=0}^{n} \tilde{K}^{qn+r}\tilde{\rho}(\lambda^{-r}\varphi)$$

$$= \frac{1}{n}\sum_{r=0}^{n-1}\tilde{K}^r\left\{\frac{1}{n+1}\sum_{q=0}^{n}\tilde{K}^{qn}\right\}\tilde{\rho}(\lambda^{-r}\varphi)$$

Lets now calculate $\frac{1}{n+1}\sum_{q=0}^{n}\tilde{K}^{qn}$ which kills the nontrivial representations of $\mathbb{Z}/n+1$.

$$K^n = 1+\lambda_{n+1}^{-1}sb \qquad = 1+\lambda^n sb$$

$$K^{2n} = K^n + \lambda_{n+1}^{n}\lambda_{n+1}^{-1}sb = 1+(\lambda^{n-1}+\lambda^n)sb$$

$$\cdot$$

$$K^{qn} = 1+(\lambda^{n-q+1}+\cdots+\lambda^n)sb$$

$$K^{n^2} = 1+(\lambda+\cdots+\lambda^n)sb$$

$$\frac{1}{n+1}\sum_{q=0}^{n}K^{qn} = 1+\frac{1}{n+1}\sum_{j=0}^{n} j\lambda^j sb$$

$$\frac{1}{n+1}\sum_{q=0}^{n}\tilde{K}^{qn} = \frac{1}{n+1}\sum_{0}^{n}\left(K^{qn}+\frac{qA}{n(n+1)}bB\right)$$

$$= 1+\frac{1}{n+1}\sum_{j=0}^{n} j\lambda^j sb + \frac{1}{(n+1)\cancel{n}}\frac{n\cancel{(n+1)}}{2}(-Nsb)$$

$$= 1+\frac{1}{n+1}\sum_{j=0}^{n}\left(j-\frac{n}{2}\right)\lambda^j sb = 1-G(\lambda)sb$$

$$\therefore\quad \boxed{\frac{1}{n+1}\sum_{q=0}^{n}\tilde{K}^{qn} = 1-G(\lambda)sb \quad \text{on } C^n}$$

Let's recall that $G\tilde{s} = Gs$ and

$$1-P = G(1-\tilde{K}) = G(b\tilde{s}+\tilde{s}b) = b(G\tilde{s})+(G\tilde{s})b$$

Thus we have $1 - Gsb = P + bG\tilde{s}$

$$\frac{1}{n+1}\sum_{q=0}^{n} \tilde{\kappa}^{qn} = P + bG\tilde{s}$$

So our lifting of $\varphi \in sC^n$ is

$$A(\varphi) = \frac{1}{n}\sum_{r=0}^{n-1} \tilde{\kappa}^{r}\left(P + bG\tilde{s}\right)\tilde{\rho}(\lambda^{-r}\varphi)$$

Now $\tilde{\kappa}^r P = P$ and $\tilde{s}\tilde{\rho} = id$ on sC^n, so

$$A(\varphi) = \frac{1}{n}\sum_{r=0}^{n-1} P\tilde{\rho}(\lambda^{-r}\varphi) + \frac{1}{n}\sum_{r=0}^{n-1} \tilde{\kappa}^{r} bG\lambda^{-r}\varphi$$

$$\boxed{A(\varphi) = P\tilde{\rho}P\varphi + bG\varphi}$$

This looks suspicious because we should have

$$\nabla\varphi = -\tilde{s}bA(\varphi) = -\tilde{s}bP\tilde{\rho}P\varphi = b\tilde{s}P\tilde{\rho}P\varphi = bP\varphi$$

and this depends only on $P\varphi$. However, note:

$$\begin{aligned}
\tilde{s}A(\varphi) &= \tilde{s}P\tilde{\rho}P\varphi + \tilde{s}bG\varphi \\
&= P\tilde{s}\tilde{\rho}P\varphi + (1 - \tilde{\kappa} - \cancel{b\tilde{s}})G\varphi \\
&= P\varphi + (1-P)\varphi = \varphi
\end{aligned}$$

$$\begin{aligned}
A(\tilde{\kappa}\varphi) &= P\tilde{\rho}P\tilde{\kappa}\varphi + bG\tilde{\kappa}\varphi \\
&= P\rho P\varphi + \tilde{\kappa}bG\varphi \\
&= \tilde{\kappa}\left(P\rho P\varphi + bG\varphi\right) = \tilde{\kappa}A(\varphi)
\end{aligned}$$

Thus it is an invariant lifting.

Let's check this.

$$\nabla\varphi = -\tilde{s}bA(\varphi)$$
$$= -\tilde{s}b\{P\tilde{\rho}P\varphi + \cancel{bG\varphi}\}$$
$$= -(\cancel{1-\tilde{\kappa}} - b\tilde{s})P\tilde{\rho}P\varphi$$
$$= bP\tilde{s}\tilde{\rho}P\varphi = bP^2\varphi = bP\varphi$$
$$\therefore \boxed{\nabla\varphi = bP\varphi}$$

Thus ∇ is b on cyclic cochains and 0 on $(1-\lambda)\sigma\mathcal{C}$.

March 26, 1990

Let's consider $\tilde{\kappa}$ on $\Omega^1 A$. One has

$$\kappa(x\,dy) = dy\,x = -y\,dx + d(yx)$$

$$\kappa^2(x\,dy) = -dx\,y + d(yx)$$
$$= x\,dy + d(yx - xy)$$

so

$$\tilde{\kappa}(x\,dy) = \kappa(x\,dy) + \tfrac{1}{2}d([x,y])$$
$$= -y\,dx + d(yx) + \tfrac{1}{2}d(xy-yx)$$

$$\boxed{\tilde{\kappa}(x\,dy) = -y\,dx + d\left(\frac{xy+yx}{2}\right)}$$

$\tilde{\kappa}$ is of order 2 on $\Omega^1 A$; it is an automorphism of the exact sequence

$$0 \longrightarrow \bar{A} \xrightarrow{d} \Omega^1 A \longrightarrow \bar{A}^{\otimes 2} \longrightarrow 0$$

which is the identity on $\bar{A}$ and λ on $\bar{A}^{\otimes 2}$. Taking invariants gives a canonical exact sequence

$$0 \longrightarrow \bar{A} \longrightarrow (\Omega^1 A)_{inv} \longrightarrow \bar{A}^{\otimes 2}_{\lambda} \longrightarrow 0$$

There must be a canonical way to lift $S^2(\bar{A}) = \bar{A}^{\otimes 2}_{\sigma}$ into $\Omega^1 A$. We can find it by looking at the image of $\frac{1-\tilde{\kappa}}{2}$. One calculates

$$\frac{x\,dy - \tilde{\kappa}(x\,dy)}{2} = \frac{1}{4}(x\,dy + y\,dx - dx\,y - dy\,x)$$

This is what you get by polarizing the quadratic map

$$\boxed{x \longmapsto \frac{x\,dx - dx\,x}{2}, \quad \bar{A} \longrightarrow \Omega^1 A}$$

For normalized 1-cochains

$$(\tilde{\kappa} f)(a_0, a_1) = -f(a_1, a_0) + \tfrac{1}{2} f(1, a_0 a_1 + a_1 a_0)$$

For 1-cocycles $\tilde{\kappa} = \kappa = \mathrm{id}$, since in general $\kappa^n - 1 = \lambda^{-1} s b$ on C^n.

March 27, 1990

Use the isomorphism $\Omega^n A = A \otimes \bar{A}^n$ to define $b : \Omega^n A \longrightarrow \Omega^{n-1} A$ by

[redacted] $\boxed{b(\omega\, da) = (-1)^{|\omega|}[\omega, a]}$

Then $b^2(\omega\, da_1\, da_2) =$ [redacted]

$$(-1)^{|\omega|} b\{-[\omega\, da_1, a_2]\} = (-1)^{|\omega|} b\{a_2 \omega\, da_1 - \omega\, da_1 a_2\}$$

$$= (-1)^{|\omega|} b\{a_2 \omega\, da_1 - \omega\, d(a_1 a_2) + \omega a_1\, da_2\}$$

$$= [a_2\omega, a_1] - [\omega, a_1 a_2] + [\omega a_1, a_2]$$

$$= \begin{matrix} a_2\omega a_1 - \omega a_1 a_2 + \omega a_1 a_2 \\ - a_1 a_2 \omega + a_1 a_2 \omega - a_2 \omega a_1 \end{matrix} = 0$$

[redacted] Next

$$db(\omega\, da) = (-1)^{|\omega|} d[\omega, a]$$
$$= (-1)^{|\omega|}[d\omega, a] + [\omega, da]$$
$$bd(\omega\, da) = b(d\omega\, da) = (-1)^{|d\omega|}[d\omega, a]$$

$$\therefore (db + bd)(\omega\, da) = [\omega, da]$$

[redacted] Define $\overbrace{K}^{\text{the Karoubi operation}}$ so that $1 - K = db + bd$:

$$\boxed{K(\omega\, da) = (-1)^{|\omega|} da\, \omega}$$

and $K = 1$ on $\Omega^0 A$. Since $b^2 = d^2 = 0$ K commutes with b, d. One has

$$\boxed{\begin{aligned} K(a_0\, da_1 \cdots da_n) = {} & (-1)^n a_n\, da_0 \cdots da_{n-1} \\ & + (-1)^{n-1} d(a_n a_0)\, da_1 \cdots da_{n-1} \end{aligned}}$$

Consider the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & d\Omega^{n-1}A & \hookrightarrow & \Omega^n A & \xrightarrow{d} & d\Omega^n A & \longrightarrow & 0 \\ & & \| \wr & & \| \wr & & \| \wr & & \\ & & \bar{A}^{\otimes n} & & A \otimes \bar{A}^{\otimes n} & & \bar{A}^{\otimes (n+1)} & & \end{array}$$

K is an endomorphism of this exact sequence which induces λ_n on $d\Omega^{n-1}A$ and λ_{n+1} on $d\Omega^n A$. Thus K (on $\Omega^n A$) is an automorphism such that $K^{n(n+1)} = 1 +$ square zero; thus K is "quasi-unipotent".

K^{-1} is given by

$$\boxed{\begin{array}{l} K^{-1}(da\,\omega) = (-1)^{|\omega|}\,\omega\, da \\ \text{or} \\ K^{-1}(a_0 da_1 \cdots da_n) = (-1)^n a_1 da_2 \cdots da_n\, da_0 \\ \qquad\qquad + (-1)^{n-1} da_2 \cdots da_n\, d(a_0 a_1) \end{array}}$$

Let's iterate K

$$K^i(\omega\, da_1 \cdots da_i) = (-1)^{|\omega| i}\, da_1 \cdots da_i\, \omega$$

$$\begin{aligned} K^n(a_0 da_1 \cdots da_n) &= da_1 \cdots da_n\, a_0 \\ &= a_0 da_1 \cdots da_n + \underbrace{[da_1 \cdots da_n, a_0]}_{\underbrace{(-1)^n b(da_1 \cdots da_n\, da_0)}_{b K^{-1}(da_0 \cdots da_n)}} \end{aligned}$$

$$\boxed{(K^n - 1) = \underset{\text{on } \Omega^n}{b K^{-1} d}}$$

$$K^{n+1} - K = K b K^{-1} d = bd = 1 - K - db$$

$$\boxed{\text{on } \Omega^n \quad K^{n+1} = 1 - db \quad \text{or} \quad 1 - K^{n+1} = db}$$

Define $B\omega_n = \sum_{i=0}^{n} K^i d\omega_n$. Then

$$\boxed{dB = Bd = 0} \qquad \boxed{B^2 = 0} \qquad \boxed{KB = BK = B}$$

$$1 - K^{n(n+1)} = \sum_{i=0}^{n-1} (K^{n+1})^i (1 - K^{n+1})$$

$$= \sum_0^{n-1} \underbrace{(K^{n+1})^i}_{\lambda_n^{(n+1)i}} db = \sum_0^{n-1} K^i db = Bb$$

Also

$$K^{n(n+1)} - 1 = \sum_{i=0}^{n} K^{ni}(K^n - 1) = \sum_{i=0}^{n} K^{ni} b K^{-1} d$$

$$= \sum_{i=0}^{n} b \underbrace{K^{ni} K^{-1}}_{\lambda_{n+1}^{-i-1}} d = bB$$

Thus $\boxed{K^{n(n+1)} - 1 = -Bb = +bB}$ on $\Omega^n A$

Let $\tilde{K} = K + \frac{1}{n(n+1)} Bb$ so that $\tilde{K}^{n(n+1)} = 1$

Let D be a derivation of A; it induces a bimodule map $\tilde{D} : \Omega^1 A \longrightarrow A$ such that $\tilde{D} d = D$. Since

$$\Omega^n A = \Omega^1 A \otimes_A \cdots\cdots \otimes_A \Omega^1 A \qquad n \text{ times}$$

there are bimodule maps $\iota_D^{(j)} : \Omega^n A \longrightarrow \Omega^{n-1} A$ $j = 1, \ldots, n$ given by

$$\boxed{\iota_D^{(j)}(a_0 da_1 \ldots da_n) = (-1)^{j-1} a_0 da_1 \ldots da_{j-1} Da_j \, da_{j+1} \ldots da_n}$$

One has for $\omega \in \Omega^1 A$, $\eta \in \Omega^{j-1} A$, $\eta' \in \Omega^{n-1-j} A$

$$\iota_D^{(j)}(\eta \omega \eta') = (-1)^{j-1} \eta \tilde{D} \omega \, \eta'$$

Claim $\boxed{\iota_D^{(j)} = K^{j-1} \iota_D^{(1)} K^{-j+1} \qquad j = 1, \ldots, n}$

To prove it suffices to check this on $\eta \omega \eta'$ where $\eta = da_1 \ldots da_{j-1} \in \Omega^{j-1} A$ is <u>closed</u>

$\dagger$ + where $\omega \in \Omega^1 A$ and $\eta' \in \Omega^{n-j-1}A$.

Then

$$\kappa^{j-1}\iota_D^{(1)}\kappa^{-j+1}(\eta\,\omega\,\eta')$$
$$= (-1)^{(j-1)(n-1)}\kappa^{j-1}\iota_D^{(1)}(\omega\eta'\eta)$$
$$= (-1)^{(j-1)(n-1)}\kappa^{j-1}(\tilde{D}\omega\;\eta'\eta)$$
$$= (-1)^{(j-1)(n-1)+(j-1)(n-2)}\;\eta\,\tilde{D}\omega\,\eta'$$
$$= (-1)^{j-1}\eta\tilde{D}\omega\,\eta' = \iota_D^{j}(\eta\omega\eta')$$

proving the claim.

Write ι_D for $\iota_D^{(1)}$ so that

$$\boxed{\iota_D(a_0 da_1 \cdots da_n) = a_0 Da_1 \cdots da_n}$$

and put

$$\boxed{I_D = \sum_{j=1}^{n} \iota_D^{(j)} = \sum_{j=1}^{n} \kappa^{j-1}\iota_D\,\kappa^{-j+1} \quad \text{on } \Omega^n A}$$

Then

$$\boxed{I_D(a_0 da_1 \cdots da_n) = \sum_{j=1}^{n}(-1)^{j-1} a_0\, da_1 \cdots da_{j-1}\, Da_j\, da_{j+1} \cdots da_n}$$

This we recognize as the unique (anti) derivation of degree -1 of ΩA such that $I_D(da) = Da$.

Then

$$L_D = [d, I_D] = dI_D + I_D d$$

is the unique degree zero derivation of ΩA such that $L_D(a) = Da$ and $[L_D, d] = 0$:

$$\boxed{L_D(a_0 da_1 \cdots da_n) = Da_0\, da_1 \cdots da_n + \sum_{j=1}^{n} a_0\, da_1 \cdots da_{j-1}\, d(Da_j)\, da_{j+1} \cdots da_n}$$

Let's check $d I_D + I_D d = L_D$ by calculating.

$$d\,\iota_D^{(j)}(a_0 da_1 \cdots da_n) = d\left((-1)^{j-1} a_0 da_1 \cdots da_{j-1} Da_j \cdots\right)$$

$$= (-1)^{j-1} da_0 \cdots da_{j-1} Da_j \, da_{j+1} \cdots da_n$$

$$+ \; a_0 da_1 \cdots da_{j-1} d(Da_j) da_{j+1} \cdots da_n$$

$$\iota_D^{(j+1)} d(a_0 da_1 \cdots da_n) = (-1)^j da_0 \cdots da_{j-1} Da_j \, da_{j+1} \cdots da_n$$

$$\therefore \quad \left(d\,\iota_D^{(j)} + \iota_D^{(j+1)} d\right)(a_0 da_1 \cdots da_n)$$

$$= + a_0 da_1 \cdots da_{j-1} d(Da_j) da_{j+1} \cdots da_n$$

If we add these for $j = 1, \ldots, n$ we get all the terms in $L_D(a_0 da_1 \cdots da_n)$ except

$$Da_0 da_1 \cdots da_n = \iota_D d(a_0 da_1 \cdots da_n)$$

so it checks.

Next $\quad \iota_D b(\omega\, da) = \iota_D (-1)^{|\omega|}(\omega a - a\omega)$

$$= (-1)^{|\omega|}\left((\iota_D \omega) a - a(\iota_D \omega)\right) = -b\left((\iota_D \omega) da\right) = -b\,\iota_D(\omega\, da)$$

$$\therefore \quad \boxed{\iota_D b + b\,\iota_D = 0}$$

Unfortunately $I_D = \sum_{j=0}^{n-1} \kappa^j \iota_D \kappa^{-j}$ on Ω^n doesn't anti-commute with b because of the n. Thus on Ω^n we have

$$b I_D = \sum_{j=0}^{n-1} \kappa^j b\, \iota_D \kappa^{-j}$$

$$I_D b = \sum_{j=0}^{n-2} \kappa^j \iota_D b\, \kappa^{-j}$$

so

$$bI_D + I_D b = \underbrace{K^{n-1}}_{\text{on } \Omega^{n-2} \text{ is } 1-db} b\iota_D K^{-n+1}$$

so $$bI_D + I_D b = b\iota_D K^{-n+1}$$

Our next project will be to average I_D and ι_D with respect to $\tilde{K}$. Averaging a map meanings conjugating by $\tilde{K}^j$ and then averaging.

$$I_D^{\natural} = \frac{1}{N}\sum_{j=0}^{N-1} \tilde{K}^j I_D \tilde{K}^{-j}$$

where $N = (n-1)n(n+1)$ should suffice for the map $I_D : \Omega^n \to \Omega^{n-1}$. Recall

$$\left.\begin{aligned} \tilde{K}^j &= K^j + \frac{j}{n(n+1)} Bb \\ K^j &= \tilde{K}^j - \frac{j}{n(n+1)} Bb \end{aligned}\right\} \text{ on } \Omega^n$$

$$K^j \iota_D K^{-j} = \left(\tilde{K}^j - \frac{j}{(n-1)n} Bb\right) \iota_D \left(\tilde{K}^{-j} + \frac{j}{n(n+1)} Bb\right)$$

$$= \tilde{K}^j \iota_D \tilde{K}^{-j} - \frac{j}{(n-1)n} Bb\iota_D \tilde{K}^{-j} + \frac{j}{n(n+1)} \tilde{K}^j \iota_D Bb \qquad (b\iota_D Bb = \iota_D Bb^2 = 0)$$

$$= \tilde{K}^j \iota_D \tilde{K}^{-j} + \left\{ \frac{j}{(n-1)n} B\iota_D \tilde{K}^{-j} b + \frac{j}{n(n+1)} \tilde{K}^j \iota_D Bb \right\}$$

When we average because $\tilde{K}^i B = B = B\tilde{K}^i$ the average of a term like $B\iota_D \tilde{K}^j$ is $B\iota_D P$. So we have

$$K^j \iota_D^\natural K^{-j} = \iota_D^\natural$$

$$+ \left\{ \frac{j}{(n-1)n} B\iota_D P b + \frac{j}{n(n+1)} P\iota_D B b \right\}$$

Adding for $j = 0, \dots, n-1$ gives

$$I_D^\natural = n\iota_D^\natural + \frac{1}{2} B\iota_D P b + \frac{n-1}{2(n+1)} P\iota_D B b$$

Now the important part of $\iota_D^\natural$ is $P\iota_D^\natural P = P\iota_D P$ so apply P to both sides

$$P I_D P = n P\iota_D P + \frac{1}{2} B\iota_D P b + \frac{n-1}{2(n+1)} P\iota_D B b$$

?

March 28, 1990

Analysis of Goodwillie's theorems about derivations and maybe Reinhart's formulas.

Let D be a derivation on A. It is an infinitesimal automorphism, hence it acts on things natural associated to an algebra. On cochains the action is

$$(L_D f)(a_0, \ldots, a_n) = \sum_{i=0}^{n} f(\ldots, Da_i, \ldots)$$

If f_n is a cyclic cochain of degree n, then

$$(L_D f_n)(a_0, \ldots, a_n) = \sum_{i=0}^{n} (-1)^{in} f(Da_i, a_{i+1}, \ldots, a_n, a_0, \ldots, a_{i-1})$$

On the other hand we have the operation on Hochschild cochains

$$(\iota_D f_n)(a_0, \ldots, a_{n+1}) = f_n(a_0 Da_1, a_2, \ldots, a_{n+1})$$

(anti-) commuting with b. Thus for f_n cyclic one has

$$(L_D f_n)(a_0, \ldots, a_n) = \sum_{i=0}^{n} (-1)^{in} (\iota_D f_n)(1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1})$$

$$= (B \iota_D f)(a_0, \ldots, a_n)$$

$$\boxed{L_D = B \iota_D I}$$

$$\begin{tikzcd} C_\lambda \arrow[r, hook, "I"] \arrow[d, "L_D"'] & C \arrow[d, "\iota_D"] \\ C_\lambda & C \arrow[l, "B"] \end{tikzcd}$$

This factorization shows immediately that $L_D S = S L_D = 0$, which is Goodwillie's theorem on cyclic cohomology.

Note that if we use $C_{inv} \subset C$ for the Hochschild cohomology, then one has a factorization

$$\begin{tikzcd} C_\lambda \arrow[r, hook] \arrow[d, "L_D"] & C_{inv} \subset C \arrow[d, "\iota_D"] \\ C_\lambda & C_{inv} \arrow[l, "\tilde{S}"'] & C \arrow[l, "P"'] \arrow[ll, bend left, "B"] \end{tikzcd}$$

and so $L_D = \tilde{S}(P\iota_D) I$, so $P\iota_D P$ on C_{inv} is the ~~oper~~ appropriate operator.

Now we would like to understand well the significance of this result. I think the good viewpoint is to emphasize the periodic theory, which is a filtered theory. The Goodwillie theorem asserts that L_D is homotopic to zero of periodic theory, but the homotopy increases filtration by one.

General discussion. We have seen that there is a canonical extension of the cyclic cochains by itself shifted

$$0 \longrightarrow C_\lambda \longrightarrow C_{inv} \longrightarrow C_\lambda[-1] \longrightarrow 0$$

To such an extension one has as invariant a homotopy class of maps: $C_\lambda[-2] \longrightarrow C_\lambda[0]$

obtained by choosing a section h of $C_{\text{inv}} \longrightarrow C_\lambda[-1]$ and then looking at $[b,h]$. Changing h alters $[b,h]$ up to homotopy.

Prop. Let K, L be complexes. Then isomorphism classes of extensions

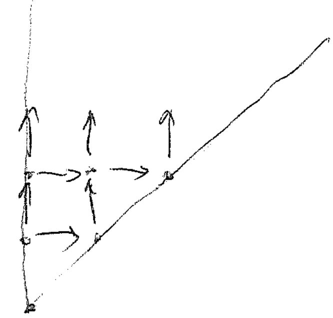
$$0 \longrightarrow K \longrightarrow E \longrightarrow L \longrightarrow 0$$

in the category of complexes correspond bijectively to elements of $H^1(\mathrm{Hom}(L,K))$.

Proof. Map extensions to this cohomology by choosing a lifting h of L and taking the class of $[d,h] : L \to K$. This map is well-defined on isomorphism classes. It is onto by the mapping cone construction. It is 1-1 because if two extensions give the same class, then we adjust the liftings until both associated maps $L[-1] \to K$ agree, and E is <u>isomorphic</u> to the mapping cone.

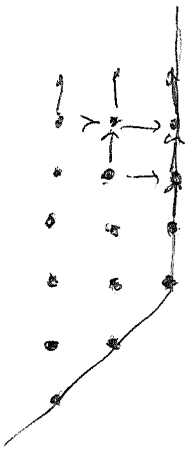
This proposition tells us that cyclic formalism depends only on the homotopy class of S on the cyclic complex. Thus we can think of the cyclic theory of A as a complex $C_\lambda(A)$ with an endomorphism of degree 2 in the derived category.

Of course it is natural to replace $C_\lambda(A)$ by an equivalent complex on which S is injective (or surjective). This is done by the double complexes. If we want S to be injective we use the "equivariant cohomology" complex



$C_{\mathrm{inv}}[u]$

If we want S surjective we use the completed negative version



$C_{\mathrm{inv}}[[u^{-1}]][u] \,/\, C_{\mathrm{inv}}[u]$

Now let us consider Goodwillie's result. We look at the periodic theory $C_{\mathrm{inv}}[u^{-1},u]$ where $S =$ multiplication by u, where the differential is $b - u\tilde{S}$. Then L_D is to homotopic to zero. The filtration is by $u^k C_{\mathrm{inv}}[u]$, the wedges to the right. The simplest form for a homotopy H would be

$$\begin{tikzcd} \cdot & \bullet \arrow[l, "I_D"'] \arrow[d, "J_D"] \\ & \cdot \end{tikzcd}$$

which means that it increases filtration by 1. Changing to $C[u^{-1},u]$ with differential $b + uB$ ~~the~~ the homotopy condition is

$$[b + uB, u^{-1}I_D + J_D] = L_D$$

so $\boxed{[b, I_D] = [B, J_D] = 0 \quad , \quad [B, I_D] + [b, J_D] = L_D}$

which is exactly what the operators constructed by Goodwillie + Rinehart satisfy.*

A natural question at this point is the following. Since the whole structure is determined by the original extension

$$(*) \quad 0 \longrightarrow C_\lambda \longrightarrow C_{inv} \longrightarrow C_\lambda[-1] \longrightarrow 0$$

how much has to be done in order to obtain the hard Goodwillie-Rinehart result (operators I_D, J_D) from the easy part ($L_D S \sim 0$)

* Rinehart has stronger formula $BJ_D = J_D B = 0$

Caution: The extension (*) is determined up to isomorphism by the homotopy class of the S-operator, but not determined up to canonical isomorphism. In other words this extension has automorphisms: degree 1 maps from the cyclic complex to itself.

March 29, 1990

More on derivations. Let us recall why an inner derivation acts trivially on cyclic cohomology. First recall the general picture behind Goodwillie's thm.

$$\begin{tikzcd}
 & & H_{n+1}(A,A) \arrow[d, "\partial"] \arrow[dd, bend left=60, "L_D"] \\
HC_n(A) \arrow[r, "1+d"] \arrow[urr, bend left, "B"] \arrow[d, "L_D"] & HC_n(A\oplus\Omega^1A) \arrow[r] & H_n(A,\Omega^1A) \arrow[d, "\tilde{D}"] \\
HC_n(A) & & H_n(A,A) \arrow[ll, "I"]
\end{tikzcd}$$

Here ∂ is the connecting homomorphism associated to $a \mapsto 1\otimes a - a\otimes 1$

$$0 \longrightarrow \Omega^1A \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0$$

and because $A\otimes A$ is projective over $A\otimes A$, ∂ is an isomorphism for $n>0$ and injective for $n=0$.

We need formulas for ∂ on the chain level: Choose a lifting for $A\otimes A \to A$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1A\otimes\bar{A}^{\otimes n} \arrow[r] & (A\otimes A)\otimes\bar{A}^{\otimes n} \arrow[r] & A\otimes\bar{A}^{\otimes n} \arrow[r] & 0 \\
 & & (1\otimes a_0, a_1,\dots,a_n) \arrow[d, "b"] & (a_0,\dots,a_n) \arrow[l, mapsto, "h"'] \arrow[d, "b"] & \\
 & & \begin{matrix}(1\otimes a_0a_1, a_2,\dots,a_n)\\ -(1\otimes a_0, b'(a_1,\dots,a_n))\\ +(-1)^n(a_n\otimes a_0, a_1,\dots,a_{n-1})\end{matrix} & \begin{matrix}(a_0a_1, a_2,\dots,a_n)\\ -(a_0, b'(a_1,\dots,a_n))\\ +(-1)^n(a_na_0, a_1,\dots,a_{n-1})\end{matrix} &
\end{tikzcd}$$

Thus
$$(bh-hb)(a_0,\dots,a_n) = (-1)^n(a_n\otimes a_0 - 1\otimes a_na_0, a_1,\dots,a_{n-1})$$
$$=(-1)^{n-1}(da_n\,a_0)\otimes(a_1,\dots,a_{n-1})$$

Thus when we compose with $\tilde{D}$ we obtain the map

$$(a_0, \dots, a_n) \longmapsto (-1)^{n-1}(Da_n a_0, a_1, \dots, a_{n-1})$$

of degree -1 on the normalized Hochschild complex. On the other hand if we lift via $a \longmapsto a \otimes 1$ we get

$$\begin{array}{ccc} (a_0 \otimes 1, a_1, \dots, a_n) & \overset{h}{\longleftarrow\!\shortmid} & (a_0, \dots, a_n) \\ \downarrow b & & \downarrow b \\ (a_0 \otimes a_1, a_2, \dots, a_n) & & (a_0 a_1, a_2, \dots, a_n) \\ -(a_0 \otimes 1, b'(a_1, \dots, a_n)) & & -(a_0, b'(a_1, \dots, a_n)) \\ +(-1)^n(a_n a_0 \otimes 1, a_1, \dots, a_{n-1}) & & +(-1)^n(a_n a_0, a_1, \dots, a_{n-1}) \end{array}$$

Thus $(bh - hb)(a_0, \dots, a_n) = (a_0 \otimes a_1 - a_0 a_1 \otimes 1, a_2, \dots, a_n)$

$$= +(a_0\, da_1, a_2, \dots, a_n)$$

This is why we have used the other sign $da = 1 \otimes a - a \otimes 1$ for $\Omega^1 A \hookrightarrow A \otimes A$. Thus the two maps realizing ι_D

$$H_n(A, A) \xrightarrow{\partial} H_{n-1}(A, \Omega^1 A) \xrightarrow{\tilde{D}} H_{n-1}(A, A)$$

are

$$(a_0, \dots, a_n) \overset{\iota_D}{\longmapsto} (a_0 Da_1, a_2, \dots, a_n)$$

$$\longmapsto (-1)^{n-1}(Da_n a_0, a_1, \dots, a_{n-1})$$

Here ι_D is Kassel's ι_D

These have to be homotopic. One can see this also because

$$\iota_D K(a_0, \dots, a_n) = \iota_D\{(-1)^n(a_n, a_0, \dots, a_{n-1}) + (-1)^{n-1}(1, a_n a_0, a_1, \dots, a_{n-1})\}$$

$$= (-1)^n(a_n Da_0, a_1, \dots, a_{n-1}) + (-1)^{n-1}(D(a_n a_0), a_1, \dots, a_{n-1})$$

$$= (-1)^{n-1}(Da_n\, a_0, a_1, \dots, a_{n-1})$$

or more simply

$$\iota_D K(a_0 da_1 \cdots da_n) = (-1)^{n-1} \iota_D (da_n a_0 da_1 \cdots a_{n-1})$$
$$= (-1)^{n-1} Da_n\, a_0 da_1 \cdots a_{n-1}$$

Summarize. The map

$$H_{n+1}(A, A) \xrightarrow{\partial} H_n(A, \Omega^1 A) \xrightarrow{\tilde{D}} H_n(A, M)$$

is realized by

$$\iota_D(a_0, \dots, a_{n+1}) = (a_0 Da_1, a_2, \dots, a_n)$$

Here $D : A \longrightarrow M$ is any derivation.

Next take the case where $D = d : A \to \Omega^1 A$ followed by $\Omega^1 A \subset A \otimes A$; this is the universal inner derivation $d(a) = 1 \otimes a - a \otimes 1 = [1\otimes 1, a]$. Then h above becomes a homotopy operator

$$(a_0(1 \otimes a_1 - a_1 \otimes 1), a_2, \dots, a_n) = (bh - hb)(a_0, \dots, a_n)$$

So we have

$$\boxed{\begin{aligned} (a_0 [m, a_1], a_2, \dots, a_n) &= (bh - hb)(a_0, \dots, a_n) \\ h(a_0, \dots, a_n) &= (a_0 m, a_1, \dots, a_n) \end{aligned}}$$

This shows why ι_D in the case of $D(a) = [m, a]$ is null homotopic.

Let's return to the cyclic complex. One has

$$(a_0, \dots, a_n) \xmapsto{\ B\ } \sum_{i=0}^{n} (-1)^{in} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1})$$
$$\xmapsto{\ \iota_D\ } \sum_{i=0}^{n} (-1)^{in} (Da_i, a_{i+1}, \dots, a_n, a_0, \dots, a_{i-1})$$
$$\xmapsto{\ I\ } \sum_{i=0}^{n} (a_0, \dots, a_{i-1}, Da_i, a_{i+1}, \dots, a_n) = L_D(a_0, \dots, a_n)$$

Thus on the cyclic complex when $D = [m,a]$, $m \in A$ we have

$$L_D = I\iota_D B = I(bh - hb)B$$
$$= b(IhB) + (IhB)b$$

where

$$(IhB)(a_0,\dots,a_n) = Ih\sum_{i=0}^{n}(-1)^{in}(1,a_i,\dots,a_{i-1})$$
$$= I\sum_{i=0}^{n}(-1)^{in}(m,a_i,\dots,a_n,a_0,\dots,a_{i-1})$$

$$\boxed{IhB = \sum_{i=0}^{n}(-1)^{i}(a_0,\dots,a_{i-1},m,a_i,\dots,a_n)}$$

(Earlier work p. 300 June 1989)

Our next project should be to obtain a contracting homotopy for $\overbrace{L_D}^{D_a=[m,a]}$ on the Hochschild complex, or at least the $\tilde{K}$-invariant part.

Actually the above formula for IhB is a bit misleading since the expression is not obviously cyclic in $a_0,\dots,a_n$. Also one notes the obvious term missing

$$(-1)^{n+1}(a_0,\dots,a_n,m)$$

is the same as the first term $(m,a_0,\dots,a_n)$.
Maybe the best formula is

$$\boxed{IhB = I\Big\{\sum_{i=0}^{n}(-1)^{in}(m,a_i,\dots,a_n,a_0,\dots,a_{i-1})\Big\}}$$

Actually the good formula appears to be

$$\boxed{H_x(a_0,\dots,a_n) = \sum_{i=1}^{n+1} (-1)^i (a_0,\dots,a_{i-1}, x, a_i,\dots,a_n)}$$

since it seems to work for the Hochschild complex where a_0 is singled out.

Let's compute in degree 1.

$$H_x(a_0,a_1) = -(a_0, x, a_1) + (a_0, a_1, x)$$

$$\begin{aligned} bH_x(a_0,a_1) &= -(a_0x, a_1) + (a_0a_1, x) \\ &\quad + (a_0, xa_1) - (a_0, a_1x) \\ &\quad - (a_1a_0, x) + (xa_0, a_1) \\ &= ([x,a_0],a_1) + (a_0,[x,a_1]) + ([a_0,a_1],x) \end{aligned}$$

$$H_x b(a_0,a_1) = -([a_0,a_1],x)$$

$$\therefore (bH_x + H_x b)(a_0,a_1) = ([x,a_0],a_1) + (a_0,[x,a_1])$$

Let's try to derive the formula

$$bH_x + H_x b = [redacted]\ L_{ad\,x}$$

proved in Loday's book I.3

[redacted] on the Hochschild complex $C_\cdot^N(A,M) = (M \otimes \bar{A}^{\otimes n}, b)$. The method would be to use the fact that $C_\cdot^N(A,M)$ is the M-degree $= 1$ part of $\bar{C}C(A \oplus M)$. We use the derivation $ad(x)$ on $A \oplus M$, where $x \in A$. Consider $(m, a_1, \dots, a_n)$ in the cyclic complex of $A \oplus M$ and apply the above formula for H_x at the top of this page:

$$H_x(m, a_1, \dots, a_n) = \sum_{i=1}^{n+1} (-1)^i (m, a_1, \dots, a_{i-1}, x, a_i, \dots, a_n)$$

We know that $L_{adx} = [b, (IH_x B)]$ on $\overline{CC}(A \oplus M)$, so we want to see that $IH_x B = H_x$ on $C^N_\cdot(A, M)$. Too confusing!

In any case since L_{adx} on $H(A, M)$ is part of L_{adx} on $\overline{HC}(A, M)$, which is zero, it ought to work.

Here is something else one can do. Introduce

$$\boxed{h_x(a_0, a_1, \ldots, a_n) = (a_0 x, a_1, \ldots, a_n)}$$

Then one has

$$(bh_x - h_x b)(a_0, \ldots, a_n) = (a_0 [x, a_1], a_2, \ldots, a_n)$$

i.e.

$$\boxed{bh_x - h_x b = \iota_{adx}.}$$

We can understand this as follows. Consider cup product of normalized cochains with values in a bimodule. If $f : \bar{A}^{\otimes n} \longrightarrow M$ and D is a derivation, that is, a 1-cocycle with values in A, then we have the cup product

$$(D \cdot f)(a_1, a_2, \ldots, a_{n+1}) = Da_1 \, f(a_2, \ldots, a_{n+1})$$

If $M = A^*$ and we identify $f(a_1, \dots, a_n)$ with the scalar valued cochain $\tilde{f}(a_0, a_1, \dots, a_n) = \langle a_0, f(a_1, \dots, a_n)\rangle$ then

$$\widetilde{D\cup f}(a_0, \dots, a_{n+1}) = \langle a_0 | Da_1 \cdot f(a_2, \dots, a_{n+1})\rangle$$
$$= \langle a_0 Da_1 | f(a_2, \dots, a_{n+1})\rangle$$
$$= \tilde{f}(a_0 Da_1, a_2, \dots, a_{n+1})$$

Here we use left mult. on A^* is given by right multiplication on A. Thus

$$\iota_D \tilde{f} = \widetilde{D\cup f}$$

Also if $x \in A$ is viewed as a 0-cochain, one has

$$\widetilde{x\cup f}(a_0, \dots, a_n) = \langle a_0 | x f(a_1, \dots, a_n)\rangle$$
$$= \langle a_0 x | f(a_1, \dots, a_n)\rangle$$
$$= \tilde{f}(a_0 x, a_1, \dots, a_n)$$

so $\quad h_x \tilde{f} = \widetilde{x\cup f}$. Then we have

$$(bh_x - h_x b)\tilde{f} = (\delta(x\cup f) - x\cup \delta f)^\sim$$
$$= (\delta x \cup f)^\sim$$

Now $\quad (\delta x)(a) = ax - xa = -(\operatorname{ad} x)(a)$, so we have

$$\boxed{h_x b - b h_x = \iota_{\operatorname{ad} x}}$$

on cochains. The change in sign comes from transposing.

March 30, 1990

$$f \in (A \otimes \bar{A}^{\otimes n})^* \text{ is } \tilde{K}\text{-invariant}$$
$$\iff sf \text{ and } sbf \text{ are } \lambda\text{-invariant}$$

Proof: ($\Longrightarrow$) is clear since $\tilde{K}$ commutes with b, s and since $\tilde{K} = \lambda$ on $\operatorname{Im} s$.

($\Longleftarrow$) We know $B = \tilde{s}$ maps C^n_{inv} onto $sC^{n\otimes}_\lambda$, hence, since sf is assumed cyclic, we can write it $sf = \frac{1}{n} Bg = sg$ with $g \in C^n_{inv}$. Thus we can suppose $sf = 0$. We have then $(1-\lambda)f = (1-K)f = sbf$. But $\operatorname{Im}(1-\lambda) \cap \operatorname{Ker}(1-\lambda) = 0$, so as sbf is assumed $\in \operatorname{Ker}(1-\lambda)$, we have $f = \lambda f$ so f is invariant.

Problem: In the augmented case show Pf is given by the obvious diagram chasing?

Let $f_n = (\psi_{n+1}, \varphi_n)$ as usual. To find Pf, or really what should turn out to be Pf one splits φ_n: $\varphi_n = P\varphi_n + (1-\lambda)G\varphi_n$. Now remove $b(G\varphi_n, 0)$ from f to obtain

$$f' = f - bGsf = (\psi, \varphi) - (bG\varphi, (1-\lambda)G\varphi)$$
$$= (\psi - bG\varphi, P\varphi)$$

At this point we have sf' cyclic. Now we turn to making $sbf' = sbf$ cyclic. Let us remove

$$Gsbf' = Gsbf = Gs(b\psi, (1-\lambda)\psi - b'\varphi) = (G(1-\lambda)\psi - Gb'\varphi, 0)$$

form f' to obtain

$$(\psi - bG\varphi - G(1-\lambda)\psi + Gb'\varphi, P\varphi)$$
$$= (P\psi - bG\varphi + Gb'\varphi, P\varphi).$$

Thus

$$\boxed{P(\psi,\varphi) = \left(P\psi + (Gb' - bG)\varphi, P\varphi\right)}$$

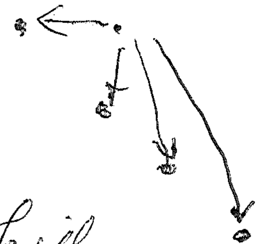
is the projection on [redacted] C_{inv}. What we have done is to use the formula

$$Pf = 1 - (1-P)f = 1 - bGsf - Gsbf$$

Let's discuss derivations in general. Let D be a derivation of A and L_D the corresponding endomorphism on cochains. The goal is to show L_D is homotopic to zero in a suitable sense. In the case of an inner derivation should be homotopic to zero on the cyclic [redacted] and Hochschild complexes. But for a general derivation it is homotopic to zero on the periodic cyclic theory with a shift of filtration by 1. This means that in the double complex the homotopy is of the form

$$\begin{array}{ccc} \cdot & \xleftarrow{I_D} & \cdot \\ & & \big\downarrow J_D \end{array}$$

at least. Presumably ~~it~~ might ~~be~~



A good viewpoint is to work in the mapping complex $\mathrm{Hom}(C_{inv}[u], C_{inv}[u])$ which is a double complex with differential $ad(b) + ad(B)$.

Then the homotopy operator is supposed to satisfy

$$[b+uB,\ u^{-1}I_D+J_D] = L_D$$

$$\therefore [b, I_D] = 0 \qquad [B, I_D] + [b, J_D] = L_D$$

$$[B, J_D] = 0$$

But we are interested in projecting things onto the invariant complex where B is exact. So if we think in the Ham complex:

$$\begin{tikzcd} 0 & & \\ I_D \arrow[u] \arrow[r, "{[B,}"] & L_D & \\ & J_D \arrow[u, "{[b,}"] \arrow[r, "{[B,}"] & 0 \end{tikzcd}$$

and use the exactness of the rows, we can get rid of J_D.

Thus the <u>conjecture</u> is that there is an I_D on invariant cochains satisfying

$$[b, I_D] = 0 \qquad [B, I_D] = L_D$$

It would be nice also from the DG Lie viewpoint to have

$$I_D^2 = 0$$

Possibility: Start with ι_D which satisfies $[b, \iota_D] = 0$, whence $[b, P\iota_D P] = 0$. Is it true that

$$[\underset{B}{\tilde{S}}, P\iota_D P] = PL_D P \ ?$$

Here's a pleasant consequence. Suppose $D = ad(x)$, $x \in A$. We know that

$$[b, h_x] = -\iota_{ad\,x}$$

(see p 331). Then

$$[b, [\tilde{s}, h_x]] = [1-\tilde{\kappa}, h_x] - \overbrace{[\tilde{s}, [b, h_x]]}^{-[\tilde{s},\, \iota_{ad\,x}]}$$

$$[b, [\tilde{s}, Ph_xP]] = 0 + [\tilde{s}, P\iota_{ad\,x}P]$$

$$= PL_{ad(x)}P$$

Thus we see $L_{ad(x)}$ on C_{inv} is homotopic to zero, where the homotopy commutes with $\tilde{s}$.

So now we must try in earnest to ~~establish~~ decide whether

$$[\tilde{s}, P\iota_D P] = PL_D \quad ?$$

We review what we did earlier (pp 315-319), working on ΩA.

$$\iota_D(a_0\,da_1 \cdots da_n) = a_0\,Da_1\,da_2 \cdots da_n$$

$$[b, \iota_D] = 0$$

$$(\kappa^j \iota_D \kappa^{-j})(a_0\,da_1 \cdots da_n) = (-1)^j (a_0\,da_1 \cdots da_j\,Da_{j+1}\,da_{j+2} \cdots da_n)$$

$$I_D = \sum_{j=0}^{n-1} \kappa^j \iota_D \kappa^{-j} \quad \text{app to } (a_0\,da_1 \cdots da_n)$$

$$= \sum_{j=0}^{n-1} (-1)^j a_0\,da_1 \cdots da_j\,Da_{j+1}\,da_{j+2} \cdots da_n \quad \text{on } \Omega^n A$$

degree -1 derivation of ΩA such that $I_D(da) = Da$

We have $[d, I_D] = L_D$

Now we propose to shift to cochains where d becomes s. Really $d_n^t = s_{n+1}$ and $(I_D^t)_n : (\Omega^n A)^* \longrightarrow (\Omega^{n+1} A)^*$ is

$$(I_{D,n+1})^t = \sum_{j=0}^{n} \kappa^{-j} \iota_D^t \kappa^{+j}$$

Thus the transpose of $[d, I_D] = L_D$ becomes on $(\Omega^n A)^*$

$$s\left(\sum_{j=0}^{n} \kappa^{-j} \iota_D \kappa^{j}\right) + \left(\sum_{j=0}^{n-1} \kappa^{-j} \iota_D \kappa^{j}\right) s = L_D$$

Now use $\kappa^j = \tilde\kappa^j - \frac{j}{n(n+1)} bB$ wrong sign - should use Bb and $s\tilde\kappa = s\kappa$

and we get

$$\sum_{j=0}^{n} \tilde\kappa^{-j} s \iota_D \left(\tilde\kappa^j - \frac{j}{n(n+1)} bB\right) + \sum_{j=0}^{n-1} \left(\tilde\kappa^{-j} + \frac{j}{n(n+1)} bB\right) \iota_D s \tilde\kappa^j = L_D$$

Apply P on both sides

$$(n+1) P s \iota_D P + n P \iota_D s P - \frac{1}{n(n+1)}\left(\sum_{j=0}^{n} j\right) P s \iota_D bB + \frac{1}{n(n+1)}\left(\sum_{j=0}^{n-1} j\right) bB \iota_D s P = P L_D$$

Recall that $\tilde s P = B = n s P$ on $(\Omega^n A)^*$, so this is

$$P \tilde s \iota_D P + P \iota_D \tilde s P - \frac{1}{n(n+1)} \frac{n(n+1)}{2} P \frac{\tilde s}{n+1} \iota_D b \tilde s P + \frac{1}{n(n+1)} \frac{\cancel{n}(n-1)}{2} bB \iota_D \frac{\tilde s}{\cancel{n}} P$$

the error term is

$$-\frac{1}{2(n+1)} B \iota_D bB + \frac{1}{2} \frac{n-1}{(n+1)n} bB \iota_D B$$

$$= b(B\iota_D B)\left(\frac{1}{2n(n+1)}\right)(\cancel{n}-1-\cancel{n})$$

$$= +\frac{1}{2n(n+1)}(B\iota_D B)\,b$$

Thus it seems we have

$$\boxed{[\tilde{s}, P\iota_D P] + \frac{1}{2n(n+1)}\,\tilde{s}P\iota_D P\tilde{s}\,b = PL_D}$$

April 2, 1990

Consider an exact sequence of complexes

$$\circledast \qquad 0 \longrightarrow X \xrightarrow{i} E \xrightarrow{p} Y \longrightarrow 0$$

Choose a map $h: Y \to E$ not necessarily compatible with differentials such that $ph = 1$, and define $k: E \to X$ so that $ik = 1 - hp$. Then we have $i(ki) = i$ so $ki = 1$. Summarizing: we have the splitting

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"'] & E \arrow[l, dashed, bend right, "k"'] \arrow[r, "p"'] & Y \arrow[l, dashed, bend right, "h"'] \arrow[r] & 0 \end{tikzcd}$$

$$\boxed{\begin{array}{lll} ph = 1 = ki & ik + hp = 1 & kh = 0 \\ [d,i] = [d,p] = 0 & & \end{array}}$$

Define $S: Y \to \Sigma X$ so that $iS = [d,h]$; this is possible since $p[d,h] = [d,ph] = [d,1] = 0$. Alternatively, put

$$S = k[d,h] = k \cdot d \cdot h$$

Then

$$iS = ikdh = (1-hp)dh = dh - hdph$$
$$= dh - hd$$

Also

$$Sp = kdhp = kd(1-ik) = kd - kidk$$
$$= kd - dk$$

$$\boxed{S = k \cdot d \cdot h . \qquad iS = [d,h] \qquad Sp = -[d,k]}$$

Next let L be an endomorphism of the exact sequence $\circledast$.

Add $[d,S] = [d,k[d,h]] = [d,k][d,h] = -SpiS = 0$ and that $E \cong \mathrm{Cone}(S)$.

Then L commutes with S up to homotopy:

$$\cancel{[L,S] = [L, k\,d\,h] = [L, k[d,h]] = k[L,[d,h]] + [L,k][d,h] = k[d,[L,h]]}$$

$$[d, kLh] = [d,k]Lh + kL[d,h]$$
$$= -SpLh + kL\iota S$$
$$= -SLph + k\iota LS = LS - SL$$

$$\therefore \boxed{[L,S] = [d, kLh]}$$

Suppose now that $LS \sim 0$ say $LS = [d,g]$. We then propose to construct maps of complexes

$$X \xleftarrow{I} E \xleftarrow{J} Y$$

such that $pJ = L$, $\iota I + Jp = L$, $I\iota = L$ on Y, E, X respectively.

~~To get the formula suppose J such that $pJ = L$ on Y. Then~~

$$\cancel{\iota LS = L\iota S = pJ[d}$$

The obvious first choice for J is Lh but this must be modified so that it commutes with d. As $[d, Lh] = L[d,h] = L\iota S = i[d,g] = [d,\iota g]$, our candidate for J is

$$J = Lh - \iota g$$

Then $[d,J] = 0$ and $pJ = L$. Moreover, $p(L - Jp) = pL - Lp = 0$ so there's a unique I with $\iota I = L - Jp$, namely $I = k(L - Jp)$.

Then $\quad i[d,I] = [d, L - Jp] = 0$

$i(Ii) = (L-Jp)i = iL$ imply

I is a map of complexes with the required properties.

So we see that given $LS = [d,g]$, then we obtain $\boxed{J = Lh - ig}$ which is a map of complexes $\ni$ $pJ = L$, and then there is a unique I to go with J.

Conversely given $pJ = L$, $[d,J] = 0$, define g by $ig = Lh - J$; this is possible as $p(Lh - J) = L - pJ = 0$. Then

$$i[d,g] = [d,ig] = [d, Lh - J] = L[d,h] = iLS$$

and we have $LS = [d,g]$. So we obtain

<u>Proposition</u>: Given an exact sequence of complexes $0 \to X \xrightarrow{i} E \xrightarrow{p} Y \to 0$, let $S : Y \to \Sigma X$ be the classifying ~~map~~ associated to a lifting h of p. Let L be an endomorphism of the exact sequence. Then there is an equivalence between

1) maps $J : Y \to E$ with $pJ = L$, $[d,J] = 0$

2) maps $J : Y \to E$, $I : E \to X$ $\ni$ $[d,J] = [d,I] = 0$
$pJ = L$, $iI + Jp = L$, $Ii = L$

3) homotopies: $[d,g] = LS$

Note that $\exists J \ni pJ = L, [d,J] = 0 \iff LS \sim 0$ follows from

$$\mathrm{Hom}(Y,E) \xrightarrow{p_*} \mathrm{Hom}(Y,Y) \xrightarrow{S_*} \mathrm{Hom}^1(Y,X)$$

and the fact that $SL \sim LS$.

Summarize formulas giving equivalence.

$$[d,g] = LS \longmapsto \begin{cases} J = Lh - cg \\ I = Lk + gp \end{cases}$$

$$\begin{matrix} \rho L = J \\ [d,J]=0 \end{matrix} \longmapsto \quad g = \iota^{-1}(Lh - J) = kLh - kJ$$

$$g = Ih$$

April 3, 1990

Continue Goodwillie–Rinehart.

Suppose we have an exact sequence of complexes

$$(*) \qquad 0 \longrightarrow X \xrightarrow{i} E \xrightarrow{p} Y \longrightarrow 0$$

Let's view it as a double complex K with horizontal differential $\partial = (i, p)$ and vertical differential d. We then have a double mapping complex $\mathrm{End}^{\bullet,\bullet}(K)$ with differentials $[\partial, ?]$, $[d, ?]$. Picture this as a sequence of complexes

$$0 \longrightarrow \mathrm{End}^{-2,\bullet} \longrightarrow \mathrm{End}^{-1,\bullet} \longrightarrow \mathrm{End}^{0,\bullet} \longrightarrow \cdots \longrightarrow \mathrm{End}^{2,\bullet} \longrightarrow 0$$

which is exact because $(*)$ is exact. In effect, if we choose a splitting of $(*)$, this is a horizontal contracting homotopy h for ∂, then h will be a contracting homotopy for the second exact sequence:

$$[d, h\xi] = \underbrace{[d,h]}_{I}\xi - h[d,\xi]$$

Ignoring the vertical differentials, any endomorphism L of $(*)$, this is an $L \in \mathrm{End}^{0,\bullet}$ such that $[\partial, L] = 0$ is of the form $[\partial, I]$, that is

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "L"'] & E \arrow[r, "p"] \arrow[d, "L"'] \arrow[dl, "I'"'] & Y \arrow[d, "L"] \arrow[dl, "I''"'] \\ X \arrow[r, "i"'] & E \arrow[r, "p"'] & Y \end{tikzcd} \qquad \begin{aligned} pI'' &= L \\ iI'' + I'p &= L \\ I'i &= L \end{aligned}$$

where I is unique up to ∂g, $g \in \mathrm{Hom}^{\bullet}(Y, X)$:

$$\text{l'arbitraire}\begin{pmatrix} I' \\ I'' \end{pmatrix} = \begin{pmatrix} gp \\ ig \end{pmatrix}$$

This arbitrariness is eliminated once we choose either I' or I''.

~~Next consider the situation where we want maps to commute with d.~~

More precisely once I'' chosen such that $pI'' = L$, then there is a unique I' such that $I = (I', I'')$ satisfies $[\partial, I] = L$. Similarly if I' is such that $I'\iota = L$, there is a unique I'' such that $I = (I', I'')$ satisfies $[\partial, I] = L$.

Next consider what happens when we suppose $[d, L] = 0$ and want I also to satisfy $[d, I] = 0$. We only need to find I'' $\ni$ $pI'' = L$, $[d, I''] = 0$ and the obstruction is given by:

$$
\begin{array}{ccccc}
H^0\{\mathrm{Hom}^\bullet(Y,E)\} & \longrightarrow & H^0\{\mathrm{Hom}^\bullet(Y,Y)\} & \xrightarrow{\ S\ } & H^1\{\mathrm{Hom}^\bullet(Y,X)\} \\
 & & \cup & & \\
 & & L & \longmapsto & SL
\end{array}
$$

where $S : Y \longrightarrow \Sigma X$ is the "connecting" map associated to $\circledast$. Thus I exists with the required properties iff $SL \sim 0$.

Now we apply this to $L = L_D$ acting on

$$0 \longrightarrow \mathcal{C}_{bas} \xrightarrow{\ \iota\ } \mathcal{C}_{inv} \xrightarrow{\ p\ } \mathcal{C}_{bas} \longrightarrow 0$$

ι = inclusion

$p = \tilde{\delta} = B$

The easy part of Goodwillie-Rinehart tells us that we can find J, I $\ni$

$$
\begin{tikzcd}
\mathcal{C}_{bas} \arrow[r, "\iota"] \arrow[d, "L"'] & \mathcal{C}_{inv} \arrow[dl, "J"] \\
\mathcal{C}_{bas} &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & \mathcal{C}_{bas} \arrow[dl, "I"'] \arrow[d, "L"] \\
\mathcal{C}_{inv} \arrow[r, "p"'] & \mathcal{C}_{bas}
\end{tikzcd}
$$

commute. In fact we have

$$\begin{tikzcd} C^n_{bas} \arrow[r, "\iota"] \arrow[d, "L_D"] & C^n \arrow[d, "\iota_D"] \\ C_{bas} & C^{n+1} \arrow[l, "B"'] \end{tikzcd}$$

$$(\iota_D f_n)(a_0, \ldots, a_{n+1}) = f_n(a_0 Da_1, a_2, \ldots, a_{n+1})$$

commuting: $(B\iota_D \iota f_n)(a_0, \ldots, a_n)$

$$= \sum_{i=0}^{n} (-1)^{in} (\iota_D \iota f_n)(1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1})$$

$$= \sum_{i=0}^{n} (-1)^{in} (\iota f_n)(Da_i, a_{i+1}, \ldots, a_n, a_0, \ldots, a_{i-1})$$

$$= \sum_{i=0}^{n} f_n(a_0, \ldots, a_{i-1}, Da_i, a_{i+1}, \ldots, a_n)$$

$$= (L_D f_n)(a_0, \ldots, a_n)$$

Thus we can take $\boxed{J = B\iota_D \text{ restricted to } C_{inv}}$

and $\boxed{I = P\iota_D \iota}$: $C_{bas} \xrightarrow{\iota} C \xrightarrow{\iota_D} C \xrightarrow{P} C_{inv}$.

Note that for f_n cyclic (reduced)

$$(\iota_D \iota f_n)(a_0, \ldots, a_{n+1}) = f_n(a_0 Da_1, a_2, \ldots, a_{n+1})$$

is a Hochschild cochain without any apparent symmetry properties, hence it is necessary to apply P.

Now the question is whether these maps I, J are compatible, that is, whether

$$L_D \overset{?}{=} \iota J + Ip = B\iota_D P + P\iota_D B$$

Thus we need to know whether

$$L_D = B(P\iota_D P) + (P\iota_D P)B \quad \text{on } C_{inv}?$$

April 4, 1990

Observation: $1 = P + bGs + Gsb$ is a decomposition into mutually annihilating idempotents. In degree n

[redacted]

$$\mathrm{Im}(Gsb) = \mathrm{Im}(1-\lambda) \subset sC^{n+1}$$

and $\mathrm{Im}(bGs) \xrightarrow[\sim]{s} \mathrm{Im}(1-\lambda) \subset sC^n$ is the canonical lift of $(1-\lambda)sC^n$ into the subspace where $K^n - 1 = \lambda^{-1}sb = 0$.

Suppose we define C_{inv} to consist of cochains f such that sf and sbf are cyclic, or equivalently such that $Gsf = Gsbf = 0$. Note that $Gs = G(\lambda)s$ is defined using λ alone. One has $sbs = (1-\lambda)s$, hence $sb = 1-\lambda$ on completely reduced cochains. Hence

$$(Gsb)(Gsb) = G(1-\lambda)Gsb = Gsb$$
$$(bGs)(bGs) = bG(1-\lambda)Gs = bGs$$
$$(bGs)(Gsb) = 0 = (Gsb)(bGs)$$

showing Gsb, bGs are annihilating idempotents. Thus if P is defined by $1 = P + bGs + Gsb$, then P is an idempotent.

One has $(1-P)b = bGsb = b(1-P)$, so $[P, b] = 0$ and $s(1-P) = sbGs = (1-\lambda)Gs = (1-P(\lambda))s$, $(1-P)s = Gsbs$ $= G(1-\lambda)s = (1-P(\lambda))s$, so $[P, s] = 0$. Clearly $f \in C_{inv}$ $\Longrightarrow Pf = f$. Conversely $Pf = f \Longrightarrow b(Gs)f + (Gs)bf = 0$
$\Longrightarrow sbGsf = (1-\lambda)Gsf = (1-P(\lambda))sf = 0 \Longrightarrow sf$ cyclic
$\Longrightarrow Gsf = 0 \Longrightarrow Gsbf = 0 \Longrightarrow sbf$ cyclic.

<u>Program:</u> Let us consider A such that b is exact in degrees ≥ 2. For example A could be free, or the group algebra on a free group, (or maybe even some graph or quiver type algebra?). Thus the reduced cyclic cohomology in degrees ≥ 1 is stable and needn't be trivial. One has

$$0 \longrightarrow \bar{H}C^1 \longrightarrow \bar{H}H^1 \xrightarrow{B} \bar{H}C^0 \longrightarrow \bar{H}C^2 \longrightarrow \underset{0}{\overset{\parallel}{\bar{H}H^2}} \longrightarrow$$

showing that the stable odd and even cyclic groups are the kernel & cokernel of $B: \bar{H}H^1 \to \bar{H}C^0$.

The assumption that b is exact in degrees ≥ 2 says one has a quis

$$\begin{tikzcd}
C^0 \arrow[r, "b"] \arrow[d, equal] & ZC^1 \arrow[r] \arrow[d, phantom, "\cap"] & 0 \arrow[r] \arrow[d] & {} \arrow[d] \\
C^0_{inv} \arrow[r] & C^1_{inv} \arrow[r] & C^2_{inv} \arrow[r] & {}
\end{tikzcd}$$

Consider the corresponding inclusion of big periodic complexes

$$\left(\begin{tikzcd}
 & & & {} \\
 & & Z^1 \arrow[r] & C^0 \arrow[u] \\
 & Z^1 \arrow[r] & C^0 \arrow[u] & \\
{} \arrow[r] & C^0 \arrow[u] & &
\end{tikzcd}\right) \subset \left(\begin{tikzcd}
 & {} & {} & {} \\
{} \arrow[r] & C^2_{inv} \arrow[r] \arrow[u] & C^1_{inv} \arrow[r] \arrow[u] & C^0_{inv} \arrow[u] \\
{} \arrow[r] & C^1_{inv} \arrow[r] \arrow[u] & C^0_{inv} \arrow[u] & \\
{} \arrow[r] & C^0_{inv} \arrow[u] & &
\end{tikzcd}\right)$$

If we form the quotient, the result has exact columns, so the quotient has zero cohomology. This means the inclusion of big periodic complexes is a quis. In fact this is true also for the first quadrant complexes giving the cyclic cohomology.

Now suppose we have a derivation D on A. We know the effect L_D on the big periodic complex is homotopic to zero, hence it follows that L_D ~~on~~ the little periodic complex is homotopic to zero.

Our program is to construct an explicit homotopy for L_D on the little periodic complex.

One method is to find an explicit homotopy equivalence of the big and little periodic complexes. Suppose we denote the inclusion by $i : C' \longrightarrow C$, the endomorphism by L, and the homotopy by: $L = [d, H]$ — this is on the big complex. Then

$$[d, pHi] = p[d,H]i = pLi = piL = L$$

shows that pHi is a homotopy for L on the little complex, assuming that there is a retraction p of C onto C'.

Now we have $C = \mathcal{C}[u]$ with $d = b + u\tilde{s}$ and $\mathcal{C} = \mathcal{C}_{inv}$ and similarly $C' = \mathcal{C}'[u]$, and we are given $i : \mathcal{C}' \to \mathcal{C}$ commuting with $b, \tilde{s}$ which is a homotopy equivalence with respect to b. Here $\mathcal{C}' = \{\mathcal{C}^0 \to Z^1 \to 0 \to \cdots\}$. Thus we have columnwise homotopy equivalences which we want to refine to heq's of the total complex. This is the sort of task homological perturbation theory is used for.

Recall the formulas:

$$\begin{array}{c} E \\ u\big\downarrow\big\uparrow v \\ E' \end{array} \qquad \begin{array}{l} [d,u] = [d,v] = 0 \qquad vu = 1 + [d,h] \\ (d+\theta)^2 = [d,\theta] + \theta^2 = 0 \end{array}$$

Then the perturbed differential on E' and modified maps are

$$\theta' = u\,\theta\,\frac{1}{1-h\theta}\,v \qquad V = \frac{1}{1-h\theta}\,v$$

$$U = u\,\frac{1}{1-\theta h} \qquad H = h\,\frac{1}{1-\theta h} = \frac{1}{1-h\theta}\,h$$

Special case: $hv = h^2 = uh = 0$, $uv = 1$

Then the same is true for U, V, H.

In the case we wish to apply these we have start say with $\mathcal{C}' \underset{\iota}{\overset{p}{\rightleftarrows}} \mathcal{C} \circlearrowleft h$ satisfying $[b,\iota] = [b,p] = 0$, $p\iota = 1$, $\begin{cases} \iota p = 1 + [b,h]. \\ h\iota = \iota^2 = ph = 0 \end{cases}$ Then we consider

$$d = b \qquad \theta = \hat{u}\tilde{s}$$

$$\begin{array}{c} E = \mathcal{C}[\hat{u}] \\ u = p \big\downarrow\big\uparrow v = i \\ E' = \mathcal{C}'[\hat{u}] \end{array}$$

We assume that $\tilde{s}\mathcal{C}' \subset \mathcal{C}'$ so that $\tilde{s}$ can be defined on $\mathcal{C}'$ so that $\tilde{s}\iota = \iota\tilde{s}$. Then

$$V = \frac{1}{1-h\theta}\,v = \frac{1}{1-h\hat{u}\tilde{s}}\,i = \iota$$

since $h\tilde{s}\iota = h\iota\tilde{s} = 0$. Also

$$\theta' = u\,\theta\,\frac{1}{1-h\theta}\,v = p(\hat{u}\tilde{s})\,\frac{1}{1-h\hat{u}\tilde{s}}\,i = \hat{u}\,p\,\tilde{s}\,\iota$$

$$= \hat{u}\,\tilde{s}\,p\,i = \hat{u}\,\tilde{s}$$

Thus $d + \theta' = b + \hat{u}\tilde{s}$ on $\mathcal{C}'[u]$ and $V = i$.

One has the new retraction

$$U = p\,\frac{1}{1-\tilde{s}\hat{u}h}.$$

and ~~[illegible]~~ one knows this and $V = i$ are compatible with the total differential $b + \hat{u}\tilde{s}$.

Now apply this to $\hat{u}^{-1}I$ on $C[u]$ which satisfies

$$[b, I] = 0 \qquad [\tilde{s}, I] = L.$$

Then $U\hat{u}^{-1}I\iota = p\dfrac{1}{1-\hat{u}\tilde{s}h}\hat{u}^{-1}I\iota$ is the homotopy for L we want on $C'[u]$. [redacted] See what it is in even degrees

$$\begin{tikzcd} C^1 & C^0 \arrow[l, "I^0"'] \\ C^0 \arrow[u, "h"] \arrow[r, "\tilde{s}"] & 0 \end{tikzcd}$$

so you get $pI^0 : C^0 \longrightarrow Z^1$. In odd degrees

$$\begin{tikzcd} C^2 \arrow[d, "h"] & C^1 \arrow[l, "I^1"'] \\ C^1 \arrow[r, "\tilde{s}"] & C^0 \end{tikzcd} \qquad p(C^2) = 0$$

and you get $\tilde{s}hI^1\iota : Z^1 \longrightarrow C^0$. [redacted] It should now be possible to check these formulas directly.

Let's check the formulas needed.

$$\begin{tikzcd} C^0 \arrow[r, "b"] \arrow[d, equal] & Z^1 \arrow[d, shift right, "\iota"'] \\ C^0 \arrow[r, "b"] & C^1 \arrow[u, shift right, "p"'] \arrow[r, "b"] & C^2 \arrow[l, bend left, "h"] \arrow[r, "b"] & {} \arrow[l, bend left, "h"] \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "\tilde{s}"'] & C^2 \arrow[l, bend right, "I^2"'] \arrow[r, "\tilde{s}"'] & C^1 \arrow[l, bend right, "I^1"'] \arrow[r, "\tilde{s}"'] & C^0 \arrow[l, bend right, "I^0"'] \end{tikzcd}$$

$p\iota = 1$

$\iota p = 1$ [redacted] hb $\qquad ph = 0$

$1 = hb + bh$ on C^2

$\tilde{s}I^0 = L$

$\tilde{s}I^1 + I^0\tilde{s} = L$

$bI^0 + I^1 b = 0$

$bI^1 + I^2 b = 0$

Let's see if these formulas suffice.

Let $\xi \in Z^1 \subset C^1$

$$L\xi = \tilde{s} I^1 \xi + I^0 \tilde{s} \xi$$

Note $b(I^0 \tilde{s}\, \xi) = +I^1 \tilde{s}\, b\xi = 0$ so $I^0 \tilde{s}\, \xi \in Z^1$

and so $I^0 \tilde{s}\, \xi = (pI^0)\tilde{s}\, \xi$. Next

$$I^1 \xi \in C^2 \quad \text{and} \quad bI^1\xi = -I^2 b\xi = 0$$

and $I^1\xi = bhI^1\xi + h\cancel{bI^1}\xi = bhI^1\xi$

Thus

$$L\xi = \tilde{s} bhI^1\xi + pI^0 \tilde{s}\xi$$

$$= -b(\tilde{s}hI^1)\xi + (pI^0)\tilde{s}\xi$$

proving the homotopy formula at Z^1.

Next let $\eta \in C^0$. Then

$$L\eta = \tilde{s} I^0 \eta = \tilde{s}(pI^0)\eta + \tilde{s}(\underbrace{I^0\eta - pI^0\eta}_{hbI^0\eta})$$

$$= \tilde{s}(pI^0)\eta - (\tilde{s}hI^1)b\eta$$

proving the homotopy formula at C^0.

The next stage is to get the formulas for I^0, I^1, I^2 straight.

April 5, 1990

Review formulas

$$d\,\iota_D^* + \iota_D^*\,d = L_D \qquad \text{on } \Omega A$$

where $$\iota_D^* = \sum_{j=0}^{n-1} \kappa^j \iota_D \kappa^{-j} \qquad \text{on } \Omega^n A$$

Check: ι_D^* is the degree -1 derivation of ΩA with $\iota_D^* da = Da$. Thus

$$\iota_D^*(a_0 da_1 \dots da_n) = \sum_{j=1}^{n} \underbrace{(-1)^{j-1} a_0 da_1 \dots da_{j-1}\, Da_j\, da_{j+1} \dots da_n}_{\iota_D^{(j)}(a_0 da_1 \dots da_n)}$$

Claim $\iota_D^{(j)}\omega = \kappa^{j-1}\iota_D\kappa^{-j+1}\omega$. To prove this one can suppose $\omega = \omega_1\omega_2\omega_3$ with $\omega_1 \in \Omega^{j-1}A$, $\omega_2 \in \Omega^1 A$, $\omega_3 \in \Omega^{n-j}A$, and further that ω_1 is of the form $d\eta$. Then

$$\kappa^{j-1}\iota_D\kappa^{-j+1}(\omega_1\omega_2\omega_3) = (-1)^{(j-1)(n-1)}\kappa^{j-1}\iota_D(\omega_2\omega_3\omega_1)$$

$$= (-1)^{(j-1)(n-1)}\kappa^{j-1}(\iota_D(\omega_2)\,\omega_3\,\omega_1)$$

$$= \underbrace{(-1)^{(j-1)(n-1)}(-1)^{(j-1)(n-2)}}_{(-1)^{j-1}}\,\omega_1\,\iota_D(\omega_2)\,\omega_3 = \iota_D^{(j)}(\omega_1\omega_2\omega_3)$$

Other formulas needed are

$$\kappa^n(a_0 da_1 \dots da_n) = da_1 \dots da_n\, a_0$$

$$= a_0 da_1 \dots da_n + \underbrace{[da_1 \dots da_n, a_0]}_{(-1)^n b(da_1 \dots da_n\, da_0)} = (b\lambda^{-1}d)(a_0 da_1 \dots da_n)$$

$$\therefore \boxed{\kappa^n = 1 + b\lambda^{-1}d \qquad \text{on } \Omega^n}$$

$$\boxed{\kappa^{n+1} = \kappa + bd = 1 - db}$$

$$\boxed{\kappa^{n(n+1)} = 1 - Bb}$$

(Bb is the preferred order on ΩA, since b lowers degree).

$$\tilde{K} = K + \frac{1}{n(n+1)} Bb$$

$$K^j = \tilde{K}^j - \frac{j}{n(n+1)} Bb$$

$$\boxed{K^j \tilde{K}^{-j} = 1 - \frac{j}{n(n+1)} Bb}$$

Return to

$$d\left(\sum_{j=0}^{n-1} K^j \iota_D K^{-j}\right) + \left(\sum_{j=0}^{n} K^j \iota_D K^{-j}\right) d = L_D \quad \text{on } \Omega^n A$$

and apply P to both sides. Put $\iota_D^\# = P\iota_D P$. Use that $dK^j = d\tilde{K}^j$ since $dB = 0$. Thus we obtain

$$d\iota_D^\#\left(\sum_{j=0}^{n-1} \underbrace{\tilde{K}^{-j} K^j}_{1+\frac{j}{n(n+1)}Bb}\right) + \left(\sum_{j=0}^{n} \underbrace{K^j \tilde{K}^{-j}}_{1-\frac{j}{n(n+1)}Bb}\right) \iota_D^\# d = PL_DP$$

$$= d\iota_D^\#\left(n + \frac{\cancel{n}(n-1)}{2\cancel{n}(n+1)} Bb\right) + \left(n+1 - \frac{\cancel{n(n+1)}}{2\cancel{n(n+1)}} Bb\right)\iota_D^\# d$$

Next use $B = (n+1)Pd$ on $\Omega^n A$

$$PL_DP = B\iota_D^\# + \iota_D^\# B + \left(\frac{n-1}{2n(n+1)}\right) B\iota_D^\# Bb + \left(-\frac{1}{2(n+1)} Bb\,\iota_D^\# B\right)$$

$$= B\iota_D^\# + \iota_D^\# B + \frac{1}{2n(n+1)}(n-1-n)\, B\iota_D^\# Bb$$

$$\boxed{PL_DP = B\iota_D^\# + \iota_D^\# B - \frac{1}{2n(n+1)} B\iota_D^\# Bb}$$

$$\boxed{\text{where } \iota_D^\# = P\iota_D P}$$

Here's what I've been missing the past week. Start from

$$d\,\iota_D^* + \iota_D^*\,d = L_D \qquad \text{where}$$

$$\iota_D^* = \sum_{j=0}^{n-1} \kappa^j \iota_D \kappa^{-j} \qquad \text{on } \Omega^n A$$

Set $\quad I_D = \frac{1}{n} P \iota_D^* P \quad$ on $(\Omega^n A)_{inv}$.

Since $B = (n+1)Pd = (n+1)dP$ on $\Omega^n A$, we have $\quad BI_D + I_D B = L_D \quad$ on $(\Omega^\bullet A)_{inv}$. We claim $\quad bI_D + I_D b = 0$. It suffices to show that $\quad b(I_D - P\iota_D P) = (I_D - P\iota_D P)b = 0,$ since we know $[b, \iota_D] = 0$. But

$$b\,P(\kappa^j \iota_D \kappa^{-j})P = P\,\kappa^j b \iota_D \kappa^{-j} P$$
$$= P\,\tilde{\kappa}^j b\,\iota_D \kappa^{-j} P = P\,b\,\iota_D \kappa^{-j} P$$
$$= -P\,\iota_D b \kappa^{-j} P = -P\,\iota_D b\,\tilde{\kappa}^{-j} P = -P\iota_D b P$$
$$= b\,(P\iota_D P)$$

so this is clear.

The point of this proof is that

$$b\,\iota_D^* = b \sum_{j=0}^{n-1} \kappa^j \iota_D \kappa^{-j}$$

$$= b \sum_{j=0}^{n-1} \tilde{\kappa}^j \iota_D \tilde{\kappa}^{-j} \qquad \text{since } b \text{ kills } \tilde{\kappa} - \kappa.$$

Hence $\quad b\,P\iota_D^* P = b\,n\,P\iota_D P. \quad \therefore \quad bI_D = b(P\iota_D P)$

Similarly $\quad I_D b = (P\iota_D P)b.$

Problem. Consider the endom. L_D of the basic exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \bar{A}_\lambda^{\otimes n} \arrow[r, hook, "i"] \arrow[d, "L_D"] & (A\otimes\bar{A}^{\otimes n})_{inv} \arrow[r, two heads, "j"] \arrow[d, "L_D"] \arrow[dl, dashed, "\beta"'] & \bar{A}_\lambda^{\otimes n+1} \arrow[r] \arrow[d, "L_D"] \arrow[dl, dashed, "\alpha"'] & 0 \\ 0 \arrow[r] & \bar{A}_\lambda^{\otimes n} \arrow[r, hook, "i"] & (A\otimes\bar{A}^{\otimes n})_{inv} \arrow[r, two heads, "j"] & \bar{A}_\lambda^{\otimes n+1} \arrow[r] & 0 \end{tikzcd}$$

We have seen that a map α such that $j\alpha = L_D$ can be completed uniquely to pair (α,β) which is a null-homotopy with respect to the differential (i,j). The easy part of Goodwillie-Rinehart gives obvious candidates for α and β and the problem is to see whether these obvious candidates are compatible.

Note that $ij = B$, and j is the obvious map from the Hochschild complex to cyclic complex, while i is the B map essentially.

Recall the easy choice for α

$$(a_0,\dots,a_n) \xmapsto{B=i} \sum_{i=0}^{n} (-1)^{in} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1})$$

$$\xmapsto{L_D} \sum_{i=0}^{n} (-1)^{in} (Da_i, a_{i+1}, \dots, a_n, a_0, \dots, a_{i-1})$$

$$\xmapsto{j} \sum_{i=0}^{n} (a_0, \dots, a_{i-1}, Da_i, \dots, a_n)$$

so $\alpha = P\iota_D B = \iota_D^{\#} B$. Actually the nice diagram is

$$\begin{tikzcd} (A\otimes\bar{A}^{\otimes n-1})_{inv} \arrow[r, two heads, "j"] \arrow[d, "L_D"] \arrow[rr, bend left, "B"] & \bar{A}_\lambda^{\otimes n} \arrow[r, hook, "i"] & (A\otimes\bar{A}^{\otimes n})_{inv} \arrow[r, two heads, "j"] \arrow[d, "L_D"] \arrow[dll, dashed] \arrow[rr, bend left, "B"] & \bar{A}_\lambda^{\otimes n+1} \arrow[r, hook, "i"] \arrow[dl, dashed, "\alpha"'] & (A\otimes\bar{A}^{\otimes n+1})_{inv} \arrow[d, "L_D"] \arrow[dll, dashed, "1"] \arrow[dl, dashed, "\beta"] \\ (A\otimes\bar{A}^{\otimes n-1})_{inv} \arrow[r, two heads, "j"] & \bar{A}_\lambda^{\otimes n} \arrow[r, hook, "i"] & (A\otimes\bar{A}^{\otimes n})_{inv} \arrow[r, "j"] & \bar{A}_\lambda^{\otimes n+1} \arrow[r, hook, "i"] & (A\otimes\bar{A}^{\otimes n+1})_{inv} \end{tikzcd}$$

The easy choices for α and β are obtained by taking the dotted arrow to be $P \iota_D P = \iota_D^{\#}$ and then

$$\alpha = \iota_D^{\#} \iota \qquad (\text{so } \alpha j = \iota_D^{\#} B)$$

$$\beta = j \iota_D^{\#}$$

April 6, 1990

Recall $I_D = \frac{1}{n} P \iota_D^* P = \frac{1}{n}\sum_{j=0}^{n-1} K^j \iota_D^{\#} K^{-j}$ on Ω^n satisfies $\qquad \iota_D^{\#} = P\iota_D P$

$$PL_D = BI_D + I_D B \qquad\qquad bI_D + I_D b = 0$$

It is the good homotopy for the Goodwillie-Rinehart theorem. One has

$$\boxed{I_D - \iota_D^{\#} = \frac{1}{2n}(B\iota_D^{\#})b + \frac{n-1}{2n(n+1)}(\iota_D^{\#}B)b} \qquad \text{on } \Omega^n$$

$$\begin{tikzcd}
\Omega^{n-1}_{inv} \arrow[r, two heads, "j"] \arrow[d] \arrow[rr, bend left, "B"] & C^\lambda_{n-1} \arrow[r, hook, "\iota"] & \Omega^n_{inv} \arrow[r, two heads, "j"] \arrow[d] \arrow[dll, "I_D"'] \arrow[rr, bend left, "B"] & C^\lambda_n \arrow[r, hook, "\iota"] & \Omega^{n+1} \arrow[d] \arrow[dll, "I_D"'] \\
\Omega^{n-1}_{inv} \arrow[r, two heads, "j"] \arrow[rr, bend right, "B"'] & C^\lambda_{n-1} \arrow[r, hook, "\iota"] & \Omega^n_{inv} \arrow[r, two heads, "j"] \arrow[rr, bend right, "B"'] & C^\lambda_n \arrow[r, hook, "i"] & \Omega^{n+1}_{inv}
\end{tikzcd}$$

Note $ij = B$. Consider the maps $\underbrace{\quad}_{}^{I_D i}: C^\lambda_n \to \Omega^n_{inv}$ and $jI_D: \Omega^n_{inv} \longrightarrow C^\lambda_{n-1}$ induced by I_D.

Since

$$I_D B = \iota_D^{\#} B + \frac{1}{2(n+1)} B\iota_D^{\#} B$$

and j is surjective, one has

$$I_D i = \left(\iota_D^{\#} + \frac{1}{2(n+1)} B\iota_D^{\#} b\right) i = \left(1 - \frac{1}{2(n+1)} Bb\right)\iota_D^{\#} i$$

Since

$$BI_D = B\iota_D^{\#} + \frac{n-1}{2n(n+1)}(B\iota_D^{\#}B)b$$

and ι is injective, one has

$$jI_D = j\left(\iota_D^{\#} + \frac{n-1}{2n(n+1)}\iota_D^{\#}Bb\right)$$

$$= j\iota_D^{\#}\left(1 + \frac{n-1}{2n(n+1)}Bb\right)$$

On the other hand suppose we want ~~the~~ to find the map $C^\lambda_n \to \Omega^n_{\natural}$ corresponding to $\jmath\,\iota_D^\#: \Omega^n_{\natural} \to C^\lambda_{n-1}$

Compute: $PL_D - B\iota_D^\# = B(I_D - \iota_D^\#) + I_D B$

$$= \frac{n-1}{2n(n+1)} B\iota_D^\# Bb + \iota_D^\# B + \frac{1}{2(n+1)} B\iota_D^\# bB$$

$$= \iota_D^\# B + \underbrace{\left(\frac{-1}{2(n+1)} + \frac{n-1}{2n(n+1)}\right)}_{\frac{-1}{2n(n+1)}} Bb\,\iota_D^\# B$$

$$\therefore \boxed{PL_D - B\iota_D^\# = \left(1 - \frac{1}{2n(n+1)}Bb\right)\iota_D^\# B}$$

so we get

$$\begin{tikzcd}
\Omega^{n-1}_{\natural} \arrow[rr, "B"] \arrow[d, "L_D"] & & \Omega^{n}_{\natural} \arrow[rr, "B"] \arrow[d, "L_D"] \arrow[dll, "\iota_D^\#"'] & & \Omega^{n+1}_{\natural} \arrow[d, "\left(1-\frac{1}{2n(n+1)}Bb\right)\iota_D^\#"] \arrow[dll] \\
\Omega^{n-1}_{\natural} \arrow[rr, "B"'] & & \Omega^{n}_{\natural} \arrow[rr, "B"'] & & \Omega^{n+1}_{\natural}
\end{tikzcd}$$

Check: $B\iota_D^\# + \iota_D^\# B - \frac{1}{2n(n+1)} \underbrace{Bb\,\iota_D^\# B}_{B\iota_D^\# Bb} = L_D$? OK

Let's now return to the problem of showing L_D is nullhomotopic on the little periodic complex

$$\xrightarrow{b} \bar{A} \xrightarrow{d} (\Omega^1 A)_\natural \xrightarrow{b} \bar{A} \xrightarrow{d}$$

assuming $\Omega^1 A$ is projective as A-bimodule. I claim that it suffices to use ι_D instead of I_D:

Suppose $T \in (\Omega^1 A_\natural)^*$. Then

$$(\iota_D BT)(a_0 da_1) = (BT)(a_0 Da_1) = T(d(a_0 Da_1))$$
$$= T(da_0 Da_1 + a_0 d(Da_1))$$
$$(B\iota_D T)(a_0 da_1) = (\iota_D T)(da_0 da_1 - da_1 da_0)$$
$$= T(Da_0 da_1 - Da_1 da_0)$$

Because T is a trace on $\Omega^1 A$ one has $T(da_0 Da_1) = T(Da_1 da_0)$, hence

$$((\iota_D B + B\iota_D)T)(a_0 da_1) = T(Da_0 da_1 + a_0 d(Da_1))$$
$$= (L_D T)(a_0 da_1).$$

The assumption that $\Omega^1 A$ is projective implies that one has an exact sequence with splitting

$$0 \longrightarrow Z^1 \underset{i}{\overset{p}{\rightleftarrows}} C^1 \underset{b}{\overset{h}{\rightleftarrows}} Z^2 \longrightarrow 0$$

Now I want to show that L_D on

$$\begin{tikzcd} {} & (\bar{A})^* \arrow[l] \arrow[r, dashed, bend left, "p\iota_D"] & (\Omega^1 A)^*_\natural \arrow[l, "B"] \arrow[r, dashed, bend left, "-Bh\iota_D"] & (\bar{A})^* \arrow[l, "b"] & {} \arrow[l] \end{tikzcd}$$

is null-homotopic with the null-homotopy given by the dotted arrows.

Let $\tau \in (\bar{A})^*$. Then

$$\iota_D \tau = p\iota_D \tau + hb\iota_D \tau$$

$$L_D \tau = B\iota_D \tau = B(p\iota_D \tau) + Bhb\iota_D \tau$$
$$= B(p\iota_D)\tau + (-Bh\iota_D) b\tau$$

Next let $T \in (\Omega^1 A)_\natural^*$. Then

$$L_D T = B\,\iota_D T + \iota_D B T$$

Now $b(\iota_D T) = -\iota_D(bT) = 0$, so $\iota_D T \in Z^2$

so $\iota_D T = bh\iota_D T$. [redacted] So

$$L_D T = Bbh\iota_D T + \iota_D BT$$
$$= +b(-Bh\iota_D)T + \iota_D BT$$
$$= b(-Bh\iota_D)T + (p\iota_D)BT$$

where we use that $\iota_D BT$ is already in Z^1, either because $b\iota_D B = \iota_D Bb$, or because $L_D T$ [redacted] and $b(-Bh\iota_D)T$ [redacted] are in Z^1.

Actually to carry the above construction one only needs to assume the exactness of

$$0 \longrightarrow Z^1 \xrightarrow{i} C^1 \xrightarrow{b} Z^2 \longrightarrow 0$$

that is, vanishing of $H^2(A, A^*)$ or $HH_2(A)$.

Note that we have exact sequences

$$0 \to HH_n(A) \longrightarrow (\Omega^n A)_\natural \xrightarrow{b} \Omega^{n-1} A \longrightarrow (\Omega^{n-1} A)_\natural \longrightarrow 0$$

$$b(a_0 da_1 \cdots da_n) = (-1)^{n-1}[a_0 da_1 \cdots da_{n-1}, a_n]$$

In particular ~~[illegible]~~ when $HH_n(A) = 0$, one has

$$0 \longrightarrow (\Omega^2 A)_\natural \xrightarrow{b} \Omega^1 A \longrightarrow (\Omega^1 A)_\natural \longrightarrow 0$$

Recall that the A-bimodule sequence

$$\xrightarrow{b'} A\otimes\bar{A}^{\otimes 2}\otimes A \xrightarrow{b'} A\otimes\bar{A}\otimes A \xrightarrow{b'} A\otimes A$$

gives the complex $C^{\cdot}(A,M)$ of normalized cochains with values in the bimodule M upon applying $\mathrm{Hom}_{A\otimes A^{op}}(?, M)$. This sequence is a free A-bimodule resolution of A, which is why the cohomology $H^{\cdot}(A,M)$ of the complex $C^{\cdot}(A,M)$ coincides with $\mathrm{Ext}^{\cdot}_{A\otimes A^{op}}(A,M)$.

The cokernel of b' in degree n represents n-cocycles. Here's a canonical n-cocycle. Start with $d: A \longrightarrow \Omega^1 A$ which is a 1-cocycle and take its n-fold cup product

$$d^{\cup n}: A^n \longrightarrow \overbrace{\Omega^1 A\otimes_A \cdots \otimes_A \Omega^1 A}^{n\text{-times}} = \Omega^n A$$

$$d^{\cup n}(a_1,\dots,a_n) = da_1\cdots da_n$$

Thus we have a canonical bimodule map

$$A\otimes\bar{A}^{\otimes n}\otimes A \longrightarrow \Omega^n A$$

$$(a_0,\dots,a_{n+1}) \longmapsto a_0\, da_1\cdots da_n\, a_{n+1}$$

which kills the image of b'. It turns out that this map gives an isomorphism $\mathrm{Coker}\, b' \xrightarrow{\sim} \Omega^n A$; equivalently $d^{\cup n}$ is a universal n-cocycle.

Consider $s: A\otimes\bar{A}^{\otimes n}\otimes A \longrightarrow A\otimes\bar{A}^{\otimes n+1}\otimes A$

$s(a_0,\dots,a_{n+1}) = (1,a_0,\dots,a_{n+1})$. s is a right A-module map such that $b's+sb'=1$ in degrees ≥ 1. (It's also true in degree 0 provided one includes $A\otimes A \xrightarrow{b'=\text{mult}} A$ and $A \xrightarrow{s=1\otimes ?} A\otimes A$.) Also $s^2=0$. This means the b'-sequence is exact and that $b's$ is a projector with image $= \mathrm{Im}(b')$.

Consider the ~~situation~~ in degree 1. Because 1-cocycles are the same as derivations we have $\mathrm{Coker}(b') \xrightarrow{\sim} \Omega^1 A$.

$$\begin{tikzcd} 1\otimes a\otimes 1 \arrow[rr, mapsto] \arrow[d, mapsto, bend right] & & a\otimes 1 - 1\otimes a = [a, 1\otimes 1] \\ A\otimes \bar{A}\otimes A \arrow[rr, "b'"] \arrow[dr] & & A\otimes A \\ da \arrow[urr, mapsto] & \Omega^1 A \arrow[ur, hook] & \end{tikzcd}$$

Notice that
$$A\otimes \Omega^1 A \longrightarrow A\otimes \bar{A}\otimes A \quad \text{is bijective.}$$
$$a\otimes a_0 da_1 \longmapsto a\, s(a_0 da_1)$$

Consider next the map of A-bimodules
$$\Omega^n A = \Omega^1 A \otimes_A \Omega^{n-1}A \longrightarrow (A\otimes A)\otimes_A \Omega^{n-1}A \cong A\otimes \Omega^{n-1}A$$
$$da_1 \cdots da_n \longmapsto (a_1\otimes 1 - 1\otimes a_1)\otimes_A da_2\cdots da_n \longmapsto$$
$$a_1\otimes da_2\cdots da_n - 1\otimes a_1 da_2\cdots da_n$$

This sends $d^{\cup n}$ to $(a \mapsto [a, 1\otimes 1]) \cup d^{\cup(n-1)}$. Since the exact sequence
$$0 \longrightarrow \Omega^1 A \longrightarrow A\otimes A \xrightarrow{\text{mult}} A \longrightarrow 0$$
splits as right A-modules with the splitting given by s, we obtain by applying $? \otimes_A \Omega^{n-1}A$ an exact sequence of A-bimodules

$$(**) \qquad 0 \longrightarrow \Omega^n A \longrightarrow A\otimes \Omega^{n-1}A \xrightarrow{\text{mult}} \Omega^{n-1}A \longrightarrow 0$$
$$da_1\cdots da_n \longmapsto a_1\otimes da_2\cdots da_n - 1\otimes a_1 da_2 \cdots da_n$$

. Let's check one has a commutative diagram

$$\begin{tikzcd}
A\otimes\bar{A}^{\otimes n}\otimes A \arrow[rr, "b'"] \arrow[d, equal] & & A\otimes\bar{A}^{\otimes n-1}\otimes A \arrow[d, equal] \\
A\otimes\Omega^n A \arrow[dr, "\text{mult}"'] & & A\otimes\Omega^{n-1}A \\
& \Omega^n A \arrow[ur, hook, "\text{from } (**)"'] &
\end{tikzcd}$$

Here the vertical maps are obtained from the standard right A-module isom. $\Omega^n A = \bar{A}^{\otimes n}\otimes A$, $da_1\cdots da_n a \leftrightarrow (a_1,\dots,a_n,a)$.

The path is

$$(1,a_1,\dots,a_n,1)\longmapsto 1\otimes da_1\cdots da_n \longmapsto da_1\cdots da_n$$

$$\longmapsto a_1\otimes da_2\cdots da_n - 1\otimes a_1 da_2\cdots da_n$$

The path is

$$(1,a_1,\dots,a_n,1)\longmapsto b'(1,a_1,\dots,a_n,1)$$

$$=(a_1,\dots,a_n,1)+\sum_{i=1}^{n-1}(-1)^i(1,\dots,a_ia_{i+1},\dots,1)+(-1)^n(1,a_1,\dots,a_n)$$

$$\downarrow$$

$$a_1\otimes da_2\cdots da_n+\sum_{i=1}^{n-1}(-1)^i\,1\otimes da_2\cdots d(a_ia_{i+1})\cdots da_n+(-1)^n\,1\otimes da_1\cdots da_{n-1}a_n$$

$$=a_1\otimes da_2\cdots da_n - 1\otimes a_1da_2\cdots da_n$$

Thus one sees that $\Omega^n A$ is the cokernel of b' in degree n, hence it represents n-cocycles.

The significant thing about the above discussion is that the higher degree cocycles are naturally understood inductively.

April 7, 1990

Let $\partial : A \to A \otimes A$, $\quad \partial a = a\otimes 1 - 1 \otimes a$,

and consider

$$\Omega^n A \xrightarrow{\iota_\partial^{(n)}} \Omega^{n-1}A \otimes_A (A\otimes A) = \Omega^{n-1}A \otimes A$$

$$\omega_{n-1}\, da \longmapsto (-1)^{n-1}\omega_{n-1} \otimes_A (a\otimes 1 - 1\otimes a) = (-1)^{n-1}(\omega a \otimes 1 - \omega \otimes a)$$

Claim

$$\begin{tikzcd} A\otimes \bar{A}^{\otimes n}\otimes A \arrow[rr, "b'"] \arrow[d, equal] & & A\otimes \bar{A}^{\otimes n-1}\otimes A \arrow[d, equal] \\ \Omega^n A\otimes A \arrow[r, "\text{mult}"] & \Omega^n A \arrow[r, "\iota_\partial^{(n)}"] & \Omega^{n-1}A\otimes A \end{tikzcd}$$

commutes.

$$a_0 da_1 \cdots da_n \otimes a_{n+1} \xmapsto{\text{mult}} a_0 da_1 \cdots da_n a_{n+1}$$

$$\xmapsto{\iota_\partial^{(n)}} (-1)^{n-1} a_0 da_1 \cdots da_{n-1}(a_n \otimes 1 - 1\otimes a_n)\, a_{n+1}$$

$$= \underbrace{(-1)^{n-1} a_0\, da_1 \cdots da_{n-1} a_n}_{} \otimes a_{n+1} + (-1)^n a_0 da_1 \cdots da_{n-1} \otimes a_n a_{n+1}$$

$$\left((-1)^{n-1}\theta\, d\theta^{n-1}\theta\right)(a_0, \ldots, a_n)$$

$$b'(\theta d\theta^{n-1}) = \cancel{\theta^2 d\theta^{n-1}} - \theta\left(\cancel{\theta d\theta^{n-1}} - (-1)^{n-1} d\theta^{n-1}\theta\right)$$

$$= (-1)^{n-1}\theta\, d\theta^{n-1}\theta$$

so it checks.

Next

$\eta \otimes a \mapsto a\eta$

$$\begin{tikzcd} \Omega^n A\otimes A \arrow[r, "m"] \arrow[d] & \Omega^n A \arrow[r, "\iota_\partial^{(n)}"] \arrow[d] & \Omega^{n-1}A\otimes A \arrow[d] \\ \Omega^n A \arrow[r] & (\Omega^n A)_\natural \arrow[r] & \Omega^{n-1}A \end{tikzcd}$$

$\deg \omega = n-1$

$$\omega\, da \otimes a' \longmapsto \omega\, da\, a' \longmapsto (-1)^{n-1}(\omega a \otimes a' - \omega \otimes a a')$$

$$\begin{tikzcd} \omega\, da\otimes a' \arrow[d, mapsto] & (-1)^{n-1}(\omega a \otimes a' - \omega \otimes a a') \arrow[d, mapsto] \\ a'\omega\, da \arrow[r, mapsto, "b"] & (-1)^{n-1}[a'\omega, a] \end{tikzcd}$$

showing this version of b' is compatible with the formula

$$\boxed{b(\omega da) = (-1)^{|\omega|}[\omega, a]}$$

Thus we have

$$\boxed{b'(\omega da \otimes a') = (-1)^{|\omega|}(\omega a \otimes a' - \omega \otimes aa')}$$

Consider now the case of a free algebra $A = T(V)$. We have the exact sequence

$$(a_0, a_1, a_2) \longmapsto a_0 da_1 a_2$$

$$0 \longrightarrow \Omega^2 A \longrightarrow A \otimes \bar{A} \otimes A \longrightarrow \Omega^1 A \longrightarrow 0$$

$$A \otimes \bar{A} \otimes A = \Omega^1 A \otimes A \xrightarrow{\text{mult}} \Omega^1 A$$

$$a_0 da_1 \otimes a_2$$

$$da_1 da_2 \longmapsto (a_1, a_2, 1) - (1, a_1a_2, 1) + (1, a_1, a_2)$$

$$\updownarrow$$

$$a_1 da_2 \otimes 1 - d(a_1a_2) \otimes 1 + da_1 \otimes a_2$$

$$= -da_1 a_2 \otimes 1 + da_1 \otimes a_2$$

of A-bimodules. Write it again

$$(*) \qquad 0 \longrightarrow \Omega^2 A \longrightarrow A \otimes \bar{A} \otimes A \longrightarrow \Omega^1 A \longrightarrow 0$$

When we take bimodule homomorphisms into M we get

$$0 \longrightarrow Z^1(A, M) \longrightarrow C^1(A, M) \xrightarrow{\delta} Z^2(A, M)$$

which will be short exact when A is free, or more generally iff $\Omega^1 A$ is a projective bimodule.

In the free case we have a canonical splitting of $(*)$ as a sequence of bimodules. In effect a 2-cocycle f with values in M

corresponding to an algebra extension E of A by M with linear lifting ρ given. One can construct a unique [redacted] lifting θ which is an algebra homom. and which agrees with ρ on the space V of generators. The difference $\theta - \rho = g$ is then a 1-cochain whose coboundary is f:

$$\rho(a_1 a_2) - \rho(a_1)\rho(a_2) = f(a_1, a_2)$$

$$(\rho + g)(a_1 a_2) = (\rho + g)(a_1)\,(\rho + g)(a_2) \quad \text{so}$$

$$f(a_1, a_2) = a_1 g(a_2) - g(a_1 a_2) + g(a_1) a_2$$

Let's apply this to the universal 2-cocycle $(a_1, a_2) \longmapsto da_1 da_2 \in \Omega^2 A$. The corresponding extension is $RA/IA^2 \simeq \Omega^0 A \oplus \Omega^2 A$ with multiplication given by $*$ product:

$$a_1 * a_2 = a_1 a_2 - da_1 da_2$$

Then

$$\rho(v_1 \cdots v_n) + g(v_1 \cdots v_n) = (\rho + g)(v_1) \cdots (\rho + g)(v_n)$$

$$= \rho(v_1) \cdots \rho(v_n) \qquad \text{since } g(V) = 0$$

$$= v_1 * \cdots * v_n$$

$$= v_1 \cdots v_n - \sum_{1 \le i < j \le n} v_1 \cdots v_{i-1}\, dv_i\, v_{i+1} \cdots v_{j-1}\, dv_j\, v_{j+1} \cdots v_n$$

Thus

$$\boxed{g(v_1 \cdots v_n) = -\sum_{1 \le i < j \le n} v_1 \cdots v_{i-1}\, dv_i\, v_{i+1} \cdots v_{j-1}\, dv_j\, v_{j+1} \cdots v_n}$$

is the unique 1-cochain on $A = T(V)$ such that $\boxed{g(v) = 0}$ and $\boxed{da_1 da_2 = a_1 g(a_2) - g(a_1 a_2) + g(a_1) a_2}$

As a check [redacted] note that [redacted] these two conditions imply

$$\boxed{g(v a) = v g(a) - dv\, da}$$

which leads by induction to the same formula for g.

So one has $gi = 1$ in

$$\begin{tikzcd} 0 \arrow[r] & \Omega^2 A \arrow[r, "i"] & A\otimes\bar{A}\otimes A \arrow[r] \arrow[l, bend right, "g"'] & \Omega^1 A \arrow[r] \arrow[l, bend right, dashed] & 0 \end{tikzcd}$$

[redacted] The dotted arrow corresponding to g is the unique lifting killed by g, and as $\Omega^1 A = A\otimes V\otimes A$, it must be the bimodule map [redacted] sending dv to $(1, v, 1)$.

Things become much nicer if we use $\Omega^1 A\otimes A$ instead of $A\otimes\bar{A}\otimes A$. Consider

$$\begin{tikzcd} 0 \arrow[r] & \Omega^2 A \arrow[r, "i"] & \Omega^1 A\otimes A \arrow[r, "\mathrm{mult}"] \arrow[l, bend right, dashed, "g"'] & \Omega^1 A \arrow[r] \arrow[l, bend right, dashed, "h"'] & 0 \end{tikzcd}$$

$$da_1\, da_2 \longmapsto -da_1 a_2\otimes 1 + da_1\otimes a_2$$

and for $A = T(V)$ define g by $g(dv a_1\otimes 1) = -dv\, da_2$, i.e.

$$\boxed{g(a_0\, dv\, a_1\otimes a_2) = -a_0\, dv\, da_1\, a_2}$$

and $h(dv) = dv\otimes 1$, i.e.

$$\boxed{h(a_0\, dv\, a_1) = a_0\, dv\otimes a_2.}$$

Then $ig(dva\otimes 1) = -i(dv\, da) = dva\otimes 1 - dv\otimes a$

$hm(dva\otimes 1)$ [redacted] $= g(dva) = dv\otimes a$

$\therefore\ ig + gm = id.$ Also $mh(dv) = m(dv\otimes 1) = dv$

and

$$gi(a_0 dv_1 a_1 dv_2 a_2) = g\{-a_0 dv_1 a_1 v_2 \otimes a_2 + a_0 dv_1 a_1 \otimes v_2 a_2\}$$

$$= + a_0 dv_1 d(a_1 v_2) a_2 - a_0 dv_1 da_1 v_2 a_2$$

$$= a_0 dv_1 a_1 dv_2 a_2$$

General case

$$\begin{tikzcd} 0 \arrow[r] & \Omega^{n+1}A \arrow[r, "b'"'] & \Omega^n A \otimes A \arrow[l, bend right, dashed, "g"'] \arrow[r, "m"'] & \Omega^n A \arrow[r] \arrow[l, bend right, dashed, "h"'] & 0 \end{tikzcd}$$

$$g(\omega dv\, a \otimes a') = (-1)^{|\omega|+1}\, \omega dv\, da\, a'$$

$$b'(\omega\, da\, a') = (-1)^{|\omega|}(\omega a \otimes 1 - \omega \otimes a)\, a'$$

$$m = \text{mult}$$

$$h(\omega dv\, a) = \omega dv \otimes a \qquad \therefore mh = 1.$$

$$gh = 0.$$

$$\begin{tikzcd} & \omega dv a \otimes 1 \arrow[dl, "g"'] \arrow[dr, "m"] & \\ (-1)^{|\omega|+1} \omega dv\, da \arrow[d, "b'"] & & \omega dv a \arrow[d, "h"] \\ \omega dv a \otimes 1 - \omega dv \otimes a & & \omega dv \otimes a \end{tikzcd}$$

$$\therefore b'g + hm = 1$$

Finally we want $gb'(\omega dv)$. Take $\omega = \omega_1 dv_1 a$

Then $gb'(\omega_1 dv_1 a dv) = (-1)^{|\omega|+1} g(\omega_1 dv_1 a v \otimes 1 - \omega_1 dv_1 a \otimes v)$

$= \omega_1 dv_1 d(av) - \omega_1 dv_1 da\, v = \omega_1 dv_1 a\, dv. \qquad \therefore gb' = 1$

Next take commutator ~~quotient~~ spaces of these A-bimodule maps and we obtain

$$\begin{tikzcd} \Omega^{n+1}A_\natural \arrow[r, "b"'] & \Omega^n A \arrow[l, bend right, "g"'] \arrow[r, "\natural"'] & \Omega^n A_\natural \arrow[l, bend right, "h"'] \end{tikzcd}$$

$$b(\omega\, da) = (-1)^{|\omega|}[\omega, a]$$

$$g(\omega\, dv\, a) = (-1)^{|\omega|+1}\omega\, dv\, da$$

$$h(\omega\, dv) = \omega\, dv$$

$\therefore \natural h = 1$
$gh = 0$

$$\begin{tikzcd} & \omega\, dv\, a \arrow[dl, "g"'] \arrow[dr, "\natural"] & \\ (-1)^{|\omega|+1}\omega\, dv\, da \arrow[d, "b"] & & a\omega\, dv \arrow[d, "h"] \\ \omega\, dv\, a - a\omega\, dv & & a\omega\, dv \end{tikzcd}$$

$\therefore bg + h\natural = 1$

$$gb(\omega_1 dv_1 a\, dv) = (-1)^{|\omega_1|+1} g(\omega_1 dv_1 av - v\omega_1 dv_1 a)$$

$$= \omega_1 dv_1 d(av) - v\omega_1 dv_1 da$$

$$= \omega_1 dv_1 a\, dv + \underbrace{[\omega_1 dv_1 da, v]}_{0 \text{ in } (\Omega A)_\natural}.$$

$\therefore gb = 1$

So I now have a contracting homotopy for the Hochschild complex in degrees ≥ 1.

April 9, 1990

Suppose we have a split exact sequence

$$0 \longrightarrow (\Omega^2 A)_\natural \underset{b}{\overset{g}{\rightleftarrows}} \Omega^1 A \underset{\pi}{\overset{\rho}{\rightleftarrows}} (\Omega^1 A)_\natural \longrightarrow 0.$$

Let D be a derivation of A. Then we obtain a null homotopy for L_D on the little periodic complex of A as follows

$$\begin{tikzcd}
& \Omega^1 A_\natural \arrow[rr, "b"] \arrow[dl, "I_D\rho"'] & & \bar{A} \arrow[rr, "\pi d"] \arrow[dl, "-I_D g d"'] & & \Omega^1 A_\natural \arrow[dl, "I_D\rho"] \\
\bar{A} \arrow[rr, "\pi d"'] & & \Omega^1 A_\natural \arrow[rr, "b"'] & & \bar{A} &
\end{tikzcd}$$

Check:

$$(-I_D g d)b = +I_D g b B = I_D B$$

$$\pi d(I_D \rho) = d I_D$$
$$= B I_D$$

$$\begin{tikzcd}
\Omega^1 A \arrow[r, "\pi"] \arrow[d, "I_D"] & \Omega^1 A_\natural \arrow[d, "I_D"] \\
\bar{A} \arrow[r, "\pi"] & \bar{A}_\natural
\end{tikzcd}$$
exists as $[b, I_D] = 0$

$\therefore$ get $I_D B + B I_D = P L_D = L_D$ at $\Omega^1 A_\natural$

$$b(-I_D g d) = I_D b g\, d = I_D(1 - \rho\pi) d$$

$$\therefore b(-I_D g d) + (I_D\rho)(\pi d) = I_D d = I_D B = P L_D = L_D \quad \text{at } \bar{A}$$

So what's important in using I_D, as opposed to L_D, is that $[B, I_D] = P L_D$ and $P = 1$ on $\bar{A}$ and $\Omega^1 A_\natural$.

Next we want to work out the formulas in the case where $A = T(V)$.

But first we should understand $I_D : \Omega^2 A_\natural \to \Omega^1 A_\natural$. Recall this is defined ~~because~~ $[b, I_D] = 0$. $I_D = \frac{1}{2} P L_D^* P$ on $\Omega^2 A$

Recall $I_D - P L_D P$ is killed by b.

Therefore $I_D = \iota_D P : \Omega^2 A_\natural \longrightarrow \Omega^1 A_\natural$ so

$$I_D(a_0 da_1 da_2) = \frac{1}{2}\iota_D(a_0 da_1 da_2 - da_2 a_0 da_1)$$
(modulo Im bB)

$$= \frac{1}{2}(a_0 Da_1 da_2 - Da_2 a_0 da_1)$$ [redacted]

$$= \frac{1}{2}(a_0 Da_1 da_2 - a_0 da_1 Da_2) \in \Omega^1 A_\natural$$

$$\therefore \quad I_D = \frac{1}{2}\iota_D^* \quad \text{on } \Omega^2 A_\natural$$

What is $I_D = \iota_D P$ on $\Omega^1 A$?

On $\Omega^1 A$ one has $\tilde{K} = K + \frac{1}{2}Bb$ so

$$P = \frac{1+K}{2} + \frac{1}{4}Bb$$

$$P(a_0 da_1) = \frac{a_0 da_1 + da_1 a_0}{2} + \frac{1}{4}d[a_0, a_1]$$

$$\boxed{P(a_0 da_1) = \frac{a_0 da_1 - a_1 da_0}{2} + \frac{1}{4}d(a_0 a_1 + a_1 a_0)}$$

$$\tilde{K}(a_0 da_1) = da_1 a_0 + \frac{1}{2}d(a_0 a_1 - a_1 a_0)$$

$$\boxed{\tilde{K}(a_0 da_1) = -a_1 da_0 + \frac{1}{2}d(a_0 a_1 + a_1 a_0)}$$

$$\boxed{(\tilde{K}f)(x,y) = -f(y,x) + \frac{1}{2}f(1, xy + yx)}$$

So now consider $A = T(V)$ with

$$\begin{tikzcd} 0 \arrow[r] & (\Omega^2 A)_\natural \arrow[r] & \Omega^1 A \arrow[r] \arrow[l, bend right, "g"'] & \Omega^1 A_\natural \arrow[r] \arrow[l, bend right, "f"'] & 0 \end{tikzcd}$$

$$g(a_0 dv a_1) = -a_0 dv da_1$$

$$f(a dv) = a dv$$

Then the homotopy for ι_D consists of the operators

$$I_D \rho = \iota_D P \rho : \Omega^1 A_\natural \longrightarrow \bar{A}$$

$$-I_D g d = -\tfrac{1}{2}\iota_D^* g d : \bar{A} \longrightarrow \Omega^1 A_\natural$$

$$-\tfrac{1}{2}\iota_D^* g d(\sigma_1\cdots\sigma_n) = -\frac{1}{2}\iota_D^* g \sum_{i=1}^{n} \sigma_1\cdots\sigma_{i-1}\, d\sigma_i\, \sigma_{i+1}\cdots\sigma_n$$

$$= \frac{1}{2}\iota_D^* \sum_{i=1}^{n} \sigma_1\cdots\sigma_{i-1}\, d\sigma_i\, d(\sigma_{i+1}\cdots\sigma_n)$$

$$= \frac{1}{2}\iota_D^* \sum_{1\le i<j\le n} \sigma_1\cdots\sigma_{i-1}\, d\sigma_i\, \sigma_{i+1}\cdots\sigma_{j-1}\, d\sigma_j\, \sigma_{j+1}\cdots\sigma_n$$

$$= \frac{1}{2}\sum_{1\le i<j\le n} \left\{ \begin{array}{l} \sigma_1\cdots\sigma_{i-1}\, D\sigma_i\, \sigma_{i+1}\cdots\sigma_{j-1}\, d\sigma_j\, \sigma_{j+1}\cdots\sigma_n \\ -\sigma_1\cdots\sigma_{i-1}\, d\sigma_i\, \sigma_{i+1}\cdots\sigma_{j-1}\, D\sigma_j\, \sigma_{j+1}\cdots\sigma_n \end{array} \right\}$$

For example suppose $D\sigma = \sigma$. This is

$$= \frac{1}{2}\sum_{1\le i<j\le n} \left\{ \begin{array}{l} \sigma_1\cdots\sigma_{j-1}\, d\sigma_j\, \sigma_{j+1}\cdots\sigma_n \\ -\sigma_1\cdots\sigma_{i-1}\, d\sigma_i\, \sigma_{i+1}\cdots\sigma_n \end{array} \right\}$$

$$= \frac{1}{2}\sum_{1\le j\le n} (j-1)\, \sigma_{j+1}\cdots\sigma_n\sigma_1\cdots\sigma_{j-1}\, d\sigma_j$$
$$- \frac{1}{2}\sum_{1\le i<n} (n-i)\, \sigma_{i+1}\cdots\sigma_n\sigma_1\cdots\sigma_{i-1}\, d\sigma_i$$

$$= \sum_{1\le j\le n} \left(\frac{j-1-n+j}{2}\right) \sigma_{j+1}\cdots\cdots\sigma_{j-1}\, d\sigma_j$$

$$= \sum_{1\le j\le n} \underbrace{\left(j - \frac{n+1}{2}\right)}_{} \sigma_{j+1}\cdots\sigma_n\sigma_1\cdots\sigma_{j-1}\, d\sigma_j$$

sums to zero so we have the Green's operator for $1-\sigma$ times $\neq n$.

Next we have

$$
\begin{aligned}
I_{D\rho}(a\,dv) &= \iota_D P_\rho(a\,dv) \\
&= \iota_D P(a\,dv) \\
&= \iota_D\left\{\frac{a\,dv + dv\,a}{2} + \tfrac{1}{4} d[a,v]\right\} \\
&= \frac{a\,Dv + Dv\,a}{2} + \tfrac{1}{4} D(av - va)
\end{aligned}
$$

When $Dv = v$ we get

$$
= \frac{av + va}{2} + \tfrac{1}{4}(n+1)(av - va) \qquad \deg(a) = n.
$$

Thus we have for our homotopy in degree n.

$$
\begin{tikzcd}
V^{\otimes n} \arrow[r, shift right, "d = N"'] & V^{\otimes n} \arrow[l, shift right, "I_{D\rho} = \frac{1}{2}(1+\sigma) + \frac{1}{4}n(1-\sigma)"'] \arrow[r, shift right, "b = 1-\sigma"'] & V^{\otimes n} \arrow[l, shift right, "\sum_{j=1}^{n}\left(j - \frac{n+1}{2}\right)\sigma^{n-j}"']
\end{tikzcd}
$$

Check:
$$
\begin{aligned}
d I_{D\rho} + b\left(-\tfrac{1}{2}\iota_D^* g d\right) &= N\,\tfrac{1}{2}(1+\sigma) + (1-P)n \\
&= N + n - N = n.
\end{aligned}
$$

On the other hand we could have used ι_D instead of I_D, so that the homotopy is

$$
\begin{aligned}
(-\iota_D g d)(v_1 \cdots v_n) &= (-\iota_D g)\sum_{i=1}^{n} \cdots dv_i \cdots \\
&= \iota_D \sum_{1 \le i < j \le n} \cdots dv_i \cdots dv_j \cdots \\
&= \sum_{1 \le i < j \le n} v_1 \cdots v_{i-1}\, Dv_i\, v_{i+1} \cdots v_{j-1}\, dv_j\, v_{j+1} \cdots v_n
\end{aligned}
$$

When $Dv = v$

$$
= \sum_{j=1}^{n} (j-1)\, v_{j+1} \cdots v_n\, v_1 \cdots v_{j-1}\, dv_j
$$

$$(\iota_D \rho)(a\,dv) = \iota_D(a\,dv) = aDv$$
$$= av \quad \text{when} \quad Dv = v.$$

Thus in degree n

$$\begin{tikzcd} V^{\otimes n} \arrow[r, "d=N"'] & V^{\otimes n} \arrow[l, bend right, "1"'] \arrow[r, "b=1-\sigma"'] & V^{\otimes n} \arrow[l, bend right, "\sum_{j=1}^{n}(j-1)\sigma^{n-j} \;*"'] \end{tikzcd}$$

Check: $$(1-\sigma)\sum_{j=1}^{n}(j-1)\sigma^{n-j} = \sum_{j=1}^{n}(j-1)(\sigma^{n-j}-\sigma^{n-j+1})$$

$$= (n-1)(1-\sigma) + (n-2)(\sigma-\sigma^2) + (n-3)(\sigma^2-\sigma^3)$$
$$+ \cdots + 2(\sigma^{n-3}-\sigma^{n-2}) + (\sigma^{n-2}-\sigma^{n-1})$$

$$= n-1 - \sigma - \sigma^2 - \cdots - \sigma^{n-2} - \sigma^{n-1}$$
$$= n - N.$$

Thus I conclude that neither ι_D nor I_D produces the good contracting homotopy for L_D on the little periodic complex.

* Note that $\frac{1}{n}\sum_{j=1}^{n}(j-1)\sigma^{n-j} = \frac{1}{n}\sum_{j=0}^{n-1}(n-j-1)\sigma^j$ is ~~the~~ my first attempt at a Greens operator for $1-\sigma$:

$$1 - \frac{1}{n}N = \frac{1}{n}\sum_{i=0}^{n-1}(1-\lambda^i) = (1-\lambda)\frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\lambda^j$$
$$= (1-\lambda)\frac{1}{n}\sum_{j=0}^{n-1}(n-j-1)\lambda^j$$

April 11, 1990

Given D a derivation of A we have on ΩA derivations L_D, ι_D^* satisfying $[d, \iota_D^*] = L_D$, $\iota_D^{*2} = 0$. In fact we also have

$$[L_{D_1}, L_{D_2}] = L_{[D_1, D_2]}$$

$$[L_{D_1}, \iota_{D_2}^*] = \iota^*_{[D_1, D_2]}$$

$$[\iota_{D_1}^*, \iota_{D_2}^*] = 0$$

by the usual proofs: These are derivations of ΩA, and it suffices to check the formulas on the generators.

Now we would like the same sort of formulas to hold on $(\Omega A)_{inv}$. The idea up to now has be to use $I_D = \frac{1}{n} P \iota_D^* P$ on Ω^n, and we have a nice proof that $[b, I_D] = 0$ and $[B, I_D] = PL_D$. However we haven't been able to prove that $I_D^2 = 0$.

Idea: Instead of modifying d to $\tilde{d}$ satisfying $[b, \tilde{d}] = 1 - \tilde{\kappa}$, modify b or perhaps both b and d should be simultaneously modified.

Try $\qquad \tilde{b} = b + c_n P b \qquad$ on Ω^n

Then
$$\begin{aligned} \tilde{b} d &= bd + c_{n+1} P b d \\ &= bd + c_{n+1} b P d \\ &= bd + (c_{n+1}) \frac{1}{n+1} bB \end{aligned}$$

$$d\tilde{b} = db + c_n P d b = db + c_n \frac{1}{n} Bb$$

Thus

$$[\tilde{b}, d] = 1 - K + \left(\frac{c_n}{n} - \frac{c_{n+1}}{n+1}\right) Bb$$

$$\overset{?}{\|}$$

$$1 - \tilde{K} = 1 - \left(K + \frac{1}{n(n+1)} Bb\right)$$

We want

$$\frac{c_n}{n} - \frac{c_{n+1}}{n+1} = -\frac{1}{n(n+1)} = \frac{1}{n+1} - \frac{1}{n}$$

for $n \geq 1$, which has the solution

$$\frac{c_n}{n} + \frac{1}{n} = c \qquad \boxed{c_n = -1 + nc}$$

where c is a constant. On Ω^1 one has

$$\tilde{b} = (1 + c_1) b = (c) b$$

so we probably want to take $c = 1$. If we do then

$$\tilde{b} = (1 - P + nP) b \qquad \boxed{\text{In general } \tilde{b} = (1 - P + ncP) b}$$

and

$$\tilde{b} P = n \cdot Pb \qquad \text{on } \Omega^n$$

Now the hope is that $[\tilde{b}, \iota_D^*] = 0$?

$$\tilde{b}\, \iota_D^* = (1 - P + (n-1)P) b\, \iota_D^*$$

$$= (1 - P + (n-1)P) b \sum_{j=0}^{n-1} K^j \iota_D K^{-j}$$

$$= (1-P) b \sum_{j=0}^{n-1} K^j \iota_D K^{-j} + (n-1) \sum_{j=0}^{n-1} Pb\, \iota_D \underbrace{K^{-j}}_{\text{on } \Omega^n}$$

But $K^n = 1 + b\lambda^{-1} d$ on Ω^n

so $bK^n = b$ on Ω^n

$$\therefore \tilde{b}\, \iota_D^* = (1-P) b \sum_{j=0}^{n-1} K^j \iota_D K^{-j} + (n-1) n\, Pb\, \iota_D P$$

$$\iota_D^* \tilde{b} = \sum_{j=0}^{n-2} \kappa^j \iota_D \kappa^{-j} (1-P+nP) b$$

$$= \sum_{j=0}^{n-2} \kappa^j \iota_D \kappa^{-j} (1-P) b + n \sum_{j=0}^{n-2} \underbrace{\kappa^j \iota_D \cancel{\kappa^{-j}}}_{\text{on } \Omega^{n-2}} P b$$

On $\kappa^{n-1} = 1 - db$ on Ω^{n-2} so we have $\kappa^{n-1} \iota_D P b = -\kappa^{n-1} b \iota_D P = -b \iota_D P$, whence

$$\iota_D^* \tilde{b} = \sum_{j=0}^{n-2} \kappa^j \iota_D \kappa^{-j} (1-P) b + n(n-1) P \iota_D b P$$

So we end up with

$$\tilde{b} \iota_D^* + \iota_D^* \tilde{b} = (1-P) b \iota_D^* + \iota_D^* b (1-P)$$

Notice that the above calculations gives

$$P b \iota_D^* = P b \sum_{j=0}^{n-1} \cancel{\kappa^j} \iota_D \kappa^{-j} = n P b \iota_D P = n b \iota_D^\#$$
$$\kappa^n = 1 + b \lambda^{-1} d$$

$$\iota_D^* b P = \sum_{j=0}^{n-2} \kappa^j \iota_D \cancel{\kappa^{-j}} b P = (n-1) P \iota_D b P = (n-1) \iota_D^\# b$$
$$\kappa^{n-1} = 1 - db$$

Thus one has

$$[\tilde{b}, \iota_D^*] = b \iota_D^* + \iota_D^* b - n b \iota_D^\# + (n-1) b \iota_D^\#$$

$$= b \iota_D^* + \iota_D^* b - b \iota_D^\#$$