September 28, 1989

Notes concerning $R\otimes S$ and the tensor product of mixed complexes
$$(\Omega^1_R \rightleftarrows R)\otimes(\Omega^1_S \rightleftarrows S).$$

Suppose we start with the ∂ operator as basic. Then we are inclined to write the square

$$\begin{tikzcd} R\otimes S \arrow[r, "1\otimes\partial"] \arrow[d, "\partial\otimes 1"'] & R\otimes\Omega^1_{S\natural} \arrow[d, "\partial\otimes 1"] \\ \Omega^1_{R\natural}\otimes S \arrow[r, "-1\otimes\partial"] & \Omega^1_{R\natural}\otimes\Omega^1_{S\natural} \end{tikzcd}$$

rather than the square with $R\otimes S$ in the lower right corner. This square we recognize as a quotient of the multiple DGA
$$\Omega_R/\Omega_R^{\geq 2}\otimes\Omega_S/\Omega_S^{\geq 2},$$
so it natural to try to construct the cycle we seek generalizing e^ω, $\natural(\partial\rho e^\omega)$ by working in this DGA.

First let's review the case of R alone. Then we have the bigraded DGA
$$\Omega_R/\Omega_R^{\geq 2}$$
with differentials δ, ∂. Here ∂ will be treated as an operator of degree 1 resulting in some sign changes.

Given $\rho\in R^1$, we have
$$\tilde\omega = (\delta+\partial+\rho)^2 = \delta\rho+\rho^2+\partial\rho = \omega+\partial\rho$$

$$e^{\tilde\omega} = e^{\omega} + \underbrace{\int_0^1 e^{(1-s)\omega}\,\partial\rho\, e^{s\omega}\,ds}_{\mu}$$

$$[\delta+\rho+\partial,\ e^{\omega}+\mu] = 0$$

$$[\delta+\rho, e^{\omega}] = 0 \qquad \partial e^{\omega} + [\delta+\rho,\mu] = 0$$

These two equations lead to the proof that $e^{\omega} \in R$, $\natural(\partial\rho\, e^{\omega}) \in \Omega^1_{R\natural}$ ~~constitute~~ a cocycle. One uses: $\natural(\mu) = \natural(\partial\rho\, e^{\omega})$

We proceed similarly in the double case. Let's write d, ∂ for the differentials $\partial_R\otimes 1$, $1\otimes\partial_S$ on $\Omega_R\otimes\Omega_S$.

$$\tilde\omega = (\delta+d+\partial+\rho)^2 = \omega + d\rho + \partial\rho$$

$$e^{\tilde\omega} = e^{\omega} + \mu + \nu + \lambda$$

$$\mu = \int_0^1 e^{(1-s)\omega}\,d\rho\, e^{s\omega}\,ds \ \in\ \Omega^1_R\otimes S$$

$$\nu = \int_0^1 e^{(1-s)\omega}\,\partial\rho\, e^{s\omega}\,ds \ \in\ R\otimes\Omega^1_S$$

$$\lambda = \iint_{s+t\le 1} \left(e^{(1-s-t)\omega}\,d\rho\, e^{s\omega}\,\partial\rho\, e^{t\omega} + e^{(1-s-t)\omega}\,\partial\rho\, e^{s\omega}\,d\rho\, e^{t\omega}\right) ds\,dt$$

$$\in \Omega^1_R\otimes\Omega^1_S$$

$$[\delta+\rho+d+\partial,\ e^{\omega}+\mu+\nu+\lambda] = 0$$

$$[\delta+\rho, e^{\omega}] = 0 \qquad [\delta+\rho,\mu] + de^{\omega} = 0$$

$$[\delta+\rho,\nu] + \partial e^{\omega} = 0$$

$$[\delta+\rho,\lambda] + \partial\mu + d\nu = 0$$

The idea is now to use these ~~identities~~ to establish that the images of $e^{\omega}, \mu, \nu, \lambda$ in tensor product mixed complex form a cycle. This idea apparently doesn't work. For example we want

$$(\beta\otimes 1)\mu + (1\otimes\beta)\nu = [\rho, e^{\omega}]$$

To arrange this one wants to use

$$\begin{tikzcd}
R\otimes S \arrow[r] \arrow[d] & R\otimes\Omega^1_{S,\natural} \arrow[d] \\
\Omega^1_{R\natural}\otimes S \arrow[r] & \Omega^1_{R\otimes S,\natural} \arrow[ul]
\end{tikzcd}$$

i.e. that $\beta\otimes 1$ and $1\otimes\beta$ combine to factor through the β-map for $R\otimes S$.

In order to get something going into $\Omega^1_{R\otimes S,\natural}$ I found it necessary to proceed as follows. Start with ρ map it via

$$R\otimes S \xrightarrow{\tilde{d}} S\otimes\Omega^1_R\otimes S$$

multiply on both sides to get $\omega^{i-1}\tilde{d}\rho\,\omega^{n-i} \in S\otimes\Omega^1_R\otimes S$ and then map to $\Omega^1_{R\natural}\otimes S$. $\rho = r\otimes s$, $\omega^{i-1} = r_1\otimes s_1$, $\omega^{n-i} = r_2\otimes s_2$

$$\begin{aligned}
\omega^{i-1}\tilde{d}\rho\,\omega^{n-i} &= (r_1\otimes s_1)(1\otimes dr\otimes s)(r_2\otimes s_2) \\
&= s_1\otimes r_1\,dr\,r_2\otimes ss_2 \in S\otimes\Omega^1_R\otimes S
\end{aligned}$$

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
r_1\,dr\,r_2\otimes ss_2s_1 & & \Omega^1_{R\natural}\otimes S
\end{array}$$

$$= dr\,r_2r_1\otimes ss_2s_1$$

similarly

$$\omega^{i-1}\,\partial\rho\,\omega^{n-i} = (r_1\otimes s_1)(r\otimes\partial s\otimes 1)(r_2\otimes s_2)$$

$$= r_1 r\otimes s_1\,\partial s\, s_2\otimes r_2 \quad \in \quad R\otimes\Omega^1_S\otimes R$$

$$\downarrow$$

$$r_2 r_1 r\otimes s_1\,\partial s\, s_2 \quad \in \quad R\otimes\Omega^1_{S\natural}$$

$$= r_2 r_1 r\otimes s_2 s_1\,\partial s$$

Check: $(\beta\otimes 1)(dr\, r_2 r_1\otimes s s_2 s_1) + (1\otimes\beta)(r_2 r_1 r\otimes s_2 s_1\,\partial s)$

$$= -[r, r_2 r_1]\otimes s s_2 s_1 + r_2 r_1 r\otimes[s_2 s_1, s]$$

$$= (\cancel{r_2 r_1 r} - r r_2 r_1)\otimes s s_2 s_1 \otimes r_2 r_1 r\otimes(s_2 s_1 s - \cancel{s s_2 s_1})$$

$$= [(r_2\otimes s_2)(r_1\otimes s_1), (r\otimes s)]$$

The problem if we apply this process to interpret $\mu = \int e^{(1-s)\omega}\, d\rho\, e^{s\omega}\, ds$ in $\Omega^1_{R\natural}\otimes S$ then what we get is $d\rho\, e^{\omega}$ and similarly ν becomes $e^{\omega}\partial\rho$. So we still don't have a link between $\omega^{n-1}d\rho$ and μ_{2n-1} which works.

October 14, 1989

Let V be a real vector space of dimension n with volume $\omega \in \Lambda^n V^*$, let X be the radial vector field, let f be a function on $V - \{0\}$ which is homogeneous of degree $-n$. Then $f\iota_X\omega$ is basic relative to the principal bundle

$$V - \{0\} \longrightarrow SV = V - \{0\}/\mathbb{R}^\times_{>0}$$

hence it descends to an $(n-1)$-form on SV which can be integrated.

If $V = \mathbb{R}^n$, $\omega = dx_1 \cdots dx_n$, then $X = \sum x_i \partial_i$ and $\iota_X\omega = \sum_1^n (-1)^{i-1} x_i \, dx_1 \cdots \widehat{dx_i} \cdots dx_n$ is homogeneous of degree $+n$. Thus $f\iota_X\omega$ is homog. of degree 0, i.e. invariant as well as being killed by ι_X

Fredholm module $A \longrightarrow \mathcal{L}(H) \ni F$.

Get induced homomorphism $QA \longrightarrow \mathcal{L}(H)$,
$a \rightrightarrows a, FaF$. $\quad a^+ = \dfrac{a + FaF}{2} \qquad a^- = \frac{1}{2}F[F,a]$.

In [redacted] the p-summable case, get $(qA)^p \longrightarrow \mathcal{L}^1(H)$.

Ungraded case: Get even strong supertrace $\tau(x) = \operatorname{tr}(Fx)$ on $(qA)^p$. Get cocycle

$$\varphi_{2n+1}(a_0, \ldots, a_{2n}) = \tau(a_0^+ a_1^- \cdots a_{2n}^-)$$

$$= \frac{(-1)^n}{2^{2n}} \operatorname{tr}\left(F\left(\frac{a_0 + Fa_0F}{2}\right)[F,a_1] \cdots [F,a_{2n}]\right)$$

$$= \frac{(-1)^n}{2^{2n}} \operatorname{tr}\left(Fa_0[F,a_1] \cdots [F,a_{2n}]\right)$$

$$\varphi_{2n}(a_1,\dots,a_{2n}) = \tau(a_1^- \cdots a_{2n}^-)$$
$$= \frac{(-1)^n}{2^{2n}} \operatorname{tr}\big(F[F,a_1]\cdots[F,a_{2n}]\big)$$
$$= \frac{(-1)^n}{2^{2n-1}} \operatorname{tr}\big(a_1[F,a_2]\cdots[F,a_{2n}]\big)$$

Circle case: ψ_3, φ_2 obviously defined but $\psi_1(a_0) = \operatorname{tr}(Fa_0^+)$ is not

$$\begin{tikzcd}
\psi_3 \arrow[r, "1-\lambda"] & \cdot \arrow[r] & 0 & \\
& \varphi_2 \arrow[u, "b'"] \arrow[r] & N\varphi_2 \arrow[u] \arrow[r] & 0 \\
& & \text{“}\psi_1\text{”} \arrow[u] &
\end{tikzcd}$$

Graded case: Get odd supertrace = odd trace on $(QA)^\rho$ given by $\tau(x) = \operatorname{tr}(\gamma x)$. (Note: $x \in QA$ commutes with γ so τ is a trace; it is odd because $\operatorname{tr}(\gamma x) = \operatorname{tr}(F^2\gamma x) = \operatorname{tr}(F\gamma x F) = -\operatorname{tr}(\gamma F x F)$).
Get cocycles

$$\psi_{2n}(a_0,\dots,a_{2n-1}) = \tau(a_0^+ a_1^- \cdots a_{2n-1}^-)$$
$$= \frac{(-1)^{n-1}}{2^{2n-1}} \operatorname{tr}\Big(\gamma F \underbrace{\Big(\frac{a_0 + Fa_0F}{2}\Big)}_{\text{can be replaced by } a_0} [F,a_1]\cdots[F,a_{2n-1}]\Big)$$

$$\varphi_{2n-1}(a_1,\dots,a_{2n-1}) = \frac{(-1)^{n-1}}{2^{2n-1}} \operatorname{tr}\big(\gamma F[F,a_1]\cdots[F,a_{2n-1}]\big)$$
$$= \frac{(-1)^{n-1}}{2^{2n-2}} \operatorname{tr}\big(\gamma a_1[F,a_2]\cdots[F,a_{2n-1}]\big)$$

Case of a surface: ψ_2 not defined but $\psi_3, \psi_4, \ldots$ are

$$\begin{tikzcd} \psi_4 \arrow[r, "1-\lambda"] & {} \arrow[r] & 0 \\ & \psi_3 \arrow[u] \arrow[r] & N\psi_3 \arrow[u] \arrow[r] & 0 \\ & & \text{"}\psi_2\text{"} \arrow[u] \end{tikzcd}$$

Next we want to ψDO's and non commutative residue over a torus $M = \mathbb{R}^n/\Gamma$. The approach to be used involves cross products: We have constant coefficient operators, symbols, etc. which are functions of p or ξ and on which the exponential functions e^{ikx}, $k \in \Gamma'$ operate by translation:

$$e^{ikx}\, p\, e^{-ikx} = p + \hbar k .$$

Then variable coefficient operators, symbols, etc are elements in the cross product of the constant coefficient one with the group algebra of Γ' (identified with functions on M).

Let's consider ψDO symbols, classical ψDO's, where the symbol means the complete asymptotic expansion. Constant coefficients symbols are therefore formal series $P(\xi) = \sum_{n \le N} p_n(\xi)$ where $p_n(\xi)$ is a homogeneous (smooth) function of degree n on the dual $\mathbb{R}^n$. The action of Γ' = character group of M is by translation

$$e^{ikx}\, P(\xi)\, e^{-ikx} = P(\xi + k)$$

Hence it is necessary to see why this translation makes sense on formal series of the type being considered. But by Taylor series

$$p_n(\xi + k) = e^{(k\cdot\nabla_\xi)} p_n(\xi)$$

$$= \sum_\alpha \frac{1}{\alpha!} k^\alpha \partial_\xi^{(\alpha)} p_n(\xi)$$

and $\partial_\xi^{(\alpha)} p_n(\xi)$ is homogeneous of degree $n - |\alpha|$.

Next we want the noncommutative residue which is supposed to be a trace on the cross product of Γ' with the constant coefficient symbols. One way to obtain such a trace is from a linear functional on the constant coefficient symbols which is Γ' invariant. Thus we seek a linear functional

$$P(\xi) = \sum p_n(\xi) \longmapsto \tau\left(\sum p_n(\xi)\right)$$

such that

$$\tau\left(\sum_n p_n(\xi + k)\right) = \tau\left(\sum p_n(\xi)\right) \qquad (*)$$

for all $k \in \Gamma'$. We want this linear ful to be continuous, which means that it should vanish for $P(\xi)$ with $\deg(P) \leq N$ for some N. Then it can be written as a sum $\tau = \sum_{n>N} \tau_n$ where τ_n is a linear functional on homogeneous functions of order n.

Let's take the infinitesimal form of $(*)$

which says

$$\tau\Big(\sum_{n} \partial_{\xi_j} p_n(\xi)\Big) = 0$$

The idea here we are something which locally defined, so independent of letting the torus get bigger, hence Γ' getting denser. It should also be possible using continuity [redacted] to get from the invariance under the discrete Γ' to ~~invariance~~ invariance under [redacted] all translations as follows. Suppose one has established that τ vanishes on P of degree $< n$. Then we look at the condition when g_{n+1} is homogeneous of degree $n+1$.

$$\begin{aligned} \tau(g_{n+1}(\xi)) &= \tau(g_{n+1}(\xi + k)) \\ &= \tau\Big(g_{n+1}(\xi) + \sum_{j=1}^{r} k_j\, \partial_{\xi_j} g_{n+1}(\xi)\Big) \end{aligned}$$

and so one gets $\tau(\partial_{\xi_j} g_{n+1}(\xi)) = 0$.

~~Next assuming [illegible] that we have [illegible] homogeneous of degree [illegible]~~

Next assume that $\tau(\partial_{\xi_j} g_{n+1}(\xi)) = 0$ for all j and $g_{n+1}(\xi)$ homogeneous of degree $n+1$. Then take $f_n(\xi)$ and use Euler's theorem in the form

$$\sum_{j=1}^{r} \partial_{\xi_j}(\xi_j f_n) = \sum_{j=1}^{r} (\xi_j \partial_{\xi_j} + 1) f_n = (r+n) f_n$$

Then

$$(r+n)\,\tau(f_n) = \sum_{j=1}^{n} \tau\Big(\partial_{\xi_j}\big(\underbrace{\xi_j f_n}_{\text{degree } n+1}\big)\Big) = 0$$

showing that the sort of invariant trace we are after can only depend upon the homogeneous component of degree $-r$.

Better approach: Given a real vector space V of dimension r, let's look at its DR complex and how it decomposes under the action of $\mathbb{R}_+^\times$. Recall that the polynomial DR complex $S(V^*) \otimes \Lambda(V^*)$ decomposes

$$S_n(V^*) \xrightarrow{d} S_{n-1}(V^*) \otimes V^* \xrightarrow{d} S_{n-2}(V^*) \otimes \Lambda^2(V^*) \xrightarrow{d} \cdots$$

and these sequences are exact for $n \neq 0$, because the Euler vector field gives a homotopy ι_X satisfying $d\iota_X + \iota_X d = \mathcal{L}_X$ ($= n$ for the above sequence.)

Let's now enlarge polynomials to homogeneous smooth functions. Let F_n be the space of smooth functions on $V - \{0\}$ which are homogeneous of degree n. Here n can be any complex number; it is to be identified with the character $t \mapsto t^n$ of $\mathbb{R}_+^\times$. Notice that $|\xi|^n$ can be used to identify F_0 and F_n, so all these spaces F_n can be identified in this way. The degree n part of the DR complex of $V - \{0\}$ is the complex

$$\to 0 \to F_n \xrightarrow{d} F_{n-1} \otimes V^* \longrightarrow \cdots \longrightarrow F_{n-r} \otimes \Lambda^r V^* \longrightarrow 0.$$

As before the Euler vector field shows this is acyclic for $n \neq 0$. What is the cohomology for $n = 0$?

Use a diffeom $V - \{0\} = S \times \mathbb{R}$ given by an inner product. The invariant forms for

the translation action of $\mathbb{R}$ on itself
will be $\Omega(S) \otimes \{ \mathbb{C} \xrightarrow{d=0} \mathbb{C}\frac{dt}{t} \}$. Thus
we get two copies of the cohomology of
the sphere S. We therefore have

<u>Prop.</u> <u>The sub-complex of the DR complex of</u>
$V - \{0\}$ <u>which is homogeneous of degree</u> $n \in \mathbb{C}$:

$$\xrightarrow{0} F_n \longrightarrow F_{n-1} \otimes V^* \longrightarrow \cdots \xrightarrow{d} F_{n-r} \otimes \Lambda^r V^* \xrightarrow{0}$$

<u>is acyclic for</u> $n \neq 0$ <u>(because of</u> L_X,
X <u>the Euler vector field</u>), <u>and for</u> $n = 0$
<u>its cohomology is</u> $H^\bullet(S) \oplus H^\bullet(S)[1]$. <u>Thus</u>

$$\mathrm{Coker}\{ F_{-r+1} \otimes \Lambda^{r-1} V^* \xrightarrow{d} F_{-r} \otimes \Lambda^r V^* \}$$

<u>is 2-dimensional for $r = 1$ and 1-dimensional</u>
<u>for</u> $r > 1$.

This explains why one has two independent traces when $r = 1$ and only one when $r > 1$. The point is the highest degree cohomology is $H^0(S^0) = \mathbb{C}^{\oplus 2}$ when $r = 1$ and $H^{r-1}(S^{r-1}) = \mathbb{C}$ when $r > 1$. At this point we have achieved the construction of the non commutative residue at least over a torus.

October 15, 1989

$M = \mathbb{R}^r/\Gamma$ torus; view $C^\infty(M)$ as the group algebra of the dual lattice Γ'. Constant coefficient operators, symbols, etc are described by "functions" $f(\xi)$ and variable coefficient operators, symbols, etc are elements of the cross product of the constant coefficient ones with Γ' using the rule

$$e^{ikx} f(\xi)\, e^{-ikx} = f(\xi + k).$$

We consider the index thm. over M. The index is a function on homotopy (or K-equivalence) classes of symbols of elliptic operators; these classes form

$$K_c^0(T^*M) = K^0(D^*M, S^*M) = K_0(\mathcal{S}(T^*M)).$$

The index map can be obtained via "quantizing", more precisely via homomorphisms

$$\mathcal{S}(T^*M) \xleftarrow{0 \leftarrow\!\shortmid h} C \xrightarrow{h \mapsto \neq 0} \begin{pmatrix}\text{smoothing}\\ \text{operators}\end{pmatrix}$$

where C the convolution algebra of the tangent groupoid to M. We take C to be a little larger, namely the cross product with Γ' of functions $f(h,p)$ smooth in h, rapidly decreasing in p, with rule

$$e^{ikx} f(h,p)\, e^{-ikx} = f(h, p+hk)$$

We have seen that to evaluate the index or more generally the cyclic cohomology class on $\mathcal{S}(T^*M)$ it suffices to treat h formally, to order $r = \dim M$.

We will be concerned with elliptic symbol classes of "Toeplitz" type, i.e. in the image of

$$K^{-1}(S^*M) \xrightarrow{\partial} K^0(D^*M, S^*M)$$

For such classes we have another defn. of index, namely using the extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Psi^{-1} & \longrightarrow & \Psi^0 & \longrightarrow & C^\infty(S^*M) & \longrightarrow & 0 \\ & & \cap & & \cap & & & & \\ & & \mathcal{K}(H) & & \mathcal{L}(H) & & & & \end{array}$$

which induces $K^{-1}(S^*M) \cong K_1(C^\infty(S^*M)) \xrightarrow{\partial} K_0(\mathcal{K}(H)) = \mathbb{Z}$

Our goal is to link these two indices.

The idea is that Ψ^0 is a quantized version of $C^\infty(D^*M)$. So we wish to find an $\mathcal{A}$ fitting into the diagram

$$\begin{array}{ccccccc} 0 & & \mathcal{S}(T^*M) & \longrightarrow & C^\infty(D^*M) & \longrightarrow & C^\infty(S^*M) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ h & & \mathcal{C} & \longrightarrow & \mathcal{A} & \longrightarrow & ? \\ h & & \downarrow & & \downarrow & & \downarrow \\ \downarrow & & & & & & \\ \neq 0 & & \begin{pmatrix}\text{smoothing}\\ \text{operators}\end{pmatrix} & \longrightarrow & \Psi^0 & \longrightarrow & C^\infty(S^*M) \end{array}$$

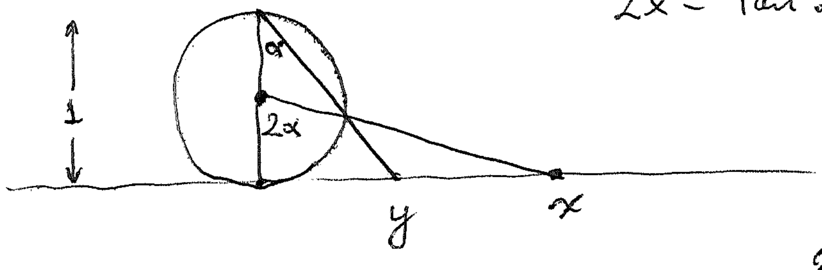
The top + bottom rows, although not exact, are exact from the K-theory viewpoint. In fact we have ideals I = fns. vanishing on S^*M, $\Psi^{-1} \subset \Psi^0$ such that $\bigcap_n (\Psi^{-1})^n = \text{smoothing ops}$, $\Psi^0/\Psi^{-1} = C^\infty(S^*M)$, and similarly for the top row. Thus we also want to find an appropriate ideal $\mathcal{I}$ in $\mathcal{A}$.

What are the sort of things required to evaluate the cyclic class on $C^\infty(S^*M)$ [redacted] asymptotically as $h \downarrow 0$? Now on $\mathcal{C}$ we have different traces for each $h \neq 0$ coming from the different representations of $\mathcal{C}$ ~~on~~ $H = L^2(M)$, and we know how to take the limit of these as $h \to 0$ and obtain a trace with values in $h^{-1}\mathbb{C}[h]/h\mathbb{C}[h]$. So the obvious thing to do it seems is to generalize the construction of so as to allow asymptotic behavior ~~at~~ $p \to \infty$. This $\mathcal{A}$ should relate to Ψ^0 in the same way $\mathcal{C}$ relates to (smoothing operators). If done properly we will have a family of traces τ_h defined on $\mathcal{I}^{r+1}$ with a limiting trace with values in $h^{-r}\mathbb{C}[h]/h\mathbb{C}[h]$ (better in $h^{-r}\mathbb{C}[[h]]$). Then all we need is a linear [redacted] map $C^\infty(S^*M) \longrightarrow \mathcal{A}$ which is a homomorphism modulo $\mathcal{I}$, in order to obtain a cyclic cocycle.

The puzzle is whether the noncommutative residue appears.

Digression: It should be possible to identify [redacted] smooth functions $f(\xi)$ on $\mathbb{R}^n$ having an [redacted] asymptotic expansion $\sum_{n\le 0} f_n(\xi)$ as $|\xi|\to\infty$ with smooth functions on D^n. One uses a suitable embedding $\mathbb{R}^n$ as the interior of D^n. One such embedding comes from stereographic projection

$$y = \tan\alpha$$



$$2x = \tan 2\alpha = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2y}{1-y^2}$$

Thus $\quad x = \dfrac{y}{1-y^2} \quad$ and $\quad y = \dfrac{2x}{1+\sqrt{1+4x^2}} = \dfrac{-1+\sqrt{1+4x^2}}{2x}$

Let us assume that there is a well-defined algebra R consisting of smooth functions $f(\xi)$ on $\mathbb{R}^n$ having an asymptotic expansion $\sum_{n\le 0} f_n(\xi)$ as $|\xi|\to\infty$, and that moreover we have a translation action $f(\xi)\mapsto f(\xi+k)$ of $\mathbb{R}^n$ on this algebra. Change ξ to p and consider [redacted] the cross product algebra $\mathcal{a}_h = \mathbb{C}[\Gamma]\,\tilde{\otimes}_h\, R$ where $e^{-ikx}f(p)e^{+ikx} = f(p+hk)$. We have a homomorphism $\theta_h : \mathcal{a}_h \to \mathcal{L}(H)$, $H = L^2(M)$ sending $f(p)$ to the operator

$$\theta_h(f(p))(e^{i\xi x}) = f(h\xi)e^{i\xi x}$$

Check this: $\left(e^{-ikx}\theta_h(f(p))e^{+ikx}\right)e^{i\xi x} =$

$$e^{-ikx}\,\mathcal{O}_h(f(p))\,e^{i(\xi+k)x} = e^{-ikx} f(h(\xi+k))\,e^{i(\xi+k)x}$$
$$= f(h\xi + hk)\,e^{i\xi x} = \mathcal{O}_h(f(p+hk))\,e^{i\xi x}$$

Next consider the trace on $\mathcal{A}_h$ induced by $\mathcal{O}_h$ from the trace on $\mathcal{L}^1(H)$. One has $\operatorname{tr}\left(\mathcal{O}_h(f(p))\right) = \sum_{\xi\in\Gamma'}' f(h\xi)$ and this will be defined on those f of degree $< -n$.

By Poisson summation

$$\sum_{\xi\in\Gamma'} f(h\xi) = \frac{c}{h^n}\sum_{\gamma\in\Gamma}\hat{f}\left(\frac{\gamma}{h}\right)$$

$\hat{f}$ will decay exponentially as f is smooth; $\hat{f}$ is smooth except at 0 where it is continuous. Thus we have

$$\sum_{\xi\in\Gamma'}' f(h\xi) = \frac{c}{h^n}\int f(p)\,d^np + O(h^\infty)$$

At this point we have a candidate for the algebra $\mathcal{A}$ and ideal $\mathcal{J}$. We consider smooth $f(h,p)$ with $|h|\le 1$ and $p\in\mathbb{R}^n$ which have asymptotic expansions $\sum_{n\le 0} f_n(h,p)$ as $|\xi|\to\infty$, where $f_n(p)$ is homogeneous of degree n. We take the cross product with $\mathbb{C}[\Gamma']$ to obtain $\mathcal{A}$. $\mathcal{J}$ is the cross product where the f's have degree ≤ -1. Thus $\mathcal{A}/\mathcal{J}$ = functions $f_0(h,x,p)$ on S^*M depending on h, and $\mathcal{J}^\infty = \mathcal{C}$ discussed before.

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \longrightarrow C^\infty(S^*M)[h] \longrightarrow 0$$

[redacted] We should have a trace defined on $\mathcal{I}^{r+1}$ with values in $h^{-r}\mathbb{C}[h]$ defined by the integral over T^*M. Therefore a cyclic cohomology class of degree $2r+1$ on $A = C^\infty(S^*M)$ should be defined. [redacted]

[redacted] Since $H^{2r+1}(A, A^*) = \left(\Omega_A^{2r+1}\right)^* = 0$ it should be possible to descend the class to $HC^{2r-1}(A) = H^{odd}(S^*M)$.

We should carry this out in the case of the circle. The first step is to choose a lifting: $C^\infty(S^*M) \xrightarrow{\ell} \mathcal{a}$. Using cross product this means defining ρ on $f_0(p)$, and the result should be a smooth function $f(p) \in R$ [redacted] (maybe [redacted] $f(h,p) \in R\{h\}$) which has $f_0(p)$ for the leading term of its asymptotic expansion. So in the case of the circle this means joining smoothly the two values of $f_0(p)$ for $p = \pm 1$.

This introduces a non-canonical element into the calculation, the choice of a C^∞ approximation to the δ function. It does not seem to be worth pursuing, since [redacted] some metric should be used with Gaussians maybe instead.

Let's consider the noncommutative residue. Here we deal with formal symbols, things like $\Psi^0/\Psi^{-\infty}$, which is the cross product of Γ' acting on formal series $\sum_{n\le 0} f_n(\xi)$ of homogeneous functions.

The noncommutative residue gives a trace on Ψ^0/Ψ^{-r-1}, which is a nilpotent extension of $A = C^\infty(S^*M)$ of length $r+1$, so there is determined a cyclic cohomology class in $HC^{2r}(A)$. However as $H^{2r}(A, A^*) = (\Omega^{2r}_A)^* = 0$, this class lifts back via S to a class in $HC^{2r-2}(A)$.

The next point is that we have a nice lifting $A = C^\infty(S^*M) \xrightarrow{\rho} \Psi^0/\Psi^{-\infty}$, namely take $f_0(\xi)$ to itself. Therefore we really have a big cyclic cocycle defined and a Chern-Simons form. The only arbitrariness comes from the normal ordering prescription in writing $g(x)f(p)$ etc.

Note that we also can lift $\rho : A \to \mathcal{A}/\mathcal{J}^\infty$ by $\rho : f_0(p) \longmapsto f_0(p)$. As

$$e^{-ikx} f_0(p) e^{+ikx} = f_0(p + hk) \qquad = f_0(\xi + k)$$

$$= \sum \frac{h^{|\alpha|} k^\alpha}{\alpha!} (\partial^\alpha f_0)(p) \qquad = \sum \frac{k^\alpha}{\alpha!} (\partial^\alpha f_0)(\xi)$$

we see the subalgebra of $\mathcal{A}/\mathcal{J}^\infty$ generated by the image of ρ is somewhat special. This may have some significance.

Guess: Gaussian Thom form methods should be appropriate to the $\mathcal{S}(T^*M)$ cyclic class, and transgression form methods should be appropriate to the non commutative residue.

October 17, 1989 (Cindy is 9)

Let V be a real symplectic vector space of dim $n = 2m$, let $A_0 = \mathcal{S}(V)$ be the algebra of Schwartz functions on V under multiplication. There is a deformation of A_0; A_h, $h \in \mathbb{R}$ defined as follows.

We regard the exponential functions $e^{i\xi x}$, $\xi \in V^*$ as a sort of basis for A_0 via the Fourier transform

$$f(x) = (2\pi)^{-n} \int e^{i\xi x} \hat{f}(\xi)\, d\xi$$

$$\hat{f}(\xi) = \int e^{-i\xi x} f(x)\, dx$$

We define a twisted product on the exponential functions

$$e^{i\xi x} * e^{i\eta x} = e^{ihB(\xi,\eta)} e^{i(\xi+\eta)x}$$

where B is symplectic form transported to V^*. Then we extend this twisted product to $\mathcal{S}(V)$ by linearity

$$\cancel{(f*g)(x) = \int e^{i\xi x} \hat{f}(\xi) \frac{d\xi}{(2\pi)^n} * \int e^{i\eta x} \hat{g}(\eta) \frac{d\eta}{(2\pi)^n}}$$

$$\cancel{= \iint \hat{f}(\xi)\, \hat{g}(\eta)\, e^{ihB(\xi,\eta)} e^{i(\xi+\eta)x} \frac{d\xi\, d\eta}{(2\pi)^n (2\pi)^n}}$$

$$\cancel{= \int e^{i\xi x} \left\{ \int e^{ih\,[\text{illegible}]} \hat{f}(\xi-\eta)\, \hat{g}(\eta) \frac{d\eta}{(2\pi)^n} \right\} \frac{d\xi}{(2\pi)^n}}$$

$$(f*g)(x) = \int e^{i\xi x}\hat{f}(\xi)\frac{d\xi}{(2\pi)^n} * \int e^{i\eta x}\hat{g}(\eta)\frac{d\eta}{(2\pi)^n}$$

$$= \iint e^{i(\xi+\eta)x}\, e^{ihB(\xi,\eta)}\hat{f}(\xi)\hat{g}(\eta)\frac{d\xi}{(2\pi)^n}\frac{d\eta}{(2\pi)^n}$$

$\xi \mapsto \xi-\eta$

$$= \iint e^{i\xi x}\, e^{ihB(\xi-\eta,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)\frac{d\xi}{(2\pi)^n}\frac{d\eta}{(2\pi)^n}$$

$$= \int e^{i\xi x}\underbrace{\left\{\int e^{ihB(\xi-\eta,\eta)}\hat{f}(\xi-\eta)\hat{g}(\eta)\frac{d\eta}{(2\pi)^n}\right\}}_{\text{twisted convolution on the Fourier transforms}}\frac{d\xi}{(2\pi)^n}$$

(The above holds for any bilinear form B)

A_h denotes with the $*$ product, which depends on h.

Next we define a linear functional τ on A_h by the rule

$$\tau(e^{i\xi x}) = (2\pi)^n\delta(\xi)$$

that is

$$\tau(f) = \tau\left(\int e^{i\xi x}\hat{f}(\xi)\frac{d\xi}{(2\pi)^n}\right) = \hat{f}(0)$$

$$= \int f(x)\,dx$$

Then

$$\tau(f*g) = \int e^{ihB(-\eta,\eta)}\hat{f}(-\eta)\hat{g}(\eta)\frac{d\eta}{(2\pi)^n}$$

$$= \int \hat{f}(-\eta)\hat{g}(\eta)\frac{d\eta}{(2\pi)^n} = \int f(x)g(x)\,dx$$

is symmetric in f, g, and hence τ is a trace on A_h.

There seems to be a canonical way to normalize τ for $h \neq 0$, because A_h has a unique irreducible Hilbert space representation and one can use the Hilbert space trace.

Let us consider the scaling transformation $(\theta_t f)(x) = f(tx)$ for $t \neq 0$. We can use this to show the algebras A_h are isomorphic for $h \neq 0$. (?) Let us write $*_h$ to distinguish the different twisted products. One has

$$\theta_t\left(e^{i\xi x} *_{h_1} e^{i\eta x}\right) = \theta_t\left(e^{ih_1 B(\xi,\eta)}\, e^{i(\xi+\eta)x}\right)$$

$$= e^{ih_1 B(\xi,\eta)}\, e^{it(\xi+\eta)x}$$

$$\theta_t\left(e^{i\xi x}\right) *_{h_2} \theta_t\left(e^{i\eta x}\right) = e^{ih_2 B(t\xi,\, t\eta)}\, e^{it(\xi+\eta)x}$$

which shows that

$\theta_t : A_{h_1} \longrightarrow A_{h_2}$ is an isomorphism when $h_1 = t^2 h_2$

This shows only that the A_h with $h > 0$ are isomorphic. Perhaps the F.T. gives an isomorphism between A_h and A_{-h}.

We have

$$\tau(\theta_t f) = \int f(tx)\,dx = \int f(x)\frac{dx}{t^n} = t^{-n}\tau(f).$$

Thus if we want traces to be compatible with the scaling isomorphism $\theta_t : A_{t^2 h} \xrightarrow{\sim} A_h$ we want to use

$$\boxed{\tau_h(f) = h^{-m}\int f(x)\,dx \quad \text{on } A_h}$$

Check $\tau_h(\theta_t f) = t^{-n}\tau_h(f) = t^{-n}h^{-m}\tau_1(f) = (t^2 h)^{-m}\tau_1(f)$

$$= \tau_{t^2 h}(f).$$

Now we want to use the algebra A_h and trace τ_h to obtain cyclic cohomology classes on A_0 as follows. We have a linear map $\rho_h : A_0 \longrightarrow A_h$, $\rho_h(f) = f$ which one can view as quantizing a function on phase space to an operator. To this linear map and the trace τ_h is associated a (big) cocycle on A_0. So one has a family of cocycles on A_0 depending on the parameter h.

Question: Are these cocycles entire cocycles?

Let's show next that the cocycles for different h are cohomologous. One has a commutative diagram

$$\begin{tikzcd} A_0 \arrow[r, "\rho_{t^2h}"] \arrow[d, "\theta_t", "\wr"'] & A_{t^2h} \arrow[dr, "\tau_{t^2h}"] \arrow[d, "\theta_t", "\wr"'] & \\ & & \mathbb{C} \\ A_0 \arrow[r, "\rho_h"] & A_h \arrow[ur, "\tau_h"'] & \end{tikzcd}$$

Take $t = h^{-1/2}$

$$\begin{tikzcd} A_0 \arrow[r, "\rho_1"] \arrow[d, "\theta_{h^{-1/2}}", "\wr"'] & A_1 \arrow[dr, "\tau_1"] \arrow[d, "\theta_{h^{-1/2}}", "\wr"'] & \\ & & \mathbb{C} \\ A_0 \arrow[r, "\rho_h"] \arrow[uur, dashed, "\rho_1\theta_{h^{1/2}}"] & A_h \arrow[ur, "\tau_h"'] & \end{tikzcd}$$

Thus the cocycle assoc. to (ρ_h, τ_h) is the cocycle associated to $(\rho_1 \theta_{h^{1/2}}, \tau_1)$. Thus we are deforming the linear map and we know this leads to cohomologous cocycles.

But there is a special feature here, namely

The deformation proceeds via a 1-parameter group of automorphisms of A_0.

This is not very promising since if you have a fixed cocycle on A_0 and you transform it under the one-parameter group $\theta_{h^{1/2}}$, then it is not clear why the cocycle should have a limit. Look at the first cochain

$$\psi_1(a) = \tau_h(\rho_h(a)) = h^{-m}\int a(x)\,dx$$

Check $$\psi_1(a) = \tau_1(\rho_1\theta_{h^{1/2}}a) = \int a(h^{1/2}x)\,dx.$$

This blows up.

On the other hand we know that the cyclic cohomology class is unchanged under the deformation, so we don't have the right cocycles to take the limit.

October 18, 1989

Change notation in order to work with the convolution algebras. [redacted]

Let us interchange V and V^*.

Let V be a real symplectic vector space, let A_h be the algebra given by $\mathcal{S}(V)$ with twisted convolution

$$(f *_h g)(x) = \int e^{ih [redacted] B(x-y,y)} f(x-y)\,g(y)\,dy$$

where B is the given symplectic form on V.

Let $\mathcal{A} = C^\infty(\mathbb{R}, \mathcal{S}(V))$ be the algebra [redacted] consisting of smooth functions $h \mapsto f_h$ with values in $\mathcal{S}(V)$, equipped with the multiplication

$$(f*g)_h = f_h *_h g_h$$

On $\mathcal{A}$ we define the scaling transformation θ_t by

$$(\theta_t f)_h(x) = \frac{1}{t^n} f_{t^2h}\left(\frac{x}{t}\right).$$

Let's check that this is an automorphism of $\mathcal{A}$.

$$\left(\theta_t f * \theta_t g\right)_h(x) = \int e^{ihB(x,y)} (\theta_t f)_h(x-y)\,(\theta_t g)_h(y)\,dy$$

$$= \int e^{ihB(x,y)} \frac{1}{t^n} f_{t^2h}\left(\frac{x-y}{t}\right) \frac{1}{t^n} g_{t^2h}\left(\frac{y}{t}\right) dy \qquad y \mapsto ty$$

$$= \frac{1}{t^n} \int e^{i(ht^2)B(\frac{x}{t},y)} f_{(t^2h)}\left(\frac{x}{t} - y\right) g_{(t^2h)}(y)\,dy$$

$$= \frac{1}{t^n}\left(f_{t^2h} *_{t^2h} g_{t^2h}\right)\left(\frac{x}{t}\right) = \frac{1}{t^n}(f*g)_{t^2h}\left(\frac{x}{t}\right)$$

$$= \left(\theta_t(f*g)\right)_h(x)$$

Next we define

$$\tau : \mathcal{A} \longrightarrow h^{-m} C^\infty(\mathbb{R})$$

by

$$\tau(f)_h = h^{-m} f_h(0)$$

Then

$$\tau(f * g)_h = h^{-m} (f_h *_h g_h)(0)$$

$$= \frac{1}{h^m} \int f_h(-y)\, g_h(y)\, dy$$

Since this is symmetric in f, g, τ is a trace on $\mathcal{A}$. Next we check that τ is equivariant for the action of the scaling transformation θ_t.

$$\tau(\theta_t f)_h = \frac{1}{h^m} (\theta_t f)_h (0)$$

$$= \frac{1}{h^m} \frac{1}{t^n} f_{t^2 h}\left(\frac{0}{t}\right) = \frac{1}{(t^2 h)^m} f_{t^2 h}(0)$$

$$= \tau(f)_{t^2 h} = \big(\theta_t(\tau f)\big)_h$$

Next we discuss how to apply the above to cyclic theory. In general given a linear map $\rho : A \longrightarrow R$ and a trace τ on R we have big cocycle on A defined. This has no cohomological significance unless we restrict the growth of the component cochains. Algebraically one does this by giving an ideal $I \subset R$ such that ρ is a homomorphism modulo I and τ vanishes on I^{m+1}. In this case we obtain a well-defined element in $HC^{2m}(A)$.

This cohomology class is trivial if τ is null cobordant.

One way this happens is when there is a derivation $D: R \to R$ such that $\tau D = c\tau$ with $c \neq 0$. Because then D induces $\tilde{D}: \Omega^1 R \longrightarrow R$ and

$$\tau = \frac{1}{c}\tau D = \left(\frac{1}{c}\tau \tilde{D}\right) d,$$

showing τ is null cobordant.

Let us now treat the variable h formally. This means that we replace $\mathcal{A}$ by the algebra $\hat{\mathcal{A}}$ consisting of formal power series $f_h(x) = \sum_{j\geq 0} h^j f_j(x)$ with coefficients in $\mathcal{S}(V)$, and we replace $C^\infty(\mathbb{R})$ by formal power series in h. Then we have still a action of $\mathbb{R}_{>0}$ by the scaling transformation θ_t and a trace τ from $\hat{\mathcal{A}}$ to $h^{-m}\mathbb{C}[[h]]$ which is invariant under θ_t.

Since θ_t is a 1-parameter group of automorphism (in the variable $\log t$), we consider its infinitesimal generator

$$t\partial_t(\theta_t f)\Big|_{t=1}(h,x) = t\partial_t\Big|_{t=1}\left\{\frac{1}{t^n} f_{t^2 h}\left(\frac{x}{t}\right)\right\}$$

$$= \left(-n + 2h\partial_h - \sum x_\mu \partial_{x_\mu}\right) f(h,x)$$

As a check we have

$$(2h\partial_h)\tau(f)(h) = (2h\partial_h)\left(\tfrac{1}{h^m} f_h(0)\right)$$
$$= \tfrac{1}{h^m}\left(-2m + 2h\partial_h\right) f_h(0).$$

$$\tau\left((-n + 2h\partial_h - x_\mu \partial_\mu) f\right)_h$$
$$= \frac{1}{h^m}\left((-n + 2h\partial_h - x_\mu\partial_\mu) f\right)_h (0)$$
$$= \frac{1}{h^m}\left(-n + 2h\partial_h\right) f_h(0)$$

which shows $\tau : \hat{a} \longrightarrow h^{-m}\mathbb{C}[[h]]$ intertwines the derivations $-n + 2h\partial_h - x_\mu\partial_\mu = 2h\partial_h - \partial_\mu \circ x_\mu$ and $2h\partial_h$.

Let's now look at the ordinary traces τ^k on $\hat{a}$ we obtain from the coefficient in τ:

$$\tau(f) = \sum_{k \geq -m} h^k \cdot \tau^k(f)$$

From $\tau\{(2h\partial_h - x_\mu\partial_\mu - n)f\} = 2h\partial_h\, \tau(f)$

we obtain

$$\tau^k\{(2h\partial_h - x_\mu\partial_\mu - n)f\} = 2k\,\tau^k(f).$$

Therefore we conclude that the traces τ^k for $k \neq 0$ are null cobordant.

Now the trace τ^0 is defined on $\hat{a}/h^{m+1}\hat{a} = \left\{\sum_{j=0}^{m} h^j f_j(x)\right\}$ which is a nilpotent extension of order m of $\hat{a}/h\hat{a} = A_0$.

So we have a well-defined cyclic cohomology class in $HC^{2m}(A_0)$. Suppose we try to calculate it using the linear lifting $\rho : A_0 \to \mathcal{a}$ [redacted] given by $\rho(f)_h = f$. Note that ρ commutes with θ_t:

$$\rho(\theta_t f)(h,x) = (\theta_t f)(x) = \frac{1}{t^n} f\left(\frac{x}{t}\right)$$

$$\theta_t(\rho(f))(h,x) = \frac{1}{t^n}(\rho(f))_{t^2 h}\left(\frac{x}{t}\right) = \frac{1}{t^n} f\left(\frac{x}{t}\right)$$

One has the commutative diagram

$$\begin{tikzcd}
A_0 \arrow[r, "\rho"] \arrow[d, "\theta_t"] & \hat{\mathcal{a}}/h^{m+1}\hat{\mathcal{a}} \arrow[d, "\theta_t"] \arrow[dr, "\tau^0"] & \\
 & & \mathbb{C} \\
A_0 \arrow[r, "\rho"'] & \hat{\mathcal{a}}/h^{m+1}\hat{\mathcal{a}} \arrow[ur, "\tau^0"'] &
\end{tikzcd}$$

We conclude that the big cocycle on A_0 attached to (ρ, τ^0) is <u>invariant</u> under θ_t.

Now this [redacted] should be a key point. It's here that we have [redacted] to find an understanding of the cyclic cocycle game.

What is the "sphere" analogue of the above. We [redacted] started with $A_0 = \mathcal{S}(V)$ under convolution, or $A_0 = \mathcal{S}(V^*)$ under multiplication. Now suppose we enlarge A_0 to the algebra

B_0 consisting of smooth functions $\hat{f}(\xi)$ on V^* have integral degrees asymptotic expansion as $|\xi| \to \infty$.

Equivalently B_0 can be described as distributions $f(x)$ on V smooth and rapidly decreasing for $x \neq 0$ with the singularity of "classical type" at $x = 0$. B_0 is filtered by B_0^n, where the degree of $\hat{f}(\xi)$ is $\leq n$. ~~[illegible]~~ Let's assume that the deformation A_h, $\mathcal{A}$, etc. we have constructed for A_0 can be extended to obtain B_0, $\mathcal{B}$, etc. Thus $\mathcal{B}$ should be a filtered algebra consisting of smooth functions $h \mapsto \hat{f}_h(\xi)$ with values in B_0.

We should have $\mathcal{B}^{-\infty} = \bigcap \mathcal{B}^n = \mathcal{A}$ and the trace τ on $\mathcal{A}$ with values in $h^{-m}C^\infty(\mathbb{R})$ should extend to $\mathcal{B}^{-n-1}$. As before we should obtain a scale invariant $\mathbb{C}$-valued trace τ^0 defined on $\mathcal{B}^{-n-1}/h^{m+1}\mathcal{B}^{-n-1}$. Another thing we expect is that the associated graded algebra $\bigoplus_n \mathcal{B}^n/\mathcal{B}^{n-1}$ is $\bigoplus_n C^\infty(\mathbb{R}, C^\infty(V^*)^{(n)})$, where $C^\infty(V^*)^{(n)}$ is the smooth functions on $V^* - 0$ which are homogeneous of degree n. This is because when we compute $f *_h g$ the correction terms from fg involves derivatives which decrease order.

So we should have a situation where a cyclic cohomology class on $C^\infty(S)$ is defined, namely an extension $C^\infty(S) = \mathcal{B}^0/\mathcal{B}^{-1}$ with trace τ^0 on $\mathcal{B}^{-n-1}$.

October 19, 1989

Yesterday we digressed to consider the sphere case and we [redacted] described (without checking details) an extension with trace: $\qquad S = SV^*$

$$\begin{tikzcd}
 & & & C^\infty(S) \arrow[d, phantom, "\cap" description] & \\
0 \arrow[r] & \mathcal{B}^{-1} \arrow[r] & \mathcal{B}^0 \arrow[r] & C^\infty(\mathbb{R}, C^\infty(S)) \arrow[r] & 0 \\
 & \mathcal{B}^{-n-1} \arrow[u, phantom, "\cup" description] \arrow[d, "\tau^0"] & & & \\
 & \mathbb{C} & & &
\end{tikzcd}$$

Moreover there is an action of scaling transformations on the whole picture which is trivial on $C^\infty(S)$ and $\mathbb{C}$. There is a cyclic cohomology class in $HC^{2n+1}(C^\infty(S))$ associated to the above diagram. It [redacted] is [redacted] possible to realize this class in $HC^{2n-1}(C^\infty(S))$, but [redacted] it is not obvious how to do so.

To get cocycles on $C^\infty(S)$ we have to choose a lifting $\rho : C^\infty(S) \to \mathcal{B}^0$. This amounts to extending a homogeneous function of degree 0: $f_0(\xi)$ to a smooth function $\psi(\xi) f_0(\xi)$, where ψ vanishes to infinite order at $|\xi| = 0$ and $\psi(\xi) \equiv 1$ outside some nbd of 0. The action of scaling transformations shows that the resulting cocycles are independent of the change $\psi(\xi) \mapsto \psi(t\xi)$. One expects that the cocycles are completely independent of ψ.

Observe: Suppose we look at the extension

$$0 \longrightarrow \mathcal{B}^{-1}/\mathcal{B}^{-\infty} \longrightarrow \mathcal{B}^{0}/\mathcal{B}^{-\infty} \longrightarrow C^\infty(\mathbb{R}, C^\infty(S)) \rightarrow 0$$

In this case we have an obvious choice for ρ namely $\rho(f_0(\xi)) = f_0(\xi)$. The curvature is

$$f_0 * g_0 - f_0 g_0 = \sum_{|\alpha|=|\beta| \,[\text{redacted}]\, >0} \frac{h^{|\alpha|}}{|\alpha|!} B_{\alpha\beta}\, \partial^\alpha f_0\, \partial^\beta g_0$$

This belongs to $\mathcal{B}^{-2}/\mathcal{B}^{-\infty}$; the leading term is $h\{f_0, g_0\}$. [redacted] To get h^m one takes the product of m curvatures, landing in $\mathcal{B}^{-2m}/\mathcal{B}^{-\infty}$.

The relation with noncommutative residue remains obscure. Here's an idea: The fact is that the noncomm residue is a trace on arbitrary order ΨDO's $_{A}$ symbols. Suppose we view $\Psi^\infty/\Psi^{-\infty}$ as the analogue of a local field F with $\Psi^0/\Psi^{-1} = C^\infty(S)$ as the residue field k and Ψ^0 as the d.v.r. A. Then one might expect an even K-class on F to [redacted] comes from an odd K-class on k via ∂:

$$\longrightarrow K_i(k) \longrightarrow K_i(A) \longrightarrow K_i(F) \xrightarrow{\partial} K_{i-1}(k) \longrightarrow$$

So maybe it is possible that the residue on F [redacted] is canonically "trivial" when restricted to A and hence "descends" to an odd [redacted]cyclic class on k.

Let us return to our twisted convolution algebra defined by the rule

$$\delta_x * \delta_y = e^{ihB(x,y)}\,\delta_{x+y}$$

and the trace τ^0 which evaluates the coefficient of h^m at 0. Thus

$$\tau^0\left(\delta_{x_1} * \cdots * \delta_{x_k}\right) = \tau^0\left(e^{ih\sum_{i<j} B(x_i,x_j)}\,\delta_{x_1+\cdots+x_k}\right)$$

$$= \frac{1}{m!}\left(i\sum_{i<j} B(x_i,x_j)\right)^m \delta(x_1+\cdots+x_k)$$

Notice that τ^0 picks out the scale-invariant part of the trace $\delta_x \longrightarrow \delta(x)$, since $\delta(x)$ is of degree $-n$ in x:

$$\delta(tx)\,t^n dx = \delta(x)\,dx \implies \delta(tx) = t^{-n}\delta(x).$$

We have the following situation: We have a group algebra A_0 consisting of certain distributions on V under convolution. We have a twisted group algebra A_h (better to write A_B to emphasize the bilinear form B), and a trace τ on A_h. Thus we have a linear map $\rho : A_0 \longrightarrow A_h$ which is the identity on vector spaces and a trace on A_h. In good cases there will be some kind of even cyclic cohomology class on the group algebra A_0.

Our present approach consists of using the induced map $RA_0 \longrightarrow A_h$ to get a trace which we then convert to a big cocycle via $\int \omega^n, \omega^n$. Our problem is to get directly from the extension with trace to the answer, which is a definite cyclic cocycle of degree $2m = n$ on A_0. It is essentially $\int_V f_0\, df_1 \dots df_n$.

Let's generalize the setup a bit. We obtain a twisted group algebra when we have a 2-cocycle on the group with values in S^1. Such a cocycle is given by $e^{ihB(x,y)}$ when B is bilinear. We obtain a trace which picks out the coefficient of the identity when

$$\tau\left(\delta_x * \delta_y\right) = e^{ihB(x,y)}\,\delta(x+y)$$

is symmetric in x, y. This is the case when $B(x,y) = B(y,x)$, when $x+y=0$, i.e. $B(x,-x) = B(-x,x)$, which is the case when B is bilinear.

Apparently we obtain something interesting when we replace V by a torus??

We have been studying the Weyl quantization map ρ from the Schwartz space $A_0 = \mathcal{S}(V)$ under convolution to the corresponding smooth Weyl algebra A_h. We have run into difficulty trying to compute the cyclic class associated to ρ and the canonical trace on A_h. According to the general formalism of the GNS construction and Cuntz algebra it is natural to dilate ρ to a homomorphism. Also there are natural geometric dilations of some sort around. We know that, on choosing a polarization of V, it is possible to realize the irreducible representation of A_h as a subspace of functions of V (or V^*). So it's natural to inquire whether the canonical Weyl quantization map ρ is obtained from a polarization.

Let's calculate in the case of the holomorphic function model with $n=1$. Here $V^* = \mathbb{C}$ and the irreducible repn. is the holomorphic function subspace $\mathcal{H}$ of $L^2(\mathbb{C}, e^{-|z|^2}\frac{d^2z}{\pi})$. Functions $\varphi(z,\bar{z})$ on $\mathbb{C}$ give operator on $\mathcal{H}$ by the Toeplitz process $T_\varphi f = P(\varphi f)$, $P =$ proj. on $\mathcal{H}$. Recall that $T_z = z$, $T_{\bar{z}} = \partial_z$. We are interested in calculating T_φ when φ is an exponential function: $\varphi = e^{\bar{c}\bar{z} - cz}$ with $c \in \mathbb{C}$. We have

$$T_\varphi f = P\{e^{\bar{c}\bar{z}}(e^{-cz}f(z))\} = e^{\bar{c}\partial_z}(e^{-cz}f(z))$$

$$= e^{-\frac{1}{2}|c|^2} e^{\bar{c}\partial_z - cz} f(z)$$

Recall $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$

On the other hand the Weyl quantization map ρ is given by

$$\rho\left(e^{\bar{c}\bar{z}-cz}\right) = e^{\bar{c}\partial_z - cz}$$

Thus it's clear that complex polarizations don't yield the Weyl map ρ.

On the other hand if we use the usual ΨDO way of representing functions $P(x,\xi)$ as operators on $L^2(\mathbb{R})$, then

$$e^{i(ax+b\xi)} \longmapsto e^{iax} e^{b\partial_x} = e^{-\frac{i}{2}ab} e^{iax+b\partial_x}$$

which is also not the Weyl quantization

Remarks on $GNS(\rho) = \Gamma(\rho)$.

① Modules over Γ are equivalent to quadruples (M, N, i, j), where M is an A-module, N is a B-module, and $i: N \to M$, $j: M \to N$ are linear maps such that

$$j a i(n) = \rho(a) n$$

The equivalence is given by sending a Γ module X to $M = X$, $N = eX$, i = inclusion $eX \hookrightarrow X$, $j(x) = ex$. Recall Γ is the semi-direct product $A \oplus A \otimes B \otimes A$, $e = 1 \otimes 1 \otimes 1$ and $e\Gamma e = 1 \otimes B \otimes 1$ can be identified with B. The inverse equivalence sends (M, N, i, j) to $X = M$ with obvious A action and with $(a' \otimes b \otimes a'')m = a' i b j a''(m)$.

② Question: Is Γ a superalgebra in a natural way? This is true in the universal case, but not in general.

Recall one has a decomposition

$$\Gamma = \begin{pmatrix} e\Gamma e & e\Gamma(1-e) \\ (1-e)\Gamma e & (1-e)\Gamma(1-e) \end{pmatrix}$$

where $\Gamma e \Gamma$ (which is everything but the lower right block) is $A \otimes B \otimes A$. One has

$$\Gamma e = A \otimes B \otimes 1$$
$$e\Gamma = 1 \otimes B \otimes A.$$

$$e\Gamma e = 1 \otimes B \otimes 1 \cong B$$

$$(1-e)\Gamma e = \{a \otimes b - 1 \otimes \rho(a) b\} \otimes 1$$
$$\cong \bar{A} \otimes B$$

$$e\Gamma(1-e) \cong B \otimes \bar{A}$$

$$(1-e)\Gamma(1-e) = (1-e)(A \oplus A\otimes B\otimes A)(1-e)$$

$$\cong A \oplus \bar{A}\otimes B\otimes \bar{A}$$

Thus given $\rho : A \longrightarrow B$, one obtains another linear map $\rho' : A \longrightarrow B'$ where $B' = (1-e)\Gamma(1-e)$. In the universal case B, B' are isomorphic, but not in general.

October 25, 1989 (in Paris)

I want to explain where d on ΩA comes from if one were to define ΩA as $\mathrm{gr}\, QA$. In fact there is an odd derivation d defined on QA with $d^2=0$. One sees from the formula for the product in QA:

$$\omega * \eta = \omega\eta - (-1)^{|\omega|} d\omega\, d\eta$$

that the differential d on ΩA is an odd derivation with respect to this $*$ product:

$$d(\omega * \eta) = d(\omega\eta) = d\omega\, \eta + (-1)^{|\omega|}\omega\, d\eta$$

$$= d\omega * \eta + (-1)^{|\omega|}\omega * d\eta$$

But we would like an abstract proof based on the universal property of QA as the superalgebra generated by the ordinary algebra A. We want an odd derivation d of QA which is the same as a superalgebra morphism

$$QA \xrightarrow{1+\varepsilon d} k[\varepsilon]\otimes QA = QA \oplus \varepsilon QA$$

where ε is odd with $\varepsilon^2=0$. This is ~~the same as~~ equivalent to an alg morphism

$$A \xrightarrow{1+\varepsilon d} QA \oplus \varepsilon QA.$$

So starting with a superalgebra S and an algebra map $A \xrightarrow{u} S$ we want to lift to an algebra map $A \xrightarrow{u+\varepsilon v} S\oplus\varepsilon S$. So given a superalgebra S we want a canonical algebra map $S \xrightarrow{1+\varepsilon V} S\oplus\varepsilon S$.

Now we know what $d: QA \longrightarrow QA$ does: $d(a_0^+ a_1^- \cdots a_n^-) = a_0^- \cdots a_n^-$. So $\left(\begin{array}{l} da^+ = a^- \\ da^- = 0 \end{array}\right.$

which tells us $da = a^-$.

Let's check that

$$S \longrightarrow S \oplus \varepsilon S \qquad x \longmapsto x + \varepsilon x^-$$

is an algebra morphism

$$(x + \varepsilon x^-)(y + \varepsilon y^-) = xy + \varepsilon x^- y + x \varepsilon y^-$$

$$= xy + \varepsilon(x^- y^+ + \cancel{x^- y^-} + x^+ y^- - \cancel{x^- y^-})$$

$$= xy + \varepsilon (xy)^-$$

Put another way, given a superalgebra S, the map $x \longmapsto x^-$ is a derivation of the underlying algebra with values in S considered as a bimodule with left multiplication twisted by γ, and the right multiplication untwisted:

$$(xy)^- \overset{?}{=} [redacted]\ x^- y + x^\gamma y^-$$

$$= x^-(y^+ + \cancel{y^-}) + (x^+ \cancel{- x^-}) y^- = (xy)^-$$

In elementary terms to define d on QA, one starts with the algebra morphism

$$A \longrightarrow QA \oplus \varepsilon QA \qquad a \longmapsto a + \varepsilon a^-$$

extends it to QA as a superalgebra morphism and takes the coefficient of ε. Thus there is a unique odd derivation d of QA such that $d(a^+) = \bar{a}$, $d(a^-) = 0$.

Consider the complex $\Omega^\cdot A$ with ~~left~~ A-multiplication and [redacted] differential d. Consider the [redacted] subalgebra of endomorphisms of $\Omega^\cdot A$ generated by A and d. This subalgebra should be

1) $\Omega A \oplus \Omega A d \underset{a}{=} A * \mathbb{C}[d]$

Proof of this isomorphism. Consider a graded algebra $R^{\cdot}$ with a homomorphism of algs $A \longrightarrow R^0$ and $d \in R^1$ with $d^2 = 0$. This is the same as a graded alg. morphism $A * \mathbb{C}[d] \longrightarrow R^{\cdot}$. Then $R^{\cdot}$ is a DGA with $dr \overset{\text{def}}{=} d \circ r - (-1)^{|r|} r . d$. Hence $A \to R^0$ induces $\Omega A \longrightarrow R$, a morphism of DGA's. But this we get $\Omega A \oplus \Omega A d \longrightarrow R$, a ~~map~~ morphism of graded algebras. Thus ~~one~~ an equivalence of graded algebra morphisms to an arbitrary $R^{\cdot}$

The above isomorphism is ~~=~~ analogous to the isomorphism

2) $$A * \mathbb{C}[F] = QA \oplus QAF$$

Perhaps 1) 2) should be written

$$A * \mathbb{C}[d] = \Omega A \,\tilde{\otimes}\, \mathbb{C}[d]$$

$$A * \mathbb{C}[F] = QA \,\tilde{\otimes}\, \mathbb{C}[F]$$

Now QA is also a subalgebra of the ~~graded~~ algebra $\mathcal{E}$ of endos. of ΩA generated by A, d, but it's not a graded subalgebra. One has

$$a^+ \longmapsto a - [d,a]d$$
$$a^- \longmapsto [d,a]$$

In fact we can lift QA into $A * \mathbb{C}[d]$ by

$$a \longmapsto (1+d)\, a (1-d)$$
$$\longmapsto (1-d)\, a (1+d)$$

October 28, 1989

Consider $\Omega = \Omega A$ with the operators d, and left multiplication by elements of Ω. We have maps of $\mathbb{Z}$-graded algebras

$$A * \mathbb{C}[d] \xrightarrow{\sim} \Omega \tilde{\otimes} \mathbb{C}[d] \longrightarrow \mathrm{Hom}_k(\Omega, \Omega)$$

$$\Omega \tilde{\otimes} \mathbb{C}[d] = \Omega \oplus \Omega d \quad \text{with} \quad d \cdot \omega = d\omega + (-1)^{|\omega|} \omega d$$

The first map we have seen is an isomorphism: $A * \mathbb{C}[d]$ is a DGA with differential ad d, so there is a canonical DGA map $\Omega A \longrightarrow A * \mathbb{C}[d]$; this extends to a map $\Omega \tilde{\otimes} \mathbb{C}[d] \longrightarrow A * \mathbb{C}[d]$ which is clearly inverse to the first map.

We claim the 2nd map is injective: This map sends $\omega_1 + \omega_2 d$ to the operator $\eta \longmapsto (\omega_1 + \omega_2 d)\eta$. If this operator is zero, one sees upon taking $\eta = 1$ that $\omega_1 = 0$, and then taking η to be any $a \in A - \mathbb{C}$ we see $\omega_2 = 0$.

<u>Thus</u> $A * \mathbb{C}[d]$ <u>is the algebra of operators on Ω generated by d and by left multiplication by elts of A.</u> Notice that these operator commute with right multiplication by $T(dA)$, so $A * \mathbb{C}[d]$ is contained in

$$\mathrm{Hom}_{T(dA)^{op}}(\Omega, \Omega) = \mathrm{Hom}_k(A, \Omega)$$

Recall that we have a natural action of $Q = QA$ on Ω given by
$$a^+ \longmapsto a-(da)d$$
$$a^- \longmapsto da$$
This gives us an embedding
$$Q \hookrightarrow A * \mathbb{C}[d] = \Omega \oplus \Omega d$$
Notice that Q is stable under ad d, since
$$d\cdot(a-dad) - (a-dad)\cdot d = d\cdot a - a\cdot d = da$$
$$d\cdot(da) + (da)\cdot d = 0$$
and the induced derivation d on Q is given by $d(a^+) = a^-$, $d(a^-) = 0$, and hence agrees with what we defined before. It's clear that we have
$$Q \oplus Qd = \Omega \oplus \Omega d \qquad \text{i.e.}$$
$$Q \tilde{\otimes} \mathbb{C}[d] = \Omega \tilde{\otimes} \mathbb{C}[d]$$

Next we take up $A * \mathbb{C}[F] = Q \tilde{\otimes} \mathbb{C}[F]$. We recall this is the GNS algebra associated to the universal linear map $\rho : A \to R = RA$, and that as a GNS algebra it operates on $A \otimes R$ and $\mathrm{Hom}(A, R)$ compatibly with right R-multiplication. Recall we have a canonical ~~an embedding~~ map
$$A \otimes R \longrightarrow \mathrm{Hom}(A,R) \qquad a\otimes r \mapsto (\alpha \mapsto \rho(\alpha a) r)$$
compatible with right R-multiplication and left GNS multiplication.

On the other hand we have inclusion and projection of R as the even part of Q:
$$R \overset{i}{\hookrightarrow} Q \xrightarrow{i^* = \frac{1+F}{2}} R = Q^+$$

and $\quad i^* a i r = (ar)^+ = a^+ r = \rho(a) r$

so we also have GNS acting on Q as right R-module. It is natural to ask about the relation of Q to $A \otimes R$ or $\mathrm{Hom}(A,R)$. One has a canonical map

$$(*) \qquad A \otimes R \longrightarrow Q \qquad\qquad a \otimes r \mapsto ar$$

compatible with left A, right R-multiplication and with projection back to R:

$$\begin{tikzcd}
a \otimes r \arrow[r, mapsto] \arrow[dr, mapsto, "i^*"'] & ar \arrow[d, mapsto] \\
 & \rho(a) r = (ar)^+
\end{tikzcd}$$

We check that $(*)$ is an isomorphism

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\iota = 1 \otimes ?"] \arrow[d, equal] & A \otimes R \arrow[r] \arrow[d, "(*)"] & \bar{A} \otimes R \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & R \arrow[r, "i"] & Q \arrow[r] & \bar{Q} \arrow[r] & 0
\end{tikzcd}$$

Let's compute β. Given $\bar{a} \otimes r$, lift it to $a \otimes r$, apply $(*)$ to get ar, project to get $(ar)^- = a^- r$. So $\beta(\bar{a} \otimes r) = a^- r$, and hence β is an isomorphism.

Thus we have identified the $A * \mathbb{C}[F]$ module given by Q, with a acting by left mult. by $a = \iota a$ and with F the $\mathbb{Z}/2$-grading involution, with the canonical module $A \otimes B$ of GNS theory. Alternatively the canonical module $A \otimes B$ is given by Ω with $a \mapsto a + da - dad$ and with $\quad F\omega = (-1)^{|\omega|}\omega.$

The above discussion shows that we want to consider the algebra of operators on Ω generated by left multiplication by $a \in A$, by d, and by F above. Thus we consider the alg map

$$(A * \mathbb{C}[d]) \tilde{\otimes} \mathbb{C}[F] \longrightarrow Hom(\Omega, \Omega)$$

$$\wr\wr$$

$$\Omega \oplus \Omega d \oplus \Omega F \oplus \Omega dF$$

We now show this map is injective. Suppose

$$(\omega_0 + \omega_1 d + \omega_2 F + \omega_3 dF)(\eta) = 0$$

for all η. Taking $\eta = 1$ and $\eta = da$ gives

$$\omega_0 + \omega_2 = 0 \qquad (\omega_0 - \omega_2)\, da = 0 \qquad \forall a \in A$$

whence we conclude $\omega_0 = \omega_2 = 0$. Taking $\eta = a$ gives then

$$\omega_1 da + \omega_3 da = 0 \implies \omega_1 + \omega_3 = 0.$$

and with $\eta = a_0 da_1$ get

$$\omega_1 \, da_0 da_1 - \omega_3 \, da_0 da_1 = 0 \implies \omega_1 - \omega_3 = 0$$

October 29, 1989

Recall one has an isomorphism

$$QA \tilde{\otimes} \mathbb{C}[d] = \Omega A \tilde{\otimes} \mathbb{C}[d]$$

hence an embedding

$$1) \qquad \begin{array}{ccc} Q & \hookrightarrow & \Omega \tilde{\otimes} \mathbb{C}[d] \\ \cup & & \cup \\ Q^+ & \hookrightarrow & \Omega^{ev} \oplus \Omega^{odd} d \end{array} \qquad a^+ \mapsto a - dad$$

On the other hand we also have an embedding

$$2) \qquad \Omega \hookrightarrow Q \tilde{\otimes} \mathbb{C}[F] = Q \oplus FQ$$

given by $a \mapsto a$ and $da \mapsto Fa^- = \frac{1}{2}F(a - FaF) = \frac{1}{2}[F, a]$.

To be more precise, the right side is a differential superalgebra with differential $ad(\frac{F}{2})$ and one has an alg homomorphism from A into it, and 2) comes from the universal property.

This map 2) is also the one Connes uses to obtain a super trace on Ω from a Fredholm modules. One has an action of $Q \tilde{\otimes} \mathbb{C}[F]$ on the Hilbert space in the ungraded case and an action of $(Q \tilde{\otimes} \mathbb{C}[F]) \tilde{\otimes} \mathbb{C}[\gamma] = M_2(Q)$ in the graded case.

In the Connes-Cuntz paper 2) is described as the map $\Omega \longrightarrow M_2(Q)$

$$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & a^F \end{pmatrix} \qquad da \longmapsto \begin{pmatrix} 0 & -a^- \\ a^- & 0 \end{pmatrix}$$

This is clearly the composition

$$\Omega \longrightarrow Q \tilde{\otimes} \mathbb{C}[F] \longrightarrow (Q \tilde{\otimes} \mathbb{C}[F]) \tilde{\otimes} \mathbb{C}[\gamma] = M_2(Q)$$

with $F \mapsto \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad \gamma \mapsto \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

and $a \mapsto \begin{pmatrix}a&0\\0&a^F\end{pmatrix}$. One has

$$da \mapsto Fa^- \mapsto \frac{F}{2}\begin{pmatrix}a&0\\0&a^F\end{pmatrix} - \frac{F}{2}\begin{pmatrix}a^F&0\\0&a\end{pmatrix} = F\begin{pmatrix}a^-&0\\0&-a^-\end{pmatrix}$$
$$= \begin{pmatrix}0&-a^-\\a^-&0\end{pmatrix}$$

Apparently we have various ways of transfering traces on Q to Ω and conversely. It is therefore natural to ask whether there is an equivalence of classes of traces.

Note that 2) induces an embedding of $\Omega^{ev}A$ into QA:

$$\begin{tikzcd} \Omega A \arrow[r, hook] & QA \tilde{\otimes} \mathbb{C}[F] \\ \Omega^{ev}A \arrow[u, phantom, "\cup" rotate=90] \arrow[r, hook] & QA \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

since $\quad da_1 da_2 \longmapsto Fa_1^- Fa_2^- = -a_1^- a_2^-$.

Note that the algebras we consider sit inside $\Omega A \tilde{\otimes} (\mathbb{C}[d] \tilde{\otimes} \mathbb{C}[F])$. The algebra $\mathbb{C}[d] \tilde{\otimes} \mathbb{C}[F]$ is <u>not semi-simple</u> because d generates a nilpotent ideal.

October 30, 1989

Consider a unital algebra A, its normalized Hochschild complex with coefficients in A:

$$\longrightarrow A\otimes\bar{A}^{\otimes 2} \xrightarrow{b} A\otimes\bar{A} \xrightarrow{b} A$$

and the operator s or d. One has

$$bs+sb = \underbrace{b's+sb'}_{1} + cs+sc$$

where c is the crossover term. Compute

$$(sc)(a_0,\dots,a_n) = s(-1)^n(a_na_0,a_1,\dots,a_{n-1}) = (-1)^n(1,a_na_0,\dots,a_{n-1})$$

$$(cs)(a_0,\dots,a_n) = c(1,a_0,\dots,a_n) = (-1)^{n+1}(a_n,a_0,\dots,a_{n-1})$$
$$= -\lambda(a_0,\dots,a_n)$$

Thus we get the strange operator on the Hoch. cx

$$-(cs+sc): (a_0,\dots,a_n) \longmapsto \lambda(a_0,\dots,a_n) + (-1)^{n-1}(1,a_na_0,\dots,a_{n-1})$$

which is homotopic to the identity. On the differential form picture this operator becomes

$$a_0da_1\dots da_n \longmapsto (-1)^{n-1} da_n\, a_0da_1\dots da_{n-1}$$
$$= (-1)^n a_n da_0 \dots da_{n-1}$$
$$+ (-1)^{n-1} d(a_na_0)da_1\dots da_{n-1}$$

Karoubi Asterisque §2.12

which Karoubi (I think YES) defined. Call this operator K. One has a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & d\Omega^{n-1}A \arrow[r] \arrow[d, "\lambda_n"] & \Omega^n A \arrow[r] \arrow[d, "K"] & d\Omega^n A \arrow[r] \arrow[d, "\lambda_{n+1}"] & 0 \\
0 \arrow[r] & d\Omega^{n-1}A \arrow[r] & \Omega^n A \arrow[r] & d\Omega^n A \arrow[r] & 0
\end{tikzcd}$$

So one sees K is an automorphism of $\Omega^n A = A \otimes \bar{A}^{\otimes n}$. The smallest order it could have is $n(n+1)$.

Take $n = 1$.

$$a_0 da_1 \xrightarrow{K} da_1 a_0 = d(a_1 a_0) - a_1 da_0$$
$$\xrightarrow{K} d(a_1 a_0) - da_0 a_1$$
$$= d(a_1 a_0 - a_0 a_1) + a_0 da_1$$

This shows K^2 is unipotent not the identity, hence of infinite order.

This operation K is strange and it is not clear that it is canonical.

Let's consider it's effect on normalized Hochschild cochains $f_{n+1} \in (A \otimes \bar{A}^{\otimes n})^*$. Let's pass to the non unital situation $A = \tilde{a}$ whence f_{n+1} becomes a pair (ψ_{n+1}, φ_n). We have

$$K f_{n+1}(a_0, \ldots, a_n) = (-1)^n f_{n+1}(a_n, a_0, \ldots, a_{n-1})$$
$$+ (-1)^{n+1} f_{n+1}(1, a_n a_0, \ldots, a_{n-1})$$

so that $\boxed{K\varphi_n = \lambda_n \varphi_n}$ also

$$\boxed{\psi_{n+1} - K\psi_{n+1} = (1-\lambda_{n+1})\psi_{n+1} + \text{Crossover for } b\varphi_n}$$

Suppose we assume $b'\varphi_n = (1-\lambda)\psi_{n+1}$. Then we find

$$(1-K)\varphi_n = (1-\lambda)\varphi_n$$
$$(1-K)\psi_{n+1} = \underbrace{(1-\lambda)\psi_{n+1}}_{b'\varphi_n} + \underset{\text{for } b\varphi_n}{\text{crossover}} = b\varphi_n$$

which shows that f_{n+1} and Kf_{n+1} differ by the Hochschild coboundary of φ_n. This is not surprising. One feels that this K operation is really well-defined on Hochschild cochains (ψ_{n+1}, φ_n) satisfying $b'\varphi_n = (1-\lambda)\psi_{n+1}$ for the following reason. We know that this cochain can be extended to a [redacted] (weak) trace on J^n, i.e. to a τ on $J^n/[Q, J^n] = (J \otimes_Q)^n$, and there is an obvious $\mathbb{Z}/n$ action on this cyclic tensor product.

As a check it should be the case that $K^n = 1$ on such a Hochschild cochain.

$$\psi - K\psi = b\varphi \qquad\qquad K\varphi = \lambda\varphi$$

$$K\psi - K^2\psi = b\lambda\varphi$$

$$+ \quad K^{n-1}\psi - K^n\psi = b\lambda^{n-1}\varphi \qquad\qquad K^n\varphi = \lambda^n\varphi = \varphi$$

$$\psi - K^n\psi = bN\varphi = Nb'\varphi = N(1-\lambda)\psi = 0$$

Thus $\psi = K^n\psi$, $\varphi = K^n\varphi$ showing that $K^n = 1$. Notice also that if $\lambda\varphi = \varphi$, then $b\varphi = \frac{1}{n} bN\varphi = \frac{1}{n} Nb'\varphi = \frac{1}{n} N(1-\lambda)\psi = 0$ and so $\psi = K\psi$.

November 2, 1989

There are many puzzles about the K-operation. This suggests that I don't understand yet an important idea in cyclic formalism.

Let us review. Consider $A = \tilde{a}$ unital augmented. Recall that we have interpreted the cyclic double complex in terms of the bar construction. This "explains" the maps B, $1-\lambda$ and leads one to distinguish between bar and Hochschild cochains. But I have recently seen lots of examples of $b\varphi$, where b is applied to something viewed as a bar cochain.

Before I had to treat the b, b' complexes separately, but now I am working with them together. I seem to be finding out about the normalized Hochschild complex.

Important is the operator s which on normalized cochains $f(a_0, a_1, \ldots, a_n) \in (A \otimes \bar{A}^{\otimes n})^*$ is

$$(sf)(a_1, \ldots, a_n) = f(1, a_1, \ldots, a_n)$$

The operator s is essentially equivalent to $d: \Omega^{n-1}A \longrightarrow \Omega^n A$, and is related to B.

Recall that we have

$$\boxed{(bs + sb)f = (1-K)f}$$

where
$$(Kf)(a_0, \ldots, a_n) = (-1)^n f(a_n, a_0, \ldots, a_{n-1}) + (-1)^{n-1} f(1, a_n a_0, a_1, \ldots, a_{n-1})$$

s is an operation on normalized Hochschild cochains f_n on A, which in the case of $A = \tilde{a}$ can

be identified with a pair
$f_n = (\psi_{n+1}, \varphi_n)$ where $\psi_{n+1} \in (\mathcal{A}^{\otimes n+1})^*$, $\varphi_n \in (\mathcal{A}^{\otimes n})^*$

Then $\quad sf_n = (\varphi_n, 0)$.

Note that

$$(sKf)(a_1,\dots,a_n) = (Kf)(1, a_1, \dots, a_n)$$
$$= (-1)^{n-1} f(1, a_n, a_1, \dots, a_{n-1})$$
$$= (\lambda s f)(a_1, \dots, a_n)$$

$\boxed{sK = \lambda s}$

Assertions: Assume $sbf_n = 0$, i.e. that $b'\varphi_n = (1-\lambda)\psi_{n+1}$. Then

1) $\xRightarrow{\lambda sf = sf} \quad Kf = f$

2) $\quad K^n f = f$

Proof 1). $\quad f - Kf = (bs + sb)f = bsf \overset{\text{if } \lambda sf = sf}{=} b\frac{1}{n}Nsf$

$$= \frac{1}{n} N b'\varphi = \frac{1}{n} N(1-\lambda)\varphi = 0.$$

(Notice that the general unital algebra A is a quotient of $\tilde{A} = \mathbb{C} \oplus A$. Consequently a normalized $f_n \in (A \otimes \bar{A}^{\otimes n})^*$ can be lifted to a normalized Hoch n-cochain on $\tilde{A}$, which is the pair (ψ_{n+1}, φ_n), $\quad \psi_{n+1} = f_n \quad$ and $\varphi_n = sf_n$.
The condition $sbf = 0$ over A lifts to $b'\varphi = (1-\lambda)\psi$ over $\tilde{A}$; note that b' is not defined over A.)

$\boxed{bK = Kb}$

2) $\quad f - K^n f = \sum_{i=0}^{n-1} (1-K)K^i f = \sum_{i=0}^{n-1} (bs+sb)K^i f$

$$= \sum_{i=0}^{n-1} \left(b\lambda^i sf + \lambda^i \underbrace{sbf}_{0} \right) = bBf = -Bbf$$
$$= -Nsbf = 0.$$

Actually a better proof is

$$f - K^n f = (1-K)\sum_0^{n-1} K^i f$$

and you apply i) to $\sum_0^{n-1} K^i f$. Thus

$$sb\sum K^i f = s\sum K^i bf = \sum \lambda^i sbf = 0$$

so 1) applies, and also

$$s\sum K^i f = \sum \lambda^i sf = Nsf$$

is λ invariant $\Longrightarrow \sum K^i f$ is K-invariant.

Notice that in general the first proof gives on f_n

$$\begin{aligned} 1 - K^n &= \sum_0^{n-1} (b\lambda^i s + \lambda^i sb) \\ &= bNs + (N - \lambda_{n+1}^n)sb \\ &= bB + Bb - \lambda_{n+1}^n sb = -\lambda_{n+1}^n sb \\ &= -\lambda^{-1} sb \end{aligned}$$

Actually it ought to be possible to turn this around to get a proof of $bB + Bb = 0$. The reason is that

$$K^n(a_0 da_1 \ldots da_n) = da_1 \ldots da_n a_0$$

is known

November 3, 1989

Review discussion with Jacek last week. My understanding of the Goodwillie result $L_D S = S L_D = 0$ uses the diagram

$$\begin{tikzcd}
 & & & HC_n(A\oplus\Omega^1 A) = HC_n(A)\oplus H_n(A,\Omega^1_A)\oplus\cdots \arrow[d, "pr_1"] \\
HC_{n+2}(A) \arrow[r, "S"] \arrow[d, "L_D"] & HC_n(A) \arrow[urr, "(1+d)_*"] \arrow[r, "B"] \arrow[d, "L_D"] & H_{n+1}(A,\tilde A) \arrow[r, "\partial"] \arrow[d, "\iota_D"] & H_n(A,\Omega^1_A) \arrow[dl, "(\tilde D)_*"] \\
HC_{n+2}(A) \arrow[r, "S"] & HC_n(A) & H_n(A,\tilde A) \arrow[l, "I"'] &
\end{tikzcd}$$

which tells us that L_D factors into $I\,\iota_D\,B$ and so $L_D S = 0$ as $BS = 0$.

Lets go over the arrows in the diagram. ∂ is the boundary map in Hochschild homology associated to the exact sequence

$$0 \longrightarrow \Omega^1 A \longrightarrow \tilde A\otimes\tilde A \longrightarrow \tilde A \longrightarrow 0$$

Thus we have the exact sequence of Hochschild chain complexes:

$$0 \longrightarrow \Omega^1 A\otimes A^{\otimes n} \longrightarrow \tilde A\otimes\tilde A\otimes A^{\otimes n} \underset{\xleftarrow{\;s\;}}{\longrightarrow} \tilde A\otimes A^{\otimes n} \longrightarrow 0$$

(in degree n). We choose a lifting s; then $bs - sb$ is a map of complexes of degree -1 from $\tilde A\otimes A^{\otimes n}$ to $\Omega^1 A\otimes A^{\otimes n}$. Choose

$$s(\tilde a_0, a_1, \dots, a_n) = (\tilde a_0\otimes 1, a_1, \dots, a_n)$$

Then $(bs - sb)(\tilde a_0, a_1, \dots, a_n)$ has lots of terms cancelling; the terms that don't cancel are

$$(\tilde a_0\otimes a_1, a_2, \dots, a_n) - (\tilde a_0 a_1\otimes 1, a_2, \dots, a_n)$$

which comes from

$$(\tilde{a}_0 da_1, a_2, \ldots, a_n)$$

Thus ∂ is induced by the (almost) isomorphism of complexes

$$\begin{array}{ccc} \tilde{A} \otimes A^{\otimes n} & \longrightarrow & \Omega^1 A \otimes A^{\otimes n-1} \\ (\tilde{a}_0, a_1, \ldots, a_n) & \longmapsto & (\tilde{a}_0 da_1, a_2, \ldots, a_n) \end{array}$$

Now we can check commutativity of the pentagon at the top right.

$$\begin{tikzcd}[column sep=small] & (a_0 + da_0, \ldots, a_n + da_n) \arrow[r, squiggly] & {} \arrow[d, "pr_1"] \\ (a_0, \ldots, a_n) \arrow[ur] \arrow[d, "B"] & & \sum_{i=0}^{n} (a_0, \ldots, a_{i-1}, da_i, a_{i+1}, \ldots, a_n) \\ \sum_{i=0}^{n} (-1)^{in} (1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1}) \arrow[rr, mapsto, "\partial"] & & \sum_{i=0}^{n} (-1)^{in} (da_i, a_{i+1}, \ldots, a_n, a_0, \ldots, a_{i-1}) \arrow[u, equal] \end{tikzcd}$$

Next

$$\begin{tikzcd} (\tilde{a}_0, a_1, \ldots, a_n) \arrow[r, mapsto] \arrow[dr, mapsto, "\text{Kassel's formula for } \iota_D"'] & (\tilde{a}_0 da_1, a_2, \ldots, a_n) \arrow[d, "(\tilde{D})_*"] \\ & (\tilde{a}_0 Da_1, a_2, \ldots, a_n) \end{tikzcd}$$

Finally the verification of

$$L_D = I \, \iota_D B$$

is easy, just like the path $\urcorner$ in the pentagon.

November 4, 1989

Variation maps. These are related to the evaluation map

$$S \times M^S \longrightarrow M$$

If $S = \mathbb{R}$, then for each t we have an evaluation $ev_t : M^S \longrightarrow M$. Given a contravariant functor F such as the DR complex $\Omega^\bullet(?)$, one has a family $ev_t^* : F(M) \longrightarrow F(M^S)$ and one can take the derivative $\partial_t ev_t^*$ at some point, say $t = 0$. This derivative is the variation map.

The variation ought to depend on the first infinitesimal nbd of $t = 0$. Thus we want to take $S = \mathrm{Spec}\, k[\varepsilon]$ as in algebraic geometry.

Claim: $M^{\mathrm{Spec}\, k[\varepsilon]} = TM$

Justifications: Here are three arguments

1) $S = \mathrm{Spec}\, k[\varepsilon]$ is a limiting case of $\{$two points$\}$ and TM is the limit of $M \times M$.

2) In algebraic geometry M^X is smooth when M is smooth no matter whether X is singular. This follows from the infinitesimal criterion for smoothness. So M^S should be a manifold and hence it can found from its points; the points of M^S form TM.

3) $\mathrm{Hom}_{Comm}(A, k[\varepsilon] \otimes R) = \mathrm{Hom}_{Comm}(B, R)$ if $A \leftrightarrow M$, $B \leftrightarrow M^S$. But an element of the former is a map $u + \varepsilon \dot{u} : A \longrightarrow R + \varepsilon R$, i.e. a pair $(u, \dot{u})$ with $u : A \to R$ a homomorphism

and $u: A \to R$ a derivation wrt u. This is the same as a morphism $(A, \Omega^1_A) \to (R, R)$ i.e. a homom. $S_A(\Omega^1_A) \to R$. So $B = S_A(\Omega^1_A)$ which corresponds to TM.

Alt. approach: Given $f_t : N \to M$ $t \in \mathbb{R}$ want $\partial_t f_t^* \big|_{t=0} : F(M) \to F(N)$. This depends only on the first order in t behavior of f_t at $t=0$ which is a map

$$j_1(f)_{t=0} : N \longrightarrow TM$$

The universal case is $N = TM$.

If we take $F = \Omega^i(\ )$, we see there ought to be a DGA homom. and derivation

$$\pi^* : \Omega(M) \longrightarrow \Omega(TM)$$

$$D : \Omega(M) \longrightarrow \Omega(TM)$$ This the <u>variation map</u>

In coords $\Omega(M)$ generated by x_μ, dx_μ

$\Omega(TM)$ generated by $x_\mu, dx_\mu, \xi_\mu, d\xi_\mu$

π^* is obvious, and

$$D(x_\mu) = \xi_\mu \qquad D(dx_\mu) = d\xi_\mu$$

D is obtained from the canonical family of tangent vectors on M parametrized by TM.

In this setup we also have interior product with this canonical vector field, which gives a derivation h of degree -1 such that

$$h(x_\mu) = 0 \qquad h(dx_\mu) = \xi_\mu$$

Check: $$(dh + hd)(x_\mu) = \xi_\mu = D(x_\mu)$$
$$(dh + hd)(dx_\mu) = d\xi_\mu = D(dx_\mu).$$

Next we want to discuss the variation map in the case of a Lie group G. This time $G^S = TG$ is the Lie group $G \tilde{\times} \mathfrak{g}$. One thus gets on left invariant forms a DGA homomorphism and derivation

$$\Lambda^\bullet \mathfrak{g}^* \longrightarrow \Lambda^\bullet (\mathfrak{g} \tilde{\times} \mathfrak{g})^* \qquad \text{obvious}$$

$$D : \Lambda^\bullet \mathfrak{g}^* \longrightarrow \Lambda^\bullet (\mathfrak{g} \tilde{\times} \mathfrak{g})^* \qquad \text{variation map}$$

which I want to calculate.

Recall $\quad \theta = X_a \theta^a \qquad$ (θ^a basis for $\mathfrak{g}^*$ dual to basis X_a of $\mathfrak{g}$)

$$d\theta + \tfrac{1}{2}[\theta, \theta] = 0$$

$$d(X_a \theta^a) + \tfrac{1}{2}[\theta_b X^b, \theta_c X^c] = 0$$

$$d\theta^a + \tfrac{1}{2} f^a_{bc}\, \theta^b \theta^c = 0$$

Let θ^a, η^a be the basis for $(\mathfrak{g} \tilde{\times} \mathfrak{g})^*$ dual to $(X_a, 0), (0, X_a)$. Then we should have also

$$d\eta^a + f^a_{bc}\, \theta^b \eta^c = 0.$$

The θ^a, η^a should be like the dx^μ_μ, $d\xi^\mu_\mu$ before, so the variation map should be

given by
$$D(\theta^a) = \eta^a .$$
Since D is a derivation we never get more than [redacted] linear terms in η^a. The variation map thus appears as a derivation of degree -1
$$H^\bullet(\mathfrak{g}) \longrightarrow H^{\bullet-1}(\mathfrak{g}, \mathfrak{g}^*).$$

If $\mathfrak{g}$ is abelian, then this variation map appears to be polarization
$$\Lambda^n(\mathfrak{g}^*) \longrightarrow \left(\Lambda^{n-1}\mathfrak{g}^*\right) \otimes \mathfrak{g}^*$$
which is injective for $n \geq 1$.

If $\mathfrak{g}$ is semi simple, the variation map is supposed to be zero. This is because nontrivial irreducible representations of $\mathfrak{g}$ have trivial cohomology. An alternative proof might be that the Killing form might give rise to a map of degree -2 which could be used as homotopy operator.
[redacted]

November 5, 1989

Aim is to understand homotopy for supertraces on the Cuntz algebra. Let's recall the old approach where we have a Fredholm module

$$A \longrightarrow \mathcal{L}(H) \ni F$$

say ungraded, whence we have a homom.

$$Q = QA \longrightarrow \mathcal{L}(H) \qquad \begin{array}{l} a \longmapsto a \\ a^F \longmapsto FaF \end{array}$$

and an even supertrace $\tau(x) = tr(Fx)$ for $x \in J^m \subset Q$. Letting $\alpha(a) = \frac{a - FaF}{2}$, $\rho(a) = \frac{a + FaF}{2}$ we have the cocycle

$$\varphi_{2n} = tr(F\alpha^{2n})$$

$$\psi_{2n+1} = tr(F\,\partial\rho\,\alpha^{2n}).$$

Now suppose we have a variation $\dot{F}$ of F. We have

$$\dot{\varphi}_{2n} = tr\Big(\dot{F}\alpha^{2n} + F\sum_1^{2n}\alpha^{i-1}\dot{\alpha}\,\alpha^{2n-i}\Big)$$

$$= tr\Big(F\sum_1^{2n}\alpha^{i-1}(\dot{\alpha})^-\alpha^{2n-i}\Big)$$

where we use that $tr(x) = 0$ if x anti-commutes with F, e.g. $tr(\dot{F}\alpha^{2n}) = 0$ because $\dot{F}$ anti-comm. w. F and α^{2n} commutes with F. Next

$$\dot{\alpha}(a) = \alpha(a)^{\cdot} = \left(\frac{a - FaF}{2}\right)^{\cdot} = -\frac{1}{2}\left(\dot{F}aF + Fa\dot{F}\right)$$

$$\dot{\alpha}(a)^- = -\frac{1}{2}\left(\dot{F}a^+F + Fa^+\dot{F}\right) = -\frac{1}{2}\left(-F\dot{F}a^+ + a^+F\dot{F}\right)$$

$$\dot{\alpha}(a)^- = [\underbrace{\tfrac{1}{2}F\dot{F}}_{\text{call this } L}, \rho(a)] = -[\delta+\rho, L]$$

Then $$\dot{\varphi}_{2n} = tr\Big(-F\,[\delta+\rho, \underbrace{\sum_{i=1}^{2n} (-1)^{i-1}\alpha^{i-1} L \alpha^{2n-i}}_{\mu}]\Big)$$

$$= -\delta\, tr(F\mu) + \beta\, tr^{\natural}(F\partial_\rho \mu)$$

which is the first equation of our old homotopy formula. The second equation is derived [redacted] similarly using the formulas

$$\dot{\alpha}^- = [L, \rho]$$

$$\dot{\rho}^+ = [L, \alpha]$$

However this procedure is awkward and not really consistent with the super viewpoint. We can improve things as follows

~~Let's [redacted] extend our variation of F to the 1-parameter family~~

$$\cancel{F_t = e^{-tL} F_0 e^{tL}}$$

~~where $F_0 = F$ and $L = \frac{1}{2}F\dot{F}$. Better: Consider~~

$$\cancel{F_t = e^{-tL} F e^{tL} \quad \text{with } L = \tfrac{1}{2}F\dot{F}}$$

~~Since $FLF = -L$ we have $F_t = Fe^{2tL}$~~

$$\cancel{F_t \dot{F}_t = [redacted](Fe^{2tL})(Fe^{2tL}2L) = 2L}$$

~~so $L = \frac{1}{2}F_t\dot{F}_t$ at~~

Consider

$$F_t = e^{-tL} F e^{tL} \qquad \text{where } L = \tfrac{1}{2} F \mathring{F}$$

Then F_t is a 1-parameter family of involutions such that

$$F_t = F e^{2tL} \qquad \text{as } FL = -LF$$

$$\mathring{F}_t = F e^{2t} 2L \implies \mathring{F}_0 = F 2\left(\tfrac{1}{2} F\mathring{F}\right) = \mathring{F}$$

so F_t is a 1-parameter family realizing the variation $\mathring{F}$ of F at $t=0$.

Instead of the family of homos.

$$QA \longrightarrow \mathcal{L}(H) \quad \text{with} \quad \iota a \longmapsto a$$

$$(\iota a)^F \longmapsto F_t a F_t = e^{-tL} F e^{tL} a e^{-tL} F e^{tL}$$

let us conjugate and consider instead the family of homos $QA \longrightarrow \mathcal{L}(H)$ with

$$\iota a \longmapsto e^{tL} a e^{-tL}$$

$$(\iota a)^F \longmapsto F e^{tL} a e^{-tL} F = e^{-tL} F a F e^{tL}$$

Then

$$a^+ \longmapsto \rho(a) = \frac{e^{tL} a e^{-tL} + e^{-tL} F a F e^{tL}}{2}$$

$$a^- \longmapsto \alpha(a) = \frac{\cdots\cdots}{2}$$

Hence we have the symmetric equations

$$\boxed{\dot{\rho} = [L, \alpha] \quad , \quad \dot{\alpha} = [L, \rho]}$$

for a variation.

What do we learn? We want to prove a homotopy formula for the cocycles on A attached to supertraces on QA. This means that we want to consider a superalgebra homomorphism

$$QA \longrightarrow S \oplus M$$

where S is a superalgebra and M is a superbimodule over S. Also we want a strace on M. We can suppose $S = QA$, in which case the above homomorphism is of the form $1+\partial$, where ∂ is a derivation of $Q = QA$ with values in the bimodule M. [redacted] Furthermore we want to have an element L in M such that

$$\partial a^+ = [L, a^-] \qquad \partial a^- = [L, a^+].$$

[redacted] Since [redacted] ∂ is supposed to be of degree zero, L must be of odd degree. [redacted] Let's see if this is consistent with the relations in Q. One has

$$(a_1 a_2)^+ = a_1^+ a_2^+ + a_1^- a_2^-$$

So

$$\partial (a_1 a_2)^+ \overset{?}{=} \partial a_1^+ a_2^+ + a_1^+ \partial a_2^+ + \partial a_1^- a_2^- + a_1^- \partial a_2^-$$

$$\begin{array}{ccc} \parallel & & \parallel \\ [L, (a_1 a_2)^-] & & [L, a_1^-] a_2^+ + a_1^+ [L, a_2^-] + [L, a_1^+] a_2^- + a_1^- [L, a_2^+] \\ \parallel & & \end{array}$$

$$[L, a_1^+ a_2^- + a_1^- a_2^+] = [L, a_1^+] a_2^- + a_1^+ [L, a_2^-] + [L, a_1^-] a_2^+ - a_1^- [L, a_2^+]$$

This is a puzzle. Everything seems to work if both ∂ and $[L, ?]$ are strict derivations, but then the parity is not preserved.

November 6, 1989

Let us try to prove a homotopy formula for the [redacted] cocycles attached to [redacted] Fredholm modules using the method of my cochain paper. Recall that situation where we have a family $\rho : A \longrightarrow R[t]$. [redacted] We consider cochains $\mathrm{Hom}(B(A), k[t,dt]\otimes R)$ with values in the DGA $k[t,dt]\otimes R$. Then one has the curvature of ρ ~~for~~ the total differential $dt\,\partial_t + \delta$:

$$\tilde{\omega} = (dt\,\partial_t + \delta)\rho + \rho^2 = \underbrace{\delta\rho + \rho^2}_{\omega} + dt\,\dot{\rho}$$

$$\tilde{\omega}^n = \omega^n + \underbrace{dt \sum_{i=1}^{n} \omega^{i-1}\dot{\rho}\,\omega^{n-i}}_{\mu_{2n-1}}$$

etc. This device of introducing $k[t,dt]$ corresponds geometrically to proving homotopy properties of characteristic forms over M by working on $\mathbb{R}\times M$.

Let us next consider the situation arising from Fredholm modules:

$$A \xrightarrow{\theta} \mathcal{L}(H) \ni F$$

$$\theta = \frac{\theta + F\theta F}{2} + \frac{\theta - F\theta F}{2} = \rho + \alpha$$

$$[\delta + \rho, \alpha] = (\delta + \rho)^2 + \alpha^2 = 0$$

We now want to consider a family $F \in \mathcal{L}(H)[t]$. We will work to first order at $t=0$, which means we can assume $F = e^{-tL}F_0\,e^{tL}$ where $F_0 L = -LF_0$ and hence $\frac{1}{2}F\dot{F} = \frac{1}{2}F_0 e^{2tL}F_0 e^{2tL}2L = L$

Now working in $\mathrm{Hom}(B(A), k[t,dt]\otimes \mathcal{L}(H))$ we consider the flat connection

$$D = dt\,\partial_t + \delta + \theta$$

and the involution $F \ni \delta(F) = 0$.

We have $D = D_+ + D_-$ where

$$D_+ = \frac{dt\,\partial_t + F.dt\,\partial_t.F}{2} + \delta + \frac{\theta + F\theta F}{2}$$

$$= dt\,(\partial_t + L) + (\delta + \rho)$$

$$D_- = \frac{dt\,\partial - F(dt\,\partial_t)F}{2} + \frac{\theta - F\theta F}{2}$$

$$= -dt\,L + \alpha$$

We have $(D_+ + D_-)^2 = \underbrace{[D_+, D_-]}_{0} + \underbrace{(D_+^2 + D_-^2)}_{0}$

$$[D_+, D_-] = [dt(\partial_t + L) + (\delta + \rho), -dt\,L + \alpha] = 0$$

$$= [\delta + \rho, \alpha] + dt\left([\delta + \rho, L] + [\partial_t + L, \alpha]\right)$$

so $\quad [\rho, L] + \dot{\alpha} + [L, \alpha] = 0$

which yields since $FL = -LF$.

$$(\dot{\alpha})^- = [L, \rho]$$

$$(\dot{\rho})^+ = -(\dot{\alpha})^+ = [L, \alpha]$$

This means we still have to apply a gauge transformation to get the simpler equations.

In general suppose given a 1-parameter family $F = F_t$ of involutions, and let's try to find a 1-parameter family of autos u such that $F = u^{-1} F_0 u$ where F_0 is constant. Then

$$(uFu^{-1})^\circ = \mathring{u} F u^{-1} + u \mathring{F} u^{-1} - u F u^{-1} \mathring{u} u^{-1}$$

$$0 = (u^{-1}\mathring{u}) F + \mathring{F} - F (u^{-1}\mathring{u})$$

$$[F, u^{-1}\mathring{u}] = \mathring{F}$$

Solution is $u^{-1}\mathring{u} = \frac{1}{2} F \mathring{F} \; (= L)$.

Go back to $D = dt\,\partial_t + \delta + \Theta$ which we split into even and odd components relative to F. Conjugation by u leads to splitting

$$uDu^{-1} = \underbrace{u\cdot(dt\,\partial_t)\cdot u^{-1}}_{dt(\partial_t - \cancel{uu}^{-1}\mathring{u}u^{-1}) = dt(\partial_t - uLu^{-1})} + \delta + u\Theta u^{-1}$$

relative to F_0.

$$uD_+u^{-1} = dt\left(\partial_t - \tfrac{1}{2}\left(u \overset{u^{-1}\mathring{u}}{L} u^{-1} + F_0 u \overset{u^{-1}\mathring{u}}{L} u^{-1} F_0\right)\right) + \delta + \frac{u\Theta u^{-1} + F_0 u \Theta u^{-1} F_0}{2}$$

But
$$u^{-1}(\mathring{u}u^{-1} + F_0 \mathring{u} u^{-1} F_0) u = u^{-1}\mathring{u} + \underbrace{u^{-1}F_0}_{Fu^{-1}} \mathring{u} u^{-1} \underbrace{F_0 u}_{uF}$$

$$= u^{-1}\mathring{u} + F(u^{-1}\mathring{u})F$$

$$= \tfrac{1}{2} F\mathring{F} + F\left(\tfrac{1}{2} F\mathring{F}\right) F = 0$$

So the effect of the gauge transformation is to make

$$uD_+u^{-1} = dt\,\partial_t + \delta + u\left(\frac{\theta + F\theta F}{2}\right)u^{-1}$$

$$uD_-u^{-1} = dt(-\dot{u}u^{-1}) + u\left(\frac{\theta - F\theta F}{2}\right)u^{-1}$$

So if we now replace D, F by uDu^{-1}, uFu^{-1} then we reach a situation where F is constant and

$$D_+ = dt\,\partial_t + \delta + \rho$$

$$D_- = -dt\,L + \alpha$$

where L is now an operator anticommuting with F. Then we have

$$[D_+, D_-] = [dt\,\partial_t + \delta + \rho, \alpha - dt\,L] = 0$$

$$\boxed{[\delta + \rho, \alpha] = 0 \qquad \dot{\alpha} = [L, \rho]}$$

Also $0 = D_+^2 + D_-^2 = (\delta + \rho)^2 + dt\,\dot{\rho} + \alpha^2 - dt[L, \alpha]$

gives $\boxed{(\delta + \rho)^2 + \alpha^2 = 0 \qquad \dot{\rho} = [L, \alpha]}$

Remark that in the case $F = e^{-tL}F_0e^{tL}$ then

$$\begin{matrix}\rho \\ \alpha\end{matrix} = \frac{e^{tL}\theta e^{-tL} \pm e^{-tL}F_0\theta F_0 e^{tL}}{2}$$

so that the equations $\dot{\rho} = [L, \alpha]$ $\dot{\alpha} = [L, \rho]$ are ~~[illegible]~~ obvious.

<u>November 15, 1989</u>

Homotopy formula for $Q = QA$.

Recall that a superalg morphism $Q \to S$ is equivalent to an alg morphism $A \to S$, which is equivalent to a pair of linear maps $\rho : A \to S^+$, $q : A \to S^-$ satisfying

1) $$(\delta+\rho)^2 = q^2 \quad , \quad [\delta+\rho, q] = 0$$

in the differential superalgebra $\mathrm{Hom}(B(A), S)$. Also if τ is a supertrace on S, then we obtain a big cocycle:

$$\delta\,\tau(q^n) = \tau(\delta(q^n)) = \tau([-\rho, q^n]) = \beta\tau^\natural(\partial\rho\, q^n)$$

$$\delta\,\tau^\natural(\partial\rho\, q^n) = \tau^\natural([\delta+\rho, \partial\rho\, q^n]) = \tau^\natural(\underbrace{[\delta+\rho, \partial\rho]}_{\partial(\delta\rho+\rho^2) = \partial q^2}\, q^n)$$

$$= \tau^\natural((\partial q\, q + q\,\partial q)\, q^n)$$

$$= 2\tau^\natural(\partial q\, q^{n+1}) = \frac{2}{n+2}\,\bar\partial\,\tau(q^{n+2})$$

In the above τ is supposed to be even; in the odd case there is an extra sign when δ is moved past τ.

~~Next we wish to consider homotopy. We consider a family of (ρ, q) i.e. $\rho : A \to k[t]\otimes Q$, $q : A \to k[t]\otimes Q$ satisfying 1), and $L \in k[t]\otimes Q^-$ such that~~

~~$\boxed{\dot\rho a = [L, a^-] \qquad \dot q a = [L, a^+]}$.~~

Next we take up homotopy. We consider a family of (ρ, q), that is, a pair of linear maps

$$\rho : A \longrightarrow k[t]\otimes S^{+}$$
$$q : A \longrightarrow k[t]\otimes S^{-},$$

together with an $L \in k[t]\otimes S^{-}$ such that 1) $(\delta+\rho)^2 = q^2$, $[\delta+\rho, q]=0$ and

$$(\rho a)^{\circ} = La^{-} - a^{-}L = (Lq+qL)(a)$$
$$(q a)^{\circ} = La^{+} - a^{+}L = (L\rho+\rho L)(a)$$

Thus

$$2)\begin{cases} \mathring{q} = L\rho + \rho L = [L,\rho] \\ \mathring{\rho} = Lq + qL \quad \underline{\text{not}} \text{ the bracket of } L, q \end{cases}$$

We can organize this data as follows. Let us first handle 1). We have a connection $\delta+\rho$ and we have written its curvature $(\delta+\rho)^2$ as a square q^2 where q satisfies the Bianchi property $[\delta+\rho, q] = 0$.

To handle 2) we consider the DR complex $k[t,dt]$ and the differential superalgebra $k[t,dt]\otimes S$, as well as $\mathrm{Hom}(B(A), k[t,dt]\otimes S)$ with differential $dt\,\partial_t + \delta$. We have the connection

$$dt\,\partial_t + \delta + \rho$$

and the endomorphism-valued 1-form

$$\tilde{q} = q + dt\,L$$

Then

$$[dt\,\partial_t + \delta + \rho, q + dt\,L] = [\delta+\rho, q] + dt\{\mathring{q} - \underbrace{[\delta+\rho, L]}_{\rho L + L\rho}\} = 0$$

$$(dt\,\partial_t + \delta + \rho)^2 = (\delta+\rho)^2 + dt\,\mathring{\rho}$$
$$(q + dt\,L)^2 = q^2 + dt(Lq + qL)$$

so the relations are satisfied.

Thus given a supertrace $\tau : S \to V$ if we extend it to $\tau : k[t,dt]\otimes S \longrightarrow k[t,dt]\otimes V$ linearly over $k[t,dt]$, then we obtain a cocycle $\tau(\tilde{q}^n)$, $\tau^{\natural}(\partial\rho\,\tilde{q}^n)$:

$$(dt\,\partial_t + \delta)\,\tau(\tilde{q}^n) = \beta\,\tau^{\natural}(\partial\rho\,\tilde{q}^n)$$

$$(dt\,\partial_t + \delta)\,\tau^{\natural}(\partial\rho\,\tilde{q}^n) = \frac{2}{n+2}\,\bar{\partial}\tau(\tilde{q}^{n+2})$$

This gives the following relations by taking the coefficient of dt: (use $\tilde{q}^n = q^n + dt\,\mu_{n-1}$)

$$\boxed{\begin{aligned} \partial_t \tau(q^n) - \delta\tau(\mu_{n-1}) &= \beta\,\tau^{\natural}(-\partial\rho\,\mu_{n-1}) \\ \partial_t \tau^{\natural}(\partial\rho\, q^n) + \delta\,\tau^{\natural}(\partial\rho\,\mu_{n-1}) &= \frac{2}{n+2}\,\bar{\partial}\tau(\mu_{n+1}) \end{aligned}}$$

November 17, 1989

More on homotopy for Q.

I want to develop the proof of the homotopy formula given above which uses $dt\,\partial_t + \delta + \rho$, $dt L + g$ and $\mathrm{Hom}(B(A), k[t,dt]\otimes S)$. The idea is that

$$k[t,dt]\otimes S = S[t] \oplus dt\, S[t]$$

and $\rho, \delta : A \longrightarrow S^{\pm}[t]$ give $Q \longrightarrow S[t]$, whereas $dt\,\partial_t : S[t] \longrightarrow dt\,S[t]$ is an odd derivation. I would like to carry out the proof with $S[t]$ replaced by Q and $dt\,S[t]$ replaced by the Q-bimodule $dt\,QLQ$ which should be the free Q-bimodule with even generator $Y = dt\,L$.

So we transform

$$\partial_t a^+ = La^- - a^-L$$
$$\partial_t a^- = La^+ - a^+L$$

by multiplying by dt to get

$$da^+ = Ya^- + a^-Y \qquad\qquad d\rho = Yg + gY$$
$$da^- = Ya^+ - a^+Y \qquad\qquad dg = Y\rho - \rho Y$$

We claim these formulas define an odd derivation $d : Q \longrightarrow QYQ \simeq Q\otimes Q$:

$$d(a_1a_2)^+ \overset{?}{=} d(a_1^+a_2^+ + a_1^-a_2^-)$$

$$\begin{aligned}
&\parallel && \parallel\\
&Y(\overset{2}{a_1^+a_2^-} + \overset{1}{a_1^-a_2^+}) && (\overset{1}{Y}a_1^- + \cancel{a_1^-Y})a_2^+ + a_1^+(\cancel{Ya_2^-} + \overset{3}{a_2^-Y})\\
&+(\underset{3}{a_1^+a_2^-} + \underset{4}{a_1^-a_2^+})Y && (\overset{2}{Y}a_1^+ - \cancel{a_1^+Y})a_2^- - a_1^-(\cancel{Ya_2^+} - \underset{4}{a_2^-Y})
\end{aligned}$$

$$d(a_1 a_2)^- \overset{?}{=} d(a_1^+ a_2^- + a_1^- a_2^+)$$

$$\Vert \qquad\qquad\qquad\qquad \Vert$$

$$Y(\overset{2}{a_1^+ a_2^+} + \overset{1}{a_1^- a_2^-}) \qquad (\overset{1}{Y a_1^-} + \cancel{a_1^- Y}) a_2^- + a_1^+ (\cancel{Y a_2^+} - \overset{3}{a_2^+ Y})$$

$$-(\underset{3}{a_1^+ a_2^+} + \underset{4}{a_1^- a_2^-}) Y \qquad (\overset{2}{Y a_1^+} - \cancel{a_1^+ Y}) a_2^+ - a_1^- (\cancel{Y a_2^-} + \underset{4}{a_2^- Y})$$

Put another way the formulas

$$d a^+ = 1 \otimes a^- + a^- \otimes 1$$

$$d a^- = 1 \otimes a^+ - a^+ \otimes 1$$

define an odd derivation $Q \longrightarrow Q \otimes Q$. Note that $a \mapsto d\iota a$ will not be a derivation for the obvious ιA - bimodule structure.

[redacted] Consider $\mathrm{Hom}(B(A), Q \oplus QYQ)$ with the total differential $d + \delta$. We have

$$d + \delta + \rho \qquad \text{connection}$$

$$Y + q \qquad \text{even element}$$

$$[d+\delta+\rho, Y+q] = [\delta+\rho, q] + (dq + \overset{\rho Y - Y\rho}{[\rho, Y]}) = 0$$

$$(d+\delta+\rho)^2 - (Y+q)^2 = (\delta+\rho)^2 - q^2 + (d\rho - qY - Yq) = 0$$

Another way to describe the formalism: The equations [redacted] describing a superalg. hom. $QA \longrightarrow S$ are

$$[\delta+\rho, q] = (\delta+\rho)^2 - q^2 = 0$$

The first variation is

$$[\mathring{\rho}, q] + [\delta+\rho, \mathring{q}] = 0$$

$$[\delta+\rho, \mathring{\rho}] = q\mathring{q} + \mathring{q}q \text{ [redacted]}$$

and the sort of variations we need to prove a homotopy formula are of the form

$$\mathring{\rho} = Lq + qL \qquad \mathring{q} = L\rho + \rho L$$

for an odd L. If we change the even derivation $\cdot = \partial$ to the odd derivation $dt\,\partial = d$ then the variation equations become

$$d\rho = Yq + qY \qquad dq = Y\rho - \rho Y$$

November 19, 1989

Problem: Given a trace τ on $R = RA$ we know that there is another trace $\kappa\tau$ with $\kappa^2\tau = \tau$. We also know that $\tau - \kappa\tau$ is nullcobordant in a definite way. There is apparently a factorization

$$\begin{tikzcd} & \Omega^1 R_\natural \arrow[dr, dashed] & \\ R_\natural \arrow[ur, "d"] \arrow[rr, "1-\kappa"'] & & R_\natural \end{tikzcd}$$

and the problem is to construct it in a natural way.

Here are the formulas. We have $(\kappa\tau)(a_0^+) = \tau(a_0^+)$ and

$$(\kappa\tau)(a_0^+ a_1^- \cdots a_{2n}^-) = -\tau(a_{2n}^- a_0^+ a_1^- \cdots a_{2n-1}^-)$$

for $n \geq 1$. Hence

$$(\tau - \kappa\tau)(a_0^+) = 0$$
$$(\tau - \kappa\tau)(a_0^+ a_1^- \cdots a_{2n}^-) = \tau([a_0^+ a_1^- \cdots a_{2n-1}^-, a_{2n}^-])$$

If f is the cocycle of τ [redacted]:

$$f_{2n}(a_0, \ldots, a_{2n}) = \tau(a_0^+ a_1^- \cdots a_{2n}^-)$$

then the cocycle of $(\tau - \kappa\tau)$ is

$$(1-\kappa)f = bsf$$

since we know that $sbf = 0$. (This follows from $bf_{2n} = \frac{1}{n+1} Bf_{2n+2}$ and the fact that $sB = 0$.) Notice this implies that $(1-\kappa)f_{2n}$ is a Hochschild cocycle.

We define a trace τ' on $\Omega' R$ by

$$\boxed{\tau'(a_0^+ a_1^- \cdots a_{2n-2}^- \, da_{2n-1}^+) = -\frac{1}{n}\tau(a_0^- \cdots a_{2n-1}^-)}$$

Notice that $(sf_{2n})(a_1, \dots, a_{2n}) = \tau(a_1^- \cdots a_{2n}^-)$ is λ^2-invariant, or better $K^2 sf_{2n} = sf_{2n}$. τ' above is defined as the trace on $\Omega' R$ whose g-cochain is $g_{(2n)} = (-\frac{1}{n} sf_{2n})$. Its h-cochain is

$$h_{(2n)} = \left(\sum_{i=0}^{n-1} K^{2i}\right) g_{(2n)} = -sf_{2n}.$$

The cocycle associated to $\tau' d$ is the coboundary of the h-cochain. This is degree $2n$ in this nonunital notation (which needs checking) should be

$$-bh_{(2n)} + \text{const}\cdot \underbrace{Bh_{(2n+2)}}_{0} = bsf_{2n} = (1-K)f_{2n}$$

as $Bs = 0$.

Thus we have the identity

$$\boxed{(1-K)\tau = \tau' d}$$

τ' defined above. In the above paragraph $g_{(2n)}$, $h_{(2n)}$ are Hochschild $2n-1$ cochains which are defined on $\bar{A}^{\otimes 2n}$ considered as quotient of $A \otimes \bar{A}^{\otimes(2n-1)}$.

Check

$$\tau' d(a_0^+ a_1^- a_2^-) = \tau'(da_0^+ a_1^- a_2^- + a_0^+ d(a_1 a_2)^+ - a_0^+ da_1^+ a_2^+ - a_0^+ a_1^+ da_2^+)$$

gives 0 $\quad (a_2 a_0)^+ - a_2^+ a_0^+$

$$= \underbrace{\tau'(a_1^- a_2^- da_0^+)}_{0} + \tau'(a_0^+ d(a_1 a_2)^+ - a_2^+ a_0^+ da_1^+ - a_0^+ a_1^+ da_2^+)$$

$(a_0 a_1)^+ - a_0^- a_1^-$ gives 0

$$= -\tau(a_0^-(a_1 a_2)^- - (a_2 a_0)^- a_1^- - (a_0 a_1)^- a_2^-)$$

$$= -\tau\big(a_0^- a_1^+ a_2^- - a_2^+ a_0^- a_1^- - a_0^+ a_1^- a_2^-$$
$$a_0^- a_1^- a_2^+ - a_2^- a_0^+ a_1^- - a_0^- a_1^+ a_2^- \big)$$
$$= \tau\left(a_0^+ a_1^- a_2^- + a_2^- a_0^+ a_1^- \right)$$
$$= (1-K)\,\tau\left(a_0^+ a_1^- a_2^- \right)$$

Remark: The presence of the $\frac{1}{n}$ in the formula for $\bar{\tau}'$ indicates something subtle like integration. It's unlikely you will find a simple map

$$\circledast \qquad (\Omega^1 R)_\natural \longrightarrow R_\natural$$

whose composition with d is $1-K$.

Idea: Use exactness

$$0 \longrightarrow \bar{R}_\natural \xrightarrow{\;\;} (\Omega^1 R)_\natural \xrightarrow{\beta} R$$

But $\circledast$ is supposed to give

$$-\rho\,\omega^{n-1} d\rho \longmapsto \left(-\tfrac{1}{n}\right)\omega^n$$

Applying d to $\left(-\frac{1}{n}\right)\omega^n$ does not seem to remove the $\left(-\frac{1}{n}\right)$, as

$$d(\omega^n) = [\delta + \rho, \mu_{2n-1}].$$

November 26, 1989

Let M be [redacted] a compact odd dim'l manifold, let E be a vector bundle on M with inner product, let $\mathcal{A}, \mathcal{G}$ be the corresponding spaces of connections + gauge transformations.

On $\mathcal{A}$ there is a canonical 1-form with values in $\Omega^1(M, \operatorname{End} E)$; denote it δA. We also have [redacted] a function on $\mathcal{A}$ with values in $\Omega^2(M, \operatorname{End} E)$ given by $A \mapsto F = D_A^2$, the curvature. If $\dim(M) = 2n+1$ we then have defined a 1-form on $\mathcal{A}$ by

$$\int_M \operatorname{tr}(\delta A\, F^n)$$

This one form is invariant under the action of $\mathcal{G}$ because one knows that $g \in \mathcal{G}$ acts by conjugation on δA and F.

Let's check that the 1-form is closed

$$\delta \int \operatorname{tr}(\delta A\, F^n) = + \int \operatorname{tr}\Big(\delta A \sum_{i=1}^{n} F^{i-1}\, \delta F\, F^{n-i}\Big) \qquad \begin{pmatrix}\delta A \\ \text{even}\end{pmatrix}$$

Now $\quad \delta F = \delta(D_A^2) = \delta A\, D_A - D_A\, \delta A = -D_A(\delta A)$

so this is

$$-\int \operatorname{tr}\Big(\delta A \sum_{i=1}^{n} F^{i-1} D_A(\delta A)\, F^{n-i}\Big)$$

This is

$$-\int \operatorname{tr}\Big(\sum F^{n-i}\, \delta A\, F^{i-1}\, D_A(\delta A)\Big)$$

$$\| $$

$$-\int \operatorname{tr}(\mu\, D_A(\delta A))$$

$$= \int \operatorname{tr}\Big(-\delta A\, D_A\big(\overbrace{\textstyle\sum F^{i-1}\, \delta A\, F^{n-i}}^{\mu}\big)\Big)$$

$$= \int \operatorname{tr}\big(\cancel{-D_A(\delta A)\mu} + \underline{D_A(\delta A)}\, \mu\big)$$

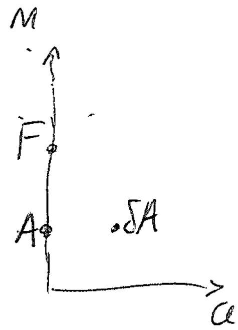
opposite sign

We should give a more convincing proof. Consider the canonical connection on $\mathcal{A}\times E$ over $\mathcal{A}\times M$. Recall this can be described by giving the partial connections in the $\mathcal{A}$ and M directions. There's the obvious trivial connection in the $\mathcal{A}$-direction since $\mathcal{A}\times E = pr_2^*(E)$, and there's a tautological connection in the M direction. Let's work locally & assume E trivial (or else upstairs in the principal bundle $\mathcal{A}\times P$.) Let δ, d be the partial differential in $\mathcal{A}$ and M-directions. Then the connection on $\mathcal{A}\times E$ is

$$\delta + d + A = \delta + D_A$$

where A is the canonical element in

$$\Omega^0(\mathcal{A}, \Omega^1(M, \mathrm{End}\, E))$$



Total curvature

$$\tilde{F} = (\delta + d + A)^2 = \delta A + \overbrace{dA + A^2}^{F}$$

Bianchi (or Russian formula)

$$0 = [\delta + d + A, \ \delta A + F]$$
$$= \left([d+A, \delta A] + \delta F\right) + [d+A, F]$$

gives $\boxed{\delta F = - D_A(\delta A)}$

$$\tilde{F}^{n+1} = F^{n+1} + \sum_0^n F^i \delta A F^{n-i} + \cdots$$

$$\int_M tr(\tilde{F}^{n+1}) = (n+1)\int_M tr(\delta A\, F^n)$$

is closed on $\mathcal{A}$ because

$$\delta \int_M - = (-1)^{\dim M} \int_M (\delta + d) -$$

Summarizing we have this closed $\mathcal{G}$-invariant 1-form on $\mathcal{A}$.

Next we show it is horizontal for the $\mathcal{G}$-action. Let $X \in \tilde{g} = Lie(\mathcal{G})$. Then

$$\iota_X \int_M tr(\delta A\, F^n) = \int_M tr(D_A(X) F^n)$$

$$= \int_M tr(D_A(XF^n)) = \int_M d\, tr(XF^n) = 0.$$

Let us now suppose $\dim M = 2n+2$. In this case we can obtain a closed 2-form on $\mathcal{A}$ by integrating $tr(\tilde{F}^{n+2})$ over the fibre for $\mathcal{A}\times M \longrightarrow \mathcal{A}$. We have

$$\tilde{F}^{n+2} = (F + \delta A)^{n+2} = F^{n+2} + \sum_0^{n+1} F^i\, \delta A\, F^{n+1-i}$$

$$+ \sum_{i+j+k=n} F^i\, \delta A\, F^j\, \delta A\, F^k + \cdots\cdots$$

so

$$\int_M tr(\tilde{F}^{n+2}) = \int_M \sum_{i+j+k=n} tr(F^i\, \delta A\, F^j\, \delta A\, F^k)$$

$$= \int_M \sum_j \sum_{i+k=n-j} tr\left(\delta A\, F^j \delta A\, F^{\overbrace{k+i}^{n-j}}\right)$$

$$= \int_M \sum_{j=0}^{n} (n-j+1)\, tr\left(\delta A\, F^j\, \delta A\, F^{n-j}\right)$$

$$= \int_M \sum_{j=0}^{n} (j+1)\, tr\left(\delta A\, F^j\, \delta A\, F^{n-j}\right)$$

$$= \int_M \sum_{j=0}^{n} \left(\frac{n}{2}+1\right) tr\left(\text{———}\right)$$

$$= \left(\frac{n+2}{2}\right) \int_M \sum_{j=0}^{n} tr\left(\delta A\, F^j \delta A\, F^{n-j}\right)$$

Thus we obtain a closed 2-form on $\mathcal{A}$

$$\omega = \left(\frac{n+2}{2}\right)\int_M \sum_{j=0}^{n} tr\left(\delta A\, F^j\, \delta A\, F^{n-j}\right)$$

Then we can ask about the moment map relative to this closed invariant 2-form. This means finding a map $X \longmapsto f_X$, a function on $\mathcal{A}$, such that

$$\iota_X(\omega) = df_X$$

Now

$$\iota_X \int_M \sum_0^n tr\left(\delta A\, F^j\, \delta A\, F^{n-j}\right)$$

$$= \int_M \sum_0^n tr\left\{D_A(X)\, F^j\, \delta A\, F^{n-j} + \delta A\, F^j\, D_A(X)\, F^{n-j}\right\}$$

$$= \int_M \sum_0^n \operatorname{tr}\Big\{ D_A\big(\cancel{XF^j\,\delta A\,F^{n-j}} + \cancel{\delta A\,F^j X F^{n-j}}\big) - XF^j \overbrace{D_A(\delta A)}^{-\delta F} F^{n-j} - D_A(\delta A)\,F^j X F^{n-j} \Big\}$$

$$= \int_M \sum_0^n \operatorname{tr}\{ XF^j\,\delta F\,F^{n-j} + \delta F\,F^j X F^{n-j} \}$$

$$= \int_M \sum_0^n \operatorname{tr}\{ X (F^j\,\delta F\,F^{n-j} + F^{n-j}\,\delta F\,F^j) \}$$

$$= 2\int_M \operatorname{tr}\{ X\,\delta(F^{n+1}) \} = \delta\; 2\int_M \operatorname{tr}(X F^{n+1})$$

The last step needs more care:

$$\delta \operatorname{tr}(X F^{n+1}) = \operatorname{tr}(X\,\delta F^{n+1})$$

and $\delta \int_M = \int_M \delta$ as dim M is even. Thus we have

$$\iota_X \omega = \delta f_X \qquad f_X = (n+2)\int_M \operatorname{tr}(X F^{n+1})$$

Actually things become nicer if one puts in the factorials. Start with

$$\operatorname{tr}\left(\frac{\tilde F^{n+2}}{(n+2)!}\right)$$

then one obtains

$$\omega = \int_M \frac{1}{(n+1)!} \sum_{j=0}^n \frac{1}{2} \operatorname{tr}(\delta A\, F^j\, \delta A\, F^{n-j})$$

$$f_X = \int_M \operatorname{tr}\left(X \frac{F^{n+1}}{(n+1)!} \right)$$

November 27, 1989

Ideas to be listed for future ref.

1) Old project of deriving cyclic theory using the derived category of A-bimodules with operations

$M \otimes_A N$ "composition"

$M \otimes_A$ "trace"

2) Standard normalized resolution $\mathcal{B}_n = A \otimes \bar{A}^{\otimes n} \otimes A$ of A as $A \otimes A^{op}$ module.

$$\mathcal{B} \xrightarrow{\Delta} \mathcal{B} \otimes_A \mathcal{B} \underset{1 \otimes \varepsilon}{\overset{\varepsilon \otimes 1}{\underset{\uparrow h}{\rightrightarrows}}} \mathcal{B}$$

$$\Delta(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (a_0, \ldots, a_i, 1) \otimes (1, a_{i+1}, \ldots, a_{n+1})$$

$$h\{(a_0, \ldots, a_{p+1}) \otimes (a'_0, \ldots, a'_{q+1})\} = (a_0, \ldots, a_{p+1} a'_0, a'_1, \ldots, a'_{q+1})$$

I had ideas about h as a differential on $T_A(A \otimes A)/\text{ideal}$ in connection work late summer 89 on F's $\ni$ $[F, a]$, $F^2 - 1$ compact.

Now $\mathcal{B} \otimes_A$ is the normalized Hochschild complex, so it has operators b, s, K, B. Can these be explained using homological algebra?

Apparently $\underline{B \text{ is obtained from}}$

$$\mathcal{B} \otimes_A \xrightarrow{\Delta} (\mathcal{B} \otimes_A)^2 \xrightarrow{\text{flip}} (\mathcal{B} \otimes_A)^2 \xrightarrow{h} \mathcal{B} \otimes_A$$

but I don't see where s comes from, K also.

3) The formalism b, s, κ, B is reminiscent of Getzler-Jones-Petrack where they considered a circle action. Analogy

$$bs + sb = 1 - \kappa \quad \longleftrightarrow \quad d\iota_X + \iota_X d = L_X$$

$$B = \sum \kappa^i s \quad \longleftrightarrow \quad P = A\iota_X \qquad A = \text{averaging}$$

They consider $d + P$ on all forms and because ι mucks up the ~~[illegible]~~ product this involves A^∞ algebras.

4) Equivalence of periodic complexes constructed from $B(A)$ and RA and maybe QA

December 2, 1989

We want to calculate the map

$$(\Omega^1 Q)_\natural \longrightarrow (QLQ)_\natural \simeq LQ$$

Recall that we have a description of supertraces τ on QLQ via the cochains

$$f_n = \tau(L\rho\delta^n)$$

$$f_n(a_0,\dots,a_n) = \tau(La_0^+ a_1^- \dots a_n^-)$$

Cuntz describes supertraces T on $\Omega^1 Q$ via cochains of the form $T(a_0 da_1 a_2^- \dots a_n^-)$, and only half of these are needed for supertraces of a given parity.

I want to first do the calculations using the cochain calculus, and then check it by hand.

Consider an even supertrace T on $\Omega^1 Q$, and the cochains $T^\natural(d\theta\,\partial\theta\,\alpha^n)$. Here $\theta = \rho + \alpha$, where I treat Q as an algebra and not superalgebra in defining the product on bar cochains. Then

$$0 = (\delta+\theta)^2 = \underbrace{((\delta+\rho)^2 + \alpha^2)}_{\delta\rho + \rho^2 + \alpha^2 = 0} + \underbrace{[\delta+\rho, \alpha]}_{0}$$

The even supertrace identity is $T(xy) = T(y\bar{x}) = T(\bar{y}x)$ so that we have

$$T^\natural(d\theta\,\partial\theta\,\alpha^n) = (-1)^{n+1} T^\natural(\partial\theta\,\alpha^n\,d\bar{\theta})$$

where the sign comes from because $d\theta$ is odd and $\partial\theta\,\alpha^n$ has parity $(-1)^{n+1}$.

One has

$$\delta\alpha + \rho\alpha + \alpha\rho = 0$$

$$\delta\alpha + (\rho \pm \alpha)\alpha + \alpha(\rho \mp \alpha) = 0$$

$$\delta\alpha + \theta\alpha + \alpha\bar{\theta} = 0$$

$$\delta\alpha + \bar{\theta}\alpha + \alpha\theta = 0$$

$$\delta(\alpha^2) = \delta\alpha\,\alpha \mp \alpha\,\delta\alpha$$

$$= (-\theta\alpha - \alpha\bar{\theta})\alpha + \alpha(\bar{\theta}\alpha + \alpha\theta)$$

$$= -\theta\alpha^2 + \alpha^2\theta$$

Also by applying $-$ in QA we get

$$\delta(\alpha^2) = -\bar{\theta}\alpha^2 + \alpha^2\bar{\theta}$$

Thus $$\boxed{[\delta + \theta, \alpha^{2n}] = 0 \qquad [\delta + \bar{\theta}, \alpha^{2n}] = 0}$$

and $$\delta(\alpha^{2n+1}) = \delta(\alpha^{2n})\alpha + \alpha^{2n}\delta\alpha$$

$$= (-\theta\alpha^{2n} + \alpha^{2n}\theta)\alpha + \alpha^{2n}(-\theta\alpha \mp \alpha\bar{\theta})$$

$$\boxed{\delta(\alpha^{2n+1}) = -\theta\alpha^{2n+1} - \alpha^{2n+1}\bar{\theta}}$$

$$\boxed{\delta(\alpha^{2n+1}) = -\bar{\theta}\alpha^{2n+1} - \alpha^{2n+1}\theta}$$

Recall that from $\delta\theta + \theta^2 = 0$ we obtain

$$[\delta + \theta, d\theta] = [\delta + \theta, \partial\theta] = 0$$

Now consider the cochains $T^{\natural}(d\theta\,\partial\theta\,\alpha^n)$ in the case of an even supertrace T. One has

$$\delta T^{\natural}(d\theta\,\partial\theta\,\alpha^{2n}) = T^{\natural}(\delta(d\theta\,\partial\theta\,\alpha^{2n}))$$

$$= +T^{\natural}\Big(-\underbrace{\theta(d\theta\,\partial\theta\,\alpha^{2n})}_{\bar\theta}+(d\theta\,\partial\theta\,\alpha^{2n})\theta\Big)$$

$$= T^{\natural}\big(d\theta\,\partial\theta\,\alpha^{2n}(\theta-\bar\theta)\big) \quad \text{so}$$

$$\boxed{\delta T^{\natural}(d\theta\,\partial\theta\,\alpha^{2n}) = 2T^{\natural}(d\theta\,\partial\theta\,\alpha^{2n+1})}$$

even case

In the case of an odd supertrace we have

$$\delta T^{\natural}(d\theta\,\partial\theta\,\alpha^{2n+1})$$

$$= T^{\natural}\Big(-\underbrace{\theta(d\theta\,\partial\theta\,\alpha^{2n+1})} - (d\theta\,\partial\theta\,\alpha^{2n+1})\bar\theta\Big)$$

$$= T^{\natural}\big(d\theta\,\partial\theta\,\alpha^{2n+1}(\theta-\bar\theta)\big)$$

$$\boxed{\delta T^{\natural}(d\theta\,\partial\theta\,\alpha^{2n+1}) = 2T^{\natural}(d\theta\,\partial\theta\,\alpha^{2n+2})}$$

odd case

Digress: Recall that for Hochschild cochains $f_n(a_0,\dots,a_n)$ we have

$$\boxed{\delta f_n = (-1)^n b f_n}$$

because f_n is multilinear of degree $n+1$. Let's define

$$\boxed{\sigma f_n = (-1)^{n-1} s f_n}$$

Then $\delta\sigma f_n = (-1)^{n-1}\delta s f_n = (-1)^{n-1}(-1)^{n-1} b s f_n$

and so

$$\boxed{\begin{aligned} \delta\sigma &= bs \\ \sigma\delta &= sb \end{aligned}}$$

Next consider a Hochschild n-cochain of the form

$$\tau^\natural(\partial\rho\,\eta_n)$$

where $\eta_n \in (\bar{A}^{\otimes n})^*$. Then

$$\tau^\natural(\partial\rho\,\eta)(a_0,\dots,a_n) = (-1)^n\,\tau\big(\rho(a_0)\,\eta(a_1,\dots,a_n)\big)$$

$$\sigma\big(\underbrace{\tau^\natural(\partial\rho\,\eta)}_{f_n}\big) = (-1)^{n-1}\,s\,\tau^\natural(\partial\rho\,\eta)$$

$$= (-1)^{n-1}(-1)^n\,\tau(\eta)$$

so

$$\boxed{\sigma\big(\tau^\natural(\partial\rho\,\eta)\big) = -\tau(\eta)}$$

Let's return to the even supertrace T on $\Omega' Q$. We propose to link Cuntz's cochains to ones more consistent with the superpicture.

Let
$$h_{2n+1} = T^\natural(\partial\theta\,\alpha^{2n}\,d\bar{\theta})$$
and note that

$$\delta h_{2n+1} = T^\natural\big(\delta(\partial\theta\,\alpha^{2n})\,d\bar{\theta} - (\partial\theta\,\alpha^{2n})\,d(-\bar{\theta}^2)\big)$$

$$= T^\natural\big(-\theta\,\partial\theta\,\alpha^{2n}\,d\bar{\theta} - \partial\theta\,\alpha^{2n}\,\theta\,d\bar{\theta} + \partial\theta\,\alpha^{2n}\,d\bar{\theta}\,\bar{\theta} + \partial\theta\,\alpha^{2n}\,\bar{\theta}\,d\bar{\theta}\big)$$

d of degree 0

$$= -2\,T^\natural(\partial\theta\,\alpha^{2n+1}\,d\bar{\theta})$$

$$\tfrac{1}{2}\,\sigma\delta h_{2n+1} = T^\natural(\partial\alpha\,\alpha^{2n}\,d\bar{\theta}) = -T^\natural(\partial\alpha\,\alpha^{2n}\,d\alpha)$$

$$\left(1-\tfrac{1}{2}\sigma\delta\right)h_{2n+1} = T^{\natural}(\partial\rho\,\alpha^{2n}\,d\bar\theta)$$
$$= T^{\natural}(\partial\rho\,\alpha^{2n}\,d\rho)$$

$$\sigma h_{2n+1} = -T^{\natural}(\partial\alpha\,\alpha^{2n-1}\,d\bar\theta)$$
$$= -T^{\natural}(\partial\alpha\,\alpha^{2n-1}\,d\rho)$$

$$-\tfrac{1}{2}\delta h_{2n+1} + \sigma h_{2n+1} = T^{\natural}(\partial\theta\,\alpha^{2n-1}\,d\bar\theta) - T^{\natural}(\partial\alpha\,\alpha^{2n-1}\,d\bar\theta)$$
$$= T^{\natural}(\partial\rho\,\alpha^{2n-1}\,d\bar\theta)$$
$$= -T^{\natural}(\partial\rho\,\alpha^{2n-1}\,d\alpha)$$

Formulas

$$\boxed{\begin{aligned}
T^{\natural}(\partial\rho\,\alpha^{2n}\,d\rho) &= \left(1-\tfrac{1}{2}\sigma\delta\right)h_{2n+1}\\
T^{\natural}(\partial\alpha\,\alpha^{2n-1}\,d\rho) &= -\sigma h_{2n+1}\\
T^{\natural}(\partial\rho\,\alpha^{2n-1}\,d\alpha) &= -\sigma h_{2n+1} + \tfrac{1}{2}\delta h_{2n-1}\\
T^{\natural}(\partial\alpha\,\alpha^{2n}\,d\alpha) &= -\tfrac{1}{2}\sigma\delta\,h_{2n+1}
\end{aligned}}$$

The basic cochains I need to describe things are

$T^{\natural}(\partial\rho\,\alpha^{2n}\,d\rho)$ $\quad n \geq 0$

$T^{\natural}(\partial\rho\,\alpha^{2n-1}\,d\alpha)$ $\quad n \geq 1$

and Cuntz needs simply $h_{2n+1} = T^{\natural}(\partial\theta\,\alpha^{2n}\,d\bar\theta)$ $\quad n \geq 0$

Next we compute the map

$$(\Omega^1 Q)^+_\natural \longrightarrow (QLQ)^+_\natural \simeq LQ^-$$

really we compute the effect on linear functionals. Let τ be an even supertrace on QLQ; we know τ is equivalent to the cochains

$$f_{2n+1} = \tau^\natural(L\partial\rho\,\alpha^{2n+1}) \qquad n \geq 0.$$

Let T be the induced trace on $\Omega^1 Q$.
We compute Cuntz's cochains (rather my modifications)

$$h_{2n+1} = T^\natural(\partial\theta\alpha^{2n}\,d\bar\theta) = \tau^\natural(\partial\theta\alpha^{2n}(-L\bar\theta + \bar\theta L))$$

$$= \tau^\natural(-L\theta\partial\theta\alpha^{2n} - L\,\partial\theta\alpha^{2n}\bar\theta)$$

Use $\quad \delta(\partial\theta\alpha^{2n}) + \underline{\theta(\partial\theta\alpha^{2n})} + (\partial\theta\alpha^{2n})\theta = 0$

$$h_{2n+1} = \tau^\natural(-L\{-\delta(\partial\theta\alpha^{2n}) - \partial\theta\alpha^{2n}\theta\} - L\partial\theta\alpha^{2n}\bar\theta)$$

$$= \delta\tau^\natural(L\partial\theta\alpha^{2n}) + L\partial\theta\alpha^{2n}(\theta - \bar\theta))$$

$$= \delta\underbrace{\tau^\natural(L\partial\rho\alpha^{2n})} + 2\tau^\natural(L\partial\theta\alpha^{2n+1})$$

$$\tau(L\alpha^{2n+1}) = -\sigma\tau^\natural(L\partial\rho\,\alpha^{2n+1})$$

$$h_{2n+1} = (2 - \delta\sigma)\tau^\natural(L\partial\rho\,\alpha^{2n+1})$$

$$\therefore \boxed{h_{2n+1} = 2\left(1 - \tfrac{1}{2}bs\right)f_{2n+1}}$$

Next let's compute the map

$$\Omega^1 R_\natural \longrightarrow (\Omega^1 Q)^+_\natural$$

Given T ~~an~~ even supertrace on $\Omega^1 Q$ we want to find the cochains $T^\natural(-\partial\rho\,\omega^n d\rho)$.

Set

$$g_{2n+1} = T^\natural(\partial\rho\,\alpha^{2n} d\rho)$$

Then we have seen that

$$g_{2n+1} = \left(1 - \tfrac{1}{2} s b\right) h_{2n+1}$$

Now if ~~T came from~~ τ on QLQ°, then

$$\boxed{h_{2n+1} = 2\left(1 - \tfrac{1}{2} bs\right) f_{2n+1}}$$

and we have

$$2\left(1-\tfrac{1}{2}sb\right)\left(1-\tfrac{1}{2}bs\right) = 2\left(1-\tfrac{1}{2}(sb+bs)\right)$$

$$= 2-(1-K) = 1+K.$$

Thus we have

$$T^\natural(\partial\rho\,\alpha^{2n} d\rho) = (1+K)\,\tau^\natural(L\partial\rho\,\alpha^{2n+1})$$

which agrees with our earlier formulas, since the sign deviation is the same

$$T^\natural(\partial\rho\,\alpha^{2n} d\rho)(a_0, \dots, a_{2n+1})$$

$$= (-1)^{\frac{(2n+2)(2n+1)}{2}} \quad T(a_0^+ a_1^- \cdots a_{2n}^- \, da_{2n+1}^+)$$

$$= (-1)^{n+1} \; T^\natural(-\rho\,\omega^n d\rho)(a_0, \dots, a_{2n+1})$$

$$\tau^\natural(L\partial\rho\,\alpha^{2n+1})(a_0, \dots, a_{2n+1})$$

$$= (-1)^{n+1} \; \tau(L a_0^+ a_1^- \cdots a_{2n+1}^-)$$

$$= (-1)^{n+1} \; \tau^\natural(L\partial\rho\, g^{2n+1})\,(a_0, \dots, a_{2n})$$

Next consider an $\boxed{\text{odd supertrace}}$ T on $\Omega^1 Q$, and form the cochains

$$h_{2n} = T^\natural(d\theta\,\partial\theta\,\alpha^{2n-1}) = T^\natural(\partial\theta\,\alpha^{2n-1}\,d\theta)$$

We have

$$\delta h_{2n} = T^\natural\Big(\delta(\partial\theta\,\alpha^{2n-1})\,d\theta + (\partial\theta\,\alpha^{2n-1})\,d(-\theta^2)\Big)$$

$$= T^\natural\Big(\big(-\theta(\partial\theta\,\alpha^{2n-1}) + (\partial\theta\,\alpha^{2n-1})\bar\theta\big)\,d\theta - \partial\theta\,\alpha^{2n-1}(d\theta\,\theta + \theta\,d\theta)\Big)$$

$$= T^\natural\Big\{-\theta\,\partial\theta\,\alpha^{2n-1}\,d\theta\,(-\theta) + \partial\theta\,\alpha^{2n-1}\bar\theta\,d\theta - \partial\theta\,\alpha^{2n-1}\,d\theta\,\theta - \partial\theta\,\alpha^{2n-1}\,\theta\,d\theta\Big\}$$

$$= -2\,T^\natural(\partial\theta\,\alpha^{2n}\,d\theta)$$

Check: $\delta\,\tau^\natural(d\theta\,\partial\theta\,\alpha^{2n-1})$

$$= \tau^\natural\Big(\delta(d\theta\,\partial\theta\,\alpha^{2n-2})\,\alpha + (d\theta\,\partial\theta\,\alpha^{2n-2})(-\theta\alpha - \alpha\bar\theta)\Big)$$

$$= \tau^\natural\Big(-\theta\,d\theta\,\partial\theta\,\alpha^{2n-2}\alpha + \cancel{(d\theta\,\partial\theta\,\alpha^{2n-2})\theta\,\alpha} - \cancel{d\theta\,\partial\theta\,\alpha^{2n-2}\theta\,\alpha} - d\theta\,\partial\theta\,\alpha^{2n-1}\bar\theta\Big)$$

$$= \tau^\natural\Big(+d\theta\,\partial\theta\,\alpha^{2n-1}\theta - d\theta\,\partial\theta\,\alpha^{2n-1}\bar\theta\Big)$$

$$= 2\tau^\natural(d\theta\,\partial\theta\,\alpha^{2n}) = -2\tau^\natural(\partial\theta\,\alpha^{2n}\,d\theta)$$

$$h_{2n} = T^\natural(\partial\theta\,\alpha^{2n-1}\,d\theta)$$

$$-\sigma h_{2n} = T(\alpha^{2n-1}\,d\theta) = T^\natural(\partial\alpha\,\alpha^{2n-2}\,d\theta)$$

$$-\sigma h_{2n} = T^\natural(\partial\alpha\,\alpha^{2n-2}\,d\rho)$$

$$-\tfrac{1}{2}\delta h_{2n} = T^\natural(\partial\theta\,\alpha^{2n}\,d\theta)$$

$$\tfrac{1}{2}\sigma\delta h_{2n} = T(\alpha^{2n}\,d\theta) = T^\natural(\partial\alpha\,\alpha^{2n-1}\,d\theta)$$

$$\tfrac{1}{2}\sigma\delta h_{2n} = T^4(\partial\alpha\,\alpha^{2n-1}d\alpha)$$

$$\left(1-\tfrac{1}{2}\sigma\delta\right)h_{2n} = T^4(\partial\rho\,\alpha^{2n-1}d\theta) = T^4(\partial\rho\,\alpha^{2n-1}d\rho)$$

$$-\sigma h_{2n+2} = T^4(\partial\alpha\,\alpha^{2n}d\theta)$$

$$-\tfrac{1}{2}\delta h_{2n} + \sigma h_{2n+2} = T^4(\partial\rho\,\alpha^{2n}d\theta) = T^4(\partial\rho\,\alpha^{2n}d\alpha)$$

$$\boxed{\begin{aligned} T^4(\partial\rho\,\alpha^{2n-1}d\rho) &= \left(1-\tfrac{1}{2}\sigma\delta\right)h_{2n} \\ T^4(\partial\alpha\,\alpha^{2n-2}d\rho) &= -\sigma h_{2n} \\ T^4(\partial\rho\,\alpha^{2n}d\alpha) &= \sigma h_{2n+2} - \tfrac{1}{2}\delta h_{2n} \\ T^4(\partial\alpha\,\alpha^{2n-1}d\alpha) &= \tfrac{1}{2}\sigma\delta h_{2n} \end{aligned}}$$

better to write as bar cochains

Next we compute $(\Omega' Q)^-_4 \longrightarrow (QLQ)^-_4 = LQ^+$

$$\begin{aligned} h_{2n} &= T^4(\partial\theta\,\alpha^{2n-1}d\theta) \\ &= \tau^4\left(\partial\theta\,\alpha^{2n-1}(L\theta-\theta L)\right) \\ &= \tau^4\Big(\underbrace{L\theta\,\partial\theta\,\alpha^{2n-1}}_{-\delta(\partial\theta\alpha^{2n-1}) + (\partial\theta\alpha^{2n-1})\bar\theta} - L\,\partial\theta\,\alpha^{2n-1}\theta\Big) \\ &= -\delta\,\tau^4(L\partial\theta\,\alpha^{2n-1}) + \tau^4\left(L\partial\theta\,\alpha^{2n-1}(\bar\theta-\theta)\right) \\ &= -\delta\,\tau^4(L\,\partial\alpha\,\alpha^{2n-1}) - 2\tau^4(L\,\partial\theta\,\alpha^{2n}) \\ &= -\delta\,\tau^4(L\,\alpha^{2n}) - 2\tau^4(L\,\partial\rho\,\alpha^{2n}) \\ &= \delta\sigma\,\tau^4(L\,\partial\rho\,\alpha^{2n}) - 2\tau^4(L\,\partial\rho\,\alpha^{2n}) \\ &= (\delta\sigma-2)\,\tau^4(L\,\partial\rho\,\alpha^{2n}) \end{aligned}$$

Thus we have

$$\boxed{h_{2n} = -2\left(1-\tfrac{1}{2}bs\right) f_{2n}}$$

better

$$T^\natural(\partial\theta\,\alpha^{2n-1}d\theta) = -2\left(1-\tfrac{1}{2}bs\right)\tau^\natural(L\partial_\rho\,\alpha^{2n})$$

which is the same formula up to sign as in the even case.

There is perhaps a problem at the lowest degrees which should be checked. Also we need to understand $1-\frac{1}{2}bs$.

Direct calculation: Even case

Put

$$h'_{2n+1}(a_0,\dots,a_{2n+1}) = T(a_0 a_1^- \cdots a_{2n}^-\, da^F_{2n+1})$$

so that

$$h'_{2n+1} = \underbrace{(-1)^{\frac{(2n+2)(2n+1)}{2}}}\, T^\natural(\partial\theta\,\alpha^{2n}\,d\bar\theta)$$

$$= (-1)^{n+1} h_{2n+1}$$

Similarly let

$$f'_{2n+1}(a_0,\dots,a_{2n+1}) = \tau(L\,a_0^+ a_1^- \cdots a_{2n+1}^-)$$

so that

$$f'_{2n+1} = (-1)^{n+1}\tau^\natural(L\partial_\rho\,\alpha^{2n+1}) = (-1)^{n+1} f_{2n+1}$$

Then

$$h'_{2n+1}(a_0,\dots,a_{2n+1}) = \tau(a_0a_1^-\cdots a_{2n}^-(-La^F_{2n+1} + a^F_{2n+1}L))$$

$$= \tau(La_{2n+1}a_0a_1^-\cdots a_{2n}^- - La_0a_1^-\cdots a_{2n}^-a^F_{2n+1})$$

On the other hand we have

$$bsf'_{2n+1}(a_0,a_1,\dots,a_{2n+1}) =$$

$$\left.\begin{array}{l} f'_{2n+1}(1, a_0a_1, \dots, a_{2n+1}) \\ -f'_{2n+1}(1, a_0, a_1a_2, \dots\quad) \\ \\ +f'_{2n+1}(1, a_0 \qquad, a_{2n}a_{2n+1}) \\ -f'_{2n+1}(1, a_{2n+1}a_0, \dots, a_{2n}) \end{array}\right\} = \left\{\begin{array}{l} \tau(L(a_0a_1)^{-} a_2^{-} \dots a_{2n+1}^{-}) \\ -\tau(L a_0^{-}(a_1a_2)^{-} \dots \quad) \\ \\ \tau(L a_0^{-} \dots (a_{2n}a_{2n+1})^{-}) \\ -\tau(L(a_{2n+1}a_0)^{-} a_1^{-} \dots a_{2n}^{-}) \end{array}\right.$$

$$= \tau\{ L a_0^{+} a_1^{-} \dots a_{2n+1}^{-} + L a_0^{-} \dots a_{2n}^{-} a_{2n+1}^{+}$$
$$- L(a_{2n+1}a_0)^{-} a_1^{-} \dots a_{2n}^{-} \quad \}$$

can be removed as τ is even

Thus $\left(bs\, f'_{2n+1} + h'_{2n+1}\right)(a_0, \dots, a_{2n})$

$$= \tau\left\{ L a_0^{+} a_1^{-} \dots a_{2n-1}^{-} + L a_0^{-} \dots a_{2n}^{-} a_{2n+1}^{+} - L a_0 a_1^{-} \dots a_{2n}^{-} a_{2n+1}^{F} \right)$$

$$= \tau\{ L a_0^{+} a_1^{-} \dots a_{2n+1}^{-} + \cancel{L a_0^{-} \dots a_{2n}^{-} a_{2n+1}^{+}}$$
$$- L a_0^{+} a_1^{-} \dots a_{2n}^{-}(-a_{2n+1}^{-}) - \cancel{L a_0^{-} a_1^{-} \dots a_{2n}^{-} a_{2n+1}^{+}} \}$$

$$= 2\tau(L a_0^{+} a_1^{-} \dots a_{2n}^{-}) = 2 f'_{2n+1}.$$ Thus we have

$$\boxed{h'_{2n+1} = (2 - bs) f'_{2n+1} = 2(1 - \tfrac{1}{2} bs) f'_{2n+1}}$$

which checks.

Next suppose τ is odd and put

$$\underbrace{h'_{2n}}_{(a_0,\dots,a_{2n})} = \tau(a_0 a_1^{-} \dots a_{2n-1}^{-}\, d a_{2n})$$

so that $\quad h'_{2n} = (-1)^n\, \tau(\partial\theta\, \alpha^{2n-1} d\theta)$. Then

$$h'_{2n}(a_0, \dots, a_{2n}) = \tau(a_0 a_1^{-} \dots a_{2n-1}^{-}(L a_{2n} - a_{2n} L))$$

$$= \tau\left(L a_{2n} a_0 a_1^- \cdots a_{2n-1}^- - L a_0 a_1^- \cdots a_{2n-1}^- a_{2n} \right)$$

$$bs f'_{2n}(a_0, \ldots, a_{2n}) =$$

$$\left.\begin{aligned} & f'_{2n}(1, a_0a_1, \ldots, a_{2n}) \\ & -f'_{2n}(1, a_0, a_1a_2, \ldots) \\ & \quad \vdots \\ & -f'_{2n}(1, a_0, \ldots, a_{2n-1}a_{2n}) \\ & +f'_{2n}(1, a_{2n}a_0, a_1, \ldots, a_{2n-1}) \end{aligned}\right\} = \left\{\begin{aligned} & \tau(L(a_0a_1)^+ a_2^- \cdots a_{2n}^-) \\ & -\tau(L a_0^- (a_1a_2)^- \cdots) \\ & \quad \vdots \\ & -\tau(L a_0^- \cdots (a_{2n-1}a_{2n})^-) \\ & +\tau(L(a_{2n}a_0)^- a_1^- \cdots a_{2n-1}^-) \end{aligned}\right.$$

$$= \tau\left\{ L a_0^+ a_1^- \cdots a_{2n}^- - L a_0^- \cdots a_{2n-1}^- a_{2n}^+ + L(a_{2n}a_0)^- a_1^- \cdots a_{2n-1}^- \right\}$$

can be removed as τ is odd

$$\left(bs f'_{2n} - h'_{2n}\right)(a_0, \ldots, a_{2n})$$

$$= \tau\left\{ L a_0^+ a_1^- \cdots a_{2n}^- - \cancel{L a_0^- \cdots a_{2n-1}^- a_{2n}^+} + \underbrace{L a_0 a_1^- \cdots a_{2n-1}^- a_{2n}}_{L a_0^+ a_1^- \cdots a_{2n-1}^- a_{2n}^- + \cancel{L a_0^- a_1^- \cdots a_{2n-1}^- a_{2n}^+}} \right\}$$

$$= 2\tau(L a_0^+ a_1^- \cdots a_{2n}^-) = 2 f'_{2n}(a_0, \ldots, a_{2n})$$

Thus

$$\boxed{h'_{2n} = (bs-2) f'_{2n} = -2\left(1 - \tfrac{1}{2} bs\right) f'_{2n}}$$

which checks with our previous calculation

December 3, 1989

Let's begin by understanding the operator $1-\frac{1}{2}bs$ on normalized Hochschild cochains of degree n. Recall the notation

$$0 \longrightarrow \bar{A}^{\otimes n} \xrightarrow{\ \iota\ } A\otimes \bar{A}^{\otimes n} \xrightarrow{\ j\ } \bar{A}^{\otimes n+1} \longrightarrow 0$$

where $s = \iota j$. Dually we have $s = j^*\iota^*$

$$\begin{tikzcd}
0 \arrow[r] & (\bar{A}^{\otimes n+1})^* \arrow[r, "j^*"] \arrow[d, "1"] & (A\otimes\bar{A}^{\otimes n})^* \arrow[r, "\iota^*"] \arrow[d, "1-\frac{1}{2}bs"] \arrow[dd, bend left, "\frac{1+K}{2}"] & (\bar{A}^{\otimes n})^* \arrow[r] \arrow[d, "\frac{1+\lambda}{2}"] & 0 \\
0 \arrow[r] & (\bar{A}^{\otimes n+1})^* \arrow[r, "j^*"] \arrow[d, "\frac{1+\lambda}{2}"] & (A\otimes\bar{A}^{\otimes n})^* \arrow[r, "\iota^*"] \arrow[d, "1-\frac{1}{2}sb"] & (\bar{A}^{\otimes n})^* \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & (\bar{A}^{\otimes n+1})^* \arrow[r, "j^*"] & (A\otimes\bar{A}^{\otimes n})^* \arrow[r, "\iota^*"] & (\bar{A}^{\otimes n})^* \arrow[r] & 0
\end{tikzcd}$$

because $\quad s(1-\frac{1}{2}bs) = s(1-\frac{1}{2}(1-K-sb)) = s\left(\frac{1+K}{2}\right)$

so ($j^*\iota^* = s$ with j^* injective) gives

$$\iota^*(1-\tfrac{1}{2}bs) = \iota^*\left(\frac{1+K}{2}\right) = \left(\frac{1+\lambda}{2}\right)\iota^*$$

Now $\frac{1+\lambda}{2}$ on $(\bar{A}^{\otimes n})^*$ is an isomorphism for n odd. Conclude that $1-\frac{1}{2}bs$ is an isomorphism on odd normalized Hochschild coch. and $1-\frac{1}{2}sb$ is an isomorphism on even normalized Hochains.

$1-\frac{1}{2}bs$	bijective on	$(A\otimes\bar{A}^{\otimes 2n+1})^*$
$1-\frac{1}{2}sb$	——————	$(A\otimes\bar{A}^{\otimes 2n})^*$

This should imply (subject to checking low degrees) that the map

$$(\Omega^1 Q)^+_\natural \longrightarrow (QLQ)^+_\natural = LQ^-$$

is an <u>isomorphism</u>, since the formula for the cochains of supertraces is

$$h_{2n+1} = -2\left(1 - \tfrac{1}{2} bs\right) f_{2n+1}.$$

Next we want to discuss the odd case.

First we go over the low degrees carefully in the odd case. Given an odd supertrace T on $\Omega^1 Q$, we can associate to T the cochain

$$T^\natural(\partial\theta\, d\theta)$$

which is a Hochschild 1-cocycle. When T comes from τ on QLQ, we have

$$\begin{aligned}
T^\natural(\partial\theta\, d\theta) &= \tau^\natural(\partial\theta(L\theta - \theta L)) \\
&= \tau^\natural(-L\theta\partial\theta - L\partial\theta\theta) = \tau^\natural(L\partial(-\theta^2)) \\
&= \delta\tau^\natural(L\partial\theta) = \delta\tau^\natural(L\partial\rho) = \delta f_0.
\end{aligned}$$

Thus the class of $T^\natural(\partial\theta d\theta)$ in $H^1(A, A^*)$ is an obstruction to T coming from a τ.

This discussion dually amounts to a commutative square

$$\begin{tikzcd}
a_0 da_1 \arrow[rr, mapsto] \arrow[d, mapsto] & & (a_0 da_1)^- \arrow[d, mapsto] \\
-[a_0, a_1] & & (a_0 L a_1 - a_0 a_1 L)^- \equiv -L[a_0,a_1]^+
\end{tikzcd}$$

$$\begin{tikzcd}
(\Omega^1 A)_\natural \arrow[r] \arrow[d] & (\Omega^1 Q)^-_\natural \arrow[d] \\
A \arrow[r] & (QLQ)^-_\natural = LQ^+
\end{tikzcd}$$

$$a \longmapsto La^+,$$

Let's try to [redacted] split $(\Omega^1 A)_\natural$ off $(\Omega^1 Q)^-_\natural$. In the following we use that odd supertraces are the same as odd traces. We have

$$\Omega^1 Q = \left(Q \otimes_{\iota A} \Omega^1 \iota A \otimes_{\iota A} Q\right) \oplus \left(Q \otimes_{\bar\iota A} \Omega^1 \bar\iota A \otimes_{\bar\iota A} Q\right)$$

Consider the folding homomorphism $Q \to A$ and make the bimodule induced by this homom.

$$A \otimes_Q \Omega^1 Q \otimes_Q A = \underbrace{\Omega^1 A}_{\substack{\text{gen. by} \\ d\iota a}} \oplus \underset{\substack{\text{gen by} \\ d\bar\iota a}}{\Omega^1 A}$$

In other words given an A-bimodule M, there is an equivalence between derivations $Q \to M$ and pairs of derivations $D, \bar D : A \to M$. Thus we have [redacted] a map of Q-bimodules compatible with the folding hom.

(*)

$$\Omega^1 Q \longrightarrow \Omega^1 A$$

$$\begin{array}{lll} \iota a \mapsto a & d(\iota a) \mapsto da & da^+ \mapsto 0 \\ \bar\iota a \mapsto a & d(\bar\iota a) \mapsto -da & da^- \mapsto da \\ & & a^+ \mapsto a \\ & & a^- \mapsto 0 \end{array}$$

and the composition

$$\Omega^1 A \longrightarrow \Omega^1 Q \longrightarrow \Omega^1 A$$

$$a_0 da_1 \longmapsto \iota a_0 d\iota a_1 \longmapsto a_0 da_1$$

is the identity. This means that $(\Omega^1 A)_\natural$ is a direct summand [redacted] of the usual (not superalg) commutator quotient of $\Omega^1 Q$. On the other hand the map intertwines the involution $^-$ on $\Omega^1 Q$ with -1 on $\Omega^1 A$, so we [redacted] see $(\Omega^1 A)_\natural$ is a direct summand of $(\Omega^1 Q)^-_\natural$.

Next take an odd supertrace T on $\Omega^1 Q$ coming from a trace T_0 on $\Omega^1 A$; thus

$$T^\natural(\partial\theta d\theta) = T_0^\natural(\partial\theta d\theta)$$

$$T^\natural(\partial\theta\alpha^n\partial\theta) = 0 \qquad n \geq 1.$$

and let us compute the cocycle of the odd trace Td on Q. Then

$$T^\natural(d(\partial\rho\,\alpha^{2n+1})) = 0 \qquad \text{for } n > 0$$

since there is at least one α present after applying d.

Also $$T^\natural(d(\partial\rho\,\alpha)) = T^\natural(\partial\rho\, d\alpha)$$

$$= T_0^\natural(\partial\theta\, d\theta) \qquad \begin{matrix} a^+ \mapsto a \\ da^- \mapsto da \end{matrix}$$

Thus the cocycle consists of the single Hochschild 1-cocycle $T_0^\natural(\partial\theta d\theta)$.

Interesting point: When we study odd supertraces on Q, the associated cocycle

$$f_1 : \tau(a_0^+ a_1^-),\ f_3 : \tau(a_0^+ a_1^- a_2^- a_3^-),\ \text{etc.}$$

does <u>not</u> satisfy the bottom condition $Bf_1 = 0$.

An odd supertrace on Q and one on J are the same.

December 4, 1989

We continues with the map

$$(\Omega' Q)^-_{\natural} \longrightarrow (QLQ)^-_{\natural} = LQ^+$$

which we have seen is given by

h[redacted]$_{2n} = (-2)(1-\frac{1}{2}bs) f_{2n} \qquad f_{2n} = \tau' \natural (L \partial \rho \alpha^{2n})$

on cochains, at least for $n \geq 1$.

Suppose that h[redacted] $= 0$; then the induced odd trace $\tau = \tau' D = Td$ on Q is zero, hence my homotopy formula should give a cocycle. Let's check this.

$$\left(1-\frac{1}{2}bs\right) f_{2n} = 0 \implies b f_{2n} = 0$$

$$\kappa f_{2n} = (1 - bs - sb) f_{2n} = f_{2n} - 2 f_{2n} = -f_{2n}$$

The coboundary cochain is given by

$$f'_{2n} = \left(\sum_{i=0}^{2n} \kappa^i\right) f_{2n} = \left(\sum_{i=0}^{2n} (-1)^i\right) f_{2n} = f_{2n}$$

We can suppose all $f_{2i} \neq 0$ for $i \neq n$, so that f_{2n} by itself should be a b, B cocycle. We have $b f_{2n} = 0$ and

$$B f_{2n} = \sum_{i=0}^{2n-1} \kappa^i s f_{2n} = s \sum_{i=0}^{2n-1} (-1)^i f_{2n} = 0$$

so it checks.

Thus we can conclude in the odd case that Cuntz's homotopy formula is different from mine.

Let's next consider odd traces on $\mathcal{I}'Q$ in low degrees. To such a T we can associated

$$T^\natural(\partial\theta\, d\theta)$$

which is a Hochschild 1-cocycle. If T comes from an odd trace τ' on QLQ we, have

$$T^\natural(\partial\theta\, d\theta) = \tau'^\natural(\partial\theta(L\theta - \theta L))$$

$$= \tau'^\natural(-L(\partial\theta\theta + \partial\theta\theta)) = \delta\tau'^\natural(L\theta)$$

so this Hochschild 1-cocycle is a coboundary.

Next consider the odd trace $\tau = Td$ on Q, and let g be its cocycle. $g: (g_1, g_2, \cdots)$ satisfies $bg_1 = \frac{2}{3}Bg_3$ etc + K-invariance but not $Bg_1 = 0$. So we have to consider g a cocycle in a truncated complex. Put another way $Q^- = J^-$ so that there is no distinction between ~~odd~~ traces on Q and on J.

The cohomology class of g is the cyclic cohomology class of the cyclic 1-cocycle

$$\sigma g_1 = \sigma T^\natural(d(\partial\rho\alpha)) = T(d\alpha) = T(d\theta)$$

and this can be a non-trivial trace since any Hoch 1-cocycle occurs as $T^\natural(\partial\theta\, d\theta)$.

To summarize we have

$$\begin{array}{ccc} H^1(A, A^*) & \xrightarrow{B} & HC^0(A) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [T^\natural(\partial\theta d\theta)] & \longmapsto & T(d\theta) = \text{class of } Td \text{ on } Q \end{array}$$

obstruction to T coming from QLQ

Here's an argument to suggest Cuntz's homotopy formula should be OK. Let's start with an odd trace T on $F_J^{2m}(\Omega^1 Q)$, whence we have the odd trace Td defined on J^{2m+1}. Let's compute the cyclic classes in $HC^{2n}(A)$ n large and see that they are trivial.

The cocycle of Td is made of the cochains

$$g_{2n+1} = T\,d(\partial\rho\ \alpha^{2n+1}) \qquad n \geq m$$

The cyclic $2n$ cocycle associated to Td is

$$Bg_{2n+1} = Nsg_{2n+1} = -NT(d(\alpha^{2n+1})). \text{ Now}$$

$$NT(d(\alpha^{2n+1})) = NT\Big(\sum_0^{2n} \alpha^i \, d\alpha \, \alpha^{2n-i}\Big)$$

$$= (2n+1)\, NT(\alpha^{2n} d\alpha)$$

But from Dec 2. formulas we have

$$T(\alpha^{2n} d\alpha) = \frac{1}{2}\sigma\delta h_{2n} = \frac{1}{2} sb\, h_{2n}$$

so

$$NT(\alpha^{2n} d\alpha) = \frac{1}{2} Nsb\, h_{2n} = \frac{1}{2} Bb\, h_{2n}$$

$$= -\frac{1}{2} bBh_{2n}$$

But this is the coboundary of the cyclic $(2n-1)$ cochain $-\frac{1}{2}Bh_{2n}$.

December 5, 1989

Let T be an even supertrace on $\Omega^1 Q$. We have

$$T(d(\rho\alpha^{2n})) = T(d\rho\,\alpha^{2n}) + \sum_0^{2n-1} T(\rho\alpha^i d\alpha\,\alpha^{2n-1-i})$$

Up to the fixed sign $(-1)^{(2n+1)2n/2} = (-1)^n$ the i-th cochain in the sum is

$$T(a_0^+ a_1^- \cdots a_i^- \, da_{i+1}^- \, a_{i+2}^- \cdots a_{2n}^-)$$

$$= (-1)^{i+1} T(a_{i+2}^- \cdots a_{2n}^- \, a_0^+ a_1^- \cdots a_i^- \, da_{i+1}^-)$$

It should be possible to identify this cochain with κ^{2n-1-i} applied to the cochain

$$T(a_0^+ a_1^- \cdots a_{2n-1}^- \, da_{2n}^-).$$

Assuming this we have

$$T(\rho\alpha^i d\alpha\,\alpha^{2n-1-i}) = \kappa^{2n-1-i}\, T(\rho\alpha^{2n-1} d\alpha)$$

for $i = 0, \ldots, 2n-1$. Thus we have

$$T(d(\rho\alpha^{2n})) = \underbrace{T(d\rho\,\alpha^{2n})} + \Big(\sum_0^{2n-1}\kappa^j\Big)\underbrace{T(\rho\alpha^{2n-1}d\alpha)}_{-\sigma h_{2n+1} + \frac12 \delta h_{2n-1}}$$

$$\lambda^{-1} T(\alpha^{2n} d\rho) = \lambda^{-1}(-\sigma h_{2n+1}) = \lambda^{2n}(-\sigma h_{2n+1})$$

$$T(d(\rho\alpha^{2n})) = \Big(\sum_0^{2n-1}\kappa^j\Big)\tfrac12 \delta h_{2n-1} + \sum_0^{2n}\lambda^j(-\sigma h_{2n+1})$$

better maybe to have κ

$$= \delta\Big(\frac12\sum_0^{2n-1}\kappa^j\Big)h_{2n-1} \pm B h_{2n+1}$$

But $\quad B\left(\frac{1}{2}\sum_0^{2n+1} K^j h_{2n+1}\right) = \frac{2n+1}{2} B h_{2n+1}$

whence the cocycle of Td on Q is the coboundary of the cochain

$$\left(\frac{1}{2}\sum_0^{2n-1} K^j\right) h_{2n-1}$$

We therefore seem to have a homotopy formula different from Cuntz's.

Let's compare with the homotopy formula we have already proved when T comes from T' on QLQ. In this case

$$h_{2n+1} = 2\left(1-\tfrac{1}{2}bs\right) f_{2n+1}$$

so the cobounding cochain is

$$\left(\sum_0^{2n+1} K^j\right)\left(1-\tfrac{1}{2}bs\right) f_{2n+1}$$

$$= \left(\sum_0^{2n+1} K^j\right) f_{2n+1} - \overbrace{\tfrac{1}{2} bs \left(\sum_0^{2n+1} K^j\right)}^{bs + bB} f_{2n+1}$$

because $b(1+N)s$

Now one has $\quad b(bsf') = 0 \,, \quad B(bsf) = -bBsf = 0$

so any cochain in the form bsf is a b,B cocycle. Thus the cobounding cochains differs by a cochain killed by both b and B.

(Better: The cobounding cochains differ by a cochain in the form bsf' which is the b,B coboundary of sf'.)

December 6, 1989

Consider an <u>inner derivation</u> on Q [redacted] $Dx = Lx - xL$ where L is [redacted] even. The universal case is $Dx = 1\otimes x - x\otimes 1 \in Q\otimes Q$, where $L = 1\otimes 1$. We want to compute the induced map $(\Omega^1 Q)_\natural \longrightarrow (Q\otimes Q)_\natural = Q$.

We describe odd supertraces T on $\Omega^1 Q$ via the cochains $h_{2n} = T^\natural(\partial\theta\, \alpha^{2n-1} d\theta)$ and odd supertraces τ on $Q\otimes Q$ by the cochains $f_{2n-1} = \tau^\natural(L\partial\rho$ [redacted] $\alpha^{2n-1})$. Then where T is τD we have

$$h_{2n} = T^\natural(\partial\theta\,\alpha^{2n-1}d\theta) = \tau^\natural\big(\partial\theta\,\alpha^{2n-1}(L\theta - \theta L)\big)$$
$$= \tau^\natural(\,L\theta\partial\theta\,\alpha^{2n-1} - L\,\partial\theta\,\alpha^{2n-1}\theta)$$

But $\quad \delta(\partial\theta\,\alpha^{2n-1}) + \theta\partial\theta\,\alpha^{2n-1} - \partial\theta\,\alpha^{2n-1}\bar\theta = 0$

so
$$h_{2n} = \tau^\natural\big(L(-\delta(\partial\theta\,\alpha^{2n-1})) + L\partial\theta\,\alpha^{2n-1}\bar\theta - L\partial\theta\,\alpha^{2n-1}\theta\big)$$
$$= -\delta\,\tau^\natural(L\partial\theta\,\alpha^{2n-1}) - 2\,\tau^\natural(L\partial\theta\,\alpha^{2n})$$
$$= -\delta\,\tau^\natural(L\partial\rho\,\alpha^{2n-1}) - 2\underbrace{\tau^\natural(L\partial\alpha\,\alpha^{2n})}_{\tau^\natural(L\alpha^{2n+1}) = \sigma\tau^\natural(L\partial\rho\,\alpha^{2n+1})}$$

$$\therefore \quad \boxed{h_{2n} = -\delta f_{2n-1} + 2\sigma f_{2n+1}}$$

Next consider an even supertrace T on $\Omega^1 Q$ obtained from an even supertrace τ on $Q\otimes Q$. We have
$$h_{2n+1} = T^\natural(\partial\theta\,\alpha^{2n}d\bar\theta) = \tau^\natural\big(\partial\theta\,\alpha^{2n}(L\bar\theta - \bar\theta L)\big)$$

$$= \tau^4\left(-L\theta\,\partial\theta\alpha^{2n} - L\partial\theta\alpha^{2n}\bar{\theta}\right)$$

$$\delta(\partial\theta\alpha^{2n}) + \theta\partial\theta\alpha^{2n} + \partial\theta\alpha^{2n}\theta = 0$$

$$\therefore\ h_{2n+1} = \tau^4\left(L(\delta(\partial\theta\alpha^{2n})) + L\partial\theta\alpha^{2n}\theta - L\partial\theta\alpha^{2n}\bar{\theta}\right)$$

$$= \delta\tau^4(L\partial\theta\alpha^{2n}) + 2\tau^4(L\partial\theta\alpha^{2n+1})$$

$$= \delta\tau^4(L\partial\rho\alpha^{2n}) + \underbrace{2\tau(L\alpha^{2n+2})}_{=-2\sigma\tau^4(L\partial\rho\alpha^{2n+2})}$$

$h_{2n+1} = \delta f_{2n}$ [redacted] $-2\sigma f_{2n+2}$

December 7, 1989

Comparison of my + Cuntz's homotopy formula.

Cuntz's P operation on normalized Hochschild cochains h_n is defined as follows. Let's first consider the nonunital case where $h_n = (\psi_{n+1}, \varphi_n)$ and $bh_n = (b\psi_{n+1}, (1-\lambda)\psi_{n+1} - b'\varphi_n)$. He does the simplest thing to h_n so as to make its coboundary $(b + \text{const } B)h_n$ κ-invariant. He takes $sbh_n = ((1-\lambda)\psi_{n+1} - b'\varphi_n, 0)$ and splits it into components relative to the direct sum decomposition $\operatorname{Im} N \oplus \operatorname{Im}(1-\lambda)$, and then lifts the latter component back via $1-\lambda$. Thus

$$P(\psi_{n+1}, \varphi_n) = \left(\psi_{n+1} - W_{n+1}(\lambda)\big((1-\lambda)\psi_{n+1} - b'\varphi_n\big), \varphi_n\right)$$

$$= \left(\frac{1}{n+1} N\psi_{n+1} + W_{n+1}(\lambda) b'\varphi_n, \varphi_n\right)$$

where $W_n(\lambda) = \left(1 - \frac{1}{n+1}N\right)\frac{1}{1-\lambda}\left(1 - \frac{1}{n+1}N\right)$ is the inverse of $1-\lambda$ on $\operatorname{Im}(1-\lambda)$. Specifically

$$W_{n+1}(\lambda) = \left(1 - \frac{1}{n+1}N\right)\frac{1}{1-\lambda}\left(\sum_{i=0}^{n}(1-\lambda^i)\right)/n+1$$

$$= \left(1 - \frac{1}{n+1}N\right)\frac{1}{n+1}\sum_{i=0}^{n}(1 + \cdots + \lambda^{i-1})$$

(note that $1 - \frac{1}{n+1}N$ is the projector onto $\operatorname{Im}(1-\lambda)$ with kernel $\operatorname{Im} N$).

We can write P in terms of κ as

$$\boxed{Ph_n = h_n - W_{n+1}(\kappa)\, sbh_n}$$

At this point because I want to apply W_{n+1} to κ which is not of order $n+1$, I have to make precise $W_{n+1}(x)$ as a polynomial in x. It will be unique if its degree is $\leq n$.

$$W_{n+1}(\lambda) = \left(1 - \frac{1}{n+1} N\right) \frac{1}{n+1} \sum_{i=0}^{n} (1+\lambda+\cdots+\lambda^{i-1})$$

$$= \frac{1}{n+1}\left(\sum_{i=0}^{n} \left(1+\lambda+\cdots+\lambda^{i-1} - \frac{i}{n+1}N\right)\right)$$

$$= \frac{1}{n+1}\left(\left(\sum_{i=0}^{n} \sum_{0\leq j<i} \lambda^{j}\right) - \frac{1}{n+1}\frac{(n+1)n}{2} N\right)$$

$$= \frac{1}{n+1}\left(\sum_{j=0}^{n} (n-j)\lambda^{j} - \frac{n}{2}\sum_{j=0}^{n} \lambda^{j}\right)$$

$$= \frac{1}{n+1} \sum_{j=0}^{n} \left(\frac{n}{2}-j\right)\lambda^{j}$$

Thus $$\boxed{W_{n+1}(x) = \frac{1}{n+1} \sum_{j=0}^{n} \left(\frac{n}{2}-j\right)x^{j}}$$

Then

$$(1-x)\, W_{n+1}(x) = \frac{1}{n+1}\left\{ \frac{n}{2}(1-x) + \left(\frac{n}{2}-1\right)(x-x^2) + \cdots \right.$$

$$\left. \cdots + \left(-\frac{n}{2}\right)(x^{n}-x^{n+1})\right\}$$

$$= \frac{1}{n+1}\left\{\frac{n}{2} - x - \cdots - x^{n} + \frac{n}{2}x^{n+1}\right\}$$

$$= \frac{1}{n+1}\left\{+n - x - \cdots - x^{n} + \frac{n}{2}(x^{n+1}-1)\right\}$$

$$= \left(1 - \frac{1}{n+1} N_{n+1}(x)\right) + \frac{n}{2(n+1)}(x^{n+1}-1)$$

~~which is a little inconvenient. Let's write~~

$$\cancel{W_{n+1}(x) = \frac{1}{n+1}\sum_{j=0}^{n}(n-j)x^{j} - \frac{n}{2(n+1)}\sum_{j=0}^{n} x^{j}}$$

Let's write

$$W_{n+1}(x) = \underbrace{\frac{1-\frac{1}{n+1}N_{n+1}(x)}{1-x}}_{\frac{1}{n+1}\sum_{i=0}^{n}(1+\cdots+x^{i-1})} - \frac{n}{2(n+1)}N_{n+1}(x)$$

call this $W'_{n+1}(x)$.

so that $(1-x)\,W'_{n+1}(x) = 1-\frac{1}{n+1}N_{n+1}(x)$

Then we have

$$Ph_n = h_n - W_{n+1}(K)\,sbh_n$$

$$= h_n - W'_{n+1}(K)(1-K-bs)h_n + \frac{n}{2(n+1)}\sum_0^n K^i sbh_n$$

$$= h_n - \left(1-\frac{1}{n+1}N_{n+1}(K)\right)h_n + bs\,W'_{n+1}(K)h_n$$

$$+ \frac{n}{2(n+1)}\underbrace{\left(\sum_0^n K^i s\right)bh_n}_{= Bbh_n = -bBh_n = -b\sum_0^{n-1}K^i s h_n}$$

$$= \frac{1}{n+1}N_{n+1}(K)h_n + b\underbrace{\left(W'_{n+1}(K) - \frac{n}{2(n+1)}\sum_0^{n-1}K^i\right)}_{\left(\frac{1}{n+1}\sum_{j=0}^{n-1}(n-j)K^j - \frac{n}{2}\sum_0^{n-1}K^i\right)} sh_n$$

$$\parallel$$

$$\frac{1}{n+1}\sum_{j=0}^{n-1}\left(\frac{n}{2}-j\right)K^j \qquad \text{not } W_n(K)$$

Conclude in any case that

$$\boxed{Ph_n = \underbrace{\frac{1}{n+1}\sum_{i=0}^{n}K^i h_n}_{\frac{2}{n+1}\text{ times my homotopy cochain}} + bs\underbrace{\left(\frac{1}{n+1}\sum_{j=0}^{n-1}\left(\frac{n}{2}-j\right)K^j\right)h_n}_{\text{killed by both } b,\,B}}$$

which means that Cuntz's homotopy formula and mine are equivalent

A technically useful fact about Cuntz's homotopy cochain Ph is that if the coboundary of h is K-invariant then $Ph = h$. Note that the coboundary in degree n [redacted] is $bh_{n-1} + \text{const } Bh_{n+1}$ and that Bh_{n+1} is automatically K-invariant. Thus the condition is that

$$0 = (1-K)bh = (bs+sb)bh = bsbh \text{ [redacted]}$$

[redacted]

[redacted] We have the equivalent conditions

1) bh K-invariant
2) $(b + cB)h$ K-invariant
3) $bsbh = 0$
4) sbh is K-invariant (or λ-invariant)

In effect the equivalence of 1), 2), 3) has been demonstrated. Since $sb(bh) = 0$, we know that bh is K-invariant $\Longleftrightarrow$ sbh is K (or λ)-invariant. (Recall the proof $h_n - Kh_n = bsh$ is K invariant when sh is, so

$$\underbrace{\sum_0^{n-1} K^i(h_n - Kh_n)}_{= n(h - Kh)} = b\sum_0^{n-1} K^i s h_n = bBh_n = -\underbrace{Bb}_{Nsb}h_n = 0.)$$

This demonstrates 1) $\Leftrightarrow$ 4).

But the same sort of fact holds for my homotopy cochain. Suppose h such that coboundary of h is K-invariant. Then each bh_n is K-invariant and

$$b\left(\frac{1}{n+1}\sum_0^n K^i h_n\right) = bh_n$$

$$B\left(\text{———}\right) = Bh_n$$

so that h and my homotopy cochain have the same coboundary.
The ~~same~~ thing holds with K^2-invariances.
This point will be useful in doing the diagram chasing, that is, showing that if the cocycle of a trace is a coboundary then it comes from a trace on Ω'.

Further formulas in the non unital case
If $f_n = (\psi_{n+1}, \varphi_n)$ then

$$bf_n = (b\psi_{n+1}, (1-\lambda)\psi_{n+1} - b'\varphi_n)$$

$$sbf_n = ((1-\lambda)\psi_{n+1} - b'\varphi_n, 0)$$

$$\left(1-\tfrac{1}{2}sb\right)f_n = \left(\frac{1+\lambda}{2}\psi_{n+1} + \frac{1}{2}b'\varphi_n, \varphi_n\right)$$

$$bsf_n = (b\varphi_n, (1-\lambda)\varphi_n)$$

$$\left(1-\tfrac{1}{2}bs\right)f_n = \left(\psi_{n+1} - \tfrac{1}{2}b\varphi_n, \frac{1+\lambda}{2}\varphi_n\right)$$

$$Kf_n = (\lambda\psi_{n+1} - \mathrm{cross}(\varphi_n), \lambda\varphi_n)$$

Consider the map $\Omega'Q_\natural \longrightarrow (QLQ)_\natural = LQ$. In the ~~xxxx~~ case of even supertraces it is an isom., but not for odd traces. The map is given on cochains by $h_{2n} = -2(1-\frac{1}{2}bs)f_{2n}$ and $1-\frac{1}{2}bs$ has kernel & cokernel isomorphic to ~~xx~~ those of $\frac{1+\lambda}{2}$. On the other hand we have the map

$$\Omega'Q_\natural \xrightarrow{\beta} (Q\otimes Q)_\natural = Q$$

whose effect on cochains is $h_{2n} = -\delta f_{2n+1} + \sigma f_{2n+1}$.

The question is whether

$$\Omega^1 Q_{\neq} \longrightarrow LQ \oplus Q$$

is injective. Let us consider then h_{2n} and try to write it as the sum of something on the image of $1-\frac{1}{2}bs$ and something in the form $-\delta f_{2n-1} + \sigma f_{2n+1}$

$$\begin{tikzcd}
0 \arrow[r] & (\bar{A}^{\otimes 2n+1})^* \arrow[r] \arrow[d, "1"] & (A \otimes \bar{A}^{\otimes 2n})^* \arrow[r] \arrow[d, "1-\frac{1}{2}bs"] & (\bar{A}^{\otimes 2n})^* \arrow[r] \arrow[d, "\frac{1+\lambda}{2}"] & 0 \\
0 \arrow[r] & {} \arrow[r] & {} \arrow[r] & {} \arrow[r] & 0
\end{tikzcd}$$

We are interested in the cokernel of $1-\frac{1}{2}bs$ which is isomorphic to the cokernel of $\frac{1+\lambda}{2}$. On the other hand we know that the composition

$$\mathrm{Ker}(1+\lambda) \subset (\bar{A}^{\otimes 2n})^* \longrightarrow \mathrm{Coker}(1+\lambda)$$

is an isomorphism, since $(\bar{A}^{\otimes 2n})^* = \mathrm{Ker}(1+\lambda) \oplus \mathrm{Im}(1+\lambda)$. Thus given an elt of $\mathrm{Coker}(1-\frac{1}{2}bs)$, we move it to an element in $\mathrm{Coker}(\frac{1+\lambda}{2})$, then represent it by an element of $\mathrm{Ker}(1+\lambda)$, which we lift to an element of $\mathrm{Ker}(1-\frac{1}{2}bs)$. Thus it seems we have

$$(A \otimes \bar{A}^{\otimes 2n})^* = \mathrm{Ker}(1-\tfrac{1}{2}bs) \oplus \mathrm{Im}(1-\tfrac{1}{2}bs)$$

So we can assume any element in the cokernel of $1-\frac{1}{2}bs$ is represented by an h_{2n} such that

$$h_{2n} - \tfrac{1}{2}bsh_{2n} = 0$$

But then take $f_{2n-1} = sh_{2n}$, $f_{2n+1} = 0$. One has $\delta f_{2n-1} = 0$ and $bsf_{2n-1} = 2h_{2n}$ so this h comes from an odd trace on $Q \otimes Q$.

Let's check the homotopy formula for a supertrace on $\Omega^1 Q$ obtained from one in $Q \otimes Q$. This means we have (for even supertraces)

$$h_{2n+1} = \delta f_{2n} - 2\sigma f_{2n+2} .$$

First we want to derive the homotopy formula to get the appropriate notation.

$$h_{2n+1} = T^{\natural}(\partial\theta\, \alpha^{2n}\, d\bar{\theta})$$

$$T(d(\rho\alpha^{2n})) = T(d\rho\,\alpha^{2n}) + \sum_0^{2n-1} T(\rho\alpha^i\, d\alpha\, \alpha^{2n-1-i})$$

$$= \kappa^{-1}\underbrace{T(\alpha^{2n} d\rho)}_{-\sigma h_{2n+1}} + \sum_0^{2n-1} \kappa^{2n-1-i}\, \underbrace{T(\rho\alpha^{2n-1} d\alpha)}_{-\sigma h_{2n+1} + \frac{1}{2}\delta h_{2n-1}}$$

$$= \underbrace{-\left(\sum_0^{2n} \kappa^i\right)\sigma h_{2n+1}}_{B h_{2n+1}} + \delta\left(\frac{1}{2}\sum_0^{2n-1} \kappa^i\right) h_{2n-1}$$

where $B = \sum \kappa^i \sigma$
$B = \sum \kappa^i s$

$$B h_{2n+1} = \frac{2}{2n+2} B\left(\frac{1}{2}\sum_0^{2n+1} \kappa^i\right) h_{2n+1} .$$

$$\therefore \boxed{T(d(\rho\alpha^{2n})) = \delta\left(\frac{1}{2}\sum_0^{2n-1}\kappa^i\right) h_{2n-1} - \frac{2}{2n+2} B\left(\frac{1}{2}\sum_0^{2n+1}\kappa^i\right) h_{2n+1}}$$

Now take $h_{2n-1} = \delta f_{2n-2} - 2\sigma f_{2n}$. The homotopy cochain is

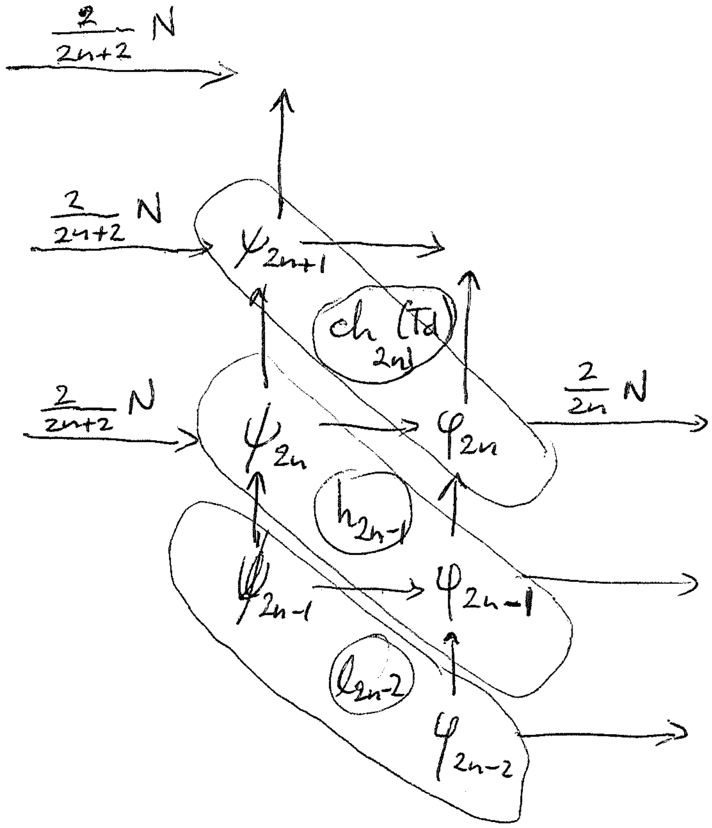
$$\frac{1}{2}\sum_0^{2n-1} \kappa^i \left(\delta f_{2n-2} - 2\sigma f_{2n}\right)$$

$$= \delta\left(\frac{1}{2}\sum_0^{2n-1} \kappa^i f_{2n-2}\right) - \sum_0^{2n-1} \kappa^i \sigma f_{2n}$$

$$= \delta\left(\tfrac{1}{2}\sum_0^{2n-1} K^i f_{2n-2}\right) - B f_{2n}$$

$$= \delta\left(\tfrac{1}{2}\sum_0^{2n-1} K^i f_{2n-2}\right) - \frac{2}{2n+2} B\left(\tfrac{1}{2}\sum_0^{2n+1} K^i f_{2n}\right)$$

and this is a cocycle, as it should be, since the induced trace on Q is zero. The constants in front of B fit into the following picture



$$l_{2n-2} = \tfrac{1}{2}\sum_0^{2n-1} K^i f_{2n-2}$$

Let A be a unital algebra; we work now in the unital category. We adjoin an indeterminate P to A obtaining the algebra $A * k[P]$. Let $k[F]$ denote the algebra $k \oplus kF$ with $F^2 = 1$. Consider the linear map

$$\rho : k[F] \longrightarrow A * k[P] \qquad \rho(1) = 1 \quad \rho(F) = P$$

and let $\Gamma = k[F] \oplus \left(k[F] \otimes (A * k[P]) \otimes k[F] \right)$ be the GNS algebra associated to ρ.

A homomorphism $\Gamma \longrightarrow R$ is equivalent to a triple consisting of a homomorphism

$$u : k[F] \longrightarrow R \qquad \text{i.e. an involution } F \in R$$

an idempotent $e \in R$, and a homomorphism

$$v : A * k[P] \longrightarrow eRe \qquad \text{satisfying}$$

$$v(\rho(c_1 + c_2 F)) = e(c_1 + c_2 F)e \qquad \text{i.e. } v(P) = eFe.$$

Thus $\Gamma \longrightarrow R$ is equivalent to a triple (F, e, v) where F is an involution in R, e is an idempotent in R and $v : A * k[P] \rightarrow eRe$ is a homomorphism with $v(P) = eFe$. Such a v is equivalent to a homom. $A \rightarrow eRe$.

Thus a homomorphism $\Gamma \longrightarrow R$ is equivalent to a triple (F, e, v'), where $F, e \in R$ are as above and v' is a homomorphism $A \rightarrow eRe$. But the pair (e, v') where e is an idempotent in R and $v' : A \rightarrow eRe$ is a homomorphism is equivalent to a homomorphism $\tilde{A} \longrightarrow R$. Thus $\Gamma \rightarrow R$ is equivalent to a pair $F \in R$, $\tilde{A} \longrightarrow R$, which is the same as a homomorphism $\tilde{A} * k[F] \longrightarrow R$.

Thus we have proved that there is a canonical isomorphism

$$\Gamma = k[F] \oplus \left(k[F] \otimes (A * k[P]) \otimes k[F]\right) \cong \tilde{A} * k[F]$$

Picture: A pre Fredholm module consists of a homomorphism $A \longrightarrow \mathcal{L}(H)$ together with an operator P satisfying compactness conditions. Thus we get a homomorphism

$$A * \mathbb{C}[P] \longrightarrow \mathcal{L}(H)$$

with certain properties relative to the ideal $\mathcal{K}(H)$ of compact operators ($[P,a]$, $P^2-1 \in \mathcal{K}(H)$). Let's ignore the compactness conditions and concentrate on the GNS business.

We have then a module H over $A * \mathbb{C}[P]$, and we have the linear map ρ_n from $k[F]$ to $A * \mathbb{C}[P]$.

Recall that given $\rho : \mathcal{A} \to \mathcal{B}$ a linear map with $\rho(1) = 1$, the GNS construction associated to ρ is the algebra whose modules are equivalent to data

$$\begin{array}{ccc} N & \underset{i^*}{\overset{\iota}{\rightleftarrows}} & M \\ | & & | \\ \mathcal{B}\text{-mod} & & \mathcal{A}\text{-mod.} \end{array} \qquad i^* a \iota(n) = \rho(a) n$$

Also if $N = \mathcal{B}$ and the data has a right $\mathcal{B}$-module structure, then there are two candidates for M:

$$\mathcal{A} \otimes \mathcal{B} \longrightarrow \operatorname{Hom}(\mathcal{A}, \mathcal{B})$$

and a canonical map between them.

Maybe a better way to say it is that given a B-module N there are two natural ways to obtain a Γ-module and [redacted] a canonical map between them:

$$\mathcal{a} \otimes N \longrightarrow \mathrm{Hom}(\mathcal{a}, N)$$

[redacted] So now apply this to $\mathcal{a} = k[F]$, ρ above, $N = H$. We have calculated $\mathcal{a} \otimes H$ in August 89. Recall

$$\cancel{\mathcal{a}\otimes H \xleftarrow{\sim} H^{\oplus 2}}$$

$$\cancel{1\otimes x + \longleftarrow\!\!\shortmid \begin{pmatrix} x \\ y \end{pmatrix}}$$

$$\cancel{F\otimes y - 1\otimes Py}$$

~~Better if we use~~ $\cancel{H^{\oplus 2} \xrightarrow{\sim} \mathcal{a}\otimes H, \ \begin{pmatrix} x \\ y\end{pmatrix} \mapsto}$

$\cancel{1\otimes x + F\otimes y}$ ~~? then~~

$$H^{\oplus 2} \xrightarrow{\sim} \mathcal{a}\otimes H \xleftarrow{\sim} H^{\oplus 2}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto 1\otimes x + F\otimes y$$

$$1\otimes x + F\otimes y - 1\otimes Py \longleftarrow\!\!\shortmid \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \longleftarrow\!\!\shortmid \begin{pmatrix} x \\ y \end{pmatrix}$$

Using the second isomorphism we have that $\iota(x) = 1\otimes x$ and $\iota^*(1\otimes x) = x$, $\iota^*(F\otimes x) = Px$ become

$$\iota = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \iota^* = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 1 & P \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix} \text{[redacted]}$$

$$= \begin{pmatrix} 1 & P \\ 0 & 1 \end{pmatrix} \text{[redacted]} \begin{pmatrix} 0 & 1 \\ 1 & -P \end{pmatrix} = \begin{pmatrix} P & 1-P^2 \\ 1 & -P \end{pmatrix}$$

Similarly $Hom(a, H)$ becomes $H^{\oplus 2}$ with i, i^* the same but with

$$F = \begin{pmatrix} P & 1 \\ 1-P^2 & -P \end{pmatrix}.$$

and the map $a \otimes H \longrightarrow Hom(a, H)$ [redacted] in these coordinates becomes the operator

$$\begin{pmatrix} 1 & 0 \\ 0 & 1-P^2 \end{pmatrix}$$

[redacted] Let's check this. Let $1^*, F^*$ denote the basis of a^* dual to $1, F$. Then one has

$$a \otimes H \longrightarrow Hom(a, H) = a^* \otimes H$$

$$1 \otimes x \longmapsto \left\{ \begin{matrix} 1 \mapsto x \\ F \mapsto Px \end{matrix} \right\} = 1^* \otimes x + F^* \otimes Px$$

$$F \otimes y \longmapsto \left\{ \begin{matrix} 1 \mapsto Py \\ F \mapsto y \end{matrix} \right\} = 1^* \otimes Py + F^* \otimes y$$

and we want the isomorphism $a^* \otimes H \xrightarrow{\sim} H^{\oplus 2}$ sending $1^* \otimes x + F^* \otimes Px$ to $\begin{pmatrix} x \\ 0 \end{pmatrix}$ which is unipotent. Thus we want

$$1^* \otimes u + F^* \otimes v \longmapsto \begin{pmatrix} u \\ v - Pu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Thus the map $H^{\oplus 2} \cong a \otimes H \longrightarrow Hom(a, H) \cong H^{\oplus 2}$ becomes

$$\begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix} \begin{pmatrix} 1 & P \\ P & 1 \end{pmatrix} \begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -P & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ P & 1-P^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1-P^2 \end{pmatrix}$$

as claimed.

So far given $A \longrightarrow \mathcal{L}(H) \ni P$
we dilate to $\tilde{A} \longrightarrow \mathcal{L}(H^{\oplus 2}) \ni F$ where

$$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} P & 1-P^2 \\ 1 & -P \end{pmatrix}$$

Let's transform F to standard diagonal form:

$$\begin{pmatrix} P & 1-P^2 \\ 1 & -P \end{pmatrix}\begin{pmatrix} 1+P & 1-P \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+P & P-1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+P & 1-P \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then we have

$$\begin{pmatrix} 1+P & 1-P \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1+P & 1-P \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1-P}{2} \\ \frac{1}{2} & -\frac{1+P}{2} \end{pmatrix}\begin{pmatrix} a(1+P) & a(1-P) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a\frac{1+P}{2} & a\frac{1-P}{2} \\ a\frac{1+P}{2} & a\frac{1-P}{2} \end{pmatrix}$$

whence

$$a^+ \longmapsto \begin{pmatrix} a\frac{1+P}{2} & 0 \\ 0 & a\frac{1-P}{2} \end{pmatrix} \qquad a^- \longmapsto \begin{pmatrix} 0 & a\frac{1-P}{2} \\ a\frac{1+P}{2} & 0 \end{pmatrix}$$

and we recover our earlier approach to dilation

December 19, 1989

Consider an unbounded Fredholm module

$$A \longrightarrow \mathcal{L}(H) \ni X$$

and the JLO cocycle associated to it. This is based on the superconnection $\delta + \theta + \sigma X$ with curvature $X^2 + [\sigma X, \theta]$. It is the ~~normalized inhomogeneous~~ Hochschild cochain [redacted]

$$f = \tau^{\natural}\left(\partial\theta\, e^{X^2 + [\sigma X, \theta]}\right)$$

where τ is appropriately defined according to whether the Fredholm module is graded or ungraded.

Consider the ungraded case. Then f is odd, that is, it has components f_{2n+1} for $n \geq 0$. The associated sequence of cyclic cocycles Nsf_{2n+1} are of even degree.

At first sight there is a puzzle: If we consider the geometric superconnection $\delta + \theta + \sigma X$ on the trivial Hilbert bundle with fibre $H^{\oplus N}$ over $\mathcal{G} = U_N(A)$, these even cyclic cocycles correspond to the homogeneous superconnection character forms

$$tr^{\natural}_{s}\left(e^{X^2 + \sigma[X,\theta]}\right)$$

which [redacted] are of odd degree. Yet we know that the important K-class on $\mathcal{G}$ is even, hence where do we get even forms on $\mathcal{G}$ describing it's character?

The point is [redacted] the following. [redacted] By diagram chasing all of these odd cocycles are cyclic coboundaries. On the other hand, at least in the p-summable case one has the classical limit deformation $X \mapsto hX$ with $h \downarrow 0$ which expresses the high degree cyclic cocycles as coboundaries. This is clear in the Dirac operator situation by Getzler asymptotics, but I think I checked this following Connes-Moscovici's transgression paper.

Thus there are two reasons why the high degree even ~~odd~~ cyclic cocycles are coboundaries and the difference of these reasons is an odd cyclic class.

Now all of this is fairly explicit and it should become much clearer if one brings in the Laplace Transform to study the superconnection forms.

GIHC: geometric interpretation of Hochschild cochains
Start with a Fredholm module:

$$A \longrightarrow \mathcal{L}(H) \quad F \in \mathcal{L}(H)$$

which is p-summable. Consider two consecutive Chern character cyclic cocycles, and the Hochschild cochain f satisfying $sbf = 0$ which links them. Find formulas

Let us consider $H^N = H \otimes \mathbf{C}^N$. We have an action by $N \times N$-matrices

$$M_N(A) = A \otimes M_N \longrightarrow \mathcal{L}(H) \otimes M_N = \mathcal{L}(H^N)$$

Write F for the induced involution $F = F \otimes 1$. The group

$$\mathcal{G} = U(A \otimes M_N) = U_N(A)$$

acts on H^N by unitary operators, and (assuming A unital acting unitally on H) the subgroup $G = U_N$ fixes F.

We have a map

$$\mathcal{G}/G \longrightarrow Grass(H^N) \qquad g \mapsto gFg^{-1}$$

which is equivariant for the left translation action of $\mathcal{G}$ on $\mathcal{G}/G$ and the action on H^N. Over the Grassmannian there is a canonical splitting of the trivial bundle with fibre H^N. Thus over $\mathcal{G}/G$ we have the trivial bundle $(\mathcal{G}/G) \times H^N$ with the splitting gFg^{-1}. With respect to the diagonal action of $\mathcal{G}$ the splitting and the canonical connection d on this trivial bundle are invariant. Hence over $\mathcal{G}/G$ we have an equivariant Hilbert space bundle equipped with an invariant connection and an invariant splitting.

We have an isomorphism

$$\mathcal{G} \times^G H^N \cong (\mathcal{G}/G) \times H^N \qquad (g, \xi) \mapsto (g, g\xi)$$

Using this we obtain an alternative description of the equivariant bundle with invariant connection and splitting over $\mathcal{G}/G$ as follows.

Suppose we lift up to $\mathcal{G}$. Then we have the trivial bundle $\mathcal{G} \times H^N$ with the diagonal action, the connection d, and the splitting gFg^{-1}. If we use the isomorphism

$$\mathcal{G} \times H^N \cong \mathcal{G} \times H^N \qquad (g, \xi) \mapsto (g, g\xi)$$

then we have the trivial bundle $\mathcal{G} \times H^N$ with $\mathcal{G}$ acting trivially on the second factor, with the connection

$$g^{-1} \cdot d \cdot g = d + g^{-1}dg = d + \theta$$

and the splitting F. Here θ is the Maurer-Cartan form of $\mathcal{G}$, which can be viewed as an operator valued form since the complexified Lie algebra of $\mathcal{G}$ is $M_N(A)$.

Thus we see that our equivariant bundle with invariant connection and splitting over $\mathcal{G}/G$ is obtained from the trivial bundle $\mathcal{G} \times H^N$, where $\mathcal{G}$ acts trivially on the second factor, by dividing out by the action $u(g, \xi) = (gu^{-1}, u\xi)$. The invariant connection and splitting are obtained from $d + \theta$ and F on $\mathcal{G} \times H^N$.

Note that $\mathcal{G} \times^G H^N$ is the vector bundle associated to the principal G-bundle $\mathcal{G} \to \mathcal{G}/G$ and the representation of G on H^N. This associated bundle has a splitting, because F is G-invariant. It is interesting why the associated bundle has a connection - normally one would expect to start from a connection in the principal bundle $\mathcal{G}$, and combine this with the representation of $Lie(G)$ on H^N to get a connection on $\mathcal{G} \times H^N$ which descends. This shows that connections in associated bundles can occur even without being given a connection in the principal bundle.

Let E denote the vector bundle $\mathcal{G} \times^G H^N$ over $\mathcal{G}/G$. Because it is an associated bundle, we know that

$$\Omega(\mathcal{G}/G, \mathrm{End}E) = \Omega(\mathcal{G}, \mathcal{L}(H) \otimes M_N)_{bas}$$

where basic refers to the right action of G on $\mathcal{G}$ and to the conjugation action of G on M_N; this is the adjoint representation as M_N is the complexified Lie algebra of G. If we want forms invariant under the $\mathcal{G}$-action, we get

$$\Omega(\mathcal{G}/G, \mathrm{End}E)^{\mathcal{G}} = \mathrm{Hom}(\Lambda(\tilde{g}/g), \mathcal{L}(H) \otimes M_N)^G$$

$\tilde{g} = Lie(\mathcal{G})_c$
$g = Lie(G)_c$

Is this is a relative Lie algebra cochain complex?

Review:

If E is an equivariant vector bundle over a Lie group G for the left translation action, then an invariant flat connection on E is equivalent to a representation of the Lie algebra on the fibre V of E over the identity.

Furthermore suppose H is a Lie subgroup of G and that one is given an extension of $Lie(H) \to Lie(G) \to \mathrm{End}(V)$ to a representation of H on V such that $Lie(G) \to \mathrm{End}(V)$ is compatible with the H-action. Then E and its flat invariant connection descend to G/H.

Example: If H is a discrete subgroup of G, and we have compatible representations of H and $Lie(G)$ on V, then we get a flat bundle $G \times_H V$ over G/H.

Let us apply this to our previous situation. We have the group $\mathcal{G}$ acting on H^N, and this gives compatible representations of its Lie algebra and of the subgroup G. Thus it should follow that for the invariant flat connection on $\mathcal{G} \times^G H^N$ over $\mathcal{G}/G$ the complex of $\mathcal{G}$-invariant forms with values in this bundle is a relative Lie algebra cochain complex.

We have

$$\Omega(\mathcal{G}/G, \mathrm{End}E)^{\mathcal{G}} = \mathrm{Hom}(\Lambda(\underbrace{\tilde{g}/g}_{\bar{A} \otimes M_N}), \mathcal{L}(H) \otimes M_N)^G$$

and if we use invariant theory we should find inside this DGA an algebra of cochains

$$\mathrm{Hom}(\bar{A}^{\otimes \bullet}, \mathcal{L}(H))$$

which should be the DGA of normalized Hochschild cochains having values in the A-bimodule $\mathcal{L}(H)$. The differential is essentially $\delta + ad(\theta)$.

Recap: Given $A \longrightarrow \mathcal{L}(H) \ni F$ we form an equivariant bundle with flat connection + splitting

$$E = \mathcal{G} \times^G H^N \longrightarrow \mathcal{G}/G$$

, and identify the primitive part of the DGA of End-valued $\mathcal{G}$-invariant forms with standard normalized Hochschild cochains having values in $\mathcal{L}(H)$ considered as A-bimodule. Something strange happens when we consider the functional $\operatorname{tr} F$ on $\mathcal{L}^1(H)$. This is not a trace on the A-bimodule $\mathcal{L}^1(H)$. If it were, then we would get Hochschild cocycles. Instead

$$\delta \operatorname{tr}^{\natural}(F\partial\theta[F,\theta]^{2n}) = \operatorname{tr}^{\natural}\big(\underbrace{[\delta+\theta, F]}_{-[F,\theta]}\, \partial\theta\, [F,\theta]^{2n}\big)$$

$$= \operatorname{tr}^{\natural}\big(\partial\theta\,[F,\theta]^{2n+1}\big)$$

$$\sigma \operatorname{tr}^{\natural}\big(F\partial\theta[F,\theta]^{2n+2}\big) = -\operatorname{tr}\big(F[F,\theta]^{2n+2}\big)$$

$$= -2\operatorname{tr}\big(\theta[F,\theta]^{2n+1}\big)$$

The case to study this mystery is perhaps with $f_0 = \operatorname{tr}^{\natural}(F\partial\theta)$ which is not a trace:

$$\delta \operatorname{tr}^{\natural}(F\partial\theta) = \operatorname{tr}^{\natural}(\partial\theta[F,\theta])$$

Let look at the unbounded Fredholm situation. Recall if u, v are two alg homom. $A \rightrightarrows R$, then

$$(u-v)(xy) = u(x)u(y) - v(x)v(y)$$

$$= \big(u(x)-v(x)\big)u(y) + v(x)\big(u(y)-v(y)\big)$$

$$= u(x)\big(u(y)-v(y)\big) + \big(u(x)-v(x)\big)v(y)$$

So if $gx = \frac{1}{2}(u(x) - v(x))$, we have

$$
\begin{aligned}
g(xy) &= g(x)u(y) + v(x)g(y) \\
&= u(x)g(y) + g(x)v(y) \\
&= \rho(x)g(y) + g(x)\rho(y)
\end{aligned}
$$

where $\rho = \frac{u+v}{2}$. Also

$$g(xy) = u(x)g(y) + g(x)\big(u(y) - 2g(y)\big)$$

$$\boxed{g(xy) = u(x)g(y) + g(x)u(y) - 2g(x)g(y)}$$

The latter formula is useful if one wants somehow to view u as being the identity.

Example: $u:\ a \longmapsto a$

$v:\ a \longmapsto g^{-1}ag = \frac{1-X}{1+X}\, a\, \frac{1+X}{1-X}$

Then
$$g(a) = \frac{1}{2}\left\{a - a - \frac{1-X}{1+X}\Big[a, \underbrace{\frac{1+X}{1-X}}_{-1+\frac{2}{1-X}}\Big]\right\}$$

$$= \frac{1}{2}\,\frac{1-X}{1+X}\,\frac{2}{1-X}\,[a, 1-X]\,\frac{1}{1-X} = \frac{1}{1+X}[X, a]\frac{1}{1-X}$$

Recall the JLO cocycle

$$f = \tau^4\left(\partial\theta\, e^{X^2 + [\sigma X, \theta]}\right)$$

where in the ungraded case τ is the trace applied to a term with an odd number of σ's. Thus f consists of odd cochains

Note: $\mathcal{H}$

$$\theta : a \longmapsto a$$

$$\bar{\theta} : a \longmapsto g^{-1} a g$$

$$q = \frac{\theta - \bar{\theta}}{2} : a \longmapsto \frac{1}{1+x} [x, a] \frac{1}{1-x}$$

then
$$\delta q + \theta q + q \bar{\theta} = 0$$
$$\delta q + \bar{\theta} q + q \theta = 0$$
$$\delta q^2 + \theta q^2 - q^2 \theta = 0$$

Unfortunately we do not consider combinations like $a_0 q a_1 \cdots q a_n$ in the JLO situation