<u>July 27, 1989</u>

The spectral flow concept leads to considering the subvariety of codimension one consisting of the operators X having zero as eigenvalue. If we have a path X_t we want to count properly the net number of times the path crosses this subvariety.

The topology in this situation is interesting. Consider first a closed submanifold $N \subset M$ of real codimension 1, and ask when we can assign to any loop $S^1 \longrightarrow M$ an integer, which is the intersection number of the loop with N. We need to assign a class in $H^1(M,\mathbb{Z})$ to N. We seem to need the normal bundle of N in M to be <u>oriented</u>. The cohomology class is $\iota_* 1$ where

$$\iota_* : H^0(N,\mathbb{Z}) \longrightarrow H^1(M,\mathbb{Z})$$

is the Gysin homomorphism.

Now in our ~~situation~~ N has singularities, but there is apparently a nice resolution $\tilde{N}$. If the map $f: \tilde{N} \longrightarrow M$ is oriented, then we will obtain $f_* 1 \in H^1(M,\mathbb{Z})$ as desired.

Digression: A (real) divisor in M in analogy with alg geometry can be identified with a real line bundle L together with a section s. Since

$$H^1(M,\mathbb{Z}/2) = [M, \mathbb{R}P^\infty] = \mathrm{Pic}_{\mathbb{R}}(M)$$

divisors represent mod 2 cohomology of degree 1.

In our operator ~~situation~~ we have ~~ad~~ class in
$$H^1(M,\mathbb{Z}) = [M, S^1]$$
which by Thom-Pontryagin are represented by submanifold of codim 1 with trivial normal bundle. We have the exact sequence

$$\begin{array}{ccccccc} H^1(M,\mathbb{Z}) & \longrightarrow & H^1(M,\mathbb{Z}/2) & \longrightarrow & H^2(M,\mathbb{Z}) & \xrightarrow{2} & H^2(M,\mathbb{Z}) \\ & & \| & & \| & & \| \\ & & Pic_{\mathbb{R}}(M) & & Pic_{\mathbb{C}}(M) & & Pic_{\mathbb{C}}(M) \end{array}$$

Geometrically this can be understood as follows.

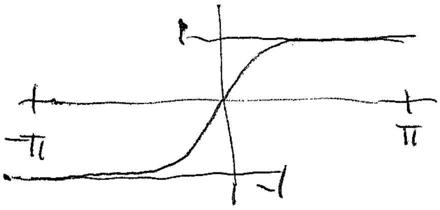
1) Given a complex line bundle L' with a trivialization $L' \otimes L' \xrightarrow{\sim} \tilde{\mathbb{C}}$, L' is the complexification of the real line bundle L consisting of $\xi \in L'$ such that $\xi \otimes \xi \longmapsto$ non-negative real no.

2) Given a real line bundle L whose complexification is trivialized, this gives two sections of L which do not simultaneously vanish, hence L is induced by a map from M to $P_1(\mathbb{R}) = S^1$.

Our next project must be to understand the geometry of the subvariety of the singular operators, that is, its resolution and how to represent the cohomology class in $H^1(\ ,\mathbb{Z})$ by something supported on the subvariety.

July 28, 1989

Idea from yesterday: Choose a function $\varphi : S^1 - \{-1\} \longrightarrow [-1, 1]$ which has the form



Consider $\underbrace{\det \varphi(g)}_{f(g)=} = \prod_{i=1}^{n} \varphi(\lambda_i)$ if λ_i are the eigenvalues of $g \in U_n$. This is ambiguous up to sign if g has the eigenvalue -1.

For each $-1 \leq x \leq 1$, let W_x be the open set of U_n consisting of g which do not have $x \pm i\sqrt{1-x^2}$ as eigenvalues. Define f_x on W_x by

$$f_x(g) = \prod_{Re(\lambda_i) > x} \varphi(\lambda_i)$$

Now restrict to $-1 < x < a$ where $\varphi(\lambda) = \pm 1$ for $Re(\lambda) \leq a$. Then $f_x(g) = \pm f(g)$ up to sign. Moreover ~~if~~ $-1 < x, y < a$, then f_x and f_y on $W_x \cap W_y$ agree up to sign. Thus we have ~~a~~ Čech 1-cocycle with values in the [redacted] constant sheaf $\{\pm 1\}$, and so a real line bundle on U_n with a section f.

$f(g)$ vanishes iff g has the eigenvalue 1. Moreover $f(g)$ is close to zero iff g has some eigenvalue close to 1. In effect [redacted]

$$\left| \prod_{i=1}^{n} \varphi(\lambda_i) \right| < \varepsilon \implies \text{some } |\varphi(\lambda_i)| < \varepsilon^{1/n}$$

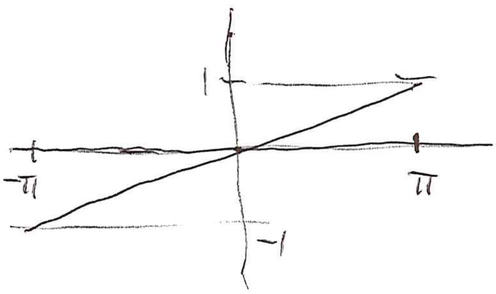
In this way we obtain a real line bundle with section on $U(V)$ whose zero set is the subvariety of unitaries having the eigenvalue $+1$.

Call L this real line bundle.

If we complexify L, then we obtain a trivial line bundle, because we have the function $\det(1-g) = \prod_{i=1}^{n}(1-\zeta_i)$ having the same divisor as f, that is, coinciding with f up to invertible functions.

Then it should be easy to construct a map $U_n \longrightarrow S^1$. The idea is to write $\det(1-g)$ as $(a+ib)f$, then use $\frac{b}{a} : U_n \longrightarrow \mathbb{P}_1(\mathbb{R}) = S^1$. Equivalently one takes twice the phase of $a+ib$.

Here's the simplest version. Take $\varphi(e^{i\theta}) = \sin\frac{\theta}{2}$:



and replace $1-\zeta$ by $i\left(\frac{1-\zeta}{2}\right) = \frac{\zeta-1}{2i} = e^{\frac{i\theta}{2}}\sin(\theta/2)$

Then
$$\left(\frac{\det\left(\frac{g-1}{2i}\right)}{\varphi(g)}\right)^2 = \prod_j \left(e^{i\theta/2}\right)^2 = \det(g).$$

Cayley transform version:

$$g = \frac{1+X}{1-X} \qquad g^{1/2} = \frac{1+X}{\sqrt{1-X^2}}$$

$$\varphi(g) = \frac{-iX}{\sqrt{1-X^2}} \qquad g = 1 + \frac{2X}{1-X}$$

$$\frac{g-1}{2} = \frac{X}{1-X}$$

$$\left(\frac{\frac{g-1}{2i}}{\varphi(g)}\right)^2 = \left(\frac{\frac{-iX}{1-X}}{\frac{-iX}{\sqrt{1-X^2}}}\right)^2 = \frac{1-X^2}{(1-X)^2} = \frac{1+X}{1-X} = g$$

We seem therefore to find something interesting. Suppose we work on the real t line and consider the distributions

$$\frac{1}{t+i0} = P\left(\frac{1}{t}\right) + i\pi\,\delta(t) \qquad ?$$

Let's summarize. We have been considering the real codimension one subvariety of U_n where g has the eigenvalue 1. We have described this as a real divisor, that is, in terms of a real line bundle with section. We have also described the trivialization of the complexification and the associated map $U_n \longrightarrow S^1$.

Consider next the group G of unitary operators $g \equiv -1 \mod \mathcal{L}'$. Then we have the complex-valued function

$$\det\left(\frac{1-g}{2}\right)$$

defined on G, which vanishes exactly on the subvariety of g with eigenvalue 1.

We can obtain the corresponding map $G \to S^1$ as follows. First we describe the real divisor. Given g_0 suppose its +1-eigenspace has dim k. For g near g_0, there is a unique k-dimensional subspace stable under g on which g is near 1. Convert g on this subspace to a self-adjoint operator using either

$$\frac{1}{i}\log g \qquad \text{or} \qquad \frac{1}{i}\frac{g-1}{g+1}$$

or some local orientation-preserving diffeomorphism of S^1 near 1 with $\mathbb{R}$ near 0. Then take the determinant. This gives a real function defined near g_0 whose divisor is the subvariety being considered.

Next take $\det\left(\frac{1-g}{2}\right)$ divide by this real function and take the square of the phase. This doesn't depend on the real function used to remove the zeroes of $\det\left(\frac{1-g}{2}\right)$. It gives the desired map $G \to S^1$.

To calculate it in the present case we can look at each eigenvalue separately.

$$\frac{1-e^{i\theta}}{2} = -\frac{1}{2}(e^{i\theta}-1) = -\frac{1}{2}e^{i\theta/2}\left(e^{i\theta/2}-e^{-i\theta/2}\right)$$

$$= e^{i\theta/2}\left(-\frac{1}{2}\right)2i\sin\theta/2$$

Thus

$$(\text{phase})^2 = \left(e^{i\theta/2} i\right)^2 = -e^{i\theta}$$

and so therefore we obtain the function

$$g \longmapsto \det(-g) \;:\; G \longrightarrow S^1$$

If we work over the affine spase of $X = X_0 + B$ where X_0 is invertible, then we have available the real determinant function

$$\det(X_0^{-1}X)$$

which satisfies

$$d \log \det(X_0^{-1}X) = \operatorname{tr}(X^{-1}dX)$$

July 28, 1989

Consider $G = U(\mathbb{C}^n)$, $Z = \{g \in G \mid \det(g-1) = 0\}$
L = real line bundle with section having Z for zero set. Then we know $L \otimes \mathbb{C}$ can be trivialized giving a map $\psi : G \to S^1$.

We are missing the link between Z and a generic level submanifold for ψ.

Example: $\varphi(g) = \det\left(\frac{1-g}{2}\right)$ This is a $\mathbb{C}$-valued function with zero set Z.

$$\psi(g) = (\text{phase } \varphi(g))^2 = \frac{\varphi(g)^2}{|\varphi(g)|^2} = \frac{\varphi(g)}{\overline{\varphi(g)}}$$

(This can be modified to remove the eigenvalues = 1)

$$= \frac{\det\left(\frac{1-g}{2}\right)}{\det\left(\frac{1-g^{-1}}{2}\right)} = \frac{\det\left(\frac{1-g}{2}\right)}{\det(-g^{-1})\det\left(\frac{1-g}{2}\right)} = \det(-g)$$

The level sets of $\det(-g)$ are the cosets of SU_n.

Now here's the difficulty with our present approach. We want to use maps $\psi : G \to S^1$ of a special form. For example the superconnection character 1-form gives a ψ with

$$d \log \psi = \text{const } \operatorname{tr}\left(e^{X^2} dX\right)$$

and $\quad d \log \det(-g) = \operatorname{tr} g^{-1}dg = \operatorname{tr}\left(\frac{2}{1-X^2} dX\right)$

such a ψ results by considering each eigenvalue separately, applying a function and taking the product. It seems this process will yield the same result for the diagonal matrix $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ and the identity, since the function applied to the eigenvalues has opposite sign for ζ and ζ^{-1}. NO

Thus none of the ψ's we expect to encounter will give Z as level set. NO.

Let's return to the index question. Review. We start with H, X_0 and assume $\frac{1}{\Lambda + X_0}$ trace class. Suppose also that X_0 invertible. We consider the affine space $\mathcal{Q}$ of bounded perturbations $X = X_0 + B$, B any bdd skew-adjoint operator.

The first goal should be to define the index. One possibility is to map the invertible X into involutions $\frac{-iX}{|X|}$. This should give a ~~[illegible]~~ map from the invertible operators to a restricted Grassmannian. Thus given two invertible operators we should be able to define an integer giving the relative index.

July 29, 1989

Suppose A_0, A_1 are bounded invertible self-adjoint operators such that $A_0 - A_1$ is compact. Then

$$\frac{1}{\lambda - A_1} - \frac{1}{\lambda - A_0} = \frac{1}{\lambda - A_0}(A_0 - A_1)\frac{1}{\lambda - A_1}$$

is compact, so if γ is a circle in the RHP $Re(\lambda) > 0$ enclosing the spectrum of A_0 and A_1 in the RHP we have

$$\underbrace{\frac{1}{2\pi i}\int_\gamma \frac{d\lambda}{\lambda - A_1}}_{\substack{\text{proj. on} \\ +\text{ spectrum for } A_1}} \equiv \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{d\lambda}{\lambda - A_0}}_{\substack{\text{same for} \\ A_0}} \qquad \text{mod compacts}$$

So we have two projections congruent modulo compacts, which means an index is defined.

Next let D_0, D_1 be unbounded self-adj operators such that the resolvents are compact and such that $D_0 - D_1$ is bounded. Put $A_j = \frac{D_j}{\sqrt{1+D_j^2}}$. Then I would like to prove that $A_0 - A_1$ is compact. More generally I would like to do this when $\frac{x}{\sqrt{1+x^2}}$ is replaced by other approximate sign functions.

The philosophy is the following. We know the C.T.'s $g_j = \frac{1+iD_j}{1-iD_j}$ are congruent modulo compacts, and we want to conclude that $\varphi(g_0)$ and $\varphi(g_1)$ differ by a compact operator for a function φ on the unit circle which has a jump discontinuity at $\zeta = -1$.

To be more specific, put $X_j = iD_j$.

Then
$$g_j = -1 + \frac{2}{1-X_j}$$

and
$$\frac{1}{\lambda - X_0} - \frac{1}{\lambda - X_1} = \frac{1}{\lambda - X_0}(X_0 - X_1)\frac{1}{\lambda - X_1}$$

The typical function φ is $\varphi(\zeta) = \zeta^{1/2}$. Thus
$$g^{1/2} = \varphi(g) = \frac{1+X}{\sqrt{1-X^2}} \qquad \text{if} \qquad g = \frac{1+X}{1-X}$$

Notice that $g_0, g_1 \equiv -1 \mod \mathcal{K}$ iff the resolvents $\frac{1}{\lambda - X_j}$ are compact; this doesn't require $X_1 - X_0$ bounded. Yet this is needed for $g_0^{1/2}$ and $g_1^{1/2}$ to be congruent modulo $\mathcal{K}$.

Central question. Start with X_0 invertible. Then for B small we have a path of invertible operators $X_0 + tB$. Find conditions which imply that the involutions associated to X_0 and $X_0 + tB$ differ by a compact operator. Actually we want a path in the restricted Grassmannian, and we should begin by understanding the tangent vector at the beginning

July 30, 1989

Let A be self-adjoint invertible and let

$$P = \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda - A}\,d\lambda$$

be the projection on the subspace where $A>0$; thus γ is some contour enclosing the positive spectrum. Suppose we have a variation $\dot{A}$ of A; the induced variation of P is

$$\dot{P} = \frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda - A}\dot{A}\frac{1}{\lambda - A}\,d\lambda$$

Let ψ_i be eigenvectors of A with eigenvalues λ_i. We have

$$\langle \psi_2 | \dot{P} | \psi_1 \rangle = \frac{1}{2\pi i}\int \frac{1}{\lambda-\lambda_2}\langle \psi_2|\dot{A}|\psi_1\rangle \frac{1}{\lambda-\lambda_1}\,d\lambda$$

If $\lambda_1, \lambda_2 > 0$, the integrand is holomorphic outside γ, and there is no pole at ∞, so the contour can be shoved to ∞ showing that the integral is zero. Similarly if $\lambda_1,\lambda_2<0$, the integrand is analytic inside γ and we get zero. If $\lambda_1>0$, $\lambda_2<0$ we get

$$\frac{1}{\lambda_2-\lambda_1}\langle\psi_2|\dot{A}|\psi_1\rangle$$

The problem is to control the operator $\dot{P} = (1-P)\dot{P}P + P\dot{P}(1-P)$ knowing something about A and $\dot{A}$. For example, assuming that A has compact resolvant (means eigenvalues of $A \to \pm\infty$) and that $\dot{A}$ is bounded, does it follow that $\dot{P}$ is compact?

Let's consider a matrix a_{mn} $m,n \geq 1$ and sequences λ_m, μ_n $m \geq 1$ of positive numbers. Then

$$\frac{1}{\lambda_m + \mu_n} a_{mn} = \int_0^\infty e^{-t\lambda_m} a_{mn} e^{-t\mu_n} \, dt$$

is the matrix of the operator

$$\int_0^\infty e^{-tL} A e^{-tM} \, dt$$

where $L = \mathrm{diag}(\lambda_m)$, $M = \mathrm{diag}(\mu_n)$, and $\dot{A} = (a_{mn})$

Suppose that A is bounded on ℓ^2 and that $\lambda_m \longrightarrow +\infty$. Then the operator e^{-tL} is compact for $t > 0$, so the operator $e^{-tL} A e^{-tM}$ is compact for $t > 0$. On the other hand the integral converges in norm since if $\lambda_m \geq \varepsilon > 0$, then

$$\int_0^\infty \| e^{-tL} \dot{A} e^{-tM} \| \, dt \leq \int_0^\infty e^{-t\varepsilon} \|A\| dt = \frac{1}{\varepsilon} \|A\|.$$

Suppose now we consider the general situation. Let A and A' be invertible self-adjoint operators such that $A' = A + B$ with B bounded. Assume A^{-1} compact (hence A'^{-1} also).

Then for the projections on the + eigenspaces we have

$$P' - P = \frac{1}{2\pi i} \int_\gamma \frac{1}{\lambda - A'} (B) \frac{1}{\lambda - A} d\lambda$$

Note that $P'(P'-P)P = (P')^2 P - P'P^2 = P'P - P'P = 0$

$(1-P')(P'-P)(1-P) = 0$, so the nonzero blocks of $P'-P$ viewed as an operator from $PH \oplus (1-P)H$ to $P'H \oplus (1-P')H$ are the off-diagonal ones.

Look at $(1-P')(P'-P)P$. Let $A\psi_n = \mu_n \psi_n$ be the pos eigenvalue + eigenvectors for A and similarly $A'\varphi_m = (-\lambda_m)\varphi_m$ be the negative eigenvalue + eigenvectors for A'. Then $P'-P$ is the matrix

$$\langle \varphi_m | P'-P | \psi_n \rangle = \frac{1}{2\pi i} \int_\gamma \frac{1}{\lambda + \lambda_m} \langle \varphi_m | B | \psi_n \rangle \frac{1}{\lambda - \mu_n} d\lambda$$

pole inside γ

$$= \frac{1}{\lambda_m + \mu_n} \langle \varphi_m | B | \psi_n \rangle$$

$$(1-P')(P'-P)P = \int_0^\infty e^{-tL} B e^{-tM} dt$$ is compact by the previous argument.

~~[illegible]~~

We don't need to use the contour integral to get the matrix elements of $P'-P$. We have

$$\langle \varphi_m | B | \psi_n \rangle = \langle \varphi_m | A' - A | \psi_n \rangle = (-\lambda_m - \mu_n) \langle \varphi_m | \psi_n \rangle$$

$$\langle \varphi_m | P'-P | \psi_n \rangle = (0-1) \langle \varphi_m | \psi_n \rangle$$

QED

Notice also that

$$\left\| \int_0^\infty e^{-tL} B e^{-tM} \, dt \right\|_1$$

$$\leq \int_0^\infty \|e^{-tL}\|_1 \, \|B\|_\infty \, \|e^{-tM}\|_\infty \, dt$$

$$= \int_0^\infty tr\left(e^{-tL}\right) dt \, \|B\| = tr\left(L^{-1}\right) \|B\|$$

This shows that $P'-P$ should be trace class when A^{-1} is trace class.

Let's try to calculate the index as $tr\,(P'-P)$. Let $A'\varphi_m = \lambda_m \varphi_m$, $A\psi_n = \mu_n \psi_n$ where $\{\varphi_m\}$, $\{\psi_n\}$ are orthonormal bases. Then

$$P'-P = \sum_{m,n} |\varphi_m\rangle\langle\varphi_m| P'-P |\psi_n\rangle\langle\psi_n|$$

$$(1-P')(P'-P)P = \sum_{\lambda_m<0,\,\mu_n>0} |\varphi_m\rangle \underbrace{\langle\varphi_m| P'-P |\psi_n\rangle}_{-\langle\varphi_m|\psi_n\rangle} \langle\psi_n|$$

$$tr\left((1-P')(P'-P)P\right) = -\sum_{\lambda_m<0,\,\mu_n>0} \langle\varphi_m|\psi_n\rangle\langle\psi_n|\varphi_m\rangle$$

$$= -\sum_{\lambda_m<0,\,\mu_n>0} |\langle\varphi_m|\psi_n\rangle|^2$$

$$= -\sum_{\lambda_m<0,\,\mu_n>0} \frac{|\langle\varphi_m|B|\psi_n\rangle|^2}{(-\lambda_m+\mu_n)^2}$$

Similarly

$$\cancel{P(P'-P)(1-P') = \sum_{\mu_n>0,\ \lambda_m<0} |\psi_n\rangle \underbrace{\langle\psi_n|P'-P|\varphi_m\rangle}_{\langle\psi_n|\varphi_m\rangle} \langle\varphi_m|}$$

~~and~~ $\cancel{\langle\psi_n|B|\varphi_m\rangle =}$

$$P'(P'-P)(1-P) = \sum_{\lambda_m>0,\ \mu_n<0} |\varphi_m\rangle \underbrace{\langle\varphi_m|P'-P|\psi_n\rangle}_{\langle\varphi_m|\psi_n\rangle} \langle\psi_n|$$

$$\langle\varphi_m|B|\psi_n\rangle = \langle\varphi_m|A'-A|\psi_n\rangle = (\lambda_m-\mu_n)\langle\varphi_m|\psi_n\rangle$$

Thus

$$\mathrm{tr}\left(P'(P'-P)(1-P)\right) = \sum_{\substack{\lambda_m>0\\ \mu_n<0}} \left|\langle\varphi_m|\psi_n\rangle\right|^2$$

$$= \sum_{\substack{\lambda_m>0\\ \mu_n<0}} \frac{|\langle\varphi_m|B|\psi_n\rangle|^2}{(\lambda_m-\mu_n)^2}$$

so

$$\text{Index} = \mathrm{tr}\,(P'-P)$$

$$= \left\{ \sum_{\substack{\lambda_m>0\\ \mu_n<0}} - \sum_{\substack{\lambda_m<0\\ \mu_n>0}} \right\} \frac{|\langle\varphi_m|B|\psi_n\rangle|^2}{(\lambda_m-\mu_n)^2}$$

Same formula comes out of the contour integral. This formula is probably not useful since it is not clear why this should be an integer.

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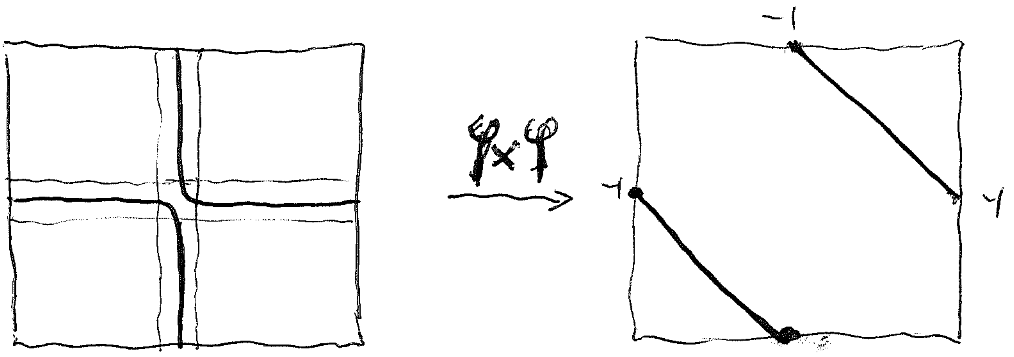
Correction: Let us consider the function on U_n

$$f: g \longmapsto \det(\varphi(g)) = \prod_{j=1}^{n} \varphi(\zeta_j)$$

where $\varphi: S^1 \longrightarrow S^1$ is a degree 1 map which wraps a small arc containing 1 onto S^1 so that $\varphi(1) = -1$ and such that φ takes the endpoints of the arc to $+1$, and such that φ takes the complement of the arc to $+1$.

If $f(g)$ is close to -1, then g has to have at least one eigenvalue close to 1. It seems that -1 is a regular value for f, and that $f^{-1}\{-1\}$ is a smoothing of the divisor Z.

Picture in case of the 2-torus



where $f = -1$ where $\det = -1$. where $\zeta_1 \zeta_2 = -1$.

In any case we see that the 1-form

$$d\log f = \operatorname{tr} d(\log \varphi) = \operatorname{tr} \frac{d\varphi}{\varphi}$$

on U_n is supported in the nbd of Z where some eigenvalue ζ belongs to the little arc around 1.

The issue now is find a formula for the index. The setting is that we have two skew adjoint operators X' X differing by a bounded operator, and both are invertible. ∎

July 31, 1989

Yesterday I corrected an earlier misconception, and I now understand the link between the singular divisor and the 1-forms representing the determinant class.

Our main problem is to understand the relative index of two invertible skew adjoint operators X_0, X_1 such that X_0^{-1}, X_1^{-1} are compact and $X_1 - X_0$ is bounded. We know this is well defined because we showed that the two spectral projections for the spectrum in the UHP differ by a compact operator. Our problem is now to find a means to calculate it, say an analogue of the McKean–Singer formula.

Let's consider a path X_t joining X_0, X_1. The index should be the intersection number of the path with the singular divisor. To calculate it we can take a closed 1-form coming from the unitary group supported in a nbd of the singular divisor not containing X_0, X_1, and then integrate this 1-form over the path. Such a 1-form is

$$tr\left(\varphi\left(\tfrac{X}{i}\right) d\tfrac{X}{i}\right)$$

where $\varphi(x)$ is a smooth approximation to $\delta(x)$.

Such a compactly supported φ is not likely to yield a good analytical formula. Instead we want to work with something like

$$\frac{e^{-x^2}}{\sqrt{\pi}} \quad \text{or} \quad \frac{1}{\pi}\frac{1}{(1+x^2)}$$

The problem then becomes the fact that there are endpoint contributions - we don't have something topologically defined.

Let $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(x)\,dx$.

Formally we have

$$\operatorname{tr}\left(\Phi\left(\tfrac{1}{i}X_1\right) - \Phi\left(\tfrac{1}{i}X_0\right)\right)$$

$$= \int_0^1 \operatorname{tr}\left(\partial_t \Phi\left(\tfrac{1}{i}X_t\right)\right) dt$$

$$= \int_0^1 \operatorname{tr}\left(\varphi\left(\tfrac{1}{i}X_t\right)\tfrac{1}{i}\dot{X}_t\right) dt = \int_0^1 \operatorname{tr}\, \varphi\left(\tfrac{1}{i}X_t\right)\tfrac{1}{i}dX_t$$

Now the last expression is well-defined at least if φ has compact support. However we have seen already that

$$\Phi\left(\tfrac{1}{i}X_1\right) - \Phi\left(\tfrac{1}{i}X_0\right) \quad \text{and} \quad \partial_t \Phi\left(\tfrac{1}{i}X_t\right)$$

need not be of trace class. In fact if the support of φ is a very small interval around 0 and X is invertible, then $\Phi(\frac{1}{i}X)$ is the spectral projection for the UHP spectrum of X.

An obvious question is the following. We know that the ~~[illegible]~~ relative index for a pair of projections e', e with $(e'-e)^m$ of trace class is

$$\operatorname{tr}(e'-e)^{2n+1} \qquad 2n+1 \geq m$$

So is there a more general formula for the index, something multilinear in X_1, X_0?

Suppose given two ~~idempotents~~ e', e with $e'-e$ compact. How many ways can one calculate the index? Assume $e'-e \in \mathcal{L}^p$. We have a trace on the Cuntz algebra, really on a power of the folding ideal, and we are pairing this trace with the canonical K_0-class represented by $[e']-[e]$ on the folding ideal. The trace is given by a family of cochains on $A = \mathbb{C}e$ which constitute a big cyclic cocycle of some sort. It would seem then that we just have to look at the different representatives for the cyclic cycle class corresponding to the K-class.

August 1, 1989

Let's consider the the affine space $\mathcal{A}$ of skew-adjoint operators $X = X_0 + B$ with B bdd and where X_0 is invertible with X_0^{-1} of trace class.

On $\mathcal{A}$ we have various determinant functions. First consider some intrinsically defined ones. Let $g_t = \frac{t+X}{t-X}$ $\qquad -g_t = \frac{t+X}{-t+X}$

$$\frac{1-g_t}{2} = \frac{1 - \frac{t+X}{t-X}}{2} = \frac{-X}{t-X}$$

$$t > 0$$

$$\frac{1-g_t^{-1}}{2} = \frac{1 - \frac{t-X}{t+X}}{2} = \frac{X}{t+X}$$

These are 1+ trace class, so have determinants

$$\det(-g_t) = \det\left(\frac{t+X}{-t+X}\right)$$

$$\det\left(\frac{1-g_t}{2}\right) = \det\left(\frac{X}{-t+X}\right)$$

which should satisfy

$$d \log \det(-g_t) = \operatorname{tr}\left(\frac{1}{t+X}dX + \frac{1}{t-X}dX\right)$$
$$= \operatorname{tr}\left(\frac{2t}{t^2-X^2}dX\right)$$

etc.

Next we have determinants defined using the basepoint X_0, hence which are well-defined up to scalar factors:

$$\det(X_0^{-1}X) \qquad\qquad d\log\det(X_0^{-1}X) = \operatorname{tr}\left(\frac{1}{X}dX\right)$$

$$d \log \det\left(X_0^{-1}(t+X)\right) = \operatorname{tr}\left(\frac{1}{t+X}\,dX\right)$$

Here is the geometry of the singular set. First of all we have the complex-valued function

$$\det\left(\frac{1-g_t}{2}\right) = \det\left(\frac{X}{-t+X}\right)$$

describing the divisor. We have the real function

$$\det(X_0^{-1}X)$$

describing the same divisor. Dividing gives

$$\frac{\det\left(\frac{X}{-t+X}\right)}{\det(X_0^{-1}X)} = \det\left(\frac{1}{-t+X}X_0^{-1}\right)$$

and if we divide by its conjugate, we get

$$\det\left(\frac{1}{-t+X}X_0^{-1}\right)\Big/\det\left(\frac{1}{+t+X}X_0^{-1}\right) = \det\left(\frac{t+X}{-t+X}\right)$$

$$= \det(-g_t)$$

which gives the map $\mathcal{a} \longrightarrow S^1$.

Here is how to obtain the index of an invertible operator X_1 in $\mathcal{a}$ relative to X_0. We consider a path X_s $0 \le s \le t$ joining X_0 to X_1. We integrate

$$\operatorname{tr}\left(\frac{1}{t+X}\,dX\right)$$

over this path and obtain

$$\log \det\left(X_0^{-1}(t+X_1)\right) - \log \det\left(X_0^{-1}(t+X_0)\right)$$

with a definite branch of the logarithm chosen.

~~Now let $t \downarrow 0$ and the limit should be $\pm(\pi i)$ Index. This is roughly because $\det(X_0^{-1}X)$ is real.~~

Now let $t \downarrow 0$ and we should obtain

$$\log\left|\det(X_0^{-1}X_1)\right| + (\pi i)\,\mathrm{Index}$$

The reason is that if we took $t<0$ and let $t \uparrow 0$, then we should obtain the same real part but opposite imaginary part. The difference should be $[\log(-g_s)]_0^1 = (2\pi i)$ Index.

The next stage might be to obtain some kind of formula for the index.

Having discussed the $\mathcal{L}^1$-case where the determinants $\det\left(\frac{X}{t+X}\right)$ exist, we now ~~wish~~ to consider the $\mathcal{L}^p$-case and find renormalized determinants. First note that

$$d\log\det\left(\frac{X}{t+X}\right) = \operatorname{tr}\left\{\left(\frac{1}{X}-\frac{1}{t+X}\right)dX\right\}$$

where $\frac{1}{X}-\frac{1}{t+X} = \frac{t}{X(t+X)}$. The latter is trace class when X^{-1} is Hilbert-Schmidt, so it ought to be possible to construct in the $\mathcal{L}^2$-case functions unique up to scalar factors

$$\det\left(\frac{X}{t+X}\right) \qquad \det\left(\frac{X}{-t+X}\right)$$

whose ratio would satisfy

$$d\log\det\left(\frac{t+X}{-t+X}\right) = \operatorname{tr}\left(\frac{1}{t+X}-\frac{1}{-t+X}\right)dX$$

$$= \operatorname{tr}\left(\frac{2t}{t^2-X^2}dX\right)$$

Next let's try to find a determinant for. corresponding to the form

$$\operatorname{tr}\frac{4t^3}{(t^2-X^2)^2}dX$$

which also represents the determinant class in $H^1(U,\mathbb{Z})$. We have

$$\frac{4t^3}{(t^2-X^2)^2} = \frac{t}{(t-X)^2}+\frac{1}{t-X}+\frac{t}{(t+X)^2}+\frac{1}{t+X}$$

which suggests looking for a determinant fn $\widetilde{\det}(X)$ satisfying

$$d\log\left(\widetilde{\det}(X)\right) = \operatorname{tr}\left(\frac{1}{X} - \frac{1}{t+X} - \frac{t}{(t+X)^2}\right) dX$$

But notice the geometric series

$$\frac{1}{t+X} + \frac{t}{(t+X)^2} + \cdots = \frac{1}{t+X} \cdot \frac{1}{1-\frac{t}{t+X}} = \frac{1}{X}$$

so that

$$\frac{1}{X} - \left(\frac{1}{t+X} + \cdots + \frac{t^n}{(t+X)^{n+1}}\right) = \frac{t^{n+1}}{(t+X)^{n+2}} \cdot \frac{1}{1-\frac{t}{t+X}}$$

$$= \frac{t^{n+1}}{X(t+X)^{n+1}}$$

Here is the meaning of this determinant function. Integrating

$$\frac{1}{z} = \frac{1}{t+z} + \frac{t}{(t+z)^2} + \cdots$$

gives

$$\log z = \log(t+z) \underbrace{- \frac{t}{t+z} - \frac{1}{2}\frac{t^2}{(t+z)^2} - \cdots}_{-\log\left(1-\frac{t}{t+z}\right)}$$

or

Thus

$$\widetilde{\det}(X) = \prod_k \left\{\left(\frac{z_k}{t+z_k}\right) e^{-\frac{t}{t+z_k} - \frac{1}{2}\left(\frac{t}{t+z_k}\right)^2 - \cdots - \frac{1}{n}\left(\frac{t}{t+z_k}\right)^n}\right\}$$

and we recognize the Weierstrass trick.

August 3, 1989

Link with the η-invariant. This is the traditional phase type function [redacted] used in studying spectral flow. Formulas: $\eta_A(s) = tr\left(\frac{A}{|A|}|A|^{-s}\right)$

$$\delta\eta_A(s) = (-s)\, tr\left(\delta A\, |A|^{-s-1}\right)$$

$$= -s\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty tr\left(\delta A\, e^{-tA^2}\right) t^{+\left(\frac{s+1}{2}\right)}\frac{dt}{t}$$

Normally $tr\left(\delta A\, e^{-tA^2}\right)$ has an asymptotic expansion [redacted] with powers $t^{k+\frac{1}{2}}$, this is the case for Dirac operators on odd manifolds. If $a_{-1/2}t^{-1/2}$ is the term with $t^{-1/2}$, then we have

$$\delta\eta_A(0) = -\frac{2}{\sqrt{\pi}}\, a_{-1/2}$$

Now [redacted] $\eta_A(0)$ jumps by 2 as an eigenvalue crosses 0, so $\frac{1}{2}\eta_A(0)$ is better. Then we have

$$\boxed{d\left(\frac{\eta_A(0)}{2}\right) = -\frac{1}{\sqrt{\pi}} \,\text{"}\lim_{t\downarrow 0}\text{"}\, tr\left(e^{-tA^2}\sqrt{t}\,dA\right)}$$

where "$\lim_{t\downarrow 0}$" means the constant term in the asymptotic expansion. Over the circle we really do have a limit.

[redacted] Recall how this η-invariant looks for $\frac{1}{2\pi i}\frac{d}{d\theta}+a$ on S^1. If $a=0+$ then $\eta_A(0)$ is $+1$, then it decreases linearly to -1 for $a=1-$. It is zero for $a=\frac{1}{2}$.

In the general case of arbitrary Diracs on S^1, one has the monodromy map to the group U_n, which describes equivalence classes of connection modulo gauge transformations fixed at the basepoint. One then pulls back the determinant information we have on U_n. So we should have a complex determinant given by $\det(M-1)$ where M is the monodromy.

Graeme points out that the absolute value of the determinant can be defined by zeta function methods. Recall some formulas

$$\zeta_A(s) = \operatorname{tr}\left(|A|^{-s}\right) = \operatorname{tr}\left((A^2)^{-s/2}\right)$$

$$= \frac{1}{\Gamma(s/2)} \int_0^\infty \operatorname{tr}\left(e^{-tA^2}\right) t^{s/2} \frac{dt}{t}$$

$$\delta \zeta_A(s) = \frac{1}{\Gamma(s/2)} \int_0^\infty \operatorname{tr}\left(e^{-tA^2}(-t)(\delta A\, A + A\,\delta A)\right) t^{s/2} \frac{dt}{t}$$

$$= \frac{-2}{\Gamma(s/2)} \int_0^\infty \underbrace{\operatorname{tr}\left(e^{-tA^2} A\, \delta A\right)}_{\frac{d}{dt}\left(-\operatorname{tr}\left(e^{-tA^2}A^{-1}\delta A\right)\right)} t^{s/2}\, dt$$

$$= \frac{-2}{\Gamma(s/2)}(s/2) \int_0^\infty \operatorname{tr}\left(e^{-tA^2} A^{-1} \delta A\right) t^{s/2} \frac{dt}{t}$$

In good cases

$$\delta \zeta_A'(0) = \left.\frac{\delta \zeta_A(s)}{s}\right|_{s=0} = \left[-\frac{1}{\Gamma(s/2)} \int_0^\infty \operatorname{tr}\left(e^{-tA^2}A^{-1}\delta A\right) t^{s/2}\frac{dt}{t}\right]_{s=0}$$

$$= -\operatorname{tr}\left(e^{-tA^2}A^{-1}\delta A\right)_{\text{coeff of } t^0}$$

On the circle with $A = \frac{1}{2\pi i}\partial_\theta + a$ we should have

$$\operatorname{tr}\left(e^{-tA^2}A^{-1}\delta A\right)\Big|_{\substack{\text{coeff}\\ \text{of } t^0}} = \sum_{n\in\mathbb{Z}}{}' \frac{1}{a+n}\,da \qquad \text{Eisenstein normalized}$$

$$= \frac{\pi\cos\pi a}{\sin\pi a}\,da = d\log|\sin\pi a|$$

> **Formulas**
>
> $$-d\left(\zeta_A'(0)\right) = \operatorname{tr}\left(e^{-tA^2}A^{-1}dA\right)_{\text{coeff } t^0}$$
>
> $$-d\left(\frac{\eta_A(0)}{2}\right) = \operatorname{tr}\left(\sqrt{\frac{t}{\pi}}\,e^{-tA^2}dA\right)_{\text{coeff } t^0}$$

Next we want to examine the Hilbert-Schmidt case more thoroughly. Here we can define a real-valued determinant by means of the 1-form

$$\operatorname{tr}\left(\frac{1}{X} - \frac{1}{X_0}\right)dX$$

Notice that changing basepoint leads to the difference

$$\operatorname{tr}\left(\left(\frac{1}{X_1} - \frac{1}{X_0}\right)dX\right) = d\operatorname{tr}\left\{\left(\frac{1}{X_1} - \frac{1}{X_0}\right)B\right\}$$

which modifies the determinant by the exponential of a linear function on the affine space.

Notice that the 1-form is real since

$$\overline{\operatorname{tr}\left(\frac{1}{X} - \frac{1}{X_0}\right)dX} = \operatorname{tr}(dX)^*\left(\frac{1}{X} - \frac{1}{X_0}\right)^* = \operatorname{tr}(-dX)\left(-\frac{1}{X} + \frac{1}{X_0}\right)$$

$$= \operatorname{tr}\left(\frac{1}{X} - \frac{1}{X_0}\right) dX$$

Let's review the basic geometry again. The issue is that we have a real divisor which leads to a class mod 2 normally, but in the present case there is more structure and the cohomology class is integral. The extra structure is a sort of framing adapted somehow to allow for the singularities of the divisor.

The study of spectral flow for one-param. families of unbdd skew adjoint operators leads via Cayley transform to asking about the link between the determinant class in $H^1(U_n, \mathbb{Z})$ and the divisor $\det(1-g) = 0$.

Elements of $H^1(M, \mathbb{Z})$ can be represented by maps $f : M \to S^1$ or by framed submanifolds $N \subset M$ of codimension 1. The link between these pictures is given by Pontryagin-Thom: a submanifold N can be obtained as the inverse image under f of a regular value.

The class in $H^1(U_n, \mathbb{Z})$ we want is represented by $\det : U_n \to S^1$ whose inverse images are the cosets of SU_n. How do we get to the divisor Z given by $\det(1-g) = 0$?

We can use spectral deformation of U_n.

If we take $\varphi : S^1 \longrightarrow S^1$ homotopic to the identity, then extending φ to U_n in the obvious way using functional calculus, we obtain a map $\varphi : U_n \to U_n$ homotopic to the identity. The det coh. class is also represented by $g \mapsto \det \varphi(g)$.

Let take $\varphi(\xi)$ to satisfy

$$\overline{\varphi(\xi)} = \varphi(\bar{\xi}) \qquad \varphi(1) = -1, \quad \varphi(-1) = 1$$

whence $\frac{1}{2\pi i} d \log \varphi = \omega$ is a real 1-form on the unit circle of total integral 1 such that

$$\varphi(\xi) = e^{2\pi i \int_{-1}^{\xi} \omega}$$

and also ω is invariant wrt $\xi \leftrightarrow \bar{\xi}$. Now if I replace S^1 by the line in the C.T. way: $\xi = \frac{1+ix}{1-ix}$, then

$$\varphi(\xi) = e^{2\pi i \int_{-\infty}^{x} \omega}$$

where $\int_{-\infty}^{x} \omega$ is a kind of approximation to the Heaviside function.

Next suppose ω concentrated near $\xi = 1$ or $x = 0$. I'll suppose $\omega \geq 0$. If $\det \varphi(g)$ is close to -1, then because g has only n eigenvalues, at least one of these eigenvalues must be near $\xi = 1$ where $\omega > 0$, so varying this eigenvalue to first order will produce a first order change in $\det \varphi(g)$.

Then -1 is a regular value for $\det \varphi(g)$, and the inverse image is a hypersurface contained in a nbd of $Z = \{ g \mid \det(1-g) = 0 \}$.

Simplest example

$$\varphi(\xi) = \exp\left\{ i \int_{-\infty}^{x} \frac{2t}{t^2+x^2}\, dx \right\}$$

$$= \exp\left\{ \int_{-\infty}^{x} \left(\frac{1}{t+ix} + \frac{1}{t-ix} \right) i\, dx \right\}$$

$$= \frac{t+ix}{-t+ix} = \frac{t + \frac{\xi-1}{\xi+1}}{-t + \frac{\xi-1}{\xi+1}}$$

$$= \frac{t(\xi+1) + \xi - 1}{-t(\xi+1) + \xi - 1} = \frac{(1+t)\xi - (1-t)}{(1-t)\xi - (1+t)}$$

$$= \frac{\xi - a}{a\xi - 1} \quad \text{where} \quad a = \frac{1-t}{1+t} \uparrow 1 \text{ as } t \downarrow 0$$

August 4, 1989

I am trying to give a thorough discuss of the topology + geometry of the determinant class in $H^1(U_n, \mathbb{Z})$ and the subvariety $Z = \{g \in U_n \mid \det(g-1) = 0\}$

So far I have linked the usual representative $g \longmapsto \det(g)$ for the determinant class to Z via the deformation

$$r_a(g) = \det\left(\frac{g-a}{ag-1}\right) \qquad 0 \le a < 1.$$

The level set $r_a(g) = -1$ is the transform of $\det(g) = -1$ (coset of SU_n), under the Cayley transform flow. As $a \uparrow 1$, the submanifold $r_a(g) = -1$ approaches Z.

However we want to deal directly with Z and with determinants.

Z itself is a real divisor in U_n, that is, it is the zero set of a section of a real line bundle. The real line bundle and section are unique up to canonical isomorphism. These are the determinant line bundle and canonical section in the present situation.

This real line bundle determines a class in $H^1(U_n, \mathbb{Z}/2)$. Here we have an integral class. From the exact sequence

$$H^1(U_n, \mathbb{Z}) \longrightarrow H^1(U_n, \mathbb{Z}/2) \longrightarrow H^2(U_n, \mathbb{Z})$$

we learn that the complexification of this real line bundle is trivial, at least it has a trivialization unique up to homotopy.

What is the meaning of such a trivialization? First of all it gives a complex valued function when applied to the canonical section of the determinant line bundle. This complex determinant is unique up to exponential functions of the form e^f. Also the trivialization determines a map $U_n \longrightarrow S^1$, because it means that the real determinant line bundle is spanned by two sections, hence is induced from the canonical line bundle over $\mathbb{P}_1(\mathbb{R}) = S^1$. In concrete terms given the complex determinant function $D(g)$ one forms the function $D(g)/\overline{D(g)}$, which has a smooth extension over Z.

August 6, 1989

Determinant class $\in H^1(U, \mathbb{Z})$.

Up to homotopy there is a unique map $U \longrightarrow S^1$ representing this class. Because of the fibration

$$S^1 \longrightarrow B(\mathbb{Z}/2) \longrightarrow B(S^1)$$
$$\{\pm 1\}$$

the class is represented by a real line bundle whose complexification is trivial.

If we realize U (the space rep. K^1) by unitaries g on a Hilbert space such that $1 \notin$ essential spectrum (i.e. $1-g$ Fredholm), then the real line bundle along with a canonical section may be constructed as follows. For $0 < a < \pi$, let W_a be the open set of g having essentially spectrum outside the arc from e^{-ia} to e^{ia} and which do not have the endpoints $e^{\pm ia}$ as eigenvalue. Define a real-valued function f_a on W_a by letting its value at g be the product

$$\prod_j x_j = \text{“}\det \tfrac{1}{i} \log g \text{ rest. to } (e^{-ia}, e^{ia})\text{”}$$

where e^{ix_j} are the eigenvalues of g in the arc from e^{-ia} to e^{ia} (counted according to multiplicity). Then on $W_a \cap W_{a'}$ we see that $\frac{f_{a'}}{f_a}$ is invertible. Thus we have a Cech 1-cocycle defining a real line bundle, and the f_a's give a canonical section.

Now this determinant line bundle

is unique up to canonical isomorphism, because of the canonical section. It does not seem that its complexification is canonically trivial. To trivialize the complexification we construct a complex-valued determinant function f such that f/f_a is invertible on W_a.

Suppose we work with unitaries g such that $g+1 \in \mathcal{L}^p$. We construct a determinant fn. in the form

$$\det f(g) = \prod f(\xi_j)$$

where $f: S^1 \longrightarrow \mathbb{C}$ is chosen suitable. We want f to be smooth, to have a simple zero at $\xi = 1$ and no other zeroes, and such that $1-f$ ~~vanishes~~ to high order ($>>p$) at $\xi = -1$. Thus if $f = 1-h$ we want h to have a high order zero at $\xi = -1$ and to satisfy $h(0) = 1$, $h'(0) \neq 0$.

The simplest example seems to be

$$h(\xi) = \left(\frac{1+\xi}{2}\right)^n \qquad n \geq 1.$$

$$h'(\xi) = n\left(\frac{1+\xi}{2}\right)^{n-1} \frac{1}{2} \qquad h'(1) = \frac{n}{2} > 0.$$

Note that $\left|\frac{1+\xi}{2}\right| < 1$ unless $\xi = 1$, so that $f(\xi) = 1 - \left(\frac{1+\xi}{2}\right)^n$ vanishes only at $\xi = 1$.

In the C.T. picture $\xi = \dfrac{1+ix}{1-ix}$

$$\frac{1+\xi}{2} = \frac{1}{1-ix} = \frac{i}{i+x}$$

and

$$f(\xi) = 1 - \left(\frac{i}{i+x}\right)^n = 1 - \left(\frac{t}{t+x}\right)^n$$

if $t = i$. Recall the other Weierstrass candidate:

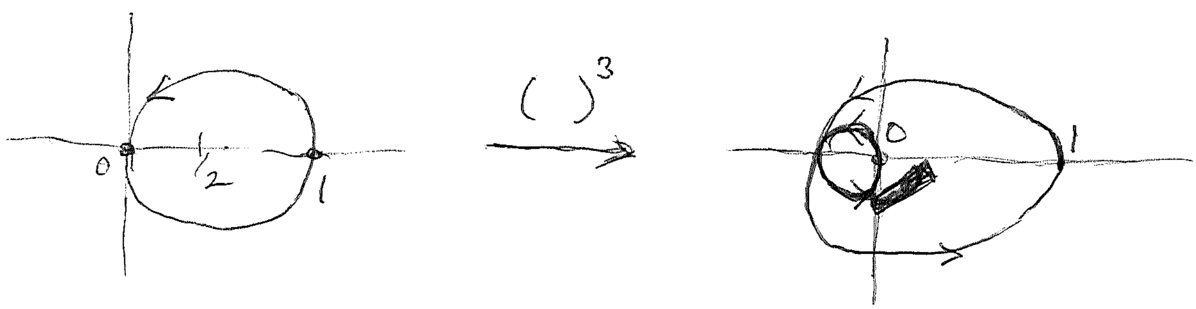
$$f(x) = \frac{x}{t+x}\, e^{\frac{1}{2}\left(\frac{t}{t+x}\right) + \frac{1}{3}\left(\frac{t}{t+x}\right)^2 + \cdots + \frac{1}{n}\left(\frac{t}{t+x}\right)^{n-1}}$$

satisfying

$$\frac{f'}{f}(x) = \frac{1}{x} - \frac{1}{t+x} - \frac{t}{(t+x)^2} - \cdots - \frac{t^{n-1}}{(t+x)^n}$$

$$= \cancel{\frac{1}{x}} - \frac{\left(\cancel{\frac{1}{t+x}} - \frac{t^n}{(t+x)^{n+1}}\right)}{1 - \frac{t}{t+x}} = \frac{t^n}{x(t+x)^n}$$

Disadvantage with $f(\xi) = 1 - \left(\frac{1+\xi}{2}\right)^n$ is that as $\xi \longrightarrow -1$ the phase isn't monotone for $n \geq 3$. For example



<u>Discussion</u>: In order to understand all this stuff, the important example is $U(1) = S^1$. The determinant line bundle can be described as follows. There is the real line bundle whose sections are real smooth functions on the circle vanishing at 1. The dual line bundle is the determinant line bundle. Its sections are smooth real functions h on $S^1 - \{1\}$ such that near 1, the function θh extends smoothly across 1. The canonical section is given by the function $h \equiv 1$.

To trivialize the determinant line bundle $\otimes \mathbb{C}$ one gives a smooth complex-valued function f on S^1 with a simple zero at 1 and no other zeros. Such a trivialization is unique up to an invertible complex function on S^1, and these fall into contractible components indexed by the degree.

How do I explain which sort of trivialization is correct? The condition is that $f/\bar{f} : S^1 \longrightarrow S^1$ be of degree 1. For example if $f(\zeta) = \frac{1-\zeta}{2}$, then

$$\frac{f(\zeta)}{\bar{f}(\zeta)} = \frac{1-\zeta}{1-\zeta^{-1}} = \zeta\,\frac{1-\zeta}{\zeta-1} = -\zeta$$

has degree 1.

Thus the other condition on $f(\zeta)$ is this degree condition, which means that

$$\frac{2f(\zeta)}{1-\zeta} = e^{w(\zeta)}$$

for w a smooth complex function on S^1!

Now for $f(\zeta) = 1 - \left(\frac{1+\zeta}{2}\right)^n$. This is a polynomial whose roots are

$$\frac{1+\zeta}{2} = e^{2\pi i \frac{k}{n}} \qquad \zeta = 2e^{2\pi i \frac{k}{n}} - 1$$

Except for the root $\zeta = 1$, these lie outside the unit circle, so

$$\frac{2f(\zeta)}{1-\zeta}$$

is a polynomial with roots outside S^1, so it is of degree zero. Ex. $n=2$

$$1 - \left(\frac{1+\zeta}{2}\right)^2 = \frac{(1-\zeta)}{2}\frac{(\zeta+3)}{2}$$

So we have seen that a trivialization of the complexification of the determinant line bundle is given by a function

$$f(\zeta) = \left(\frac{1-\zeta}{2}\right) e^{w(\zeta)}$$

where w is a smooth complex function on S^1. Now we want $f(-1) = 1$ to hight order. Since

$$\frac{1-\zeta}{2} = 1 - \left(\frac{1+\zeta}{2}\right)$$

the obvious Weierstrass choice for f is

$$\boxed{f(\xi) = \left(\frac{1-\xi}{2}\right) e^{\left(\frac{1+\xi}{2}\right) + \frac{1}{2}\left(\frac{1+\xi}{2}\right)^2 + \cdots + \frac{1}{n}\left(\frac{1+\xi}{2}\right)^n}}$$

~~[illegible]~~ e.g. $\xi = \dfrac{1+ix}{1-ix}$

$$\frac{1-\xi}{2} = \frac{-ix}{1-ix} = \frac{x}{i+x} \quad \Big| \quad \frac{1+\xi}{2} = \frac{1}{1-ix} = \frac{i}{i+x}$$

$$\boxed{f(x) = \left(\frac{x}{i+x}\right) \exp\left\{\left(\frac{i}{i+x}\right) + \frac{1}{2}\left(\frac{i}{i+x}\right)^2 + \cdots + \frac{1}{n}\left(\frac{i}{i+x}\right)^n\right\},}$$

August 7, 1989

Consider $X_t = X_0 + tB$. Then

$$\frac{1}{\lambda - X_t}\dot{X}_t = \frac{1}{\lambda - X_0 - tB}B$$

$$= \frac{1}{1 - t\frac{1}{\lambda - X_0}B}\,\frac{1}{\lambda - X_0}B = \sum_{n\geq 1}\left(\frac{1}{\lambda - X_0}B\right)^n t^{n-1}$$

so $$\int_0^t \frac{1}{\lambda - X_t}\dot{X}_t\,dt = \sum_{n\geq 1}\frac{t^n}{n}\left(\frac{1}{\lambda - X_0}B\right)^n$$

$$= -\log\left(1 - t\frac{1}{\lambda - X_0}B\right)$$

$$= -\log\left(\frac{1}{\lambda - X_0}(\lambda - X_t)\right)$$

where these series converge for $\left\|\frac{t}{\mathrm{Re}\lambda}B\right\| < 1$.

This is intriguing because normally one doesn't expect to have for a path g_t starting with $g_0 = 1$

$$g_t = \exp\left\{\int_0^t g_t^{-1}\dot{g}_t\,dt\right\}$$

rather there should be a time ordered integral.

What happens for the above is ~~we~~ have

$$-\log(1 - Z) = \sum_{n\geq 1}\frac{1}{n}Z^n$$

where $Z = tC$ with $C = \frac{1}{\lambda - X_0}B$ constant.

Thus we are in a commutative situation.

Recall that for $\lambda \neq 0$

$$tr\left(\int_0^1 \frac{2\lambda}{\lambda^2 - X_t^2}\dot{X}_t\,dt\right)/2\pi i$$

should given in the $\mathcal{L}^2$ case the index of X_1 relative to X_0, assuming X_1 is invertible. More we expect the limit to be unnecessary where $X_1 = g^{-1}X_0 g$. Therefore in this case ~~we~~ seem to ~~[illegible]~~ be able to compute the index by taking the imaginary part of

$$\int_0^1 \frac{1}{\lambda - X_t}\dot{X}_t\,dt = \sum_{n\geq 1}\frac{1}{n}\left(\frac{1}{\lambda - X_0}B\right)^n$$

and then the trace. This series will converge for $\lambda \gg 0$.

We have
$$B = g^{-1}X_0 g - X_0$$
$$= g^{-1}[X_0, g] = -[X_0, g^{-1}]g$$

We recall that the cyclic cocycle formula which we think ought to give the index should involve terms like

$$tr\left\{g\frac{1}{\lambda - X_0^2}[X_0, g^{-1}]\frac{1}{\lambda - X_0^2}\cdots\cdots[X_0, g^{-1}]\frac{1}{\lambda - X_0^2}\right\}$$

$$tr\left\{g^{-1}\frac{1}{\lambda - X_0^2}[X_0, g]\frac{1}{\lambda - X_0^2}\cdots\cdots[X_0, g]\frac{1}{\lambda - X_0^2}\right\}$$

where g, g^{-1} alternate.

August 8, 1989

Consider a self-adjoint operator A_0 which is invertible and such that A_0^{-1} is compact, and $A_1 = A_0 + B$ a bounded perturbation which is invertible. We want the index of A_1 relative to A_0. This is the number of times counted properly that the path

$$(1-t)A_0 + tA_1$$

crosses the singular divisor. Now

$$\big((1-t)A_0 + tA_1\big)v = 0$$

$$\left(\frac{1-t}{t} + A_0^{-1}A_1\right)v = 0$$

as t goes from 0 to 1, $\frac{1-t}{t}$ goes from $+\infty$ to 0. This links the index to the number of negative eigenvalues of $A_0^{-1}A_1$. Note that

$$\left(A_0^{-1}A_1\right)^* = A_1A_0^{-1} = A_0\left(A_0^{-1}A_1\right)A_0^{-1}$$

so $\left(A_0^{-1}A_1\right)^*$ is conjugate to $A_0^{-1}A_1$, which implies that the spectrum is invariant under complex conjugation. One can also see this because $A_0^{-1}A_1$ is hermitian with respect to an indefinite hermitian form

$$\langle x \,|\, A_0(A_0^{-1}A_1)y\rangle = \langle x \,|\, A_1 y\rangle = \langle A_1 x \,|\, y\rangle$$

$$= \langle (A_0^{-1}A_1)x \,|\, A_0 y\rangle$$

On the other hand suppose we consider the path of self-adjoint ops:

$$\begin{pmatrix} A_0 & t \\ t & -A_1 \end{pmatrix} \qquad t > 0$$

This is singular when

$$\begin{pmatrix} A_0 & t \\ t & -A_1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad \begin{matrix} A_0 u + tv = 0 \\ tu - A_1 v = 0 \end{matrix}$$

$$u = \frac{1}{t} A_1 v \qquad A_0\left(\frac{1}{t} A_1 v\right) + tv = 0$$

$$\begin{cases} A_0 A_1 v + t^2 v = 0 \\ A_1 A_0 u + t^2 u = 0. \end{cases}$$

Thus we encounter the negative eigenspaces of $A_0 A_1$ or $A_1 A_0$.

Let us consider an example where A_0, A_1 commute:

$$A_j = \frac{1}{2\pi i}\partial_\theta - c_j \qquad j = 0, 1.$$

Then we have the simultaneous eigenvectors $e^{in\theta}$ on which A_j has the eigenvalue $n - c_j$.

$$\cancel{\det\left[\lambda - \begin{pmatrix} a_0 & t \\ t & -a_1 \end{pmatrix}\right] = \lambda^2 - (a_0 - a_1)\lambda + (-a_0 a_1 - t^2)}$$

~~This will be singular for some t positive if and only if a_0, a_1 have opposite sign, in which case $t = \sqrt{-a_0 a_1}$~~

Thus $\begin{pmatrix} A_0 & t \\ t & -A_1 \end{pmatrix}$ is made of blocks

$$\begin{pmatrix} a_0 & t \\ t & -a_1 \end{pmatrix} \quad \text{where } a_j = n - c_j$$

We have

$$\det \begin{pmatrix} a_0 & t \\ t & -a_1 \end{pmatrix} = -a_0 a_1 - t^2$$

If a_0, a_1 have the same sign, this remains < 0. If a_0, a_1 have opposite sign, then it starts positive and becomes negative. Thus the spectral flow is always in the same direction; it is ~~independent~~ the number of negative eigenvalues of $A_0^{-1} A_1$ (or $A_0 A_1$), and it doesn't change sign if A_0, A_1 are interchanged. So we can't expect the flow for

$$\begin{pmatrix} A_0 & t \\ t & -A_1 \end{pmatrix}$$

to give the index.

August 9, 1989

Recall the index formula

$$\text{Index}(u) = \text{tr}\left\{(1-qp)^n - (1-pq)^n\right\}$$

where u is an invertible modulo I, p, q lift u, u^{-1} respectively, and the trace is defined on I^n. Observe that this expression involves [redacted] no monomials in p, q containing p^2 or q^2. Thus [redacted] it appears appropriate to the situation where u goes from one space to another, i.e. a graded situation.

So let us consider $\mathbb{C}[\gamma]$, where γ is an odd involution. Thus $\mathbb{C}[\gamma]$ is the Clifford algebra C_1. Thus we consider [redacted] superalgebras. A supermodule is a $\mathbb{Z}/2$-graded vector space $H = H^+ \oplus H^-$ with γ of the form $\begin{pmatrix} 0 & u^{-1} \\ u & 0 \end{pmatrix}$.

Now consider the extension of superalgebras

$$0 \longrightarrow (1-z^2)\mathbb{C}[z] \longrightarrow \mathbb{C}[z] \longrightarrow \mathbb{C}[\gamma] \longrightarrow 0$$

where z is odd. In the above index situation we have a $\mathbb{C}[z]$-supermodule, i.e. $H = H^+ \oplus H^-$ with $z = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$ such that $1 - z^2 = \begin{pmatrix} 1-qp & 0 \\ 0 & 1-pq \end{pmatrix}$ is a compact (or $\mathcal{L}^p$) operator. The supertrace $\text{tr}\,\varepsilon$ pulls back to a supertrace defined on I^n where $I = (1-z^2)\mathbb{C}[z]$. Our index formula becomes

$$\text{Index} = tr\left(\varepsilon(1-z^2)^n\right)$$

What are the supertraces on $\mathbb{C}[z]$. An odd supertrace = odd trace = any linear fnl on $(\mathbb{C}[z])^- = z\,\mathbb{C}[z^2]$. An even supertrace is [redacted] a linear functional on $\mathbb{C}[z^2]$ which vanishes on

$$[z^{2k+1}, z^{2l+1}] = 2z^{2k+2l+2}$$

for all k, l. Thus it is a linear functional on $\mathbb{C}[z^2]/z^2\mathbb{C}[z^2]$, and so there is up to scalars only one even supertrace, namely evaluating at $z=0$.

However we want supertraces on I^m vanishing on $[R, I^m]$. An even such supertrace is a linear functional on

$$(I^m) \cap (\mathbb{C}[z]^+) = (1-z^2)^m\,\mathbb{C}[z^2]$$

vanishing on $[z, (1-z^2)^m z\,\mathbb{C}[z^2]] = (1-z^2)^m z^2\,\mathbb{C}[z^2]$.
So again there is only one possibility [redacted] which is evaluating at $z=0$.

Let's consider by analogy the superalgebra $\mathbb{C}[g, g^{-1}]$ where [redacted] $\varepsilon(g) = g^{-1}$.
The algebra $\mathbb{C}[g, g^{-1}] \tilde{\otimes} \mathbb{C}[\varepsilon]$ is the Cuntz algebra $\mathbb{C}[F] * \mathbb{C}[\varepsilon]$, but there is no superalgebra structure on the former. [redacted]
A supermodule over $\mathbb{C}[g, g^{-1}]$ is a $\mathbb{Z}/2$ graded $H = H^+ \oplus H^-$ together with an ~~[illegible]~~ involution $g\varepsilon$

which does not respect the grading. Thus the standard Cuntz algebra setup is perhaps not appropriate for understanding the infinite Grassmannian. Rather it seems that we want to consider the superalgebra $\mathbb{C}[g,g^{-1}]$. The sort of thing giving an index is

$$tr\left(\varepsilon\, f(g)\right) \qquad \begin{aligned} f(1)&=1\\ f(-1)&=0. \end{aligned}$$

This is an even supertrace on $\mathbb{C}[g,g^{-1}]$. It's a supertrace because [redacted] $tr\,\varepsilon$ is a supertrace on $\mathrm{End}(H)$, and it's even because it is supported on f such that $f(g)=f(g^{-1})$.

Since

$$\left[\frac{g-g^{-1}}{2},\frac{g-g^{-1}}{2}\right] = 2\,\frac{(g-g^{-1})^2}{4} = 2\left(\frac{g-g^{-1}}{2}\right)^2$$

the even supertraces on $\mathbb{C}[g,g^{-1}]$ are just linear functionals on

$$\mathbb{C}[g,g^{-1}]^+ \Big/ \left(\frac{g-g^{-1}}{2}\right)^2 \mathbb{C}[g,g^{-1}]^+$$

There are essentially two of these – evaluation at $g=1$ and at $g=-1$, because

$$\mathbb{C}[g,g^{-1}] = \mathbb{C}\left[\frac{g+g^{-1}}{2}\right] \qquad \begin{aligned}\left(\frac{g-g^{-1}}{2}\right)^2 &= \left(\frac{g+g^{-1}}{2}\right)^2 - 1\\ &= \left[\left(\frac{g+g^{-1}}{2}\right)+1\right]\left[\left(\frac{g+g^{-1}}{2}\right)-1\right]\end{aligned}$$

August 10, 1989

Yesterday I noticed that in pairing odd K-classes we use just u, u^{-1} instead of the whole group $\{u^n\}$. Put another way, we can work with a unitary $u: H_0 \xrightarrow{\sim} H_1$ between different spaces. Thus we replace the group algebra $\mathbb{C}[\mathbb{Z}]$ by the superalgebra $C_1 = k[\gamma]$, where γ is an odd involution.

There should be a canonical "K^1" class on C_1, which should pair with elements of "K_1". The latter should be elements of $KK(C_1, C_1)$, so should be represented by the following data: A right Hilbert C_1-module E with an F, and a left C_1-mod structure on E quasi commuting with F. Now a graded C_1-module with F is an ungraded Hilbert space (equipped) with F tensored on the right with C_1. Thus we have $E = H \oplus H$, $F = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$. The left C_1-module then gives an operator $\begin{pmatrix} 0 & u^{-1} \\ u & 0 \end{pmatrix}$ such that $F \equiv u^{-1}Fu$ modulo compacts.

It seems therefore that the KK picture gives a single F and unitary u such that $F - u^{-1}Fu$ is compact. I find this significant for the following reasons.

The index is defined for an ordered pair of invertible self-adjoint operators A_0, A_1 which are congruent modulo compacts. Or it can be defined for a self-adjoint self-adjoint Fredholm A and a unitary u such that $A \equiv u^{-1}Au \pmod{\mathcal{K}}$.

The problem is to find a good analytical formula for the index. It may only exist in the latter case. The reason is that if one proceeds by integrating over a path from A_0 to A_1, there may be endpoint contributions which might cancel if one knew that A_0 and A_1 have the same eigenvalues.

Connes-Moscovici-Bismut have defined cyclic cocycles for Dirac operators as transgressed Chern character forms. These are finite degree cyclic cocycles, so we might try to do index theory in the odd case using their formulas.

In the case of an invertible skew-adjoint X, Connes-Moscovici define cyclic cocycles as follows. One needs differential forms on the space of invertible skew-adjoint operators which are invariant under conjugation by unitaries.

Consider first the finite dimensional situation. Here the space Inv of invertible skew-adjoint operators admits as strong deformation retract the space of iF where F is an involution. Thus $\mathrm{Inv} \sim \mathrm{Grass}(V)$ by the map $X \longmapsto \frac{-iX}{|X|}$, ~~so Inv has~~ Chern character classes. We can obtain them by the Bott map as in my paper on superconnections.

Thus to X associate the path

$$\frac{1+tX}{1-tX} \qquad 0 \leq t \leq \infty$$

in $U(V)$ starting at 1 and ending at -1. This gives a map

$$[0,\infty] \times \mathrm{Inv} \longrightarrow U(V)$$

collapsing the ends. We then can pull back the odd character forms on $U(V)$ integrate out the t-variable and obtain even forms on Inv.

August 12, 1989

Idea: Before you look at index matters for unbounded operators, you should understand how to generalize from $F^2 = 1$ to $F^2 - 1 \in \mathcal{K}$. (The goal is to find analytical formulas for the index.)

Let's consider the standard index situations.

In the even case we consider an odd involution $z = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}$ and an even idempotent $e = \begin{pmatrix} e_+ & 0 \\ 0 & e_- \end{pmatrix}$ such that $[z, e] \in \mathcal{K}$. Setting $F = 2e - 1$, we have two involutions z, F with z odd and F even such that $[F, z] \in \mathcal{K}$. Put $F = \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix}$, then

$$z^{-1}[F, z] = z^{-1}Fz - F = \begin{pmatrix} p^{-1}F_-p - F_+ & 0 \\ 0 & pF_+p^{-1} - F_- \end{pmatrix}$$

To simplify suppose $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$g = Fz = \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & F_+ \\ F_- & 0 \end{pmatrix}$$

satisfies $g \equiv g^{-1} \mod \mathcal{K}$.

Notice that the index we want $\operatorname{tr}(e^+ - e^-)^{\text{odd}}$ can not be obtained as the trace of a polynomial in z, F. In effect conjugation by z preserves z and interchanges $F_\pm$.

In the <u>odd case</u> we obtain similar algebra but with different interpretation. One version is to have an invertible g and an F_0 with $F_0^2 = 1$ such that $[g, F_0] \in \mathcal{K}$. Then we consider

$$z = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \qquad F = \begin{pmatrix} F_0 & 0 \\ 0 & F_0 \end{pmatrix}$$

and we have the previous situation in the case where $F_+ = F_-$ relative to a given isomorphism $H_+ = H_-$.

Next we consider the generalization where involutions modulo compacts occur. In the even case we have

$$\begin{cases} \text{odd } z & z^2 - 1 \in \mathcal{K} \\ \text{even } F & F^2 = 1 \\ & [F, z] \in \mathcal{K} \end{cases}$$

In the odd case we have

$$\begin{cases} \text{ungraded } F_0 & F_0^2 - 1 \in \mathcal{K} \\ g \text{ invertible} & [F_0, g] \in \mathcal{K} \end{cases}$$

There doesn't seem to be a way to keep $z^2 = 1$ (so $z = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}$) and weaken $F^2 = 1$ to $F^2 - 1 \in \mathcal{K}$ and have an index defined unless we know that $F_+ = F_-$ relative to some other (than p) identification of H_+, H_-. This point also came out of the KK-theory, that is, the way elements of $KK(C_1, C_1)$ are represented.

Consider the following problem. Let $z = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$ be an odd operator such that $1 - z^2 = \begin{pmatrix} 1-qp & 0 \\ 0 & 1-pq \end{pmatrix} \in \mathcal{K}$. Then we have the index formula

$$\text{Index} = \operatorname{tr} \varepsilon (1 - z^2)^n$$

provided the trace makes sense. More generally we should expect

$$\text{Index} = \operatorname{tr}(\varepsilon f(z))$$

for any polynomial $f(z)$ with $f(0) = 1$, again provided the trace makes sense.

Next suppose we have an even involution $F = \begin{pmatrix} F_+ & 0 \\ 0 & F_- \end{pmatrix}$ with $[F, z] \in \mathcal{K}$, or equivalently an even idempotent $e = \begin{pmatrix} e_+ & 0 \\ 0 & e_- \end{pmatrix}$ with $[e, z] \in \mathcal{K}$. We want to find an analytical formula for the index.

There are obvious candidates:

$$\operatorname{tr} \varepsilon \{ e - (eze)^2 \}^n \qquad n \gg 0$$

but I would like to have the answer in the following form: A cyclic cocycle attached to the operator z with e substituted in.

Suppose to start that $pq = qp = 1$, whence we can assume $p = q = 1$. We consider the algebra A of $a = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}$ even operators such that $a_+ \equiv a_- \mod \mathcal{K}$. Then we have the cyclic cocycle

$$\varphi(a_0, \ldots, a_{2n}) = \mathrm{tr}\left(\left((a_0)_+ - (a_0)_-\right) \cdots \left((a_{2n})_+ - (a_{2n})_-\right)\right)$$

for n large, and the index is $\varphi(\underbrace{e, \ldots, e}_{(2n+1)\text{-times}})$.

In order to treat the general $\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$ we can try to dilate it to an odd autom. This means embedding the given super Hilbert space in a larger one with odd autom. such that $\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$ is the contraction. Thus we want

$$H^+ \underset{i^*}{\overset{i}{\rightleftarrows}} \tilde{H} \underset{j^*}{\overset{j}{\rightleftarrows}} H^-$$

with $p = j^* i$, $q = i^* j$.

We can study the dilation problem algebraically using GNS theory. We have the algebra $A = \mathbb{C}[\gamma]$ with $\gamma^2 = 1$ and the linear map $\rho : A = \mathbb{C}[\gamma] \longrightarrow \mathbb{C}[z] = B$ such that $\rho(\gamma) = z$. Clearly this ρ is universal such that $\rho(1) = 1$, so we know that the GNS algebra R is the free product $A * \mathbb{C}[F]$.

Review the GNS theory for a linear map $\rho : A \longrightarrow B$ of unital algebras such that

$\rho(1) = 1$. We consider quadruples
(E, N, ι, ι^*) where E is a (left) A-module, N is a B-module, and ι, ι^* are linear map

$$N \xrightarrow{i} E \xrightarrow{\iota^*} N$$

such that $\iota^* a i n = \rho(a) n$. These form a category and we have an equivalence of this category with the category of R-modules; the equivalence [redacted] is given by $(E, N, \iota, \iota^*) \longmapsto E$ with $R = A \oplus A \otimes B \otimes A$ acting by $a \cdot \xi = a\xi$ and $(a \otimes b \otimes a') \cdot \xi = a\iota(b\,\iota^*(a'\xi))$

In ~~the~~ situation at hand we have a B-module given by $H = H^+ \oplus H^-$ with the operator $z = \begin{pmatrix} 0 & \delta \\ \rho & 0 \end{pmatrix}$. We then seek ~~to~~ embed H inside a larger A-module E as a direct factor.

Recall that there is a canonical left R-right B bimodule E given by:

$$(*) \qquad B \xrightarrow{\iota} A \otimes B \xrightarrow{\iota^*} B$$

$$i(b) = 1 \otimes b \qquad \iota^*(a \otimes b) = \rho(a) b$$

Hence given the B-module H we can get

$$H = B \otimes_B H \longrightarrow (A \otimes B) \otimes_B H \longrightarrow H$$

$$(A \otimes B) \otimes_B H = A \otimes H$$

to obtain a dilation of z.

Recall $B = \mathbb{C}[z]$, $A = \mathbb{C}[\gamma] = \mathbb{C} \oplus \mathbb{C}\gamma$

so we can identify $A \otimes B$ with $B^{\oplus 2}$ as right $\mathbb{C}[z]$-module
We have for the maps in ⊛

$$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad i^* = \begin{pmatrix} 1 & z \end{pmatrix}$$

and multiplication by γ on the left is

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Check: $$i^* i = \begin{pmatrix} 1 & z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$i^* \gamma i = \begin{pmatrix} 1 & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = z$$

Instead of $A \otimes B$ we can use $\mathrm{Hom}(A, B) = A^* \otimes B$, where A^* has the dual basis $\hat{1}, \hat{\gamma}$ to $1, \gamma$. We have maps

$$B \xrightarrow{j} \mathrm{Hom}(A, B) \xrightarrow{j^*} B$$

with $j^*(\varphi) = \varphi(1)$ and $j(b)(a) = \rho(a) b$.
Moreover j induces a map of A, B bimods

$$A \otimes B \xrightarrow{\Phi} \mathrm{Hom}(A, B)$$
$$a \otimes b \longmapsto (\alpha \longmapsto \rho(\alpha a) b).$$

Thus it sends $1 \otimes b$ to $\hat{1} \otimes b + \hat{\gamma} \otimes zb$ and $\gamma \otimes b$ to $\hat{1} \otimes zb + \hat{\gamma} \otimes b$. So

$$\Phi = \begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix} \qquad j = \begin{pmatrix} 1 \\ z \end{pmatrix} \qquad j^* = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Check

$$\Phi i = \begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ z \end{pmatrix} = j$$

$$j^* \Phi = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix} = \begin{pmatrix} 1 & z \end{pmatrix} = i^*$$

Also left mult. by γ on $\mathrm{Hom}(A, B)$ is given by $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$j^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} j = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}$$

$$= z$$

$\blacksquare$

August 15, 1989

Problem: Understanding index theory and cyclic cocycles in the case where $F^2=1$ is replaced by $1-F^2 \in$ a suitable ideal.

This might shed light on how to handle unbounded operators.

Consider the following situation. Let R, I, τ be an algebra, ideal, and trace defined on I^n considered as R-bimodule. Let $z \in R$ satisfy $1-z^2 \in I$. Let $A \longrightarrow R$ be a homomorphism such that $[z, A] \subset I$.

The analytical examples are clear: R, I, τ are $\mathcal{L}(H)$, $\mathcal{L}^p(H)$, tr, where $H = L^2(M, E)$, $A = C^\infty(M)$, z pseudodifferential operator of order zero whose symbol is an involution.

In this situation we want to produce cyclic cocycles on A. In the ungraded case they should be odd cocycles, and they should be even cocycles in the graded case.

One way to proceed in the ungraded case is to move z to be an involution, then use Connes framework. This doesn't always work in the even case because the index of z can be $\neq 0$.

Let's consider the ungraded case first, where we have some insight from the study of extensions, We have that mod I, z is an involution commuting with elements of A. Let $e = \frac{1+z}{2}$ mod I in R/I. Then $a \longmapsto e(a \bmod I)e$ is a homomorphism from A to $e(R/I)e \subset R/I$.

We have odd cyclic classes on R/I defined by the extension R and trace τ, so we can pull these back to A. The resulting classes can be computed as Chern character forms using a lifting of the homomorphism into R. Such a lifting is

$$\rho(a) = \frac{1+z}{2}\, a\, \frac{1+z}{2}$$

Similarly we have another homomorphism

$$a \longmapsto (1-e)(a \bmod I)(1-e) \qquad A \longrightarrow R/I$$

with the lifting

$$a \longmapsto \frac{1-z}{2}\, a\, \frac{1-z}{2}$$

The resulting odd cyclic classes for the latter should have opposite sign, since their sum should correspond to the cyclic classes belonging to the homorphism $a \longmapsto a \bmod I$, and these vanish because the homomorphism lifts up into R.

It would be interesting to work out proofs of these assertions.

In any case we learn that it is possible to define cyclic classes, at least in the ungraded case, without introducing something like $\sqrt{1-z^2}$. Work out curvature of ρ.

Digression. Refining z to an involution modulo I^n.

Here are three methods:

1) From Bass' book:

$$1 = \left(\frac{1+z}{2} + \frac{1-z}{2}\right)^{2n-1}$$

$$= \sum_{k=0}^{2n-1} \binom{2n-1}{k} \left(\frac{1+z}{2}\right)^{2n-1-k} \left(\frac{1-z}{2}\right)^{k}$$

$$= \underbrace{\sum_{k=0}^{n-1}}_{p} + \underbrace{\sum_{k=n}^{2n-1}}_{q}$$

Then $p \in \left(\frac{1+z}{2}\right)^n k[z]$, $q \in \left(\frac{1-z}{2}\right)^n k[z]$

and $p(1-p) = pq = q(1-q) \in \left(\frac{1-z^2}{4}\right)^n k[z]$

2) My integral formula:

$$p(z) = \int_{-1}^{z} (1-t^2)^{n-1} dt \Big/ \int_{-1}^{1} (1-t^2)^{n-1} dt$$

allows one to work out $p(z)$ explicitly as a polynomial in z.

Actual in the above I am really interested in $2p(z) - 1 = \int_0^z (1-t^2)^{n-1} dt \Big/ \int_0^1 (1-t^2)^{n-1} dt$

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which is an approximate involution.

3) Power series: Recall binomial series

$$(1-u)^{-1/2} = \sum_{n\geq 0} \frac{1\cdot 3 \cdot (2n-1)}{2^n\, n!} u^n$$

So

$$\frac{z}{\sqrt{z^2}} = z\left(1-(1-z^2)\right)^{-1/2}$$

$$= z\sum_{n\geq 0} \frac{1\cdot 3 \cdots (2n-1)}{2^n\, n!}(1-z^2)^n$$

Program: Consider $A \xrightarrow{\text{alg hom}} R \supset I \supset I^m \xrightarrow{\tau} k$ $z \in R \ni [z, A],\ 1-z^2 \subset I$. We have the homomorphism $A \longrightarrow R/I$ given by $a \longmapsto \left(\frac{1+z}{2}\right)a + I = a\left(\frac{1+z}{2}\right) + I$, which leads to odd cyclic cocycles on I. We try now to choose a lifting ρ as ~~simple~~ simple as possible and compute these cocycles. The question is whether they can be expressed in terms of $1-z^2$, $[z,\theta]$ without using z, or possibly using only a single z.

Let's consider $\rho(a) = \frac{1+z}{2}a$. Then

$$\omega(a_1,a_2) = \rho(a_1a_2) - \rho(a_1)\rho(a_2)$$

$$= \frac{1+z}{2}a_1\left(1-\frac{1+z}{2}a_2\right) = \frac{1+z}{2}a_1\frac{1-z}{2}a_2$$

The odd cyclic cocycles associated to this choice of ρ are

$$\tau\left(\omega(a_1,a_2)\cdots\omega(a_{2n-1},a_{2n})\right) - \tau\left(\omega(a_{2n},a_1)\cdots\omega(a_{2n-2},a_{2n-1})\right)$$

$$(*) \quad = \tau\left(\frac{1+z}{2}a_1\frac{1-z}{2}a_2\cdots\frac{1-z}{2}a_{2n}\right) - \tau\left(\frac{1+z}{2}a_{2n}\frac{1-z}{2}a_1\cdots\frac{1-z}{2}a_{2n-1}\right)$$

On the other hand suppose we consider the lifting $\rho'(a) = a\frac{1+z}{2}$. The curvature is

$$\omega'(a_1,a_2) = a_1\left(1-\frac{1+z}{2}\right)a_2\frac{1+z}{2} = a_1\frac{1-z}{2}a_2\frac{1+z}{2}$$

and the cocycles are

$$\tau\left(a_1\frac{1-z}{2}a_2\frac{1+z}{2}\cdots a_{2n}\frac{1+z}{2}\right) - \tau\left(a_{2n}\frac{1-z}{2}a_1\frac{1+z}{2}\cdots a_{2n-1}\frac{1+z}{2}\right).$$

We get the same cocycles, at least provided there is enough room to move $\frac{1+z}{2}$ around in the trace.

Next consider the lifting $\rho''(a) = a\frac{1-z}{2}$ of the complementary homomorphism $a \longmapsto a(1-e)$ from A to R/I. The curvature is

$$\omega''(a_1,a_2) = a_1a_2\frac{1-z}{2} - a_1\frac{1-z}{2}a_2\frac{1-z}{2}$$

$$= a_1\left(1-\frac{1-z}{2}\right)a_2\frac{1-z}{2} = a_1\frac{1+z}{2}a_2\frac{1-z}{2}$$

The cyclic cocycles are in this case

$$\underbrace{\tau\left(a_1\frac{1+z}{2}a_2\frac{1-z}{2}\cdots\overbrace{\frac{1+z}{2}a_{2n}\frac{1-z}{2}}^{\in I}\right)} - \underbrace{\tau\left(a_{2n}\frac{1+z}{2}a_1\frac{1-z}{2}\cdots a_{2n-1}\frac{1-z}{2}\right)}$$

$$\tau\left(\frac{1+z}{2}a_{2n}\frac{1-z}{2}a_1\frac{1+z}{2}\cdots a_{2n-1}\right) \qquad \tau\left(\frac{1+z}{2}a_1\frac{1-z}{2}\cdots a_{2n-1}\frac{1-z}{2}a_{2n}\right)$$

These are exactly the cocycles ⊛ except with the opposite sign, so we obtain

<u>Prop:</u> Assume $\tau : I^n/[I, I^{n-1}] \to k$. Then the cyclic cocycles of degree $2n-1$ associated to $\rho(a) = \frac{1+z}{2}a$ and $\rho''(a) = a\frac{1-z}{2}$ are of opposite sign.

Check for $n = 1$.

$$\omega(a_1, a_2) = \frac{1+z}{2} a_1 \frac{1-z}{2} a_2 \qquad \omega''(a_1,a_2) = a_1 \frac{1+z}{2} a_2 \frac{1-z}{2}$$

For ρ: $\boxed{\tau\left(\frac{1+z}{2} a_1 \frac{1-z}{2} a_2 - \frac{1+z}{2} a_2 \frac{1-z}{2} a_1\right)}$

For ρ'': $\tau\left(a_1 \frac{1+z}{2} a_2 \frac{1-z}{2} - a_2 \frac{1+z}{2} a_1 \frac{1-z}{2}\right)$

Note that
$$\frac{1+z}{2} a \frac{1-z}{2} = \frac{1+z}{2}\left(\left[a, \frac{1-z}{2}\right] + \frac{1-z}{2} a\right)$$
$$= + \frac{1+z}{4}[z, a] + \frac{1-z^2}{4} a \in I$$

For ρ': $\boxed{\tau\left(a_1 \frac{1-z}{2} a_2 \frac{1+z}{2} - a_2 \frac{1-z}{2} a_1 \frac{1+z}{2}\right)}$

$$= \tau\left(\frac{1-z}{2} a_2 \frac{1+z}{2} a_1 - \frac{1-z}{2} a_1 \frac{1+z}{2} a_2\right)$$

The cocycles for ρ, ρ' should be cohomologous. On the other hand the latter is obtained from the former by changing $z \longmapsto -z$ and the total sign. Therefore the even part should be cohomologous to zero.

But

$$\frac{1+z}{2}a_1 \frac{1-z}{2}a_2 = \frac{1}{4}(a_1+za_1)(a_2-za_2)$$

$$= \frac{1}{4}(a_1a_2 + za_1a_2 - a_1za_2 - za_1za_2)$$

$$= \frac{1}{4}\{(a_1a_2 - za_1za_2) + [z,a_1]a_2\}$$

and so the final formula is

$$\frac{1}{4}\cancel{\text{Tr}}\left([z,a_1]a_2 - [z,a_2]a_1\right)$$

$$= \boxed{\frac{1}{4}\tau\left([z,a_1]a_2 - a_1[z,a_2]\right)}$$

We would like to check that this formula defined a cyclic 1-cocycle. It is the skew-symmetryation of

$$\tau(a_1[z,a_2])$$

which we know is a Hochschild 1-cocycle, so it it enough to ~~know~~ the latter is skew-symmetric, that is, that

$$\tau(a_2[z,a_1]) + \tau(a_1[z,a_2]) = \tau([z,a_1a_2]) = 0$$

August 16, 1989

Here's a proof that $\tau([z,a])=0$ in the situation considered yesterday. We propose to show that $\tau([z,x])=0$ for $x \in R$ such that $x \bmod I$ is in the subalgebra of R/I generated by A, z. The set $x \in R$ such that $\tau([z,x])=0$ is a subspace containing I; since in R/I z is an involution commuting with A we have to show $\tau([z,x])=0$ for

$$x \in cAc + (1-c)A(1-c) + I$$

where $c = \frac{1+z}{2}$, $1-c = \frac{1-z}{2}$. But

$$[z, cac] = (-2)\Big[\underbrace{\tfrac{1-z}{2}}_{1-c}, cac\Big]$$

and

$$\tau((1-c)cac) = \tau(ac(1-c)c) \qquad \text{as } (1-c)c \in I$$

$$= \tau(c(1-c)ca)$$

$$= \tau(cac(1-c))$$

so $\tau[z, cac] = 0$. Similarly

$$\tau([z, (1-c)a(1-c)]) = 0$$

August 17, 1989

$i = 0, 1$

Let $\theta_i : R \longrightarrow S$, be morphisms of DG algebras. In $\mathrm{Hom}(B(R), S)$ we have $\delta\theta_i + \theta_i^2 = d\theta_i = 0$.

What's the natural notion of homotopy between θ_0, θ_1? First of all it should be an ordinary homotopy

$$1) \qquad \boxed{\theta_1 - \theta_0 = dh}$$

But we also want h to satisfy something akin to the fact that $\theta_1 - \theta_0$ is a derivation relative to θ_0 on one side and θ_1 on the other: e.g.

$$\delta(\theta_1 - \theta_0) = -(\theta_1^2 - \theta_0^2)$$
$$= -\left(\theta_0(\theta_1 - \theta_0) + (\theta_1 - \theta_0)\theta_1\right)$$

or

$$(*) \qquad \delta(\theta_1 - \theta_0) + \theta_0(\theta_1 - \theta_0) + (\theta_1 - \theta_0)\theta_1 = 0$$

Thus we ask that h satisfies

$$2) \qquad \boxed{\delta h + \theta_0 h - h\theta_1 = 0}$$

Check: If this holds then applying $-d$ given

$$\delta(dh) + \theta_0(dh) + (dh)\theta_1 = 0$$

which agrees with $(*)$.

The above conditions 1), 2) I encountered before in "Guggenheim perturbation theory" in diffl. homological alg.

August 18, 1989

There is a general problem of understanding when two maps of DGA's are homotopic, or more precisely what ~~constitutes~~ a homotopy between them. One should understand this within the A_∞-algebra framework.

Given two DGA maps $R \rightrightarrows S$ let θ_i be the corresponding elements of $\mathrm{Hom}^1(B(R), S)$, so that

$$\delta\theta_i + \theta_i^2 = d\theta_i = 0 \qquad i = 0, 1.$$

Then the natural candidate for a homotopy from θ_0 to θ_1 is an element $h \in \mathrm{Hom}^0(B(R), S)$ such that

$$(*) \quad \begin{cases} \theta_1 - \theta_0 = dh \\ \delta h + \theta_0 h - h\theta_1 = 0 \end{cases}$$

In fact applying $-d$ to the latter gives

$$\begin{aligned} (-d)(\delta h + \theta_0 h - h\theta_1) &= \delta\, dh + \theta_0\, dh + dh\, \theta_1 \\ &= \delta(\theta_1 - \theta_0) + \theta_0(\theta_1 - \theta_0) + (\theta_1 - \theta_0)\theta_1 \\ &= (\delta\theta_1 + \theta_1^2) - (\delta\theta_0 + \theta_0^2) \end{aligned}$$

which shows that if θ_0 is a DGA map and $(*)$ holds, then θ_1 is also a DGA map.

The above notation is motivated by the case where R is an algebra. In the general case δ on $\mathrm{Hom}(B(R), S)$

comes from the multiplication on R, and d is $[d, \,]$ relative to the differentials on R and S.

In the A_∞ framework [redacted] DGA maps $R \longrightarrow S$ are generalized to DGC maps $B(R) \longrightarrow B(S)$, [redacted] which are the same as twisting cochains $\theta \in \mathrm{Hom}^1(B(R), S)$. These satisfy

$$(\delta+d)\theta + \theta^2 = 0$$

The natural equivalence for twisting cochains is gauge transformations:

$$g^{-1}(\delta + d + \theta_0)\, g = \theta_1$$

or

$$\boxed{(\delta+d)g + \theta_0 g = g\theta_1}$$

for $g \in \mathrm{Hom}^0(B(R), S)$. [redacted] If $g = 1+h$ this becomes

$$(\delta+d)h + \theta_0 h - h\theta_1 = \theta_1 - \theta_0$$

which is satisfied when h satisfies $\circledast$ on the preceding page.

Because the invertibility of $1+h$ is not clear, it seems that one can have a homotopy h from θ_0 to θ_1 without there being a homotopy from θ_1 to θ_0.

Review perturbation theory of Guggenheim et al.

One suppose given [redacted] a complex (E, d) and [redacted] a twisted differential $d+\theta$, i.e.

$$d\theta + \theta^2 = 0$$

Next suppose given a direct embedding up to homotopy of E into another complex E':

$$\begin{tikzcd} E \arrow[d, shift right, "u"'] \\ E' \arrow[u, shift right, "v"'] \end{tikzcd} \qquad vu - 1 = [d, h]$$

(In the interesting cases E' is a strong deformation retract of E i.e. $uv = 1$, so that E' is smaller than E, however up to homotopy one should think of transporting the twisted structure on E to E' which is possible because E is a retract up to homotopy of E'.)

~~[illegible]~~ We propose now to construct a twisted differential $d + \theta'$ on E' plus:

$$\begin{tikzcd} E \arrow[d, shift right, "U"'] \\ E' \arrow[u, shift right, "V"'] \end{tikzcd} \qquad \begin{aligned} d\theta' + \theta'^2 &= 0 \\ U(d+\theta) &= (d+\theta')U \\ (d+\theta)V &= V(d+\theta') \\ VU - 1 &= [d, H] \end{aligned}$$

First approx to θ' is $u\theta v$. But

$$d(u\theta v) + (u\theta v)^2 = u\theta\underbrace{(-1+vu)}_{dh}\theta u$$

suggesting the correction $u\theta v + u\theta h\theta v$, which eventually leads to the <u>Formulas</u>

$$\theta' = u(\theta + \theta h\theta + \theta h\theta h\theta + \cdots)v = u\theta\underbrace{\frac{1}{1-h\theta}v}_{V} = \underbrace{u\frac{1}{1-\theta h}}_{U}\theta v$$

$$H = h + h\theta h + \cdots = h\frac{1}{1-\theta h}$$

Check:

$$d\theta' = u(-\theta^2)\frac{1}{1-h\theta}v - u\theta\frac{1}{1-h\theta}(\underset{va-1}{\underbrace{dh}}\theta + h\theta^2)\frac{1}{1-h\theta}v$$

$$= -u\theta^2\frac{1}{1-h\theta}v + u\theta\cancel{\frac{1}{1-h\theta}}(\overset{\theta}{\theta \cancel{- h\theta^2}})\frac{1}{1-h\theta}$$

$$-\left(u\theta\frac{1}{1-h\theta}v\right)\left(u\theta\frac{1}{1-h\theta}v\right) = -(\theta')^2$$

Summarize what we learned before this digression. We considered the problem of defining homotopy of maps of DGA's, and then describing the relation with the intuitive notion of one-parameter family of homomorphisms.

Replacing DGA homomorphism $R \to S$ by DGC morphism $B(R) \longrightarrow B(S)$ leads to twisting cochains $\theta \in \mathrm{Hom}^1(B(R), S)$. We saw that [redacted] the natural notion of homotopy between DGA maps leads to gauge transformations between twisting cochains. Recall:

$$\delta\theta_i + \theta_i^2 = d\theta_i = 0 \qquad i=0,1$$

[redacted] $\delta h + \theta_0 h - h\theta_1 = 0$ $\qquad \overset{dh}{\text{[redacted]}} = \theta_1 - \theta_0$

$\Rightarrow$

$$(\delta + d + \theta_0)(1+h) - (1+h)(\delta + d + \theta_1)$$

$$= [\delta + d, h] + (\theta_0 - \theta_1) + \theta_0 h - h\theta_1$$

$$= (\overset{dh}{\text{[redacted]}} + \theta_0 - \theta_1) + (\delta h + \theta_0 h - h\theta_1) = 0.$$

Now let's bring in 1-parameter families.

Suppose we consider a 1-parameter family $\theta = \theta_t$ of twisting cochains. (Actually we should look more generally at a one-parameter family of connection forms.) The infinitesimal version of the idea that there is a gauge transformation from $d+\theta_0$ to $d+\theta_1$ is that $\dot\theta_t$ is tangent to the gauge orbit through θ_t, which means that there is a family $X = X_t$ such that

$$[d+\theta, X] + \dot\theta = 0 \qquad d = d_{tot}$$

If this is the case, then we can solve

$$\dot g = Xg \qquad g_0 = 1$$

to obtain a family $g = g_t$ of gauge transformations. Then

$$\begin{aligned}\left(g^{-1}(d+\theta)g\right)^{\cdot} &= (-g^{-1}Xg\,g)(d+\theta)g \\ &\quad + g^{-1}(\dot\theta)g + g^{-1}(d+\theta)Xg \\ &= g^{-1}\{[d+\theta, X] + \dot\theta\}g = 0\end{aligned}$$

Thus we have that $\boxed{g^{-1}(d+\theta)g = d+\theta_0}$ is constant in t.

Special situation: Suppose we return to bigraded notation with a family $\theta = \theta_t$ such that $d\theta = 0$, $\delta\theta + \theta^2 = 0$. Then

because $$\delta\dot\theta + [\theta, \dot\theta] = 0$$

better $$[\delta + \theta, \dot\theta] = 0$$

a natural homotopy condition is for ~~there~~ to be a family $h = h_t$ with

$$dh = \dot{\theta} \qquad [\delta + \theta, h] = 0$$

We observe that this case is included in the previous situation since

$$[\delta + d + \theta, (-h)] + \dot{\theta} = -[\delta+\theta, h] - dh + \dot{\theta} = 0$$

Next let's return to the situation

$$A \longrightarrow R \supset I, \quad z \in R, \quad [z, A], 1 - z^2 \subset I.$$

Let's suppose R is generated by A, z. Then R/I is generated by A and the image of z which is a central involution. We therefore have two ideals

$$I^{\pm} = I + \left(\tfrac{1 \pm z}{2}\right) A$$

such that $I^+ + I^- = R$ and $I^+ \cap I^- = I$. This means we are in the "Tate" example considered last spring. Recall that we form the free product of DGAs.

$$S = (I^+ \longrightarrow R) *_R (I^- \longrightarrow R)$$

and obtain a ~~contractible~~ DGA with zero homology (since $d\eta = 1$. $\eta = \frac{1+z}{2} \oplus \frac{1-z}{2} \in S_1 = I^+ \oplus I^-$, and product with η gives a null-homotopy.

One obtains cyclic cocycles on R from closed traces on S (vanishing on R). This is because $CC(S) \sim 0$, so

$$\Sigma\, CC(R) \sim CC(S)/CC(R) \longrightarrow S_\natural / R_\natural$$

(actually $S_\natural$ [redacted] is concentrated in even degrees, so all traces are closed.)

Anyway suppose we have chosen $\eta \in S_1$ with $d\eta = 1$, and let $\theta \in \mathrm{Hom}^1(B(R), S)$ be the homomorphism $R = S_0 \subset S$. Set $\boxed{h = \eta\theta.}$ We have

[redacted] $$dh = (d\eta)\theta - \eta \overbrace{d\theta}^{0} = \theta$$

$$\delta h = \underbrace{(\delta\eta)}_{0}\theta - \eta(\delta\theta) = \eta\theta^2 = h\theta.$$

Thus h is a homotopy from $\theta_0 = 0$ to $\theta_1 = \theta$.

Check:

$$-(1+h)(\delta + d + \theta) + (\delta + d)(1+h)$$
$$= [\delta + d, 1+h] - (1+h)\theta$$
$$= \delta h + dh - (1+h)\theta = 0$$

But [redacted] if tr denotes the trace [redacted] $\mathrm{Hom}(B(R), S) \longrightarrow \mathrm{Hom}(B(R)^\natural, S_\natural)$, we have

$$d\, \mathrm{tr}(h^n) = n\, \mathrm{tr}(h^{n-1}\theta)$$
$$\delta\, \mathrm{tr}(h^n) = n\, \mathrm{tr}(h^{n-1}h\theta) = n\, \mathrm{tr}(h^n\theta)$$

which means [redacted] ?

Recall that we are looking at an algebra R generated by a subalgebra A and an element z, and τ is a trace defined on $I^n/[R,I^n]$ where I is an ideal containing $[z,A]$ and $1-z^2$. We have ~~[illegible]~~ algebra homomorphisms

$$A \longrightarrow R/I \qquad\qquad a \longmapsto \frac{1\pm z}{2}a$$

~~[illegible]~~ Better we have the map $\rho(a) = \frac{1+z}{2}a$ from A to R which is a homomorphism modulo I:

$$\begin{aligned}\omega(a_1,a_2) &= \rho(a_1a_2) - \rho(a_1)\rho(a_2)\\ &= \frac{1+z}{2}a_1\left(1-\frac{1+z}{2}\right)a_2 = \frac{1+z}{2}a_1\frac{1-z}{2}a_2\\ &= \frac{1-z^2}{4}a_1a_2 + \frac{1+z}{2}\left[a_1, \frac{-z}{2}\right]a_2\end{aligned}$$

August 20, 1989

Review (2-) yesterday's discussion of homotopy for DGA maps. There are two kinds of maps between DGA's to consider:

(1) Ordinary DGA morphisms $R \longrightarrow S$

(2) A_∞-maps $R \longrightarrow S$, that is, DGC maps $B(R) \longrightarrow B(S)$ or equivalently twisting cochains in $\mathrm{Hom}(B(R), S)$.

A twisting cochain is a $\theta \in \mathrm{Hom}^1(B(R), S)$ satisfying:

$$(\delta + d)\theta + \theta^2 = 0$$

A DGA morphism $R \to S$ gives rise to a twisting cochain satisfying

(*) $\delta\theta + \theta^2 = 0 \qquad d\theta = 0.$

A homotopy between DGA morphisms θ_0, θ_1 is an $h : R \to S$ of degree -1 such that

(**) $dh = \theta_1 - \theta_0 \quad , \quad \delta\theta + \theta_0 h - h\theta_1 = 0$

The second formula is a derivation property with repect to multiplication in R.

Actually it seems that a $\theta \in \mathrm{Hom}^1(B(R), S)$ satisfying (*) is the same as a DGA map $R \to S$, and that an $h \in \mathrm{Hom}^0(B(R), S)$ satisfying (**) is the same as a homotopy. The issue is whether the δ conditions imply that the map

$$B(R) = T(\Sigma R) \longrightarrow S$$

vanishes on $(\Sigma R)^{\otimes k}$ for $k \geq 2$. Until this point is checked we should go only in one direction.

Note that if h is a homotopy from θ_0 to θ_1, then $g = (1+h)$ is a gauge transformation from $d+\delta+\theta_0$ to $d+\delta+\theta_1$:

$$(d+\delta)(g) + \theta_0 g - g\theta_1 = 0 \qquad \text{or}$$

$$(d+\delta+\theta_0)g = g(d+\delta+\theta_1)$$

Link with one-parameter families: Suppose given a family $\theta = \theta_t$ of twisting cochains, such that $\dot\theta$ is tangent to the gauge orbit, i.e. there is a family $X = X_t$ such that

$$\oplus \qquad [d+\delta+\theta, X] = \dot\theta$$

Then integrating $\dot g = gX$ for $g = g_t$ such that $g_0 = 1$, we have

$$g^{-1}(d+\delta+\theta_0)g = d+\delta+\theta$$

Check: $$\left(g(d+\delta+\theta)g^{-1}\right)^{\cdot} = gX(d+\delta+\theta)g^{-1} + g\dot\theta g^{-1} - g(d+\delta+\theta)\cancel{g^{-1}g}Xg^{-1}$$

$$= g\left\{\dot\theta - [d+\delta+\theta, X]\right\}g^{-1} = 0$$

Notice that $\oplus$ above holds if

$$dX = \dot\theta \qquad [\delta+\theta, X] = 0$$

This is the appropriate infinitesimal homotopy condition for morphisms of DGA's. It fits with $\delta\theta + \theta^2 = 0$, $d\theta = 0$. It says that we lift the derivation $\mathring{\theta}$ with respect to the alg map θ to the derivation X with respect to θ.

Next, what do these homotopy relations imply about cyclic complexes?

It appears that gauge transformations are like inner automorphisms, and their effect on cyclic homology should be trivial.

Suppose R, S are ordinary algebras considered as DGA's concentrated in degree zero. Then

$$\theta \in \mathrm{Hom}^1(B(R), S) = \mathrm{Hom}(R, S)$$

is a twisting cochain iff it is an algebra homomorphism. Similarly

$$g \in \mathrm{Hom}^0(B(R), S) = S$$

is an element of S. Note that S is the degree zero subalg of $\mathrm{Hom}(B(R), S)$. Then

$$g^{-1}(\delta + \theta)g = \delta + g^{-1}\theta g.$$

Thus in this example gauge equivalence of twisting cochains is the same as conjugation by invertible elements of S acting on algebra morphisms from R to S.

Suppose next we replace R by an algebra A

and we let S be the DGA

$$S = R \oplus sI,$$

where I is an ideal in I, and $s \in S_1$ is a central element [redacted] (commutes with elts. of R) [redacted] such that $s^2 = 0$, $ds = -1$.

We recall that twisting cochains $\theta \in \mathrm{Hom}^1(B(A), S)$ are of the form $\theta = \rho + s\omega$, where $\rho : A \to R$ is a linear map, $\omega = \delta\rho + \rho^2$ is its curvature, and where $\omega \in \mathrm{Hom}^2(B(A), I)$.

[redacted]

Consider the algebra $\mathrm{Hom}^0(B(A), S)$. Elements of this are of the form $r + sh$ with $r \in \mathrm{Hom}^0(B(A), R) = R$ and $h \in \mathrm{Hom}^1(B(A), I) = \mathrm{Hom}(A, I)$. The product is

$$(r_1 + sh_1)(r_2 + sh_2) = r_1 r_2 + s(h_1 r_2 + r_1 h_2)$$

so we have the semi-direct product of R and the bimodule $\mathrm{Hom}(A, I)$, where R acts via [redacted] left + right multiplication on I.

The group of invertible elements of $\mathrm{Hom}^0(B(A), S)$ is the semi-direct product of $R^\times$ with $\mathrm{Hom}(A, S)$.

Consider the effect of $g = 1 + sh$ on $\theta = \rho + s\omega$:

$$d + \delta + \theta_1 = (1 + sh)(d + \delta + \underbrace{\theta}_{\rho + s\omega})(1 - sh)$$

$$= (1 + sh)\left[(1 - sh)(d + \delta) - (d + \delta)sh + \rho - \rho sh + s\omega\right]$$

$$= d + \delta + (1 + sh)\left[h + s\delta h + \rho + s(\rho h + \omega)\right]$$

$$= d + \delta + h + \rho + s(h^2 + h\rho) + s(\delta h + \rho h + \overset{\delta\rho + \rho^2}{\overset{\shortparallel}{\omega}})$$

$$= d + \delta + (h + \rho) + s\{\delta(h + \rho) + (h + \rho)^2\}$$

Thus the effect of the gauge transformation $g = 1 + sh$ on $\theta = \rho + s\omega$ is to give the twisting cochain belonging to $h + \rho$. [redacted]

The conjecture based on these examples appears to be that if $\theta_0, \theta_1 \in \mathrm{Hom}(B(R), S)$ are twisting cochains which are related by gauge transformations, then the induced maps $\bar{B}(R)^{\natural} \Longrightarrow \bar{B}(S)^{\natural}$ of cyclic [redacted] complexes are homotopic.

[redacted] We can test this by seeing what happens to cyclic cocycles on S obtained from a trace on S. Consider τ [redacted] $: S \longrightarrow S_{\natural}$ the universal trace. Form the trace

$$\tau^{\natural} : \mathrm{Hom}(B(R), S) \longrightarrow \mathrm{Hom}(B(R)^{\natural}, S_{\natural})$$

It should be the case that $\tau^{\natural}(\theta_i)$ gives the composite

$$B(R)^{\natural} \xrightarrow{(\theta_i)_*} B(S)^{\natural} \xrightarrow{\tau} \Sigma S_{\natural}$$

So we would like to see whether $\tau^{\natural}(\theta_i)$ and more generally $\tau^{\natural}(\theta_i^{2k+1})$ are cohomologous, when θ_1, θ_0 are related by a gauge transformation. However if

$$g^{-1}(\hat{d} + \theta_0)g = [redacted]\ \hat{d} + \theta_1 \qquad \hat{d} = d + \delta$$

Then
$$g^{-1}\hat{d}g + g^{-1}\theta_0 g = \theta_1$$

so
$$tr(\theta_1) - tr(\theta_0) = tr(g^{-1}\hat{d}g)$$

But this last $tr(g^{-1}\hat{d}g)$ is not usually exact, unless one has some sort of deformation of g to the identity.

So we modify the conjecture to say that if g_t is a one-parameter family of invertible elements of $Hom^0(B(R), S)$, then the family of twisting cochains $g_t^{-1}(d+\delta)(g_t) + g_t^{-1}\theta g_t$ induces maps of cyclic complexes which up to homotopy are the same.

Let's now return to R generated by A, z, an ideal I in R containing $[z, A]$ and $1-z^2$, and a trace τ on I^m. We then have a linear map

$$\rho(a) = \frac{1+z}{2}a \qquad \rho : A \longrightarrow R$$

which is a homomorphism modulo I:

$$\omega(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$$

$$= \frac{1+z}{2}a_1\left(1 - \frac{1+z}{2}\right)a_2$$

$$= \frac{1+z}{2}a_1 \frac{1-z}{2}a_2 = \frac{1-z^2}{4}a_1 a_2 + \frac{1+z}{4}[z, a_1]a_2$$

So we get cyclic cocycles

$$\tau\left(\frac{1+z}{2}a_1 \frac{1-z}{2}a_2 \cdots \frac{1-z}{2}a_{2n}\right) - \tau\left(\frac{1+z}{2}a_{2n} \frac{1-z}{2}a_1 \cdots \frac{1-z}{2}a_{2n-1}\right)$$

for $n \geq m$. At least for $n > m$ we can move the $\frac{1+z}{2}$ in the first term and each of $\frac{1+z}{2}$, a_{2n}, $\frac{1-z}{2}$ in the second to obtain

$$\tau\left(a_1 \frac{1-z}{2} a_2 \cdots a_{2n} \frac{1+z}{2}\right) - \tau\left(a_1 \frac{1+z}{2} a_2 \cdots a_{2n} \frac{1-z}{2}\right)$$

This is just the same as the part of the first term which is odd under $z \mapsto -z$.

Question: Can this cocycle be written in terms of $[z, \theta]$ and $\frac{1-z^2}{4}$?

We have

$$\theta \frac{1+z}{2} = \frac{1+z}{2}\theta + [\theta, \tfrac{z}{2}]$$

when left multiplied by $\theta \frac{1-z}{2}$ the first term contributes $\theta \frac{1-z^2}{4} \theta$, and in the second contribution we can move the $\frac{1-z}{2}$ to the left making a $\pm[\frac{z}{2}, \theta]^2$ and $\frac{1-z}{2}\theta[z,\theta]$ which will then lead to a $\theta \frac{1-z^2}{4}\theta$ contribution. Thus we recognize the typical setup of a Grassmannian connection:

$$\begin{pmatrix} [\nabla, \theta] & -\theta\, i^* dj \\ j^* di\, \theta & 0 \end{pmatrix}$$

which leads to polynomials in the terms $[\nabla, \theta]$ and $\theta \nabla^2 \theta$

In order to understand this algebra better let's return to our previous analysis of the Tate [redacted] residue situation. Let

$$I^{\pm} = R\left(\tfrac{1\pm z}{2}\right) + I$$

so that $\quad I^+ + I^- = R$

$$I^+ \cap I^- = I$$

Recall also that $\quad I = I^+I^- + I^-I^+$
since if $x \in I^+ \cap I^-$, then

$$x = x\frac{1+z}{2} + x\frac{1-z}{2} \in I^-I^+ + I^+I^-.$$

Before we constructed $S = (I^+ \to R) *_R (I^- \to R)$
which is a DGA with

$$\underbrace{\left(I^+ \otimes_R I^- \otimes_R \cdots\cdots\right)}_{n} \oplus \underbrace{\left(I^- \otimes_R I^+ \otimes_R \cdots\cdots\right)}_{n}$$

in degree $n > 0$. A simpler DGA
suitable for our purposes is the following.
We use 2×2 matrices over R

maybe $S_0 = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ No

$$S_0 = R = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \,\middle|\, x \in R \right\}$$

$$S_1 = \begin{pmatrix} 0 & I^- \\ I^+ & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} I^-I^+ & 0 \\ 0 & I^+I^- \end{pmatrix} \quad \text{etc.}$$

d is superbracketing with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Set

$$\eta = \begin{pmatrix} 0 & \frac{1-z}{2} \\ \frac{1+z}{2} & 0 \end{pmatrix} \qquad d\eta = 1$$

Superbracketing with η gives a degree 1
derivation, which is not of square zero,
rather the square is bracketing with $\eta^2 = \frac{1-z^2}{4}$,
which plays the role of curvature.

Observe: Natural hypotheses do not use an $A \subset R$. Instead one takes an R, I, z where $1-z^2, [z,R] \subset I$.

Now

$$[\eta, a] = \begin{pmatrix} 0 & [\frac{1-z}{2}, a] \\ [\frac{1+z}{2}, a] & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -[z,a] \\ [z,a] & 0 \end{pmatrix}$$

However it seems I am not interested in bracketing with η, rather multiplying by it. ?

I still haven't managed to explain why something like $\tau(\varepsilon(\eta\theta)^{2n})$ should be associated to

$$\begin{array}{ccccccc} 0 \longrightarrow & CC(R) & \longrightarrow & CC(S) & \longrightarrow & Q & \longrightarrow 0 \\ & & & & & \downarrow & \\ & & & & & S_I/R_I & \end{array}$$

Note: We have to have $S_0 = R$ and not $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ because we want $d: S_1 \to S_0$ to be onto. Here $d = [\gamma, \ ]$, $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and if $x \in \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ we have

$$[\gamma, \eta x]_+ = \underbrace{[\gamma, \eta]_+}_{1} x - \eta[\gamma, x]_-$$

So to have $d(\eta x) = 1$, we want $[\gamma, x]_- = 0$ i.e. for $x = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$.

Special case of perturbation theory.

Suppose one has a double complex in which each row is acyclic, and suppose a contracting homotopy is given for each row. Then we can construct a contracting homotopy for the total complex.

Let E, d be the complex with E^n the space of total degree n and with d the horizontal differential. The vertical differential is then an operator on θ of degree $+1$ such that

$$[d,\theta] = 0 \qquad \theta^2 = 0.$$

Let h be the operator of degree -1 given by the contracting homotopies in the rows, so that

$$[d,h] = 1$$

Let $E' = 0$, and $u = v = 0$.

Then perturbation gives the following contracting homotopy for the differential $d+\theta$

$$H = h\frac{1}{1-\theta h} = h + h\theta h + \cdots\cdots$$

We have in view the following situation. Let S be a contractible DGA such as the example considered yesterday. We consider the double complex $\mathrm{Hom}(\bar{B}(S)^{\natural}, \mathbb{C})$ of cyclic cochains on S. The rows are contractible. Recall that if h is a contracting homotopy for a complex V, then extending h to a derivation on $T(V)$ gives an operator $\tilde{h}$ such that $[d,\tilde{h}] = \widetilde{[d,h]} = \tilde{1}$ = multiplication by n on $V^{\otimes n}$. Thus we conclude a contracting homotopy of S should lead to a contracting homotopy for $\mathrm{Hom}(\bar{B}(S)^{\#}, \mathbb{C})$.

Recall that in perturbation theory one finds it convenient to have side conditions appropriate to a strong deformation retract situation. Thus one adds the condition

$$\begin{array}{c} E \\ u \downarrow \uparrow v \\ E' \end{array}$$

$uv = 1$ to $vu = 1 + [d,h]$ and requires $uh = hv = h^2 = 0$. This means the complementary complex $\mathrm{Ker}(u)$ is the "support" of h and that $h^2 = 0$.

Here why one can suppose $h^2 = 0$. In general start with $dh + hd = 1$. Then $(dh)^2 = (1-hd)\,dh = dh$ showing that dh is an idempotent and hd is the complementary idempotent. Thus we have

$$* \qquad d(hdh) + (hdh)d = (dh)^2 + (hd)^2 = dh + hd = 1$$

showing hdh is also a contracting homotopy. We claim $(hdh)^2 = 0$. One proof:

$$dh^2d = (dh)(hd) = 0$$ because these are complementary idempotents

In detail $(dh)(hd) = dh(1-dh) = dh - (dh)^2 = 0$

Can also prove $*$ by expanding $(dh+hd)^2$.

Note: A unitary DGA S is acyclic iff $d\eta = 1$ for some $\eta \in S_1$. The direction $\Longrightarrow$ follows since $d1 = 0$. Conversely given such an η we have

$$d(\eta x) = (d\eta)x - \eta\, dx = 1 - \eta\, dx$$

showing that $h(x) = \eta x$ is a contracting homotopy.

Given an algebra R with 1, let us consider the DGA S obtained by freely adjoining [redacted] to R in degree zero an element η of degree 1 with $d\eta = 1$. Then $S_1 = R\eta R \simeq R \otimes R$ is the free R-bimodule generated by η. A first candidate for S is the tensor algebra of this bimodule:

$$S = T_R^{\otimes}(R \otimes R)$$

so that $S_n = R^{\otimes(n+1)} = R(\eta R)^n$. We have to check that there is a unique degree -1 (anti)-derivation of this tensor algebra which extends the multiplication $m: R \otimes R \longrightarrow R$.

Take $x_0 \eta x_1 \eta \cdots \eta x_p \in S_p$. We have

$$d(x_0 \eta x_1 \eta \cdots \eta x_p) = \sum_{i=0}^{p-1} (-1)^i (\cdots \eta x_i x_{i+1} \eta \cdots)$$

so the differential appears to be b'. Let's check the (anti) derivation property.

$$d\{(x_0,\dots,x_p)(y_0,\dots,y_q)\} = d(x_0,\dots,x_p y_0,\dots y_q)$$

$$= \sum_{i=0}^{p-2}(-1)^i (x_0,\dots,x_i x_{i+1},\dots,x_p y_0,\dots y_q)$$

$$+ (-1)^{p-1}(x_0,\dots,x_{p-2},x_{p-1}x_p y_0, y_1,\dots,y_q)$$

$$+ (-1)^p (x_0,\dots,x_p y_0 y_1, y_2,\dots,y_q)$$

$$+ \sum_{j=1}^{q-1}(-1)^{p+j}(x_0,\dots,x_p y_0,\dots,y_j y_{j+1},\dots,y_q)$$

$$= \left(\sum_{i=0}^{p-1}(-1)^i (x_0,\dots,x_i x_{i+1},\dots,x_p)\right)(y_0,\dots,y_q)$$

$$+ (-1)^p (x_0,\dots,x_p)\sum_{j=0}^{q-1}(-1)^j (y_0,\dots,y_j y_{j+1},\dots,y_q)$$

$$= d(x_0,\dots,x_p)\cdot(y_0,\dots,y_q) + (-1)^p (x_0,\dots,x_p)\, d(y_0,\dots,y_q)$$

__Check__: $d\{(x_0,x_1)(y_0,y_1)\} = d(x_0,x_1y_0,y_1)$

$$= (x_0x_1y_0,y_1) - (x_0,x_1y_0y_1)$$

$$(d(x_0,x_1))(y_0,y_1) = (x_0x_1)(y_0,y_1) = (x_0x_1y_0,y_1)$$

$$-(x_0,x_1)\,d(y_0,y_1) = -(x_0,x_1)(y_0y_1) = -(x_0,x_1y_0y_1)$$

Now the interesting thing is that when we take the commutator ~~quotient~~ space of S we obtain the cyclic complex of R augmented by the trace map to $R_\natural$. In fact it seems that we have $S_\natural/R_\natural = \Sigma CC(R)$

August 22, 1989

Let $R \longrightarrow S$ be a DGA morphism, let $\eta \in S_1$ satisfy $d\eta = 1$, and let $\tau : S \longrightarrow V$ be a trace on S. Consider the hom. $R \longrightarrow S$ as a $\theta \in \mathrm{Hom}(B(R), S)$ satisfying $\delta\theta + \theta^2 = 0$ and $d\theta = 0$. Let $tr = \tau^{\natural} : \mathrm{Hom}(B(R), S) \longrightarrow \mathrm{Hom}(B(R)^{\natural}, V)$ be the induced trace on the algebra of cochains.

We have

$$d(\eta\theta) = \theta \qquad \delta(\eta\theta) = \eta\theta^2$$

hence

$$\begin{aligned} d\, tr(\eta\theta)^n &= tr\, d(\eta\theta)^n \\ &= tr \sum_{i=1}^{n} (\eta\theta)^{i-1}\theta(\eta\theta)^{n-i} \\ &= n\, tr\, \theta(\eta\theta)^{n-1} \end{aligned}$$

$$\begin{aligned} \delta\, tr(\eta\theta)^n &= tr \sum_{i=1}^{n} (\eta\theta)^{i-1}\eta\theta^2(\eta\theta)^{n-i} \\ &= n\, tr\, \theta(\eta\theta)^n \end{aligned}$$

Thus

$$\boxed{(d+\delta)\underbrace{\sum_{n\geq 1}\frac{(-1)^n}{n}\, tr(\eta\theta)^n}_{tr\,\log(1+\eta\theta)} = tr(\theta)}$$

So we see that $\tau(R) = 0$, then $tr\{\log(1+\eta\theta)\}$ is a cocycle in $\mathrm{Hom}(B(R)^{\natural}, V)$, i.e. we have a map of complexes $B(R)^{\natural} \to V$.

To see that there should be no problems

with "convergence", it would be useful to know that we can truncate $S_\bullet$. We suppose $S_n = 0$, $n<0$. In this case we claim

$$S_{>p} + dS_{p+1}$$

is an ideal in S closed under d. It is clearly closed under d and closed under left + right multiplication by elements of $S_{>0}$. But if $x \in S_0$, $y \in S_{p+1}$ then

$$x\,dy = d(xy) \in dS_{p+1}$$
$$(dy)x = d(yx)$$

so it is also closed under multiplication by elements of S_0. Notice that also $S_{>p} + \operatorname{Ker}\{d \text{ on } S_p\}$ is an ideal closed under d.

The DGA

$$S/S_{>p} + dS_{p+1}$$

has the <u>same homology as S</u> in the range $[0,p]$

Let's consider

$$\begin{tikzcd} \bar{B}(R)^{\natural} \arrow[r] \arrow[d] & \bar{B}(S)^{\natural} \arrow[d] & \\ R_{\natural} \arrow[r] & S_{\natural} \arrow[r] & S_{\natural}/R_{\natural} \end{tikzcd}$$

There are two reasons why the composed map $\bar{B}(R)^{\natural} \to S_{\natural}/R_{\natural}$ is null-homotopic. (This composition is the cocycle $\tau^{\natural}(\theta)$, where θ corresponds to $R \to S$, and $\tau : S \to S_{\natural}/R_{\natural}$ is canonical map.) First of all it is zero, secondly $\bar{B}(S)^\natural$ is acyclic. The "difference" of these two ~~reasons~~ gives a map $\bar{B}(R)^\natural \longrightarrow R_\natural/S_\natural$ well-defined up to homotopy. To explicitly construct this map, we take the cyclic cocycle [redacted]

$$tr(\hat{\theta}) \in \mathrm{Hom}\,(\bar{B}(S)^\natural, S_\natural)$$

where $\hat{\theta}$ is the canonical universal twisting cochain, and we write it as a coboundary

$$(d+\delta)\,tr\log(1+\eta\hat{\theta}) = tr(\hat{\theta})$$

and we restrict the cochain $tr\log(1+\eta\hat{\theta})$ to get a map of complexes

$$B(R)^\natural \longrightarrow B(S)^\natural \xrightarrow{tr\log(1+\eta\hat{\theta})} S_\natural \longrightarrow S_\natural/R_\natural$$

Let A be a unital algebra. Then we can take the DGA $T_A^{\bullet}(A\otimes A)$ and form a quotient DGA

$$\begin{tikzcd}
{} \arrow[r] & A\otimes A^{\otimes 2}\otimes A \arrow[r, "d"] \arrow[d] & A\otimes A\otimes A \arrow[r, "d"] \arrow[d] & A\otimes A \arrow[r, "d"] \arrow[d, "1"] & A \arrow[d, "1"] \\
{} \arrow[r] & A\otimes \bar{A}^{\otimes 2}\otimes A \arrow[r, "d"] & A\otimes \bar{A}\otimes A \arrow[r, "d"] & A\otimes A \arrow[r, "d"] & A
\end{tikzcd}$$

which should be the quotient by the relation $\eta^2 = 0$. We already know that $d = b'$ passes to this quotient, so we only have to check that the equivalence relation is compatible with multiplication

$$(x_0,\dots,x_p)(y_0,\dots,y_q) = (x_0,\dots,x_{p-1},x_p y_0, y_1,\dots,y_q)$$

which is completely clear. As a further check we note that $d(\eta^2) = d\eta\cdot\eta - \eta\cdot d\eta = \eta-\eta = 0$ so the ideal generated by η^2 is closed under d.

Next let us consider the commutator quotient complex. Let $S = T_A(A\otimes A)$ with diffl b' and let $S' = T_A(A\otimes A)/(\eta^2=0)$. It is clear that $S'\otimes_A$ is the complex with $A\otimes \bar{A}^{\otimes(n-1)}$ in degree $n\geq 1$ and the differential b. Thus it is the normalized Hochschild complex with augmentation to $A_\natural = A/[A,A]$. As $S'/[S',S']$ is a quotient of $S/[S,S]$, which can clearly be identified with $CC(A)$ augmented to $A_\natural$, it seems clear that $S'/[S',S']$ has to be the <u>reduced cyclic complex</u> of A, except for there being an A instead of $\bar{A}$ and also an augmentation to $A_\natural$.

It is now time to consider again the original analytical situation. Let us consider an algebra A acting on a Hilbert space H, let z be an operator on H such that $[z,A]$ and $1-z^2 \in \mathcal{L}^p$.

If $z^2 = 1$, then Connes has defined cyclic cocycles of odd degree

$$\operatorname{tr}\left(\frac{z}{2}\left[\frac{z}{2}, \theta\right]^{2n}\right) \qquad 2n \geq p$$

We now wish to generalize his cocycles to the general case. We propose to introduce the DGA S generated freely by A in degree 0 and η in degree 1 with $d\eta = 1$, and then construct traces on S yielding the desired cocycles.

This means that we have to assign operators on H to elements of S. We proceed as follows.

Form $H \otimes C_1 = H \oplus H$ with $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. (Doubling seems necessary as we propose to dilate z.) $\mathcal{L}(H \oplus H)$ is a superalgebra with grading given by conjugation by ε; bracketing with γ gives an odd derivation of square zero (since $(\operatorname{ad}\gamma)^2 = \operatorname{ad}(\gamma^2) = 0$). Thus we have an acyclic $\mathbb{Z}/2$-graded differential algebra. We can spread this out to be a periodic DGA and then truncate to get a DGA $\tilde{S}$ with

$$\tilde{S}_0 = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \middle| \, x \in \mathcal{L}(H) \right\}, \quad \left.\begin{aligned} \tilde{S}_{2n-1} &= \mathcal{L}(H \oplus H)^- \\ \tilde{S}_{2n} &= \mathcal{L}(H \oplus H)^+ \end{aligned}\right\} \; n \geq 1$$

and with differential given by ad(γ).

Then
$$\eta = \begin{pmatrix} 0 & \frac{1-z}{2} \\ \frac{1+z}{2} & 0 \end{pmatrix} \in \tilde{S}_1$$

satisfies $d(\eta) = 1$.

Now we have $A \longrightarrow \tilde{S}_0$ and so we get a DGA map

$$T_A(A \otimes A) \longrightarrow \tilde{S}$$

$$(a_0, \dots, a_n) \longmapsto a_0 \eta a_1 \cdots \eta a_n$$

The only traces around on $\tilde{S}$ seems to be the supertrace $\operatorname{tr} \varepsilon$ in the even degrees. The p-summable hypothesis implies that $a_0 \eta a_1 \cdots \eta a_n$ is of trace class for $n \gg 0$. Thus we have odd cyclic cocycles

$$\frac{1}{2n} \operatorname{tr} \varepsilon (\eta \theta)^{2n} \eta$$

defined for large n.

Since $\eta\theta$ is even in $\mathrm{Hom}(B(A), S)$, there are no signs in evaluating this cocycle, and we get

$$\frac{1}{2n}\left\{ \operatorname{tr}\left(\varepsilon(\eta a_1)\cdots(\eta a_{2n})\right) + \begin{matrix}\text{cyclic perms} \\ \text{with signs}\end{matrix} \right\}$$

However the first term is already λ-invariant, so we get

$$\operatorname{tr} \varepsilon (\eta a_1)\cdots(\eta a_{2n}) = \operatorname{tr}\left(\frac{1-z}{2} a_1 \frac{1+z}{2} a_2 \cdots \frac{1+z}{2} a_{2n}\right)$$
$$- \operatorname{tr}\left(\frac{1+z}{2} a_1 \frac{1-z}{2} a_2 \cdots \frac{1-z}{2} a_{2n}\right)$$

If we want we can up to changing sign move η to the right of θ:

$$\mathrm{tr}\,\varepsilon(\eta\theta)^{2n} = -\mathrm{tr}\{\eta\,\varepsilon(\theta\eta)^{2n-1}\theta\} = -\mathrm{tr}(\varepsilon(\theta\eta)^{2n})$$

so from now on we work with $\mathrm{tr}\,\varepsilon(\theta\eta)^{2n}$.

Now we want to rearrange this cochain so that it is written in terms of the cochain $[\eta,\theta]$ and η^2 which have values in the Schatten ideal. We have

$$\theta\eta = [\theta,\eta] - \eta\theta$$

$$(\theta\eta)^2 = ([\theta,\eta]-\eta\theta)[\theta,\eta] - \theta\eta^2\theta$$

$$= [\theta,\eta]^2 + (-\theta\eta^2\theta) - \eta\theta[\theta,\eta]$$

Set $A = [\theta,\eta]$, $B = (-\theta\eta^2\theta)$. Then

$$\theta\eta = A - \eta\theta$$

$$(\theta\eta)^2 = (A^2+B) - \eta\theta A$$

$$(\theta\eta)^3 = \underbrace{A(A^2+B) + BA}_{A^3+AB+BA} - \eta\theta(A^2+B)$$

$$(\theta\eta)^4 = A(A^3+AB+BA) + B(A^2+B) - \eta\theta(A^3+AB+BA)$$

The pattern is clear. Assign A the degree 1 and B the degree 2 and let $P^{(k)}(A,B)$ be the sum of all (noncomm.) monomials in A,B of total degree k. Then

$$(\theta\eta)^n = P^{(n)}(A,B) - \eta\theta\,P^{(n-1)}(A,B)$$

For arguing by induction

$$(\theta\eta)^{n+1} = \underbrace{(AP^{(n)} + BP^{(n-1)})}_{P^{(n+1)}} - \eta\theta P^{(n)}$$

Next we compute $-\mathrm{tr}\,\varepsilon(\theta\eta)^{2n}$. Consider conjugation by γ.

This interchanges $\frac{1+z}{2}$, $\frac{1-z}{2}$ in η, so we have

$$\gamma\eta\gamma = \tilde{\eta} = \begin{pmatrix} 0 & \frac{1+z}{2} \\ \frac{1-z}{2} & 0 \end{pmatrix}$$

where $\eta + \tilde{\eta} = \gamma$. Thus we have

$$\gamma\theta\gamma = \theta \quad , \quad \gamma\varepsilon\gamma = -\varepsilon \quad , \quad \text{and}$$

$$\gamma A \gamma = \gamma[\theta,\eta]\gamma = [\theta, \gamma-\eta] = -[\theta,\eta] = -A$$

$$\gamma B \gamma = \gamma(-\theta\eta^2\theta)\gamma = B$$

since $\eta^2 = \begin{pmatrix} \frac{1-z^2}{4} & 0 \\ 0 & \frac{1-z^2}{4} \end{pmatrix}$ commutes with γ. We have

$$-\operatorname{tr} \varepsilon(\theta\eta)^{2n} = \underbrace{-\operatorname{tr}\{\varepsilon P^{(2n)}(A,B)\}} + \operatorname{tr}\{\varepsilon\eta\theta P^{(2n-1)}(A,B)\}$$

$$-\operatorname{tr}(\gamma\varepsilon P^{(2n)}(A,B)\gamma) = +\operatorname{tr}(\varepsilon P^{(2n)}(-A,B)) = +\operatorname{tr}\{\varepsilon P^{(2n)}(A,B)\}$$

so this term is zero.

$$\operatorname{tr}\{\varepsilon\eta\theta P^{(2n-1)}\} = \operatorname{tr}\{\gamma\varepsilon\eta\theta P^{(2n-1)}\gamma\} = \operatorname{tr}(-\varepsilon)\tilde{\eta}\theta(-P^{(2n-1)})\}$$

Adding, using $\eta + \tilde{\eta} = \gamma$, we get

$$\boxed{+\operatorname{tr} \varepsilon(\eta\theta)^{2n} = \frac{1}{2}\operatorname{tr}\left\{\varepsilon\gamma\theta P^{(2n-1)}([\eta,\theta], -\theta\eta^2\theta)\right\}}$$

for our cocycle.

Let's simplify this. We have

$$[\eta,\theta] = \left[\begin{pmatrix} 0 & \frac{1-z}{2} \\ \frac{1+z}{2} & 0 \end{pmatrix}, \theta\right] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left[\frac{z}{2},\theta\right]$$

$$= \gamma\varepsilon\otimes\left[\frac{z}{2},\theta\right]$$

$$\theta\eta^2\theta = 1\otimes\theta\left(\frac{1-z^2}{4}\right)\theta$$

$$p^{(2n-1)}\left(\gamma\varepsilon\otimes\left[\frac{z}{2},\theta\right], -1\otimes\theta\left(\frac{1-z^2}{4}\right)\theta\right)$$

$$= \sum_{k+2l=2n-1} P_{kl}\left(\gamma\varepsilon\otimes\left[\frac{z}{2},\theta\right], -1\otimes\theta\left(\frac{1-z^2}{4}\right)\theta\right)$$

Now when we take a monomial occurring in P_{kl}, and we move the factors $\gamma\varepsilon\otimes 1$ to the left, we encounter signs because $\gamma\varepsilon\otimes 1$ is odd and so is $1\otimes[\frac{z}{2},\theta]$. No other signs occur because $\gamma\varepsilon\otimes 1$ commutes strictly with $1\otimes\theta\left(\frac{1-z^2}{4}\right)\theta$. The sign is $(-1)^{0+1+\cdots+k-1} = (-1)^{k(k-1)/2} = (-1)^{(k-1)/2}$ since k is odd.

Thus we have

$$\operatorname{tr}\,\varepsilon(\eta\theta)^{2n} = \frac{1}{2}\operatorname{tr}\left(\varepsilon\gamma\theta\, p^{(2n-1)}(\ldots\ldots)\right)$$

$$= \frac{1}{2}\operatorname{tr}\left(\varepsilon\gamma\otimes\theta\right)\sum_{k+2l=2n-1} P_{kl}\left(\gamma\varepsilon\otimes\left[\frac{z}{2},\theta\right], -1\otimes\theta\left(\frac{1-z^2}{4}\right)\theta\right)$$

$$= \frac{1}{2}\operatorname{tr}\sum_{k+2l=2n-1}\underbrace{(-1)^{(k-1)/2}\varepsilon\gamma(\gamma\varepsilon)^k(-1)^k}\otimes\theta P_{kl}\left(\left[\frac{z}{2},\theta\right], -\theta\left(\frac{1-z^2}{4}\right)\theta\right)$$

$$(-1)^{(k-1)/2}(-1)(-1)^k\underbrace{(\gamma\varepsilon)^{k+1}}_{(-1)^{k+1/2}} = (-1)$$

$$\boxed{= \frac{-1}{2}\operatorname{tr}\left\{\theta\, p^{(2n-1)}\left(\left[\frac{z}{2},\theta\right], -\theta\left(\frac{1-z^2}{4}\right)\theta\right)\right\}}$$

Take $n=1$. Then

$$\left(\operatorname{tr}\,\varepsilon(\eta\theta)^2\right)(a_1,a_2) = \operatorname{tr}(\varepsilon\eta a_1\eta a_2) = \operatorname{tr}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&\frac{1-z}{2}\\ \frac{1+z}{2}&0\end{pmatrix}a_1\left(\quad\right)a_2\Big\}$$

$$= \operatorname{tr}\left(\frac{1-z}{2}a_1\,\frac{1+z}{2}a_2\right) - \operatorname{tr}\left(\frac{1+z}{2}a_1\,\frac{1-z}{2}a_2\right)$$

$$= \operatorname{tr}\left(\frac{a_1a_2 - za_1a_2 + a_1za_2 - za_1za_2}{4}\right) - \left(\begin{smallmatrix}\text{same but}\\ z\mapsto -z\end{smallmatrix}\right)$$

$$= \frac{1}{2}\operatorname{tr}\left(-za_1a_2 + a_1za_2\right) = -\frac{1}{2}\operatorname{tr}\left([z,a_1]\,a_2\right)$$

$$= \frac{1}{2}\operatorname{tr}\left(a_1[z,a_2]\right).$$

But $\left(\frac{1}{2}\operatorname{tr}\left(\theta\left[\frac{z}{2},\theta\right]\right)\right)(a_1,a_2) = -\frac{1}{2}\operatorname{tr}\left(a_1\left[\frac{z}{2},a_2\right]\right)$,

the sign occurring because $\left[\frac{z}{2},\theta\right]$ is odd.

Notice that once we know the above expression gives a cyclic cocycle, then we can obtain similar expressions by rescaling tricks. For example multiplying by 2^{2n-1} we can remove the 2, 4 in the denominators of $P^{(2n-1)}$. Then replacing z by $\frac{z}{t}$ and multiplying by t^{2n-1} we obtain

$$\operatorname{tr}\left\{\theta\, P^{(2n-1)}\left([z,\theta],\, -\theta(t^2-z^2)\theta\right)\right\}$$

and finally replacing z by $z+c$ we can replace t^2-z^2 by any quadratic polynomial in z with leading term $-z^2$, such as for example $z-z^2$, which is appropriate for z being an idempotent modulo the ideal.

August 24, 1989

We yesterday obtained the cochains

$$-\frac{1}{2}\,\mathrm{tr}\left\{\theta\, P^{(2n-1)}\left(\left[\frac{z}{2},\theta\right],\ -\theta\left(\frac{1-z^2}{4}\right)\theta\right)\right\}$$

which are cyclic cocycles. It's better to write them in the form

$$(*)\qquad \boxed{-\frac{1}{2^{2n}}\,\mathrm{tr}\left\{\theta\, P^{(2n-1)}\left([z,\theta],\ -\theta(1-z^2)\theta\right)\right\}}$$

Let's consider the case where $z^2=1$, in which case let's write F [redacted] instead of z. Recall [redacted] the character forms over the Grassmannian:

$$\theta = \underbrace{\frac{\theta+F\theta F}{2}}_{\rho} + \underbrace{\frac{\theta-F\theta F}{2}}_{\alpha\,=\,\frac{1}{2}F[F,\theta]} \qquad \alpha^2 = -\frac{1}{4}[F,\theta]^2$$

$$0 = (\delta+\rho+\alpha)^2 = \underbrace{[\delta+\rho,\alpha]}_{0} + \underbrace{(\rho^2+\alpha^2)}_{0}$$

[redacted] Curvature $= \rho^2 = -\alpha^2 = \left[\frac{F}{2},\theta\right]^2$

Character forms: $\frac{1}{n!}\,\mathrm{tr}\left(\frac{1+F}{2}\left[\frac{F}{2},\theta\right]^{2n}\right) = \frac{1}{n!\,2^{2n+1}}\,\mathrm{tr}\left(F[F,\theta]^{2n}\right)$

Also $\quad \mathrm{tr}\left(F[F,\theta]^{2n}\right) = \mathrm{tr}\left((\theta-F\theta F)[F,\theta]^{2n-1}\right) = 2\,\mathrm{tr}\left(\theta[F,\theta]^{2n-1}\right).$

So the character forms are

$$\frac{1}{n!\ [\text{redacted}]\ 2^{2n+1}}\,\mathrm{tr}\left(F[F,\theta]^{2n}\right) = \frac{1}{n!\,2^{2n}}\,\mathrm{tr}\left(\theta[F,\theta]^{2n-1}\right)$$

[redacted] This shows that thes [redacted] cyclic cocycles $(*)$ agree with Connes cyclic cocycles in the case of an involution. Next we want to extend these via dilations.

What this means is that we embed our setup as a direct factor of a larger one such that z is the contraction of an involution. Let

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \theta e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} z & \sqrt{1-z^2} \\ \sqrt{1-z^2} & -z \end{pmatrix} \qquad \text{or} \qquad F = \begin{pmatrix} z & 1-z \\ 1+z & -z \end{pmatrix}$$

Each of these F's is an involution such that $eFe = z$. To see these are involutions we write them as a sum of anti-commuting elements

$$\varepsilon z + \gamma^1 \sqrt{1-z^2} \qquad \text{or} \qquad \varepsilon z + \gamma^1 + \gamma^2 \frac{1}{i} z$$

The [redacted] interest of the first F is that it is nice for applications to the graded case, although one has to enlarge the algebra

Let's consider F of the form

$$F = \begin{pmatrix} z & b \\ c & -z \end{pmatrix}$$

This will be an involution provided b, c commuting with z and $bc = cb = 1 - z^2$:

$$\begin{pmatrix} z & b \\ c & -z \end{pmatrix} \begin{pmatrix} z & b \\ c & -z \end{pmatrix} = \begin{pmatrix} z^2 + bc & zb - bz \\ cz - zc & z^2 + cb \end{pmatrix}$$

We have

$$[F, \theta e] = \left[\begin{pmatrix} z & b \\ c & -z \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} [z,\theta] & -\theta b \\ c\theta & 0 \end{pmatrix}$$

Formula:

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^n = \begin{pmatrix} P^{(n)}(A,BC) & P^{(n-1)}(A,BC)B \\ CP^{(n-1)}(A,BC) & CP^{(n-2)}(A,BC)B \end{pmatrix}$$

where $P^{(n)}(A,BC)$ is the sum of all the monomials in A, BC of total degree n, where $\deg(A)=1$, $\deg(BC)=2$.

Thus

$$P^{(0)}(A,BC) = 1$$

$$P^{(1)}(A,BC) = A$$

$$P^{(2)}(A,BC) = A^2 + BC$$

and

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^2 = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} A^2+BC & AB \\ CA & CB \end{pmatrix}$$

showing the formula is correct for $n = 0, 1, 2$.

The general case then follows by induction

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} P^{(n)} & P^{(n-1)}B \\ CP^{(n-1)} & CP^{(n-2)}B \end{pmatrix} = \begin{pmatrix} AP^{(n)} + BCP^{(n-1)} & \begin{pmatrix} AP^{(n-1)} \\ + \\ BCP^{(n-2)} \end{pmatrix}B \\ CP^{(n)} & CP^{(n-1)}B \end{pmatrix}$$

$$= \begin{pmatrix} P^{(n+1)} & P^{(n)}B \\ CP^{(n)} & CP^{(n-1)}B \end{pmatrix}$$

Using this formula we get

$$tr\left(\theta e[F,\theta e]^{2n-1}\right) = tr\left\{\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}p^{(2n-1)} & *\\ * & *\end{pmatrix}\right\}$$

$$= tr\left\{\theta\, p^{(2n-1)}(\underset{[z,\theta]}{\underset{\shortparallel}{A}},\ \underset{-\theta bc\theta\ =\ -\theta(1-z^2)\theta}{\underset{\shortparallel}{BC}})\right\}$$

giving us the cocycles

$$tr\left\{\theta\, p^{(2n-1)}\left([z,\theta],\ -\theta(1-z^2)\theta\right)\right\}$$

What is the relation between the different $F = \begin{pmatrix} z & b\\ c & -z\end{pmatrix}$ and the GNS analysis we gave 44 pages ago. Recall we considered $A = \mathbb{C}[\gamma] \xrightarrow{\rho} B = \mathbb{C}[z]$, $\begin{cases}\rho(1) = 1\\ \rho(\gamma) = z\end{cases}$

$(\delta\rho + \rho^2)(\gamma,\gamma) = \rho(\gamma^2) - \rho(\gamma)^2 = 1 - z^2$. We considered the dilations

$$B \xrightarrow{i} A \otimes B \xrightarrow{i^*} B$$

$$B \xrightarrow{i} Hom(A,B) \xrightarrow{i^*} B$$

In either case the big (A,B)-bimodule E is $\mathbb{C}[z]^2$ with γ acting as $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. In the first case $i = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad i^* = (1\ \ z)$. We look for an isomorphism T carrying i, i^* into standard form

$$\begin{tikzcd}
 & \mathbb{C}[z]^2 \arrow[dr, "(1\ \ z)"] \arrow[dd, "T"] & \\
\mathbb{C}[z] \arrow[ur, "\binom{1}{0}"] \arrow[dr, "\binom{1}{0}"'] & & \mathbb{C}[z] \\
 & \mathbb{C}[z]^2 \arrow[ur, "(1\ \ 0)"'] &
\end{tikzcd}$$

Then $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, $(1\ 0)T = (1\ z)$

so $T = \begin{pmatrix}1 & z\\0 & ?\end{pmatrix}$ so we can take

$T = \begin{pmatrix}1 & z\\0 & 1\end{pmatrix}$. Then the involution F becomes

$$F = T\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}T^{-1} = \begin{pmatrix}1 & z\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & -z\\0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & z\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & -z\end{pmatrix} = \begin{pmatrix}z & 1-z^2\\1 & -z\end{pmatrix}$$

Similarly in the second case we have

$i = \begin{pmatrix}1\\z\end{pmatrix}$ $\quad i^* = (1\ \ 0)$ $\quad T = \begin{pmatrix}1 & 0\\-z & 1\end{pmatrix}$ and

$$F = \begin{pmatrix}1 & 0\\-z & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\z & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\-z & 1\end{pmatrix}\begin{pmatrix}z & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}z & 1\\1-z^2 & -z\end{pmatrix}$$

August 27, 1989

Given a Riemannian manifold M one has Brownian motion on it. This gives ~~a~~ measures depending on t, the Wiener measures, on the space P_{xy} of continuous paths from x to y, for any $x, y \in M$. Given a bundle E, a connection ∇ in E, and an endomorphism V of E (called the "potential"), one can considered parallel transport along smooth paths from x to y [redacted] ~~twisted~~ modified by the potential V. This gives an invertible operator [redacted]

$$X_{txy}(\gamma) \in \mathrm{Hom}(E_y, E_x)$$

where γ is the path. The time is relevant because of the potential V. The theory of stochastic differential equations extends the definition of $X_{txy}(\gamma)$ to a.e. $\gamma \in P_{xy}$. When one integrates X_{txy} with respect to the Wiener measure one obtains the kernel $\langle x | e^{-tH} | y \rangle$ where $H = -\nabla^2 + V$. [redacted]

We emphasize that all the heat kernels associated to generalized Laplaceans on M are obtained by integration of suitable modified parallel transport operators with respect to the same measure. This is [redacted] apparently related to the fact that one can control lower order perturbations of [redacted] the Laplacean in terms of the Laplacean itself. (Ezra's estimates for $\langle A_0, \dots, A_k \rangle = \int_{\Delta_k} \mathrm{str}(A_0 e^{t_0 D^2} \cdots A_k e^{t_k D^2}) dt_1 \dots dt_k$

where A_i are first order ΨDO's.)

The theory of stochastic DE's tells us that although we must work with continuous paths to use the Wiener measure, certain kinds of expressions associated to differentiable paths have a meaning a.e. for continuous paths. Such expressions are things like parallel transport with respect to a connection.

August 28, 1989

Let's discuss some of Ezra's ideas.

Witten current. From the original Witten-Atiyah work one learns that for supersymmetric quantum mechanics on M the path integral should be a linear functional on differential forms on the free loop space LM. This is the Witten current. Getzler proposes that it should be given on a cylinder form $e^*_{t_1}(a_1)\dots e^*_{t_n}(a_n)$, $0\le t_1<\dots<t_n<1$, by

$$tr_s\left(e^{t_1 D^2}c(a_1)\dots c(a_n)e^{(1-t_n)D^2}\right)$$

where $a_i\in\Omega(M)$ and $c:\Omega(M)\simeq\Gamma(C(M))$ is skew-symmetrization isomorphism. This should be reasonable I think for $a_i\in\Omega^0(M)$ or $\Omega^1(M)$ because one has $c(df)=[D,f]$. The path integral should give n point functions at distinct times. Apparently Lott CMP 108(1987)605 has rigorously constructed this path integral.

Notation: $e_t : LM \longrightarrow M$ evaluation at time t,

$$(*) \qquad \langle \mu_{\not D}, e_{t_1}^* a_1 \cdots\cdots e_{t_n}^* a_n \rangle = tr_s\left(e^{t_1 \not D^2} c(a_1) \cdots c(a_n) e^{(1-t_n)\not D^2}\right)$$

A basic difficulty is this is discontinuous as two t_i coallesce, since $c(a_1)c(a_2) \neq c(a_1 a_2)$

<u>Witten current is a density wrt Wiener measure.</u>

The heat kernel for $\not D^2$ is obtained by integration wrt Wiener measure a parallel transport operator

$$X(t_1, t_2)(\gamma) \in Iso(E_{\gamma_{t_2}}, E_{\gamma_{t_1}})$$

defined for a.e. paths. We have

$$tr_s\left(e^{t_1 \not D^2} c(a_1) \cdots c(a_n) e^{(1-t_n)\not D^2}\right)$$

$$= \int_{LM} tr_{s, E_{\gamma_0}} \left(X(0,t_1)(\gamma) c(a_1)(\gamma_{t_1}) \cdots X(t_n, 1)(\gamma) \right) d\nu(\gamma)$$

In other words we assign to the cylinder form $e_{t_1}^* a_1 \cdots e_{t_n}^* a_n$ a function on paths and then integrate with respect to Wiener measure. Presumably this function is obtained from some section of the exterior algebra of the tangent bundle.

This is a good question, namely whether the Witten current, whatever it is, is given by pairing with a section of the exterior algebra of the tangent bundle followed by integration with respect to Wiener measure.

Ezra considers iterated integrals

$$\circledast \qquad \int_{0<t_1<\cdots<t_n<1} \iota e_{t_1}^* a_1 \cdots \iota e_{t_n}^* a_n \; dt_1 \cdots dt_n$$

where $\iota = i(\partial_t)$. He talks about the "value" of an iterated integral at a loop γ

The "value" of $e_{t_1}^* a_1 \cdots e_{t_n}^* a_n$ at γ appears to be the collection of multilinear functionals $a_1(\gamma_{t_1}), \ldots, a_n(\gamma_{t_n})$. The "value" of $\iota e_t^* a$ at γ appears to be the multilinear functional $\iota(\dot\gamma_t) a(\gamma_t)$ on the tangent space at γ_t.

He proposes to interpret the "value" of the iterated integral $\circledast$ at γ to be the operator on E_{γ_0} given by

$$X(0,t_1)\; c\big(i(\dot\gamma_1)\, a_1(\gamma_1)\big)\, X(t_1,t_2) \cdots \qquad X(t_n,1).$$

What is interesting about this operator is that as he shows it can be interpreted in terms of parallel transport with respect to a connection $\nabla_D + c(A)$ on some auxiliary bundle. This means that when the integral is done with respect to Wiener measure one gets the heat kernel for $D\!\!\!/^{2} + C$, where C is a curvature built out of

$$[D, c(a)] - c(da) \qquad c(a_1 a_2) - c(a_1)c(a_2)$$

Ezra is obviously doing something correct and interesting, but it is not clear what it is.

It seems that the real [redacted] problem [redacted] from his viewpoint of starting with cylinder forms is how to relate $\iota(\partial_t)e_t^* a$ to cylinder forms.

A good question is how to interpret the Witten current applied to this ~~thing~~ form.