April 16, 1989

I want to discuss the basic K-theory and index theory associated to Fredholm modules. What I want to describe is the elementary, better primitive, things one should keep in mind. I want things of "topological" character like K-classes, indices. The theme in any cohomology theory is one first studies the classes and then may ways of representing them.

I think I am trying to describe elements of $KK(A, \mathbb{C})$, i.e. K-"homology". According to Kasparov one looks at representations of A on a Hilbert space H together with an F on H such that $F = F^*$, $F^2 - I \in \mathcal{K}$, $[F, a] \in \mathcal{K}$.

Thus the universal algebra is the following: One starts with H and an idempotent in the Calkin algebra $\mathcal{Q}$, a nontrivial selfadjoint idempotent. Then A is the subalgebra of $\mathcal{L}(H)$ consisting of operators commuting with this idempotent. In this way we describe the universal ungraded Fredholm setup.

In the ungraded case there is exactly one possibility since up to isomorphism there is only one projector in $\mathcal{Q}$ which is nontrivial.

In the graded case F is given by a Fredholm operator and its adjoint $H^+ \rightleftarrows H^-$ and there is an obvious index invariant. But the universal algebra will be pairs of operators $a^\pm \in \mathcal{L}(H^\pm)$ which modulo compacts coincide with respect to the isomorphism $\mathcal{L}(H^+) \simeq \mathcal{L}(H^-)$ given by the Fredholm operator.

Now consider the pairing with K-theory. In the ungraded case we have to define an index map from $K_1 A$ to $\mathbb{Z}$. In the graded case we need to define an index map from $K_0 A$ to $\mathbb{Z}$.

April 17, 1989

Discuss the JLO cocycle to see if we can obtain it via the Cuntz algebra. Set

$$R = (\delta + \theta + \sigma X)^2 = X^2 + \sigma[X,\theta]$$

We recall that the JLO is given by the various homogeneous components of the cochains

$$\varphi = tr_s(e^{uR})$$

$$\psi = tr_s(\partial\theta\, e^{uR})$$

I think that these cochains satisfy

$$\delta\varphi = \beta\psi$$

$$\delta\psi = \frac{1}{u}\bar{\partial}\varphi$$

where one really has to normalize the homogeneous components properly.

Now the viewpoint we want to adopt is that at least for the finite dimensional Dirac case the failure of the L.T. of φ, namely $tr_s\left(\frac{1}{\lambda - R}\right)$ to be defined is due to the ambiguity of extending φ, ψ for $u > 0$ to distributions supported on $[0,\infty)$. In other words a sufficiently high derivative

$$(-\partial_\lambda)^N tr_s\left(\frac{1}{\lambda - R}\right) = tr_s\left(\frac{N!}{(\lambda - R)^{N+1}}\right)$$

is well-defined, or equivalently multiplying by a power of u makes φ, ψ extend to distributions. If we follow this viewpoint, then we ought to be able to construct the JLO cocycle on the resolvent side.

Thus instead of e^{uR} we consider $\frac{1}{\lambda-R}$. We have the Bianchi identity

$$\delta \frac{1}{\lambda-R} = -\left[\theta+\sigma X, \frac{1}{\lambda-R}\right]$$

and

$$\partial \frac{1}{\lambda-R} = \frac{1}{\lambda-R}\, \partial R\, \frac{1}{\lambda-R}$$

$$= \frac{1}{\lambda-R}\, \partial[\sigma X, \theta]\, \frac{1}{\lambda-R}$$

and

$$\delta\left\{\partial\theta \frac{1}{\lambda-R}\right\} = \partial \overset{-\theta^2}{\overbrace{\delta\theta}} + \partial\theta\left[\theta+\sigma X, \frac{1}{\lambda-R}\right]$$

$$= -\theta\,\partial\theta \frac{1}{\lambda-R} + \partial\theta \frac{1}{\lambda-R}\theta$$

$$-\left[\sigma X, \partial\theta\frac{1}{\lambda-R}\right] + \partial[\sigma X, \theta]\frac{1}{\lambda-R}$$

This last can be written

$$\left[\delta+\theta+\sigma X, \ \partial\theta\frac{1}{\lambda-R}\right] = [\delta+\theta+\sigma X, \partial\theta]\frac{1}{\lambda-R}$$

$$[\delta+\theta, \partial\theta] = \partial(-\theta^2) + \theta\,\partial\theta + \partial\theta\,\theta = 0$$

I think it is now clear what to expect namely we have these operator identities, and differentiating wrt λ enough times puts them into the trace class ideal.

So now it should be clear what we have. We have an algebra of operators depending holomorphically on λ and some sort of ideal.

Some difficulties: The problem with trying to find big cocycles via the resolvent $\frac{1}{\lambda-R}$ is that we must differentiate, i.e. the identi~~ties~~ will be

$$\delta\varphi = \beta\psi$$

$$(-\partial_\lambda)\delta\psi = \bar{\partial}\varphi$$

and we must take L-transform to [redacted] convert the $(-\partial_\lambda)$ to u. Thus it seems that only the JLO cocycle will be obtained

One should note that the JLO cocycle does not satisfy the symmetry condition since

$$\varphi(a_1,\dots,a_n) = \int_{\Delta(n)} tr_s\left(e^{t_0X^2}[X,a_1]\cdots[X,a_n]e^{t_nX^2}\right)dt_1\cdots dt_n$$

is apparently not cyclically symmetric. Thus it seems that the JLO cocycle does not come from a supertrace on the Cuntz algebra.

Another point: In the ~~situation~~

$$\theta = \rho + \alpha = \theta^+ + \theta^-$$

where $[\delta+\rho,\alpha]=0$, $(\delta+\rho)^2+\alpha^2=0$ we have

$$[\delta,-\alpha^2] = -\cancel{[\delta+\rho,\alpha^2]} + [\rho,\alpha^2]$$

$$= [\theta,\alpha^2] \qquad \text{since } [\alpha,\alpha^2]=0.$$

April 18, 1989

In my cochain paper I noted that the JLO cocycle can be viewed as a superconnection character form in the cochain formalism. It seems worthwhile to go over this carefully and to try to obtain some sort of geometric insight.

Suppose given A, H, X as usual, [redacted] we form $M_n(A) = A \otimes M_n$, $H \otimes \mathbb{C}^n_0$, $X = X \otimes 1$ and let $G = U_n$, $\mathcal{G} = U_n(A)$. [redacted] I recall that I got a lot of insight in the Fredholm situation (where one has F instead of X) by using the map $g \mapsto gFg^{-1}$ to the Grassmannian. [redacted] Pulling back via this map the character forms gives [redacted] cyclic cocycles, and by pulling back the subbundle, connection form, curvature we were led to algebra cochains.

Let's try the same process here. We have the trivial [redacted] Hilbert space bundle over $\mathcal{G}$ with fibre H^n and a non-trivial family $g \mapsto gXg^{-1}$. Thus we have the superconnection $d + \sigma gXg^{-1}$ with curvature

$$(d+\sigma gXg^{-1})^2 = gX^2g^{-1} - \sigma\{dg\,Xg^{-1} - gXg^{-1}dg\,g^{-1}\}$$

$$= g\{X^2 + \sigma[X, g^{-1}dg]\}g^{-1}$$

On the other hand we can use the tautological gauge transformation "g" in this bundle to make the family of operators constant. This gives the superconn. $d + g^{-1}dg + \sigma X$ with curvature $R = X^2 + \sigma[X, g^{-1}dg]$.

Consider e^{uR} or $\frac{1}{\lambda - R}$. These are differential forms on $\mathcal{G}$ with values in $\mathcal{L}(H^n)[\sigma]$; the former is in $\mathcal{L}^1(H^n)[\sigma]$ and the latter in $\mathcal{L}^p(H^n)[\sigma]$ for some p in the usual cases. They are also left-invariant forms on $\mathcal{G}$, with $\mathcal{G}$ acting trivially on H^n. Also they are G-invariant where $G = U_n$ acts by conjugation.

From invariant theory we know the space or DG algebra of $\mathcal{G}$, G-invariant forms contains as subalgebra $\mathrm{Hom}(B(A), \mathcal{L}(H)[\sigma])$. It's clear that e^{uR} comes from the similar gadget in this algebra of cochains.

Recall $\theta \in \mathrm{Hom}^1(B(A), A)$ is a flat connection form: $\delta\theta + \theta^2 = 0$, and hence one can use it to construct twisted differentials. Thus if M is a left A-module we have the differential $\delta + \rho_\ell(\theta)$ on $\mathrm{Hom}(B(A), M)$. Similarly there's something for a right A-module and there is $\delta + \mathrm{ad}\,\theta$ in the case of a bimodule.

Now e^{uR}, $\frac{1}{\lambda - R}$ in $\mathrm{Hom}(B(A), \mathcal{L}(H)[\sigma])$ satisfy

$$[\delta + \theta + \sigma X, e^{uR}] = 0 \qquad \text{etc.}$$

April 21, 1987

Consider an unbdd skew-adjoint operator X_0 on H. To fix the ideas suppose X_0 is a Dirac operator on $L^2(M, S\otimes E)$. Consider the coset $X_0 + \mathcal{L}(H)$ of bdd perturbations of X_0. [redacted] Let A be an algebra acting on H such that $[X_0, a] \in \mathcal{L}(H)$ for all a.

Now consider $M_n A$ acting on $H \otimes \mathbb{C}^n$, (bounded ops) and the skew-[redacted] adjoint operator $X_0 \otimes 1$. If $\alpha = (\alpha_{ij}) \in M_n A$, then

$$[X_0 \otimes 1, \alpha] = \left([X_0, \alpha_{ij}]\right) \in \mathcal{L}(H) \otimes M_n$$

I claim that the coset $X_0 \otimes 1 + \mathcal{L}(H) \otimes M_n$ is stable under conjugation by $GL_n(A)$. In effect $\mathcal{L}(H) \otimes M_n$ is stable under conjugation by $GL_n(A)$, since we are assuming [redacted] $A \to \mathcal{L}(H)$. Also

$$g^{-1}(X_0 \otimes 1) g - (X_0 \otimes 1) = g^{-1}[X_0 \otimes 1, g] \in \mathcal{L}(H) \otimes M_n.$$

Better: $\mathcal{L}(H) \otimes M_n = \mathcal{L}(H^n)$, so this is all clear.

So we have this affine space $X_0 \otimes 1 + \mathcal{L}(H) \otimes M_n$. Notice that it is stable under conjugation by U_n. Let us put $\mathcal{A} = X_0 \otimes 1 + \mathcal{L}(H) \otimes M_n$. Then we consider operator valued differential forms

$$\Omega^{\circ}(\mathcal{A}, \mathcal{L}(H) \otimes M_n)$$

Because $\mathcal{A}$ is an affine space, this is roughly a polynomial ring on $R^* = (\mathcal{L}(H) \otimes M_n)^*$, tensored with an exterior algebra on R^*, tensored with R.

So it is interesting to contemplate what happens when we take U_n-invariants. For the moment we concentrate on exterior part. By invariant theory we can sort of understand part of this algebra as cochains. These will be multilinear functions on $\mathcal{L}(H)$ (more generally the tangent space to a) with values in $\mathcal{L}(H)$.

Some examples of these cochains come from the C.T. map from a to unitaries:

$$u^{-1}du = \frac{2}{1+X}\,dX\,\frac{1}{1-X} \qquad ?$$

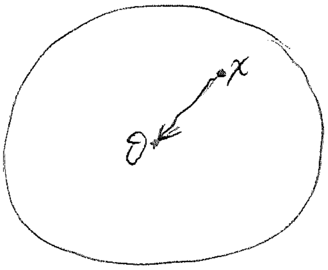
April 28, 1989

Connes described a way to pass from group cohomology of Γ to entire cyclic cocycles on $\mathbb{C}\Gamma$ for Γ hyperbolic, at least for Γ acting freely on a symmetric space. Take the symmetric space to be the [redacted] Poincare disk H. He defines a map

$$\text{currents on } M = \Gamma\backslash H \longrightarrow \text{entire cocycles on } \mathbb{C}\Gamma$$

using superconnections.

The point is that there is a natural superconnection here. To each point $m \in M$ we consider ℓ^2 of the Γ-orbit corresponding to M, probably with coefficients in spinors. This gives a $\mathbb{Z}_2$-graded Hilbert bundle over M. But, if ~~fix~~ an origin in the symmetric space, then we have the geodesics from an arbitrary point x to O. This can be [redacted] converted to a tangent vector at x, and we can Clifford multiply by this tangent vector. This defines dual Dirac on the symmetric space, and its restricts to an odd operator X on the Hilbert bundle $\Gamma x \longmapsto \ell^2(\Gamma x, S)$ over M.



There is apparently also a natural connection ∇ in the Hilbert bundle obtained from the standard connection on spinors. So we have a superconnection

$$\tilde{d} = \varepsilon\nabla + X \qquad \tilde{d}^2 = \underbrace{\nabla^2}_{\text{curvature of symmetric space}} + \varepsilon\underbrace{(\nabla X + X\nabla)}_{\nabla d(r^2/2)} + r^2$$

Here the superconnection is viewed as the analogue of the (H, X) in constructing the JLO cocycle. An appropriate supertrace is obtained from a current on M. However one still has to give $A = \mathbb{C}\Gamma$ on which the entire cocycle is to appear, and this means A acts on the whole setup.

April 30, 1984

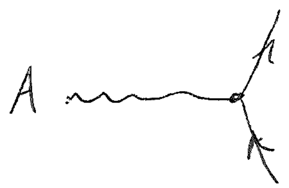
According to Stora the BRS analysis of anomalies can be understood in the case, where the gauge field is treated as an external field, and only the fermion field is a quantum field.

Consider then the quadratic Lagrangian $\tilde{\psi} \not{\partial}_A \psi$ where ψ is a "chiral" fermion. Here $\not{\partial}_A$ stands for half the Dirac operator from + spinors to − spinors. We know the functional integrals attached to this Lagrangian are equivalent to things associated to the determinant line of $\not{\partial}_A$. More precisely, one gives a meaning to the functional integrals

$$(*) \qquad \int D\tilde{\psi}\, D\psi \; e^{\int \tilde{\psi} \not{\partial}_A \psi} \; \psi .. \psi \, \tilde{\psi} .. \tilde{\psi}$$

by trivializing the determinant line.

If one expands around a gauge field A_0 with $\not{\partial}_{A_0}$ non-singular the diagrams have only vertices



and the propagator is given by a geometric series.

The infinities arise from loops. We have

$$\det(\not{\partial}_A) = \det(\not{\partial}_{A_0}) \exp\left\{ -\sum_{n \geq 1} \frac{1}{n} \operatorname{tr}(K^n) \right\}$$

where $K = \pm \not{\partial}_{A_0}^{-1}(A - A_0)$. The traces $\operatorname{tr}(K^n)$ for small n are infinite and one must choose a finite value for them.

In renormalized perturbation theory there is a technique of adding counter-terms to the Lagrangian to remove the infinities. Problem: Explain what these counter terms are in the present situation.

Here is an attempt to make some sense out of the above situation:

First of all we have the determinant line bundle $\mathcal{L}$ over the space $\mathcal{A}$ of gauge fields A. The functional integrals $(*)$ are canonical sections of $\mathcal{L}$, or better perhaps is to think of $(*)$ as linear maps on $\mathcal{L}^{-1}$, so that picking a nonzero elts of $\mathcal{L}_A^{-1}$ gives a meaning to these integrals as numbers.

The issues in making sense of this QFT is just to construct a trivialization of the determinant line bundle.

Secondly there are the ideas from Weinberg's paper on pseudo-particles, which are based in part on the Fredholm determinant theory. These ideas suggest that there are complex analytic trivializations of $\mathcal{L}$ unique up to a factor of the form $\exp\{\text{polynomial on } \mathcal{A}\}$, where the degree is bounded by the traces which have to be regularized.

We saw this is true over a Riemann surface. Here the ambiguity was $\exp\{\text{linear fn. on } \mathcal{A}\}$.

There are apparently technical difficulties caused by the presence of zero modes. However Fredholm really showed how to represent $\frac{1}{1-\lambda K}$ as the meromorphic operator function $\frac{Cof}{det}$, and I have the example of Riemann surfaces.

These considerations lead to the following conjectural picture. Over the space $\mathcal{A}$ of gauge fields there should be a principal bundle for the additive group of polynomial functions of degree $\leq d$, where d bounds the traces which have to be regularized. The idea is that near each A we should have a well-defined trivialization of $\mathcal{L}$ up to exp of such a polynomial. Moreover we should have a flat connection on this bundle.

May 1, 1989

Let us consider smooth functions $F(x)$ on a vector space V and put

$$(\partial_v F)(x) = \frac{\partial}{\partial\varepsilon} F(x+\varepsilon v)\Big|_{\varepsilon=0}$$

as usual. Given $f(x;v)$ smooth and linear in V, we have the necessary and sufficient condition for it to be in the form $(\partial_v F)(x)$, namely:

$$(*) \qquad \partial_{v_1} f(x; v_2) = \partial_{v_2} f(x; v_1)$$

Let's recall how to find F. We have

$$F(x) - F(0) = \int_0^1 \frac{d}{dt} F(tx)\, dt$$

$$= \int_0^1 \frac{d}{d\varepsilon} F(tx+\varepsilon x)\Big|_{\varepsilon=0} dt = \int_0^1 (\partial_x F)(tx)\, dt$$

Now given $f(x;v)$ satisfying the symmetry condition $(*)$, we ~~put~~

$$F(x) = \int_0^1 f(tx; x)\, dt$$

Then

$$(\partial_v F)(x) = \int_0^1 \frac{d}{d\varepsilon} f(t(x+\varepsilon v); x+\varepsilon v)\Big|_{\varepsilon=0} dt$$

$$= \int_0^1 \Big\{ \underbrace{t \frac{d}{d\eta} f(tx+\eta v; x)\Big|_{\eta=0}} + f(tx, v) \Big\} dt$$

$$\partial_v f(tx; x) = \partial_x f(tx; v) \qquad \text{(symm. cond.)}$$

$$= \frac{d}{d\varepsilon} f(tx+\varepsilon x; v)\Big|_{\varepsilon=0}$$

$$= \frac{d}{dt} f(tx, v)$$

Thus

$$(\partial_v F)(x) = \int_0^1 \left\{ t \frac{d}{dt} f(tx; v) + f(tx; v) \right\} dt$$

$$= \Big[t f(tx; v) \Big]_0^1 = f(x; v)$$

Let us now try to characterize smooth functions of the form

$$f(x; v_1, .., v_n) = \partial_{v_1} \cdots\cdots \partial_{v_n} F(x)$$

Such an f is multilinear and symmetric in $v_1, .., v_n$. Moreover

$$\partial_{v_0} f(x; v_1, .., v_n) = \partial_{v_0} \cdots \partial_{v_n} F(x)$$

is also symmetric in $v_0, .., v_n$.

Suppose given then $f(x; v_1, .., v_n)$ multilinear and symmetric in $v_1, .., v_n$ and such that $\partial_{v_0} f(x; v_1, .., v_n)$ is symmetric in $v_0, .., v_n$. Put

$$g(x; v_2, .., v_n) = \int_0^1 f(tx; x, v_2, .., v_n)\, dt$$

Clearly g is multilinear and symmetric in $v_2, .., v_n$. Furthermore we have by the above calculations, treating $v_2, .., v_n$ as parameters, that

$$\partial_{v_1} g(x; v_2, .., v_n) = f(x; v_1, .., v_n)$$

is symmetric in $v_1, .., v_n$. So by induction we have a smooth function $F(x)$ with

$$g(x; v_2, .., v_n) = \partial_{v_2} \cdots\cdots \partial_{v_n} F(x)$$

and so $f(x; v_1, .., v_n) = \partial_{v_1} \cdots \partial_{v_n} F(x)$. It is clear that F is unique up to a polynomial fn. of degree $< n$.

May 4, 1989

Problem (Discussion) from yesterday: We know that the canonical homomorphism

$$A \xrightarrow{1+d} A \oplus \Omega^1_A$$

induces a map on cyclic homology

$$\begin{tikzcd} HC_n(A) \arrow[r] & HC_n(A\oplus\Omega^1_A) \arrow[d, equal] \\ & HC_n(A) \oplus H_n(A,\Omega^1_A) \oplus \cdots \end{tikzcd}$$

and that the component of degree 1 in Ω^1_A is essentially the B operator:

$$\begin{tikzcd} HC_n(A) \arrow[rr] \arrow[dr, "B"'] & & H_n(A,\Omega^1_A) \\ & H_{n+1}(A,\tilde{A}) \arrow[ur, "\partial"', "\sim"] & \end{tikzcd}$$

∂ is an isom. for $n>0$ and is injective for $n=0$.

Let's be more specific. Given a derivation $D: A \longrightarrow M$, we have the homomorphism $1+D$: $A \longrightarrow A\oplus M$, whose effect to first order ~~on~~ the cyclic complex is relative to the Goodwillie isom

$$\begin{tikzcd} (a_0,\ldots,a_n) \arrow[r, mapsto] \arrow[d, phantom, sloped, "\in"] & \sum_{i=0}^{n}(-1)^{in}(Da_i, a_{i+1},\ldots,a_n,a_0,\ldots,a_{i-1}) \arrow[d, phantom, sloped, "\in"] \\ C^{\lambda}_n(A) \arrow[r] & C_n(A;M) \end{tikzcd}$$

Now when $M=\Omega^1_A$ we can identify $C.(A,\Omega^1_A)$ with $C.(A,\tilde{A})/\tilde{A}$ by

$$(a_0,\ldots,a_n) \longleftrightarrow (a_0da_1, a_2,\ldots,a_n)$$

This should be checked and perhaps understood better; it's not clear why this identification is related to ∂ for $0 \longrightarrow \Omega^1_A \longrightarrow \tilde{A}\otimes\tilde{A} \longrightarrow \tilde{A} \longrightarrow 0$.

In any case this identification is consistent with the formula from Kassel + Husemoller

$$\iota_D(a_0, \dots, a_n) = (a_0 Da_1, a_2, \dots, a_n)$$

Assertion: Given $D : A \longrightarrow M$ there is an induced map $HC_n(A) \xrightarrow{D_*} H_n(A, M)$ such that

$$\begin{tikzcd}
HC_n(A) \arrow[r, "B"] \arrow[rrd, "D_*"'] & H_{n+1}(A, \tilde{A}) \arrow[r, "\partial"] \arrow[rd, "\iota_D"] & H_n(A, \Omega^1_A) \arrow[d, "\tilde{D}_*"] \\
& & H_n(A, M)
\end{tikzcd}$$

commutes.

Assertion: Given $D : A \longrightarrow A$ one has a commutative diagram

$$\begin{tikzcd}
HC_n(A) \arrow[r, "B"] \arrow[d, "L_D"] & H_{n+1}(A, \tilde{A}) \arrow[d, "\iota_D"] \\
HC_n(A) & H_n(A, \tilde{A}) \arrow[l, "I"]
\end{tikzcd}$$

Proof by simple calculation:

$$(a_0, \dots, a_n) \xmapsto{B} \sum_0^n (-1)^{in} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1})$$

$$\xmapsto{\iota_D} \sum_0^n (-1)^{in} (Da_i, a_{i+1}, \dots, a_n, a_0, \dots, a_{i-1})$$

$$\xmapsto{I} \underline{\hspace{4cm}}$$

$$= \sum_0^n (a_0, \dots, Da_i, \dots, a_n) = L_D(a_0, \dots, a_n)$$

Cor: $SL_D = L_D S = 0$.

c

So we learn that the Hochschild homology enters naturally when we bring in derivations to cyclic homology. [redacted] Specifically we get the map B when we consider $A \xrightarrow{1+d} A \oplus \Omega^1_A$.

However I learned in the cochain paper that the Hochschild complex, or at least the b complex, which computes $H_\cdot(A, A)$ as opposed to $H_\cdot(A, \tilde{A})$, is obtained from the coalgebra analogue of differentials $\Omega^{B(A)}$. The b complex is $\Omega^{B(A)} \otimes^{B(A)}$.

Conclusion: It would be interesting to find a direct link between $\Omega^{B(A)} \otimes^{B(A)}$ and $C_\cdot(A, \Omega^1_A)$ based [redacted] deformations, or perhaps first order [redacted] variations, of twisting cochains.

[redacted]

Notice that the ∂ map on the coalg side

$$\begin{tikzcd}
\Omega^B \arrow[r, "\partial"] & B \\
\Omega^B \otimes^B \arrow[u, hook] \arrow[r, "\bar{\partial}"] & B \otimes^B \arrow[u, hook]
\end{tikzcd}$$

goes from the b complex to the cyclic complex, so it seems to lead[redacted] to [redacted] Connes I map.

d

Let's return to the idea that a Hochschild n-cocycle

$$\psi(a_0,\dots,a_n), \quad \varphi(a_1,\dots,a_n) \qquad \begin{matrix} b'\varphi=(1-\lambda)\psi \\ b\psi=0 \end{matrix}$$

is to be viewed as a null-homotopy of the cyclic $(n-1)$ cocycle $N\varphi$. Furthermore if $N\varphi=0$, i.e. (ψ,φ) is a self-homotopy of the 0 $(n-1)$-cocycle, then (ψ,φ) is equivalent to a cyclic n-cocycle. This equivalence is done by changing (ψ,φ) to a Hochschild cocycle $(\psi',0)$, whence ψ' is cyclic.

I want to check that if we have a 1-parameter family $u_t : A \longrightarrow B$ and a cyclic cocycle f on B, then for each t the cyclic cocycle on A

$$\frac{d}{dt}\left(f\circ u_t\right)$$

comes from a Hochschild cocycle.

Argument: The family u_t can be viewed as a homomorphism $A \xrightarrow{u} B\otimes S$, where $S=k[t]$ if the family is polynomial in A. What is $f\circ u_t$?

$$(f\circ u_t)(a_0,\dots,a_n) = f(u_t a_0,\dots,u_t a_n)$$

This is obvious $\tilde{f}u$ where $\tilde{f}$ is the cyclic cocycle on $B\otimes S$ defined by

$$\tilde{f}(b_0\otimes s_0,\dots,b_n\otimes s_n) = f(b_0,\dots,b_n)\otimes(s_0\cdots s_n)$$

Thus we need the map

$$Bar(B\otimes S) \longrightarrow Bar(B)\otimes S \qquad S \text{ comm.}$$

which we encountered with the Dennis traces.

Now in this form we can take $S = k[\varepsilon] = k[t]/(t^2)$ to analyze the derivation. So we reach the Gelfand-Fuks variation situation. This is strange because I associate the I map to the G-F situation.

However we have

$$\begin{tikzcd} A \arrow[r, "u+\varepsilon\mathring{u}"] \arrow[d, "1+d"] & B\otimes k[\varepsilon] \\ A\oplus\Omega^1_A \arrow[ur, "(u,\tilde{\mathring{u}})"'] & \end{tikzcd}$$

so f cyclic n cocycle on $B \longmapsto \tilde{f}$ on $B\otimes k[\varepsilon]$

$\xmapsto[\text{of } \varepsilon]{\text{coeff}} \mathring{\tilde{f}}$ Hochschild n-cocycle on B

$\xmapsto[\text{by } u]{\text{pull back}} \mathring{\tilde{f}}\circ(u,\tilde{\mathring{u}})$ Hochschild $\overset{+1}{n}$-cocycle on A.

What happens it seems is that we have generalized to

$$\begin{tikzcd} HC_n(A) \arrow[r, "\partial_t(u_{t*})"] \arrow[d, "B"] & HC_n(B) & \\ H_n(A,\Omega^1_A) \arrow[r, "(u,\mathring{u})"] & H_n(B,B) \arrow[r] & H_n(B,\tilde{B}) \arrow[ul, "I"'] \end{tikzcd}$$

1

May 10, 1989 back from Germany May 5

Recall that associated to an extension $A = R/I$ is an exact sequence

$$0 \to I/I^2 \longrightarrow A\otimes_R \Omega^1_R \otimes_R A \longrightarrow \Omega^1_A \longrightarrow 0$$

of A-bimodules. The right exactness is an easy consequence of the exactness of

$$0 \to \mathrm{Der}(A, M) \longrightarrow \mathrm{Der}(R, M) \longrightarrow \mathrm{Hom}_A(I/I^2, M)$$

for any A-bimodule M. The delicate point is why I/I^2 embeds into $A\otimes_R \Omega^1_R \otimes_R A$. This is equivalent to the existence of an embedding $I/I^2 \hookrightarrow N$ of A-bimodules which extends to a derivation $R \longrightarrow N$. We can suppose $I^2=0$.

By the fact that the extension

$$0 \to I \to R \longrightarrow A \longrightarrow 0$$

is classified by an elt of $H^2(A,I)$ which is is a derived functor on the category of A-bimods, we know there is an embedding $I \hookrightarrow N$ ~~which~~ such that the induced extension E splits:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "i"] & R \arrow[r, "\pi"] \arrow[d, "j"'] & A \arrow[r] \arrow[d, equal] \arrow[dl, dashed, "s"'] & 0 \\
0 \arrow[r] & N \arrow[r] & E \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

We then have two homomorphisms $j, s\pi : R \longrightarrow E$ whose difference is a derivation $R \longrightarrow N$ restricting to $i : I \to N$. Conversely if $D: R \to N$ restricts to i, then $j - D$ is a homomorphism from R to E which vanishes on I, so E splits.

May 15, 1989

Morita invariance.

Let $\mathcal{P}(R)$ be the category of finitely generated projective right modules over the unital algebra R. Let $P \in \mathcal{P}(R)$ and let $A = \mathrm{End}(P)$. Then we have a canonical map

$$H_*(GL(A)) \longrightarrow H_*(GL(R))$$

described as follows.

First we have canon. maps

$$\mathrm{Aut}(P) \times \mathrm{Aut}(P') \longrightarrow \mathrm{Aut}(P \oplus P')$$

which induce

$$H_*(\mathrm{Aut}(P)) \otimes H_*(\mathrm{Aut}(P')) \longrightarrow H_*(\mathrm{Aut}(P \oplus P'))$$

and $$H_*(\mathrm{Aut}(P)) \longrightarrow H_*(\mathrm{Aut}(P \oplus P'))$$

In general given $P \oplus P' \xrightarrow{\sim} P''$ we get

$$\text{(*)} \qquad H_*(\mathrm{Aut}(P)) \longrightarrow H_*(\mathrm{Aut}(P''))$$

A better way to say it is: Given a direct embedding $P \underset{i}{\overset{i^*}{\rightleftarrows}} P''$, $i^* i = id$ in $\mathcal{P}(R)$, there is an associated map (*). This map depends only on the isomorphism class of the complement $P' = \mathrm{Ker}(i^*)$. In effect two isomorphisms

$$P \oplus P' \xrightarrow{\sim} P'' \xleftarrow{\sim} P \oplus P'$$

are linked by an automorphism of P'' which acts trivially on $H_*(\mathrm{Aut}(P''))$.

Now consider direct embeddings

$$P \underset{i}{\overset{i^*}{\rightleftarrows}} R^m$$

for different m. Each gives us a map

$$H_*(\mathrm{Aut}(P)) \longrightarrow H_*(GL_m(R))$$

which depends on the isomorphism class of the complement.

Next point is that if

$$P \oplus P_1 = R^m \qquad P \oplus P_2 = R^n$$

then

$$\begin{tikzcd} P_1 \oplus (P \oplus P_2) \arrow[r, phantom, "="] \arrow[d, equal] & (P_1 \oplus P) \oplus P_2 \arrow[d, equal] \\ P_1 \oplus R^n & R^m \oplus P_2 \end{tikzcd}$$

Thus the two maps

$$\begin{tikzcd} H_*(\mathrm{Aut}(P)) \arrow[r] \arrow[dr] & H_*(GL_m R) \\ & H_*(GL_n R) \end{tikzcd}$$

becomes equal in $H_*(GL_{m+n} R)$.

Conclusion: There's a canonical map

$$H_*(\mathrm{Aut}\, P) \longrightarrow H_*(GL(R))$$

obtained from any direct embedding $P \rightleftarrows R^n$.

The same should be true on the Lie algebra level but the proof will be harder as $GL_n(R)$ probably doesn't act trivially on $H_*(\mathfrak{gl}_n(R))$, although this should be true stably.

So we are led to the following problem for cyclic homology. Given a direct embedding $P \underset{i}{\overset{i^*}{\rightleftarrows}} R^m$, consider the induced maps on cyclic complexes

$$CC(\mathrm{End}(P)) \longrightarrow CC(M_n R) \longrightarrow CC(R)$$

where the first is induced by the non-unital homomorphism

$$\alpha \longmapsto i\alpha\iota^* \qquad \mathrm{End}(P) \longrightarrow M_n R = \mathrm{End}(R^n)$$

and the second is the trace map. Show that up to homotopy this composite map is independent of the choice of direct embedding.

May 16, 1989

Let $\mathcal{P}$ be the category of finitely generated projective modules over some algebra. It can be characterized as a k-linear additive Karoubian category with a generator. Choosing a generator P one obtains an equivalence of $\mathcal{P}$ with $\mathcal{P}(\text{End}\, P)$ (right End P modules).

Let P, Q be generators, let $A = \text{End}\, P$, $B = \text{End}\, Q$. Choose direct embeddings $P \underset{i}{\overset{i^*}{\leftrightarrows}} Q^m$ $\quad Q \underset{j}{\overset{j^*}{\leftrightarrows}} P^m$. Then we have maps

$$CC(A) \longrightarrow CC(M_n B) \xrightarrow{tr} CC(B)$$

$$CC(B) \longrightarrow CC(M_m A) \xrightarrow{tr} CC(A)$$

which we would like to show are homotopy inverses of each other. We compose:

$$P \underset{i}{\overset{i^*}{\leftrightarrows}} Q^n \underset{j^n}{\overset{j^{*n}}{\leftrightarrows}} (P^m)^n$$

and we have the diagram

$$\begin{tikzcd}
CC(A) \arrow[r] & CC(M_n B) \arrow[r] \arrow[d, "tr_n"] & CC(M_n(M_m A)) \arrow[d, "tr_n"] \\
 & CC(B) \arrow[r] & CC(M_m A) \arrow[d, "tr_m"] \\
 & & CC(A)
\end{tikzcd}$$

It is clear the square commutes, and it should be clear that $tr_m \circ tr_n = tr_{mn}$. So we are reduced to showing that for any e

direct embedding

$$\begin{tikzcd} P \arrow[r, shift right, "\iota"'] & P^n \arrow[l, shift right, "\iota^*"'] \end{tikzcd}$$

the map

$$CC(A) \longrightarrow CC(M_n A) \xrightarrow{tr} CC(A)$$

is homotopic to the identity. Since we have an obvious direct embedding such that this composition is the identity, it suffices to show that up to homotopy the composite map is independent of the choice of direct embedding.

What happens in the group case? We have a canonical map

$$H_*(Aut(P)) \longrightarrow H_*(Aut(Q))$$

for each isom. class of P' such that $P \oplus P' \cong Q$. Stably it is reasonable to expect that homotopies which are elementary in the spirit of Bass & Whitehead should link two direct embeddings.

Given $\begin{tikzcd} P \arrow[r, shift right, "\iota"'] & Q \arrow[l, shift right, "\iota^*"'] \end{tikzcd}$ with $\iota^*\iota = 1$, suppose we try to vary this. We have

$$\delta\iota^* \iota + \iota^* \delta\iota = 0$$

for any variation. If we ask that the variation only move P perpendicular to itself, then we have $\iota^*\delta\iota = 0$, whence both $\delta\iota^*\iota = 0 = \iota^*\delta\iota = 0$. Thus the natural tangent space to consider is the sum $Hom(P, P') \oplus Hom(P', P)$ where P' is f

the complement Ker ι^*.

An elementary homotopy should therefore involve changing i by a map in Hom(P,P') but [redacted] keeping ι^* fixed, or keeping ι fixed and changing ι^* by a map in Hom(P',P).

Relative to the decomposition $Q = \begin{matrix}P\\ \oplus \\ P'\end{matrix}$, we have

$$i = \begin{pmatrix}1\\0\end{pmatrix} \qquad \iota^* = \begin{pmatrix}1 & 0\end{pmatrix}$$

and the deformation gives the family

$$\iota = \begin{pmatrix}1\\0\end{pmatrix} \qquad \iota^* = \begin{pmatrix}1 & z\end{pmatrix}$$

with z ranging over Hom(P',P). Thus we have the homomorphism

$$u_z : \text{End}(P) \longrightarrow \text{End}(Q)$$

$$a \longmapsto \begin{pmatrix}1\\0\end{pmatrix} a \begin{pmatrix}1 & z\end{pmatrix} = \begin{pmatrix}a & az\\ 0 & 0\end{pmatrix}$$

Observe that

$$\begin{pmatrix}1 & -z\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} a \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} a \begin{pmatrix}1 & z\end{pmatrix}$$

Thus u_z is related by an [redacted] inner automorphism to $u_0(a) = \iota a \iota^*$.

[redacted]

I ought to consider the effect of inner automorphisms. Let's start with inner derivations.

Let's recall that a derivation $D : A \longrightarrow M$

induces a map of complexes

$$CC(A) \xrightarrow{D_\#} C(A, M)$$

Specifically

$$(a_0, \dots, a_n) \overset{D_\#}{\longmapsto} \sum_{i=0}^{n} (-1)^{in} (Da_i, a_{i+1}, \dots, a_n, a_0, \dots, a_{i-1}).$$

This is just the effect of $1+D : A \to A \oplus M$ on the cyclic complex to first order in M.

Now take D to be the universal derivation $d : A \to \Omega^1_A$ and identify $C(A, \Omega^1_A)$ with a shift of $C(A,A)$ and the degree 0 A part killed. Then we have maps of complexes

$$\begin{tikzcd}
CC_n(A) \arrow[r, "B"] \arrow[dr, "d_\#"] \arrow[ddr, "D_\#"'] & C_{n+1}(A,A) \arrow[d, "\wr", "\text{for } n \geq 0"'] \\
& C_n(A, \Omega^1_A) \arrow[d, "\tilde{D}"] \\
& C_n(A, M)
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[dl, "d"'] \arrow[dr, "D"] & \\
\Omega^1_A \arrow[rr, "\tilde{D}"'] & & M
\end{tikzcd}$$

bimod map

Now suppose D is an inner derivation, i.e. $Da = ma - am$ for some $m \in M$. Then

$$\begin{tikzcd}
da \arrow[r, mapsto] & \cdot\, 1 \otimes a - a \otimes 1 \arrow[ddr] & \\
\Omega^1_A \arrow[r, hook] \arrow[dr, "\tilde{D}"'] & A \otimes A \arrow[d] & \\
& M & +(ma - am)
\end{tikzcd}$$

commutes. On the other hand

$$\Sigma^{-1} C(A,A) \longrightarrow C(A, \Omega^1_A) \longrightarrow C(A, A \otimes A)$$

should be null-homotopic, being essentially h

the triangle of Hochschild cxs. associated ~~to~~ the exact sequence

$$0 \longrightarrow \Omega^1_A \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0$$

So it should follow that $D_\#$ is null-homotopic when D is an inner derivation.

The next step is to show how this implies that inner autos act trivially on cyclic homology. Let $u \in A^\times$ and consider the homomorphisms $a \mapsto a,\ uau^{-1}$. Let's use the fact that

$$CC(A) \longrightarrow CC(M_2A) \xrightarrow{tr} CC(A)$$

is the identity where the first map is induced by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. We have

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} uau^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

which reduces us to the case of the invertible $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ in M_2A. We have

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & u \\ -u^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}$$

which reduces us to the case of the invertibles $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$ in $M_2(A)$. But these belong to 1-parameter groups $\begin{pmatrix} 1 & tu \\ 0 & 1 \end{pmatrix}$ and so by differentiation we are reduced to the case of an inner derivation.

All this is essentially in Connes paper. He considers Morita invariance in the following form. Let $p \in M_r A$, $q \in M_s A$ be projectors and suppose we have

$$p = uv \qquad q = vu$$

with $u : A^s \longrightarrow A^r$, $v : A^r \longrightarrow A^s$. Then u, v give isos $pA^r \underset{u}{\overset{v}{\rightleftarrows}} qA^s$. In effect

$$uq = uvu = pu$$

showing that $uqA^s \subset pA^r$ and similarly that $vpA^r \subset qA^s$. Also $vuq = q^2 = q$ and $uvp = p^2 = p$, showing these are inverse isos.

One wants to show that

$$\begin{array}{cccc} R = & \mathrm{End}(pA^r) & = & \mathrm{End}(qA^s) \\ & \shortparallel & & \shortparallel \\ & pM_r(A)p & & qM_s(A)q \\ & \cap & & \cap \\ & M_r(A) & & M_s(A) \end{array}$$

lead to the same map up to homotopy from $CC(R)$ to $CC(A)$. So we have a projective P and two isoms.

$$P \oplus P' = A^r \qquad P \oplus P'' = A^s$$

j

You want to show that the two homomorphisms from End(P) to $M_r(A) \subset M_{r+s}(A)$ and $M_s(A) \subset M_{r+s}(A)$ are conjugate by an inner auto. So we need an isomorphism of

$$P \oplus P' \oplus A^s \quad \text{with} \quad P \oplus P'' \oplus A^r$$

which is $id_P \oplus$ something. But

$$P' \oplus A^s = P' \oplus P \oplus P'' \simeq A^r \oplus P'' \simeq P'' \oplus A^r$$

so it works.

The conclusion we draw from the above is that the essential point is the fact that inner automorphisms act trivially on cyclic homology.

However we learn something more which might be useful, namely that the basic derivation map

$$CC(A) \xrightarrow{D_\#} C(A,M)$$

is (or should be) null homotopic for an inner derivation

May 17, 1988

Let A be a unital algebra and let us consider A as a nonunital algebra. A nonunital A-module M is a unital $\tilde{A}$-module. The identity in A becomes an idempotent operator on M, call it e, so we have $M = eM \oplus (1-e)M$ where eM is a unital A-module and $(1-e)M$ is a vector space on which A acts ~~by~~ zero multiplication.

Similarly a nonunital bimodule will be a direct sum of four pieces - a unital bimodule, a unital left A-module with zero right mult., a unital right A-module with zero left mult, and a vector space with zero left + right mult. by elements of A.

Let $D : A \longrightarrow M$ be a derivation, and let's consider the four types of bimodule, ~~separately~~. If M is a vector space with zero left + right A-mult, then $D(a) = D(1a) = (D1)a + 1(Da) = 0$. If M is a unital left A-module with zero right A-multiplication then

$$D(a) = D(a1) = \cancel{Da \cdot 1} + aD1 = a(D1)$$

so D is equivalent to the element $D1$ of M. Similarly for unital right A-module with zero left mult.

From this description we see that the

nonunital Ω^1 which is the unital $\Omega^1_{\tilde{A}}$ has the description

$$\Omega^1_{\tilde{A}} = \Omega^1_A \oplus {}_{id}A_0 \oplus {}_0A_{id}$$

as A-bimodule, and that

$$da = (da, a, a)$$

It might be better to say that

$$d1 = (0, d'1, d''1)$$

or $$d1 = 1d1 + \underset{\sim}{d1 \cdot 1}.$$ Better:

$$de = ede + de\,e \qquad \text{where} \qquad de\,e = (1-e)de$$

I would like eventually to understand Connes curious S-operation defined by non-comm differential forms.

May 19, 1989

Suppose A unital. Consider the Connes exact sequence

$$0 \longrightarrow \overline{HC}^1(A) \longrightarrow H^1(A, A^*) \longrightarrow \overline{HC}{}^0_{[\text{redacted}]}(A) \longrightarrow$$

We know that an element of $\overline{HC}^1(A)$ can be represented by an extension $A = R/I$ (unital) together with a trace $\tau :$ [redacted] $I/[R,I] \longrightarrow k$. Also an element of $H^1(A,A^*)$ is represented by an A-bimodule extension $0 \to M \to E \to A \to 0$ together with a trace on the bimodule M. The trace gives rise to a canonical bimodule map $M \longrightarrow A^*$ and we can suppose $M = A^*$ with the trace equal to the canonical trace on A^* (evaluation at 1). In this case the element of $H^1(A,A^*)$ is equivalent to the bimodule extension up to isomorphism.

The problem is to explicitly go from an algebra extension $A = R/I$ with $\tau : I/[R,I] \to k$ to a bimodule extension $0 \to A^* \to E \to A \to 0$.

Let's recall that the algebra extension with trace determines a cyclic 1-cocycle class as follows. One takes the trace τ on I, extends it to a linear fnl f on R and considers $f([x,y])$. This is a cyclic 1-cocycle on R, which vanishes if either $x, y \in I$ by the assumption that τ vanishes on $[R,I]$. Thus it [redacted] descends to give a cyclic 1-cocycle on A. [redacted] [redacted] f is unique up to a linear functional on A, so the cyclic 1-cocycle

$f([\bar{x},\bar{y}])$ is unique up to a 1-coboundary.

To construct E we start with the idea the elements of E lying over $1 \in A$, which form a torsor under A^*, should be the linear functionals f on R extending τ on I. Then we want to bring in some sort of left and right multiplication [redacted] by elements of A, and we note that the linear functional $\alpha \longmapsto \tau(r\alpha) = \tau(\alpha r)$ on I for $r \in R$ is independent of left + right.

Let $E' = \{(x,f) \mid x \in R, f \in R^*, \quad f(\alpha) = \tau(x\alpha) \ \forall \alpha \in I\}$

Then E' is an R-bimodule with

[redacted]

$$y(x,f) = (yx, r \longmapsto f(ry))$$

$$(x,f)\,y = (xy, r \longmapsto f(yr))$$

Check: $f(\alpha y) \overset{?}{=} \tau(yx\alpha) = \tau(\alpha y x)$ OK

$f(y\alpha) \overset{?}{=} \tau(xy\alpha)$ OK.

[redacted]

Clearly E' is an R-bimodule extension:

[redacted] $0 \longrightarrow A^* \longrightarrow E' \longrightarrow R \longrightarrow 0$

If we restrict to $I \subset R$, then this extension splits because if $x \in I$ there is canonical extension of $\alpha \longmapsto \tau(x\alpha)$ to R namely

[redacted] $r \longmapsto \tau(xr) = \tau(rx)$

Thus set $E'' = \{(\alpha, r \mapsto \tau(r\alpha)) \mid \alpha \in I\}$ and let $E = E'/E''$. Then we have an R-bimodule extension

$$0 \longrightarrow A^* \longrightarrow E \longrightarrow A \longrightarrow 0$$

and all we have to check is that I acts trivially on E. Let $(x,f) \in E'$ and $\beta \in I$. Then $\beta(x,f) = (\beta x, r \mapsto f(r\beta))$. But $f(r\beta) = \tau(x r \beta) = \tau(\beta x r)$, so $\beta(x,f) = (\beta x, r \mapsto \tau(r\beta x)) \in E''$.

May 20, 1989

Yesterday we described how to go from an element of $\overline{HC}^1(A)$ represented by an extension and trace $A = R/I$, $\tau : I/[R,I] \to k$ to its image under I in $H^1(A, A^*)$ viewed as a bimodule extension $0 \to A^* \longrightarrow E \to A \to 0$. In essence we obtain from τ a square

$$\begin{tikzcd} I \arrow[r] \arrow[d] & R \arrow[d] \\ R^* \arrow[r] & I^* \end{tikzcd}$$

and $E = R^* \times_{I^*} R / I$. Better to think of the above complex as the mapping cylinder complex of [illegible] its two rows, and E as its only homology group. Then [illegible] the long exact sequence in homology associated to

$$\{R^* \to I^*\} \longrightarrow \left\{\begin{tikzcd} I \arrow[r] \arrow[d] & R \arrow[d] \\ R^* \arrow[r] & I^* \end{tikzcd}\right\} \longrightarrow \{I \to R\}$$

gives the exact sequence

[illegible] $$0 \to A^* \longrightarrow E \longrightarrow A \longrightarrow 0$$

Question: Is there a natural map $E \longrightarrow E^*$?

Given a bimodule extension [illegible]

$$0 \longrightarrow A^* \longrightarrow N \longrightarrow A \longrightarrow 0$$

we can dualize to get a bimodule extension

$$0 \longrightarrow A^* \longrightarrow N^* \longrightarrow A^{**} \longrightarrow 0$$

and then pull back via the canonical map $A \longrightarrow A^{**}$ to get another bimodule extension of A by A^*. This gives some sort of operation on $H^1(A, A^*)$, which appears to be of order 2.

Calculation shows this map is -1.

Check: We start with an A-bimodule extn.

$$① \qquad 0 \longrightarrow A^* \xrightarrow{i} N \xrightarrow{\pi} A \longrightarrow 0$$

and pick $n \in \pi^{-1}(1)$. Then

$$Da = i^{-1}[n, a]$$

is a derivation of A with values in A^* and its class in $H^1(A, A^*)$ classifies the extn.

We now consider

$$\begin{tikzcd}
①^*: & 0 \arrow[r] & A^* \arrow[r, "\pi^t"] & N^* \arrow[r, "i^t"] & A^{**} \arrow[r] & 0 \\
② : & 0 \arrow[r] & A^* \arrow[r, "i'"] \arrow[u, equal] & N' \arrow[r, "\pi'"] \arrow[u, hook] & A \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

and look for an elt f of N' lying over $1 \in A$. Thus f is an element of N^* such that

$$f(i\lambda) = \lambda(1) \qquad \forall \lambda \in A^*$$

To construct f we use the isomorphism

$$A^* \oplus A \xrightarrow{\sim} N$$
$$\lambda + a \longmapsto i(\lambda) + na$$

of right A-modules. We define

$$f(\iota\lambda + na) = \lambda(1)$$

Now we compute the derivation

$$a \mapsto D'a = \iota^{-1}[f,a] \qquad D' : A \to A^*$$

~~We have $\iota'(D'a) = \pi^t D'a$~~

~~[illegible] $(\pi^t D'a)(a_0) = (D'a)(\pi^*$ [illegible]$)$~~

$[f,a]$ is an element of $N' \subset N^*$ which vanishes on $\iota(A^*)$ and so is a linear functional $D'a$ on A. To calculate this linear fnl at a_0 we can lift a_0 to either $a_0 n$ or $n a_0$ and apply $[f,a]$.

$$(D'a)(a_0) = [f,a](a_0 n) = f(a a_0 n) - f(a_0 n a)$$

This doesn't work. Instead use $n a_0$

$$(D'a)(a_0) = [f,a](n a_0) = f(a n a_0) - \underbrace{f(n a_0 a)}_{0 \text{ by defn.}}$$

$$= f(a n a_0) - \overbrace{f(n a a_0)}^{0}$$

$$= f([a,n]a_0) = (\iota^{-1}[a,n]a_0)(1) = (\iota^{-1}[a,n])(a_0)$$

$$= -(Da)(a_0).$$

Thus $D'a = -Da$ which proves apparently that the duality map is -1.

The problem we wish to study is whether [redacted] or to what extent a cyclic [redacted] 1-cocycle can be canonically represented [redacted] by an extension $A = R/I$ with trace $\tau : I/[R,I] \longrightarrow k$. From Cuntz we know that there is a [redacted] diagram chasing construction of a trace on the free algebra $R = T_{red}(A)$. In my earlier investigations of Cuntz's letter I understood this process as lifting the cyclic 1-cocycle [redacted] on A up to R, then using the triviality of $HC_1(R)$ to write the 1-cocycle as $f([x,y])$ with $f \in R^*$, whence f restricts to a trace on I.

Viewpoint: In the spirit of Grothendieck let's try to understand 1-dim'l cycle cohomology as the groupoid defined by the complex

$$A^* \longrightarrow Z^1_\lambda(A) \qquad ?$$

I want some sort of canonical model for $\overline{HC}^1(A)$, and one possibility is to use the exact sequence

$$0 \longrightarrow \overline{HC}^1(A) \longrightarrow H^1(A, A^*) \longrightarrow \overline{HC}^0(A)$$

plus the fact that we have the [redacted] groupoid of bimodule extensions of A by A^* to understand $H^1(A, A^*)$. The problem now is whether I can obtain a similarly nice model using extensions.

Let's look at [redacted] my result

$$HC^1(A) \xleftarrow{\sim} \varprojlim_{A=R/I} \left(I/[R,I]\right)^*$$

May 21, 1989

The problem is to start with a cyclic 1-cocycle $\psi(a_0,a_1)$ on A and to construct an extension $A = R/I$ with $f \in R$ such that $f([x,y]) = \psi(\pi x, \pi y)$ in some quasi-canonical way.

I think this is a sort of rigidification problem for an element of $\bar{HC}^1(A)$. The hope is that by lifting the class to a specific cyclic 1-cocycle, then we can solve the problem.

Consider an extension $A = R/I$. From the exact sequence

$$\bar{HC}_1(R) \to \bar{HC}_1(A) \longrightarrow I/[R,I] \longrightarrow \bar{R}/[R,R] \longrightarrow \bar{A}/[A,A] \to 0$$

we learn that the element $[\psi]$ of $\bar{HC}^1(A)$ comes from a $\tau \in (I/[R,I])^*$ iff it dies in $\bar{HC}^1(R)$, which means there is an $f \in \bar{R}^*$ with $f([x,y]) = \psi(\pi x, \pi y)$.

Consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & (\bar{A}/[A,A])^* \arrow[r] \arrow[d] & (\bar{R}/[R,R])^* \arrow[d] \arrow[dr, bend left] & & \\
0 \arrow[r] & \bar{A}^* \arrow[r] \arrow[dr, bend right] & (\bar{R}/[R,I])^* \arrow[r] \arrow[d] & (I/[R,I])^* \arrow[r] \arrow[d] & 0 \\
 & & \bar{Z}^1_\lambda(A) \arrow[r] \arrow[d] & \bar{HC}^1(A) \arrow[r] \arrow[d] & 0 \\
 & & \bar{HC}^1(R) \arrow[r, equal] & \bar{HC}^1(R) &
\end{tikzcd}$$

If $\bar{H}C^1(R)=0$, this shows we have quasi-isomorphisms of complexes

$$\begin{tikzcd}
\left(\bar{R}/[R,R]\right)^* \arrow[r] & \left(I/[R,I]\right)^* \\
\bar{A}^* \oplus \left(\bar{R}/[R,R]\right)^* \arrow[r] \arrow[u] \arrow[d] & \left(\bar{R}/[R,I]\right)^* \arrow[u] \arrow[d] \\
\bar{A}^* \arrow[r] & \bar{Z}^1_\lambda(A)
\end{tikzcd}$$

This is supposed to support my feeling that rigidifying a cyclic 1-diml class to a cyclic one-cocycle is similar to equipping the extension $A = R/I$ with an $f \in \left(\bar{R}/[R,I]\right)^*$.

Let us start with a cyclic 1-cocycle [redacted] on A, [redacted] and let us use the [redacted] notation $\int_\gamma$ [redacted] $f\,dg$ suggestive of manifolds and loop groups. The problem is still to convert this into a trace on an extension.

The idea is that we have an explicit family of cyclic cocycles ~~starting~~ with this one, namely

$$c_n \cdot tr \int_\gamma \theta^{2n+1} d\theta$$

which are all related by the S-operation as far as their classes are concerned. ~~[illegible]~~

Moreover we have explicit formulas proving the S-relations, that is writing [redacted] the cyclic 1-cocycle as a b+B coboundary. These formulas are of Chern-Simons type, and the hope might be that the Chern-Simons formulas [redacted] explicitly do the transformation to a trace [redacted] on the universal extension.

~~It is a bit confused and~~

May 22, 1989

Review yesterday's calculations.

Recall the general Chern-Simons type cyclic cocycle with values in non-comm. differential forms:

$$(d+\delta)\underbrace{\int_0^1 tr\left(\theta e^{td\theta+(t^2-t)\theta^2}\right)dt}_{\textstyle\sum_{k,l\geq 0} tr\{\theta P(\overset{k}{\theta^2},\overset{l}{d\theta})\}\underbrace{\frac{1}{(k+l)!}\int_0^1 (t^2-t)^k t^l\,dt}_{c_{kl}}} = tr\left(e^{d\theta}-1\right)$$

If γ_l is an l-diml closed current, then we have a sequence of cyclic cocycles

$$\varphi_k = \int_{\gamma_l} c_{kl}\, tr\{\theta P(\overset{k}{\theta^2},\overset{l}{d\theta})\} \qquad k=0,1,2,\dots$$

In cyclic cohomology we have $S[\varphi_k]=[\varphi_{k+1}]$.

If we [redacted] form $A\otimes\mathbb{C}e \subset \Omega^{\cdot}_A\otimes\Omega^{\cdot}_{\mathbb{C}e}$, then $\int_{\gamma_l} tr(\theta P(\overset{k}{\theta^2},\overset{l}{d\theta}))$ is the result of combining $\int_{\gamma_l}$ on $\Omega^{\cdot}_A$ with the trace on $\Omega^{2k}_{\mathbb{C}e}$ which sees $e(de)^{2k}$.

Review how we establish the formula $S[\varphi_k]=[\varphi_{k+1}]$. We work in $Hom(B(A),\Omega_A)$ with the 1-param. family of connections $d+\delta+t\theta$. Put $\rho=t\theta$ and $\omega=(d+\delta)\rho+\rho^2 = td\theta+(t^2-t)\theta^2$. Recall the S-relation formulas

$$(d+\delta)\{e^{\omega}\} = -[\rho,e^{\omega}] = \beta\{\natural(\partial\rho\, e^{\omega})\}$$

$$(d+\delta)\{\natural(\partial\rho\, e^{\omega})\} = \natural[d+\delta+\rho,\partial\rho\, e^{\omega}] = \natural(\partial\omega\, e^{\omega}) = \bar{\partial}\{e^{\omega}\}$$

These express the fact that e^{ω} and $\natural(\partial_\rho e^{\omega})$ constitute a big cyclic cocycle.

Next we differentiate with respect to the parameter t, and the result is we obtain a big cyclic coboundary

$$\boxed{\begin{aligned}(e^{\omega})^{\bullet} &= (d+\delta)(\mu) + \beta\{-\natural(\partial_\rho \mu)\} \\ \{\natural(\partial_\rho e^{\omega})\}^{\bullet} &= (d+\delta)\{-\natural(\partial_\rho\mu)\} + \bar\partial\{\text{[redacted]}\,\mu\}\end{aligned}}$$

where $\mu = \int_0^1 e^{(1-s)\omega}\,\dot\rho\, e^{s\omega}\,ds.$

In the ~~situation~~ $\rho = t\theta$ $\quad\omega =$ [redacted] $(t^2-t)\theta^2 + td\theta$

we have

$$tr(\mu) = tr(\dot\rho e^{\omega}) = tr\left(\theta e^{(t^2-t)\theta^2+td\theta}\right)$$

so $\int_0^1 tr(\mu)dt =$ [redacted] $\sum_{k,l} c_{kl}\, tr\{\theta P(\overset{k}{\theta^2}, \overset{l}{d\theta})\}$ is our ~~basic~~ Chern–Simons cyclic cochain.

[redacted]

The point: ~~Starting~~ from a closed current γ_ℓ we obtain a big cyclic cocycles. If this satisfies the symmetry (or normalization) condition, then we obtain a supertrace on the Cuntz algebra. The question [redacted] is whether the symmetry condition is satisfied, i.e whether $\int_\gamma\int_0^1 \mu\, dt$ which is a bar cochain is actually cyclic.

Now we can [redacted] actually modify $\varphi = \int_\gamma\int_0^1 \mu\, dt$ without changing the sequence of cyclic cocycles $N\varphi$, but perhaps, as in the case of the JLO cocycle, this is an ugly and unnatural process.

Let's check yesterday's calculations when $\dim(\mathfrak{g}) = 1$.

$$\mu = \int_0^1 e^{(1-s)\omega} \mathring{\rho}\, e^{s\omega}\, ds = \sum_{i,j} \frac{\omega^i \mathring{\rho}\, \omega^j}{i!\, j!} \underbrace{\int_0^1 (1-s)^i s^j\, ds}_{\beta(i+1,\, j+1)}$$

$$= \sum_{i,j \geq 0} \omega^i \mathring{\rho}\, \omega^j \frac{1}{(i+j+1)!} = \sum_{n \geq 0} \frac{1}{(n+1)!} \underbrace{\sum_{i+j=n} \omega^i \mathring{\rho}\, \omega^j}_{\mu_{n+1}}$$

$$\mu_{n+1} = \sum_{i+j=n} \left((t^2-t)\theta^2 + t\, d\theta \right)^i \theta \left((t^2-t)\theta^2 + t\, d\theta \right)^j$$

$$= \sum_{i \geq 1} \sum_{a+b=i-1} (\theta^2)^a d\theta\, (\theta^2)^b\, \theta\, (\theta^2)^j \cdot (t^2-t)^{\overset{n-1}{\boxed{a+b+j}}}\, t$$

$$+ \sum_{j \geq 1} \sum_{c+d=j-1} (\theta^2)^i \theta\, (\theta^2)^c\, d\theta\, (\theta^2)^d \cdot (t^2-t)^{\overset{n-1}{\boxed{i+c+d}}}\, t$$

first sum is $(t^2-t)^{n-1} t$ times

$$\boxed{2(b+j) = 2(n-a-1)}$$

$$\sum_{a=0}^{n-1} (\theta^2)^a\, d\theta\, \theta^{2(n-a)-1} \underbrace{\sum_{a+1 \leq i \leq n} 1}_{(n-a)}$$

second term is $(t^2-t)^{n-1} t$ times

$$c + i = n-d-1$$

$$\sum_{d=0}^{n-1} \theta^{2(n-d)-1}\, d\theta\, (\theta^2)^d \underbrace{\sum_{d+1 \leq j \leq n} 1}_{n-d}$$

Thus

$$\int_0^1 \frac{\mu_{n+1}}{(n+1)!}\,dt = \left(\int_0^1 (t^2-t)^{n-1}\,t\,dt\right) \times$$

$$\sum_{a=0}^{n-1} (n-a)\left\{ \theta^{2a}\,d\theta\,\theta^{2(n-a)-1} + \theta^{2(n-a)-1}\,d\theta\,\theta^{2a} \right\}$$

Take $n=1$. get $(d\theta\,\theta + \theta\,d\theta)$

$n=2$ get $\boxed{2\,d\theta\,\theta^3 + 2\theta^3 d\theta + \theta^2 d\theta\,\theta + \theta\,d\theta\,\theta^2}$

May 23, 1989 (Alice is 27)

The first mystery with the S-operation is why Connes can describe it so easily using non-commutative differential forms and idempotents. In general one has a map

$$\Omega^\bullet_{A\otimes B} \longrightarrow \Omega^\bullet_A \otimes \Omega^\bullet_B$$

of DGA's (non unital setting), so taking $B = \mathbb{C}e$ one gets a DGA map

$$\Omega^\bullet_A = \Omega^\bullet_{A\otimes\mathbb{C}e} \longrightarrow \Omega^\bullet_A \otimes \Omega^\bullet_{\mathbb{C}e}$$

so a map of complexes.

$$\Omega^\bullet_A/[\,,\,] \longrightarrow \Omega^\bullet_A/[\,,\,] \otimes \Omega^\bullet_{\mathbb{C}e}/[\,,\,]$$

Now $\Omega^\bullet_{\mathbb{C}e}/[\,,\,]$ consists of $\overline{\mathbb{C}e(de)^{2n}}$ in degree $2n$ and 0 in odd degrees. Also a cyclic n-cocycle on A is a closed n-trace on $\Omega^\bullet_A$, so given such a closed n-trace, one can combine it with the closed $2k$-trace on Ω_A which carries $e(de)^{2k}/k!$ to 1 ~~to~~ get a closed $(n+2k)$-trace on $\Omega^\bullet_A$. This is the operation S^k on cyclic cocycles, and indeed one can prove that $S^k = (S^1)^k$.

Recall the proof. If φ_n is a cyclic n-cocycle on A, let $\int_{\varphi_n}$ denote the corresp. closed n-trace on Ω_A:

$$\int_{\varphi_n} a_0\,da_1 \ldots da_n = \varphi_n(a_0, \ldots, a_n)$$

Let γ_{2k} be the closed $2k$-trace on $\Omega_{\mathbb{C}e}$ such that

$$\int_{\gamma_{2k}} e(de)^{2k}/k! = 1$$

Then $S^k\varphi_n$ is the cyclic $(n+2k)$-cocycle on A given by

$$(S^k\varphi_n)(a_0, \ldots, a_{n+2k}) = \int_{\varphi_n\otimes\gamma_{2k}} (a_0e)\,d(a_1e)\cdots d(a_{n+2k}e)$$

Suppose we now want to compute $S^k\circ S^l$. Then we have

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[dd] & A\otimes\mathbb{C}e_1 \arrow[r, equal] \arrow[dd] & A\otimes\mathbb{C}e_1\otimes\mathbb{C}e_2 \arrow[d] \\
 & & \Omega_{A\otimes\mathbb{C}e_1}\otimes\Omega_{\mathbb{C}e_2} \arrow[d] \\
\Omega_A \arrow[r] & \Omega_A\otimes\Omega_{\mathbb{C}e_1} \arrow[r] \arrow[d] & \Omega_A\otimes\Omega_{\mathbb{C}e_1}\otimes\Omega_{\mathbb{C}e_2} \arrow[d, "\int_{\gamma_{2l}\otimes\gamma_{2k}}"] \\
 & {} & ?
\end{tikzcd}$$

We have

$$\int_{S^k\{S^l\varphi_n\}} a_0\,da_1\cdots da_{n+2l+2k} = \int_{S^l\varphi_n\otimes\gamma'_{2k}} a_0e'\,d(a_1e')\cdots d(a_{n+2l+2k}e')$$

$$= \int_{\varphi_n\otimes\gamma_{2l}\otimes\gamma_{2k}} a_0ee'\,d(a_1ee')\cdots d(a_{n+2l+2k}ee')$$

Now

$$d(ee') = de\,e' + e\,de'$$

$$(ee')d(ee')^2 = ee'\left(de^2e'^2 + de\,e\,e'de' + e\,de\,de'e' + e^2de'^2\right)$$

$$= e(de)^2\cdot e' + e\cdot e'(de')^2$$

We need to know what becomes of $\gamma_{2l}\otimes\gamma_{2k}$ under the map $\Omega_{\mathbb{C}\tilde{e}} \longrightarrow \Omega_{\mathbb{C}e}\otimes\Omega_{\mathbb{C}e'}$ sending $\tilde{e}\longmapsto ee'$. But

$$\int_{\gamma_{2l}\otimes\gamma_{2k}} (ee')\,d(ee')^{2(k+l)} = \int_{\gamma_{2l}\otimes\gamma_{2k}} \left(e\,de^2\cdot e' + e\cdot e'de'^2\right)^{k+l}$$

$$= \int_{\gamma l\otimes\gamma_{2k}} \frac{(k+l)!}{k!\,l!}\, e(de)^{2k}\, e'(de')^{2l} = (k+l)!$$

Thus $\gamma_{2l}\otimes\gamma_{2k}$ pulls back to $\tilde{e}(de)^{2(k+l)} \longmapsto (k+l)!$ which is $\gamma_{2(l+k)}$.

Check. Compute $(S\circ S)\,tr$ and compare with $S^2 tr$. One has

$$\int_{S^2tr} a_0\,da_1\cdots da_4 = \int_{tr\otimes\gamma_4} a_0e\,d(a_1e)\cdots d(a_4e)$$

$$= tr(a_0\cdots a_4)\int_{\gamma_4} e(de)^4 = 2!\,tr(a_0\cdots a_4)$$

But $(S\,tr)(a_0,a_1,a_2) = tr(a_0a_1a_2)$. Let $f(a_0,a_1,a_2)$ be a cyclic 2-cocycle. Then

$$(S_2f)(a_0,\dots,a_4) = \int_{f\otimes\gamma_2} (a_0e)\,d(a_1e)\,d(a_2e)\,d(a_3e)\,d(a_4e)$$

$$= \int_f a_0a_1a_2\,da_3da_4 + a_0da_1\,a_2a_3\,da_4 + a_0da_1da_2\,a_3a_4$$

$$= \int_f a_0 a_1 a_2\, da_3\, da_4 + a_0\, d(a_1 a_2 a_3) a_4 - a_0 a_1\, d(a_2 a_3) a_4 + a_0\, da_1\, d(a_2 a_3 a_4) - a_0\, d(a_1 a_2)\, d(a_3 a_4) + a_0 a_1\, da_2\, d(a_3 a_4)$$

$$(S f_2)(a_0, \ldots, a_4) = f(a_0 a_1 a_2, a_3, a_4) + f(a_0, a_1 a_2 a_3, a_4) + f(a_0, a_1, a_2 a_3 a_4) - f(a_0 a_1, a_2 a_3, a_4) - f(a_0, a_1 a_2, a_3 a_4) + f(a_0 a_1, a_2, a_3 a_4)$$

Taking $f_2(a_0, a_1, a_2) = tr(a_0 a_1 a_2)$ gives

$$(S f_2)(a_0, \ldots, a_4) = 2\, tr(a_0 \cdots a_4)$$

which checks.

So far we have considered Connes approach to the S operation based on noncommutative differential forms and the isomorphism $A = A \otimes \mathbb{C}e$ (which one might say is related to [illegible] A and $M_2(A)$?)

But there is another approach based on Chern-Simons:

$$(d+\delta) \int_0^1 tr\{\theta\, e^{t d\theta + (t^2 - t)\theta^2}\}\, dt = tr(e^{d\theta} - 1)$$

which produces the same cocycles up to apparently different numerical constants.

NO see p 332

June 5, 1989

In writing the paper for Florence Tsou I was leds to consider the non-commutative version of the Weil algebra. This is the DG algebra W freely generated by an element A of degree 1. It can be identified with $\mathbb{C}\{A,F\}$ = tensor alg generated by A, F of degrees 1, 2 resp. with d defined by $dA = F - A^2$ $\quad dF = -[A,F]$. The universal Chern character and Chern-Simons forms lie in $W_\natural = W/[W,W]$.

Here's the geometric picture. Suppose given a principal bundle P for $G = U(N)$ N large with base M. Then a connection in P is an element A of degree 1 in $\Omega(P) \otimes M_N$ which is a DGA, so we get a homomorphism of DGA's $W \longrightarrow \Omega(P) \otimes M_N$, and hence $W_\natural \longrightarrow \Omega(P)$. The Chern-Simons forms cs_{2n-1} become closed in $\Omega(P)$ for $2n > \dim(M)$. Thus we get a certain class $[cs_{2n-1}]$ in $H^*(P)$ which ought to be independent of the choice of connection provided $2n > \dim(M) + 1$. Moreover these are related by the S-operation, which must be understood geometrically.

My feeling is that what I am looking at is the geometric situation needed to understand the cyclic theory of $\mathbb{C}$ itself.

suppose that the bundle admits a trivialization. Then our bundle P over M comes by pull-back from G over pt. If we pick a trivialization $P \longrightarrow G$, then the usual odd forms on G will give the Chern-Simons forms. One thing to [redacted] see as a kind of check is whether ~~the~~ classes of high degree are independent of the trivialization.

So suppose we have over M the trivial G-bundle $M \times G$ and an automorphism of it. This is the same as a map $M \longrightarrow G$ and so we may suppose that $M = G$. Thus we have $P = G \times G$ with G acting by right multiplication on the second factor. Then we have the two trivializations of P given by

$$\begin{tikzcd} G \times G \arrow[r, shift left, "pr_2"] \arrow[r, shift right, "m"'] & G \end{tikzcd}$$

Now take the [redacted] odd character form on G, or really the odd class $e_{2n-1} \in H^{2n-1}(G)$. The two pull backs of this class are

$$pr_2^*(e_{2n-1}) = 1 \otimes e_{2n-1}$$

$$m^*(e_{2n-1}) = e_{2n-1} \otimes 1 + 1 \otimes e_{2n-1}$$

But now if we go back to M

mapping to G, then these two will agree on $P = M \times G$ provided $2n-1 > \dim M$.

At this point we see the BRS algebra it seems. Let us consider the two maps $P \times G \underset{m}{\overset{pr_1}{\rightrightarrows}} P$. Then given a connection form A in P we have

$$m^*(A) = pr_1^*(A) + pr_2^*(\Theta)$$

where Θ is the Maurer-Cartan form on G.

?. Let's consider this carefully.

Let $Y^{f,M}$ be a space over M and let $s: Y \longrightarrow P$ be a map such that

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[dr, "f"'] & P \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

commutes. Then we have an isomorphism

$$f' : Y \times G \xrightarrow{\sim} P \qquad (y,g) \longmapsto s(y)g$$

that is, a trivialization of P. We know then that $(f')^*(A)$ and $pr_2^*(\Theta)$ are two connection forms in $Y \times G / Y$ so one has

$$(f')^*(A) = pr_1^*(\omega) + pr_2^*(\Theta)$$

for a $\mathfrak{g}$-valued 1-form ω on Y. Restricting to the 1-section of $Y \times G$ over Y, we find $\omega = s^*(A)$.

Thus we find on taking $Y = P$ with $s = id_P$ that

$$\boxed{m^*(A) = pr_1^*(A) + pr_2^*(\Theta)}$$

on $P \times G$.

Consider next the BRS algebra of DV-V-T, or ~~[illegible]~~ really the non-commutative version. This is the bigraded differential algebra generated by fields A, F, χ, φ of degrees $(1,0)$, $(2,0)$, $(0,1)$, $(1,1)$ respectively with anti commuting differentials d, δ of degrees $(1,0)$ and $(0,1)$ resp., such that one has the ~~[illegible]~~ following formulas for these differentials on the generators:

$$dA = F - A^2 \quad ✓$$

$$\delta A = -\overset{\varphi}{\cancel{d\chi}} - [A,\chi] \quad ✓$$

$$dF = -[A,F] \quad ✓$$

$$\delta F = -[\chi, F] \quad ✓$$

$$d\chi = \varphi \quad ✓$$

$$\delta\chi = -\chi^2 \quad ✓$$

$$d\varphi = 0 \quad ✓$$

$$\delta\varphi = -[\chi,\varphi] \quad ✓$$

These equations can be obtained by taking components of

$$(d+\delta)\chi + \chi^2 = \varphi \qquad [d+\delta+\chi, \varphi] = 0$$

$$(d+\delta)(A+\chi) + (A+\chi)^2 = F \qquad [d+\delta+A+\chi, F] = 0$$

We have that $\Omega(P\times G)\otimes M_N$ is bigraded. We can't use $pr_1^*(A) \leftarrow A$ and $pr_2^*(\Theta) \leftrightarrow \chi$ because then $d\chi = 0$. ?

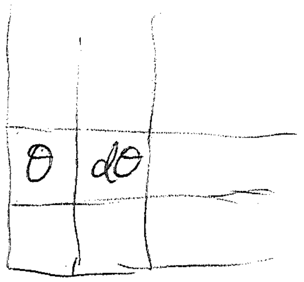
It seems that we have something different: the BRS algebra seems to be different from $\Omega(P\times G)$. In fact the total cohomology is wrong, because the BRS algebra has trivial cohomology whereas $\Omega(P\times G)$ has the cohomology of G.

It seems that we get something related to $\Omega(P\times G)$ by taking the quotient of the BRS algebra by the relation $\varphi = 0$. Thus $\chi = pr_2^*(\Theta)$ satisfies $(d+\delta)\chi + \chi^2 = 0$. and χ is of degree $(1,0)$. Also $m^*(A) = pr_1^*(A) + pr_2^*(\Theta) = A + \chi$ satisfies $(d+\delta)(A+\chi) + (A+\chi)^2 = F$ since the curvature comes from M.

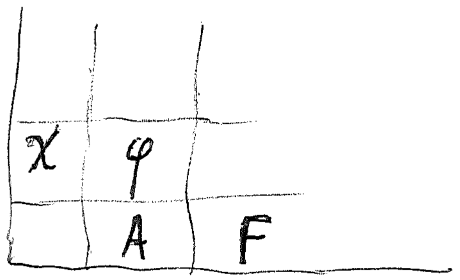
Conclude we don't understand their BRS algebra. It is some sort of model for $P\times P$ maybe? As opposed to $P\times_M P = P\times G$

Curious fact about the BRS algebra is that it has the same δ-cohomology as the δ cohomology of the Weil algebra $\mathbb{C}\{\theta, d\theta\}$. The proof of DV-T-V is to use that we can take the generators to be

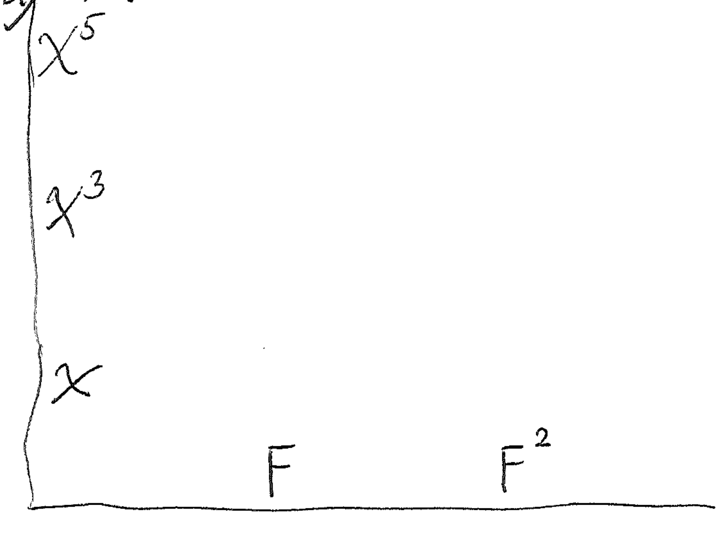
$\chi, F, A, \delta A$ and that the subalgebra generated by $A, \delta A$ has trivial cohomology. In the end the basic difference appears to be one of the indexing:



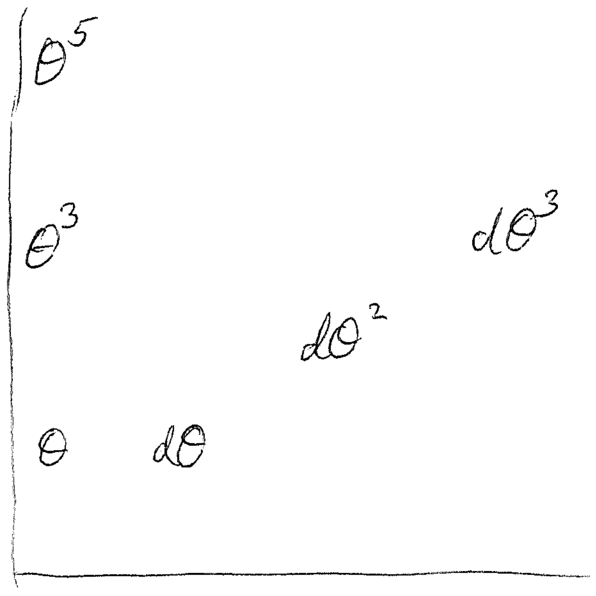
vs.



So the δ cohomology of the latter (non-commutative primitive version) is



instead of



I get the impression that one has something analogous to the different cyclic bicomplexes. Basically one has the same filtered gadget but is describing it as double complexes in different ways.

Question: Is it possible that there BRS algebra with generators

χ	φ		
	A	F	

is really much closer to the Leray spectral sequence for $PG \longrightarrow BG$? Look at concrete case with the Grassmannian.

June 6, 1989

δ mod d cohomology is by definition the (co)homology of C/dC for the differential δ. In degree p,q it is represented by elements $x \in C^{pq}$ such that $\delta x = dy$ for some $y \in C^{p-1,q+1}$ such an x represents 0 when it is of the form $x = \delta z + dw$ with $z \in C^{p,q-1}$ and $w \in C^{p-1,q}$:

$$\begin{tikzcd}
y \arrow[r, "d"] & {} \\
w \arrow[r] & x \arrow[u, "\delta"'] \\
& z \arrow[u]
\end{tikzcd}$$

In the case C has trivial d(co)homology, one has exactly sequences

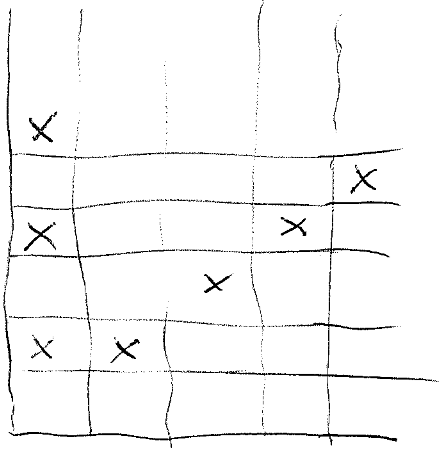
$$0 \longrightarrow C^{p-1,\bullet}/dC^{p-2,\bullet} \longrightarrow C^{p,\bullet} \longrightarrow C^{p,\bullet}/dC^{p-1,\bullet} \longrightarrow 0$$

which give a long exact sequence

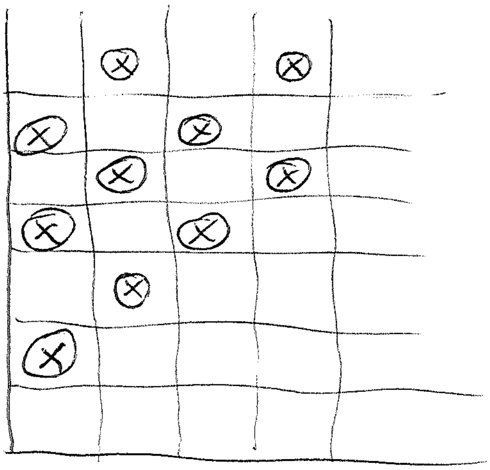
$$\longrightarrow H^q(C^{p,\bullet},\delta) \longrightarrow H^q(C^{p,\bullet}/dC^{p-1},\delta) \longrightarrow H^{q+1}(C^{p-1,\bullet}/dC^{p-2,\bullet},\delta)$$

which can sometimes be used to get the δ (mod d) cohomology from the δ cohomology.

In the case of the Weil algebra: $C = \mathbb{C}\{\theta, d\theta\}_g$ we can successively obtain the δ mod d cohomology one column at a time.



δ coh.

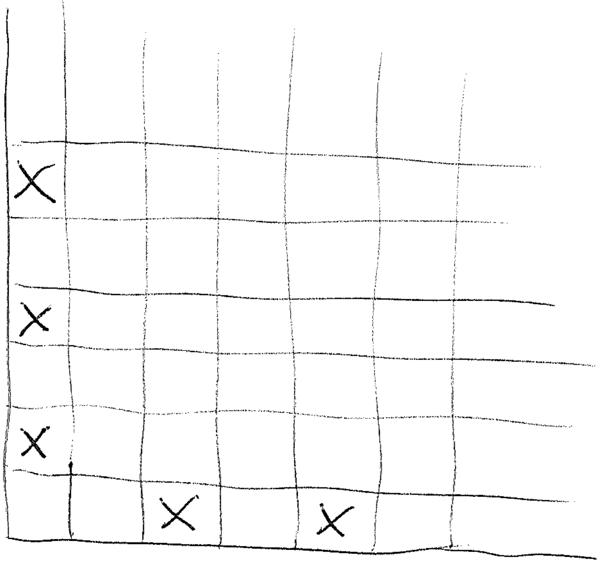


δ mod d

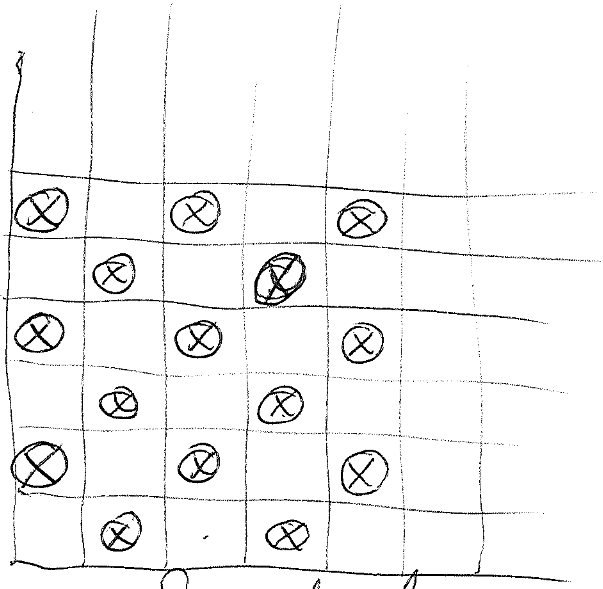
What happens is that the δ mod d for column $p-1$ shifts [redacted] down basically and at the bottom edge there is cancellation.

The conclusion is that the components of the Chern-Simons classes [redacted] represent the δ mod d cohomology of $\mathbb{C}\{\theta, d\theta\}_{\#}$.

As for the DVTV algebra or BRS alg we have



δ coh



δ mod d

Problem: Find representatives for the δ mod d coh.

June 11, 1989

The calculations of Connes S-operation using differential forms (p318-321) left me with the impression that his construction gives the same cocycles as Chern-Simons but with different numerical constants. This appears wrong. It is necessary to check things carefully.

Let's start with the fact that the map $\Omega_{A\otimes B} \longrightarrow \Omega_A \otimes \Omega_B$ in the nonunital setting gives a way to combine (closed) traces τ on Ω_A and τ'' on Ω_B to get a trace $\tau\#\tau''$ on $\Omega_{A\otimes B}$. Let's see how this works when $B = \mathbb{C}e$.

τ is completely determined by the (bar) cochain $\tau(\theta d\theta^p)$ if τ is homogeneous of degree p. Notice that this cochain happens to be cyclic. This is clear from

$$\tau(\theta\, d\theta^p)(a_0, \dots, a_p) = \tau(a_0\, da_1 \dots da_p)$$

and the fact that this is a cyclic p-cocycle.

Let τ_{2k} be the trace on $\Omega^{2k}_{\mathbb{C}e}$ such that $\tau_{2k}\{e(de)^{2k}\} = k!$. Let's calculate $\tau\#\tau_{2k}$ on $\Omega_{A\otimes\mathbb{C}e} \simeq \Omega_A$.

$$(\tau\#\tau_{2k})(\theta(d\theta)^{p+2k})$$

$$= (\tau\otimes\tau_{2k})\left(\theta\otimes e(d\theta\otimes e - \theta\otimes de)^{p+2k}\right)$$

When we expand this out and use $ede\,e = 0$, we have to use $-\theta \otimes de$ in consecutive pairs

Note
$$(-\theta \otimes de)^2 = (\theta \otimes de)(\theta \otimes de) = -\theta^2 \otimes de^2$$

Thus we get
$$(\tau \# \tau_{2k})(\theta (d\theta)^{p+2k}) = \tau(\theta P_{pk}(d\theta, -\theta^2))\, k!$$
$$= (-1)^k k!\ \tau(\theta P_{pk}(d\theta, \theta^2))$$

Notice this implies that the (bar) cochain $\tau(\theta P_{pk}(d\theta, \theta^2))$ is cyclic.

But the important lesson is that we should assign to a trace τ homogeneous of degree p the cyclic cochain
$$\tau(\theta\, d\theta^p)/p! = \tau(\theta\, d\theta^p/(p+1)!)N,$$
because then the [redacted] cyclic cochain belonging to $\tau \# \tau_{2k}$ is
$$(\tau \# \tau_{2k})(\theta\, d\theta^{p+2k}/(p+2k)!) = \frac{(-1)^k k!}{(p+2k)!} \tau(\theta P_{pk}(d\theta, \theta^2))$$
$$= \frac{(-1)^k k!}{(p+2k+1)!} \tau(\theta P_{pk}(d\theta, \theta^2))N$$

This shows the consistency of his approach with the Chern-Simons approach.

Some other things worth checking are
$$\tau_{2\ell} \# \tau_{2k} = \tau_{2(\ell+k)} \qquad \text{for } \mathbb{C}e$$
and that [redacted] the cyclic cocycle belonging to τ_{2k} paired with the canonical generator for

cyclic homology, namely $\frac{(2k)!}{k!}(e)_{2k+1}$, gives 1. Here's the latter:

$$\tau_{2k}\left(\theta(d\theta)^{2k}/(2k)!\right)\left(\frac{(2k)!}{k!}(e)_{2k+1}\right)$$

$$= \tau_{2k}\left(\frac{e(de)^{2k}}{k!}\right) = 1.$$

and the former was done on p320. So everything looks consistent.

June 21 - Aug 28, 1989 334-472

88 - 89
CYCLIC STUFF

Stanford trip: Stasheff's version of BRS 337
homological perturbation theory 342
free loop space on BG 360

Index theory
determinant divisor
$[\varepsilon, A]$, $1-\varepsilon^2 \subset A$ 423
"Tate residue" cocycles.

Obtaining standard A bimodule resolutions of A via DG algebras like $T_A(A\otimes A)$ 451, 456
Getzler's Witten current 469

June 20, 1989 (in NJ on way to Calif with Erica)

In my ChSi paper I found a geometric situation where the S-formalism appears. Let us consider a principal $G = U(r)$ bundle P over M. Then in the cohomology of P there are Chern-Simons classes $[CS_{2n-1}] \in H^{2n-1}(P)$ defined for $2n \geqslant \dim M + 1$. The ones for $2n > \dim M + 1$ are independent of the choice of connection, but the one for $2n-1 = \dim M$ can depend on the connection.

To construct these we choose a connection $A \in \Omega^1(P) \otimes \text{Lie}\, U(r) = \Omega^1(P) \otimes M_r$ and we calculate in the DGA

$$R = \Omega(P) \otimes M_r$$

We construct the CS forms using the connection and the matrix trace

$$R \longrightarrow \Omega(P)$$

Note: as $\Omega(P)$ is unital comm we have $R_\natural = \Omega(P)$.

Then we can establish the S relations using the sequence

$$\text{⊛} \quad 0 \longrightarrow R_\natural \longrightarrow \Omega^1_{R\natural} \longrightarrow R \longrightarrow R_\natural \longrightarrow 0$$

This sequence is not exact. The homology at the point $R_\natural$ is $H_0^{DR}(R) = \text{Im}\{S : HC_2 \to HC_0\}$ and at the point $\Omega^1_{R\natural}$ is $HC_1(R)$.

Because ⊛ is not exact it is not clear what the S-relations mean. Let us consider the exact sequence

(**) $$0 \longrightarrow H_1(R,R) \longrightarrow \Omega^1_{R\natural} \xrightarrow{\beta} R \longrightarrow \overset{R_\natural}{\overset{\parallel}{H_0(R,R)}} \longrightarrow 0$$

where here we use the unital setting (otherwise we would have $H_*(R,\tilde{R})$.)

This exact sequences determines a map in the derived category

(***) $$R_\natural \xrightarrow{\deg +2} H_1(R,R)$$

The S-relation asserts that two CS classes have the same image

$$\begin{tikzcd}
 & CS_{2n-1} \text{ in } R_\natural \arrow[d, "\deg +2"] \\
CS_{2n+1} \text{ in } R_\natural \arrow[r, "B=\partial_\natural"] & H_1(R,R)
\end{tikzcd}$$

But in the present example (***) is zero.

We check this. First recall facts about $A \otimes B$ unital setting. Write $\hat{\Omega}_R$ for $\Omega^1_{R\natural}$. We have for $B = M_n$

$$\begin{tikzcd}
0 \arrow[r] & H_1(B,B) \arrow[r] \arrow[d, equal] & \hat{\Omega}_B \arrow[r] & B \arrow[r] \arrow[dr, "tr"'] & B_\natural \arrow[r] \arrow[d, equal] & 0 \\
 & 0 & & & \mathbb{C} &
\end{tikzcd}$$

and general exact sequences

$$0 \longrightarrow \Omega^1_A \otimes \Omega^1_B \longrightarrow \left(\Omega^1_A \otimes B \otimes B \oplus A \otimes A \otimes \Omega^1_B\right) \longrightarrow \Omega^1_{A\otimes B} \longrightarrow 0$$

$$\hat{\Omega}_A \otimes \hat{\Omega}_B \longrightarrow \hat{\Omega}_A \otimes B \oplus A \otimes \hat{\Omega}_B \longrightarrow \hat{\Omega}_{A\otimes B} \longrightarrow 0$$

In the present case, there is injectivity at the left as $\hat{\Omega}_B \hookrightarrow B$.

Better, we have a trace map

$$\begin{tikzcd}
0 \arrow[r] & H_1(A\otimes B, A\otimes B) \arrow[r] \arrow[d, "\cong"', "B=M_r"] & \hat{\Omega}_{A\otimes B} \arrow[r] \arrow[d] & A\otimes B \arrow[r] \arrow[d] & (A\otimes B)_\natural \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_1(A,A)\otimes B_\natural \arrow[r] & \hat{\Omega}_A \otimes B_\natural \arrow[r] & A\otimes B_\natural \arrow[r] & A_\natural \otimes B_\natural \arrow[r] & 0
\end{tikzcd}$$

from $(**)$ for $A\otimes B$ to $(**)$ for A. For $B=M_r$ the maps at the ends are $\cong$. For A commutative $\hat{\Omega}_A \xrightarrow{\beta} A$ is zero.

This shows that we get a meaningless statement "$0=0$" if we try to work with $R = \Omega(P) \otimes M_r$. However there is the possibility of working in a subalgebra

July 1, 1989

Notes on Stasheff's papers.
His discussion of BRS.

W symplectic manifold, V submanifold such that at each $x \in V$ the annihilator $(T_xV)^\perp$ of the subspace $T_xV \subset T_xW$ for the symplectic form is contained in T_xV. These subspaces

$$(T_xV)^\perp \subset T_xV$$

define a foliation $\mathcal{F}$ of V and the ~~quotient~~ $V/\mathcal{F}$ is a symplectic manifold. ($V/\mathcal{F}$ is defined locally).

Example. If $H : W \longrightarrow R$ is non-degenerate, better, if a is a regular value of H, then we can take $V = H^{-1}\{a\}$. The Hamiltonian vector field X_H spans $\mathcal{F}$.

The general case locally is a codimension r version of the preceding. Suppose given $\vec{H} : W \longrightarrow R^r$ by functions $H_1, \ldots, H_r$, ~~and~~ suppose 0 is a regular value, and put $V = \vec{H}^{-1}\{0\}$. Along V, $dH_1, \ldots, dH_r$ is a basis for $(TW|V)/TV$ so the vector fields $X_{H_1}, \ldots, X_{H_r}$ span the annihilator subbundle $(TV)^\perp \subset TW|V$. We want this subbundle to lie in TV which means $i(X_{H_i})\,dH_j = \{H_i, H_j\} = 0$ on V. This means that the Poisson brackets $\{H_i, H_j\}$ lie in the ideal of fns. vanishing on V, which should be generated by the H_i. Thus we have (locally)

$$\{H_i, H_j\} = \sum_k C_{ij}^k H_k$$

for some smooth functions C_{ij}^k. Dirac calls the conditions: $H_i = 0$, where the H_i satisfy the above equations, constraints of the first class.

In the case of a group action the C_{ij}^k are constants.

One should investigate nearby level sets, i.e. where $H_i = c_i$. This is probably linked in some nice way to the moment mapping in the case of a group action.

July 2, 1989

Consider the symplectic quotient situation

$$\begin{tikzcd} W \arrow[r, "\{H_i\}"] & \mathbb{R}^r \\ V \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d] & 0 \arrow[u, phantom, "\cup"] \\ V/\mathcal{F} & \end{tikzcd}$$

regular value

$$\{H_i, H_j\} = \sum_k C_{ij}^k H_k$$

$$\mathcal{F} = \bigoplus_{i=1}^r C^\infty(V) X_{H_i} \subset T(V)$$

from yesterday. (First class constraints in Dirac's sense).

The problem is to get control of $V/\mathcal{F}$, that is to have a suitable description to do "physics". One wants a description of $C^\infty(V/\mathcal{F})$ in terms of the data $(W, \{H_i\})$.

One uses the relative DR complex for $V \longrightarrow V/\mathcal{F}$ to obtain $C^\infty(V/\mathcal{F})$ from $C^\infty(V)$. This is the complex

$$\circledast \quad C^\infty(V) \longrightarrow \Gamma(V, \mathcal{F}^*) \longrightarrow \Gamma(V, \Lambda^2 \mathcal{F}^*) \longrightarrow \cdots$$

which is the quotient of the DR complex of V by the ideal generated by $\Gamma(V, Q^*)$ where $Q = TV/\mathcal{F}$ is the quotient bundle.

This complex is not necessarily a resolution of $C^\infty(V/\mathcal{F})$, but it is clearly a good substitute for the latter from the viewpoint of cohomology.

If the C_{ij}^k are constants, i.e. if we have a Lie action given by the vector fields X_{H_i}, then because $\mathcal{F}$ has basis given by $\{X_{H_i}\}$, we see that ⊛ is the complex of Lie cochains

$$C^\cdot(\mathfrak{g}, C^\infty(V))$$

The second step is to obtain $C^\infty(V)$ from $C^\infty(W)$ using the Koszul complex

(**) $$K_\cdot(H_1, \dots, H_r ; C^\infty(W))$$

which is a resolution of $C^\infty(V)$ by the regularity assumption.

Now arises the problem of how to combine the two complexes ⊛ and (**). Let's introduce some notation and terminology.

†

Let's put $\{\omega_i\}$ = basis for $\Gamma(V, \mathcal{F}^*)$ dual to $\{X_{H_i}\}$ (= basis for $\mathfrak{g}^*$ in the group action case). Then ignoring differentials

$$\circledast = C^\infty(V) \otimes \Lambda[\omega_i]$$

The ω_i are called ghosts.

Let's write $\{\rho_i\}$ for the generators of the exterior algebra occurring in the Koszul complex $\textcircled{**}$, so ignoring diffls.

$$\textcircled{**} = K_\cdot(H; C^\infty(W)) = C^\infty(W) \otimes \Lambda[\rho_i]$$

The ρ_i are called anti-ghosts.

Putting these together we get

$$\Lambda[\omega_i] \otimes C^\infty(W) \otimes \Lambda[\rho_i]$$

and the problem is to construct a suitable differential, compatible with the differentials given in $\circledast$, $\textcircled{**}$. This differential is the classical BRST generator (or charge?)

Stasheff in (Constrained Hamiltonians: a homological approach, Proceedings Winter School on Geom + Phys. SRNÍ 10-17 Jan 1987, Supp on Rendiconti del Circolo Matematico di Palermo)

(Constrained Hams; An Intro to Homol. Alg in Field Theoretic Phys., Proc. Conf Elliptic Fns in Alg Top, IAS, Sept 1986) claims that this BRS differential can be constructed using homological perturbations ~~starting~~ from a contracting homotopy of the Koszul complex (with $C^\infty(V)$ added).

The above, where the constraints H_i are independent, is called the irreducible case. The reducible case is treated in

J. Fisch, M. Henneaux, J. Stasheff, C. Teitelboim
BRST Formulation of Reducible gauge theories.
Commun. Math Phys. 120 (1989) 379-407

They use Koszul-Tate resolution (a general free comm DGA resolution of $C^\infty(V)$ over $C^\infty(W)$) and prove existence of BRS.

Perturbation Theory in diffl homological alg.

a

July 6, 1989

More on Stasheff's ideas about perturbations in diff'l graded algebra.

Milgram pointed out to me that the original idea (Guggenheim-Milgram) paper is the following. One has a double complex, say a double chain complex, whose columns are $K_{p\cdot}$ for $p \geq 0$. Suppose one has a homotopy equivalent complex $K'_{p\cdot}$ for each p, then one can find a differential on $\bigoplus_p K'_{p\cdot}$ such that the resulting complex is homotopy equivalent to the total complex of K.

The proof should proceed by using the increasing column filtration of K and inductive. The inductive step appears to be the following. ~~[illegible]~~ Consider an exact sequence of complexes

$$① \qquad 0 \longrightarrow X \longrightarrow E \longrightarrow Y \longrightarrow 0$$

and homotopy equivalences $X \sim X'$, $Y \sim Y'$. Then there is a homotopy equivalent exact sequence

$$0 \longrightarrow X' \longrightarrow E' \longrightarrow Y' \longrightarrow 0$$

To construct this we use a splitting of ① not compatible with differentials:

$$E = X \oplus Y$$

Then the differential of E can be compared with $d_X \oplus d_Y$. Write $d = d_X \oplus d_Y$; then the differential of E is

$$d + \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}$$

Thus we see d_E is a twisted differential, that is, of the form $d + \theta$ on the complex $X \oplus Y$ where $[d, \theta] + \theta^2 = 0$ In the present case both $[d, \theta]$ and θ^2 are zero, but it is interesting to consider the case of a general θ.

So thus consider the following question. Consider a homotopy equivalence of complexes $E \sim E'$ (these are now $X \oplus Y$, $X' \oplus Y'$ in the above example). Suppose we have a twisted differential $d + \theta$ on E, can we find a twisted differential $d + \theta'$ on E' such that the two complex $(E, d+\theta)$, $(E', d+\theta')$ are homotopy equivalent.

Here's a construction that might be relevant. Suppose $E \underset{V}{\overset{U}{\rightleftarrows}} E'$ are maps of complexes such that the composite VU is homotopic to the identity:

$$VU - 1 = [d, H]$$

Thus E is part of E'. Let $d + \theta$ be a twisted differential on E. Then we want c

to construct a twisted differential $d+\theta'$ on E'.

First case: suppose $H=0$. Then take $\theta' = U\theta V$.

$$d\theta' + \theta'^2 = U\,d\theta\,V + U\theta\overbrace{VU}^{1}\theta V$$
$$= U(d\theta+\theta^2)V = 0$$

In general put $\theta_0 = U\theta V$, then

$$d\theta_0 = -U\theta^2 V$$
$$\theta_0^2 = U\theta V U\theta V$$
$$d\theta_0 + \theta_0^2 = U\theta(VU-1)\theta V$$
$$= U\theta(dH)\theta V$$

This suggests the correction $\theta_0+\theta_1$ where $\theta_1 = U\theta H\theta V$. And then further calculation suggests the geometric series

$$\theta' = U(\theta+\theta H\theta+\cdots)V = U\left(\theta\frac{1}{1-H\theta}\right)V$$

$$d\theta' = -U\theta^2\frac{1}{1-H\theta}V + U\theta\frac{1}{1-H\theta}\underbrace{d(-H\theta)}\frac{1}{1-H\theta}V$$

$(-dH\theta + Hd\theta)$

$$(\theta')^2 = U\theta\frac{1}{1-H\theta}(1+dH)\cancel{\theta}\frac{1}{1-H\theta}V$$

~~$d\theta'+\theta'^2 = U\left\{-\theta^2\frac{1}{1-H\theta} + \theta\frac{1}{1-H\theta}H(-\theta^2)\frac{1}{1-H\theta}\right\}V$~~

~~$= U\theta\left\{1+\frac{1}{1-H\theta}H\theta\right\}(-\theta)\frac{1}{1-H\theta}V$~~

d

$$d\theta' + \theta'^2 =$$

$$-u\theta^2 \frac{1}{1-H\theta} V + \overparen{u\theta} \frac{1}{1-H\theta} H(-\theta^2) \frac{1}{1-H\theta} V$$

$$+ \overparen{u\theta} \frac{1}{1-H\theta} \overline{\theta \frac{1}{1-H\theta}} V$$

$$= u\theta \Big\{ -1 - \frac{1}{1-H\theta} H\theta + \theta \frac{1}{1-H\theta} \Big\} \theta \frac{1}{1-H\theta} V$$

$$= u\theta \Big\{ -1 + \frac{1}{1-H\theta}(1-H\theta) \Big\} \theta \frac{1}{1-H\theta} V$$

$$= 0.$$

July 7, 1989

Situation: (E,d) a complex with another differential $d+\theta$; $[d,\theta]+\theta^2=0$.

$$\begin{tikzcd} E \arrow[d, shift right, "u"'] \\ E' \arrow[u, shift right, "v"'] \end{tikzcd}$$

maps of complexes such that $vu-1=[d,h]$

We want to transport the twisted differential $d+\theta$ to E'. Put

$$\theta' = u\,\theta\,\frac{1}{1-h\theta}\,v = u\,\frac{1}{1-\theta h}\,\theta\,v$$

$$\tilde{u} = u\,\frac{1}{1-\theta h} \qquad \tilde{v} = \frac{1}{1-h\theta}\,v$$

$$\tilde{h} = h\,\frac{1}{1-\theta h} = \frac{1}{1-h\theta}\,h$$

assuming the operator $(1-h\theta)^{-1}$ exists. Then we claim

$$d\theta' + (\theta')^2 = 0.$$

$$(d+\theta')\,\tilde{u} = \tilde{u}\,(d+\theta)$$

$$(d+\theta)\,\tilde{v} = \tilde{v}\,(d+\theta')$$

$$\tilde{v}\tilde{u} - 1 = [d+\theta, \tilde{h}]$$

Proof. ① $$d\theta' = u\,\frac{1}{1-\theta h}\,\overbrace{d(\theta h)}^{-\theta^2 h - \theta dh}\,\frac{1}{1-\theta h}\,\theta\,v$$

$$+\, u\,\frac{1}{1-\theta h}\,(-\theta^2)\,v$$

$$(\theta')^2 = u\,\frac{1}{1-\theta h}\,\theta(1+dh)\,\frac{1}{1-\theta h}\,\theta\,v$$

$$d\theta' + \theta'^2$$

$$= u\frac{1}{1-\theta h}\theta\Big\{(-\theta h - \cancel{dh})\frac{1}{1-\theta h} - 1$$

$$+ (1+\cancel{dh})\frac{1}{1-\theta h}\Big\}\theta v$$

$$= u\frac{1}{1-\theta h}\theta\Big\{(1-\theta h)\frac{1}{1-\theta h} - 1\Big\}\theta v = 0$$

② $(d+\theta')\tilde{u} - \tilde{u}(d+\theta) = [d,\tilde{u}] + \theta'\tilde{u} - \tilde{u}\theta$

$$[d,\tilde{u}] = u\frac{1}{1-\theta h}(-\theta^2 h - \theta dh)\frac{1}{1-\theta h}$$

$$\theta'\tilde{u} = u\frac{1}{1-\theta h}\theta(1+dh)\frac{1}{1-\theta h}$$

$$-\tilde{u}\theta = -u\frac{1}{1-\theta h}\theta$$

$$= u\frac{1}{1-\theta h}\theta\Big\{(-\theta h - \cancel{dh})\frac{1}{1-\theta h} + (1+\cancel{dh})\frac{1}{1-\theta h} - 1\Big\} = 0$$

③ $(d+\theta)\tilde{v} - \tilde{v}(d+\theta') = [d,\tilde{v}] + \theta\tilde{v} - \tilde{v}\theta'$

$$[d,\tilde{v}] = \frac{1}{1-h\theta}(dh\theta + h\theta^2)\frac{1}{1-h\theta}v$$

$$\theta\tilde{v} = \theta\frac{1}{1-h\theta}v$$

$$-\tilde{v}\theta' = -\frac{1}{1-h\theta}(1+dh)\theta\frac{1}{1-h\theta}v$$

$$= \Big\{\frac{1}{1-h\theta}(\cancel{dh} + h\theta) + 1 - \frac{1}{1-h\theta}(1+\cancel{dh})\Big\}\theta\frac{1}{1-h\theta}v = 0$$

$$\text{⑦} \quad \tilde v \tilde u \ \text{[redacted]} = \frac{1}{1-h\theta}(1+dh)\frac{1}{1-\theta h}$$

$$[d,\tilde h] = \frac{1}{1-h\theta}(dh\theta + h\theta^2)\frac{1}{1-h\theta}h + \frac{1}{1-h\theta}dh$$

$$[\theta,\tilde h] = \theta h\frac{1}{1-\theta h} + \frac{1}{1-h\theta}h\theta$$

$$dh, \ \text{[redacted]} = \frac{1}{1-h\theta}dh\underbrace{\left\{\theta\frac{1}{1-h\theta}h + 1\right\}} + \frac{1}{1-h\theta}h\theta\,\theta\frac{1}{1-h\theta}h$$

$$1+\frac{1}{1-\theta h}\theta h = \frac{1}{1-\theta h}$$

$$\therefore \ 1+d\tilde h+[\theta,\tilde h] = \frac{1}{1-h\theta}dh\frac{1}{1-\theta h} + 1 + \left(\frac{1}{1-h\theta}h\theta\right)\left(\theta h\frac{1}{1-\theta h}\right)$$
$$+\left(\theta h\frac{1}{1-h\theta}\right)\text{[redacted]} + \left(\frac{1}{1-h\theta}h\theta\right)$$

$$= \frac{1}{1-h\theta}dh\frac{1}{1-\theta h} + \left(1+\frac{1}{1-h\theta}h\theta\right)\left(1+\theta h\frac{1}{1-\theta h}\right)$$

$$= \frac{1}{1-h\theta}(dh+1)\frac{1}{1-\theta h} = \tilde v\tilde u.$$

Next suppose E' is a strong deformation retract of E in some sense. Mainly I want $uv = id_{E'}$, and I would like to conclude as Guggenheim, Lambe, Stasheff do that $\tilde u\tilde v = id_{E'}$. Now

$$\tilde u\tilde v = u\frac{1}{1-\text{[redacted]}\theta h}\frac{1}{1-h\theta}v = \sum_{m,n} u(\theta h)^m(h\theta)^n v$$

and θ is fairly [redacted] arbitrary. So we have to assume $h^2 = uh = hv = 0$, it seems.

Thus we claim

$$\left.\begin{array}{l} uv = 1 \\ h^2 = uh = hv = 0 \end{array}\right\} \Longrightarrow \text{same for } \tilde{u}, \tilde{v}, \tilde{h}.$$

Check $\quad \tilde{u}\tilde{h} = u\frac{1}{1-\theta h}h\frac{1}{1-\theta h} = uh\frac{1}{1-\theta h} = 0$

June 10, 1989

Composition

$$\begin{tikzcd} E \arrow[d, shift right, "u"'] \\ E' \arrow[u, shift right, "v"'] \arrow[d, shift right, "u'"'] \\ E'' \arrow[u, shift right, "v'"'] \end{tikzcd}$$

$$vu = 1 + [d, h]$$

$$v'u' = 1 + [d, h']$$

$$U = u'u \qquad V = vv'$$

$$VU = vv'u'u = v(1+[d,h'])u$$

$$= 1 + [d, \underbrace{h + vh'u}_{H}]$$

$$\theta' = u \frac{1}{1-\theta h} \theta v \qquad \theta'' = u' \frac{1}{1-\theta' h'} \theta' v'$$

We want to check that the setup for the composition (θ, U, V, H) is obtained by composing the setup for [redacted] (u, v, h) ~~and then~~ followed by (u', v', h')

This means showing

[redacted] $\tilde{U} = \tilde{u}' \tilde{u} \qquad \tilde{V} = \tilde{v} \tilde{v}'$

$$\theta'' = U \frac{1}{1-\theta H} \theta V$$

[redacted] $\tilde{H} = \tilde{h} + \tilde{v} \tilde{h}' \tilde{u}$

① $$\tilde{u}' \tilde{u} = u' \frac{1}{1-\theta' h'} u \frac{1}{1-\theta h} = u' \frac{1}{1 - u \frac{1}{1-\theta h} \theta v h'} u \frac{1}{1-\theta h}$$

$$= u'u \frac{1}{1 - \frac{1}{1-\theta h} \theta v h' u} \frac{1}{1-\theta h} = u'u \frac{1}{1-\theta h - \theta v h' u}$$

$$= U \frac{1}{1-\theta H} = \tilde{U}$$

② $$\tilde{v}\tilde{v}' = \frac{1}{1-h\theta}v\,\frac{1}{1-h'\theta'}v'$$

$$= \frac{1}{1-h\theta}v\,\frac{1}{1-h'u\theta\frac{1}{1-h\theta}v}v'$$

$$= \frac{1}{1-h\theta}\,\frac{1}{1-vh'u\theta\frac{1}{1-h\theta}}v v' = \frac{1}{1-h\theta-vh'u\theta}vv'$$

$$= \frac{1}{1-H\theta}V = \tilde{V}$$

③ $$\theta'' = u'\frac{1}{1-\theta'h'}\theta' v' = u'\frac{1}{1-u\frac{1}{1-\theta h}\theta v h'}u\frac{1}{1-\theta h}\theta v v'$$

$$= u'u\,\frac{1}{1-\frac{1}{1-\theta h}\theta v h' u}\,\frac{1}{1-\theta h}\theta v v'$$

$$= u'u\,\frac{1}{1-\theta h-\theta v h' u}\theta v v' = U\frac{1}{1-\theta H}\theta V$$

④ ~~$\tilde{h} + \tilde{u}\tilde{h}'\tilde{v} = h\frac{1}{1-\theta h} + u\frac{1}{1-\theta h}h'\frac{1}{1-\theta' h'}\frac{1}{1-h\theta}V$~~

~~$= h\frac{1}{1-\theta h} + u\frac{1}{1-\theta h}h'\frac{1}{1-u\frac{1}{1-\theta h}\theta v h'}\frac{1}{1-h\theta}V$~~

$$\tilde{h} + \tilde{v}\tilde{h}'\tilde{u} = h\frac{1}{1-\theta h} + \frac{1}{1-h\theta}v\,h'\frac{1}{1-\theta' h'}u\frac{1}{1-\theta h}$$

$$= \frac{1}{1-h\theta}h + \frac{1}{1-h\theta}vh'\underbrace{\frac{1}{1-u\frac{1}{1-\theta h}\theta h'}}\,u\frac{1}{1-\theta h}$$

$$= \frac{1}{1-h\theta}h + \frac{1}{1-h\theta}vh'u\frac{1}{1-\theta h-\theta v h' u}$$

$$= \frac{1}{1-h\theta}\left[h(1-\theta h-\theta v h' u) + vh'u\right]\frac{1}{1-\theta H} =$$

$$\frac{1}{1-h\theta}\Big[\overset{(1-h\theta)h}{\widetilde{h(1-\theta h)}} + (-h\theta+1)vh'u\Big]\frac{1}{1-\theta H}$$

$$= (h + vh'u)\frac{1}{1-\theta H} = H\frac{1}{1-\theta H} = \tilde{H}.$$

July 11, 1989

Review composition

$$\begin{tikzcd} E \arrow[d, shift right, "u"'] \\ E' \arrow[u, shift right, "v"'] \arrow[d, shift right, "u'"'] \\ E'' \arrow[u, shift right, "v'"'] \end{tikzcd}$$

$$vu = 1 + dh$$
$$v'u' = 1 + dh'$$

$$vv'u'u = v(1+dh')u$$
$$= 1 + d(h + vh'u)$$

This defines a category whose morphisms are triples (u, v, h). Check associativity:

$$(h + vh'u) + (vv')h''(u'u)$$

$$h + v(h' + v'h''u')u \qquad ✓$$

Now suppose we restrict attention to homotopy equivalences. Then we have something like a groupoid at least after homotopy is taken into account. Philosophy: Our setup consisting of complexes E with second differential $d + \theta$. and homotopy equivalences should be analogous to the groupoid of principal bundles equipped with flat connection under isomorphism.

Let's take this viewpoint and see where it leads. Thus we consider pairs (E,θ) where θ is to be viewed as analogous to a flat connection form. Given a homotopy equivalence $E \sim E'$ we then have a flat connection form θ' on E' more or less, and the homotopy equivalence can be modified so as to respect the flat connections.

It's not a very accurate analogy. I would really like something close to super gauge transformations.

Example. Consider (E, d, θ) and an h such that $1-h\theta$ is invertible. Then take $E'=E$, $u = 1+[d,h]$, $v=1$.

With $\theta', \tilde v, \tilde u$ as before, we have

$$\tilde v = \frac{1}{1-h\theta} \qquad (d+\theta)\overset{\tilde v}{\overbrace{\frac{1}{1-h\theta}}} = \overset{\tilde v}{\overbrace{\frac{1}{1-h\theta}}}(d+\theta')$$

so $$d+\theta' = g^{-1}(d+\theta)g \qquad g=\frac{1}{1-h\theta}$$

Check:
$$\frac{1}{1-h\theta}\Big(\overbrace{d+(1+[d,h])\theta\frac{1}{1-h\theta}}^{d+\theta'}\Big)(1-h\theta)$$

$$= \frac{1}{1-h\theta}\Big(d\circ(1-h\theta) + (1+[d,h])\theta\Big)$$

$$= d + \frac{1}{1-h\theta}\Big(\underbrace{[d,-h\theta]}_{-\cancel{[d,h]}\theta - h\theta^2} + (1+\cancel{[d,h]}\theta)\Big) = d+\theta$$

Similarly if $1-\theta h$ is invertible we can take $E'=E$, $u=1$, $V = 1+[d,h]$ and we have the "gauge transformation" $\tilde{u} = \frac{1}{1-\theta h}$ linking $d+\theta$ to $d+\theta'$.

Let us next consider algebra structures on E, E' and suppose u,v are compatible with these. Thus we have a commutative square

$$\begin{tikzcd}
E\otimes E \arrow[r, "m"] \arrow[d, shift right, "u\otimes u"'] & E \arrow[d, shift right, "u"'] \\
E'\otimes E' \arrow[r, "m'"] \arrow[u, shift right, "v\otimes v"'] & E' \arrow[u, shift right, "v"']
\end{tikzcd}$$

Next we need a homotopy $(v\otimes v)(u\otimes u) = 1+[d,H]$
There are two candidates corresponding to the different compositions

$$\begin{tikzcd}
E\otimes E \arrow[d, shift right, "u\otimes 1"'] & & E\otimes E \arrow[d, shift right, "1\otimes u"'] \\
E'\otimes E \arrow[u, shift right, "v\otimes 1"'] \arrow[d, shift right, "1\otimes u"'] & & E\otimes E' \arrow[u, shift right, "1\otimes v"'] \arrow[d, shift right, "u\otimes 1"'] \\
E'\otimes E' \arrow[u, shift right, "1\otimes v"'] & & E'\otimes E' \arrow[u, shift right, "1\otimes u"']
\end{tikzcd}$$

In the former

$$H = h\otimes 1 + (v\otimes 1)(1\otimes h)(u\otimes 1) = h\otimes 1 + vu\otimes h$$

In the latter

$$H = 1\otimes h + (1\otimes v)(h\otimes 1)(1\otimes u) = h\otimes uv + 1\otimes h$$

For H to be compatible with multiplication means ~~[illegible]~~

$$mH = h\,m$$

which in the first case is

$$h(xy) = m(h\otimes 1 + uv\otimes h)(x\otimes y)$$

$$= h(x)y + (-1)^{|x|}(uv)(x)\cdot h(y)$$

and in the second case is

$$h(xy) = m(h\otimes uv + 1\otimes h)(x\otimes y)$$

$$= h(x)(uv)(y) + (-1)^{|x|} x\, h(y)$$

These are natural (derivation) conditions relative to the two algebra homomorphisms $1, uv$; which are satisfied by $vu - 1 = [d,h]$.

Anyway supposing $H = h\otimes 1 + vu\otimes h$, let's next consider θ. Let's consider $\theta\otimes 1 + 1\otimes\theta$ on $E\otimes E$. We suppose θ is a derivation whence $m(\theta\otimes 1 + 1\otimes\theta) = \theta m$.

Then the pair of maps (m, m') ~~[illegible]~~ link (better: intertwine) $u\otimes u$, $v\otimes v$, $H = h\otimes 1 + vu\otimes h$, $\theta\otimes 1 + 1\otimes\theta$), so it should follow that they intertwine $\tilde{u}\otimes\tilde{u}$ etc. This should be checked carefully. What is clear is that (m,m') intertwine $\widetilde{u\otimes u}$, $\widetilde{v\otimes v}$, $\theta\otimes 1 + 1\otimes\theta$, $(\theta\otimes 1 + 1\otimes\theta)'$ and u, v, θ, θ'. So what we need is $\widetilde{u\otimes u} = \tilde{u}\otimes\tilde{u}$

Now we are considering

$$\begin{tikzcd}
E\otimes E \arrow[d, shift right, "u\otimes 1"'] \\
E'\otimes E \arrow[u, shift right, "N\otimes 1"'] \arrow[d, shift right, "1\otimes u"'] \\
E'\otimes E' \arrow[u, shift right, "v\otimes 1"']
\end{tikzcd}$$

$u\otimes 1,\ v\otimes 1,\ h\otimes 1,\ \theta\otimes 1+1\otimes\theta$

$1\otimes u,\ 1\otimes v,\ 1\otimes h,$?

$$U=(1\otimes u)(u\otimes 1) \qquad V=(v\otimes 1)(1\otimes v) \qquad H=h\otimes 1+(v\otimes 1)(1\otimes h)(u\otimes 1)$$

~~[illegible] studying composition that~~

$$\widetilde{u\otimes 1}=(u\otimes 1)\frac{1}{1-(\theta\otimes 1+1\otimes\theta)(h\otimes 1)}$$

$$=(u\otimes 1)\frac{1}{1-\theta h\otimes 1+h\otimes\theta}$$

$$=\sum_{n\geq 0}(u\otimes 1)\left(\frac{1}{1-\theta h}\otimes 1\right)\left[(-h\otimes\theta)\left(\frac{1}{1-\theta h}\otimes 1\right)\right]^n$$

So if you assume $uh=h^2=0$, you will get

$$\widetilde{u\otimes 1}=(u\otimes 1)\left(\frac{1}{1-\theta h}\otimes 1\right)=\tilde{u}\otimes 1$$

Similarly if $h^2=hv=0$

$$\widetilde{v\otimes 1}=\frac{1}{1-(h\otimes 1)(\theta\otimes 1+1\otimes\theta)}(v\otimes 1)$$

$$=\frac{1}{1-(h\theta\otimes 1)\ h\otimes\theta}(v\otimes 1)$$

$$=\sum_{n\geq 0}\left[\left(\frac{1}{1-h\theta}\otimes 1\right)(+h\otimes\theta)\right]^n\left(\frac{1}{1-h\theta}\otimes 1\right)(v\otimes 1)=\tilde{v}\otimes 1$$

$$(\theta\otimes 1+1\otimes\theta)' = \widetilde{(u\otimes 1)}(\theta\otimes 1+1\otimes\theta)(v\otimes 1)$$

$$= (\tilde{u}\otimes 1)(\theta\otimes 1+1\otimes\theta)(v\otimes 1)$$

$$= \tilde{u}\theta v\otimes 1 + \tilde{u}v\otimes\theta$$

and $\tilde{u}v = u\frac{1}{1-\theta h}v = uv$. So

if we assume the SDR conditions

$$uv=1 \qquad uh=hv=h^2=0$$

Then $(\theta\otimes 1+1\otimes\theta)' = \theta'\otimes 1 + 1\otimes\theta$

$$\widetilde{h\otimes 1} = (h\otimes 1)\frac{1}{1-(\theta\otimes 1+1\otimes\theta)(h\otimes 1)}$$

$$= (h\otimes 1)\frac{1}{1-(\theta h\otimes 1)+h\otimes\theta}$$

$$= \sum (h\otimes 1)\left(\frac{1}{1-(\theta h\otimes 1)}(h\otimes\theta)\right)^n\frac{1}{1-(\theta h\otimes 1)}$$

$$= h\otimes 1\,\frac{1}{1-(\theta h\otimes 1)} = \tilde{h}\otimes 1$$

Next stage

$$\begin{tikzcd} E'\otimes E \arrow[d, shift right, "1\otimes u"'] \\ E'\otimes E' \arrow[u, shift right, "1\otimes v"'] \end{tikzcd}$$

$1\otimes u,\ 1\otimes v,\ 1\otimes h,\ \theta'\otimes 1+1\otimes\theta$

Then

$$\widetilde{1\otimes u} = (1\otimes u)\frac{1}{1-(\theta\otimes 1+1\otimes\theta)(1\otimes h)} = (1\otimes u)\frac{1}{1-(1\otimes\theta h)-\theta'\otimes h}$$

$$\widetilde{1\otimes v} = \frac{1}{1-(1\otimes h)(\theta'\otimes 1+1\otimes\theta)}(1\otimes v)$$

$$= \frac{1}{1-(1\otimes h\theta)+\theta'\otimes h}(1\otimes v)$$

$$= \frac{1}{1-(1\otimes h\theta)}\sum\left\{(-\theta'\otimes h)\frac{1}{1-(1\otimes h\theta)}\right\}^n(1\otimes v)$$

$$= \frac{1}{1-(1\otimes h\theta)}(1\otimes v) = 1\otimes\tilde{v}$$

$$(\theta'\otimes 1+1\otimes\theta)' = (1\otimes\tilde{u})(\theta'\otimes 1+1\otimes\theta)(1\otimes v)$$

$$= \theta'\otimes\tilde{u}v + 1\otimes\tilde{u}\theta v$$

$$= \theta'\otimes 1 + 1\otimes\theta'.$$

$$\widetilde{1\otimes h} = (1\otimes h)\frac{1}{1-(\theta\otimes 1+1\otimes\theta)(1\otimes h)} = (1\otimes h)\frac{1}{1-\cancel{\theta}(1\otimes\theta h)-\theta'\otimes h}$$

$$= (1\otimes h)\sum\left(\frac{1}{1-(1\otimes\theta h)}(\theta'\otimes h)\right)^n\frac{1}{1-(1\otimes\theta h)} = (\otimes h)\frac{1}{1-(1\otimes\theta h)}$$

$$= 1\otimes\tilde{h}$$

$$\tilde{H} = \widetilde{h\otimes 1} + (\widetilde{v\otimes 1})(\widetilde{1\otimes h})(\widetilde{u\otimes 1})$$

$$= \tilde{h}\otimes 1 + (\tilde{v}\otimes 1)(1\otimes\tilde{h})(\tilde{u}\otimes 1)$$

$$= \tilde{h}\otimes 1 + (\tilde{v}\tilde{u}\otimes 1)(1\otimes\tilde{h})$$

$$= \tilde{h}\otimes 1 + \tilde{v}\tilde{u}\otimes\tilde{h}$$

'4

The really key point is that in the SDR situation, tensoring (u, v, h) with the identity and using $\theta\otimes 1 + 1\otimes(?)$ gives the tensor product of $(\tilde{u}, \tilde{v}, \tilde{h})$ with the identity and

$$\left(\theta\otimes 1 + 1\otimes(?)\right)' = \theta'\otimes 1 + 1\otimes(?)$$

At this point we basically understand Stasheff. Further topics

1) Why the restriction to a SDR situation? If we give an explicit homotopy equivalence of a E with a DGA, don't we get a A_∞ structure on E?

Can any homotopy equivalence be nicely represented by two SDR maps? And is there any link between any of this and the theory of regular neighborhoods?!? I have the eeling that much remains to be understood about embeddings — Narasimhan-Ramanan, ipe. thm.

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Perturbation Theory in Diff'l Homological Algebra I
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July 18, 1989

Milgram's version of the free loop space on BG. For BG we take the standard model which is the quotient of $\coprod_{n\geq 0} \Delta_n \times G^n$. Points are therefore represented by $(t_1, \dots, t_n, g_1, \dots, g_n)$ with $0 \leq t_1 \leq \dots \leq t_n \leq 1$. There is one vertex $(\cdot)$ and one simplices (t_1, g_1) for $0 < t < 1$, $g_1 \in G$. The identifications consist in dropping t_1, g_1 if $t_1 = 0$:

$$(0, t_2, \dots, t_n, g_1, \dots, g_n) = (t_2, \dots t_n, g_2, \dots, g_n)$$

or t_n, g_n if $t_n = 1$

$$(t_1, t_2, \dots, 1, g_1, \dots, g_n) = (t_1, \dots, t_{n-1}, g_1, \dots, g_{n-1})$$

and replacing g_i, g_{i+1} by $g_i g_{i+1}$ if $t_i = t_{i+1}$. Thus BG is the geometric realization of the simplicial space with

$$\begin{aligned} d_0(g_1, \dots, g_n) &= (g_2, \dots, g_n) && i=0 \\ d_i(g_1, \dots, g_n) &= (g_1, \dots, g_i g_{i+1}, \dots, g_n) && 1 \leq i \leq n-1 \\ d_n(g_1, \dots, g_n) &= (g_1, \dots, g_{n-1}) && i=n. \end{aligned}$$

Milgram says a model for $L(BG)$ is given by a space which is a quotient of

$$\coprod_{n \geq 0} G \times \sigma^n \times G^n$$

Thus given $(t_1,\dots,t_n, g_0,\dots,g_n)$ we want to assign a free loop in BG.

If $n=0$ we assign to (g) the loop

$$t \longmapsto (t,g) \quad \text{in } BG$$

If $n=0$ we assign to (t_1, g_0, g_1) the loop

start $(0, t_1, g_0, g_1) = (t_1, g_1)$

$(t, t+t_1, g_0, g_1)$ for $0 \le t \le 1-t_1$

end $(1-t_1, 1, g_0, g_1) = (1-t_1, g_0)$

start

$(t-(1-t_1), t, g_1, g_0)$ for $1-t_1 \le t \le 1$

end

$(t_1, 1, g_1, g_0) = (t_1, g_1)$

same

Notice that if $t_1 = 0$, then we get the loop $t \mapsto (t, t, g_0, g_1) = (t, g_0g_1)$ which is associated to the vertex g_0g_1. And if $t_1 =$ [illegible] we get the loop $t \longmapsto (t, t, g_1, g_0) = (t, g_1g_0)$, which is the loop assoc. to the vertex g_1g_0. Thus the boundary of the 1-simplex $\{(t_1, g_0, g_1) \mid 0 < t_1 < 1\}$ is gg_1 at 0 end and g_1g_0 at 1-end.

Further calculation shows that Milgram's model ought to be the geometric realization of the simplicial set with

$$d_i(g_0, \dots, g_n) = (\dots, g_i g_{i+1}, \dots) \qquad 0 \le i < n$$

$$d_n(g_0, \dots, g_n) = (g_n g_0, g_1, \dots, g_{n-1})$$

According to R. Cohen, this is Waldhausen's cyclic bar construction.

Notice that this "cyclic bar construction" like BG which is a "bar construction" makes sense when G is a monoid. Suppose G is a group and consider the map

$$G \times G^n \xrightarrow{\Phi} G^{n+1}$$

$$(x, g_1, \dots, g_n) \longmapsto (g_n^{-1} \cdots g_1^{-1} x, g_1, \dots, g_n)$$

Define faces on the left by

$$d_i(x, g_1, \dots, g_n) = \begin{cases} g_1^{-1} x g_1, g_2, \dots, g_n) & i = 0 \\ (x, g_1, \dots, g_i g_{i+1}, \dots) & 1 \le i \le n-1 \\ (x, g_1, \dots, g_{n-1}) & i = n \end{cases}$$

Then $(\Phi d_0)(x, g_1, \dots, g_n) = \Phi(g_1^{-1} x g_1, g_2, \dots, g_n)$

$$= (g_n^{-1} \cdots g_1^{-1} x g_1, g_2, \dots, g_n)$$

$$(d_0 \Phi)(x, g_1, \dots, g_n) = d_0(g_n^{-1} \cdots g_1^{-1} x, g_1, \dots, g_n)$$

$$= (g_n^{-1} \cdots g_1^{-1} x g_1, g_2, \dots, g_n)$$

$$\Phi d_n(x, g_1, \ldots, g_n) = \Phi(x, g_1, \ldots, g_{n-1})$$

$$= (g_{n-1}^{-1} \cdots g_1^{-1} x, g_1, \ldots, g_{n-1})$$

$$d_n \Phi(x, g_1, \ldots, g_n) = d_n(g_n^{-1} \cdots g_1^{-1} x, g_1, \ldots, g_{n-1})$$

$$= (g_n^{-1} \cdots g_1^{-1} x, g_1, \ldots, g_{n-1})$$

$$\Phi d_i(x, g_1, \ldots, g_n) = \Phi(x, \ldots, g_i g_{i+1}, \ldots)$$

$$= (g_n^{-1} \cdots (g_i g_{i+1})^{-1} \cdots g_1^{-1} x, g_1, \ldots, g_i g_{i+1}, \ldots)$$

$$d_i \Phi(x, g_1, \ldots, g_n) = (g_n^{-1} \cdots g_1^{-1} x, g_1, \ldots, g_i g_{i+1}, \ldots)$$

So Φ is an isomorphism of simplicial sets.

July 25, 1989

It is important to properly understand unbounded Fredholm modules and entire cyclic theory. Following Connes' example, we should start with an index problem. We consider the odd or ungraded situation.

Let H be a Hilbert space, let X_0 be a skew adjoint operator in H, and let u be a unitary operator on H. u makes H into a module over $A=\mathbb{C}[z,z^{-1}]$. We want a compactness condition implying that (A,H,X_0) is a Fredholm module, and in particular that there is an index in $\mathbb{Z}$ defined by pairing $[u]\in K_1A$ with the "K-homology" class associated to X_0. (The typical example is where $A=C^\infty(M)$, X_0 = Dirac over an odd dimensional manifold.)

The index can be understood by means of spectral flow. One joins the operators X_0 and $u^{-1}X_0u$ by a path and ~~counts~~ the net number of eigenvalues crossing zero.

To be precise suppose X_0 such that it's Cayley transform is $\equiv -1$ mod compacts, or equivalently that the resolvent $\frac{1}{\lambda-X_0}$ for $Re(\lambda)\neq 0$ is compact. Lets consider the affine space $\mathcal{A}$ of bounded perturbations $X=X_0+B$, where B is bounded skew-adjoint. The operator X is also skew-adjoint and has compact resolvent. This follows from the geometric series

$$\frac{1}{\lambda - X_0 - B} = \frac{1}{\lambda - X_0} + \frac{1}{\lambda - X_0} B \frac{1}{\lambda - X_0} + \cdots$$

which is convergent for $\| B \frac{1}{\lambda - X_0} \| < 1$. This is the case for $\|B\| < |Re(\lambda)|$, and hence one can establish successively that $X_0, X_0 + B, X_0 + 2B, \dots$ are skew-adjoint to obtain any bounded perturbations

Next in the space $\mathcal{A}$ we have a divisor consisting of the non invertible operators. We have to explain precisely ~~the~~ meaning of divisor. What appears to be the case is that we have a line bundle over $\mathcal{A}$ with a section whose zero set is the subset of noninvertible operators. To obtain ~~this~~ we locally construct determinant functions such that the transitions are invertible.

For each (a,b) with $a < 0 < b$, let $\mathcal{U}_{ab}$ be the open subset of $X \in \mathcal{A}$ such that ia and ib are not in the spectrum. On $\mathcal{U}_{ab}$ we have the function taking X to the product of the eigenvalues $i\lambda$ of X with $a < \lambda < b$ with i's removed. Thus if the eigenvalues are $i\lambda_j$ $j = 1, \dots, n$ we take $\prod^n \lambda_j$. This gives a smooth function on $\mathcal{U}_{ab}$ vanishing iff X is noninvertible. Given another (a', b') one sees easily that the two determinant functions $det_{ab}(X)$ and $det_{a'b'}(X)$ on $\mathcal{U}_{ab} \cap \mathcal{U}_{a'b'}$ are related by invertible functions