Here seems to be the good construction of Connes cocycles. We consider $C^\cdot(A, A*A)$ where $A*A$ is regarded as a superalgebra. Let $\theta \in C^1(A, A*A)$ be the canonical homomorphism in,. If we identify $A*A$ with Ω_A equipped with the $*$ product, then

$$\theta(a) = \underbrace{a}_{\theta^+(a)} + \underbrace{da}_{\theta^-(a)}$$

Let τ be a supertrace on $A*A$. Then we ought to be able to combine it with N to obtain a supertrace $\tilde{\tau}$ on $C^\cdot(A, A*A)$ with values in $C_\lambda^{\cdot-1}(A)$ such that $\delta\tilde{\tau} = +\tilde{\tau}\delta$.

If this is true, then the Connes cocycles are given by $\tilde{\tau}((\theta^-)^n)$. In effect we have that the "connection" form θ is flat, so we have

$$[\delta + \theta^+, \theta^-] = 0$$

hence

$$\delta\, \tilde{\tau}((\theta^-)^n) = \tilde{\tau}[\delta + \theta^+, (\theta^-)^n] = 0$$

Notice that because $\tilde{\tau}$ is a supertrace we don't have $\tilde{\tau}((\theta^-)^{2n}) = 0$ necessarily, because θ^- is even.

The signs here are really confused. It seems ridiculous to make the product in $C^\cdot(A, A*A)$ depend on the $\mathbb{Z}/2$ grading of $A*A$. ??

June 28, 1988

Let's return to extensions and to the problem of the exact sequence

$$0 \longrightarrow \bar{H}C_{2n-1}(A) \longrightarrow I^n/[I,I^{n-1}] \longrightarrow I^{n-1}\otimes_B \Omega^1_B \otimes_B$$

The goal will be to give a direct proof of this spectral sequence where $B = T(A)/(1-\rho(1_A)) = eCe$. First we should recall how a trace on I^n vanishing on $[I,I^{n-1}]$ gives a cyclic cocycle. We consider the canonical map $\rho(a) = eae$ from A to B as a "connection" form $\rho \in C^1(A,B)$. The "curvature" is $\underbrace{(\delta\rho+\rho^2)}_{K}(a_1,a_2) = \rho(a_1)\rho(a_2) - \rho(a_1a_2) \in C^2(A,I)$. Then we have

$$\delta(K^n) + [\rho, K^n] = 0$$

in $C^{2n}(A,I^n)$. This implies that the image of $K^n \in C^{2n}(A,I^n)$ in $C^{2n-1}_\lambda(A, I^n/[B,I^n])$ is a cocycle.

Let's recall that $I^n/[B,I^n] \cong (I\otimes_B)^n$ has a natural action of $\mathbb{Z}/n\mathbb{Z}$ with quotient $I^n/[I,I^{n-1}]$. It seems that $I^n/[I,I^{n-1}] = (I\otimes_B)^n_\sigma$ has a natural action of $\mathbb{Z}/2$, so perhaps $(I\otimes_B)^n$ has a natural action of $\mathbb{Z}/2n$.

Let's begin by trying to prove

$$I^n/[I,I^{n-1}] \xrightarrow{\sim} \tilde{K}^{2n}/[\tilde{K},\tilde{K}^{2n-1}]$$

where $\tilde{K} = Ce\bar{e}C + C\bar{e}CeC$ in C. Once this is proved then we will get a $\mathbb{Z}/2$-action on the LHS from the $\mathbb{Z}/2$ action ε on C.

We propose to describe C via Ω_A with the $*$ product. Recall that we have $A*A \simeq \Omega_A$ with $*$ product and where $in_1 : A \to A*A$ and $\xi \mapsto F\xi F$ can be identified with

$$a \longmapsto a + da$$

$$\omega \longmapsto (-1)^{\deg \omega}\, \omega$$

respectively. We obtain C from $A*A$ by adjoining the element F.

Here's a model for C. Consider the subalgebra of $M_2(\Omega, *)$ consisting of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that $\alpha, \delta \in \Omega^{ev}$ and $\beta, \gamma \in \Omega^{odd}$. Let $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then C can be identified with this subalgebra such that the grading ε on C is given by conjugating with ε.

The nice thing about this model is that it fits nicely with the block description of $C, \tilde{K}$ that we used before. So $\tilde{K}^{m}$ can be described as consisting of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where the forms belong to $\Omega^{\geq m}$.

We now would like to calculate the quotient $\tilde{K}^{2n} / [C, \tilde{K}^{2n}]$. Because C is generated by the image of A and F we have

$$[C, \tilde{K}^{2n}] = [A, \tilde{K}^{2n}] + [F, \tilde{K}^{2n}]$$

Now $$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right] = 2 \begin{pmatrix} 0 & \beta \\ -\gamma & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right] = \begin{pmatrix} a*\alpha - \alpha * a & \cdot\cdot \\ \cdot\cdot & a*\delta - \delta * a \end{pmatrix}$$

Hence it follows that

$$\tilde{K}^{2n}/[C, \tilde{K}^{2n}] = \begin{pmatrix} I^n/[B, I^n] & 0 \\ 0 & \bar{I}^n/[\bar{B}, \bar{I}^n] \end{pmatrix}$$

where we have used that $[A, I^n] = [B, I^n]$ since A generates B. From this we can see an action of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\tilde{K}^{2n}/[C, \tilde{K}^{2n}]$.

Next we want to divide out further by $[\tilde{K}, \tilde{K}^{2n-1}]$. The diagonal blocks for $\tilde{K}^{2n-1}$ and for $\tilde{K}^{2n}$ are the same. Thus we need only consider commutators

$$\left[\begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix}, \begin{pmatrix} 0 & \gamma' \\ \beta' & 0 \end{pmatrix} \right]$$

where $\beta, \gamma \in \Omega^{\mathrm{odd}, \geq 1}$ and $\beta', \gamma' \in \Omega^{\mathrm{odd}, \geq 2n-1}$.

Enough to look at

$$\left[\begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix} \right] = \begin{pmatrix} \omega * \eta & 0 \\ 0 & -\eta * \omega \end{pmatrix}$$

where ω, η are odd of degrees ≥ 1 and $\geq 2n-1$, or $\geq 2n-1$ and ≥ 1, respectively.

Here's how to define an action of $\mathbb{Z}/2n$ on $I^n/[B, I^n]$. Recall that

$$I = eC\bar{e}Ce = eC\bar{e} \otimes_{\bar{B}} \bar{e}Ce$$

where $\bar{B} = \bar{e}C\bar{e}$. Thus

$$I^n/[B, I^n] \overset{B \text{ free}}{=} (I \otimes_B)^n = (eC\bar{e} \otimes_{\bar{B}} \bar{e}Ce \otimes_B)^n$$

Now we have the autom. ε of C, which fixes A and changes F to $-F$; let's denote this $z \mapsto \bar{z}$.

Then we have the automorphism of $I^n/[B,I^n]$ given by

$$(\alpha_1 \otimes \beta_1 \otimes \cdots \otimes \beta_n \otimes) \in \left(eC\bar{e} \otimes_{\bar{B}} \bar{e}Ce \otimes_B\right)^n$$

$$\downarrow$$

$$\odot \qquad (\bar{\alpha}_1 \otimes \bar{\beta}_1 \otimes \cdots \otimes \bar{\beta}_n \otimes) \in \left(\bar{e}Ce \otimes_B eC\bar{e} \otimes_{\bar{B}}\right)^n$$

$$\downarrow$$

$$(\bar{\beta}_1 \otimes \bar{\alpha}_2 \otimes \cdots \otimes \bar{\alpha}_1 \otimes) \in \left(eC\bar{e} \otimes_{\bar{B}} \bar{e}C\bar{e} \otimes_B\right)^n$$

The square of this is the backward cyclic permutation of order n on $I^n/[B,I^n] = (I\otimes_B)^n$. Consequently the automorphism $\odot$ is of order $2n$.

Let τ be a linear functional on $I^n/[I,I^{n-1}]$, i.e. τ is a linear functional on $\Omega^{ev, \geq 2n}$ such that $\tau(\omega_1 * \omega_2) = \tau(\omega_2 * \omega_1)$ if $\deg(\omega_1) + \deg(\omega_2) \geq 2n$

I claim there is another linear functional $\bar{\tau}$ on $I^n/[I,I^{n-1}]$ uniquely characterized by

$$\bar{\tau}(\eta_1 * \eta_2) = \tau(\eta_2 * \eta_1)$$

for $\eta_1, \eta_2 \in \Omega^{odd}$ with $\deg(\eta_1) + \deg(\eta_2) \geq 2n$.

In effect let's define

$$\bar{\tau}(a_0 da_1 \cdots da_{2k}) = \tau(da_{2k}\, a_0\, da_1 \cdots da_{2k-1})$$

for $2k \geq 2n$. This formulas defines $\bar{\tau}$ on $\Omega^{2k} = A \otimes \bar{A}^{\otimes 2k}$, and one has

$$\bar{\tau}(\eta\, da) = \tau(da\, \eta) \qquad \begin{array}{l} a \in A \\ \eta \in \Omega^{odd, \geq 2n-1} \end{array}$$

Next suppose $\eta \in \Omega^{2k-1}$, and let $2k-1 + 2l-1 \geq 2n$.

Then

$$\bar\tau(\eta * a_0 da_1 \cdots da_{2\ell-1}) = \tau(da_{2\ell-1}(\eta * a_0 da_1 \cdots da_{2\ell-2}))$$

$$= \tau((da_{2\ell-1}\eta) * a_0 da_1 \cdots da_{2\ell-2})$$

$$= \tau(a_0 da_1 \cdots da_{2\ell-2} * da_{2\ell-1}\eta)$$

$$= \tau(a_0 da_1 \cdots da_{2\ell-1} * \eta)$$

showing $\bar\tau(\eta_1 * \eta_2) = \tau(\eta_2 * \eta_1)$ when η_1, η_2 are odd with $\deg(\eta_1) + \deg(\eta_2) \geq 2n$.

Next let ω_1, ω_2 be even forms whose sum of degrees $\deg(\omega_1) + \deg(\omega_2) \geq 2n$. I am supposing $n \geq 1$. To verify $\bar\tau(\omega_1 * \omega_2) = \bar\tau(\omega_2 * \omega_1)$ I can suppose $\deg \omega_2 > 0$, and then that $\omega_2 = a_0 da_1 \cdots da_{2k}$.

Then
$$\bar\tau(\omega_1 * a_0 da_1 \cdots da_{2k}) = \bar\tau((\omega_1 * a_0 da_1 \cdots da_{2k-1}) * da_{2k})$$

$$= \tau(da_{2k} * \omega_1 * a_0 da_1 \cdots da_{2k-1})$$

$$= \bar\tau(a_0 da_1 \cdots da_{2k-1} * da_{2k} * \omega_1) = \bar\tau(a_0 da_1 \cdots da_{2k} * \omega_1)$$

and so it works.

<u>Prop.</u> $I^n/[I, I^{n-1}] \xrightarrow{\sim} \tilde K^{2n}/[\tilde K, \tilde K^{2n-1}]$

<u>Proof:</u> Let τ be a linear functional on $I^n/[I, I^{n-1}]$. We will show it extends to $\tilde K^{2n}/[\tilde K, \tilde K^{2n-1}]$. Let $\bar\tau$ be as above and define $\tilde\tau$ on $\tilde K^{2n}$ by

$$\tilde\tau\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = \tau(\alpha) + \bar\tau(\delta)$$

Here $\alpha, \delta \in \Omega^{\text{even}, \geq 2n}$ and $\beta, \gamma \in \Omega^{\text{odd} \geq 2n+1}$. Suppose

$$\begin{pmatrix}\alpha_1 & \beta_1 \\ \gamma_1 & \delta_1\end{pmatrix} \in \tilde K \qquad \begin{pmatrix}\alpha_2 & \beta_2 \\ \gamma_2 & \delta_2\end{pmatrix} \in \tilde K^{2n-1}$$

that is $\alpha_1, \delta_1 \in \Omega^{\text{even}, \geq 2}$ and $\alpha_2, \delta_2 \in \Omega^{\text{even} \geq 2n-2}$
while $\beta_1, \gamma_1 \in \Omega^{\text{odd}, \geq 1}$ and $\beta_2, \gamma_2 \in \Omega^{\text{odd} \geq 2n-1}$.
Then

$$\tilde{\tau}\begin{pmatrix} \alpha_1\alpha_2 + \beta_1\gamma_2 & \alpha_1\beta_2 + \beta_1\delta_2 \\ \gamma_1\alpha_2 + \delta_1\gamma_2 & \gamma_1\beta_2 + \delta_1\delta_2 \end{pmatrix} = \tau(\alpha_1\alpha_2 + \beta_1\gamma_2) + \bar{\tau}(\gamma_1\beta_2 + \delta_1\delta_2)$$

$$\tilde{\tau}\begin{pmatrix} \alpha_2\alpha_1 + \beta_2\gamma_1 & \alpha_2\beta_1 + \beta_2\delta_1 \\ \gamma_2\alpha_1 + \delta_2\gamma_1 & \gamma_2\beta_1 + \delta_2\delta_1 \end{pmatrix} = \tau(\alpha_2\alpha_1 + \beta_2\gamma_1) + \bar{\tau}(\gamma_2\beta_1 + \delta_2\delta_1).$$

But we have seen that $\bar{\tau}(\delta_1\delta_2) = \bar{\tau}(\delta_2\delta_1)$
$\tau(\beta_1\gamma_2) = \bar{\tau}(\gamma_2\beta_1)$, $\bar{\tau}(\gamma_1\beta_2) = \tau(\beta_2\gamma_1)$ and so these two $\tilde{\tau}$ values coincide. Thus we see that $\tilde{\tau}$ is defined on $\tilde{K}^{2n}/[\tilde{K}, \tilde{K}^{2n-1}]$. The uniqueness of $\tilde{\tau}$ is clear, which proves the proposition.

Next let's return to the cyclic $(2n-1)$-cocycle associated to $\tau \in (I^n/[I, I^{n-1}])^*$. Actually we saw that we have a $2n$-cochain in $C^{2n}(A, I^n)$, namely

$$\varphi(a_1, \dots, a_{2n}) = da_1 \cdots\cdots da_{2n}$$

which satisfies

$$\begin{aligned}(\delta\varphi)(a_1, \dots, a_{2n+1}) = & -d(a_1a_2)\,da_3 \cdots da_{2n+1} \\ & + da_1\, d(a_2a_3)\, da_4 \cdots \\ & - \cdots \\ & + (-1)^{2n} da_1 \cdots d(a_{2n}a_{2n+1}) \\ = & -a_1\, da_2 \cdots da_{2n+1} + da_1 \cdots da_{2n}\, a_{2n+1}\end{aligned}$$

June 30, 1988

Consider a Dirac operator $\not{D}_0$ over an odd dimensional manifold and let g be a gauge transformation of the coefficient bundle. Then there is an integer defined - it is the pairing of the odd K-homology class represented by $\not{D}_0$ with the odd K-cohomology class represented by g. The problem is to find a simple analytical expression for this "index".

It seems that APS explains this index in terms of "spectral flow" as follows. Using the linear path $(1-t)\not{D}_0 + t\, g^{-1}\not{D}_0 g$, $0 \le t \le 1$, together with the gauge transformation at the ends as a clutching function, we obtain a family of Dirac operators [redacted] on our odd manifold which is parametrized by S^1. To this family is associated a spectral flow, which roughly is the net number of eigenvalues crossing 0, as t goes from 0 to 1. The spectral flow is also [redacted] the index of the family. This index is an odd K-cohomology class over the circle S^1, hence it can be identified with an integer.

The superconnection character form for the index of the family is a 1-form on S^1 whose integral [redacted] gives the index. This gives one analytical expression for this index.

Another [redacted] procedure is to compare the two operators $\not{D}_0$ and $g^{-1}\not{D}_0 g$ via the superconnection family

$$\begin{pmatrix} \not{D}_0 & 0 \\ 0 & g^{-1}\not{D}_0 g \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

[redacted]

This should give [redacted]

a 1-form on $[0,\infty)$ whose integral is the index.

It might be possible to get a geometric picture for the index using the Cayley transform interpretation of superconnection forms. Let's consider the setup abstractly. We have the skew adjoint operator X on H and the automorphism g. We use the path

$$(1-t)X + t\,g^{-1}Xg \quad = \quad X + t\,g^{-1}[X,g]$$

July 1, 1988

Problem: Let us consider a Dirac operator $\not{D}_0$ over an odd dimensional compact manifold and a gauge transformation g on the coefficient bundle. Then there is an index defined - it is the pairing of the K-homology class represented by the Dirac operator with the K-cohomology class represented by the gauge transformation. The problem is to find nice analytical expressions for this index.

A nice analytical expression of heat operator type should yield by small-time (or Planck's constant) asymptotics an expression for the index as an integral over the manifold M of a characteristic differential form. One expects the Todd or $\hat{A}$ class of M, multiplied by an odd degree character class associated to the given connection on the coeff. bundle and the gauge transf. g.

To be more specific suppose M spin, S be the module of spinors, and let E be coefficient bundle. The Dirac operator $\not{D}_0$ operates on $L^2(M, S\otimes E)$ and is obtained from the Clifford multiplication and a connection on $S\otimes E$:

$$C^\infty(M, S\otimes E) \xrightarrow[\text{connection}]{} C^\infty(M, T^*\otimes S\otimes E) \xrightarrow[\text{Cliff mult}]{} C^\infty(M, S\otimes E)$$

The connection on $S\otimes E$ is the tensor product of the connection on S obtained from the Levi-Civita connection and a given connection ∇ on E.

To the autom. g of E belongs a sequence of odd cohomology classes; these are the character classes of the odd K-class represented by g. It is possible to represent these character classes by differential forms using the connection ∇. However there are several possible ways to do this. We will now review the different methods, since each one might lead ~~to~~ a different analytical expression for the index

The main thing we have to do is to compare the two connections ∇ and $g^{-1}\cdot\nabla\cdot g$. The obvious method is to use the linear path $\nabla_t = (1-t)\nabla + t\,g^{-1}\cdot\nabla\cdot g = \nabla + t\,g^{-1}[\nabla,g]$. The curvature of ∇_t is

$$\nabla_t^2 = \nabla^2 + t\,\underbrace{[\nabla, g^{-1}[\nabla,g]]}_{[\nabla,g^{-1}][\nabla,g] + g^{-1}[\nabla^2,g]} + t^2\left(g^{-1}[\nabla,g]\right)^2$$

$$\nabla_t^2 = \nabla^2 + t g^{-1}[\nabla^2,g] + (t^2-t)\left(g^{-1}[\nabla,g]\right)^2$$

In general one has for a path of connections ∇_t the formula

$$\mathrm{tr}\left(\nabla_1^2\right)^n - \mathrm{tr}\left(\nabla_0^2\right)^n = d\left\{\int_0^1 dt\; n\,\mathrm{tr}\left(\dot{\nabla}_t\left(\nabla_t^2\right)^{n-1}\right)\right\}$$

In the present example the LHS is zero so that the odd forms for $n \geq 1$

$$\boxed{\int_0^1 dt\;\; n\,\mathrm{tr}\left\{g^{-1}[\nabla,g]\left((1-t)\nabla^2 + t g^{-1}\nabla^2 g + (t^2-t)\left(g^{-1}[\nabla,g]\right)^2\right)^{n-1}\right\}}$$

are closed.

The other way to compare two connections is to consider the superconnection family

$$\hat{\nabla}_t = \begin{pmatrix} \nabla_0 & 0 \\ 0 & \nabla_1 \end{pmatrix} + t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The curvature is

$$\tilde{\nabla}_t^2 = \begin{pmatrix} \nabla_0^2 & 0 \\ 0 & \nabla_1^2 \end{pmatrix} + t \begin{pmatrix} 0 & \nabla_1 - \nabla_0 \\ \nabla_1 - \nabla_0 & 0 \end{pmatrix} - t^2$$

Upon integrating the formula

$$\partial_t \, tr_\varepsilon\left(e^{\tilde{\nabla}_t^2}\right) = d \, tr\left(\varepsilon \dot{\tilde{\nabla}}_t e^{\tilde{\nabla}_t^2}\right)$$

from 0 to ∞ we get

$$tr\left(e^{\nabla_0^2} - e^{\nabla_1^2}\right) = d\int_0^\infty dt \left\{ tr \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix} e^{\begin{pmatrix} \nabla_0^2 & 0 \\ 0 & \nabla_1^2 \end{pmatrix} + t\begin{pmatrix} 0 & \nabla_0 - \nabla_1 \\ \nabla_1 - \nabla_0 & 0 \end{pmatrix} - t^2} \right\}$$

If we conjugate:

$$\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} \nabla & 0 \\ 0 & g^{-1}\nabla g \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$$

we get

$$\tilde{\nabla}_t = \begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix} + t\begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$$

and

$$\tilde{\nabla}_t^2 = \begin{pmatrix} \nabla^2 & 0 \\ 0 & \nabla^2 \end{pmatrix} + t\begin{pmatrix} 0 & -[\nabla, g^{-1}] \\ [\nabla, g] & 0 \end{pmatrix} - t^2$$

So ~~[illegible]~~ we obtain the closed odd form

$$\boxed{-\int_0^\infty dt \; tr\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \exp\left(-t^2 + \nabla^2 + t\begin{pmatrix} 0 & -[\nabla, g^{-1}] \\ [\nabla, g] & 0 \end{pmatrix}\right)}$$

The third idea is to use Narasimhan–Ramanan to handle the connection. Thus we embed $E \underset{i^*}{\overset{i}{\rightleftarrows}} \tilde{V}$ so that $\nabla = i^* d\, i$ and we extend g on E by 1 on $E^\perp$ to obtain $\tilde{g} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ on $\tilde{V}$.

Thus we reduce to the case where $(E,\nabla) = (\tilde{V}, d)$.

Let's calculate the odd forms where ∇ is flat. The form of degree $(2n-1)$ obtained from $\mathrm{tr}(\nabla_t)^{2n}$ for the path $\nabla_t = \nabla + t\, g^{-1}[\nabla, g]$ is

$$\mathrm{tr}\left(g^{-1}[\nabla,g]\right)^{2n-1} \underbrace{\int_0^1 dt\; n(t^2-t)^{n-1}}_{n!\,(-1)^{n-1}\frac{(n-1)!}{(2n-1)!}}$$

In the superconnection case we want

$$-\int_0^\infty dt\; \mathrm{tr}\begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix} e^{-t^2}\frac{t^{2n-1}}{(2n-1)!}\begin{pmatrix} 0 & -[\nabla,g^{-1}] \\ [\nabla,g] & 0 \end{pmatrix}^{2n-1}$$

$$= +\,\mathrm{tr}\begin{pmatrix} g^{-1}[\nabla,g][\nabla,g^{-1}]\cdots[\nabla,g] & 0 \\ 0 & g[\nabla,g^{-1}][\nabla,g]\cdots[\nabla,g^{-1}] \end{pmatrix}(-1)^{n-1}$$

$$\times \int_0^\infty e^{-t^2}\,t^{2n-1}\,dt\;\frac{1}{(2n-1)!}$$

$$= (-1)^{n-1}\,\cancel{2}\;\mathrm{tr}\left(g^{-1}[\nabla,g]\right)^{2n-1}\frac{1}{(2n-1)!}\underbrace{\int_0^\infty e^{-u}u^n\frac{du}{\cancel{2}u}}_{\Gamma(n)}$$

$$= (-1)^{n-1}\frac{(n-1)!}{(2n-1)!}\,\mathrm{tr}\left(g^{-1}[\nabla,g]\right)^{2n-1}$$

Let's now look at an embedding $E \underset{i^*}{\overset{i}{\rightleftarrows}} \tilde{V}$ with $\nabla = i^* d i$. Then

$$d = \begin{pmatrix} i^*di & i^*dj \\ j^*di & j^*dj \end{pmatrix} \qquad \tilde{g} = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

$$[d, \tilde{g}] = \begin{pmatrix} [\nabla, g] & -(g-1)(\iota^* dj) \\ (j^* di)(g-1) & 0 \end{pmatrix}$$

$$\tilde{g}^{-1}[d, \tilde{g}] = \begin{pmatrix} g^{-1}[\nabla, g] & (g^{-1}-1)(-\iota^* dj) \\ j^* di\,(g-1) & 0 \end{pmatrix}$$

July 4, 1988

Motivation: Suppose given E, ∇, g over the manifold M, we have seen that one way to attach an odd character form to this data is to use the superconnection

$$dt\,\partial_t + \begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix} + t\begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$$

over $R \times M$ and to integrate the even character form from $t=0$ to $t=\infty$. I propose to find a Dirac operator analogue when M is odd dimensional. The idea will be to produce a family of Dirac operators on M corresponding to the family of superconnections

$$\circledast \qquad \begin{pmatrix} \nabla & 0 \\ 0 & \nabla \end{pmatrix} + t\begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$$

This will then give us an odd form on R which can be integrated hopefully from $t=0$ to ∞ to obtain the index.

The obvious way to proceed is to take the Dirac operator with "potential" term associated to the superconnection $\circledast$. Thus we need to recall how to mix Dirac operators and superconnections:

<u>Even-Even</u>: S comes with γ^μ, ε_S anti-comm.
E ——— ε_E, X "

Then on $S \otimes E$ we have $\gamma^\mu \otimes 1$, $\varepsilon_S \otimes X$ which anti-commute, and which anti-commute with the total grading $\varepsilon_S \otimes \varepsilon_E$

<u>Odd-Even</u>: S comes with γ^μ, E comes with ε_E, X. On $S \otimes E$ we have $\gamma^\mu \otimes \varepsilon_E$, $1 \otimes X$ anti-commuting.

Even-Odd: S comes with $\gamma^\mu, \varepsilon_S$, E comes with X. On $S \otimes E$ we have $\gamma^\mu \otimes 1$, $\varepsilon_S \otimes X$ anti-commuting

Odd-odd: S comes with γ^μ, E with X
On $(S\otimes E)^{\oplus 2}$ we have

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \gamma^\mu \otimes 1 \\ \gamma^\mu \otimes 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix} \quad \text{anticommuting}$$

We are concerned with the odd-even case which means that on $S \otimes E$ we have

$$(\gamma^\mu \otimes \varepsilon)(\nabla^S_\mu \otimes 1 + 1 \otimes \nabla^E_\mu) + 1 \otimes X .$$

The first terms is the direct sum of the Dirac on $S\otimes E^+$ with respect to ∇^{E^+} and minus the Dirac on $S \otimes E^-$ with respect to ∇^{E^-}.

In the case of interest where we start with a Dirac $\not{D}$ on $S \otimes E^0$ and a gauge transformation g, we obtain the family of skew-adjoint operators

$$X_t = \begin{pmatrix} \not{D} & 0 \\ 0 & -\not{D} \end{pmatrix} + t \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$$

on $\Gamma(S\otimes E)^{\oplus 2}$. We have

$$X_t^2 = \begin{pmatrix} \not{D}^2 & 0 \\ 0 & \not{D}^2 \end{pmatrix} - t^2 + t \begin{pmatrix} 0 & -[\not{D}, g^{-1}] \\ -[\not{D}, g] & 0 \end{pmatrix}.$$

We want to consider the super connection $dt\,\partial_t + X_t \sigma$ over the line $\mathbb{R}$ and integrate the associated character form over $[0,\infty)$. As

$$(dt\,\partial_t + X_t\sigma)^2 = X_t^2 + dt\,\dot{X}_t\,\sigma$$

we obtain the number

$$u \int_0^\infty tr_s(\dot{X}_t \sigma\, e^{uX_t^2})\,dt = \underset{\frac{1}{(2i)^{1/2}}}{\bar{\sigma}}\, u \int_0^\infty tr(\dot{X}_t e^{uX_t^2})\,dt$$

Let's change signs slightly

$$X_t = \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix} + t\begin{pmatrix} 0 & g^{-1} \\ -g & 0 \end{pmatrix}$$

$$X_t^2 = \begin{pmatrix} \phi^2 & 0 \\ 0 & \phi^2 \end{pmatrix} + t\begin{pmatrix} 0 & [\phi, g^{-1}] \\ [\phi, g] & 0 \end{pmatrix} - t^2$$

The candidate for the index is

$$\int_0^\infty tr_s\left(e^{u\underbrace{(+dt\,\partial_t + X_t\sigma)^2}_{dt\,(\partial_t X_t)\sigma + X_t^2}}\right)$$

$$= u\int_0^\infty dt\, tr_s\left((\partial_t X_t)\sigma\, e^{uX_t^2}\right)$$

$$= u\bar\sigma \int_0^\infty dt\; tr\left\{\begin{pmatrix} 0 & g^{-1} \\ -g & 0 \end{pmatrix} e^{u\left(\phi^2 - t^2 + t\begin{pmatrix} 0 & [\phi, g^{-1}] \\ [\phi, g] & 0 \end{pmatrix}\right)}\right\}$$

We now expand the exponential obtaining

$$\sum_{n\geq 0}\left(u\bar\sigma\int_0^\infty dt\, e^{-ut^2}(ut)^{2n+1}\right)\int_{t_0+\cdots+t_{2n+1}=1} tr\begin{pmatrix} 0 & g^{-1} \\ -g & 0 \end{pmatrix} e^{ut_0\phi^2}\begin{pmatrix} 0 & [\phi, g^{-1}] \\ [\phi, g] & 0 \end{pmatrix}$$

$$\times\, e^{ut_1\phi^2}\cdots\cdots\begin{pmatrix} 0 & [\phi, g^{-1}] \\ [\phi, g] & 0 \end{pmatrix} e^{ut_{2n+1}\phi^2}$$

Now $$u\bar\sigma\int_0^\infty dt\, e^{-ut^2}(ut)^{2n+1} = \bar\sigma\int_0^\infty \frac{dt}{t}\, e^{-ut^2}(ut)^{2n+2}$$

$$= \bar\sigma\int_0^\infty \frac{dt}{2t}\, e^{-ut}(u)^{2n+2}t^{n+1} = \frac{\bar\sigma}{2}\,\frac{\Gamma(n+1)}{u^{n+1}}\,u^{2n+2}$$

$$= \frac{\bar\sigma}{2}\, n!\, u^{n+1}$$

So our formula for the index seems to be

$$\sum_{n\geq 0} \frac{\sigma}{2} n!\, u^{n+1} u^{-2n} I_n \qquad \text{where}$$

$$I_n = u^{2n}\int_{t_0+\cdots+t_{2n+1}=1} tr\left(g^{-1} e^{ut_0\phi^2}[\phi,g]\cdots[\phi,g^{-1}]e^{ut_{2n}\phi^2}[\phi,g]e^{ut_{2n+1}\phi^2}\right)$$

$$-u^{2n}\int_{t_0+\cdots+t_{2n+1}=1} tr\left(g\, e^{ut_0\phi^2}[\phi,g^{-1}]\cdots\cdots[\phi,g^{-1}]e^{ut_{2n+1}\phi^2}\right)$$

One can simplify a bit by using the L.T.

$$\int_0^\infty e^{-\lambda u} I_n\, du = tr\left\{g^{-1}\frac{1}{\lambda-\phi^2}[\phi,g]\frac{1}{\lambda-\phi^2}\left([\phi,g^{-1}]\frac{1}{\lambda-\phi^2}[\phi,g]\frac{1}{\lambda-\phi^2}\right)^n\right\}$$

$$-tr\left\{g\frac{1}{\lambda-\phi^2}[\phi,g^{-1}]\frac{1}{\lambda-\phi^2}\left([\phi,g]\frac{1}{\lambda-\phi^2}[\phi,g^{-1}]\frac{1}{\lambda-\phi^2}\right)^n\right\}$$

Note

$$\prod_{j=1}^n \int_0^\infty e^{-\lambda x_j} f_j(x_j)\,dx_j = \int_0^\infty\!\!\cdots\int_0^\infty e^{-\lambda(\overbrace{x_1+\underbrace{\cdots+x_n}_{y_2}}^{y_1})} f_1(x_1)\cdots f_n(x_n)\,dx_1\cdots dx_n$$

set $y_k = x_k+\cdots+x_n$

$$= \int_0^\infty dy_1\int_0^{y_1}dy_2\cdots\int_0^{y_{n-1}}dy_n\, e^{-\lambda y_1} f_1(y_1-y_2)\cdots f_{n-1}(y_{n-1}-y_n)f_n(y_n)$$

set $y_1 = u$ and $y_k = ut_k$

$$= \int_0^\infty du\, e^{-\lambda u} u^{n-1}\int_0^1 dt_2\int_0^{t_2}dt_3\cdots\int_0^{t_{n-1}}dt_n\; f_1(u(1-t_2))\cdots f_n(u(t_n))$$

$$= \int_0^\infty du\, e^{-\lambda u}\left(u^{n-1}\int_{t_1+\cdots+t_n=1} f_1(ut_1)\cdots f_n(ut_n)\right)$$

$$\parallel$$

$$(f_1 * \cdots * f_n)(u)$$

This seems to be too messy to deal with

Return to Kasparov theory. I want to examine the index in the odd case thoroughly, from as many angles as possible. The abstract situation is the following. One has a Hilbert space H and an involution η modulo compacts. Given a unitary g on H preserving η, there is an index defined. In fact the index is a homomorphism

$$U_{res}(H,\eta) \longrightarrow \mathbb{Z}.$$

A natural question is what is the graded analogue? One might start with a graded $H = H^+ \oplus H^-$ and an odd involution η modulo compacts. But then it is not so obvious what the analogue of g should be. Possibly one should look at even projectors commuting with η modulo compacts. To simplify suppose η has index zero, whence it can be represented by an odd involution F, i.e. $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ relative to an isom. $H^+ = H^-$. Then we are looking at the space of pairs of projectors (e, e') on H^+ which are congruent modulo compacts. We want to consider only those pairs such that the image and kernel (of e', hence also e) are infinite-dimensional. This spaces fibres over the ~~[illegible]~~ contractible space of all e' and the fibre is the restricted Grassmannian, so again we have a space of the homotopy type $\mathbb{Z} \times BU$.

It might be better to consider a more concrete situation where one is given a Dirac operator on a manifold M. This represents an element of $K(C(M), \mathbb{C})$ and the index map is the pairing with various elements of $K(\mathbb{C}, C(M))$. ~~[illegible]~~ Traditionally elements of $K(\mathbb{C}, C(M))$, that is, K-cohomology of M, are represented by maps from M to the classifying

spaces $\mathcal{U}$ or $\mathbb{Z}\times BU$. [redacted] The
usual models for these spaces are unitary matrices $\equiv 1$ modulo compacts and the restricted Grass of involutions congruent to a fixed one modulo compacts. However in the AS paper one has other Fredholm operator models which are apparently better [redacted] for the purpose of the Kasparov cup product.

Look at the ungraded case. The K-homology class is represented by a Hilbert space representation H of $C(M)$ with an F modulo compacts commuting with elements of $C(M)$ modulo compacts.

July 5, 1988

[redacted] Let's fix an F on $L^2(M,S) = H$, whence we have an element α of $KK(C(M), \mathbb{C})$, and let us consider pairing with α:

$$KK(\mathbb{C}, C(M)) \longrightarrow KK(\mathbb{C}, \mathbb{C})$$

[redacted] To fix the ideas consider the ungraded case where M is odd dimensional. Then we [redacted] can represent any element of $KK^1(\mathbb{C}, C(M))$ by a map $g : M \longrightarrow U(V)$ with V finite dimensional, or with V infinite-dimensional but with $U(V)$ replaced by unitaries $\equiv 1 \bmod \mathcal{K}$. In either of these cases we form $L^2(M, S\otimes\tilde{V}) = H \otimes V$ and compare $\tilde{F} = F \otimes 1$ with [redacted] g on $L^2(M, S\otimes\tilde{V})$. It seems that g preserves $\tilde{F}$ modulo compacts, so the index is defined.

The other way to represent an element of $KK^1(\mathbb{C}, C(M))$ is by a map $M \xrightarrow{A} \mathcal{F}_1 =$ self adj contractions ess. spectrum $\{-1, +1\}$ on V infinite-dimensional. Then we would have to construct the Kasparov cup product.

In the graded case we have to consider representations of elements in $KK^0(\mathbb{C}, C(M))$. The simplest representation is by maps $M \longrightarrow Gr(V)$, V finite-dimensional. Here one reduces $F \otimes 1$ on $L^2(M,S) \otimes V$ by the idempotent. The next kind of representation would be by maps from M to a restricted Grassmannian. Now before one can talk about the index map to $\mathbb{Z}$ from the restricted Grass one needs to have fixed at least the zero component. It seems therefore reasonable to require a basepoint ε to be given in the restricted Grass before one considers it to be a classifying space for K^0.

Then we have $V = V^+ \oplus V^-$ a graded Hilbert space, and [redacted] we have a map $x \mapsto \Phi_x$ from M to involutions congruent to ε_V modulo compacts. We form

$$L^2(M, S \otimes V) = L^2(M,S) \otimes V$$

and we would like to couple $F \otimes 1$ with the family $\{\Phi_x\}$ so as to obtain either a Fredholm operator or a point [redacted] in a restricted [redacted] Grassmannian.

On $\tilde{H} = L^2(M,S) \otimes V = H \otimes V$ we have $\tilde{F} = F \otimes 1$ and $\varepsilon = \varepsilon_H \otimes \varepsilon_V$ and the [redacted] involution Φ.

We want to combine F, Φ following the finite dimensional model. ?

Questions: 1) Fix a Fredholm module (H, F) over $C(M)$ (ungraded), + let g be a unitary autom of the trivial bundle Hilbert space $\tilde{V}$, which is $\equiv 1$ mod $\mathcal{K}$. Then [redacted] on $H \otimes V$, does $F \otimes 1$ [redacted] commute mod $\mathcal{K}$ with g? For example, when $H = L^2(S^1)$ and F is the Hilbert transform, does $F \otimes 1$ on $H \otimes V = L^2(S^1, \tilde{V})$ commute mod $\mathcal{K}$ with g?

2) Suppose instead of $M \longrightarrow U(V; 1)$ we take $M \longrightarrow Gr(V, \varepsilon)$. How can one couple this to F?

Recall the space of pairs (e,e') of projectors on H such that $e \equiv e' \bmod \mathcal{K}$ is homotopy equivalent to the restricted Grass, since it fibres over the contractible space of all projectors e' (we assume $\operatorname{Im} e'$, $\operatorname{Ker} e'$ are ∞-diml). Moreover we [redacted] know how to construct a map from this space to a restricted Grass. [redacted] Namely we take the odd almost involutive contraction $\begin{pmatrix} 0 & ee' \\ e'e & 0 \end{pmatrix}$ on $eH \oplus e'H$, we take its modified C.T. which gives an involution on $eH \oplus e'H$ which equals $-\varepsilon$ [redacted] when $e = e'$, and then we extend the by $-\varepsilon$ on the orthogonal complement $(1-e)H \oplus (1-e')H$ in $H \oplus H$.

It seems this map almost lifts into the space [redacted] $\mathcal{F}_0$ of odd almost involutive contractions. At least it does provide the almost-inv. contraction $\begin{pmatrix} 0 & ee' \\ e'e & 0 \end{pmatrix}$ on $eH \oplus e'H$. One then wants an isomorphism of the complements $(1-e)H$ [redacted] and $(1-e')H$. What we can do is to add H^{∞} and use the infinite repetition isomorphism

$$
\begin{aligned}
&(1-e)H \oplus H \oplus H \oplus \cdots \\
&= (1-e)H \oplus \big(eH \oplus (1-e)H\big) \oplus \big(eH \oplus (1-e)H\big) \oplus \cdots \\
&= \big((1-e)H \oplus eH\big) \oplus \big((1-e)H \oplus eH\big) \oplus \cdots \\
&= H \oplus H \oplus \cdots\cdots
\end{aligned}
$$

and similarly for e'.

July 7, 1988

Wave packet transform on $\mathbb{R}$. We consider $L^2(\mathbb{R})$ with the operators $q = x$ and $p = \frac{h}{i}\partial_x$. We would like a way to associate operators on $L^2(\mathbb{R})$ to functions $f(q,p)$, i.e. a quantization procedure. One method uses the holomorphic representation. Here $L^2(\mathbb{R})$ is identified with the subspace of $L^2\left(\mathbb{C}, e^{-|z|^2}\frac{d^2z}{\pi}\right)$ consisting of holomorphic functions, and $f(q,p)$ is identified with a smooth function "$f(z,\bar{z})$" on $\mathbb{C}$. Quantization is then multiplication by f and projection back onto the holom. functions.

In the holom representation the generators are the coherent ~~states~~ $u_\lambda = e^{\lambda z} = e^{\lambda a^*}|0\rangle$. ~~These states~~ are complete but not independent. We have

$$\langle u_\lambda | u_\mu \rangle = e^{\bar{\lambda}\mu}$$

$$\int \frac{d^2\lambda}{\pi} e^{-|\lambda|^2} |u_\lambda\rangle\langle u_\lambda|.$$

Let's look for similar things in $L^2(\mathbb{R})$. We consider

$$e^{-\frac{x^2}{2} + \lambda x}$$

If $\lambda = u + iv$ with $u, v \in \mathbb{R}$, then this is

$$e^{ivx}\, e^{-\frac{(x-u)^2}{2}}\, e^{-\frac{u^2}{2}}$$

which is a wave packet of wave number v centered around $x = u$. Thus to each $\lambda \in \mathbb{C}$ we have attached a wave packet.

We have

$$\left\langle e^{-\frac{x^2}{2}+\lambda x} \,\middle|\, e^{-\frac{x^2}{2}+\mu x} \right\rangle = \int e^{-x^2 + (\bar\lambda+\mu)x}\, dx$$

$$= \sqrt{\pi}\, e^{+\left(\frac{\bar\lambda+\mu}{2}\right)^2} = \sqrt{\pi}\, e^{+\frac{\bar\lambda^2}{4}+\frac{\mu^2}{4}+\frac{1}{2}\bar\lambda\mu}$$

So if we put $\boxed{\varphi_\lambda = \pi^{-1/4}\, e^{-\frac{\lambda^2}{4}}\, e^{-\frac{x^2}{2}+\lambda x}}$

then we have

$$\boxed{\langle \varphi_\lambda | \varphi_\mu \rangle = e^{\frac{1}{2}\bar\lambda\mu}}$$

We also have

$$\int d^2\lambda\; e^{-\frac{1}{2}|\lambda|^2}\, \varphi_\lambda(x)\, \overline{\varphi_\lambda(y)}$$

$$= \int \frac{d^2\lambda}{\pi^{1/2}}\, e^{-\frac{1}{2}|\lambda|^2 - \frac{\lambda^2}{4} - \frac{\bar\lambda^2}{4} - \frac{x^2}{2} + \lambda x - \frac{y^2}{2} + \bar\lambda y}$$

$$= \int \frac{d^2\lambda}{\pi^{1/2}}\, e^{-\frac{1}{4}(\lambda+\bar\lambda)^2 + \lambda x + \bar\lambda y - \frac{x^2+y^2}{2}}$$

$$= \int \frac{du\,dv}{\pi^{1/2}}\, e^{-\frac{1}{4}(2u)^2 + u(x+y) + iv(x-y) - \frac{x^2+y^2}{2}}$$

$$= 2\pi\,\delta(x-y)\, \pi^{-1/2} \left(\int du\; e^{-u^2 + u(x+y)} \right) e^{-\frac{x^2+y^2}{2}}$$

$$= 2\pi\,\delta(x-y)\, e^{\frac{(x+y)^2}{4}}\, e^{-\frac{x^2+y^2}{2}} = 2\pi\,\delta(x-y)$$

$$\boxed{\int \frac{d^2\lambda}{2\pi}\, e^{-\frac{1}{2}|\lambda|^2}\, \varphi_\lambda(x)\, \overline{\varphi_\lambda(y)} = \text{[redacted]}\; \delta(x-y)}$$

July 8, 1988

Let's return to index theory over the circle with a view toward understanding the analysis better.

We consider $L^2(S^1)$, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, Lebesgue measure $\frac{dx}{2\pi}$; it has orthonormal basis e^{inx}, $n \in \mathbb{Z}$. We consider operators on $L^2(S^1)$ built up out of multiplication by functions of x and functions of $p = \frac{h}{i}\partial_x$, where h is real. If f is a function defined on $\mathbb{R}$, then $f(p)$ is the diagonal operator

$$f(p)\, e^{inx} = f(nh)\, e^{inx}.$$

(Although this operator depends only on the values of f on the set $\mathbb{Z}h$, we want to treat h as a parameter, and so we want f to be defined on all of $\mathbb{R}$. Also we could take p to be $\frac{h}{i}(\partial_x + ia)$ with $a \in \mathbb{R}$, i.e. use a non-trivial constant coefficient connection.)

We have

$$f(p) * e^{inx} = e^{inx} * f(p+nh)$$

where $*$ denotes operator composition. Thus one is led to introduce the cross product algebra of functions of p by the integers, the integers being identified with the exponential functions $\{e^{inx}\}$.

One can generalize to a torus $M = \mathbb{R}^n/\Gamma$. One takes functions on $T^*M = M \times (\mathbb{R}^n)^*$ and associates operators to them.

July 10, 1988

We are reviewing index theory over the circle. I recall being stuck on how to couple a Dirac operator on S^1 to a loop $g : S^1 \longrightarrow U(V)$. I ran into difficulty trying to define something like

$$\frac{1}{i}\partial_x + \frac{g-1}{g+1}$$

The reason for the difficulty probably lies somewhere in the analysis — there's something ill-posed, some failure of transversality which would become clear if I had good control of the analysis.

It seems likely that a way around the difficulty can be found by working in two dimensions. We see two approaches. First one can replace the loop g by a family of Dirac's over an auxiliary circle parametrized by the given circle. Thus an element u of $U(V)$ is to be replaced by "the" connection in the trivial bundle $\tilde{V}$ over the circle with the monodromy u. After making suitable choices we find a Dirac operator on the torus $S^1 \times S^1$ which should the coupling of $\frac{1}{i}\partial_x$ and g.

Secondly we can work over the cotangent bundle $T^*(S^1) = S^1 \times \mathbb{R}$. We algebras, symbol algebras, which are constructed from functions on $T^*(S^1)$, and which operate in various ways on $L^2(S^1)$. This formalism links index theory of ΨDO's on S^1 and the K-theory of $T^*(S^1)$. Specifically it gives a map from the K-theory of $T^*(S^1)$ to

Fredholm operators on the circle.

Let's follow the second approach which assigns operators on $L^2(S^1)$ to functions on $T^*(S^1) = S^1 \times \mathbb{R}$. We consider an extension of algebras.

$$0 \xrightarrow{[redacted]} \mathcal{a} \longrightarrow \tilde{\mathcal{a}} \longrightarrow C^\infty(\underbrace{S^1 \times \{\pm 1\}}_{\substack{\text{cosphere} \\ \text{bundle of } S^1}}) \longrightarrow 0$$

$\mathcal{a}$ consists of smooth functions $f(h,x,p)$ on $\mathbb{R} \times \underbrace{S^1 \times \mathbb{R}}_{T^*(S^1)}$ which are rapidly decreasing as $p \to \infty$, whereas $\tilde{\mathcal{a}}$ [redacted] consists of $f(h,x,p)$ which tend rapidly to constant functions in p as $p \to +\infty$ or $-\infty$. We know how given a connection $\partial_x + ia$ on S^1 and $h \neq 0$ to define a homomorphism

$$\tilde{\mathcal{a}} \longrightarrow \mathcal{L}(L^2(S^1))$$

such that $\mathcal{a}$ gets mapped to smooth kernel operators. [redacted] An element of $\tilde{\mathcal{a}}$ can be expanded as a Fourier series in x

$$\sum_{n \in \mathbb{Z}} e^{inx} f_n(h,p)$$

and the algebra structure is determined by the rule

$$f_n(h,p) * e^{ikx} = e^{ikx} f_n(h, p + hk)$$

Let's assume we understand the above algebra extension and concentrate on the index theory.

Let's consider an extension of algebras

$$0 \longrightarrow I \longrightarrow R \longrightarrow A \longrightarrow 0$$

where R operates on H so that I acts as operators in a certain Schatten class. In this case we have an index map

$$K_1(A) \xrightarrow{\partial} K_0(I) \longrightarrow K_0(\mathcal{K}(H)) = \mathbb{Z},$$

and we have various trace formulas for the index. Let's review these.

Let us start with an element of $K_1(A)$; it can be represented by an invertible matrix u over A. The connecting homomorphism ∂ is defined as follows. One lifts u to $p \in R$ and u^{-1} to $q \in R$. Then we have

$$qp = 1 - x \qquad \text{with } x \in I$$
$$pq = 1 - y \qquad \text{with } y \in I$$

$$pqp = p - yp = p - px \implies px = yp$$
$$qpq = q - xq = q - qy \implies xq = qy$$

~~We have the index formula~~

$$\cancel{\text{Index} = tr(x^n) - tr(y^n)}$$

~~where n is large enough so the traces are defined. To prove this we,~~

Set $\tilde{q} = (1 + x + \cdots + x^{2n-1})q$ so that

$$\tilde{q}p = 1 - x^{2n}$$

Then we have

$$(\tilde{q} \;\; x^n)\begin{pmatrix} p \\ x^n \end{pmatrix} = 1$$

so that

$$e = \begin{pmatrix} p \\ x^n \end{pmatrix}(\tilde{q} \;\; x^n) = \begin{pmatrix} p\tilde{q} & p x^n \\ x^n \tilde{q} & x^{2n} \end{pmatrix}$$

is a projector $\equiv$ mod I^n to $e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

The connecting homomorphism takes $[u] \in K_1(A)$ into the difference $[e] - [e_0] \in K_0(I)$.

Note that $[e] - [e_0]$ is a well-defined class in $K_0(I^n)$. If trace is defined on I^n we have

$$tr(e - e_0) = tr\left(p\tilde{q} - 1 + x^{2n}\right)$$

$$p(1 + x + \cdots + x^{2n-1})\, q = (1 + y + \cdots + y^{2n-1})\overbrace{(pq)}^{1-y} = 1 - y^{2n}$$

$$\therefore \quad tr(e - e_0) = tr(x^{2n}) - tr(y^{2n}).$$

Consider an algebra extension with a map to the Calkin extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}(H) & \longrightarrow & \mathcal{L}(H) & \longrightarrow & \mathcal{Q}(H) & \longrightarrow & 0 \end{array}$$

Given an invertible matrix u over A, one lifts it to p over R. Then p is a Fredholm operator on H, so ~~there~~ is an index defined.

It seems that the fact that a Fredholm operator has closed image and finite dimensional kernel + cokernel is a basic fact from analysis. One might try to define the index via the connecting homomorphism

$$K_1(A) \xrightarrow{\partial} K_0(I) \longrightarrow K_0(\mathcal{K}(H)) = \mathbb{Z}.$$

This starts from u; one lifts u and u^{-1} to p and q respectively. Replacing q by $\tilde{q} = [1-(1-qp)]q$ if necessary one can suppose

$$qp = 1 - \beta\alpha \qquad \text{with } \beta, \alpha \in I.$$

Then

$$e = \begin{pmatrix} p \\ \alpha \end{pmatrix} \begin{pmatrix} q & \beta \end{pmatrix} \qquad \text{and} \qquad e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are projectors over $I^+ = \mathbb{C} \oplus I$ which are congruent mod I, so $[e] - [e_0]$ is a class in $K_0(I)$. But now one still has to assign an index to two projectors on H which are congruent modulo compacts. The method is to look at the projection

$$e(H^{\oplus 2}) \xrightarrow{e_0 e} e_0(H^{\oplus 2})$$

which is Fredholm, and take the index. (In the case being considered $e(H^{\oplus 2})$ and $e_0(H^{\oplus 2})$ can be identified with H and then $e_0 e$ becomes p.) Thus one still ends up using the basic fact about Fredholm operators.

Remark: Index formula: Suppose e, e_0 are two projectors on H which differ by an operator in a Schatten class. Then we have

$$\text{Index}([e]-[e_0]) = tr\,(e-e_0)^{2n+1}$$

for n large enough so the index is defined.

Proof when e, e_0 are self-adjoint. Set $F = 2e-1$, $\varepsilon = 2e_0 - 1$. We want the index of $e e_0 : e_0 H \longrightarrow e H$. One has

$$\text{Ker}(e_0 e) = \underbrace{eH}_{F=1} \cap \underbrace{(e_0 H)^\perp}_{\varepsilon=-1}$$

$$\text{Ker}(e e_0) = \underbrace{e_0 H}_{\varepsilon=1} \cap \underbrace{(e H)^\perp}_{F=-1}$$

note: $(e_0 e)^* = e e_0$; think of $e_0 e$ as $j^* i$

so

$$\text{Index} = -tr\,(\varepsilon \text{ on } g=-1 \text{ eigenspace}) \qquad g = F\varepsilon$$

$$= -tr\,(\varepsilon f(g))$$

where f is a function with $f(1)=0$, $f(-1)=1$, such that $f(g) \in \mathcal{L}^1$.

Take $f(g) =$ a power of $\dfrac{2-g-g^{-1}}{4} = \dfrac{1-g}{2}\,\dfrac{1-g^{-1}}{2}$

$$\frac{2-g-g^{-1}}{4} = \frac{2-F\varepsilon-\varepsilon F}{4} = \left(\frac{F-\varepsilon}{2}\right)^2$$

So
$$\text{Index} = +\,tr(-\varepsilon)\left(\frac{F-\varepsilon}{2}\right)^{2n} = tr\left(\frac{F-\varepsilon}{2}\right)F\left(\frac{F-\varepsilon}{2}\right)^{2n-1}$$
$$= tr\, F\left(\frac{F-\varepsilon}{2}\right)^{2n}$$

$$\therefore \quad \text{Index} = tr\left(\frac{F-\varepsilon}{2}\right)^{2n+1} = tr\,(e-e_0)^{2n+1}$$

Next we want to discuss the possible purpose of this discussion. We are interested in the problem of coupling a Dirac on S^1 with a loop $g: S^1 \to U(V)$. This means that I would like a construction which produces an element of the restricted Grassmannian of a Hilbert space close to $L^2(S^1)$.

We are going to proceed as follows. We start with an odd K-class on the cosphere bundle $S^1 \times \{\pm 1\}$ of S^1. This will be represented by a pair g_+, g_- of invertible matrices over $C^\infty(S^1)$. To simplify we suppose $g_- = 1$. Thus we have the invertible matrix over $S^*(S^1) = S^1 \times \{\pm 1\}$ given by

$$u(x,p) = \begin{cases} 1 & p=-1 \\ g(x) & p=+1. \end{cases}$$

Now we would like to investigate ways to assign representatives for the index of u. Such representatives should lie in standard spaces of the homotopy type $\mathbb{Z}\times BU$, such as the restricted Grassmannian or the space of Fredholm operators. To get started let's consider the process of lifting

u to a ΨDO of order zero. By means of the alg. extension

$$0 \to \mathcal{A} \longrightarrow \tilde{\mathcal{A}} \longrightarrow C^\infty(S^1 \times \{\pm 1\}) \to 0$$
$$\mathcal{A} = \{f(h,x,p)\}$$

we can do the lifting in $\tilde{\mathcal{A}}$. Now it would be nice to [redacted] lift u to a contraction P, so that when we come to solve

$$QP + \beta\alpha = 1$$

we can take $Q = P^*$ and $\beta = \alpha^*$. I don't know whether this is possible, however [redacted] we can look [redacted] at the case where $h = 0$ to get some ideas. In this case $\mathcal{A}_0 = \mathcal{S}(S^1 \times \mathbb{R})$ and we are using u to construct [redacted] an idempotent over $S^1 \times (\mathbb{R} \cup \{+\infty\}) = S^1 \times S^1$.

Thus we are looking at the connecting map

$$K^{-1}(S^1 \times \{\pm 1\}) \xrightarrow{\partial} K_c^0(S^1 \times \mathbb{R}),$$

and we want to see what the standard formula for ∂ yields from u.

Let's fix $\tau : \mathbb{R} \longrightarrow [-1, 1]$ a monotone function, $\tau(p) = -1$ [redacted] for $p \ll 0$, and $\tau(p) = +1$ for $p \gg 0$. Then u lifts to

$$P(x,p) = \frac{1-\tau(p)}{2} + g(x)\frac{1+\tau(p)}{2}$$
$$= \frac{g+1}{2} + \frac{g-1}{2}\tau$$

Then
$$P^* = \frac{g^{-1}+1}{2} + \frac{g^{-1}-1}{2}\tau$$

$$p^*p = \frac{2+g+g^{-1}}{4} + \frac{2-g-g^{-1}}{4}\tau^2 + \Big(\underbrace{(g^{-1}+1)(g-1) + (g^{-1}-1)(g+1)}_{1+g-g^{-1}-1+1-g+g^{-1}-1 \;=\; 0}\Big)\frac{\tau}{4}$$

$$= \frac{2+g+g^{-1}}{4} + \frac{2-g-g^{-1}}{4}(\tau^2-1+1)$$

$$= 1 - \frac{(1-g)(1-g^{-1})}{4}(1-\tau^2)$$

So if we put

$$\alpha = \frac{g-1}{2}\sqrt{1-\tau^2} \qquad \beta = \alpha^*$$

we have $p^*p + \alpha^*\alpha = 1$, and so we obtain the projector

$$e = \begin{pmatrix} p \\ \alpha \end{pmatrix} \begin{pmatrix} p^* & \alpha^* \end{pmatrix}$$

which is the orthogonal projection on the image of

$$\begin{pmatrix} p \\ \alpha \end{pmatrix} = \begin{pmatrix} \frac{1-\tau}{2} + g\,\frac{1+\tau}{2} \\ \frac{g-1}{2}\sqrt{1-\tau^2} \end{pmatrix} = \begin{pmatrix} \frac{g-1}{2}\tau + \frac{g+1}{2} \\ \frac{g-1}{2}\sqrt{1-\tau^2} \end{pmatrix}$$

We find a new embedding of the suspension of $U(V)$ into $Gr(V^{\oplus 2})$. Taking $V=\mathbb{C}$ we have the map

$$[-1,1]\times\mathbb{T} \longrightarrow \mathbb{C}P^2$$

$$(\tau,\xi) \longmapsto \frac{\tau}{\sqrt{1-\tau^2}} + \frac{1}{\sqrt{1-\tau^2}}\overbrace{\left(\frac{\xi+1}{\xi-1}\right)}^{\in i\mathbb{R}}$$

If we set

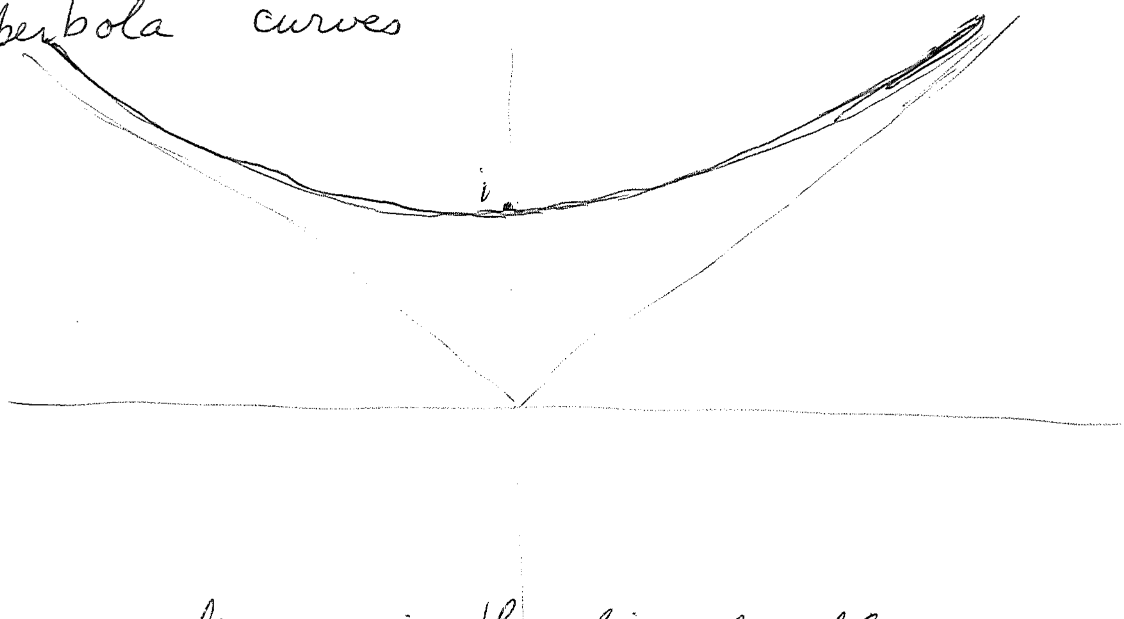
$$\tau = \frac{p}{\sqrt{1+p^2}} \qquad 1-\tau^2 = 1 - \frac{p^2}{1+p^2} = \frac{1}{1+p^2}$$

we get the map

$$\mathbb{R} \times \mathbb{T} \longrightarrow \mathbb{C}P^2$$

$$(p, \zeta) \longmapsto p + \sqrt{1+p^2}\left(\frac{\zeta-1}{\zeta+1}\right).$$

For ζ fixed, as p runs over $\mathbb{R}$ we get hyperbola curves



The monodramy in the line bundle as we follow such a curve is related to the area inside

July 12, 1988

Yesterday we found another "periodicity" map $\Sigma\, U(V) \longrightarrow Gr(V^{\oplus 2})$ using the formula for the connecting homomorphism in K theory. Like the map encountered with the operator $\frac{1}{i}\partial_x + \frac{g-1}{g+1}$, the monodromy along the suspension lines doesn't coincide exactly with g. This suggests looking at the map $\mathbb{R} \times U(V) \longrightarrow Gr(V^{\oplus 2})$ given by

$$(p,g) \longmapsto \begin{cases} \operatorname{Im}\begin{pmatrix} 1 \\ p \end{pmatrix} & p \leq 0 \\ \operatorname{Im}\begin{pmatrix} 1 \\ pg \end{pmatrix} & p \geq 0 \end{cases}$$

The reason is that the monodromy as p goes from $-\infty$ to $+\infty$ is essentially g.

Let's review this. Over $Gr(W)$ let's consider $\tilde{W}$ with the connection

$$\nabla = \frac{d + F.d.F}{2} = d + \tfrac{1}{2}FdF$$

Given a curve F_t in $Gr(W)$, the parallel transport along this curve is a family

$$g_t : \underset{\overset{\parallel}{W}}{\tilde{W}_{F_0}} \longrightarrow \underset{\overset{\parallel}{W}}{\tilde{W}_{F_t}}$$

satisfying $(\partial_t + \frac{1}{2}F_t\dot{F}_t)\, g_t = 0$. As a check let's verify that g_t preserves the involution along the curve i.e. that $g_t F_0 g_t^{-1} = F_t$. But

$$\partial_t\{g_t^{-1}F_t g_t\} = -g_t^{-1}\underbrace{\dot{g}_t}_{-\frac{1}{2}F_t\dot{F}_t g_t} g_t^{-1}F_t g_t + g_t^{-1}F_t\underbrace{\dot{g}_t}_{-\frac{1}{2}F_t\dot{F}_t g_t} + g_t^{-1}\dot{F}_t g_t$$

$$= + g_t^{-1}\left[+ \tfrac{1}{2} \underbrace{F_t \mathring{F}_t F_t}_{-\mathring{F}_t} - \tfrac{1}{2} \mathring{F}_t + \mathring{F}_t \right] g_t$$

$$= 0$$

so $g_t^{-1} F_t g_t$ is constant.

Now consider the path

$$F_\theta = (\cos\theta)\varepsilon + (\sin\theta)\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \qquad 0 \le \theta \le \pi$$

which is the path $\operatorname{Im}\begin{pmatrix} 1 \\ tg \end{pmatrix}$ $\quad 0 \le t \le \infty$

with $t = \tan\left(\frac{\theta}{2}\right)$. Then

$$\partial_\theta F_\theta = (-\sin\theta)\varepsilon + (\cos\theta)\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

$$F_\theta \partial_\theta F_\theta = (\cos^2\theta + \sin^2\theta)\,\varepsilon\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} = \begin{pmatrix} 0 & g^{-1} \\ -g & 0 \end{pmatrix}$$

$$g_\theta = e^{-\frac{\theta}{2}(F_\theta \partial_\theta F_\theta)} = e^{\frac{\theta}{2}\begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}}$$

$$g_\theta = \begin{pmatrix} \cos(\theta/2) & -g^{-1}\sin(\theta/2) \\ g\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

Thus

$$g_\pi = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix} : V^{\oplus 2} \longrightarrow V^{\oplus 2}$$

induces g on the fibres of the subbundle, i.e. g from $+1$ eigenspace of $F_0 = \varepsilon$ to the $+1$ eigenspace of $F_\pi = -\varepsilon$.

~~Suppose we take $g_+, g_- \in U(V)$ and define $R \to Gr(V^{\oplus 2})$ by~~

~~$\operatorname{Im}\begin{pmatrix} 1 \\ pu_- \end{pmatrix}$ $p \le 0$ $\quad \operatorname{Im}\begin{pmatrix} 1 \\ pu_+ \end{pmatrix}$ $p \ge 0$~~

Suppose given two loops $u_+, u_- : S^1 \to U(V)$ we define $S^1 \times (\mathbb{R} \cup \infty) \longrightarrow Gr(V^{\oplus 2})$ by

$$(x,p) \longmapsto \begin{cases} \mathrm{Im}\begin{pmatrix} 1 \\ pu_-(x) \end{pmatrix} & p \leq 0 \\ \mathrm{Im}\begin{pmatrix} 1 \\ pu_+(x) \end{pmatrix} & p \geq 0 \end{cases} \qquad (*)$$

This gives the monodromy

$$\tilde{V}^{\oplus 2}_{-\varepsilon} \xleftarrow{\begin{pmatrix} 0 & u_-^{-1} \\ -u_- & 0 \end{pmatrix}} \tilde{V}^{\oplus 2}_{\varepsilon} \xrightarrow{\begin{pmatrix} 0 & -u_+^{-1} \\ u_+ & 0 \end{pmatrix}} \tilde{V}^{\oplus 2}_{\varepsilon}$$

$$\begin{pmatrix} 0 & -u_+^{-1} \\ u_+ & 0 \end{pmatrix}\begin{pmatrix} 0 & -u_-^{-1} \\ u_- & 0 \end{pmatrix} = \begin{pmatrix} -u_+^{-1}u_- & 0 \\ 0 & -u_+u_-^{-1} \end{pmatrix}$$

which gives the monodromy $-u_+u_-^{-1}$ on the $+1$ eigenspace of $-\varepsilon$.

So the natural question is whether we can quantize the above K-class $(*)$. Thus for example it is reasonable to look for a ΨDO on the circle with the symbol

$$\begin{cases} p\,u_-(x) & p \leq 0 \\ p\,u_+(x) & p \geq 0 \end{cases} \quad ?$$

Getzler claimed one could apply heat kernel methods to this first order operator, but I am skeptical.

Let's go over the reasoning behind the above calculations. I am trying to couple a loop $S^1 \to U(V)$ with a Dirac on S^1 so as to produce either a Fredholm op. or a point in a restricted Grassmannian. I would like a construction which makes sense as $h \to 0$ in which case we obtain an even K-class over $T^*(S^1)$. The class should be obtained from an invertible matrix over $S^*(S^1) = S^1 \times \{\pm 1\}$ via the connecting map

$$K^{-1}(S^*(S^1)) \xrightarrow{\partial} K^0_c(T^*(S^1))$$

I have examined various maps $T^*(S^1) \to Gr(V)$ which should realize this even K-class.

July 13, 1988

Return to the problem of associating to an odd-diml Dirac operator $D\!\!\!/_0$ and a gauge transformation u an index $\in \mathbb{Z}$. Really the problem is to find an analytical expression for this index which is useful; hopefully, it will shed light on the cyclic class belonging to the Dirac operator.

Let's look at the topology first. We consider the linear path

$$X_t = (1-t)D\!\!\!/_0 + t\, u D\!\!\!/_0 u^{-1}$$

joining $D\!\!\!/_0$ to $uD\!\!\!/_0u^{-1}$. The choice of path is harmless since the space of connections is contractible. (Abstractly we are going to pass from X_t to the essential involution

$$A_t = \frac{-iX_t}{\sqrt{1-X_t^2}}$$

and we then stay in the contractible space of such A which mod compacts are a fixed involution in the Calkin algebra.)

We then take the Cayley transforms

$$g_t = \frac{1+X_t}{1-X_t}$$

This gives a path in $U^\infty(H,-1)$ = unitaries congruent to -1 mod compacts which starts with g_0 and ends with ug_0u^{-1}. But the space of all unitaries in H is contractible, so we can deform u to the identity in $U(H)$ in essentially one way up to homotopy. Thus we get a loop in

$\mathcal{U}^\infty(H,-1)$.

Such a loop represents an element of

$$\pi_1(\mathcal{U}^\infty(H,-1)) = \pi_1(\mathcal{U}) = \mathbb{Z}$$

Let's next look for a formula for this index. We have the C.T. map

$$t \longmapsto g_t = \frac{1+X_t}{1-X_t} \qquad 0 \le t \le 1$$

from the unit interval to $\mathcal{U}^p(H,-1) =$ unitaries congruent to -1 modulo $\mathcal{L}^p(H)$, where p is large enough. On $\mathcal{U}^p(H,-1)$ we have superconnection forms depending on a parameter u, $\overbrace{\text{of}}^{Re(u)>0}$ odd degree, which are invariant under conjugation by elements of $\mathcal{U}(H)$.

Let's look at the superconnection form of degree 1. In terms of X it is

$$u\bar{\sigma}\; tr\left(e^{uX^2}dX\right) \qquad \bar{\sigma} = (2i)^{1/2}$$

and upon dividing by $(-2\pi i u)^{1/2}$ it should have integral periods. Let's check this over $U(1)$. $X = ia$

$$u(2i)^{1/2}\int_{-\infty}^{\infty} e^{-ua^2}\, i\, da = u(2i)^{1/2}\, i\,\frac{\sqrt{\pi}}{\sqrt{u}} = (-2\pi i u)^{1/2}$$

Therefore this superconnection 1-form, when normalized, should be of the form $d\log\varphi$ where $\varphi : \mathcal{U}^p(H,-1) \longrightarrow \mathbb{T}$ is unique, if one requires $\varphi(-1) = 1$. It's clear that φ is first found as a map $\varphi_1 : U(1) \longrightarrow \mathbb{T}$ sending -1 to 1, and then

$$\varphi(g) = \prod_i \varphi_1(\lambda_i)$$

where the ζ_i are the eigenvalues of g different from -1.

So now it's clear that when we take the normalized superconnection 1-form on $U^P(H,-1)$, pull it back to $[0,1]$ via g_t, and then integrate, we are just getting a number α such that

$$\frac{\varphi(g_1)}{\varphi(g_0)} = \exp(\alpha)$$

But $g_1 = u g_0 u^{-1}$, so we have $\exp(\alpha) = 1$, which means that α is essentially the index.

Thus we have the formula for the index

$$\boxed{\text{Index} = \int_0^1 \frac{\sqrt{u}}{\sqrt{\pi}\, i}\, tr\left(e^{uX_t^2} dX_t\right) dt}$$

Check:

$$\text{Index} \underbrace{\int_0^\infty e^{-\lambda u} \sqrt{u}\, \frac{du}{u}}_{\frac{\sqrt{\pi}}{\lambda^{1/2}}} = \int_0^1 \frac{1}{\sqrt{\pi}\, i}\, tr\left(\frac{1}{\lambda - X_t^2}\, dX_t\right) dt$$

$$\therefore \text{Index} = \int_0^1 \frac{1}{2\pi i}\, tr\left(\frac{2}{1-X_t^2}\, dX_t\right) dt$$

$$= \frac{1}{2\pi i} \int_0^1 tr\left(g_t^{-1} \partial_t g_t\right) dt$$

$$g = \frac{1+X}{1-X} = -1 + \frac{2}{1-X}$$

$$dg = \frac{2}{1-X}\, dX\, \frac{1}{1-X}$$

$$g^{-1} dg = \frac{2}{1+X}\, dX\, \frac{1}{1-X}$$

$$tr(g^{-1}dg) = tr\left(\frac{2}{1-X^2}\, dX\right)$$