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May 6, 1988

Let's recall some earlier work on cyclic cocycles and left-invariant differential forms on groups of gauge transformations. Let M, E, D be a manifold, vector bundle & connection and $\mathcal{G} = \mathrm{Aut}(E)$. We consider $pr_2^*(E) = \mathcal{G} \times E$, where $pr_2 : \mathcal{G} \times M \longrightarrow M$, and we identify

$$\Omega^\cdot(\mathcal{G} \times M, \mathrm{End}(pr_2^* E)) = \Omega^\cdot(\mathcal{G}, \Omega^\cdot(M, \mathrm{End}\, E))$$

On $pr_2^*(E)$ we have the pull-back of D which we denote $\delta + D$ where $\delta = d_{\mathcal{G}}$. We also have a tautological automorphism g of $pr_2^*(E)$, and can transform this connection via g:

$$g^{-1}(\delta + D)g = \delta + \underbrace{g^{-1}\delta(g)}_{\theta} + D + g^{-1}D(g)$$

θ = the Maurer-Cartan form on $\mathcal{G}$.

To get left-invariant forms on $\mathcal{G}$, we can use the family of connections $\delta + D + t\theta$; the curvature is

$$(\delta + D + t\theta)^2 = \underbrace{D^2}_{(0,2)} + t\underbrace{D(\theta)}_{(1,1)} + (t^2 - t)\underbrace{\theta^2}_{(2,0)}$$

For $t = 0, 1$ this is flat in the $\mathcal{G}$-direction.

To simplify suppose $E = \tilde{V}$, $D = d$. Then the transgression formula becomes

$$(\delta + d)\int_0^1 \mathrm{tr}\left(e^{t d\theta + (t^2 - t)\theta^2}\theta\right) dt = \mathrm{tr}(e^{d\theta} - 1)$$

or better taking a component we see

$$\eta_{2n+1} = \int_0^1 \mathrm{tr}\left(\theta(t\,d\theta + (t^2 - t)\theta^2)^n\right) dt \in \left[C^\cdot(\tilde{\mathfrak{g}}, \Omega^\cdot(M))\right]^{2n+1}$$

satisfies $\quad \frac{1}{n!}(\delta + d)\eta = \frac{1}{(n+1)!}\,\mathrm{tr}(d\theta)^{n+1}$

In particular

$$\operatorname{tr} \Theta (d\Theta)^n \in C^{n+1}(\tilde{\mathfrak{g}}, \Omega^n(M)/d\Omega^{n-1}(M))$$

will be killed by δ, i.e. it is a cyclic n-cocycle with values in $\Omega^n/d\Omega^{n-1}$.

We can make this all algebraic as follows. Given A we can consider the double complex

$$C^{\cdot}(A, \Omega_A^{\cdot})$$

of Hochschild cochains on A with values in $\Omega_A^{\cdot}$ equipped with the zero bimodule structure. There's a natural trace on this algebra with values in the double complex

$$C_\lambda^{\cdot}(A, \Omega_A^{\cdot}/[\Omega_A^{\cdot}, \Omega_A^{\cdot}])$$

with a shift in degree.

We have $\Theta \in C^1(A, \Omega_A^0)$ given by the identity map $A \to A$, and it satisfies

$$\delta\Theta = -\Theta^2$$

So we can use the above formalism, and we obtain from the components of

$$\int_0^1 \operatorname{tr}\left(\Theta e^{t d\Theta + (t^2-t)\Theta^2}\right) dt$$

various cyclic cochains on A with values in $\Omega_A/[\ ,\ ]$. Moreover if we divide out by the image of $d\Omega_A^{\cdot}$, we get cyclic cocycles.

The goal is now to see if we can achieve an understanding of the S-operator. I want to obtain it from a non-unital homomorphism $a \longmapsto ae$, as Connes does.

Let's begin with a simple example. Let M be a manifold, let $A = C^\infty(M)$, let γ be a closed current of dim p [redacted] with compact support. Then we obtain a cyclic p-cocycle on A given by

$$\varphi(a_0, \ldots, a_p) = \int_\gamma a_0\, da_1 \cdots da_p .$$

Moreover we have cyclic p-cocycles on the matrix algebras $M_n(A)$ given by

$$\varphi(\alpha_0, \ldots, \alpha_p) = \int_\gamma tr(\alpha_0\, d\alpha_1 \cdots d\alpha_p)$$

Let's identify $M_n(A)$ with $End(\tilde{V})$ where $\tilde{V}$ is the trivial vector bundle with fibre $V = \mathbb{C}^n$. Let E be a vector bundle [redacted] which is a direct summand of $\tilde{V}$:

$$E \underset{i}{\overset{i^*}{\rightleftarrows}} \tilde{V} \qquad i^* i = id_E$$

Then we have a (non-unital) homomorphism

$$End(E) \hookrightarrow End(\tilde{V})$$

and so we obtain a cyclic p-cocycle on $End(E)$. On the other hand we have a (unital) homomorphism

$$A \longrightarrow End(E)$$

~~and hence we get another cyclic p-~~ which is an isomorphism when E is a line bundle. Thus we obtain another cyclic p-cocycle on A.

The composition

$$A \longrightarrow \mathrm{End}(E) \hookrightarrow \mathrm{End}(\tilde{V}) = M_n(A)$$

is $\quad a \longmapsto ae \quad$ where $\quad e = \iota\iota^*$, and so the new cyclic cocycle is

$$\circledast \qquad \psi(a_0, \ldots, a_p) = \int_\gamma tr\left(a_0 e \; d(a_1 e) \cdots d(a_p e)\right)$$

¶ Connes method of defining the S-operator on cyclic cocycles is to form the cochain algebra

$$\Omega^{\cdot}_{A^+} \otimes \Omega^{\cdot}_{\mathbb{C}[e]}$$

and use the universal property of $\Omega^{\cdot}_{A^+}$ to extend the homomorphism

$$A^+ \longrightarrow A^+ \otimes \mathbb{C}[e]$$

$$a \longmapsto ae$$

to a map of cochain algebras

$$\Omega^{\cdot}_{A^+} \longrightarrow \Omega^{\cdot}_{A^+} \otimes \Omega^{\cdot}_{\mathbb{C}[e]}$$

This gives a map of complexes

$$\Omega^{\cdot}_{A^+}/[\;,\;] \longrightarrow \left(\Omega^{\cdot}_{A^+}/[\;,\;]\right) \otimes \left(\Omega^{\cdot}_{\mathbb{C}[e]}/[\;,\;]\right)$$

and calculation shows

$$\Omega^{\cdot}_{\mathbb{C}[e]}/[\;,\;] \quad \text{has basis} \quad e(de)^{2i} \quad \text{for } i \geq 0.$$

Returning to ⊛ we see that it is a sum of $2p$ terms and in each we can move all the a_j's to the left and the e's to the right. If $p=2$ we have

$$tr\ a_0 e\,(da_1 \cdot e + a_1 de)(da_2 \cdot e + a_2 de)$$

$$= a_0\, da_1\, da_2\ tr(e) + a_0\, da_1\, a_2\ \cancel{tr(ede)}$$

$$+ a_0 a_1\, da_2\ \cancel{tr(edee)} + a_0 a_1 a_2\ tr(edede)$$

Thus we find that

$$\int_\gamma tr\{a_0 e\ d(a_1 e)\, d(a_2 e)\} = \left(\int_\gamma a_0\, da_1\, da_2\right) tr\, e + \int_\gamma a_0 a_1 a_2\ tr(ede^2).$$

The j-th term in general can be described as S^j applied to the cyclic $p-2j$ cocycle associated to the current

$$tr\,(e(de)^{2j}) \cap \gamma.$$

<u>General question</u>: We've seen how to associate to a closed current γ on M and to a vector bundle E which is expressed as a direct summand of a trivial bundle a cyclic cocycle on $End(E)$. Can we write this cyclic cocycle in terms of the induced connection on E?

The induced connection is

$$\nabla = i^* d\, i$$

and

$$\nabla^2 = i^* d\, i\, i^* d\, i = -i^* dj\, j^* di = -i^*[d,j]\, j^*[d,i]$$

So if $\alpha_0, \alpha_1, \alpha_2 \in \mathrm{End}(E)$

$$tr\left((\iota\alpha_0\iota^*)[d, \iota\alpha_1\iota^*][d, \iota\alpha_2\iota^*]\right)$$

$$= tr\left(\alpha_0 \iota^*[d,\iota\alpha_1\iota^*](\iota\iota^* + jj^*)[d,\iota\alpha_2\iota^*]i\right)$$

Now $\quad \iota^*[d, \iota\alpha\iota^*]i = \iota^* di\alpha - \alpha i^* di = [\nabla, \alpha]$

and

$$[d,\iota\alpha_1\iota^*]jj^*[d,\iota\alpha_2\iota^*] = \iota\alpha_1 \overset{-i^*dj}{\underset{-i^*[d,j]}{\cancel{[d,\iota^*]j}}}\, j^*[d,i]\alpha_2\iota^*$$

$$= \iota\alpha_1\nabla^2\alpha_2\iota^*$$

So the 2-cocycle is

$$tr\left(\alpha_0\left([\nabla,\alpha_1][\nabla,\alpha_2] + \alpha_1\nabla^2\alpha_2\right)\right)$$

And the 3-cocycle is

$$tr\left(\alpha_0 i^*[d,i\alpha_1 i^*](\iota i^* + jj^*)[d,\iota\alpha_2\iota^*](\iota i^* + jj^*)[d,\iota\alpha_3\iota^*]i\right)$$

$$= tr\begin{pmatrix}\alpha_0[\nabla,\alpha_1][\nabla,\alpha_2][\nabla,\alpha_3] + \alpha_0[\nabla,\alpha_1]\alpha_2\nabla^2\alpha_3 \\ + \alpha_0\alpha_1\nabla^2\alpha_2[\nabla,\alpha_3]\end{pmatrix}$$

In the case of the n-cocycle there are potentially 2^{n-1} terms however one can't choose jj^* consecutively. Thus the 4-cocycle should be

$$tr\begin{pmatrix}\alpha_0[\nabla,\alpha_1][\nabla,\alpha_2][\nabla,\alpha_3][\nabla,\alpha_4] + \alpha_0\alpha_1\nabla^2\alpha_2[\nabla,\alpha_3][\nabla,\alpha_4] \\ + \alpha_0[\nabla,\alpha_1]\alpha_2\nabla^2\alpha_3[\nabla,\alpha_4] + \alpha_0[\nabla,\alpha_1][\nabla,\alpha_2]\alpha_3\nabla^2\alpha_4 \\ + \alpha_0\alpha_1\nabla^2\alpha_2\,\alpha_3\nabla^2\alpha_4\end{pmatrix}$$

May 7, 1988

Consider a vector bundle which is a direct summand of a trivial bundle

$$\tilde{V} = E \oplus E', \qquad E \underset{i}{\overset{i^*}{\rightleftarrows}} \tilde{V} \underset{j}{\overset{j^*}{\rightleftarrows}} E'$$

Suppose we have a cyclic cocycle on $C^\infty(M)$. Then by the trace it gives a cyclic cocycle on $\mathrm{End}(\tilde{V})$ and by restriction a cyclic cocycle on $\mathrm{End}(E)$. To be specific take $\varphi(a_0, \ldots, a_p) = \int_\gamma a_0\, da_1 \ldots da_p$. More generally we should take

$$\varphi(a_0, \ldots, a_p) = a_0\, da_1 \ldots da_p \qquad \bmod d\Omega^{p-1}$$

or using Maurer-Cartan form notation

$$\varphi = \theta (d\theta)^p \qquad \bmod d\Omega^{p-1}.$$

On $\mathrm{End}\,\tilde{V}$ this becomes

$$\varphi = \mathrm{tr}\ \theta (d\theta)^p \qquad \bmod d\Omega^{p-1}$$

where tr denotes tr_V.

Now restrict to $\mathrm{End}(E)$. Let's use block notation relative to the decomposition $E \oplus E' = \tilde{V}$. Let θ now be the canonical element in $C^1(\mathrm{End}\,E, \mathrm{End}\,E)$. The restriction of φ is then

$$\mathrm{tr}_V (i\theta i^* \cdot d(i\theta i^*)^p)$$

Now

$$i\theta i^* = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}$$

$$d(i\theta i^*) = [d, i\theta i^*] = \begin{pmatrix} i^*[d, i\theta i^*]i & \overset{-\theta(i^* dj)}{\overset{\|}{i^*[d, i\theta i^*]j}} \\ \underset{(j^* di)\theta}{\underset{\|}{j^*[d, i\theta i^*]i}} & \underset{0}{\underset{\|}{j^*[d, i\theta i^*]j}} \end{pmatrix}$$

$$i^*[d, i\theta i^*]i = \underbrace{(i^* di)}_{\nabla}\theta - \theta\underbrace{(i^* di)}_{\nabla} = [\nabla, \theta] = \nabla(\theta)$$

Thus we are dealing with the cocycle

$$\circledast \qquad tr_V\left\{\begin{pmatrix}\Theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[\nabla,\Theta] & -\Theta(i^*dj)\\ (j^*di)\Theta & 0\end{pmatrix}^p\right\}$$

[redacted] This is a sum of terms which can be described as follows. Think in terms of of two state system with states E, E' where E' has "zero energy".

Actually we should probably calculate the resolvent. Recall the identity

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1} = \begin{pmatrix}1 & 0\\ -d^{-1}c & 1\end{pmatrix}\begin{pmatrix}(a-bd^{-1}c)^{-1} & 0\\ 0 & d^{-1}\end{pmatrix}\begin{pmatrix}1 & -bd^{-1}\\ 0 & 1\end{pmatrix}$$

The natural generating function for the family of cocycles $\circledast$ is

$$tr_V\left\{\begin{pmatrix}\Theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\lambda-[\nabla,\Theta] & \Theta(i^*dj)\\ -(j^*di)\Theta & \lambda\end{pmatrix}^{-1}\right\}$$

$$= tr_E\left(\Theta\ \overbrace{\frac{1}{\lambda-[\nabla,\Theta]-\Theta(i^*dj)\frac{1}{\lambda}(-j^*di)\Theta}}^{(a-bd^{-1}c)^{-1}}\right)$$

$$= tr_E\left(\Theta\ \frac{1}{\lambda-[\nabla,\Theta]-\frac{1}{\lambda}\Theta\nabla^2\Theta}\right)$$

Alternatively the p-th cocycle is the [redacted] $tr(\Theta H)$ where H is the sum of all monomials in $[\nabla,\Theta]$ and $\Theta\nabla^2\Theta$ of degree p, where $[\nabla,\Theta]$ has degree 1 and $\Theta\nabla^2\Theta$ has degree 2.

If we further restrict to $C^\infty(M) \subset End(E)$, then $[\nabla,\Theta] \longmapsto d\Theta$, $\Theta\nabla^2\Theta \longmapsto \Theta^2\nabla^2$ and we get the formula for the p-th cocycle:

$$\Theta(d\Theta)^p \, tr(1) + S\{\Theta(d\Theta)^{p-2}\} \, tr(\nabla^2) + S^2\{\Theta(d\Theta)^{p-4}\} \, tr(\nabla^4) + \cdots$$

Our next project should be the link between the S-operator and the different cyclic cocycles attached to an operator. Let's consider the odd case. We have an algebra A acting a Hilbert space H and an involution F on H such that $[F, a]$ belongs to an *ideal* I such that a trace is defined on I^p. Then we have the cyclic cocycles

$$\varphi_{2n-1}(a_0, \ldots, a_{2n-1}) = tr\left(F[F, a_0] \cdots [F, a_{2n-1}]\right)$$
$$= 2\, tr\left(a_0 [F, a_1] \cdots [F, a_{2n-1}]\right)$$

defined for n large enough.

What is needed is a way to link $S\varphi_{2n-1}$ and φ_{2n+1}. This requires an understanding of $S\varphi_{2n-1}$. According to Connes what we do is to enlarge A by adjoining an idempotent commuting with A, and then you find a suitable cyclic cocycle of degree $2n+1$ on the enlarged algebra. This gives $S\varphi_{2n-1}$.

May 8, 1988

Let R be a superalgebra: $R = R^+ \oplus R^-$ and let F be an odd involution in R. Then $[F, \ ]$ is an odd derivation of D whose square is zero, since

$$[F,[F,x]] = F(Fx - xF) + (Fx - xF)F \qquad x \in R^+$$
$$= [F^2, x] = 0$$

$$[F,[F,x]] = F(Fx + xF) - (Fx + xF)F \qquad x \in R^-$$
$$= [F^2, x] = 0$$

(Actually the argument should be for D of odd degree

$$[D,[D,x]] = [[D,D],x] - [D,[D,x]]$$

so $$[D,[D,x]] = [\tfrac{1}{2}[D,D], x] = [D^2, x].\ )$$

Thus we see that

$$R^+ \underset{[F,\ ]_+}{\overset{[F,\ ]}{\rightleftarrows}} R^-$$

is a super version of a DGA. Notice that such a superversion can be expanded to become either a chain or cochain algebra

$$R^+ \leftarrow R^- \leftarrow R^+ \leftarrow \cdots$$

$$R^+ \rightarrow R^- \rightarrow R^+ \rightarrow \cdots$$

Principle to be followed (perhaps): Use super arguments with adic filtrations.

Consider next an algebra R with involution F, ungraded. Then we can form the superalgebra $R \otimes C_1 = R \oplus R\sigma$ with the odd involution σF. This gives the super DGA:

$$R \underset{[F,\ ]_+}{\overset{[F,\ ]}{\rightleftarrows}} R$$

Example. Let $C = \varepsilon A = A * \mathbb{C}[F] = (A*A) \times \mathbb{Z}/2$. This is a superalgebra where $\varepsilon = 1$ on A and $\varepsilon = -1$ on F. In this case

$$C^+ = A*A \qquad C^- = (A*A)F$$

Note that $C^+ = A*A$ is a superalgebra and that C is the result of adjoining the grading on C^+ to C^+. This grading on $A*A$ is compatible with the isom

$$gr(A*A) = \Omega_A^{\cdot}$$

In the ungraded case we look at an algebra A and an involution F acting on a space H. Thus $C = A*\mathbb{C}[F]$ acts on H. In the graded case we ask that there be grading ε on H compatible with the grading on C. That is, we want A to be of even degree and F of odd degree with respect to their action on H. What this means is that we have an action of

$$C \hat{\otimes} \mathbb{C}[\varepsilon] = C^+ \otimes C_2$$

i.e. $H = H^+ \oplus H^+$ with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and where we have two actions of A on H^+.

It might be useful to try to describe our program. I think the main goal is to understand the Connes cocycles without being tied to $\Omega_A^{\cdot}$.

Lets now look at Connes cocycles. We start with A, F acting on H with $[F,a]$ in some Schatten ideal, so that

$$\mathrm{tr}\left(F[F,\theta]^{2n}\right) = 2\,\mathrm{tr}\left(\theta[F,\theta]^{2n-1}\right)$$

is defined. We claim this a cyclic $2n$-cocycle.

There are two proofs at least. Let's start with my proof using the connection and curvature formalisms.

Let $W \oplus W' \xrightarrow{\sim} H$ be the ± 1 eigenspaces of F and i, i^*, j, j^* as usual. Over $\mathcal{G} = GL_n(A)$ we have maps of trivial bundles

$$\tilde{W}^n \underset{g\,i}{\overset{i^* g^{-1}}{\rightleftarrows}} \tilde{H}^n \underset{g\,j}{\overset{j^* g^{-1}}{\rightleftarrows}} \tilde{W}'^n$$

where g is the tautogical autom. of $\tilde{H}^n$ over $\mathcal{G}$. Thus on $\tilde{W}^n \oplus \tilde{W}'^n$ we have the connection

$$i^* g^{-1} . d . g i \;\oplus\; j^* g^{-1} . d . g j$$

$$= d + \underbrace{\left(i^*\theta i \oplus j^*\theta j\right)}_{\text{connection form } \eta} \qquad \begin{array}{l}\theta = g^{-1}dg \\ \in C^1(A, \mathrm{End}\,H)\end{array}$$

with curvature $\omega = d\eta + \eta^2 =$

$$\left((i^*\theta i)^2 - i^*\theta^2 i\right) \oplus \left((j^*\theta j)^2 - j^*\theta^2 j\right).$$

[redacted] Now

$$(i^*\theta i)^2 - i^*\theta^2 i = i^*\theta(ii^*-1)\theta i$$
$$= -i^*\theta j j^*\theta i$$

also

$$\tfrac{1}{2}[F,\theta] = [e,\theta] = \begin{pmatrix} 0 & i^*\theta j \\ -j^*\theta i & 0 \end{pmatrix}$$

so we see the curvature of the direct sum $\widetilde{W}^n \oplus \widetilde{W}'^n$ is

$$\omega = \left(\tfrac{1}{2}[F,\theta]\right)^2$$

We have the Bianchi identity

$$d\omega + [\eta, \omega] = 0.$$

Using this one sees in the usual way that

$$tr\,(F\omega^n) \quad (\text{or } tr\,(e\omega^n) \text{ or } tr\,((1-e)\omega^n))$$

are all closed. This gives

$$[\text{redacted}]\ \delta\, tr\left(F[F,\theta]^{2n}\right) = 0$$

Next let's go over Connes proof. He considers the DGA defined by the algebra $End(H)$ and the involution F:

$$End(H) \xrightarrow{[F,\ ]} End(H) \xrightarrow{[F,\ ]_+} \cdots\cdots$$

Inside there is the [redacted] DG subalgebra

$$End(H) \longrightarrow I \longrightarrow I^2 \longrightarrow \cdots\cdots$$

where I is the Schatten ideal containing $[F,a]$. In large odd degrees the trace on I^{2n-1} is a closed supertrace on this DG subalgebra. By his theory one gets a cyclic $(2n-1)$ cocycle defined by

$$\mathrm{tr}\left(\theta[F,\theta]^{2n-1}\right).$$

Finally there his direct methods of calculating, the simplest being to compute directly

$$\delta\, \mathrm{tr}\left(F[F,\theta]^{2n}\right) \qquad \text{using } \delta\theta = -\theta^2$$

and see that the various terms cancel.

The next thing to look at is the graded case. Here we suppose that there is a grading $H = H^+ \oplus H^-$ given on H such that F is of odd degree. Then $\mathrm{End}(H)$ is a superalgebra with an odd involution, and we have the DG algebra

$$\mathrm{End}^+(H) \xrightarrow{[F,\ ]} \mathrm{End}^-(H) \xrightarrow{[F,\ ]_+} \cdots$$

We have the DG subalgebra

$$\mathrm{End}^+(H) \longrightarrow (I)^- \longrightarrow (I^2)^+ \longrightarrow \cdots$$

and in large even degrees the supertrace tr_ε on I^{2n} is a closed supertrace on this DG subalgebra. This gives by Connes theory $2n$-cocycles

$$\mathrm{tr}\left(\varepsilon\theta[F,\theta]^{2n}\right)$$

The question is now whether there is a natural connection curvature construction of this cocycle.

<u>May 9, 1988</u>

We have to understand the relation between Connes cocycles and the S-operator. Basic to his construction is the fact that to a superalgebra $R = R^+ \oplus R^-$ with involution belongs a cochain algebra

$$R^+ \xrightarrow{[F,\ ]} R^- \xrightarrow{[F,\ ]_+} R^+ \longrightarrow \cdots\cdots$$

Example: Take $R = \mathrm{End}(\mathbb{C}^2)$ with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the above cochain algebra should be the non-commutative DR algebra of $R^+ = \mathbb{C} \oplus \mathbb{C}e$, $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Recall that

$$\Omega^n_A = A \otimes \bar{A}^{\otimes n} \qquad \text{so if} \quad A = \mathbb{C}[e]$$

$$\Omega^n_A = \mathbb{C}[e]\, de^n$$

Also

$$[F,e] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$[F,e]^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

So it's clear that the canonical map

$$\Omega^\circ_{\mathbb{C}[e]} \longrightarrow \left(R^+ \longrightarrow R^- \longrightarrow R^+ \longrightarrow \cdots \right)$$

is an isomorphism.

The next ~~point~~ step is take the tensor product with this example.

Let us now consider the S-operator in my setting, by which I mean we consider left-invariant differential forms on $\mathcal{G} = GL(A)$. Thus I have A acting on V and an F on V and I construct a direct embedding

$$\tilde{W} \underset{gi}{\overset{i^*g^{-1}}{\rightleftarrows}} \tilde{V} \qquad W = +1 \text{ eigenspace of } F$$

of trivial bundles over $\mathcal{G}$. Then I use the trace on W to obtain Chern character forms on $\mathcal{G}$.

I'd like to describe the situation by saying that I have a nice description of certain kinds of cyclic cocycles using the connection and curvature formalism. In particular the cocycles associated to extensions are of this type. (However I'm still stuck on the cocycles on $C^\infty(M)$ or more generally $End(E)$. In this ~~situation~~ I haven't really understood the good way to construct these cocycles via transgression. Hence I have nothing better than Connes approach using cochain algebras and the formula $\int \theta (d\theta)^n$.)

So let us start with cocycles in the form I like and try to find their S-transforms. The idea geometrically goes as follows. We have a cocycle on A which we are viewing as a character form on $\mathcal{G}$ of a bundle $\tilde{W}$ with connection. We now lift $\tilde{W}$ up to $\mathcal{G} \times S^2$ and tensor with the Hopf line bundle. Then we get character forms over $G \times S^2$ which can be integrated to give left-invariant forms on G. Unfortunately this doesn't raise the degree of the form, so we don't have the right operation. ?

It appears that we have to use the "cycle" (fundamental class) on S^2, and this has to be multiplied with the "cycle" represented by F.

May 10, 1988

Our goal is to understand Connes thm. showing the different cyclic cocycles attached to a Fredholm module are linked by the S-operator.

Let's work in the graded case $H = H^+ \oplus H^-$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Set $R = \mathrm{End}(H)$. We have the DG algebra with traces

$$\begin{tikzcd}
R^+ \arrow[r, "{[F,\ ]}"] & R^- \arrow[r, "{[F,\ ]_+}"] & R^+ \arrow[r] & {} & {} \\
R^+ \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d, "tr_\varepsilon"] & I \arrow[u, phantom, "\cup"] \arrow[r] & I^2 \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d, "tr_\varepsilon"] & I^3 \arrow[r] & I^4 \arrow[r] \arrow[d, "tr_\varepsilon"] & {} \\
\mathbb{C} & & \mathbb{C} & & \mathbb{C}
\end{tikzcd}$$

and our problem is to show that the different cyclic cocycles attached to these traces are linked by S.

The simplest case is to relate

$$tr_\varepsilon\, \theta^3 \qquad \text{and} \qquad tr_\varepsilon\, \theta[F,\theta]^2$$

We have

$$\theta[F,\theta]^2 = \theta(F\theta - \theta F)^2 = \theta(F\theta F\theta - F\theta^2 F - \theta^2 + \theta F\theta F)$$
$$= \theta F\theta F\theta - \theta F\theta^2 F + \theta^2 F\theta F - \theta^3$$
$$\equiv 3\theta^2 F\theta F - \theta^3 \qquad (\equiv \text{ means same for } tr_\varepsilon)$$

But

$$-\delta(\theta F\theta F) = \theta^2 F\theta F - \theta F\theta^2 F \equiv 2\theta^2 F\theta F$$

consequently $tr_\varepsilon(\theta^2 F\theta F)$ is a cyclic coboundary and

$$tr_\varepsilon\, \theta[F,\theta]^2 \sim -tr_\varepsilon\, \theta^3$$

This calculation raises an interesting point, namely, we saw before in the geometric case of a manifold M and cycle γ that the S tranform of the cyclic cocycle attached to γ can be realized exactly by lifting functions on M to $M \times S^2$, then multiplying by the idempotent e describing the Hopf bundle, and then using the cycle $\gamma \times [S^2]$.

Suppose that a similar procedure is applied to the cocycles $tr_\varepsilon(\theta[F,\theta]^{2n})$, or really to the DGA with traces

$$\begin{tikzcd} R^+ \arrow[r, "{[F,\ ]}"] \arrow[d, "tr_\varepsilon"] & R^- \arrow[r, "{[F,\ ]_+}"] & R^+ \arrow[r] \arrow[d, "tr_\varepsilon"] & {} \\ \mathbb{C} & & \mathbb{C} & \end{tikzcd}$$

Call this cochain algebra $R^\cdot$. The idea is that we form

$$R^\cdot \otimes \Omega^\cdot_{\mathbb{C}[e]}$$

and take the tensor product of the traces $R^{2n} \xrightarrow{\tau_{2n}} \mathbb{C}$ and $\Omega^2_{\mathbb{C}[e]} \longrightarrow \mathbb{C}$. Clearly we then obtain the $(2n+2)$-cocycle $S\tau_{2n}$.

We see now, that, contrary to our initial impression, the S-transforms are completely analogous in the two cases. The issue is why in the "super" example the cocycles τ_{2n+2} and $S\tau_{2n}$ are cohomologous.

However lets consider some general

nonsense connected with cyclic cocycles on $C^\infty(M)$. We have a means to construct lots of [redacted] cyclic cocycles as follows. First of all we can pull-back $f^*: C^\infty(M) \longrightarrow C^\infty(N)$ with respect to a map $f: M \longrightarrow N$. Then we have the possibility of a non-unital homomorphism

$C^\infty(M) \longrightarrow \mathrm{End}(E) \hookrightarrow$ [redacted] $M_n(C^\infty(M))$

corresponding to a direct summand $E \underset{i}{\overset{i^*}{\rightleftarrows}} \tilde{\mathbb{C}}^n$. Finally we have the basic Connes [redacted] cocycle $\int_\gamma \theta(d\theta)^n$ attached to a closed current. <u>QUESTION</u> To what extent can one prove directly without using Connes' determination of $HC^\bullet(C^\infty(M))$ that [redacted] cocycles constructed geometrically in the way described [redacted] can be expressed [redacted] in terms of S-transforms of cocycles attached to closed [redacted] currents on M?

This question suggests a program, namely to completely describe $HC^\bullet(C^\infty(M))$, rather a candidate for this cyclic homology, by directly defining the S-operator and manipulating. It's necessary to prove that S applied to the cyclic cocycle attached to a closed current which is a boundary is a cyclic coboundary. I guess I would like geometric arguments where possible.

Typical question. Lets consider $E \underset{i}{\overset{i^*}{\rightleftarrows}} \tilde{V}$ over M and the corresponding cocycle $\int_\gamma \operatorname{tr}(\theta e \; d(\theta e)^n)$. We know how to write this as

$$\int_\gamma \sum_p S^p(\theta (d\theta)^{n-2p}) \cdot \operatorname{tr}(e(de)^{2p})$$

We carry this out on the level of formulas, but the question is whether there is a geometric way to proceed. The idea here is that e defines a map $M \longrightarrow Gr(V)$, and pulling back via

$$\begin{array}{c} M \times Gr(V) \\ \big\downarrow{\scriptstyle pr_1} \\ M \end{array}$$

and integrating over $\gamma \times$ (suitable $2p$ cycles in $Gr(V)$) gives $S^p \int_\gamma$.

It seems better to observe that a certain kind of data gives rise to a cyclic cocycle on $C^\infty(M)$. The data is a triple consisting of a manifold $f: N \longrightarrow M$ mapping to M, a vector bundle E with connection on N and a cycle (closed current) γ on N. We can replace the bundle with connection by an idempotent matrix, or perhaps a map $N \longrightarrow Gr(V)$. Thus we have a correspondence of some sort of M with the Grassmannian. In fact we have simply a closed current on $M \times Gr(V)$.

Let's check this. We are given a closed current γ say of dim n in $M \times Gr(V)$. Then we obtain an n-diml cyclic cocycle on $C^\infty(M)$ as follows. We have (non-unital) algebra homomorphisms

$$C^\infty(M) \longrightarrow C^\infty(M \times Gr(V)) \longrightarrow C^\infty(M \times Gr(V)) \otimes End(V)$$
$$f \longmapsto f \circ pr_1 \longmapsto (f \circ pr_1)(e \circ pr_2)$$

and we have the cyclic cocycle on the latter given by $\int_\gamma tr_V\, \theta (d\theta)^n$. Thus on $C^\infty(M)$ we have the cocycle

$$\int_\gamma tr_V \text{ [redacted] } \left(\theta e \; d(\theta e)^n\right)$$
$$= \sum_{p \geq 0} \int_\gamma pr_1^* S^p \{\theta (d\theta)^{n-2p}\} \cdot pr_2^* tr_V\, e(de)^{2p}$$
$$= \sum_{p \geq 0} \text{ [redacted] } S^p\{\theta(d\theta)^{n-2p}\} \cap pr_{1*}\left[tr(e(de)^{2p}) \cap \gamma\right]$$

Thus this cyclic cocycle can be reconstructed from the [redacted] currents on M obtained by "intersecting" γ with the character forms on the Grassmannian and then pushing down to M.

However I still need a proof that if a closed n-current γ is a boundary, say $\gamma = \partial(\eta)$, then the cyclic cocycles

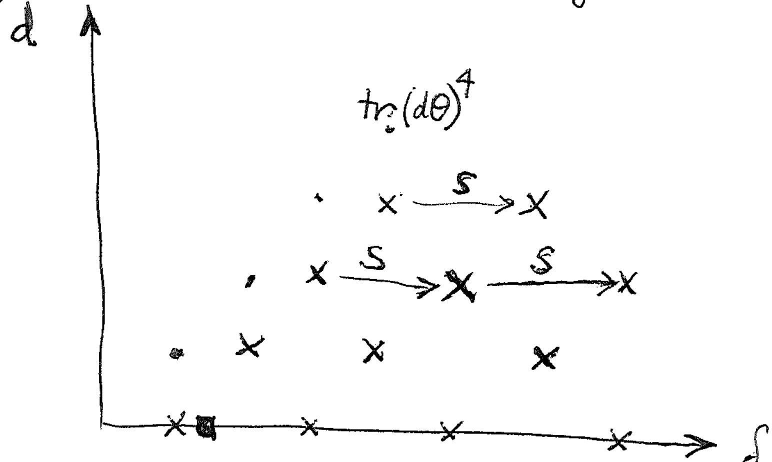
$$S^p\{\theta(d\theta)^{n \text{ [redacted]}}\} \cap \gamma$$

for $p > 0$ are cyclic coboundaries.

Here's a proof based on earlier work. We work in the double cochain algebra $C^\cdot(A, \Omega_A^\cdot)$. This has the total differential $\delta + d$ where d on $C^p(A, \Omega_A^\cdot)$ involves the sign $(-1)^p$ times the effect of d on $\Omega_A^\cdot$. Recall the formula

$$\circledast \qquad (\delta+d)\int_0^1 dt\,(n+1)\,tr\left\{\theta\left(t\,d\theta+(t^2-t)\theta^2\right)^n\right\} = tr\,(d\theta)^{n+1}$$

as well as the fact, which I think I proved, that in the diagram



the S operator transforms a term at a cross into the terms 2 steps to the right (up to some factor).

If all this is true, then the identity $\circledast$ says that (up to numerical factors)

$$(\delta+d)\left\{tr(\theta\,d\theta^n) + S\,tr\{\theta(d\theta)^{n-1}\} + S^2\,tr\{\theta(d\theta)^{n-2}\} + \cdots\right\} = tr\,(d\theta)^{n+}$$

Thus
$$\int_{\partial\eta} S^p\{tr\,\theta(d\theta)^n\} = \int_\eta d\,S^p(tr\,\theta(d\theta)^n)$$

$$= -\int_\eta \delta S^{p-1} tr\,\theta(d\theta)^{n+1} = -\delta\int_\eta S^{p-1} tr\,\theta(d\theta)^{n+1}$$

<u>May 11, 1988</u>

Let us consider finite-dimensional algebras. Consider $R = \mathrm{End}(V)$ where V is a finite-dimensional vector space. In order to get something interesting I guess we must assume V is $\mathbb{Z}/2$-graded: $V = V^+ \oplus V^-$. Suppose F is an odd involution, whence we can suppose $V^+ = V^-$ and $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we have even cyclic cocycles on R^+ defined by

$$\tau_{2n} = tr\left(\varepsilon F [F,\theta]^{2n+1}\right) = 2\, tr\left(\varepsilon \theta [F,\theta]^{2n}\right)$$

I would like to understand why these cocycles are related by Connes S-operator, and in particular why their cyclic cohomology classes are independent of F. ($\tau_0 = 2 tr(\varepsilon\theta)$ is ind. of F).

I propose to use the dilation concept together with homotopy. Homotopy means that the class of τ_{2n} doesn't change as F is deformed. I think this is fairly clear from the left-invariant form interpretation of cyclic cochain. In any case a direct proof should be possible.

Consider the ungraded situation $A, H, F = 2e-1$. I recall that if we work in the DG algebra $C^\cdot(A, \mathrm{End}(H))$, we have

connection form $\qquad \rho = e\theta e$

curvature " $\qquad \omega = d\rho + \rho^2 = e\theta e\theta e - e\theta^2 e$

$$= e\theta(e-1)\theta e = e[\theta, e]^2$$

and the $(2n-1)$-cocycle is

$$\mathrm{tr}(\omega^n) = \mathrm{tr}\left(e[\theta,e]^{2n}\right)$$

Now I would like to show that an infinitesimal change in e changes this cocycle by a cyclic coboundary. Let $\dot{e} = \partial_t e(t)|_{t=0}$ where $e(t)$ is a path of projectors with $e(0) = e$.

Then $\dot{\omega} = d\dot{\rho} + \dot{\rho}\rho + \rho\dot{\rho} = [d+\rho, \dot{\rho}]$ and

$$\mathrm{tr}(\omega^n)^{\circ} = n\,\mathrm{tr}(\dot{\omega}\,\omega^{n-1}) = n\,\mathrm{tr}\left([d+\rho, \dot{\rho}\,\omega^{n-1}]\right)$$

$$= d\, n\,\mathrm{tr}(\dot{\rho}\,\omega^{n-1})$$

Thus the cyclic cochain of which

$$\mathrm{tr}(\omega^n)^{\circ} = \mathrm{tr}\left(e[\theta,e]^{2n}\right)^{\circ}$$

is to be the coboundary is

$$n\,\mathrm{tr}\left((\dot{e}\theta e + e\theta\dot{e})\, e[\theta,e]^{2n-2}\right)$$

It might be useful to write this in terms of F for generalization to the graded case. We have

connection form $\rho = \frac{F+1}{2}\,\theta\,\frac{F+1}{2} + \frac{1-F}{2}\,\theta\,\frac{1-F}{2}$

$$= \frac{\theta + F\theta F}{2}$$

curvature $\omega = \frac{(\theta + F\theta F)^2}{4} - \frac{\theta^2 + F\theta^2 F}{2}$

$$= \frac{\theta^2 + \theta F\theta F + F\theta F\theta + F\theta^2 F - 2\theta^2 - 2F\theta^2 F}{4}$$

$$= \frac{1}{4}\left(F\theta F\theta - F\theta^2 F - \theta^2 + \theta F\theta F\right) = \frac{1}{4}[F,\theta]^2$$

We are interested in the form

$$\frac{1}{2}\operatorname{tr}(F\omega^n) = \operatorname{tr}\left(e[\theta,e]^{2n}\right)$$

(Check: $\frac{1}{2}\operatorname{tr}\left(F\frac{[F,\theta]^{2n}}{4^n}\right) = \frac{1}{2}\operatorname{tr}\left(F[\theta,e]^{2n}\right) = \operatorname{tr}\left(e[\theta,e]^{2n}\right)$
since $\operatorname{tr}\left([\theta,e]^{2n}\right) = 0$; this is true for any matrix 1-form.) Now

$$\partial_t \operatorname{tr}(F\omega^n) = \underset{\substack{\| \\ 0}}{\operatorname{tr}\left(\mathring{F}\omega^n\right)} + \operatorname{tr}\left(F\sum_{i=0}^{n-1}\omega^i\mathring{\omega}\,\omega^{n-i-1}\right)$$

because $\mathring{F}$ anti-commutes with F and ω commutes with F

$$= n\operatorname{tr}\left(F\mathring{\omega}\,\omega^{n-1}\right)$$

$$= n\operatorname{tr}\left(F\,[d+\rho,\mathring{\rho}]\,\omega^{n-1}\right)$$

$$= n\operatorname{tr}\left([d+\rho, F\mathring{\rho}\,\omega^{n-1}]\right)$$

since ρ commutes with F.

$$= d\; n\operatorname{tr}\left(F\mathring{\rho}\,\omega^{n-1}\right)$$

May 12, 1988

Let's check the transgression calculation from yesterday.

Connection $\jmath = e\theta e + (1-e)\theta(1-e)$

$$= \frac{\theta + F\theta F}{2} = \theta - \tfrac{1}{2}F[F,\theta]$$

curvature $\omega = \frac{1}{4}[F,\theta]^2 = [\theta,e]^2$. The identity to check is

$$\partial_t \underbrace{tr(F\omega^n)}_{2^{-2n}\, tr\left(F[F,\theta]^{2n}\right)} \overset{?}{=} d \underbrace{n\, tr(F\dot{\jmath}\,\omega^{n-1})}_{tr\left(F \frac{\dot F\theta F + F\theta\dot F}{2} \frac{1}{2^{2n-2}}[F,\theta]^{2n-2}\right)}$$

or

$$\boxed{\partial_t\, tr\left(F[F,\theta]^{2n}\right) = d\; 2n\, tr\left((\dot F\theta + \theta\dot F)[F,\theta]^{2n-2}\right)}$$

Compute RHS. We have from the Bianchi identity

$$d\omega^n = -\jmath\omega^n + \omega^n\jmath = -\theta\omega^n + \omega^n\theta$$

because $+\frac{1}{2}F[F,\theta]$ commutes with ω. In fact

$$d[F,\theta]^m = \sum_{i=0}^{m-1}(-1)^i[F,\theta]^i \underbrace{[F,-\theta^2]}_{-\theta[F,\theta]-[F,\theta]\theta}[F,\theta]^{m-1-i}$$

$$= -\theta[F,\theta]^m - (-1)^{m-1}[F,\theta]^m\theta$$

$$= -\theta[F,\theta]^m + (-1)^m[F,\theta]^m\theta = -[\theta,[F,\theta]^m]$$

Thus the RHS is

$$2n\, tr\left\{(-\dot F\theta^2 - \theta^2\dot F)[F,\theta]^{2n-2} - (\dot F\theta + \theta\dot F)\left(-\theta[F,\theta]^{2n-2} + [F,\theta]^{2n-2}\theta\right)\right\}$$

$$= 2n\, tr\left\{\left(\cancel{-\dot F\theta^2} \cancel{-\theta^2\dot F} + \cancel{\dot F\theta^2} + \theta\dot F\theta + \theta\dot F\theta + \cancel{\theta^2\dot F}\right)[F,\theta]^{2n-2}\right\}$$

The LHS is

$$\partial_t\, tr\left(F[F,\theta]^{2n}\right) = \underset{\substack{\parallel \\ 0}}{tr\left(\dot F[F,\theta]^{2n}\right)} + tr\left(F\sum_{i=0}^{2n-1}[F,\theta]^i[\dot F,\theta][F,\theta]^{2n-1-i}\right)$$

$$= 2n \,\mathrm{tr}\,\{F[\mathring{F},\theta][F,\theta]^{2n-1}\}$$

To see these are the same note that $\mathring{F}$ is an arbitrary operator anti-commuting with F. This suggests writing both sides as $\mathrm{tr}(\mathring{F}X)$ where X anti-commutes with F. On the left

$$\mathrm{tr}\left(2\theta\mathring{F}\theta\,[F,\theta]^{2n-2}\right) = -2\,\mathrm{tr}\left(\mathring{F}\theta[F,\theta]^{2n-2}\theta\right)$$

$$= - \,\mathrm{tr}\Big(\mathring{F}\underbrace{F\,[F,\,\theta[F,\theta]^{2n-2}\theta]}_{X \text{ as above}}\Big)$$

$$= \mathrm{tr}\Big(F\mathring{F}\big([F,\theta]^{2n-1}\theta + \theta\underbrace{[F,[F,\theta]^{2n-2}]}_{=0 \text{ as } F \text{ anticommutes with } [F,\theta]}\theta + \theta[F,\theta]^{2n-1}\big)\Big)$$

$$= \mathrm{tr}\left(F\mathring{F}\theta[F,\theta]^{2n-1} - \theta F\mathring{F}[F,\theta]^{2n-1}\right)$$

$$= \mathrm{tr}\left(F(\mathring{F}\theta - \theta\mathring{F})[F,\theta]^{2n-1}\right)$$

so it works.

Our next project should be to find a suitable graded version.

But first we can simplify the above computation. Go back to the cochain

$$\mathrm{tr}\left(\mathring{F}\theta + \theta\mathring{F}\right)[F,\theta]^{2n-2}$$

and apply the principle of writing it in the form $\mathrm{tr}(\mathring{F}X)$, where X anti commutes with F.

$$\mathrm{tr}\ \mathring{F}\left(\theta[F,\theta]^{2n-2} + [F,\theta]^{2n-2}\theta\right)$$

$$= \frac{1}{2}\mathrm{tr}\left(\mathring{F}F\,[F,\,\theta[F,\theta]^{2n-2} + [F,\theta]^{2n-2}\theta]\right)$$

$$= \mathrm{tr}\left(\mathring{F}F\,[F,\theta]^{2n-1}\right)$$

Then

$$d\, tr\left(\dot{F}F[F,\theta]^{2n-1}\right) = tr\left(\dot{F}F\left(-\theta[F,\theta]^{2n-1} - [F,\theta]^{2n-1}\theta\right)\right)$$

$$= tr\left(F(\dot{F}\theta - \theta\dot{F})[F,\theta]^{2n-1}\right)$$

So the moral is that there is a simpler transgression formula, namely

$$\boxed{\partial_t\, tr\left(F[F,\theta]^{2n}\right) = d\ 2n\ tr\left(\dot{F}F[F,\theta]^{2n-1}\right)}$$

It now begins to appear that the connection-curvature formalism you have been pushing is perhaps not fundamental.

So let us now turn to the graded case. Here one has A, H, ε, F, and $H = H^+ \oplus H^-$ where $H^+ = H^-$ and $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ rel. to this identification. Effectively this means we have two A-module structures on H^+, i.e. we are dealing with $A * A$ as emphasized by Cuntz.

A central problem will be to link the even cocycles of Connes with my even cocycles.

It would be nice to continue with the connection curvature formalism in this graded case. The problem one encounters at the start is how to make sense of the even cocycles

$$tr\left(\varepsilon F[F,\theta]^{2n+1}\right) = 2\, tr\left(\varepsilon\theta[F,\theta]^{2n}\right)$$

These are not the usual functions of the curvature $\frac{1}{4}[F,\theta]^2$. Recall that we have connection + curvature forms

$$\xi = \underbrace{\theta - \tfrac{1}{2}F[F,\theta]}_{\text{diag part of } \theta \text{ rel } F} \qquad \omega = \tfrac{1}{4}[F,\theta]^2$$

[redacted] I think one should emphasize the fact that the [redacted] Connes cocycles use the off-diagonal part of θ, namely $\frac{1}{2}F[F,\theta] = \frac{\theta - F\theta F}{2}$, [redacted] whose square is the curvature except for sign.

[redacted] Let's write

$$\alpha = \frac{\theta - F\theta F}{2} \qquad \zeta = \frac{\theta + F\theta F}{2}$$

Then

$$\alpha^2 = \frac{1}{4}F[F,\theta]F[F,\theta] = -\frac{1}{4}[F,\theta]^2 = -\omega$$

$$\alpha + \zeta = \text{[redacted]}\ \theta$$

so

$$\begin{array}{ccc} d\theta & = & d\alpha + d\zeta \\ \| & & \quad\ \ \| \\ -\theta^2 & & \quad\ \ \omega - \zeta^2 \end{array}$$

or

$$d\alpha = -\theta^2 - \omega + \zeta^2$$

$$= -(\alpha+\zeta)^2 - \omega + \zeta^2 = -\alpha\zeta - \zeta\alpha - \cancel{\alpha^2} - \cancel{\omega}$$

$\therefore$

$$d\alpha + \zeta\alpha + \alpha\zeta = 0$$

or

$$\boxed{[d + \zeta, \alpha] = 0}$$

This is a better "Bianchi identity" than $[d+\zeta, \omega] = 0$.

Check:

$$\alpha = \begin{pmatrix} 0 & i^*\theta j \\ j^*\theta i & 0 \end{pmatrix} \qquad \zeta = \begin{pmatrix} i^*\theta i & 0 \\ 0 & j^*\theta j \end{pmatrix}$$

$$-d\, i^*\theta j = +i^*\theta^2 j = (i^*\theta i)(i^*\theta j) + (i^*\theta j)(j^*\theta j)$$

$$-d\, j^*\theta i = j^*\theta^2 i = (j^*\theta i)(i^*\theta i) + (j^*\theta j)(j^*\theta i)$$

$$\alpha\zeta + \zeta\alpha = \begin{pmatrix} \text{[redacted]}\ 0 & i^*\theta j\, j^*\theta j + i^*\theta i\, i^*\theta j \\ j^*\theta i\, i^*\theta i + j^*\theta j\, j^*\theta i & 0 \end{pmatrix}$$

Here is a more intrinsic description of the formalism. Consider a vector bundle with a <u>flat</u> connection D and an involution F. Then we have the induced connections eDe, $(1-e)D(1-e)$ on the two eigenbundles of F, and we can form the direct sum connection

$$\nabla = eDe + (1-e)D(1-e) = \frac{D + FDF}{2}$$

and define α to be the difference operator 1-form:

$$\alpha = D - \nabla \quad , \qquad D = \nabla + \alpha$$

We note that $[\nabla, F] = 0$ and that F anti-commutes with α. Then flatness implies

$$0 = D^2 = \nabla^2 + [\nabla, \alpha] + \alpha^2.$$

Taking the parts commuting + anti-commuting with F we obtain

$$\begin{cases} \nabla^2 = -\alpha^2 \\ [\nabla, \alpha] = 0. \end{cases}$$

Now suppose we have a 1-parameter family of F's. Then we work over $\mathbb{R} \times M$ with the flat connection $\tilde{D} = dt\,\partial_t + D$ and the involution F on $pr_2^*(E)$ over $\mathbb{R} \times M$ corresponding to the family of involutions on E.

$$\tilde{\nabla} = \frac{1}{2}\left(\tilde{D} + F\tilde{D}F\right)$$

$$\tilde{\alpha} = \frac{1}{2}\left(\tilde{D} - F\tilde{D}F\right) = \frac{1}{2}\left(dt\,\partial_t + D - F(dt\,\partial_t + D)F\right)$$

$$= \alpha - \frac{1}{2}\,dt\,F\dot{F}$$

Then

$$\operatorname{tr}\left(F\tilde{\alpha}^{2n}\right) = \operatorname{tr}\left(F\left(\alpha - \tfrac{1}{2}dtF\dot{F}\right)^{2n}\right)$$

$$= \operatorname{tr}\left(F\alpha^{2n}\right) + \sum_{i=0}^{2n-1} \operatorname{tr}\underbrace{\left(F\alpha^{i}\left(-\tfrac{1}{2}dtF\dot{F}\right)\alpha^{2n-1-i}\right)}_{(-1)^{i}\cdot(-1)^{i(2n-i)}}$$

$$= \operatorname{tr}\left(F\alpha^{2n}\right) - \tfrac{1}{2}(2n)\, dt \operatorname{tr}\left(\dot{F}\alpha^{2n-1}\right)$$

This is a closed form over $R \times M$ and so we should obtain the transgression formula

$$\boxed{\partial_t \operatorname{tr}\left(F\alpha^{2n}\right) = -d\left\{n \operatorname{tr}\left(\dot{F}\alpha^{2n-1}\right)\right\}}$$

Let's check that this agrees with the [redacted] formula 3 pages ba[redacted]ck.

$$\alpha = \tfrac{1}{2}F[F,\theta] \quad , \quad \operatorname{tr}\left(F\alpha^{2n}\right) = \frac{1}{2^{2n}}(-1)^{n} \operatorname{tr}\left(F[F,\theta]^{2n}\right)$$

$$-n\operatorname{tr}\left(\dot{F}\alpha^{2n-1}\right) = (-1)\frac{n}{2^{2n-1}}(-1)^{n-1}\operatorname{tr}\left(\dot{F}F[F,\theta]^{2n-1}\right) \qquad \text{ok}$$

So now we have a formalism of the connection curvature type which explains the [redacted] odd Connes cocycles in the ungraded case.

Next we should check that the graded case works in essentially the same manner.

In the graded case we have M, E, D flat, F and an ε on E such that D is even and F is odd relative to ε. Then we have that

$$\alpha = \tfrac{1}{2}(D - FDF) \quad \text{[redacted] is even wrt } \varepsilon$$

$$\nabla = D - \alpha \quad \text{is even wrt } \varepsilon$$

[redacted] [redacted] Recall also that $\nabla F = F\nabla$ and $F\alpha + \alpha F = 0$.

It turns out to be possible to write down lots of closed forms using traces and these involutions, however most of them are zero. For example in the ungraded case we have

$\mathrm{tr}(\alpha^{2n}) = 0$ for any matrix 1-form

$\mathrm{tr}(\alpha^{2n+1}) = 0$ as α^{2n+1} anti-commutes with F

$\mathrm{tr}(F\alpha^{2n+1}) = 0$ same reason

so the only possibility is $\boxed{\mathrm{tr}(F\alpha^{2n})}$ In the graded case we have

$\mathrm{tr}(\varepsilon\alpha^{2n}) = 0$ for any matrix 1-form commuting with ε

$\left(\mathrm{tr}(F\alpha^{2n}) = 0 \text{ anti-commute with } \varepsilon\right)$ (also because $\varepsilon\alpha^{2n}$ anti-commutes with F.

$\mathrm{tr}(\varepsilon F\alpha^{2n}) = 0$ anti-comm. with F

$$\mathrm{tr}(\varepsilon F\alpha^{2n+1}) = \mathrm{tr}(\varepsilon F\alpha\,\alpha^{2n}) = -\mathrm{tr}(\alpha\varepsilon F\alpha^{2n}) = -\mathrm{tr}(\varepsilon F\alpha^{2n+1})$$

$$\therefore \quad \mathrm{tr}(\varepsilon F\alpha^{2n+1}) = 0.$$

Thus the only possibility is $\boxed{\mathrm{tr}(\varepsilon\alpha^{2n+1})}$. This is closed by the usual argument

$$d\,\mathrm{tr}(\varepsilon\alpha^{2n+1}) = \mathrm{tr}\,[\nabla, \varepsilon\alpha^{2n+1}] = 0$$

since $[\nabla, \varepsilon] = [\nabla, \alpha] = 0$. Note that $[\nabla,\varepsilon] = [\nabla,\varepsilon]_-$, $[\nabla,\alpha] = [\nabla,\alpha]_+$

Finally we look at the homotopy property. We set $\tilde{D} = dt\,\partial_t + D$ on $pr_2^*(E)$ over $R \times M$, and consider a family F, i.e. an involution on $pr_2^*(E)$. Then we consider

$$\tilde{\alpha} = \frac{1}{2}\left(\tilde{D} - F\tilde{D}F\right) = -\frac{1}{2}F[\tilde{D}, F]$$

$$= \alpha - \frac{1}{2}F\,dt\,\dot{F}$$

$$tr\left(\varepsilon\tilde{\alpha}^{2n+1}\right) = tr\ \varepsilon\left(\alpha^{2n+1} + \sum_{i=0}^{2n}\alpha^{i}\left(-\tfrac{1}{2}F\,dt\,\dot{F}\right)\alpha^{2n-i}\right)$$

$$= tr\left(\varepsilon\alpha^{2n+1}\right) - \sum_{i=0}^{2n} tr\left(\underbrace{\alpha^{i}\varepsilon\left(-\tfrac{1}{2}dt\,F\dot{F}\right)\alpha^{2n-i}}_{\text{sign } (-1)^{i(2n-i+1)} = +1}\right)$$

$$= tr\left(\varepsilon\alpha^{2n+1}\right) - dt\ \frac{2n+1}{2}\ tr\left(\varepsilon F\dot{F}\alpha^{2n}\right)$$

Consequently we have

$$\boxed{\partial_t\ tr\left(\varepsilon\alpha^{2n+1}\right) = -d\left\{\left(n+\tfrac{1}{2}\right)tr\left(\varepsilon F\dot{F}\alpha^{2n}\right)\right\}}$$

I would like to check this by a standard Bianchi identity proof. This means expressing $\dot{\alpha}$ in the form $[\nabla, ?]$. We have

$$\alpha = \frac{1}{2}(D - FDF) \qquad \dot{\alpha} = \left(-\tfrac{1}{2}\right)\left(\dot{F}DF + FD\dot{F}\right)$$

$$(-2)\,\dot{\alpha} = \dot{F}\nabla F + \dot{F}\alpha F + F\nabla\dot{F} + F\alpha\dot{F}$$

$$= \dot{F}F\nabla + \nabla F\dot{F} - \dot{F}F\alpha - \alpha F\dot{F}$$

$$= -(F\dot{F})\nabla + \nabla(F\dot{F}) + (F\dot{F})\alpha - \alpha(F\dot{F})$$

$$= [\nabla - \alpha, F\dot{F}]$$

$$\boxed{\dot{\alpha} = \left[\nabla - \alpha, -\tfrac{1}{2}F\dot{F}\right]}$$

Thus

$$\partial_t\ tr\left(\varepsilon\alpha^{2n+1}\right) = \sum_{i=0}^{2n} tr\left(\varepsilon\alpha^{i}\left[\nabla-\alpha, -\tfrac{1}{2}F\dot{F}\right]\alpha^{2n-i}\right)$$

continued from previous page. It is necessary to be careful because

$$[\alpha, \alpha] = 2\alpha^2$$

$$\partial_t \, tr(\varepsilon \alpha^{2n+1}) = \sum_{i=0}^{2n} tr(\varepsilon \alpha^i \underset{\underset{[\nabla-\alpha, -\frac{1}{2}F\dot{F}]}{\parallel}}{\dot{\alpha}} \alpha^{2n-i})$$

Put $\beta = -\frac{1}{2}F\dot{F}$

$$= \sum_{i=0}^{2n} tr(\varepsilon[\nabla-\alpha, \beta]\alpha^{2n})(-1)^{i(2n-i-1)}$$

or $\varepsilon[\alpha,\beta]\alpha^{2n}$ anti commutes with F $\therefore tr = 0$

$$= (2n+1)\, tr([\nabla-\alpha, \varepsilon\beta\alpha^{2n}])$$

because $[\nabla, \alpha] = 0$ and $[\alpha, \alpha^2] = 0$

$$= d\,(2n+1)\, tr(\varepsilon\beta\alpha^{2n})$$

Ungraded case

$$\partial_t \, tr(F\alpha^{2n}) = \underset{\underset{0}{\parallel}}{tr(\dot{F}\alpha^{2n})} + \sum_{i=0}^{2n-1} tr(F\alpha^i[\nabla-\alpha,\beta]\alpha^{2n-1-i})$$

or $F\alpha^i[\alpha,\beta]\alpha^{2n-1-i}$ anti-comm. with F

$$= \sum_{i=0}^{2n-1} tr([\nabla-\alpha,\beta]\alpha^{2n-1}F)\,\cancel{(-1)^i(-1)^{i(2n-i)}}$$

$$= 2n\, tr([\nabla-\alpha, \beta]\alpha^{2n-1}F)$$

$$= 2n\, tr([\nabla-\alpha, \beta\alpha^{2n-1}F])$$

Here use that
$$[\alpha, \alpha^2] = 0$$
$$[\alpha, \alpha F] = \alpha^2 F + \alpha F \alpha = \alpha^2 F - \alpha^2 F = 0.$$

$$= d\, 2n\, tr(\beta\alpha^{2n-1}F)$$

$$= d\, 2n\, tr(F\beta\alpha^{2n-1}).$$

So it all works but just barely.

Now let's continue with transgression. We consider a geometric situation where $A = C^\infty(M)$ acts on $H = L^2(M,S)$, and on H one has an involution modulo some Schatten ideal defined. I want to study the equivalence of various cocycles obtained by dilating.

The other example to keep in mind is the case of the finite-diml algebra $\mathrm{End}^{+}(V)$. The point should be to see the equivalence of the different cocycles.

Let's try to describe the goal in the geometric case. We have A acting on a Hilbert space H and an involution modulo compacts γ on H such that $[\gamma, a]$ is compact. We can lift this involution to a self-adjoint contraction and dilate to obtain an involution on a larger Hilbert space. Using this involution one obtains (under suitable summability hypotheses) cyclic cocycles. The goal is to show that these cocycles are cohomologous, better their cyclic cohomology classes are independent of the choice of contraction and dilation.

May 14, 1988

Let's consider the dilation process abstractly. We consider a vector space V with an endomorphism T, and wish to find $V \underset{i}{\overset{i^*}{\rightleftarrows}} V_1$ and an involution F on V_1 such that $T = i^* F i$. (In the C^*-setting one supposes that T is a self-adjoint contraction, and [redacted] one wants i to be an isometric embedding with adjoint i^* and F to be unitary.)

This is a special case of the GNS construction. Consider $A = \mathbb{C}[\mathbb{Z}/2]$ and form the universal algebra + map $\rho : A \to B$ such that $\rho(1) = 1$. Thus if $A = \mathbb{C} \oplus \mathbb{C}F$, then $B = \mathbb{C}[T]$ where $T = \rho(F)$. Then $C = A \oplus A \otimes B \otimes A = A * \mathbb{C}[\varepsilon] \simeq \mathbb{C}[\text{dihedral group } \langle F, \varepsilon\rangle]$ acts on V_1.

Recall that $B = eCe$ where $e = \frac{\varepsilon + 1}{2}$.
Thus we have a non unital homomorphism

$$B = eCe \longrightarrow C$$

and a B-module V and we are looking for a C-module V_1. We should probably break this non-unital homomorphism into 2 steps

$$B = eCe \underset{\substack{\text{adjoining}\\ 1}}{\hookrightarrow} \underset{\substack{\| \\ B^+}}{\mathbb{C} \oplus eCe} \underset{\text{unital}}{\longrightarrow} C$$

Since B is unital we have $B^+ = \mathbb{C} \times B$ so

$$Mod(B^+) = Mod(B) \times Mod(\mathbb{C})$$

In other words a non-unital B-module (equivalently a unital B^+-module) is uniquely the direct sum of a (unital) B-module and a vector space with zero B-multiplication.

Thus our problem is given a (unital) B-module V we wish to find a (unital) C-module V_1 such that V is the unital part of V_1 restricted to B.

Next let's look carefully at the Morita type relations between $\mathrm{Mod}(eCe)$ and $\mathrm{Mod}(C)$. We have functors

$$\mathrm{Mod}(eCe) \rightleftarrows \mathrm{Mod}(C)$$

$$X \longmapsto Ce \otimes_{eCe} X$$

$$eY = eC \otimes_C Y \longleftarrow Y$$

It's clear that the composition is the identity since

$$eCe = eC \otimes_C Ce$$

(use $C = Ce \oplus C(1-e)$). The other composition is the identity when

$$Ce \otimes_{eCe} eC \xrightarrow{\sim} C.$$

This certainly isn't the case when $CeC < C$ which is typical for the C's I am looking at.

Past work:
$$\begin{array}{ccc} Ce \otimes_{eCe} eC & \longrightarrow & CeC \\ \| & & \| \\ \tilde{I} & \xrightarrow{u} & I \end{array}$$

is the universal bimodule map $\ni$ $u(x)y = xu(y)$. see p 688, 673-674. Is $\tilde{I} = I$ in the case $C = A * \mathbb{C}[\mathbb{Z}/2]$?

Notice that the functor

$$Y \longmapsto eC \otimes_C Y = eY$$

simply takes the C-module Y restricts the C-multiplication to the non unital subalgebra eCe and then takes the unital part. Thus if we start with V over eCe, there is an obvious choice for $V_1 = Y$, namely $V_1 = Ce \otimes_{eCe} V$. This should actually amount to doubling V.

$$B = \mathbb{C}[T], \quad T = eFe$$

$$C = \mathbb{C}[F] * \mathbb{C}[\varepsilon] \qquad e = \frac{1+\varepsilon}{2}$$

$$Ce = eCe \oplus (1-e)Ce = B \oplus (1-e)Fe \cdot B$$

$$Ce \otimes_{eCe} V = eV \oplus ((1-e)Fe)V \simeq V^{\oplus 2}$$

$$Fev = eFev + (1-e)Fev \longleftrightarrow \begin{pmatrix} Tv \\ v \end{pmatrix}$$

$$\begin{aligned} F(1-e)Fev &= ev - FeFev \\ &= ev - e\underbrace{FeT v}_{T^2 v} - (1-e)FeTv \longleftrightarrow \begin{pmatrix} v - T^2 v \\ -Tv \end{pmatrix} \end{aligned}$$

So $F = \begin{pmatrix} T & 1-T^2 \\ 1 & -T \end{pmatrix}$ on $V^{\oplus 2}$.

Eigenspaces of F

$$\begin{pmatrix} T & 1-T^2 \\ 1 & -T \end{pmatrix}\begin{pmatrix} T+1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cancel{T^2}+T+1-\cancel{T^2} \\ \cancel{T}+1-\cancel{T} \end{pmatrix} = \begin{pmatrix} T+1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} T & 1-T^2 \\ 1 & -T \end{pmatrix}\begin{pmatrix} T-1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cancel{T^2}-T+1-\cancel{T^2} \\ T-1-T \end{pmatrix} = (-1)\begin{pmatrix} T-1 \\ 1 \end{pmatrix}$$

Now that I have carried the dilation process algebra, what do I do next?

Let's go back over the transgression problem with a view toward seeing whether our new ideas about superconnection forms are relevant.

Situation: M odd Riemannian compact manifold, S Clifford bundle on M, D_0 Dirac type operator on $H_0 = L^2(M,S)$. Let E be a vector bundle with inner product over M, let $\mathcal{G} = \mathrm{Aut}(E)$, $\mathcal{A}$ = connections on E. Given a connection A on E, then we have an associated Dirac D_A on $L^2(M, S\otimes E)$. The map $A \mapsto D_A$ is $\mathcal{G}$-equivariant.

There is therefore a canonical $\mathcal{G}$-equivariant family of Dirac operators on M parametrized by $\mathcal{A}$. The index of this family is a K-cohomology class on $\mathcal{A}_{\mathcal{G}} \sim B\mathcal{G}$. My goal is to apply the families index theorem to this situation. Really what I want to do is to describe the Chern character of the family index by a superconnection form.

For a superconnection form we need a bundle, connection D, and endomorphism X. From our new viewpoint the connection should come from an embedding into a trivial bundle.

Let's not worry about $B\mathcal{G}$ and instead work with a principal $\mathcal{G}$-bundle P over Y.

Our Hilbert bundle over Y is

$$P \times^G H \qquad H = L^2(M, S \otimes E)$$

and we want a skew-adjoint endom. on it, which means that we have an equivariant map from P to Dirac operators on H. We also need an embedding of $P \times^G H$ in a trivial Hilbert bundle, i.e. an equivariant map from P to embeddings of H in a fixed Hilbert space H'.

It seems the way to do this is to suppose we have an equivariant map from P to the embeddings of E ~~into~~ $\tilde{V}$ over M. Then we automatically get an embedding of $P \times^G H$ into the trivial bundle over Y with fibre $H' = L^2(M, S \otimes \tilde{V})$. Moreover given $i : E \hookrightarrow \tilde{V}$ then $i^*(\not{D}_0 \otimes 1_V) i$ is just the Dirac operator $\not{D}_A$ on $L^2(M, S \otimes E)$ where A is the connection $i^* d i$.

Thus we reach the following ~~situation~~. We have a Hilbert space $H' = L^2(M, S \otimes \tilde{V})$ with an unbounded operator $\not{D}_0 \otimes \mathrm{id}_V = \not{D}$. We consider a subbundle $\mathcal{H}$ of $\tilde{H}'_Y$. Let e be the endomorphism of $\tilde{H}'_Y$ given by the projection on $\mathcal{H}$. Then we consider the superconnection $e(d + \not{D}\sigma)e$ on $\mathcal{H}$.

May 15, 1988

We consider [redacted] a [redacted] Dirac operator $\not{D}_0$ on $H_0 = L^2(M,S)$ where M is odd-dimensional. [redacted] Let $e_y \in M_n(C^\infty(M))$ be a family of projectors over M parametrized by Y; equivalently let $e \in M_n(C^\infty(Y \times M))$. Then we have a family of Dirac operators on M parametrized by Y given by

$$\not{D}_y = e \not{D}_0 e \qquad \text{on} \qquad \mathcal{H}_y = e H_0^n$$

Moreover we have [redacted] the Grassmannian connection ∇ on the Hilbert bundle $\mathcal{H}$ over Y. Thus we have a superconnection on Y given by

$$\nabla = e d_y e \qquad \text{and} \qquad X [redacted] = e \not{D}_0 e$$

In other words one has taken the "constant" superconnection $d + \not{D}_0 \sigma$ on $\tilde{H}_0^n$ over Y and reduced it by the idempotent e.

[redacted] We have now a geometric way to look at the superconnection character forms as follows. We take the Cayley transform g of X. This gives a unitary operator on the bundle $\mathcal{H}$ and we extend it by $g=-1$ on the complementary bundle $(1-e)\tilde{H}_0^n$. This gives a map g from Y to the unitary group of H_0^n. Moreover it's a map to the space of unitaries congruent to -1 modulo some Schatten ideal. There [redacted] are certain universal superconnection character forms on this space of unitaries, and pulling these back gives the forms for $\nabla + X\sigma$ over Y.

The question arises as to whether (a why) this map

$$g : Y \longrightarrow -\mathcal{U}^p(H_0^n)$$

represents the index of the family of Dirac operators.

Now Atiyah-Singer define the index of the family as follows. One converts the family $\{\not{D}_y\}$ to a family of self-adjoint contractions with essential spectrum $\{\pm 1\}$

$$T_y = \frac{\frac{1}{i}\not{D}_y}{\sqrt{1 - \not{D}_y^2}}.$$

By Keiper's theorem the Hilbert bundle $\mathcal{H}$ is trivial so we obtain a map

$$Y \longrightarrow \mathcal{F}_1 \quad = \text{space of self-adj. contractions with essential spec. } \{+1,-1\}$$

They prove $\mathcal{F}_1$ is a classifying space for K^{odd}, so this map represents an element of $K^{odd}(Y)$, which they define to be the index.

Now it is easy to see that if we use their homotopy equivalence of $\mathcal{F}_1$ with the space of unitaries $\equiv -1 \bmod \mathcal{K}$, call this $-\mathcal{U}^\infty$, then our map $Y \longrightarrow -\mathcal{U}^\infty$ and there index map define the same element of $K^{odd}(Y)$. One uses the map $[-1,1] \longrightarrow$ unit circle

$$T \longmapsto \left(\sqrt{1-T^2} + iT\right)^2$$

to get the homotopy equivalence $\mathcal{F}_1 \longrightarrow -\mathcal{U}^\infty$, and if $T = \frac{\frac{1}{i}\not{D}}{\sqrt{1-\not{D}^2}}$, then this gives the Cayley transform of $\not{D}$.

Thus if we appeal to the AS defn of the index of the family, we see our map represents the index. Consequently our superconnection form represent the character of the index.

However [redacted] it appears that something important is missing. We started with $\not{D}_0$ on $H_0 = L^2(M,S)$, and this [redacted] operator represents an odd K-homology class of M. [redacted] Our map which assigns to e over $Y \times M$ the family $e\not{D}_0 e$ of operators over Y is the cap (or slant) product map

$$K^0(Y \times M) \longrightarrow K^1(Y)$$

with this K-homology class. Better, it's the integral in K-theory over the fundamental class of M.

At the moment I am very confused about foundations. Maybe the way to handle things is to use Kasparov theory, which is the accepted foundation for the [redacted] constructions we are using.

Kasparov theory takes [redacted] $C^\infty(M)$, $\overset{H_0}{L^2(M,S)}$, $\not{D}_0$ and replaces $\not{D}_0$ by the ΨDO of order zero

$$F_0 = \frac{\frac{1}{i}\not{D}_0}{\sqrt{1-\not{D}_0^2}}$$

and then $C^\infty(M)$ can be replaced by $C(M)$. I like to think of this geometrically as giving a splitting of H_0 modulo compacts.

Now suppose we are given e over $Y \times M$.

Geometrically we consider the Hilbert bundle $\widetilde{H}_0^n$ over Y and use the idempotent e to define a subbundle

$\mathcal{H} = e\,\widetilde{H}_0^n$.

Now we have to use F_0 on H_0^n to obtain an F on $\mathcal{H}$. The idea is that e and F_0 commute modulo compacts, and so there is a well-defined splitting of $\mathcal{H}$ modulo compacts. A specific choice of F is

$$F = eF_0e$$

We now reach the following problem. The operator $\not{D}_0$ on H_0 defines a splitting of H_0 modulo compacts represented by the contraction $F_0 = \dfrac{\frac{1}{i}\not{D}_0}{\sqrt{1-\not{D}_0^2}}$. The operator $e_y\not{D}_0e_y$ on $e_yH_0^n$ defines a splitting of $e_yH_0^n$ modulo compacts represented by the contraction

$$F_y = \frac{\frac{1}{i}\,e_y\not{D}_0e_y}{\sqrt{1-(e_y\not{D}_0e_y)^2}}$$

Is it true that $e_yF_0e_y$ and F_y are congruent modulo compacts?

This should follow just by looking at the symbols. These are both ψDO's of order zero with the same symbol.

But I would like a better version of this, and I hope that one might get an explicit link between these two using the massive Dirac operator. The idea here is that the massive Dirac:

$$\begin{pmatrix} \not{D}_0 & m \\ m & -\not{D}_0 \end{pmatrix} = \varepsilon \not{D}_0 + m\gamma^1$$

when reduced by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ always gives $\not{D}_0$, but as m becomes big its spectrum has a big gap.

May 16, 1988

Recall the formalism used in constructing Connes cocycles. One has a flat connection D and also a splitting F. Then one sets

$$\nabla = \frac{1}{2}(D + FDF) \qquad \alpha = \frac{1}{2}(D - FDF)$$

so $D = \nabla + \alpha$ [redacted] with ∇ commuting and α anticommuting with F. Then

$$0 = D^2 = \nabla^2 + \alpha^2 + [\nabla, \alpha]$$

whence [redacted] $[\nabla, \alpha] = 0$ and $\nabla^2 = -\alpha^2$. The relevant closed differential forms are

$$tr\,(F\alpha^{2n}) \qquad \text{ungraded case}$$

$$tr\,(\varepsilon\alpha^{2n+1}) \qquad \text{graded case}$$

To get cyclic cocycles we apply ~~this~~ to a representation of G on H, [redacted] where H is equipped with a splitting F_0. We consider the trivial bundle $\tilde{H}_G$ with connection $D = d$ and splitting $F = gF_0g^{-1}$, where g is the tautological autom of $\tilde{H}_G$. Equivalently we [redacted] use g as a gauge transf. and take

$$D = g^{-1}dg = d + \Theta$$

$$F = F_0$$

whence $\alpha = \frac{1}{2}(D - F_{[redacted]}DF_{[redacted]}) = \frac{1}{2}(\Theta - F\Theta F) = \frac{1}{2}F[F, \Theta]$.

Now we want to generalize this [redacted] to the case where F is replaced by a Dirac operator X. The idea is to construct a cyclic cocycle out of the operators $[X,\theta]$ and X^2. In fact we would like X^2 to enter into the formula as a smoothing type operator, e.g. $\frac{1}{\lambda - X^2}$ or e^{tX^2}. The first idea is to look at the superconnection character forms associated to $d+\theta+X\sigma$. The curvature is

$$(d+\theta+X\sigma)^2 = \theta X\sigma + \overbrace{X\sigma\theta}^{-\theta\sigma} + X^2$$
$$= [\theta, X]\sigma + X^2$$

The problem with the superconnection forms obtained in this situation is that they are the wrong parity. Let's be precise. We are looking at the superconnection on $\tilde{H}$ over $\mathcal{G}$ consisting of the connection + endomorphism $d+\theta, X$ or $d,\ gXg^{-1}$.

Thus the superconnection forms are the pull-back by the map $\mathcal{G} \longrightarrow$ unitaries, given by $g\frac{1+X}{1-X}g^{-1}$, of certain forms on the the unitary group. These **universal** forms on the unitary group are of odd degree.

The problem is related to the fact that the even forms on the Grassmannian are $tr(F\nabla^{2n})$ not $tr(\nabla^{2n})$ which is zero

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Now that we understand Connes cocycles in a fashion, let us try to prove his theorem that the consecutive forms are related by the S-operator. From his viewpoint one takes the initial situation A, H, F and tensors it with a standard 2-diml "cycle" given by $A = \mathbb{C}[e]$, $H = \mathbb{C}^2$, $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Tensoring means we consider $\tilde{H} = \mathbb{C}^2 \otimes H$ with the involution roughly

$$\tilde{F} : \quad \varepsilon \otimes F + \gamma' \otimes 1$$

To get an involution one has to pick a "mixing angle":

$$\tilde{F} = \cos(\phi)\, \varepsilon \otimes F + \sin(\phi)\, \gamma' \otimes 1$$

$$= \begin{pmatrix} \cos\phi\, F & \sin\phi \\ \sin\phi & -\cos\phi\, F \end{pmatrix}$$

A acts on $\tilde{H}$ via $ae = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Thus

$$\tilde{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$[\tilde{F}, \tilde{\theta}] = \begin{pmatrix} \cos\phi\, [F, \theta] & -\sin\phi\, \theta \\ \sin\phi\, \theta & 0 \end{pmatrix}$$

We consider $\operatorname{tr}\left(\tilde{F}[\tilde{F}, \tilde{\theta}]^{2n}\right)$. If $\phi = 0$, we get $\operatorname{tr}\left(F[F,\theta]^{2n}\right)$. It looks like the term 2nd order in ϕ is $S \operatorname{tr}\left(F[F,\theta]^{2n-2}\right)$.

Start again and put $t = \tan(\phi)$ so that

$$\tilde{F} = \cos\phi \begin{pmatrix} F & t \\ t & -F \end{pmatrix}$$

We consider in the ungraded case

$$\operatorname{tr} \tilde{F}[\tilde{F},\tilde{\theta}]^{2n} = 2 \operatorname{tr} \tilde{\theta}[\tilde{F},\tilde{\theta}]^{2n-1} \qquad \text{for } n \geq 1$$

and in the graded case

$$\operatorname{tr} \varepsilon \tilde{F}[\tilde{F},\tilde{\theta}]^{2n+1} = 2 \operatorname{tr} \varepsilon\tilde{\theta}[\tilde{F},\theta]^{2n} \qquad \text{for } n \geq 0$$

To handle both we consider the generating fn.

$$\sum_{n\geq 0} \frac{1}{\lambda^{n+1}} \tilde{\theta}[\tilde{F},\tilde{\theta}]^n = \sum_{n \geq 0} \frac{(\cos\phi)^n}{\lambda^{n+1}} \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} [F,\theta] & -t\theta \\ t\theta & 0 \end{pmatrix}^n$$

$$= \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\lambda} \frac{1}{1 - \frac{\cos\phi}{\lambda}\begin{pmatrix} [F,\theta] & -t\theta \\ t\theta & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\underbrace{\lambda - \begin{pmatrix} \cos\phi[F,\theta] & -\sin\phi\,\theta \\ \sin\phi\,\theta & 0 \end{pmatrix}}}$$

$a = \lambda - \cos\phi[F,\theta]$
$b = \sin\phi\ \theta$
$c = -\sin\phi\ \theta$
$d = \lambda$

$$\begin{pmatrix} 1 & 0 \\ +\frac{(\sin\phi)\theta}{\lambda} & 1 \end{pmatrix} \begin{pmatrix} (\lambda - \cos\phi[F,\theta] + \frac{1}{\lambda}\sin^2\phi\ \theta^2)^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\frac{\sin\phi}{\lambda}\theta \\ 0 & 1 \end{pmatrix}$$

$$\sum_{n\geq 0} \frac{1}{\lambda^{n+1}} \operatorname{tr}\left(\tilde{\theta}[\tilde{F},\tilde{\theta}]^n\right) = \operatorname{tr}\left(\theta \frac{1}{\lambda - \cos\phi[F,\theta] + \frac{1}{\lambda}\sin^2\phi\ \theta^2}\right)$$

and similarly with $\operatorname{tr}(\varepsilon \cdots)$ in the graded case

From the homotopy property established earlier we know that the cohomology class of $\mathrm{tr}(\tilde{\Theta}[\tilde{F},\tilde{\Theta}]^n)$ is independent of [redacted] ϕ. The above formula turns out not be very useful because there are two terms depending on ϕ.

Instead we should look at

$$\mathrm{tr}\left\{\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\left[\begin{pmatrix}F & t\\ t & -F\end{pmatrix},\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\right]^{2n}\right\} = (1+t^2)^n\,\mathrm{tr}\left(\tilde{\theta}[\tilde{F},\tilde{\theta}]^{2n}\right)$$

The cohomology class of the cocycle on the right is $(1+t^2)^n$ class of $\mathrm{tr}(\Theta[F,\Theta]^{2n})$. ~~[illegible]~~ The coefficient of t^{2k} on the left should be (const) $S^k\ \mathrm{tr}(\Theta[F,\Theta]^{2n-2k})$ yielding the desired relations.

Let's check this out using the generating functions

$$\sum \mathrm{tr}\left\{\begin{pmatrix}\Theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\Theta] & -\Theta t\\ t\Theta & 0\end{pmatrix}^n\right\}\frac{1}{\lambda^{n+1}}$$

$$= \mathrm{tr}\left\{\begin{pmatrix}\Theta & 0\\ 0 & 0\end{pmatrix}\frac{1}{\lambda - \begin{pmatrix}[F,\Theta] & -\Theta t\\ t\Theta & 0\end{pmatrix}}\right\}$$

$a = \lambda - [F,\Theta]$
$b = \Theta t$
$c = -t\Theta$
$d = \lambda$

$$= \mathrm{tr}\left\{\Theta\,\frac{1}{\lambda - [F,\Theta] - \Theta t\,\frac{1}{\lambda}(-t\Theta)}\right\}$$

$$= \mathrm{tr}\left\{\Theta\,\frac{1}{\lambda - [F,\Theta] + \frac{t^2}{\lambda}\Theta^2}\right\}$$

[redacted] Now if we compare this with May 7 we see that the coefficient of t^2 in this is $-S$ applied to $\mathrm{tr}\,\Theta[F,\Theta]^{2n-2}$, and so this S-transform is cohomologous to $n\, tr(\theta[F,\theta]^{2n})$.

At this point we have a mechanism for understanding the S-transforms of the Connes cocycles $tr(F[F,\theta]^{2n})$, $tr(\varepsilon F[F,\theta]^{2n+1})$

May 17, 1988

Consider a flat connection $d+\theta$ on the trivial bundle. Then we have two flat connections d, $d+\theta$ on the same bundle. The difference of the character forms for these two connection vanishes for two reasons, flatness and the general fact that the cohomology class of the character form is independent of the connection. This leads to odd forms.

The usual way of constructing the odd forms is to join the connections by a linear path

$$D_t = d + t\theta \qquad D_t^2 = (t^2-t)\theta^2$$

which gives

$$0 = tr(e^{uD_1^2}) - tr(e^{uD_0^2}) = d\int_0^1 dt\, tr\left(u\theta\, e^{u(t^2-t)\theta^2}\right)$$

The coefficient of u^{k+1} in the integral is

$$tr(\theta^{2k+1}) \underbrace{\int_0^1 \frac{1}{k!}(t^2-t)^k\, dt}$$

$$= \frac{1}{k!}(-1)^k \int_0^1 t^k(1-t)^k dt = \frac{(-1)^k}{k!} \frac{\Gamma(k+1)\Gamma(k+1)}{\Gamma(2k+2)} = \frac{(-1)^k k!}{(2k+1)!}$$

Thus $\boxed{\dfrac{(-1)^k k!}{(2k+1)!}\, tr(\theta^{2k+1})}$ are the odd forms.

Another way to obtain odd forms is to use superconnections. More generally given two connection D_1, D_0 on the same vector bundle E, we form the super vector bundle $E \oplus E$ with the connection $D = D_1 \oplus D_0$ and the odd endomorphism $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This gives the superconnection

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_0 \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma$$

depending on t. For $t=0$ we get the difference of the character forms for D_1 and D_0, and for $t=\infty$ the superconnection character forms are zero. Thus the homotopy from $t=0$ to $t=\infty$ gives an odd form whose differential is $tr(e^{uD_1^2}) - tr(e^{uD_0^2})$.

So consider this construction when $D_0 = d$ and $D_1 = d + \theta$. We have the family of superconns.

$$\tilde{D}_t = d + \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma$$

with curvatures

$$-t^2 + t\Big(\underbrace{\begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sigma + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sigma\begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}}_{\left[\begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]\sigma \;=\; -\theta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sigma}\Big)$$

$$\therefore \quad \tilde{D}_t^2 = -t^2 - t(\theta\sigma)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$e^{u\tilde{D}_t^2} = e^{-ut^2} e^{-ut(\theta\sigma)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

Now $(\theta\sigma)^2 = -\theta^2$ so the second exponential will give cosine and sine series.

$$e^{u\tilde{D}_t^2} = e^{-ut^2}\begin{pmatrix} \cos(ut\theta) & -\sin(ut\theta)\sigma \\ -\sin(ut\theta)\sigma & \cos(ut\theta) \end{pmatrix}$$

Our transgression formula is then

$$0 = tr_\varepsilon\left(e^{u\tilde{D}_\infty^2}\right) - tr_\varepsilon\left(e^{u\tilde{D}_0^2}\right) = -d\int_0^\infty dt\; tr_\varepsilon\left(u\,\text{[redacted]}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sigma\, e^{u\tilde{D}_t^2}\right)$$

And the integral on the right is the negative of $\left[\text{because } \varepsilon\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$

$$\int_0^\infty dt\; tr\left(u\,\text{[redacted]}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sigma\; e^{-ut^2}\begin{pmatrix} \cos(ut\theta) & -\sin(ut\theta)\sigma \\ -\sin(ut\theta)\sigma & \cos(ut\theta) \end{pmatrix}\right)$$

$$= \int_0^\infty dt\; e^{-ut^2}\, u\,\text{[redacted]}\; tr\Big\{ 2\underbrace{\sin(ut\theta)}_{\text{[redacted]}\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)!}(ut\theta)^{2k+1}}$$

$$= \sum_{k=0}^\infty u^{2k+2}\, tr\left(\theta^{2k+1}\right)\underbrace{\int_0^\infty dt\, \frac{(-1)^k 2}{(2k+1)!}\, t^{2k+1}\, e^{-ut^2}}$$

$$= \int_0^\infty dt\, \frac{(-1)^k}{(2k+1)!}\, t^k e^{-ut} = \frac{(-1)^k}{(2k+1)!}\,\frac{\Gamma(k+1)}{u^{k+1}}$$

$$= \sum_{k=0}^\infty u^{k+1}\, tr\left(\theta^{2k+1}\right)\frac{(-1)^k k!}{(2k+1)!}$$

Now the next step will be to go back and do the analogue of the formula

$$tr\left(e^{ud\theta} - 1\right) = (\delta + d)\int_0^1 dt\; tr\left\{u\theta\, e^{u(\ldots\theta + \text{[redacted]}(t^2-t)\theta^2)}\right\}$$

The setting is as follows. We have the trivial bundle over $G \times M$ with the two connections $\delta + d$ and $\delta + d + \theta$ where $d = d_M$, $\delta = d_{\mathfrak{g}}$. If we use the linear path $D_t = \delta + d + t\theta$, then

$$D_t^2 = t\,d\theta + (t^2 - t)\theta^2 \qquad \text{as } \delta\theta = -\theta^2$$

and we obtain the above transgression formula. Instead we want to look at the family of superconnections

$$\tilde{D}_t = \delta + d + \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma$$

$$\tilde{D}_t^2 = \begin{pmatrix} d\theta & 0 \\ 0 & 0 \end{pmatrix} - t\,\theta\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t^2$$

which will give the formula

$$-\mathrm{tr}\left(e^{u\,d\theta} - 1\right) = (\delta + d) \int_0^\infty dt \; u\, \mathrm{tr}_s \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma\, e^{u \tilde{D}_t^2} \right]$$

Calculation: Suppose we apply superconnection methods to derive a transgression form like the Chern-Simons form which is based on the path of connections $d + tA$ with curvature $t\,dA + t^2 A^2$. Then we form the family of superconnection

$$\tilde{D} = d + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma$$

with curvature

$$\tilde{D}^2 = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} + (-tA\sigma) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t^2$$

The transgression formula is

$$\operatorname{tr}\left(e^{uF}-1\right) = -d\int_0^\infty dt\, \operatorname{tr}\left\{\varepsilon u \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma e^{u\tilde{D}^2}\right\}$$

Now we divide by u and use the L.T.

$$\int_0^\infty e^{-\lambda u}\, \frac{e^{uF}-1}{u}\, du = \sum_{k\geq 1} \frac{F^k}{k!} \underbrace{\int_0^\infty e^{-\lambda u} u^{k-1} du}_{\frac{(k-1)!}{\lambda^k}}$$

$$\therefore \int_0^\infty e^{-\lambda u} \operatorname{tr}\left(\frac{e^{uF}-1}{u}\right) du = \sum_{k\geq 1} \frac{1}{k\lambda^k} \operatorname{tr}(F^k)$$

$$\int_0^\infty e^{-\lambda u} du \int_0^\infty dt\, \operatorname{tr}\left\{\varepsilon \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sigma\, e^{u\tilde{D}^2}\right\}$$

$$= -\int_0^\infty dt\, \operatorname{tr}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma \underbrace{\frac{1}{\lambda - \tilde{D}^2}}\right\}$$

$$= \frac{1}{\lambda + t^2 - \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} + tA\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \lambda + t^2 - F & tA\sigma \\ tA\sigma & \lambda + t^2 \end{pmatrix}^{-1}$$

Recall

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix} \begin{pmatrix} (a-bd^{-1}c)^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}$$

$$\operatorname{tr} \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \operatorname{tr}\left(\sigma \overbrace{(-d^{-1}c)}^{\frac{tA\sigma}{\lambda+t^2}} \frac{1}{a-bd^{-1}c}\right) + \operatorname{tr}\left(\sigma \frac{1}{a-bd^{-1}c} \overbrace{(-bd^{-1})}^{\frac{tA\sigma}{\lambda+t^2}}\right)$$

$$a - bd^{-1}c = \lambda + t^2 - F - tA\sigma \frac{1}{\lambda + t^2} tA\sigma$$

$$= \lambda + t^2 - F + \frac{t^2}{\lambda + t^2} A^2 = \lambda + t^2 - dA - \frac{\lambda}{\lambda + t^2} A^2$$

$a - bd^{-1}c$ is an even form, so it commutes with σ, and so we find

$$\mathrm{tr}\,\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix}(\lambda-\tilde{D}^2)^{-1} = 2\,\mathrm{tr}\left\{\frac{tA}{\lambda+t^2}\,\frac{1}{\lambda+t^2-dA-\frac{\lambda}{\lambda+t^2}A^2}\right\}$$

So we want

$$\int_0^\infty dt\,2t\,\mathrm{tr}\left\{A\,\frac{1}{(\lambda+t^2)^2-(\lambda+t^2)dA-\lambda A^2}\right\} \qquad s=\lambda+t^2$$

$$=\int_\lambda^\infty \frac{ds}{s}\,\mathrm{tr}\left\{A\,\frac{1\cdot s}{s^2-s\,dA-\lambda A^2}\right\}$$

$$=\int_0^{\lambda^{-1}} \frac{ds}{s}\,\mathrm{tr}\left\{A\,\frac{s^{-1}}{s^{-2}-s^{-1}dA-\lambda A^2}\right\}$$

$$=\int_0^{\lambda^{-1}} ds\,\mathrm{tr}\left(A\,\frac{1}{1-s\,dA-s^2\lambda A^2}\right) \qquad s=\lambda^{-1}t$$

$$=\int_0^1 \frac{dt}{\lambda}\,\mathrm{tr}\left(A\,\frac{1}{1-\lambda^{-1}t\,dA-\lambda^{-2}t^2\lambda A^2}\right)=\int_0^1 dt\,\mathrm{tr}\left(A\,\frac{1}{\lambda-t\,dA-t^2A^2}\right)$$

$$=\sum_{k\geq 1}\frac{1}{\lambda^k}\int_0^1 dt\,\mathrm{tr}\left\{A\left(t\,dA+t^2A^2\right)^{k-1}\right\}.$$

Thus we get exactly the Chern-Simons forms.

The previous calculation shows quite generally that if we go between a flat connection and an arbitrary connection by either the linear path or the superconnection, then we get the same transgression forms. In particular we see in the case of $\delta+d+\theta$ and $\delta+d$ that the transgression forms ~~two~~ three pages are the same, i.e.

$$-\int_0^\infty dt\ u\ \mathrm{tr}_s\left\{\varepsilon\begin{pmatrix}0&-1\\1&0\end{pmatrix}\sigma\, e^{u\overbrace{\tilde{D}^2}^{-t^2+\begin{pmatrix}d\theta & t\sigma\theta\\ t\sigma\theta & 0\end{pmatrix}}}\right\} = \int_0^1 dt\ u\ \mathrm{tr}\left\{\theta\, e^{u[t d\theta + (t^2-t)\theta^2]}\right\}$$

Let's discuss some general issues. We still have to organize lectures at Montreal. I propose to start with cyclic cocycles on $C^\infty(M)$. Exhibit Connes cocycles associated to closed currents on M. Exhibit others obtained by vector bundles with connections. This gives some feeling for the S-operator. One seems to [redacted] get formulas for the S iterates of the cocycles associated to closed currents, although it is not clear that [redacted] there is a basic operator S on cocycles.

A problem is to show that the S-transforms of [redacted] a closed current on M depend up to coboundaries only on the homology class of the current.

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to go in outline

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cyclic cocycles on $C^\infty(M)$	840
Connes cocycles assoc. to flat connection with F	849–53
Derivation of a transgression form using super-connection methods leads to usual Chern-Simons	873–75
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May 19, 1988

BRS-geometry. Atiyah has a possible interpretation of Witten's Lagrangian for topological QFT, which uses equivariant differential forms for G-manifolds. It is necessary to make precise Witten's formulas in the finite dimensional context. One one hand this should lead to an understanding of the BRS formalism, and on the other hand it might have applications to G-manifolds where the action is not free.

Let then G be a compact connected Lie group acting on a manifold M. We suppose the action is free. In this case M is a principal G-bundle over M/G. If we choose a connection $\theta \in \Omega^1(M,\mathfrak{g})^G$ in this principal bundle then we get a Chern-Weil homomorphism

$$\circledast \qquad \{S(\mathfrak{g}^*)\otimes\Omega(M)\}^G \xleftarrow{\sim} \{W(\mathfrak{g})\otimes\Omega(M)\}_{\text{basic}} \longrightarrow \Omega(M/G)$$

from equivariant forms on M to forms on M/G.

In the Witten approach one supposes a metric given on M which is G-invariant. We can then define a $\mathfrak{g}^*$ valued 1-form $\tilde{\theta}\in\Omega^1(M,\mathfrak{g}^*)^G$, which is G invariant, in the following way. Given $X \in \mathfrak{g}$, then inner product with the vector field on M associated to X by the $\mathfrak{g}$-action is a 1-form. Put another way the metric transforms the vector field to a 1-form. If $g \in \Gamma(S^2T^*)$ is the metric tensor, then the 1-form is the contraction $X \lrcorner g \in \Gamma(T^*)$. Thus we get a map $\mathfrak{g} \longrightarrow \Omega^1(M)$ which is G-invariant, whence a 1-form $\tilde{\Theta} \in \Omega^1(M, \mathfrak{g}^*)^G$.

Note that $\tilde{\Theta}$ can be defined even when the action isn't free. Thus if we can write a formula for the Chern-Weil homomorphism in terms of $\tilde{\Theta}$, it might have applications in the non-free case.

Returning to our free situation, let's ask what an invariant metric on M looks like. First of all such a metric determines a connection in the principal bundle M over M/G; the horizontal space for the connection is the orthogonal complement of the vertical space. Secondly the metric restricted to the horizontal subbundle of TM descends to a metric on M/G. Thirdly at each point of M the metric restricted to the vertical tangent space gives an inner product on $\mathfrak{g}$. Thus we get an equivariant map from M to the positive cone in $S^2(\mathfrak{g}^*)$. Let's denote this by

$$\rho \in \Omega^0(M, S^2(\mathfrak{g}^*))^G$$

Conversely given a metric on M/G, a connection $\theta \in \Omega^1(M, \mathfrak{g})$ in M over M/G, and an equivariant map ρ from M to inner products on $\mathfrak{g}$, we get an invariant metric on M.

Notice that ρ is defined in the case of a non-free action, but its values are only positive semi-definite. Also relative to the contraction map $\mathfrak{g} \otimes S^2(\mathfrak{g}^*) \longrightarrow \mathfrak{g}^*$ we have the formula

$$\tilde{\Theta} = \Theta \lrcorner \rho$$

To fix the ideas we consider the simple case where ρ is constant. In other words we fix a invariant inner product on $\mathfrak{g}$, and we use this to define the metric in the vertical direction. This inner product identifies $\mathfrak{g}$ and $\mathfrak{g}^*$, and then relative to this identification we have $\Theta = \tilde{\Theta}$. Let's fix an orthonormal basis X_a for $\mathfrak{g}$, whence the connection has the components $\Theta^a \in \Omega^1(M)$, i.e. $\Theta = X_a \Theta^a$ and the curvature is $\Omega = X_a \Omega^a$ where

$$\Omega^a = d\Theta^a + \tfrac{1}{2} f^a_{bc} \Theta^b \Theta^c \qquad [X_b, X_c] = f^a_{bc} X_a$$

Now we are after a formula for the Chern-Weil homomorphism $\circledast$ on p 877.

~~According to Atiyah-Witten there is a formula for this which uses a double integral over $\mathfrak{g} \times \mathfrak{g}$~~ What one wants to do is to start with an equivariant function $\alpha(\omega)$ ($\omega \longmapsto$) from $\mathfrak{g}$ to forms on M and then produce a basic form on M. The first stage is produce from $\alpha \in S(\mathfrak{g}^*) \otimes \Omega(M)$ an element of $W(\mathfrak{g}) \otimes \Omega(M)$ which is horizontal. I recall the formula from my paper with Mathai

$$\left(\prod_{a=1}^{n} \iota_a \hat{\Theta}^a\right) \alpha(\hat{\Omega}) = \prod_{a=1}^{n} (1 - \hat{\Theta}^a i_a)\, \alpha(\hat{\Omega})$$

where $\hat{\theta}^a$, $\hat{\Omega}^a$ refer to the generators of $W(\mathfrak{g})$. The next stage is to ~~substitute~~ $\hat{\theta}^a \longmapsto \theta^a$, $\hat{\Omega}^a \longmapsto \Omega^a$.

Now according to Atiyah, Witten has a formula for doing this which uses [redacted] a double integration over $\mathfrak{g}$. **My guess is** this double integral is essentially two Fourier transforms which [redacted] by the Fourier inversion formula amounts to the ~~substitution~~ $\omega \longmapsto \Omega$.

The problem is to figure out Witten's formula which should give the map

$$* \quad \{S(\mathfrak{g}^*) \otimes \Omega(M)\}^G \longrightarrow \Omega(M)_{\text{basic}}$$

Take the simplest case where $G = S^1$ and the metric on M is such that all orbits have same length, i.e. M is a unit circle bundle in a complex line bundle with inner product over M/G. In this case we have the Atiyah-Bott model

$$\begin{aligned}\{S(\mathfrak{g}^*) \otimes \Omega(M)\}^G &= S(\mathfrak{g}^*) \otimes \Omega(M)^G \\ &= \mathbb{C}[u] \otimes \Omega(M)^G\end{aligned}$$

and the map $*$ above sends u to the curvature $d\theta$ and $\alpha \in \Omega(M)^G$ to

$$\alpha - \theta \iota_X \alpha$$

Now it seems to me that there is no way to adjoin variables to $\Omega(M)$ and manipulate algebraically (fermion integration + integrating out the extra variables) that will give $\iota_X \alpha$ from α.

Thus we must it seems pull back α via the map $M \times G \to M$.

This leads me to suspect that in general one needs a similar action map

$$M \times \mathfrak{g} \xrightarrow{1 \times \exp} M \times G \longrightarrow M$$

and that also, as Atiyah said, one must do some sort of limiting process that concentrates the measure near the origin of $\mathfrak{g}$, where the exponential map has least distortion. (Vergne-Berline ?)

Maybe it's not so involved. What we have to understand is the operation which take a form $\alpha \in \Omega(M)$ and projects it onto the horizontal forms. This is the operation

$$\alpha \longmapsto \left(\prod_{a=1}^{n} i_a \theta^a \right) \alpha = \prod_{a=1}^{n} \left(1 - \theta^a i_a \right) \alpha$$

and it's an algebra homomorphism. To see this note we have an algebra isomorphism

$$\Omega(M) \xleftarrow{\sim} \Omega(M)_{hor} \otimes \Lambda \mathfrak{g}^*$$

and we are just picking out the degree zero component, where degree refers to the q in $\Lambda^q \mathfrak{g}^*$.

For example what is the horizontal part of $d\theta$? It's clearly the curvature, since one knows that the curvature is horizontal, and since the horizontal projection of θ is zero.

Consider the map

$$\Omega(M) \xrightarrow[\substack{\text{induced by} \\ \text{action } M \times G \to M}]{} \Omega(M \times G) = \Omega(M) \otimes \Omega(G) \xrightarrow[\substack{\text{induced by evaluating} \\ \text{a form on } G \text{ at the identity.}}]{} \Omega(M) \otimes \Lambda \mathfrak{g}^*$$

This is a ring homomorphism in fact a map of cochain algebras by construction. NO Let $\lambda^a \in \mathfrak{g}^*$ be a dual basis to $X_a \in \mathfrak{g}$. This map should be

$$\alpha \longmapsto e^{\sum \lambda^a i_a} \alpha$$

essentially. Next we have a homomorphism of Algebras

$$\Lambda \mathfrak{g}^* \longrightarrow \Omega(\mathfrak{g}) \qquad \lambda^a \longmapsto d\lambda^a$$

which is not compatible with differentials.

Note that $\Omega(G) \longrightarrow \Lambda \mathfrak{g}^*$ doesn't respective differentials.

May 20, 1988

Let M be a free G-manifold. We are trying to obtain ~~the~~ formula which Witten seems to have produced for the map (Chern-Weil homomorphism)

$$\Omega_G(M) = \{S(\mathfrak{g}^*) \otimes \Omega(M)\}^G \longrightarrow \Omega(M/G)$$

associated to a connection in M over M/G. The idea to explore is the following. Suppose we consider the G-map

$$\mu : M \times G \longrightarrow M \qquad (m,g) \longmapsto mg$$

where $M \times G$ is the trivial principal G-bundle over M (i.e. G acts trivially on M and by right multiplication on G). We have a commutative square

$$\begin{array}{ccc} \{S(\mathfrak{g}^*) \otimes \Omega(M \times G)\}^G & \longrightarrow & \Omega(M) \\ \uparrow & & \uparrow \\ \{S(\mathfrak{g}^*) \otimes \Omega(M)\}^G & \longrightarrow & \Omega(M/G) \end{array}$$

So we are reduced to finding a formula for the Chern-Weil homomorphism associated to the trivial G-bundle $M \times G$, but for the connection which is the pullback of the connection in M over M/G by the map μ.

Note

$$S(\mathfrak{g}^*) \otimes \Omega(M \times G) = S(\mathfrak{g}^*) \otimes \underbrace{\Omega(G)}_{\Lambda \mathfrak{g}^* \otimes \Omega^0(G)} \otimes \Omega(M)$$

and so

$$\{S(\mathfrak{g}^*)\otimes\Omega(M\times G)\}^G \cong S(\mathfrak{g}^*)\otimes\Lambda(\mathfrak{g}^*)\otimes\Omega(M)$$

Actually the Chern-Weil map

$$\{S(\mathfrak{g}^*)\otimes\Omega(M)\}^G \longrightarrow \Omega(M/G) = \Omega(M)_{bas}$$

can be described as follows. The connection gives an isomorphism

$$\Omega(M) \xleftarrow{\ \sim\ } \Lambda\mathfrak{g}^*\otimes\Omega(M)_{hor}$$

and then the Chern-Weil map results by taking G-invariants for the map

$$S(\mathfrak{g}^*)\otimes\Omega(M) \cong S(\mathfrak{g}^*)\otimes\Lambda\mathfrak{g}^*\otimes\Omega(M)_{hor} \xrightarrow{\ u\ } \Omega(M)_{hor}$$

where u on $S^1(\mathfrak{g}^*)$ is the curvature and u on $\Lambda^1\mathfrak{g}^*$ is zero.

Consider the Chern-Weil map

$$\{S(\mathfrak{g}^*)\otimes\Omega(P)\}^G \longrightarrow \Omega(B)$$

in the case of a principal G-bundle P/B equipped with connection and section. The section gives an isom. $P = B\times G$, and so the alg of equivariant forms is isom. to

$$\underbrace{\{S(\mathfrak{g}^*)\otimes\Omega(G)\}^G}_{W(\mathfrak{g})}\otimes\Omega(B)$$

Thus one needs a map $W(\mathfrak{g})\longrightarrow\Omega(B)$. But the difference of the two connections on P, or better just the pull-back of the connection form via the section gives us a collection η^a of 1-forms on $\Omega(B)$. Since $W(\mathfrak{g})$ is the free cochain algebra generated by the θ^a, we ~~have~~ then a map $W(\mathfrak{g}) \longrightarrow \Omega(B)$.

The question is whether there is a kind of integral formula for this sort of map.

<u>May 21, 1988:</u>

Here seems to be an example of <u>gauge-fixing</u>.

Let P be a principal G-bundle over B and let V be a representation of G. If $s : B \to P$ is a section (choice of gauge), then we have an isomorphism $B \times G \xrightarrow{\sim} P$ and so

$$\left(\Omega(P) \otimes V\right)^G = \Omega(B) \otimes \left(\Omega(G) \otimes V\right)^G = \Omega(B) \otimes \Lambda \mathfrak{g}^* \otimes V$$

Notice that $\left(\Omega(G) \otimes V\right)^G = \Lambda \mathfrak{g}^* \otimes V$ is the complex of Lie algebra cochains on $\mathfrak{g}$ with values in the $\mathfrak{g}$-module V, denoted $C(\mathfrak{g}, V)$. It is the free module over $C(\mathfrak{g}) = \Lambda \mathfrak{g}^* = \Omega(G)^G$ generated by V.

To describe the differential, let choose a basis X_a for $\mathfrak{g}$, $[X_a, X_b] = f_{ab}^c X_c$, and let χ^a denote the dual basis for $\mathfrak{g}^*$. Then $C(\mathfrak{g}) = \Omega(G)^G$ is the exterior algebra on the χ^a with differential given by

$$\chi = \chi^a X_a$$

$$\begin{aligned} d\chi &= -\chi^2 \\ &= -\tfrac{1}{2}[\chi, \chi] \end{aligned}$$

i.e.
$$d\chi^a = -\tfrac{1}{2} f_{bc}^a \chi^b \chi^c$$

The differential on $C(\mathfrak{g}, V) = C(\mathfrak{g}) \otimes V$ is determined by requiring $C(\mathfrak{g}, V)$ to be a DG module over $C(\mathfrak{g})$ such that

$$dv = \chi^a (X_a v)$$

Let's check this formula. We are using the isomorphism $(\Omega(G) \otimes V)^G \xrightarrow{\sim} \Lambda \mathfrak{g}^* \otimes V$ given by restricting a form to the tangent space at the identity. Given $v \in V$ the function on G corresponding to it under $(\Omega^0(G) \otimes V)^G = V$ is the function $f_v : x \longmapsto x^{-1} v$. In effect we must check the invariance

$$\begin{aligned}(g \cdot f_v)(x) &= g\, f_v(xg) \\ &= g\,(xg)^{-1} v \\ &= f_v(x).\end{aligned}$$

g acts to the left on V and on the right on G.

What is df_v, or rather the image of $df_v \in (\Omega^1(G) \otimes V)^G$ under the isomorphism to $\mathfrak{g}^* \otimes V$.

Unfortunately I have got left + right mixed up. The Maurer-Cartan form $\chi = g^{-1} dg$ is a left-invariant differential form, or rather the components χ^a are invariant under left multiplication.

The way to rectify this is to use the map $B \times G \longrightarrow P$, $(x g) \longmapsto s(x) g^{-1}$, so that right G-multiplication by g on P corresponds to left G-mult. by g^{-1} on G. Then $\Omega(G)^G$ is the complex of left-invariant forms on G and it is the exterior alg. on the χ^a with $d\chi^a = -\frac{1}{2} f^a_{bc} \chi^b \chi^c$.

Now consider $(\Omega(G)\otimes V)^G$ where G acts on itself by left multiplication. Given $v\in V$, let $f_v : G \longrightarrow V$ be the function $f_v(g) = gv$. This is invariant:

$$(g\cdot f_v)(x) = g\left(f_v(g^{-1}x)\right) = gg^{-1}xv = xv = f_v(x).$$

It's differential is

$$(df_v)(g) = f_v(g+dg) - f_v(g)$$

$$= dg\, v = g\,(g^{-1}dg)\,v = g\,\chi^a X_a v$$

Thus $d: \underset{\left(\Omega^0(G)\otimes V\right)^G}{\overset{\|}{V}} \longrightarrow \underset{\left(\Omega^1(G)\otimes V\right)^G}{\overset{\|}{\mathfrak{g}^*\otimes V}}$ is

$$dv = \chi^a X_a v \qquad \text{i.e. } (\delta v)(X) = Xv$$

Check

$$0 \overset{?}{=} d^2 v = d\left(\chi^a X_a v\right) = d\chi^a\cdot X_a v - \chi^a d(X_a v)$$

$$= d\chi^c X_c v - \chi^a\chi^b X_b X_a v$$

$$= d\chi^c X_c v + \tfrac{1}{2}\chi^a\chi^b [X_a, X_b] v$$

$$= \left(d\chi^c + \tfrac{1}{2}\chi^a\chi^b f_{ab}^{\;c}\right) X_c v = 0$$

Let's now apply this to the case of the equivariant forms on G. This means we are considering the ~~complex~~ cochain algebra

$$\left(S(\mathfrak{g}^*)\otimes\Omega(G)\right)^G$$

where G acts on itself by left-multiplication, and where the differential is $d = d_G - \omega^a \iota_a$.

What is i_X on $\Omega(G)^G = \Lambda \mathfrak{g}^*$. It is interior product with respect to infinitesimal left multiplication by $(1-\varepsilon X)$. Hence $i_a \chi^b = -\delta^b_a$. (According to our conventions G acts to the right and then $i_X, \mathcal{L}_X$ are defined by the transpose of multiplying by e^{tX}.)

Thus in the algebra $(S(\mathfrak{g}^*) \otimes \Omega(G))^G$ we have

$$d\chi^a = d_G \chi^a - \omega^b i_b \chi^a = -\tfrac{1}{2}\chi^b\chi^c f^a_{bc} + \omega^a$$

which means we have the Weil algebra.

May 22, 1988

Go back to the beginning. We start with a principal G-bundle $M \longrightarrow M/G$ with a connection form θ. There is then a Chern-Weil homomorphism

$$\{S(\mathfrak{g}^*) \otimes \Omega(M)\}^G \longrightarrow \Omega(M/G) = \Omega(M)_{bas}$$

which we know is given by the formula

$$\alpha(\omega) \longmapsto :e^{-\theta^a \iota_a}: \alpha(\Omega).$$

Our problem is to discover the "integral formula" for the Chern-Weil map which Witten seems to have. Here "integral formula" means that we adjoin variables which may be fermionic, that is, we lift up to some (super?) manifold over M; then we multiply by a kernel and integrate over the fibre relative to some other map to M.

Lets study the operator $:e^{-\theta^a \iota_a}:$. The bosonic analogue is

$$:e^{tx^i\partial_i}: \qquad \text{or} \qquad :e^{ta^*a}:$$

acting on polynomials. Except only part of the variables occurs. So consider polynomials in x, y and the operator $:e^{tx\partial_x}:$. We have

$$:e^{tx\partial_x}: f(x,y) = \sum_{n\geq 0} \frac{t^n x^n}{n!} \partial_x^n f(x,y)$$

$$= f((1+t)x, y)$$

so that if $t=-1$ we get the projection

$f(x,y) \longmapsto f(0,y)$. To what extent can we find a nice "integral formula" for this operator? It is a substitution operator so the Schwartz kernel should be a δ-function supported on the graph of the substitution.

We have

$$:e^{tx\partial_x}: f(x,y) = \left[e^{u\partial_x} f(x,y) \right]_{u=tx}$$

$$= \left[f(x+u,y) \right]_{u=tx}$$

$$= \int \frac{d\xi}{2\pi} e^{itx\xi} \int du \; e^{-i\xi u} f(x+u,y)$$

I guess what we learn from this calculation is that to get $:e^{-\theta^a \iota_a}: \alpha$, where $\alpha \in \Omega(M)$, we first consider the translation operator

$$e^{\lambda^a \iota_a} \alpha$$

where λ^a are auxiliary Grassmann variables, and then we substitute $\lambda^a \mapsto -\theta^a$. This substitution can be done by introducing a new set of Grassman variables μ_a and then double integrating

$$\int \mathcal{D}\mu \int \mathcal{D}\lambda \; e^{(\theta^a+\lambda^a)\mu_a}$$

Here seems to be the logical structure, or maybe it is better to say the "yoga", or way of thinking. One starts with the complex of equivariant forms $\{W(\mathfrak{g}) \otimes \Omega(M)\}_{bas}$

Here we have adjoined to $\Omega(M)$ extra fermionic and bosonic variables, then divided out by the supergroup $G + \mathfrak{g}$. Because the $\mathfrak{g}$ action is free we can divide out by it an we obtain the small model

$$\circledast \qquad \{S(\mathfrak{g}^*) \otimes \Omega(M)\}^G$$

in which only the new bosonic variables remain.

Next we want to divide out by the G in some sense, so we pull back to $M \times G$. This introduce a new set of bosonic variables and fermionic variables, but the bosonic ones disappear on taking the quotient:

$$\{S(\mathfrak{g}^*) \otimes \Omega(M \times G)\}^G = \underbrace{\{S(\mathfrak{g}^*) \otimes \Omega(G)\}^G}_{W(\mathfrak{g})} \otimes \Omega(M)$$

Thus we now have an algebra freely generated by $\Omega(M)$ and a set of bosonic and fermionic variables. Finally these have to be specialized to curvature and connection forms in $\Omega(M)$. This can be done by introducing a new set of Fourier transform bosonic + fermionic variables.

Another point: Instead of lifting back to $M \times G$ one can think of choosing (fixing a gauge) a section of M over M/G. Until one does this one can't speak about the connection forms on M/G.

Now I think we have the correct yoga or philosophy, and the next stage is to work out formulas. This means "working in a given gauge".

It's probably a good idea to ponder what I might end up with. We will start with $\alpha = (\omega \mapsto \alpha(\omega)) \in (S(\mathfrak{g}^*) \otimes \Omega(M))^G$. If a gauge is fixed we have an isomorphism

$$\left(S(\mathfrak{g}^*) \otimes \Omega(M)\right)^G = W(\mathfrak{g}) \otimes \Omega(M/G)$$

and so α becomes a polynomial in the the universal connection + curvature (χ^a, ω^a) with coefficients from $\Omega(M/G)$, specifically

$$e^{-\chi^a \iota_a} \alpha(\omega) \Big|_{\text{cross section } s(M/G)} \qquad (\pm \chi^a ?)$$

$$\parallel$$

$$\left(s^*\left(e^{-\chi^a \iota_a} \alpha\right)\right)(\omega)$$

Now introduce the dual variables to χ^a and ω^a, call them λ_a and η_a. Then the evaluation formula is something like

$$\int_{\mathfrak{g}} d\eta \int D\lambda \int_{\mathfrak{g}} d\omega \int \mathcal{D}\chi \, e^{(\Theta^a - \chi^a)\lambda_a + (\Omega^a - \omega^a)\eta_a} s^*\left(e^{-\chi^a \iota_a} \alpha(\omega)\right)$$

The idea would be to write the exponential in a really nice form as Atiyah suggested. It is tempting to identify $W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$ with the polynomial differential forms on $\mathfrak{g}$, and in some sense by using Fourier transform you are doing this. But there are problems with the grading.

Review Atiyah's idea. One starts with the connection form $\theta = X_a \theta^a$ on M, and one interprets X_a as linear ~~functions~~ η_a on $\mathfrak{g}^*$. This gives the invariant 1-form $\theta^a \eta_a$ on $M \times \mathfrak{g}$ of type $(1,0)$. An invariant 1-form can be viewed ~~as~~ as an equivariant 1-form. The differential in $\left(S(\mathfrak{g}^*) \otimes \Omega(M \times \mathfrak{g})\right)^G$ is

$$\left(d - \omega^a \iota_a\right)\left(\theta^a \eta_a\right) = \left(d\theta^a - \omega^a\right)\eta_a - \theta^a d\eta_a$$

We want something like $\quad \left(\Omega^a - \omega^a\right)\eta_a \pm \left(\theta^a - \chi^a\right)\lambda_a$

<u>May 24, 1988</u>

Consider the double cochain algebra $C^\circ(A, \Omega_A^\circ)$ with the differentials δ, d and the canonical element $\theta \in C^1(A, \Omega_A^0)$. We wish to understand the universal relations that can be generated in the double complex, which is essentially the commutator quotient of the above, namely

$$C_\lambda^\circ(A, \Omega^\circ/[\Omega^\circ, \Omega^\circ])$$

For this purpose we note the relation $\delta\theta = -\theta^2$, so that $(\delta + d + \theta)^2 = d\theta$. Let's change notation and consider the universal DG algebra generated by elements A, F of degrees 1, 2 respectively such that $F = \tilde{d}A + A^2$. We can bigraded this algebra by calling A of degree (1,0) and F of degree (1,1), since it's the tensor algebra generated by A, F. We can also define a horizontal differential δ by $\delta A = -A^2$ and $\delta F = [F, A]$ and a vertical differential d by $dA = F$ and $dF = 0$. Then we compute

$$\delta^2 A = -\delta(A^2) = -[(-A^2)A - A(-A^2)] = 0$$

$$\delta^2 F = \delta[F, A] = [[F, A], A] + [F, -A^2]$$

$$= (FA - AF)A + A(FA - AF) - FA^2 + A^2F = 0$$

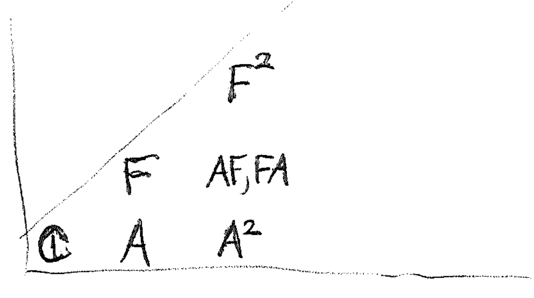
$$(\delta d + d\delta)A = \delta F + d(-A^2) = [F, A] - FA + AF = 0$$

$$(\delta d + d\delta)F = d[F, A] = [F, F] = 0.$$

What we learn then is that the universal Chern-[redacted] Simons algebra [redacted] $\mathbb{C}\langle A,F\rangle$ with $\tilde{d}A = F - A^2$ is in fact a bigraded diff. algebra, i.e. a double cochain algebra. We are interested in the double complex obtained by taking the commutator ~~quotients~~.

<u>Prop</u>. The vertical cohomology of $\mathbb{C}\langle A,F\rangle/[\,,\,]$, that is with respect to d is trivial. Hence the total cohomology is also trivial.

Proof. As far as d is concerned $\mathbb{C}\langle A,F\rangle$ with d is the tensor algebra on the complex $0 \to \mathbb{C}A \xrightarrow[\sim]{d} \mathbb{C}F \to 0 \to$. The commutator ~~quotient~~ [redacted] complex is the direct sum of the symmetric cyclic powers of this complex, and so the cohomology is trivial. The second assertion is clear from the spectral sequence which incidently sits in a sector.



An interesting question is what one can say about the δ-cohomology, and also because this is what Dubois-Violette, Talon, Viallet (Comm. Math Phys 102 (1985) 105-122) say gives the "anomalous" terms, what can one says about the δ-cohomology [redacted] mod d, that is after dividing by $\operatorname{Im} d$.

Our next step will be to get a conjecture about the δ-cohomology from the D-V, T, V results. Let's review these results.

Let $\mathfrak{g}$ be a Lie algebra; they define the BRS algebra of $\mathfrak{g}$ as a kind of bigraded generalization of the Weil algebra. It is a double cochain algebra $\mathcal{B}$ equipped with a universal "connection form" $\omega \in \mathfrak{g}\otimes(\mathcal{B}^{1,0}+\mathcal{B}^{0,1})$ such that the curvature is of type $(0,2)$ say:

$$(\delta + d)\omega + \omega^2 \in \mathfrak{g}\otimes\mathcal{B}^{0,2}.$$

One writes $\omega = \chi + A \in \mathfrak{g}\otimes(\mathcal{B}^{1,0}\oplus\mathcal{B}^{0,1})$, and then has relations

$$\delta\chi + \chi^2 = 0$$

$$d\chi + \delta A + [\chi, A] = 0.$$

The universal algebra $\mathcal{B}$ is generated by the components of the fields χ, A, $\varphi = d\chi$, $F = dA + A^2$.

There are two interesting subalgebras of $\mathcal{B}$. The first is generated by the components of A, F and is stable under d. This is the Weil algebra. The second is generated by the components of χ and $\varphi = d\chi$ and is stable under both δ, d. In fact this second algebra is really quite analogous to the non-commutative algebra I have been studying (the Chern-Simons algebra).

This suggests I should change notation immediately to χ, φ instead of A, F.

We a map from the non-commutative algebra to matrices over the commutative algebra gen. by X^a, φ^a, when we take $\mathfrak{g} = \mathfrak{gl}_N$. Further we have the trace map from the non-commutative algebra to the commutative algebra.

The commutative algebra is generated by the components of $X = X_a X^a$ $\quad \varphi = X_a \varphi^a$ where the differential is

$$\delta X = -X^2$$

$$\delta \varphi = [\varphi, X]$$

Therefore one has the complex of Lie algebra cochains of $\mathfrak{g}$ with values in $S(\mathfrak{g}^*)$. In the reductive case we know

$$H^*(\mathfrak{g}, S(\mathfrak{g}^*)) = H^*(\mathfrak{g}) \otimes S(\mathfrak{g}^*)^G$$

Now I want to take $\mathfrak{g} = \mathfrak{gl}_N$ for large N and take the primitive part. It seems that we get exactly one generator in each degree.

Thus the conjecture should be that the δ cohomology of the commutator quotient of the non-commutative Chern-Simons algebra contains a single generator in each degree. In fact they seem to be

$$\operatorname{tr}(X^{\text{odd}}) \qquad \operatorname{tr}(\varphi)^{\text{any}}$$

Check: Observe that

$$\operatorname{tr}(\theta^{2n+1}) \qquad \operatorname{tr}(d\theta^n)$$

are cyclic cocycles, i.e. killed by δ

Let's see if we can prove our conjecture about the δ-cohomology of

$$\mathbb{C}\langle \chi, \varphi \rangle / [\ ,\ ]$$

where $\delta\chi = -\chi^2$, $\delta\varphi = [\varphi, \chi]$. This splits up into rows where the qth row has φ occurring q-times interspersed with powers of χ. I propose to use the Goodwillie trick of labelling the φ's in order. This should give a q-fold cyclic covering complex where the monomials all begin with φ. So we should first look at $q = 1$.

In this case we have

$$\begin{aligned}
\delta(\varphi\chi^j) &= -\chi\varphi\chi^j + \varphi\chi\chi^j \\
&\quad + \sum_{i=1}^{j} (-1)^{i-1} \varphi \chi^{i-1}(-\chi^2)\chi^{j-i} \\
&= -\chi\varphi\chi^j + \varphi\chi^{j+1}\Big(1 + \sum_{i=1}^{j}(-1)^j\Big) \\
&\equiv \varphi\chi^{j+1}\Big(1 + \sum_{i=1}^{j}(-1)^j - (-1)^j\Big) \\
&= \varphi\chi^{j+1}\begin{Bmatrix} 0 & j \text{ even} \\ 1 & j \text{ odd} \end{Bmatrix}
\end{aligned}$$

So the conjecture works for $q = 1$. Try $q = 2$.

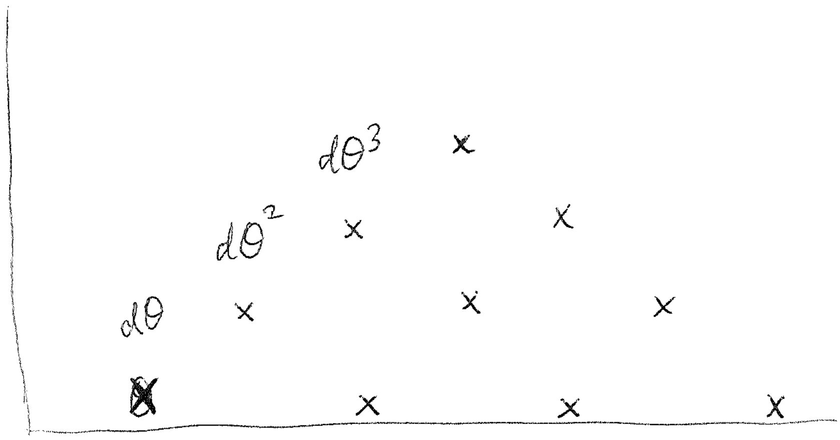
$$\begin{aligned}
\delta\{\varphi\chi^k\ \varphi\chi^l\} &= \varphi\Big(\Big(\frac{1-(-1)^k}{2}\Big)\chi^{k+1}\Big)\varphi\chi^l \\
&\quad + (-1)\chi\varphi\chi^k\varphi\chi^l \\
&\quad + (-1)^k \varphi\chi^k \varphi\chi^{l+1}\big(1 - 1 + \cdots + (-)(-)^{l-1}\big)
\end{aligned}$$

$$= \varphi\Big(\frac{1-(-1)^k}{2}\Big)\chi^{k+1}\varphi\chi^l + (-1)^k\varphi\chi^k\varphi\underbrace{\Big(1 + \sum_{i=1}^{l}(-1)^i + (-1)^{l+1}\Big)}_{\frac{1-(-1)^l}{2}}$$

Thus it seems to work

Next we should look at the δ cohomology modulo d in the sense of anomaly theory. This means that we pick a row of the commutator ~~quotient~~ complex and divide out by the image of d from the lower row. Since we know the d-cohomology, the vertical cohomology, is trivial, we ought therefore to be able to reduce things to our computation of the δ-cohomology.

This seems clear. Let's go back to our notation $\chi \mapsto \theta$, $\varphi \mapsto d\theta$ and our diagram



Notice that in the [redacted] universal ~~situation~~, but not $C_\lambda^\bullet(A, \Omega_A^\bullet/[\ ,\ ])$, the columns are acyclic. In the universal case then, we see [redacted] by diagram chasing that at each of the crosses there is δ cohomology modulo d.

Now we ought to see if we can produce Chern-Simons formulas for the relevant [redacted] cyclic [redacted] cochains, now that we know about the [redacted] Goodwillie pictures for the rows.

Let's see if we can explain how the BRS alg $\mathcal{B}$ of DVTV is related to anomalies.

An "anomalous term" is of the form

$$\Delta = \int Q$$

where the integral is over space-time, and Q is a polynomial in the fields + their derivatives, which satisfies a "consistency equation"

$$\delta \Delta = 0$$

However solutions of this equation of the form $\Delta = \delta \Delta'$ with $\Delta' = \int Q'$ are considered trivial.

We get solutions of the consistency equation from Q such that

$$\delta Q = dQ'$$

for some Q', that is, if Q is a δ-cocycle mod d.

If

$$Q = \delta L' + dL$$

then

$$\int Q = \int \delta L' = \delta \int L'$$

is a trivial solution. Thus anomalous terms arise from the Lie algebra cohomology H^1 with values in forms on space-time depending on the fields modulo $\mathrm{Im}\, d$ on these forms.

Now consider a pure gauge theory which means that are considers functionals of a variable gauge field A, e.g. YM. The sort of Q's to consider are polynomials in the 1-forms A^a and their differentials dA^a together with the components χ^a of a variable infinitesimal gauge transformation $\chi = (\chi^a)$ and their differentials $d\chi^a$. A mystery is why the χ^a should be treated as the component of the MC form of the group of gauge transformations. Thus why do the sort of terms $\Delta = \int Q$ considered have the form where Q is a polynomial in the fermionic variables A^a, χ^a and the bosonic variables $dA^a, d\chi^a$? This must have something to do with the quantum theory.

The BRST algebra has in addition to $A, \chi, dA, d\chi$ also a contractible piece involving the second Faddeev-Popov ghost $\bar{v}$ and the gauge fixing Lagrange multiplier γ satisfying $\delta \bar{v} = \gamma$.

May 25, 1988:

Chern-Simons forms. These are the components of

$$\int_0^1 dt\ tr\left(\theta\, e^{td\theta + (t^2-t)\theta^2}\right)$$

Let us look at the form of degree (p,q), so that there are q $d\theta$'s and the [redacted] θ-degree is p. I find it easier to think in terms of the number of θ^2 used; call this k. Then $p = \theta\text{-degree} = 1 + q + 2k$ and the total degree is $p+q = 1 + 2q + 2k$. We are after therefore the component of

$$\int_0^1 dt\ tr\left(\theta \frac{1}{(n-1)!}\left(td\theta + (t^2-t)\theta^2\right)^{\overset{q+k}{(n-1)}}\right)$$

of degree q in $d\theta$. Thus we get

$$tr\ \theta\left(\text{[redacted] non-commutative monomials of degree } q \text{ in } d\theta \text{ and } k \text{ in } \theta^2\right)$$

times the numerical factor

$$\frac{1}{(n-1)!}\int_0^1 dt\ t^q(t^2-t)^k = \frac{(-1)^k}{(q+k)!}\int_0^1 t^{(q+k+1)-1}(1-t)^{(k+1)-1}dt$$

$$= \frac{(-1)^k}{\cancel{(q+k)}!}\ \frac{\Gamma(\cancel{q+k}+1)\Gamma(k+1)}{\Gamma(q+2k+2)} = \frac{(-1)^k\, k!}{(q+2k+1)!}$$

So we learn that if we set

$$\gamma_{q,k} = tr\left\{\theta\left(\begin{matrix}\text{sum of the monomials}\\ \text{of degree } q,k \text{ in } d\theta, \theta^2\end{matrix}\right)\right\}$$

then $\delta\gamma_{q,k}$ and $d\gamma_{q-1,k+1}$ coincide up to numerical factors

For example

$$\delta\, tr\, \theta(d\theta)^g = const \cdot d\, tr(\theta\, Sym(d\theta^{g-1}, \theta^2))$$

It would be nice to see this directly.

May 26, 1988

There seems to be hope that the Chern-Simons transgression forms and their splitting into homogeneous components might explain most of cyclic formalism. We learned yesterday that in the usual formula

$$tr\, \frac{d\theta^n}{n!} = (\delta + d)\int_0^1 dt\ tr\, \theta \frac{(t\,d\theta + (t^2-t)\theta^2)^{n-1}}{(n-1)!}$$

all the complexity due to the factors of t and the integration disappears if one takes the homogeneous components for the bigrading. Let's now do the combinatorics as simply as possible.

Thus we consider the ~~form~~ expression

$$tr\, \theta(d\theta + \theta^2)^n$$

in the commutator quotient of $\mathbb{C}\langle\theta, d\theta\rangle$, which is a bigraded differential algebra with $\delta\theta = -\theta^2$, $\delta d\theta = [d\theta, \theta]$. We can write

$$tr\, \theta(d\theta + \theta^2)^n = \underbrace{tr\, \theta(d\theta)^n}_{\gamma_{n,0}} + \qquad + \underbrace{tr\, \theta^{2n+1}}_{\gamma_{0,n}}$$

and where $\gamma_{n-k,k}$ involves $d\theta$ $n-k$ times and θ^2 k-times.

Notice that

$$\delta(d\theta) = [d\theta, \theta] \qquad \therefore [\delta+\theta, d\theta+\theta^2]=0$$
$$d(\theta^2) = [d\theta, \theta] \qquad \therefore [d+\theta, d\theta+\theta^2]=0$$

and so

$$\delta\, tr\left(\theta(d\theta+\theta^2)^n\right) = tr\,[\delta+\theta, \theta(d\theta+\theta^2)^n]$$
$$= tr\,[\delta+\theta,\theta](d\theta+\theta^2)^n = tr\left(\theta^2(d\theta+\theta^2)^n\right)$$
$$d\, tr\left(\theta(d\theta+\theta^2)^n\right) = tr\,[d+\theta, \theta(d\theta+\theta^2)^n]$$
$$= tr\,[d+\theta,\theta](d\theta+\theta^2)^n = tr\left(d\theta(d\theta+\theta^2)^n\right)$$

Now we want to identify the components up to numerical factors.

Let's consider two non-commuting variables A, B and form in $\mathbb{C}\langle A,B\rangle/[\ ,\ ]$ (A, B both even)

$$tr(A+B)^{n+1} = tr(A^{n+1}) + (n+1)tr(A^nB) + \cdots + tr(B^{n+1})$$

We propose to calculate the term of degree k in A and $p = n+1-k$ in B in two ways; this is for $0<k<n+1$. The term of bidegree (k,p) is the sum of traces of monomials of degree (k,p) in (A,B) and their $\binom{k+p}{k}$ of these monomials, one for each card k subset of $\{1,\dots,n+1\}$. The traces are the same for monomials differing by a cyclic permutation; this is the cyclic group of order $n+1$. Thus we get a sum over orbits, and the contribution of an orbit is the trace of one of its representative monomials times the size of the orbit. So if we

divide by the order $n+1$ of the group we get a sum over the orbits where the contribution of an orbits is the trace of a representative monomial in the orbit divided by the order of the ~~stabilizer~~.

But now let us restrict to those monomials of degree (k,p) which begin with A. This set of monomials is in 1-1 correspondence with the set of monomials of degree $(k-1,p)$. There is also action of $\mathbb{Z}/k$ on this set which cyclically moves the "blocks" AB^j. So the sum of the traces of the monomials of degree (k,p) which begin with A divided by k equals the sum over the orbits where the contribution from an orbit is the trace of a representative term divided by the order of the stabizer. But now we apply the equivalence of categories used in our proof of Goodwillie's theorem and we see

$$\frac{1}{n+1}\,tr\begin{pmatrix}\text{sum of monomials of}\\ \text{degree } (k,p) \text{ in } A,B\end{pmatrix} = \frac{1}{k}\,tr\left(A\begin{pmatrix}\text{sum of monomials}\\ \text{of degree } (k-1,p)\\ \text{in } A,B\end{pmatrix}\right)$$

Reversing the roles of A, B we see this also equals

$$\frac{1}{p}\,tr\ B\begin{pmatrix}\text{sum of monomials}\\ \text{of degree } k,p-1 \text{ in } A,B\end{pmatrix}$$

We use this when $A=\theta^2$, $B=d\theta$. We have

$$\frac{1}{k+1}\,\delta\,\underbrace{tr\,\theta\begin{pmatrix}\text{mon. in } d\theta,\theta^2\\ \text{degree } n-k,k\end{pmatrix}}_{\gamma_{n-k,k}} = \frac{1}{k+1}tr\,\theta^2\begin{pmatrix}\text{mon. in } d\theta,\theta^2\\ \text{of deg } n-k,k\end{pmatrix}$$

$$= \frac{1}{(n-k)+k+1}\,tr\begin{pmatrix}\text{mon. in } d\theta,\theta^2\\ \text{of degree } n-k,k+1\end{pmatrix}$$

$$= \frac{1}{n-k} \operatorname{tr} d\theta \begin{pmatrix} \text{mon in } d\theta, \theta^2 \\ \text{degree } n-k-1, k+1 \end{pmatrix}$$

$$= \frac{1}{n-k} d \underbrace{\operatorname{tr} \theta \begin{pmatrix} \text{mon. in } d\theta, \theta^2 \\ \text{degree } n-k-1, k+1 \end{pmatrix}}_{\gamma_{n-k-1,k+1}}$$

Now that we understand the Chern-Simons we want to apply them to understand cyclic formalism.

Let's review the Connes cocycles. We suppose A acts on H and there is given an involution F of H. We then do calculations in $\operatorname{End}(H)$, but perhaps one should work in $A * \mathbb{C}[F]$. This is so for ungraded. In the graded case one supposes $H = H^+ \oplus H^-$ $\quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ such that A is even and F is odd with respect to ε. The universal algebra would be $(A * \mathbb{C}[F]) \tilde{\otimes} \mathbb{C}[\varepsilon]$ which is isomorphic to $(A * A) \otimes M_2(\mathbb{C})$.

I have two ways to understand the Connes cocycles. First of all there is his method where one forms the DG algebra

$$R^+ \xrightarrow{[F,\cdot]_-} R^- \xrightarrow{[F,\cdot]_+} R^+ \xrightarrow{[F,\cdot]_-} \cdots$$

where $R^{\pm} = \operatorname{End}(H)$ in the ungraded case and in the graded case $R^{\pm} = \operatorname{End}^{\pm}(H)$ relative to ε. In the ungraded case a trace defined on a suitable ideal of R will give closed odd degree traces on the above DGA. Then we can form the cyclic cocycles $\operatorname{tr}(\theta[F,\theta]^{2n-1})$.

In the graded case one has $\operatorname{tr}\varepsilon$ on $R^+ = \operatorname{End}^+(H)$, or some ideal say, and this gives closed even degree traces on the DGA. Thus one obtains even cyclic cocycles $\operatorname{tr}(\varepsilon\theta[F,\theta]^{2n})$

In addition to Connes method there is mine which is as follows. Suppose we have a vector bundle with a flat connection D and a splitting given by an involution F. Then we can write $D = \nabla + \alpha$ as the sum of a connection $\nabla = \frac{1}{2}(D + FDF)$ commuting with F and a 1-form $\alpha = \frac{1}{2}(D - FDF)$ anti-commuting with F. From the flatness (or even the weaker assertion that D^2 commute with F) we find that

$$D^2 = \underbrace{\nabla^2 + \alpha^2}_{\substack{\text{commutes}\\ \text{with } F}} + \underbrace{[\nabla, \alpha]}_{\substack{\text{anti-}\\ \text{commutes with } F}}$$

so that $[\nabla, \alpha] = 0$. This implies by the usual argument that $\operatorname{tr}(F\alpha^{2n})$ is closed.

In the graded case one considers instead the form $\operatorname{tr}(\varepsilon\alpha^{2n+1})$ which is closed.

To obtain cyclic cocycles one works in the DG algebra $C^{\cdot}(A, \operatorname{End} H)$ with $D = \delta + \theta$. Then

$$\alpha = \frac{1}{2}\big((\delta+\theta) - F(\delta+\theta)F\big) = \frac{1}{2}(\theta - F\theta F)$$

$$= \frac{1}{2} F[F,\theta]$$

and $\operatorname{tr}(F\alpha^{2n}) = (-1)^n 2^{-2n} \operatorname{tr}([F,\theta]^{2n}) = (-1)^n 2^{-2n}\, 2 \operatorname{tr}(\theta[F,\theta]^{2n-1})$

$\operatorname{tr}(\varepsilon\alpha^{2n+1}) = 2^{-2n-1}(-1)^n \operatorname{tr}(\varepsilon F[F,\theta]^{2n+1}) = (-1)^n 2^{-2n-1}\, 2 \operatorname{tr}(\varepsilon\theta[F,\theta]^{2n})$

Now the problem I want to get to the bottom of is to relate these different cocycles by the S-operator. As usual there is the basic problem of what the S-operator *is*.

The goal now is to bring to bear all we know about the formalism so far. We know that we ought to be able to replace End(H) by $A * \mathbb{C}[F]$ in the ungraded case.

But we also did find a proof that the different cocycles are S-related using the homotopy property established in my formalism. Let's go over this.

May 27, 1988

I want to discuss the S-operation for the odd cocycles belonging to an extension. My idea is to use the homotopy invariance in roughly the following way. Let τ_{2n-1} be the cocycle's associated to θ, F. Then by doubling one achieves another pair

$$\tilde{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \qquad \tilde{F} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & t \\ t & -F \end{pmatrix}$$

and it seems possible to relate $S^k \tau_q$ for θ, F to the coefficient of t^{2k} in τ_{q+2k} for $\tilde{\theta}, \tilde{F}$. Then the homotopy invariante should show $S^k \tau_q$ and τ_{q+2k} are cohomologous.

I would now like to see what this sort of construction amounts to when I work with the idempotent $e = \frac{1+F}{2}$. The idea is roughly that what we are doing is to deform the idempotent e, and in general such a deformation is described by a map from $\mathrm{Im}(e)$ to $\mathrm{Im}(1-e)$. If we stay within the given space H, then eH and $(1-e)H$ are unrelated, so there is nothing we can do, i.e. there is no deformation singled out. On the other hand if we double H, then there is the copy of eH in the 2nd factor.

So we should just add eH to H perhaps.

May 28, 1988

The problem is still to relate the cocycles belonging to an extension via the S operator. To do this I propose to start from A, H, F then double H and use

$$\tilde{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \qquad \tilde{F} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & t \\ t & -F \end{pmatrix}$$

Notice that

$$\tilde{F}' = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & tF \\ tF & -F \end{pmatrix} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} 1 & t \\ t & -1 \end{pmatrix} \otimes F$$

should work as well, since $\tilde{F}$ and $\tilde{F}'$ are conjugate by $\begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}$ which commutes with $\tilde{\theta}$.

We have

$$\left[\begin{pmatrix} F & t \\ t & -F \end{pmatrix}, \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} [F,\theta] & -\theta t \\ t\theta & 0 \end{pmatrix}$$

so

$$(1+t^2)^{n-\frac{1}{2}} \operatorname{tr}\left(\tilde{\theta}[\tilde{F},\tilde{\theta}]^{2n-1}\right) = \operatorname{tr} \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} [F,\theta] & -\theta t \\ t\theta & 0 \end{pmatrix}^{2n-1}$$

On the right side we have a polynomial in t^2 whose coefficients are S-transforms of the cocycles $\operatorname{tr} \theta[F,\theta]^{2k-1}$. On the left we see $\operatorname{tr} \tilde{\theta}[\tilde{F},\tilde{\theta}]^{2n-1}$ whose cohomology class is independent of t.

The problem with this proof is that one must work infinitesimally in t. Thus the "ideal of Shatter" must contain $t\theta$.

Let $C = A * \mathbb{C}[F] = (A*A) \tilde{\otimes} \mathbb{C}[F]$.
This is $\mathbb{Z}_2$-graded where A is even and F is odd. We want to set up an additive isomorphism of $A*A$ with $\Omega_A^{\bullet}$. This isomorphism is constructed by defining a module structure on $\Omega_A^{\bullet}$ over $A*A$ and then acting on 1. Another feature of the isomorphism is that $g(a) = a - FaF$ should correspond to da. In fact we want

$$* \qquad a_0\, da_1 \cdots da_n \longleftrightarrow a_0\, g(a_1) \cdots g(a_n)$$
$$= a_0\, F[F, a_1] \cdots F[F, a_n]$$

The simplest way to proceed is to define a $C = (A*A) \tilde{\otimes} \mathbb{C}[F]$-module structure on $\Omega_A^{\circ} \otimes \mathbb{C}[F]$. For this we have to have an A-module structure - this is just left multiplication of A on Ω_A° - and an F-operator. We have to move the F through Ω_A° and so must define conjugation by F on Ω_A°. But F anti-commutes with $g(a) = a - FaF$, so

$$F(a_0 g(a_1) \cdots g(a_n))F = (-1)^n\, Fa_0F\, g(a_1) \cdots g(a_n)$$
$$= (-1)^n \Big\{ a_0\, g(a_1) \cdots g(a_n) + \underbrace{(Fa_0F - a_0)}_{-g(a_0)}\, g(a_1) \cdots g(a_n) \Big\}$$

Thus if we want $*$ we must define

$$F(a_0\, da_1 \cdots da_n)F = (-1)^n (a_0\, da_1 \cdots da_n - da_0 \cdots da_n)$$
$$= \sigma(1 + d)(a_0\, da_1\, da_2 \cdots da_n)$$

where $\sigma(\omega) = (-1)^{\omega}\omega$.

Let's check $\sigma(1+d)$ on Ω_A is an involution:

$$\sigma(1+d)\,\sigma(1+d) = (1-d)(1+d) = 1.$$

So now we can make $\overset{\circ}{\Omega}_A$ into a left C-module by making F act as $\sigma(1+d)$, and A by the obvious left multiplication. One then has

$$\begin{aligned} FaF &= \sigma(1+d)\,a\,\sigma(1+d) \\ &= (1-d)\,a\,(1+d) = a-(da)-(da)d \end{aligned}$$

and $g(a) = a - FaF = da + (da)d$. Thus $g(a)$ acting on a closed form is multiplication by da. Thus

$$a_0\, g(a_1)\cdots g(a_n) \overset{*}{\text{acting on } 1} = a_0\, da_1 \cdots da_n .$$

Notice that $F * 1 = \sigma(1+d)1 = 1$

so that

$$C/\underbrace{C(1-F)}_{C\bar{e}} \xrightarrow{\sim} \overset{\circ}{\Omega}_A$$

or

$$\boxed{\begin{aligned} Ce &\xrightarrow{\ \sim\ } \overset{\circ}{\Omega}_A \\ ce &\longmapsto c*1 \end{aligned}}$$

However this is not the expected isomorphism since

$$\begin{aligned} eae &\longmapsto \frac{1+F}{2}a = \frac{1}{2}(a + \sigma(1+d)a) \\ &= \frac{1}{2}(a + a - da) = a - \frac{1}{2}da \end{aligned}$$

Idea: We are looking at the family $F_t = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} F & t \\ t & -F \end{pmatrix}$ with the A-module structure $\tilde{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}$. Then we form the cocycles $tr\left(F_t[F_t, \tilde{\theta}]^{2n}\right)$. It might be interesting to have, instead of $\tilde{\theta}$ fixed and F varying, for F to be fixed and $\tilde{\theta}$ to vary. Now we have seen that we can replace F_t above by

$$F_t = \underbrace{\frac{1}{\sqrt{1+t^2}}\begin{pmatrix} 1 & t \\ t & -1 \end{pmatrix}}_{u_t\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u_t^{-1}} \otimes F$$

where $u_t = \begin{pmatrix} \cos\phi/2 & -\sin\phi/2 \\ \sin\phi/2 & \cos\phi/2 \end{pmatrix}$ $\qquad \tan\phi = t$

It follows that

$$[F_t, \tilde{\theta}] = \left[u_t\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}u_t^{-1} \otimes F, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \theta\right]$$

$$u_t^{-1}[F_t, \tilde{\theta}]u_t = \left[\begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}, \underbrace{\begin{pmatrix} \cos & \cancel{\sin}\ 0 \\ -\sin & \cancel{\cos}\ 0 \end{pmatrix}\cancel{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}\begin{pmatrix} \cos & -\sin \\ \cancel{\sin}\ 0 & \cancel{\cos}\ 0 \end{pmatrix}}_{\begin{pmatrix} \cos^2 & -\cos\sin \\ -\sin\cos & \sin^2 \end{pmatrix}\theta}\theta\right]$$

$$= \begin{pmatrix} \cos^2 & -\cos\sin \\ \sin\cos & -\sin^2 \end{pmatrix} F\theta - \begin{pmatrix} \cos^2 & \cos\sin \\ -\sin\cos & -\sin^2 \end{pmatrix} \theta F$$

$$= \begin{pmatrix} \cos^2 [F, \theta] & -\sin\cos [F, \theta]_+ \\ \sin\cos [F, \theta]_+ & -\sin^2 [F, \theta] \end{pmatrix}$$

In other words we have made things *more* complicated.

Thus let's return to the family

$$F_t = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & t \\ t & -F \end{pmatrix}$$

$$\left[F_t, \tilde{\theta}\right] = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} [F,\theta] & -\theta t \\ t\theta & 0 \end{pmatrix}$$

Our goal should now be to [redacted]. completely understand the proof of the S-equivalence of the different cocycles.

Question: If $t=0$ then

$$tr\left(F_t [F_t, \tilde{\theta}]^{2n}\right) = tr\left(F[F,\theta]^{2n}\right)$$

and if $t=\infty$ then $F_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$tr\left(F_\infty [F_\infty, \tilde{\theta}]^{2n}\right) = tr\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}^{2n}\right) = 0$$

Is the "transgression" form equal to the Chern-Simons form in this case?

May 29, 1988

In studying the deformation $F_t = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} F & t \\ t & -F \end{pmatrix}$ it has become clear that it should be useful to work with F's such that F^2 is scalar, or more generally F^2 commutes with A.

Let's prove that $\operatorname{tr} F[F,\theta]^{2n}$ is a cyclic cocycle assuming only that $[F^2,\theta]=0$. This implies

$$0 = [F^2,\theta] = F[F,\theta] + [F,\theta]F$$

hence $[F,\theta]$ anti-commutes with F. We have

$$\delta[F,\theta] = [F,-\theta^2] = -[F,\theta]\theta - \theta[F,\theta]$$

or

$$[\delta+\theta, [F,\theta]] = 0. \qquad \text{Thus}$$

$$\delta \operatorname{tr}\left(F[F,\theta]^{2n}\right) = \operatorname{tr}\,[\delta+\theta, F[F,\theta]^{2n}]$$

$$= \operatorname{tr}\left([\delta+\theta,F][F,\theta]^{2n}\right) = -\operatorname{tr}\left([F,\theta]^{2n+1}\right) = 0$$

because $[F,\theta]$ anti-commutes with F.

Alternative elementary proof.

$$\delta \operatorname{tr}\left(F[F,\theta]^{2n}\right) = \sum_{i=0}^{2n-1}(-1)^i \operatorname{tr}\left\{F[F,\theta]^i\left(-[F,\theta]\theta-\theta[F,\theta]\right)[F,\theta]^{2n-1-i}\right\}$$

$$= +\operatorname{tr}\left(-F\theta[F,\theta]^{2n} - (-1)^{2n-1}F[F,\theta]^{2n}\theta\right)$$

$$= -\operatorname{tr}\left([F,\theta]^{2n+1}\right) = 0 \qquad \text{Same reason.}$$

Let's check it also works in the graded case

$$\delta \operatorname{tr}\left(\varepsilon F[F,\theta]^{2n+1}\right) = \operatorname{tr}\left([\delta+\theta, \varepsilon F][F,\theta]^{2n+1}\right)$$

$$= \operatorname{tr}\left(\varepsilon[F,\theta]^{2n+2}\right) = 0 \quad \text{as}$$

$\varepsilon[F,\theta]^{2n+1}$ anticommutes with F.

Notice that

$$[F^2, \theta] = F[F,\theta] + [F,\theta]F$$

so we have F anti-commutes with $[F,\theta]$ iff F^2 commutes with θ. This means that the hypothesis that F^2 commutes with A is essential to use the method employed.

I would now like to derive homotopy properties for these cocycles. As a first step let us consider the problem of generalizing the construction of forms associated to a flat bundle with splitting.

Previously we considered a bundle with flat connection D and splitting F satisfying $F^2 = 1$; the example was $D = \delta + \theta$ where $\delta(F) = 0$. Now we relax the condition $F^2 = 1$, but require $[D, F^2] = 0$ and also that F be invertible. Then we can define

$$\nabla = \frac{1}{2}(D + F^{-1}DF)$$

$$\alpha = \frac{1}{2}(D - F^{-1}DF) = -\frac{1}{2}F^{-1}[D,F]$$

so that ∇ commutes with F and α anticommutes with F. It follows by the usual argument that

$$\circledast \qquad tr(F^{m}\alpha^{2n})$$

are closed forms. For m even they are zero because $tr(F^m\alpha^{2n}) = tr(\alpha F^m \alpha^{2n-1}) = -tr(F^m\alpha^{2n-1}\alpha)$.

I think I might as well assume that F^2 is a scalar, more precisely that $tr(F^2 \,?) = F^2 tr(?)$. Then the form $\ast$ with $m = 1$ essentially gives the others. There might be some interesting play with the hypothesis that F be invertible.

Next we consider a family of $\{F_t\}$ such that $[D, F_t^2]=0$ for all t. In order to prove homotopy invariance we need to construct a similar setup over our manifold $\times \mathbb{R}$. Normally we take $\tilde{D} = dt\,\partial_t + D$, but

$$[dt\,\partial_t + D,\ F^2] = dt\,(\dot{F}F + F\dot{F})$$

Unfortunately, if F^2 is a scalar endomorphism, then there is no connection on the bundle relative to which F^2 is constant.

Thus we run into a problem with our approach based on flat connections D + endos. F such that $[D, F^2] = 0$. Actually flatness is not needed; it suffices for $[D^2, F] = 0$. This implies

$$\begin{aligned}[D, [D,F]] &= D[D,F] + [D,F]D \\ &= [D^2, F] = 0\end{aligned}$$

and so

$$\begin{aligned} d\,tr\left(F^{2k+1}[D,F]^{2n}\right) &= tr\left([D,F^{2k+1}][D,F]^{2n}\right) \\ &= tr\left(F^{2k}[D,F]^{2n+1}\right) = 0\end{aligned}$$

It's clear from the preceding that one eventually has to find a way to weaken the condition $[F^2, a] = 0$, e.g. the Kasparov condition $a(F^2-1) \in$ ideal.

Let's return to $F_t = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} F & t \\ t & -F \end{pmatrix}$

and review the proof that the cyclic cohomology class of $tr(F_t[F_t, \theta e]^{2n})$ is independent of t. Recall the proof (p 853) with some simplifications.

We have D fixed but F varying with respect to t. Then

$$\partial_t tr(F\alpha^{2n}) = tr(\underbrace{\cancel{\dot F}}_{0}\alpha^{2n}) + \sum_{i=0}^{2n} tr(F\alpha^i\dot\alpha\,\alpha^{2n-1-i})$$

$$= 2n\; tr(F\dot\alpha\,\alpha^{2n-1})$$

The point is that the latter depends only on the part of $\dot\alpha$ anti commuting with F. We have

$$\alpha = \frac{1}{2}(D - FDF)$$

$$\dot\alpha = \left(-\frac{1}{2}\right)(\dot F D F + F D \dot F)$$

$$= \left(-\frac{1}{2}\right)\Big(\underbrace{\dot F \nabla F + F\nabla\dot F}_{\underbrace{[\nabla, F\dot F]}_{\text{anti-comm. with } F.}} + \underbrace{\dot F\alpha F + F\alpha\dot F}_{\underbrace{[F\dot F, \alpha]}_{\text{commutes with } F}}\Big)$$

Thus

$$\partial_t\, tr(F\alpha^{2n}) = 2n\; tr\left(F[\nabla, -\tfrac{1}{2}F\dot F]\alpha^{2n-1}\right)$$

$$= d\;\; 2n\;\; tr\left\{\left(-\tfrac{1}{2}\dot F\right)\alpha^{2n-1}\right\}$$

Let's substitute $\alpha = \frac{1}{2}F[F,\theta]$ $\qquad \alpha^{2n-1} = \frac{(-1)^{n-1}}{2^{2n-1}}F[F,\theta]^{2n-1}$

$$\alpha^{2n} = \frac{(-1)^n}{2^{2n}}[F,\theta]^{2n}$$

$$\partial_t \frac{\cancel{(-1)^n}}{\cancel{2^{2n}}}\, tr(F[F,\theta]^{2n}) = d\;\; 2n\, tr\left\{\dot F\; \cancel{\left(\frac{-1}{2}\right)\frac{(-1)^{n-1}}{2^{2n-1}}}\; F[F,\theta]^{2n-1}\right\}$$

$$\boxed{\partial_t\left\{2\, tr(\theta[F,\theta]^{2n-1})\right\} = d\;\; 2n\; tr\left\{\dot F F[F,\theta]^{2n-1}\right\}}$$

Actually this can be simplified further because

$$\mathrm{tr}\left(\underbrace{\mathring{F}F(F\theta-\theta F)[F,\theta]^{2n-2}}\right) = 2\,\mathrm{tr}\left(\mathring{F}\theta[F,\theta]^{2n-2}\right)$$

but we don't use this because the θ does not have a parity with respect to F.

Now apply this formula to $\tilde{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}$ and

$$\tilde{F} = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} F & t \\ t & -F \end{pmatrix} \qquad [\tilde{F},\tilde{\theta}] = \frac{1}{\sqrt{}}\begin{pmatrix} [F,\theta] & -\theta t \\ t\theta & 0 \end{pmatrix}$$

Note that $\mathrm{tr}\left(\mathring{\tilde{F}}\tilde{F}[\tilde{F},\tilde{\theta}]^{2n-1}\right)$ depends only on the part of $\mathring{\tilde{F}}$ anti commuting with $\tilde{F}$. Thus we get the formula

$$\partial_t\left[2\,\mathrm{tr}\left\{\begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} [F,\theta] & -\theta t \\ t\theta & 0 \end{pmatrix}^{2n-1}\right\}\left(\frac{1}{\sqrt{1+t^2}}\right)^{2n-1}\right] =$$

$$d\, 2n\, \mathrm{tr}\left[\frac{1}{\sqrt{}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{}}\begin{pmatrix} F & t \\ t & -F \end{pmatrix}\left(\frac{1}{\sqrt{}}\begin{pmatrix} [F,\theta] & -\theta t \\ t\theta & \theta \end{pmatrix}\right)^{2n-1}\right]$$

Actually it is probably clearer if we use

$$\mathring{\tilde{F}}\tilde{F} = \frac{1}{2}[\mathring{\tilde{F}},\tilde{F}] = \frac{1}{2}\left[\partial_t\frac{1}{\sqrt{}}\begin{pmatrix} F & t \\ t & -F \end{pmatrix}, \frac{1}{\sqrt{}}\begin{pmatrix} F & t \\ t & -F \end{pmatrix}\right]$$

$$= \frac{1}{2}\frac{1}{(1+t^2)}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} F & t \\ t & -F \end{pmatrix}\right] = \frac{1}{(1+t^2)}\begin{pmatrix} 0 & -F \\ F & 0 \end{pmatrix}$$

Now

$$\left(\sqrt{1+t^2}\right)^{2n+1}\partial_t\left(\frac{1}{\sqrt{1+t^2}}\right)^{2n-1} = (1+t^2)\left(\partial_t - \frac{2n-1}{2}\frac{2t}{1+t^2}\right)$$

$$= (1+t^2)\,\partial_t - (2n-1)t.$$

So we obtain the formula

$$\boxed{\begin{aligned}&\left((1+t^2)\partial_t - (2n-1)t\right)\ 2\,\mathrm{tr}\left\{\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -\theta t\\ t\theta & 0\end{pmatrix}^{2n-1}\right\}\\ &= d\ 2n\ \mathrm{tr}\left\{\begin{pmatrix}0 & -F\\ F & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -\theta t\\ t\theta & 0\end{pmatrix}^{2n-1}\right\}\end{aligned}}$$

Notice that both sides are odd polynomials in t. Let's look at the coefficient of t on both sides. On the right we have

$$-(2n-1)2\ \mathrm{tr}\ \theta[F,\theta]^{2n-1} + 4(-1)\ \mathrm{tr}\left\{\theta\ \mathrm{Sym}\left([F,\theta]^{2n-3}, (\theta^2)^1\right)\right\}$$

and on the left we have $d\ 2n$ of coeff of t in

$$\mathrm{tr}\left\{\begin{pmatrix}0 & -F\\ F & 0\end{pmatrix}\underbrace{\begin{pmatrix}[F,\theta] & -\theta t\\ t\theta & 0\end{pmatrix}\cdots\cdots\begin{pmatrix}[F,\theta] & -\theta t\\ t\theta & 0\end{pmatrix}}_{2n-1\ \text{times}}\right\}$$

$$= \mathrm{tr}\, t\left(F[F,\theta]^{2n-2}(-\theta) - F\theta[F,\theta]^{2n-2}\right) + O(t^3)$$

$$= (-2t)\ \mathrm{tr}\left(F\theta[F,\theta]^{2n-2}\right)$$

So it would seem we have the identity

$$\boxed{\begin{aligned}&(2n-1)\ \mathrm{tr}\ \theta[F,\theta]^{2n-1} - 2\ \mathrm{tr}\left\{\theta\ \mathrm{Sym}\left([F,\theta]^{2n-3}, (\theta^2)^1\right)\right\}\\ &= d\ 2n\ \mathrm{tr}\left(F\theta[F,\theta]^{2n-2}\right)\end{aligned}}$$

Let's now work out the formulas in the graded case.

$$\partial_t\ \mathrm{tr}\left(\varepsilon\alpha^{2n+1}\right) = \sum_{i=0}^{2n}\mathrm{tr}\left(\varepsilon\alpha^i\dot{\alpha}\alpha^{2n-i}\right) = (2n+1)\mathrm{tr}\left(\varepsilon\dot{\alpha}\alpha^{2n}\right)$$

$$\mathring{\alpha} = \left(-\tfrac{1}{2}\right)\Big(\underbrace{[\nabla, F\mathring{F}]}_{\substack{\text{anti comm}\\ \text{with } F}} - \underbrace{[\alpha, F\mathring{F}]}_{\substack{\text{commun}\\ \text{with } F}}\Big)$$

$$\partial_t \operatorname{tr}(\varepsilon\alpha^{2n+1}) = (2n+1)\left(-\tfrac{1}{2}\right)\operatorname{tr}\left(\varepsilon[\nabla, F\mathring{F}]\alpha^{2n}\right)$$
$$= d\,(2n+1)\left(-\tfrac{1}{2}\right)\operatorname{tr}\left(\varepsilon F\mathring{F}\alpha^{2n}\right)$$

$$\partial_t \operatorname{tr}\left(\varepsilon F[F,\theta]^{2n+1}\right)\frac{(-1)^n}{2^{2n+1}} = d(2n+1)\left(-\tfrac{1}{2}\right)\operatorname{tr}\left(\varepsilon F\mathring{F}[F,\theta]^{2n}\right)\frac{(-1)^n}{2^{2n}}$$

$$\boxed{\partial_t \operatorname{tr}\left(\varepsilon F[F,\theta]^{2n+1}\right) = d\,(2n+1)\operatorname{tr}\left(\varepsilon\mathring{F}F[F,\theta]^{2n}\right)}$$

Now apply to $\tilde{\theta} = \begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}$ $\quad \tilde{F} = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix}F & t\\ t & -F\end{pmatrix}$ $\quad \tilde{\varepsilon} = \begin{pmatrix}\varepsilon & 0\\ 0 & -\varepsilon\end{pmatrix}$

$$\partial_t\ 2\operatorname{tr}\left(\tilde{\varepsilon}\tilde{\theta}[\tilde{F},\tilde{\theta}]^{2n}\right) = d(2n+1)\operatorname{tr}\left(\tilde{\varepsilon}\ \tfrac{1}{2}[\mathring{\tilde{F}},\tilde{F}]\,[\tilde{F},\tilde{\theta}]^{2n}\right)$$

$$\partial_t \frac{1}{(1+t^2)^n} 2\operatorname{tr}\left(\begin{pmatrix}\varepsilon\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n}\right) = d(2n+1)\operatorname{tr}\Big(\tilde{\varepsilon}\ \tfrac{1}{2}\Big[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}F & t\\ t & -F\end{pmatrix}\Big]$$
$$\times\begin{pmatrix}[F,\theta] & -\theta t\\ t\theta & 0\end{pmatrix}^{2n}\frac{1}{(1+t^2)^{n+1}}$$

So
$$\boxed{\begin{aligned}&\left((1+t^2)\partial_t - 2nt\right)2\operatorname{tr}\begin{pmatrix}\varepsilon\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -\theta t\\ t\theta & 0\end{pmatrix}^{2n}\\ &= d\,(2n+1)\operatorname{tr}\begin{pmatrix}\varepsilon & 0\\ 0 & -\varepsilon\end{pmatrix}\begin{pmatrix}0 & -F\\ F & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -\theta t\\ t\theta & 0\end{pmatrix}^{2n}\end{aligned}}$$

Now we look at the coeffs. of t. On the left

$$-4n\operatorname{tr}\varepsilon\theta[F,\theta]^{2n} - 4\operatorname{tr}\varepsilon\theta\operatorname{Sym}\left(\overset{2n-2}{[F,\theta]},\overset{1}{\theta^2}\right)$$

on the right $d(2n+1)$ applied to

$$\operatorname{tr}\left\{\varepsilon(-F)\,\theta\,[F,\theta]^{2n-1} + (-\varepsilon)F\,[F,\theta]^{2n-1}(-\theta)\right\}$$
$$= -\operatorname{tr}\left\{\varepsilon F\theta[F,\theta]^{2n-1} + \varepsilon\theta F[F,\theta]^{2n-1}\right\}$$

So it seems we get the identity

$$\boxed{\begin{aligned} &2n\ \mathrm{tr}\left(\varepsilon\theta[F,\theta]^{2n}\right) - 2\ \mathrm{tr}\left\{\varepsilon\theta\,\mathrm{Sym}\left([F,\theta]^{2n-2},\theta^2\right)\right\} \\ &= d\,(2n+1)\ \mathrm{tr}\left(\varepsilon F\theta[F,\theta]^{2n-1}\right)\end{aligned}}$$

which checks for $n=1$.

May 30, 1988

Here's a mystery. In practice when we deal with the S-operator, we go from a known cocycle of degree n and produce another cocycle of degree n which turns out to be $\hat{S}$ of a known cocycle of degree $n-2$. We still don't have an operation on cyclic cocycles, although Connes does this by means of non-commutative differential forms.

Let's review the geometric picture of the S-operation for cocycles on $C^\infty(M)$. We have seen that we obtain a cyclic n-cocycle from a n-dimensional correspondence from M to a Grassmannian. This means a closed n-diml current γ on $M \times Gr(V)$. To obtain S of this cocycle one multiplies by S^2 as follows. One has the closed current $\gamma \times [S^2]$ of dim $(n+2)$ on $M \times Gr(V) \times S^2$ and one pushes this forward under the Bott map $Gr(V) \times S^2 \to Gr(V^{\oplus 2})$.

Connes has an operator analogue of this which goes as follows. Suppose the algebra A acts on H preserving a grading ε and that F is an odd involution on H. Form

$$\tilde{H} = H \otimes \mathbb{C}^2 \qquad \tilde{\varepsilon} = \varepsilon \otimes \varepsilon_c \qquad \varepsilon_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{\theta} = \theta \otimes e \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\tilde{F} = \frac{1}{\sqrt{1+t^2}} \left(F \otimes 1 + t\varepsilon \otimes F_c \right) \qquad F_c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

I forgot to say that the cyclic cocycle of interest is $\operatorname{tr}\left(\varepsilon\, \theta [F,\theta]^{2n}\right)$; this is formally analogous to $\int_\gamma tr(\Theta d\Theta^k)$ if γ has dim k. The analogue of the geometric construction of multiplying by the 2-sphere is to consider $tr(\tilde\varepsilon\tilde\Theta[\tilde F,\tilde\Theta]^{2n+2})$. We have, on writing $H\otimes\mathbb{C}^2 = H\oplus H$,

$$[\tilde F,\tilde\Theta] = [F\otimes 1 + t\varepsilon\otimes F_c,\ \Theta\otimes e]\frac{1}{\sqrt{1+t^2}}$$

$$= \left([F,\Theta]\otimes e + t\varepsilon\Theta\otimes[F_c,e]\right)\frac{1}{\sqrt{\ }}$$

$$[F_c,e] = \left[\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix}\right] = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

$$\therefore\quad [\tilde F,\tilde\Theta] = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix}[F,\Theta] & -t\varepsilon\Theta\\ t\varepsilon\Theta & 0\end{pmatrix}$$

and so

$$tr\left(\tilde\varepsilon\tilde\Theta[\tilde F,\tilde\Theta]^{2n}\right) = (1+t^2)^{-n}\, tr\left\{\begin{pmatrix}\varepsilon&0\\0&-\varepsilon\end{pmatrix}\begin{pmatrix}\Theta&0\\0&0\end{pmatrix}\begin{pmatrix}[F,\Theta]&-t\varepsilon\Theta\\t\varepsilon\Theta&0\end{pmatrix}^{2n}\right\}$$

$$= (1+t^2)^{-n}\, tr\left\{\begin{pmatrix}\varepsilon\Theta&0\\0&0\end{pmatrix}\begin{pmatrix}[F,\Theta]&-t\Theta\\t\Theta&0\end{pmatrix}^{2n}\right\}$$

The reason we can get rid of the ε is because we can conjugate by $\begin{pmatrix}1&0\\0&\varepsilon\end{pmatrix}$. This is exactly the formula we used on p. 921 in the graded case. (see also next page)

Ungraded case: One can view an ungraded (H,F) as a graded pair $(C_1\otimes H,\ \gamma\otimes F)$. More precisely suppose given a graded C_1-module H' $H^+\oplus H^-$ with an odd F' commuting with γ strictly. Then we can identify $H^+=H^-$ so that γ becomes $\begin{pmatrix}0&1\\1&0\end{pmatrix}$, and then F' becomes $\begin{pmatrix}0&F\\F&0\end{pmatrix}=\gamma\otimes F$.

Actually it looks like in the graded case the best thing to do is to form

$$\tilde{H} = \mathbb{C}^2 \otimes H \qquad \tilde{\varepsilon} = \varepsilon_{\mathbb{C}} \otimes \varepsilon \qquad \tilde{\theta} = e \otimes \theta$$

$$\tilde{F} = \frac{1}{\sqrt{1+t^2}} \left(\varepsilon_{\mathbb{C}} \otimes F + t F_{\mathbb{C}} \otimes 1 \right)$$

Then
$$[\tilde{F}, \tilde{\theta}] = \frac{1}{\sqrt{1+t^2}} \left(e \otimes [F,\theta] + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \theta \right)$$
$$= \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} [F,\theta] & -t\theta \\ t\theta & 0 \end{pmatrix}$$

and
$$\operatorname{tr} \tilde{\varepsilon} \tilde{\theta} [F, \tilde{\theta}]^{2n} = \frac{1}{(1+t^2)^n} \operatorname{tr} \left[\begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix} \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} [F,\theta] & -t\theta \\ t\theta & 0 \end{pmatrix}^{2n} \right]$$

When we do the ungraded case, then?

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Consider a vector bundle E over M with a connection D. We have various ways to produce cyclic cocycles on $\operatorname{End} E$ using a closed current M, for example the fundamental class if M is compact and oriented.

Let's review the construction. We think of cyclic cochains as left-invariant forms on a group of gauge transformations $\mathcal{G} = \operatorname{Aut}(E^{\oplus n})$, and we work with differential forms on $\mathcal{G}\times M$. The bundle $pr_2(E^{\oplus n})$ over $\mathcal{G}\times M$ has the connection the inverse image of D which we denote $\delta + D$. To this we can add the Maurer-Cartan form θ. Then we get a connection $\delta + D + \theta$ with curvature

$$(\delta + D + \theta)^2 = D^2 + [D,\theta]$$

which is of type $(0,2)+(1,1)$ over $\mathcal{G}\times M$, hence is flat in the $\mathcal{G}$-direction.

Now we have two ways of producing odd forms. Better, when a connection is flat along the leaves of a foliation, one has the Bott observation that the characteristic forms are of a certain filtration in the DR complex, and hence that they vanish in certain cases. In the above situation one has two connections $\delta + D$ and $\delta + D + \theta$ which are partially flat in the $\mathcal{G}$-direction. Thus powers of the curvature of degree $>\dim M$ are zero. Thus we have two reasons for the characteristic classes of $pr_2^*(E^{\oplus n})$ on $G \times M$ of certain degrees [redacted] to be zero. The difference of these two reasons is an odd form on $G \times M$ which is closed.

We have two ways to construct this odd form. The first uses the linear path $(\delta + D + t\theta)$ between the two connections and leads to the formula

$$\boxed{(\delta + d)\int_0^1 dt\ tr\left\{u\theta e^{u(D^2 + t[D,\theta] + (t^2-t)\theta^2)}\right\} = tr\left(e^{u(D^2+[D,\theta])} - e^{uD^2}\right)}$$

The second uses the superconnection

$$\delta + D + \begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix} + t\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\sigma$$

whose curvature is

$$-t^2 + D^2 + \begin{pmatrix}[D,\theta] & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix} t\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\sigma + t\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\sigma\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}$$

$$t\begin{pmatrix}0 & -\theta\sigma\\ 0 & 0\end{pmatrix} + t\begin{pmatrix}0 & 0\\ \sigma\theta & 0\end{pmatrix}$$

$$= -t^2 + D^2 + \begin{pmatrix}[D,\theta] & t\sigma\theta\\ t\sigma\theta & 0\end{pmatrix}$$

and leads to the formula

$$\boxed{\begin{aligned} & tr\left(e^{u(D^2+[D,\theta])}\right) - tr\left(e^{uD^2}\right) \\ & = (\delta + d)\int_0^\infty dt\ tr\left\{ u\begin{pmatrix}0&1\\1&0\end{pmatrix}\sigma\ e^{u\left(-t^2+D^2+\begin{pmatrix}[D,\theta] & t\sigma\theta\\ t\sigma\theta & 0\end{pmatrix}\right)}\right\}\end{aligned}}$$

However there is another way to obtain odd forms, namely, one can use an embedding $i: E \longrightarrow \tilde{V}$ such that $i^* di = D$ and construct the odd forms in the case of the trivial bundle $\tilde{V}$ and then restrict [redacted] from $Aut(\tilde{V})^{\sigma}$ to $\mathcal{G} = Aut(E^{\sigma n})$. Thus we start with the odd form

$$\int_0^1 dt\ tr\left\{u\theta\ e^{u(t d\theta + (t^2-t)\theta^2)}\right\}$$

which we know [redacted] coincides with the super connection version: p 873–875

$$\int_0^\infty dt\ tr\left\{u\begin{pmatrix}0&1\\1&0\end{pmatrix}\sigma e^{u\left(-t^2+\begin{pmatrix}d\theta & t\sigma\theta\\ t\sigma\theta & 0\end{pmatrix}\right)}\right\}$$

Then we restrict this to $\iota\theta\iota^*$ and use

$$d(\iota\theta\iota^*) = \begin{pmatrix}[D,\theta] & -\theta\iota^* dj\\ (j^*di)\theta & 0\end{pmatrix}$$

Now we have seen the effect of is to take $tr\,\theta(d\theta)^n$ into $tr\begin{pmatrix}\theta&0\\0&0\end{pmatrix}\begin{pmatrix}[D,\theta] & -\theta i^* dj\\ (j^*di)\theta & 0\end{pmatrix}^n$ which is

$$tr\ \theta\begin{pmatrix}\text{words in } [D,\theta] \text{ and}\\ \theta D^2\theta \text{ of degree } n \text{ in } \theta\\ \text{and also in } D\end{pmatrix}$$

Now we can also do this ~~substituting~~ to a typical component of the Chern-Simons form. Let's take

$$\mathrm{tr}\ \theta \begin{pmatrix} \text{words in } d\theta \text{ and } \theta^2 \\ \text{of degree } n-k,\ k \end{pmatrix}$$

Then it is clear that when we ~~substitute~~ $\theta \longmapsto d(i\theta i^*)$ the $d\theta$ changes to $[D,\theta]$ and when we have consecutive $d\theta$'s in a word we are allowed to produce extra terms by $d\theta^2 \longmapsto \theta D^2 \theta$.

We notice that the sort of cyclic cochains obtained sit in the sector of the double complex $C_\lambda^p(\tilde{\mathcal{g}}, \Omega_M^q)$ where $p > q$. If we start with $\mathrm{tr}\ \theta\, S(d\theta^{n-k}, (\theta^2)^k)$ we have $p = 1 + n-k + 2k = 1+n+k$ and $q = n-k$, so $p-q = 1+2k \geq 1$. Now the extra terms produced by allowing $(d\theta)^2$ to become $\theta D^2 \theta$ have the same p, q.

Thus we conclude that the odd form

$$\int_0^1 dt\ \mathrm{tr}\{u\theta\ e^{u(t d\theta + (t^2-t)\theta^2)}\}$$

when pulled back via $\theta \longmapsto i\theta i^*$ is probably not equal to either of the odd forms produced on p 927 or 928. Let's see if this checks for the total differential, i.e. let's compare

$$\mathrm{tr}\left(e^{u(D^2+[D,\theta])} - e^{uD^2}\right) \quad \text{and} \quad \mathrm{tr}\left(e^{u\, d(i\theta i^*)} - 1\right)$$

Since $d(i\theta i^*) = \begin{pmatrix} [D,\theta] & -\theta i^* dj \\ j^* di\, \theta & 0 \end{pmatrix}$ the latter is a polynomial in $\theta D^2 \theta$ and $[D,\theta]$, whereas the former is a polynomial in D^2 and $[D,\theta]$.

The conclusion perhaps is that the former two schemes based on some method of joining $\delta + D$ and $\delta + D + \theta$ are "wrong".

Recall

$$(1 \quad 0)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{a - bd^{-1}c}$$

similarly

$$(0 \quad 1)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{d - ca^{-1}b}$$

and so

$$tr\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = tr\left(\frac{1}{a - bd^{-1}c}\right) + tr\left(\frac{1}{d - ca^{-1}b}\right)$$