1

April 11, 1988

Dear Professor Kassel,

Thank you for your letter of March 3. I am grateful for the references you provided and will add them to my paper. I also found the reference to Hochschild's paper very stimulating along with the discussion that follows it in your ~~paper~~ letter.

It seems to me highly unlikely that the condition of H-unitarity on I implies $\mathrm{Ker}(I \to M(I)) = 0$. Although I do not know a counterexample, one has the following analogy. One can consider the functor which associates to a non-unital algebra I the space $\mathrm{Tor}_*^{I^+}(k,k)$, where k is the ground field and $I^+ = k \oplus I$ the associated unital algebra, as analogous to the functor which associates to a group G its integral homology $H_*(BG)$. Then the analogue of "I H-unital $\Rightarrow$ $\mathrm{Ker}(I \to M(I)) = 0$" is "$BG$ acyclic $\Rightarrow$ the center of G is trivial". A counterexample to the latter can be produced by using the Kan-Thurston theorem to obtain a perfect group G such BG^+ is an Eilenberg-Maclane space $K(A,2)$. Then the covering group $\tilde{G}$ contains A in its center and is such that $B\tilde{G}$ is acyclic.

Your theory of bivariant cyclic groups $HC^*(A,B)$ is really elegant. I am unfortunately unable to offer any intelligent comments about it. My own work has been more or less involved with the attempt to achieve a better understanding, say in traditional homological algebra and derived category terms, of the cyclic formalism. By cyclic formalism

I mean the formulas underlying cyclic theory, i.e., the cyclic complex and the double complexes which are the ~~starting~~ point for your bivariant theory.

I shall now try to explain some of the ideas I have been involved with recently.

1, Hochschild cohomology classes. (Conventions: unital algebras over $\mathbb{C}$.) It turns out to be quite easy to produce Hochschild cohomology classes, that is, elements of $H^{\cdot}(A, A^*) = H_{\cdot}(A,A)^*$, where $*$ denotes dual. If K is an A-bimodule equipped with a trace $\tau: K/[A,K] \longrightarrow \mathbb{C}$, then τ determines a bimodule map

$$K \xrightarrow{\tilde{\tau}} A^* \qquad \tilde{\tau}(k)(a) = \tau(ka) = \tau(ak)$$

whence an induced map

$$H^{\cdot}(A, K) \longrightarrow H^{\cdot}(A, A^*).$$

Hence any element of $H^n(A,K)$ gives rise to an n-dimensional Hochschild cohomology class. To describe this explicitly we use the fact that there is a 1-1 correspondence between bimodule maps $\Omega^n_A \longrightarrow K$ and normalized Hochschild n-cocycles ψ on A with values in K given by

$$a_0\, da_1 \cdots da_n\, a_{n+1} \longmapsto a_0 \psi(a_1, \ldots, a_n) a_{n+1}.$$

Here Ω^n_A is the bimodule of non-commutative n-forms, which one can show fits into an exact sequence

$$A \otimes \bar{A}^{\otimes n+1} \otimes A \xrightarrow{b'} A \otimes \bar{A}^{\otimes n} \otimes A \longrightarrow \Omega^n_A \longrightarrow 0$$

$$(a_0, \ldots, a_{n+1}) \longmapsto a_0\, da_1 \cdots da_n\, a_{n+1}$$

If we represent an element of $H^n(A,K)$ by the normalized Hochschild n-cocycle $\phi(a_1,\dots,a_n)$, then the associated class in $H^n(A,A^*) = H_n(A,A)^*$ is represented by the Hochschild n-cocycle

$$\varphi(a_0,a_1,\dots,a_n) = \tau(a_0\phi(a_1,\dots,a_n))$$

Example. Let M be an A-module (left module) which is finite-dimensional over $\mathbb{C}$. Then we have a map

$$\begin{tikzcd}
\mathrm{Ext}^n_A(M,M) = H^n(A, \mathrm{Hom}_{\mathbb{C}}(M,M)) \arrow[d, "\text{induced by trace on } \mathrm{Hom}_{\mathbb{C}}(M,M)"] & \\
H^n(A,A^*) = H_n(A,A)^* \arrow[d, equal, "\text{if } A \text{ commutative}"] \\
(\Omega^n_A)^*
\end{tikzcd}$$

which was used by Cartier (unpublished) to construct higher-dimensional (Grothendieck) residues.

The trouble with the Hochschild classes arising this way, that is, from a diagram

$$\begin{tikzcd}
\arrow[r] & R_{n+1} \arrow[r] & R_n \arrow[r] \arrow[d, "\tau"] & \cdots \arrow[r] & R_1 \arrow[r] & R_0 \arrow[r] & A \arrow[r] & 0 \\
& & \mathbb{C} & & & & &
\end{tikzcd} \quad (*)$$

where $R.$ is an A-bimodule resolution of A and τ is a closed trace ($\tau d = 0$), is the fact that one can't tell easily when the Hochschild class can be refined to a cyclic cohomology class.

2. DG Algebra resolutions. The key idea behind my paper on cyclic homology + extensions is to view an extension

$$0 \longrightarrow I \longrightarrow R \longrightarrow A \longrightarrow 0$$

as a DG Algebra resolution of A and to apply the cyclic complex functor to it. In general, given a DG algebra resolution $R.$ of A the spectral sequence associated to the double complex $CC.(R.)$ has edge homomorphisms

$$(**) \qquad HC_n(A) \longrightarrow H_n\left(R./[R.,R.]\right).$$

Hence a closed n-dimensional trace on $R.$ yields an n-dimensional cyclic cohomology class on A.

Feigin + Tsygan developed similar ideas using simplicial algebras.

Here's a variant of $(**)$. Consider a DG chain algebra $R.$ starting with $A = R_0$, which is acyclic, or equivalently $1 \in dR_1$. Then the double complex $CC.(R.)$ has exact rows so its positive degree columns form a resolution of $CC.(A)$. This yields edge homomorphisms

$$(**)' \qquad HC_n(A) \longrightarrow H_{n+1}\left(R./[R.,R.]+R_0\right)$$

Example: Tate in his theory of residues on curves considered the situation of an algebra A which is the sum of two ideals: $A = I + J$, and such that there is a trace given on the intersection

$$\tau : I \cap J / [I, J] \longrightarrow \mathbb{C}$$

He essentially constructed a 1-dimensional cyclic cohomology class on A in this situation, although his formulas are written in terms of a commutative subalgebra of A.

Tate's construction can be understood and generalized to give higher-dimensional odd cyclic classes by using the chain algebra which is the amalgamated product

$$R_{.} = (I \longrightarrow A) \underset{A}{*} (J \longrightarrow A)$$

In degree $n \geq 1$ it is $(I \otimes_A J \otimes_A I \otimes_A \cdots) \oplus (J \otimes_A I \otimes_A J \otimes_A \cdots)$ with n-factors in each tensor product. The commutator quotient is $(I \otimes_A J \otimes_A)^n_\sigma$ in degree $2n$ for $n \geq 1$ and is zero in odd degrees ≥ 3. Thus one gets canonical homomorphisms from (**)′

$$HC_{2n-1}(A) \longrightarrow (I \otimes_A J \otimes_A)^n_\sigma$$

and consequently a trace defined on $K^n/[K, K^{n-1}]$ $K = I \cap J$ gives a $(2n-1)$-dimensional cyclic class on A.

I haven't had much success in understanding periodicity phenomena in cyclic theory using chain algebras. Even to describe the basic classes in $HC_*(\mathbb{C} \oplus \mathbb{C}e)$ this way seems hard.

3. The GNS construction. [redacted] The letters GNS stand for either the Gelfand-Neumark-Segal or generalized Stinespring theorem in the C^*-theory. What follows is a translation to a purely algebraic setting in which all positivity notions are suppressed.

3.1. Let A, B be unital algebras and let $\rho : A \to B$ be a linear map on the underlying vector spaces such that $\rho(1) = 1$. Then we define the <u>GNS algebra</u> associated to ρ to be

$$C = A \oplus A \otimes B \otimes A$$

with multiplication as follows. Firstly, $A \otimes B \otimes A$ is a non-unital algebra with the product

$$(a_1 \otimes b_1 \otimes a_1')(a_2 \otimes b_2 \otimes a_2') = a_1 \otimes b_1 \rho(a_1' a_2) b_2 \otimes a_2'.$$

Secondly, it is an A-bimodule in an obvious way and C is the semi-direct product algebra.

The motivation for C is the following. The purpose of GNS is to realize ρ as a "matrix element" in a representation of A over B. This means we seek a left A, right B bimodule E together with B°-module maps

$$B \xrightarrow{\iota} E \xrightarrow{\iota^*} B \qquad \begin{aligned} \iota^*\iota &= id \\ \iota^* a \iota(b) &= \rho(a) b \end{aligned}$$

The GNS algebra C acts naturally on any such representation

$$a \otimes b \otimes a' \longmapsto a \iota b \iota^* a' \in End_{B^\circ}(E)$$

Incidently the maps ι, ι^* extend and coextend respectively to $A\otimes B^\circ$-module maps

$$A\otimes B \longrightarrow E \longrightarrow \mathrm{Hom}_{\mathbb{C}}(A,B)$$
$$a\otimes b \longmapsto a\iota(b)$$
$$\xi \longmapsto (a \mapsto \iota^*(a\xi))$$

whose composition is the unique $A\otimes B^\circ$-module map

$$A\otimes B \xrightarrow{\tilde{\rho}} \mathrm{Hom}_{\mathbb{C}}(A,B)$$

sending $1\otimes 1$ to ρ. We can identify triples (E, ι, ι^*) with factorizations of $\tilde{\rho}$. There is obviously a smallest such factorization, namely the image of $\tilde{\rho}$, and this is the GNS representation in the algebraic setting. (In the C^*-setting one completes $\mathrm{Im}(\tilde{\rho})$ to obtain a Hilbert C^*-module.)

3.2. *Properties of the GNS algebra.* It contains A as a (unital) subalgebra and the idempotent $e = 1\otimes 1\otimes 1$. One has

$$eCe = 1\otimes B\otimes 1$$

and the non-unital subalgebra eCe can be identified with B. One has

$$\rho(a) = eae$$

relative to this identification.

Let $\tilde{e} = 1 - e$. One has

$$eC\tilde{e} = e(A + AeCeA)\tilde{e}$$
$$= eA\tilde{e} + \underbrace{eAeCe}_{=eCe}A\tilde{e} = eCe\cdot eA\tilde{e}$$

and similarly
$$\tilde{e}Ce = \tilde{e}Ae \cdot eCe.$$
Hence the "block" decomposition of C relative to the idempotent e is
$$C = \begin{pmatrix} eCe & eCe\cdot eA\tilde{e} \\ \tilde{e}Ae\cdot eCe & \tilde{e}C\tilde{e} \end{pmatrix}$$
We have the ideals $CeC = AeCeA$ and $C\tilde{e}C$ in C and they have the block decompositions
$$CeC = \begin{pmatrix} eCe & eCe\cdot eA\tilde{e} \\ \tilde{e}Ae\cdot eCe & \tilde{e}Ae\cdot eCe\cdot eA\tilde{e} \end{pmatrix}$$
$$C\tilde{e}C = \begin{pmatrix} eCe\cdot eA\tilde{e}\cdot\tilde{e}Ae\cdot eCe & eCe\cdot eA\tilde{e} \\ \tilde{e}Ae\cdot eCe & \tilde{e}C\tilde{e} \end{pmatrix}$$

Of particular interest is the ideal in $B = eCe$ given by
$$I = eC\tilde{e}Ce = eCe\cdot eA\tilde{e}\cdot\tilde{e}Ae\cdot eCe$$
It is generated by the elements
$$-ea_1\tilde{e}a_2e = e[e,a_1][e,a_2] = \rho(a_1)\rho(a_2) - \rho(a_1a_2),$$
and is the smallest ideal modulo which ρ is a homomorphism.

The expression $e[e,a_1][e,a_2]$ reminds one of the <u>curvature</u> of the Grassmannian connection, see §4 below.

The GNS algebra has the following universal property.

<u>Prop.</u> A (unital) homomorphism $C \to R$ is the same as a homomorphism $A \to R$ together with a $\mathbb{C}$-linear map $v: B \to R$ satisfying the equivalent conditions

i) $$v(b_1)\,a\,v(b_2) = v(b_1\rho(a)b_2)$$

ii) $$v(b_1)v(b_2) = v(b_1b_2) \quad \text{and}$$
$$v(1)\,a\,v(1) = v(\rho(a))$$

Let's now consider the case where A is given and we take (B,ρ) such that $\rho: A \to B$ is a universal $\mathbb{C}$-linear map from A to an algebra such that $\rho(1) = 1$. Thus

$$(*) \qquad B = T(A)\big/T(A)\left(1-\rho(1_A)\right)T(A) \cong T(\bar{A})$$

where $\rho(1_A)$ denotes the identity of A in $T_1(A) = A$. In this case one sees using ii) above that the map v is completely determined by the idempotent $v(1)$ and that this can be an arbitrary idempotent in R. Thus we have

<u>Prop.</u> When (B,ρ) is universal as above, the GNS algebra is the free product algebra

$$C = A * (\mathbb{C} \oplus \mathbb{C}e)$$

or equivalently the cross-product algebra

$$C = (A * A) \otimes \mathbb{C}[\mathbb{Z}/2]$$

where $\mathbb{Z}/2$ flips the two copies of A.

This proposition establishes a link between the Connes-Cuntz study of $A*A$ with the ideal $qA = \mathrm{Ker}(A*A \to A)$ on one hand and the the GNS algebra C with the ideal $K = (CeC) \cap (C\tilde{e}C)$ on the other hand. Corresponding to their result

$$gr_{\cdot}^{qA}(A*A) = \Omega_A^{\cdot}$$

together with an explicit vector space isomorphism between these algebras are similar results such as

$$gr_n^{I}(B) = \Omega_A^{2n}$$

$$gr_n^{I}(eC\tilde{e}) = \Omega_A^{2n+1}$$

An explicit vector space isomorphism of B with Ω_A^{even} is obtained as follows. For each $a \in A$, let $\gamma(a)$ be the operator on Ω_A given by

$$\gamma(a)\omega = a\omega + da\cdot d\omega$$

Then this extends to a left B-module structure on $\Omega_A^{\cdot}$ and acting on 1 and dA gives

$$B \xrightarrow{\sim} \Omega_A^{ev}$$

$$B \otimes \bar{A} \xrightarrow{\sim} \Omega_A^{odd}$$

I haven't had the chance to work out the consequences of these ideas very much. It seems that because

$$0 \longrightarrow I \longrightarrow B \underset{\rho}{\overset{\dashleftarrow}{\longrightarrow}} A \longrightarrow 0$$

is the universal extension of A with linear lifting, one ought to be able to replace Connes' use of Ω_A by the algebra B with its I-adic filtration and thereby obtain a better understanding of cyclic homology. I have in mind the exact sequences of my paper such as

$$0 \to HC_{2n}(A) \longrightarrow HC_0(B/I^{n+1}) \longrightarrow H_1(B, B/I^n)_\sigma \longrightarrow HC_{2n-1}(A) \to 0$$

instead of Connes' formula for the image of S in terms of non-commutative DR homology.

4. Connes homomorphisms + Chern-Simons forms.

With a certain amount of work I succeeded in putting homotopy operators on the rows of the double complex $CC(R \leftarrow I)$ and then checking that the Connes homomorphism

$$HC_{2n+1}(A) \longrightarrow I^{n+1}/[I, I^n] \qquad A = R/I$$

defined as an edge homomorphism in my paper in fact coincides with the map given by Connes' formulas. I didn't succeed in finding formulas for the even Connes homomorphisms the same way. The calculations became too hard, precisely I think because of the complexity of the Chern-Simons forms.

Let's recall how these arise. Over a manifold suppose we have a connection $d + \alpha$ on the trivial bundle with fibre W. Then the connection for α and the curvature $\beta = d\alpha + \alpha^2$ are 1- and 2-forms respectively with coefficients in $\mathrm{End}(W)$. The Chern-Simons form of degree $2n-1$ is (up to constant factors)

$$\eta = n \int_0^1 tr\left\{ \alpha\left((t^2-t)\alpha^2 + t\beta\right)^{n-1} \right\} dt$$

and it satisfies

$$(*) \qquad d\eta = tr(\beta^n)$$

This formula is purely algebraic; it holds in the free cochain algebra $\mathbb{C}\langle \alpha, d\alpha \rangle =$ tensor algebra of the complex

$$0 \to \mathbb{C}\alpha \longrightarrow \mathbb{C}\, d\alpha \longrightarrow 0 \longrightarrow \cdots$$

where tr denote the map to the commutator quotient space.

To apply this let A, B be two non-unital algebras. The reason for this change in setting is because we want to regard B as an A-bimodule with zero left and right multiplication, hence we must forget whether A has a unit. We consider the complex of Hochschild cochains

$$C^\cdot(A, B)$$

on A with values in B. This is a cochain algebra with product

$$(\varphi\psi)(a_1, \ldots, a_{p+q}) = \varphi(a_1, \ldots, a_p)\psi(a_{p+1}, \ldots, a_{p+q})$$

if $\varphi \in C^p(A,B)$, $\psi \in C^q(A,B)$), and with differential $d\varphi = -b'\varphi$, i.e.

$$(d\varphi)(a_1,\dots,a_{p+1}) = \sum_{i=1}^{p}(-1)^i\,\varphi(\dots,a_i a_{i+1},\dots)$$

There's a trace map

$$C^n(A,B) \xrightarrow{\tau} \mathrm{Hom}_{\mathbb{C}}(A_\lambda^{\otimes n}, B/[B,B])$$

defined by sending φ to the cyclic sum

$$(N\varphi)(a_1,\dots,a_n) = \varphi(a_1,\dots,a_n) + (-1)^{n-1}\varphi(a_n,a_1,a_2,\dots) + \dots$$

By the identity $Nb' = bN$ it follows that

$$C^{\cdot}(A,B) \xrightarrow{\tau} \mathrm{Hom}(CC_{\cdot}(A), B/[B,B])$$

satisfies $\tau d = -b\tau$, and so is a map of complexes if we put in the suspension.

Consider a linear map $\rho : A \to B$, i.e. an element $\rho \in C^1(A,B)$. Then

$$(d\rho + \rho^2)(a_1,a_2) = -\rho(a_1 a_2) + \rho(a_1)\rho(a_2)$$

Now apply the Chern-Simons formula η on p. 12 to $C^{\cdot}(A,B)$ with the trace τ and with $\alpha = \rho$, $\beta = d\rho + \rho^2$. This gives a cyclic $2n-2$ - cochain

$$\tau(\eta) : A_\lambda^{\otimes(2n-1)} \longrightarrow B/[B,B]$$

such that $\quad b\tau(\eta) = -\tau(d\eta) = -\tau[(d\rho+\rho^2)^n]$

But if I is an ideal in B containing the elements $\rho(a_1)\rho(a_2) - \rho(a_1 a_2)$, then $(d\rho + \rho^2)^n$ has values in I^n and so we obtain

<u>Prop.</u> Let $\rho : A \longrightarrow B$ be a linear map between non-unital algebras which is a homomorphism modulo the ideal I in B. Then the Chern-Simon expression of degree $2n-1$ applied to the cochain algebra $C^\bullet(A, B)$ with the trace τ, and with the "connection form" $\alpha = \rho$ and the "curvature form" $\beta = d\rho + \rho^2$, gives a cyclic $(2n-2)$-cocycle on A with values in $HC_0(B/I^n)$.

<u>Motivation</u> for $C^\cdot(A,B)$ and the Chern-Simon forms. This is related to the link between cyclic cochains and left invariant forms on matrix groups. One way to produce left-invariant forms on a Lie group G is to start with a representation of G on a vector space V and an idempotent operator e on V. By acting on this idempotent $g \longmapsto geg^{-1}$ we obtain a map from G to the space of idempotents, and we can pull back the canonical character forms on the latter to obtain left-invariant forms on G.

To be specific let $W = eV$ and let $\iota_0 : W \longrightarrow V$ be the inclusion and $\iota_0^* = e : V \longrightarrow W$ the projection. Over G we have the trivial bundle $\tilde{W}$ with fibre W embedded as a direct factor of $\tilde{V}$ by the maps

$$\tilde{W} \xrightarrow{g\iota_0} \tilde{V} \xrightarrow{\iota_0^* g^{-1}} \tilde{W}$$

where g here denotes the tautological automorphism of $\tilde{V}$ over G associated to the G-action on V. Associated to this 'direct' embedding of $\tilde{W}$ in $\tilde{V}$ is a Grassmannian connection having the connection form

$$\iota_0^* g^{-1} dg\, \iota_0 \in \Omega^1(G, \operatorname{End} W)$$

which is left-invariant. From the connection form we obtain other left invariant forms such as the curvature, character forms, and Chern-Simons forms.

In doing all this we work in the cochain algebra

$$\Omega^\cdot(G, \operatorname{End} \tilde{W})^G = C^\cdot(\mathfrak{g}, \operatorname{End} W)$$

of Lie algebra cochains with values in the algebra $\operatorname{End} W$ with the trivial action of $\mathfrak{g}$.

Now when we come to take G to be a matrix group $G = GL_n(A)$ acting on $V = \mathbb{C}^n \otimes M$, M an A-module, the complex of Lie cochains is replaced by the much simpler algebra cochain complex $C^\cdot(A, B)$, where A acts trivially on B.

This letter was written over the past two weeks and the last part about $C^\cdot(A, B)$ was only discovered yesterday. As you can see it starts with the goal of understanding the formulas of cyclic theory via traditional homological algebra and ends with more formulas. I would be very interested in your comments.

Please make a copy for Loday of this letter.

Best regards

Daniel G. Quillen

April 13, 1988

Let's recall the equivalence between profinite sets and compact (Hausdorff) totally disconnected spaces, where a profinite set is a pro object in the category of finite sets. Specifically given a compact totally disconnected space Ω we can consider partitions of Ω into open (hence closed) sets. Such a partition is equivalent to a continuous map from Ω onto a finite set (with the discrete topology), up to isomorphism. Moreover Ω is the projective limit over the directed set of these partitions of the corresponding finite quotient spaces.

It's worth noting that given two finite quotients

$$\begin{tikzcd} & \Omega \arrow[dl] \arrow[dr] & \\ X & & X' \end{tikzcd}$$

corresponding to partitions $\{U_x, x \in X\}$ $\{U'_{x'}, x' \in X'\}$, then the finite quotient

$$\mathrm{Im}\{\Omega \longrightarrow X \times X'\}$$

is the partition consisting of the non-empty intersections $U_x \cap U_{x'}$.

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Given a profinite set Ω we can identify its finite quotient sets with partitions of Ω into clopen sets. The set of partitions is a lattice. To be specific, it is partially ordered where a map $(\Omega \twoheadrightarrow X) \longrightarrow (\Omega \twoheadrightarrow X')$ is a commutative triangle

$$\begin{tikzcd} & \Omega \arrow[dl] \arrow[dr] & \\ X \arrow[rr, two heads] & & X' \end{tikzcd}$$

and the l.u.b. (inf) of $(\Omega \twoheadrightarrow X)$ and $(\Omega \twoheadrightarrow X'')$ is

$$\mathrm{Im}\{\Omega \longrightarrow X \times X'\}.$$

The g.l.b. (sup) is less easy to describe, because it involves the equivalence relation on Ω generated by two equivalence relations.

Important for the sequel are squares in the category of finite quotients of Ω

$$\begin{tikzcd} & \Omega \arrow[d, two heads] & \\ & X' \arrow[dl, two heads] \arrow[dr, two heads] & \\ X \arrow[dr, two heads] & & Y' \arrow[dl, two heads] \\ & Y & \end{tikzcd}$$

which are cartesian, that is, such that $X' \xrightarrow{\sim} X \times_Y Y'$. Let's say that two arrows $X \to Y$ and $Y' \to Y$ in our category of partitions are *transversal* when they lead to such a square that is, when the map

$$\Omega \longrightarrow X \times_Y Y'$$

is surjective.

Now suppose Ω is a profinite set equipped with an automorphism σ. A partition $\Omega \xrightarrow{f} X$ is called a Markov partition provided the following condition holds. Let $\Omega \twoheadrightarrow X^\sigma$ denoted the partition with the map

$$\Omega \xrightarrow{\sigma} \Omega \xrightarrow{f} X$$

Thus if the first partition consists of the open sets $U_x = f^{-1}(x)$ for $x \in X$, then X^σ consists of the open sets

$$(f\sigma)^{-1}(x) = \sigma^{-1}(U_x).$$

Let

$$X_1 = \operatorname{Im}\left\{ \Omega \xrightarrow{(f, f\sigma)} X \times X \right\}$$

be the intersection partition of X, X^σ. Thus X_1 is the subset of $X \times X$ consisting of pairs (x, x') such that there is some $\omega \in \Omega$ with $f(\omega) = x$ and $f(\sigma x) = x'$. Then $f : \Omega \to X$ is a Markov partition provided the map

$$\Omega \xrightarrow{(f\sigma^n,\ n \in \mathbb{Z})} X^{\mathbb{Z}}$$

is an isomorphism of Ω with the subset of sequence (x_n) such that $(x_n, x_{n+1}) \in X_1$ for all n.

Put another way, X is a Markov partition if all the squares in the diagram

$$\begin{tikzcd}
 & & & {} \arrow[dl] \arrow[dr] & & {} \arrow[dr] & & & \\
 & & {} \arrow[dl] \arrow[dr] & & X_2^{\sigma^{-1}} \arrow[dl] \arrow[dr] & & X_2 \arrow[dl] \arrow[dr] & & \\
 & X_1^{\sigma^{-2}} \arrow[dl] \arrow[dr] & & X_1^{\sigma^{-1}} \arrow[dl] \arrow[dr] & & X_1 \arrow[dl] \arrow[dr] & & X_1^{\sigma} \arrow[dl] \arrow[dr] & \\
X^{\sigma^{-2}} & & X^{\sigma^{-1}} & & X & & X^{\sigma} & \cdots & X^{\sigma^2}
\end{tikzcd}$$

are cartesian and if Ω is the inverse limit ~~of~~ this diagram of finite sets.

A pair (Ω,σ) is called a <u>subshift of finite type</u> provided there exists a Markov partition. Example: Suppose we start with a finite set S, and for $k=1,\dots,p$ suppose we are given a subset T_k of S^k. Let Ω be the subset of $S^{\mathbb{Z}}$ consisting of sequences (x_n) such that $(x_{n+1},\dots,x_{n+k}) \in T_k$ for $\forall n$. Let σ be the shift 1-step backward on $S^{\mathbb{Z}}$. Then (Ω,σ) should be a subshift of finite type provided $\Omega \neq \phi$. To see this we can replace the subsets T_k for $k=1,\dots,p$ by a single subset Γ of S^n. Namely Γ consists of all $(x_1,\dots,x_p) \in S^p$ such that $(x_{n+1},\dots,x_{n+k}) \in T_k$ for $n=0,\dots,p-k$.

So we have Ω is the subset of $S^{\mathbb{Z}}$ consisting of sequences such that any segment of length p

belongs to Γ. We next replace Γ by the image of Ω in S^p ??

Review: We let $\Omega \subset S^{\mathbb{Z}}$ consist of all sequences such that any segment of length p belongs to the subset Γ of S^p. Let X_0 ~~be~~ the ~~subset~~ of S^p consisting of $(x_0, \ldots, x_{p-1})$ which can be prolonged to an element of Ω. Let X_1 be the subset of S^{p+1} consisting of $(x_0, \ldots, x_p)$ which can be prolonged to an element of Ω. Then we have two maps

$$X_1 \rightrightarrows X_0 \qquad (x_0, \ldots, x_p) \begin{array}{l} \longmapsto (x_0, \ldots, x_{p-1}) \\ \longmapsto (x_1, \ldots, x_p) \end{array}$$

which are surjective. Moreover $X_1 \hookrightarrow X_0 \times X_0$ since $S^{p+1} \hookrightarrow S^p \times S^p$ in this way. Thus X_1 is a relation on the set X_0 such that the two projections are surjective. We have a map

$$\Omega \longrightarrow \cdots\cdots \times_{X_0} X_1 \times_{X_0} X_1 \times_{X_0} X_1 \times_{X_0} \cdots\cdots$$

which associates to a sequence in Ω the sequence of its segments of length p. This map is compatible with the shifts on both sides. It's clearly injective because

$$\cdots \times_{S^p} S^{p+1} \times_{S^p} S^{p+1} \times_{S^p} \cdots = S^{\mathbb{Z}}$$

Similarly it's surjective.

The above is fairly awkward. Review: Suppose $\Omega \subset S^{\mathbb{Z}}$ is specified by conditions on its segments of $\leq p$. Let X_0 be the set of segments of length $p-1$ occurring in Ω, and X_1 the set of segments of length p occurring in Ω. Then the two arrows $X_1 \rightrightarrows X_0$ are

surjective, and we have a map

$$\Omega \longrightarrow \quad \cdots \times_{X_0} X_1 \times_{X_0} X_1 \times_{X_0} \cdots$$

which assigns to any ω in Ω its sequence of length p-segments. The infinite product on the left is contained in

$$S^{\mathbb{Z}} = \cdots \times_{S^{p-1}} S^p \times_{S^{p-1}} S^p \times_{S^{p-1}} \cdots$$

so the above map is injective. It's surjective because an element of Ω is determined by conditions on its segments of length $\leq p$.

Note that if $p = 2$ in the above argument then $X_0 =$ those elements of S occurring in Ω and $X_1 =$ those pairs occurring consecutively, so that we recover the standard description of a subshift of finite type as the infinite product of the correspendense.

This can be improved in clarity. The logical point is that any subshift of finite type (this means an $\Omega \subset S^{\mathbb{Z}}$, stable under the shift, defined by finitely many conditions — $\Omega = \bigcap \sigma^n U$ where U is a cylinder set) has a Markov partition. This gives a presentation where the conditions are given on consecutive pairs only

April 15, 1988

Let's define a subshift of finite type to be a subspace Ω of $S^{\mathbb{Z}}$ with S finite of the form $\bigcap_{n\in\mathbb{Z}} \sigma^n U$ where U is a clopen subset. Clopen sets should be the same as cylinder sets. To see this it is enough to show that any partition $f: S^{\mathbb{Z}} \to X$ finite factors through $S^{[-n,n]}$ for some n. This in turn should be part of the identification of compact totally-disconnected spaces with profinite sets.

(Suppose $\Omega = \varprojlim_{\alpha} \Omega/\mathcal{U}_\alpha$, where $\mathcal{U}_\alpha$ is a directed set of partitions. Let A be a clopen set in Ω, and consider the partition $(A, \Omega - A)$. Each point of Ω has a nbd basis consisting of the members of the partitions $\mathcal{U}_\alpha$ containing it. We can cover A by finitely many such open sets contained in A, use directedness, and see that there is an α such that A is the union of members of $\mathcal{U}_\alpha$.)

We saw yesterday that any subshift of finite type has Markov partitions. What I need is a better feeling for these among all partitions. The natural program would be to start with a pair (Ω, σ) and ask when it can be identified with a subshift of finite type. Such an identification is determined by a map $f: \Omega \to S$. In fact

$$\mathrm{Hom}_{\mathbb{Z}\text{-spaces}}(\Omega, S^{\mathbb{Z}}) = \mathrm{Hom}_{\text{spaces}}(\Omega, S)$$

And we can suppose $f : \Omega \to S$
i.e. f is essentially a partition. Thus starting with (Ω, σ) we can consider partitions which are <u>σ-separating</u> in the sense that the map $\Omega \to S^{\mathbb{Z}}$ is injective.

Next we would like to show that the image of this map is a subshift of finite type. What seems to be true is that if there is one separating partition with this property then any separating partition has this property.

<u>Idea:</u> Define (Ω, σ) to be a <u>subshift of finite type</u> when it admits a presentation

$$(*) \qquad \Omega \longrightarrow S_1^{\mathbb{Z}} \rightrightarrows S_0^{\mathbb{Z}}$$

where S_0, S_1 are finite sets.

For example: Start with a relation $S_1 \subset S_0 \times S_0$ such that the two projections $S_1 \rightrightarrows S_0$ are surjective. Let $\Omega \subset S_0^{\mathbb{Z}}$ be the set of sequences (x_n) such that $(x_n, x_{n+1}) \in S_1$ for all n. Then we have such a presentation when $\Omega \to S_1^{\mathbb{Z}}$ corresponds to $\Omega \to S_1$ sending (x_n) to (x_0, x_1), and where $S_1^{\mathbb{Z}} \rightrightarrows S_0^{\mathbb{Z}}$ corresponds to $S_1^{\mathbb{Z}} \rightrightarrows S_0$ sending (γ_n) where $\gamma_n = (x_n', x_n'') \in S_1 \subset S_0 \times S_0$ to x_0'' and x_1' respectively.

Once we have a subshift of finite type then we can show there exist Markov partitions as follows. Starting with the presentation $(*)$ we look at the two maps $S_1^{\mathbb{Z}} \rightrightarrows S_0^{\mathbb{Z}}$. These correspond to maps $S_1^{\mathbb{Z}} \rightrightarrows S_0$ which in turn come from maps $S_1^{[-N,N]} \rightrightarrows S_0$ for some N. Thus Ω is the largest $\mathbb{Z}$-invariant subspace of $S_1^{\mathbb{Z}}$ whose $(-N,N)$ segments lie in the subspace $\Gamma = \mathrm{Ker}\{S_1^{[-N,N]} \rightrightarrows S_0\}$. Now let $X_0 \subset S_1^{p-1}$, $X_1 \subset S_1^{p}$ be the subsets of finite sequences which extend to sequences in Ω, where $p = 2N+1$. Then we have the Markov presentation

$$\Omega \longrightarrow X_1^{\mathbb{Z}} \rightrightarrows X_0^{\mathbb{Z}}$$

In effect we are describing sequences ~~with~~ values in S_1 such that each length p-segment agrees with a length p-segment of an element of Ω; but such a sequence satisfies the conditions defining a sequence in Ω.

So the remaining question is whether given an embedding $\Omega \hookrightarrow X^{\mathbb{Z}}$ where Ω is a subshift of finite type, does it follow that this embedding can be extended to a presentation

$$\Omega \longrightarrow X^{\mathbb{Z}} \rightrightarrows Y^{\mathbb{Z}}$$

In other words if we glue two copies of $X^{\mathbb{Z}}$ along Ω, can we find a separating partition for the resulting $\mathbb{Z}$-space?

It seems that the sequence space $S^{\mathbb{Z}}$ is injective in the category of profinite $\mathbb{Z}$-spaces. Because of

$$\mathrm{Hom}_{\mathbb{Z}\text{-spaces}}(\Omega, S^{\mathbb{Z}}) = \mathrm{Hom}_{\text{spaces}}(\Omega, S)$$

it suffices to show that S is injective in the category of profinite sets. But if $\Omega \hookrightarrow \Omega'$ is injective, then we can write it as the filtered induction limit of injective maps of finite sets $S_\alpha \hookrightarrow S'_\alpha$, and then

$$\begin{array}{ccc} \operatorname{Hom}_{\text{spaces}}(\Omega, S) & = & \varinjlim_\alpha \operatorname{Hom}(S_\alpha, S) \\ \uparrow & & \uparrow \\ \operatorname{Hom}_{\text{spaces}}(\Omega', S) & = & \varinjlim_\alpha \operatorname{Hom}(S'_\alpha, S) \end{array}$$

so by exactness of filtered inductive limits we see any map $\Omega \longrightarrow S$ can be extended to Ω'.

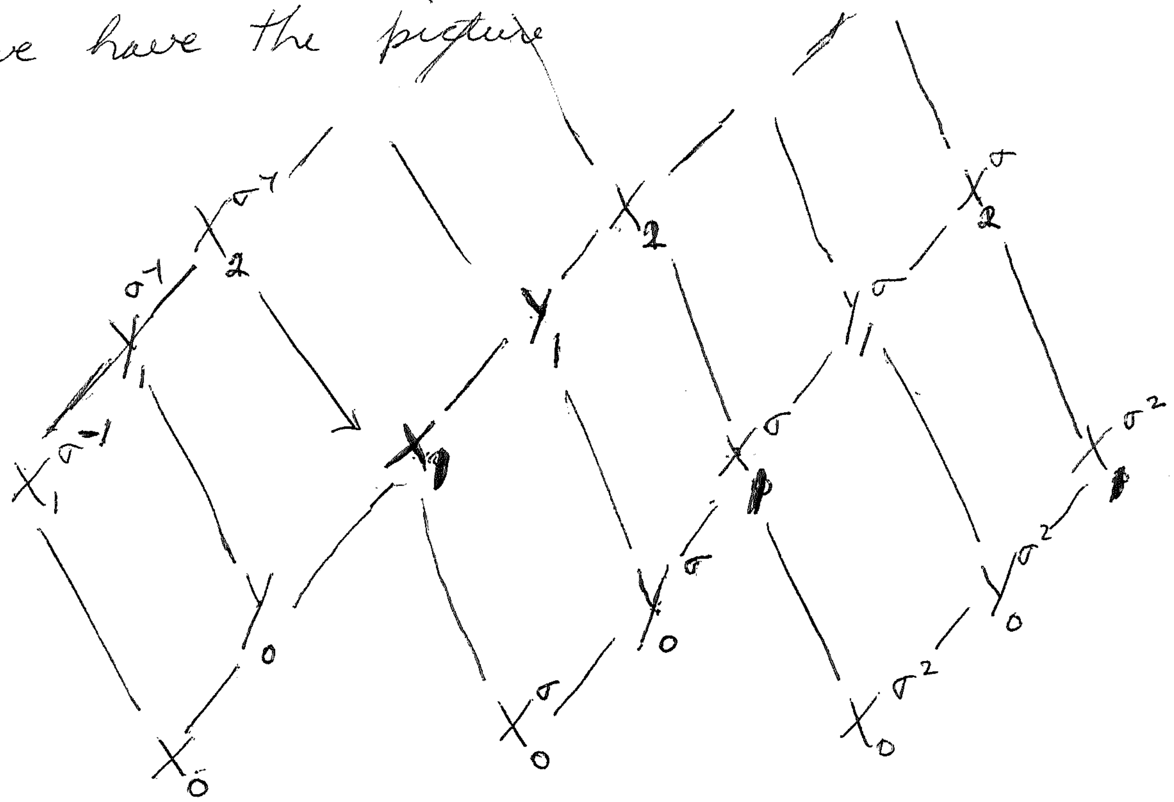
Let's start with a subshift Ω of finite type and a separating partition $\Omega \twoheadrightarrow X$. We know we can find a Markov presentation of Ω which dominates X. This gives us a diagram

$$\begin{tikzcd} \Omega \arrow[r, hook, "i"] \arrow[dr, "j"'] & X_0^{\mathbb{Z}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X_1^{\mathbb{Z}} \\ & X^{\mathbb{Z}} \arrow[u, dashed, bend left, "s"] & \end{tikzcd}$$

By the injectivity of $X_0^{\mathbb{Z}}$ we can find the dotted arrow s such that $sj = i$. Unfortunately s is not a lifting of $X^{\mathbb{Z}}$ into $X_0^{\mathbb{Z}}$. ?

So let's turn next to relating different Markov partitions on the same subshift of finite type. Williams showed that any two Markov partitions could be joined by a chain of

Markov [redacted] partitions [redacted] such that [redacted] any two consecutive M.P. in the chain are related [redacted] in an elementary fashion. The fundamental lemma seems to be that if $\Omega \twoheadrightarrow X_0$ is a MP then any partition between X_0 and $X_1 = \mathrm{Im}(\Omega \to X_0 \times X_0^\sigma)$ is also a Markov partition. Call such a partition Y_0, so that in the poset of partitions we have the picture



Here Y_1 is the sup of Y_0 and Y_0^σ, or really it would be better to say the intersection. Since Y_0 is between X_0 and $X_1 = X_0 \cap X_0^\sigma$, it is clear that $X_1 = Y_0 \cap X_0^\sigma$, so

$$X_1 \cap Y_0^\sigma = Y_0 \cap X_0^\sigma \cap Y_0^\sigma = Y_0 \cap Y_0^\sigma = Y_1$$

But we know that

$$X_2 \xrightarrow{\sim} X_1 \times_{X_0^\sigma} X_1^\sigma$$

as sets, hence we can conclude that

all squares in

$$\begin{tikzcd}
 & & X_2 \arrow[dl] \arrow[dr] & \\
 & Y_1 \arrow[dl] \arrow[dr] & & X_1^{\sigma} \arrow[dl] \\
X_1 \arrow[dr] & & Y_0^{\sigma} \arrow[dl] & \\
 & X_0^{\sigma} & &
\end{tikzcd}$$

are cartesian. The same arguments works for all the subdivided squares of the X-diagram. Thus we can conclude Y is Markov.

Another proof starts from

$$Y_1 = X_1 \times_{X_0^{\sigma}} Y_0^{\sigma}$$

Then

$$\cdots \times_{Y_0^{\sigma^{-1}}} Y_1^{\sigma^{-1}} \times_{Y_0} Y_1 \times_{Y_0^{\sigma}} Y_1^{\sigma} \times_{Y_1^{\sigma^2}} \cdots$$

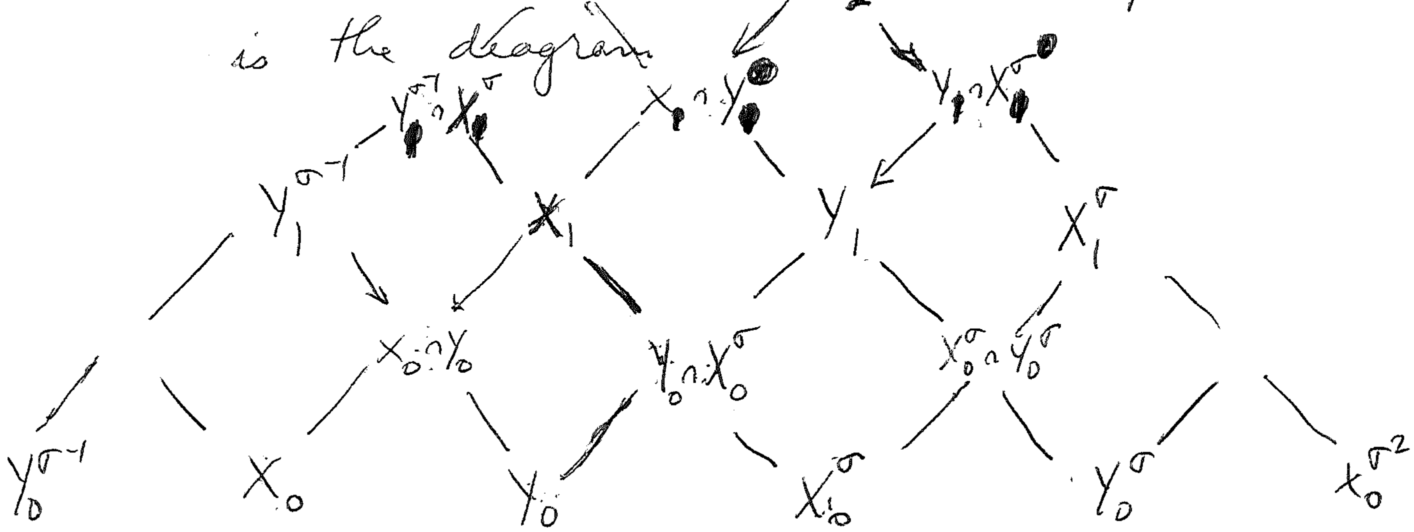
$$= \cdots \times_{Y_0^{\sigma^{-1}}} \left(X_1^{\sigma^{-1}} \times_{X_0} \cancel{Y_0}\right) \times_{\cancel{Y_0}} \left(X_1 \times_{X_0^{\sigma}} \cancel{Y_0^{\sigma}}\right) \times_{\cancel{Y_0^{\sigma}}} \times \cdots$$

$$= \cdots \times_{X_0^{\sigma^{-1}}} X_1^{\sigma^{-1}} \times_{X_0} X_1 \times_{X_0^{\sigma}} X_1^{\sigma} \times_{X_0^{\sigma^2}} \cdots = \Omega$$

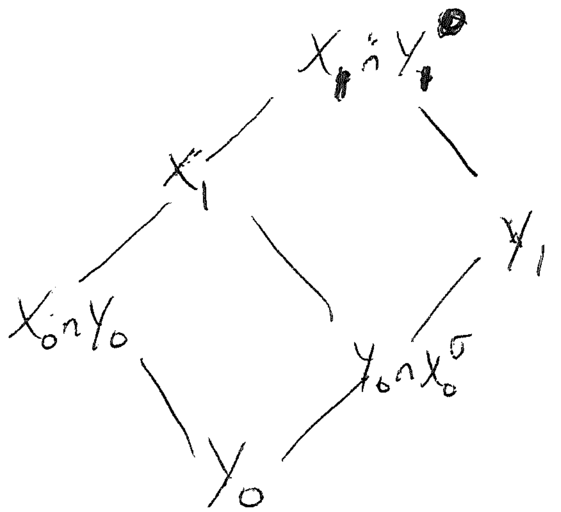
In a similar way one can see that any partition between X_0^{σ} and X_1 is Markov.

Next we come to Williams's key notion of an elementary strong shift equivalence. These are two MPs, call them X_0, Y_0 such that $X_0 \leftarrow X_0 \cap Y_0 \leftarrow X_1$ and $Y_0 \leftarrow X_0 \cap Y_0 \leftarrow Y_1^{\sigma^{-1}}$.

Associated to ~~X_2~~ such a pair is the diagram



From our previous example we know that since Y_0 is Markov the big square



is cartesian as a diagram of sets. Reasoning as before we conclude the little squares are also cartesian.

Now we want to understand why we can go from one MP to another by a chain of these elementary equivalences.

Let (Ω, σ) be a subshift of finite type, and let $\mathcal{P}$ be the set of its Markov partitions. If X, Y are Markov partitions say they are elementary (strong shift) equivalent if in the lattice of partitions we have

$$\begin{tikzcd}[column sep=small]
Y \cap Y^{\sigma^{-1}} \arrow[dr, dash] & & X \cap X^{\sigma} \arrow[dl, dash] \\
& X \cap Y \arrow[dl, dash] \arrow[dr, dash] & \\
X & & Y
\end{tikzcd}$$

Wagoner takes this as the definition of ordered 1-simplex. Evidently he defines the structure of an ordered simplicial complex on $\mathcal{P}$ by defining a simplex to be a sequence of MPS $(X_0, \ldots, X_p)$ such that for $0 \le i < j \le p$ the pair (X_i, X_j) are elementary equivalent.

What I find confusing is the sense of two directions. The really elementary moves are of two types from X up to something bounded by $X \cap X^{\sigma}$ or from X up to something bounded by $X \cap X^{\sigma^{-1}}$.

It seems that the natural structure one has on the set of Markov partitions is some sort of bisimplicial complex. We have two kinds of 1-simplices which move upward either to the left or to the right. To be specific a right 1-simplex is an ordered pair (X, Y) such that $X \geq Y \geq X \cap X^{\sigma}$ whereas a left

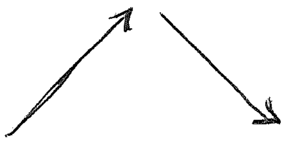
one-simplex is a pair (X, Y) such that
$$X \geq Y \geq X \cap X^{\sigma^{-1}}.$$
It's clear what one should mean by a right p-simplex, namely, a chain

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_p \leftarrow X_0 \cap X_0^{\sigma}$$

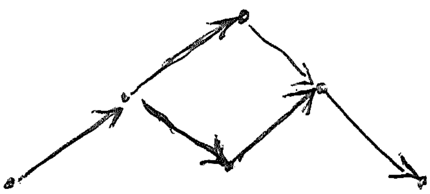
Similarly we have left p-simplices.

Wagener's 1-simplices are pairs consisting of a right followed by a left 1-simplex.

so if we reverse the direction of the arrows for the left 1-simplices we get the picture



which is reminiscent of the Artin-Mazur construction of a simplicial set associated to a bisimplicial set. The Artin-Mazur 2-simplices are diagrams of the form



April 17, 1988

Recall in the poset of Markov partitions we have 1-simplices rising to the right

$$\begin{tikzcd} & Y \\ X \arrow[ur, dash] & \end{tikzcd}$$

this means $X \leftarrow Y \leftarrow X \cap X^{\sigma}$

and 1-simplices rising to the left

$$\begin{tikzcd} Y \arrow[dr, dash] & \\ & X \end{tikzcd}$$

means: $X \leftarrow Y \leftarrow X^{\sigma^{-1}} \cap X$

Consider now two of these simplices in succession:

$$\begin{tikzcd} & Z \arrow[dl, dash] \arrow[dr, dash] & \\ X & & Y \end{tikzcd}$$

Question: Is $Z = X \cap Y$?

Certainly Z maps to $X \cap Y$ and we know ~~[illegible]~~ we obtain a diagram of Markov partitions with cartesian squares

$$\begin{tikzcd}[column sep=small] & & (X\cap Y)_1^{\sigma^{-1}} \arrow[dl, dash] \arrow[dr, dash] & & (X\cap Y)_1 \arrow[dl, dash] \arrow[dr, dash] & & & \\ & Y_1^{\sigma^{-1}} \arrow[dl, dash] \arrow[dr, dash] & & X_1 \arrow[dl, dash] \arrow[dr, dash] & & Y_1 \arrow[dl, dash] \arrow[dr, dash] & & \\ {} \arrow[dr, dash] & & X\cap Y \arrow[dl, dash] \arrow[dr, dash] & & Y \cap X^{\sigma} \arrow[dl, dash] \arrow[dr, dash] & & X^{\sigma} \cap Y^{\sigma} \arrow[dl, dash] \arrow[dr, dash] & \\ & X & & Y & & X^{\sigma} & & Y^{\sigma} \end{tikzcd}$$

So $X \cap Y$ is a MP, and the arrow $Z \to X \cap Y$ would be both a left and right 1-simplex. This forces $Z = X \cap Y$ as we have seen.

Let's check this [redacted] as follows.
We know that the square

$$\begin{tikzcd}[column sep=small]
& Y_1^{\sigma^{-1}} \cap X_1 \arrow[dl, dash] \arrow[dr, dash] & \\
Y_1^{\sigma^{-1}} \arrow[dr, dash] & & X_1 \arrow[dl, dash] \\
& X \cap Y &
\end{tikzcd}$$

is cartesian. Since the arrows are surjective it is also cocartesian. Since Z is dominated by $Y_1^{\sigma^{-1}}$ and X_1, it follows that Z must equal $X \cap Y$.

The next thing we want to do is to describe Wagner's 2-simplices. This means we [redacted] have three of his one simplices

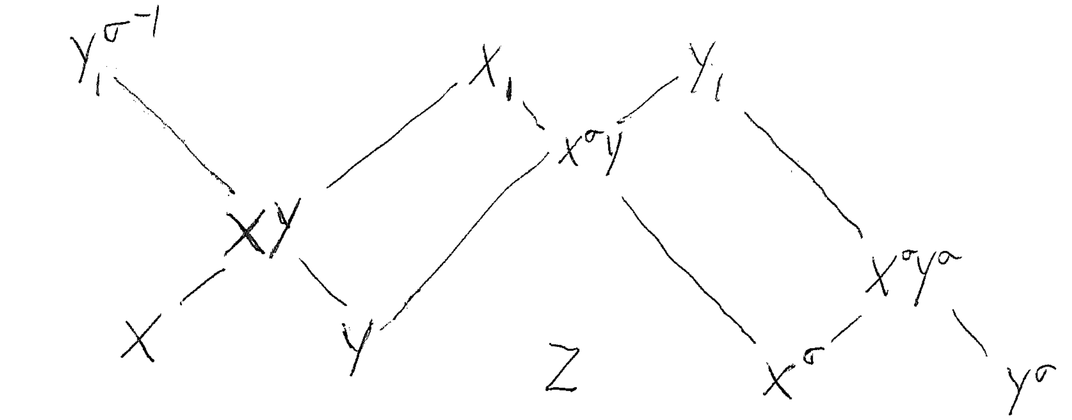
$$\begin{tikzcd}[column sep=small]
& & XZ \arrow[dll, dash] \arrow[drr] & & \\
& XY \arrow[dl, dash] \arrow[dr, dash] & & YZ \arrow[dl, dash] \arrow[dr, dash] & \\
X & & Y & & Z
\end{tikzcd}$$

[redacted] Our goal is to show that [redacted] XZ dominates Y so that we have

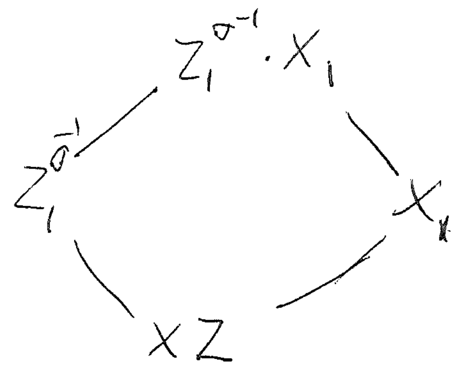
$$\begin{tikzcd}[column sep=small]
& & & X_1\text{[redacted]} \arrow[dl, dash] \arrow[dr, dash] & & & \\
& & XZ \arrow[dl, dash] \arrow[dr, dash] & & YX^{\sigma} \arrow[dl, dash] \arrow[dr, dash] & & \\
& XY \arrow[dl, dash] \arrow[dr, dash] & & YZ \arrow[dl, dash] \arrow[dr, dash] & & ZX^{\sigma} \arrow[dl, dash] \arrow[dr, dash] & \\
X & & Y & & Z & & X^{\sigma}
\end{tikzcd}$$

Let's start drawing the diagram

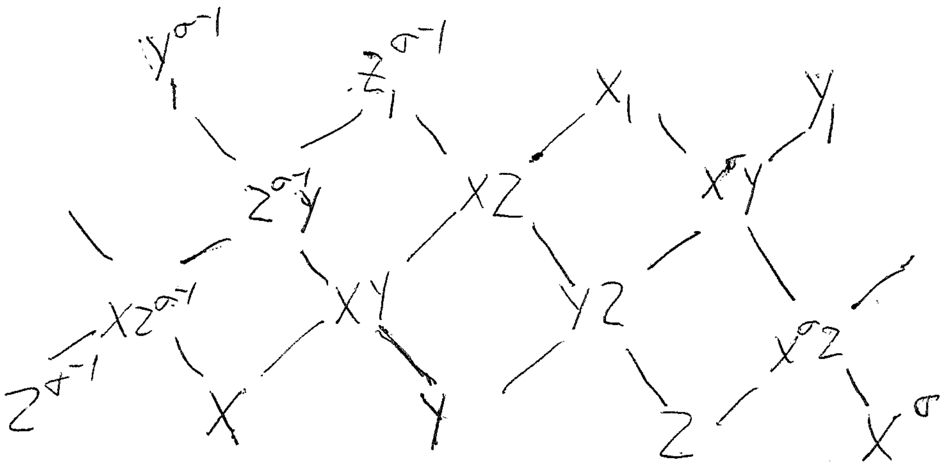
We start with



then add Z. By assumption $Y \le YZ \le Z_1^{\sigma^{-1}}$ and X_1. But we know that



is also cocartesian. Thus since $Y \le Z_1^{\sigma^{-1}}$ and X_1 it follows that $Y \le XZ$. So we can fill in



It appears that the key step is

$X \le Y_1^{\sigma^{-1}}$ and $Z_1^{\sigma^{-1}}$ so $X \le Z^{\sigma^{-1}}Y$

$Y \le Z_1^{\sigma^{-1}}$ and X_1 so $Y \le XZ$

$$Z \leq X_1 \text{ and } Y_1 \text{ so } Z \leq X \circ Y$$

So at this point I think I understand Wagnoner's RS triangle ~~identities~~. It would be nice to understand the rest of his arguments.

Let's review what we learned about the Cuntz-Krieger algebra associated to a zero-one matrix. Notation: Let $\Gamma \subset I \times I$ project surjectively onto both factors, where I is a finite set. Let $\Omega \subset I^{\mathbb{Z}}$ be the subspace of sequences whose consecutative pairs lie in Γ, and let

$$\Omega_{\geq 0} = \Gamma \times_I \Gamma \times_I \cdots$$

be the space of sequences in $I^{\mathbb{N}}$ with consecutive pairs in Γ. σ is the backward shift on Ω and $\Omega_{\geq 0}$.

Look at various subsets of $\Omega_{\geq 0}$. First of all we have the partition given by the map $\Omega_{\geq 0} \xrightarrow{\pi_0} I$ giving the initial element of a sequence. Call the members of this partition

$$U_i = \pi_0^{-1}\{i\}.$$

Then we have the inverse images of these sets under σ, i.e. the partition

$$\Omega_{\geq 0} \xrightarrow{\pi_1 = \pi_0 \sigma} I$$

Set $V_i = \pi_1^{-1}\{i\} = \sigma^{-1}\{U_i\}$. Then V_i consists of sequences $(x_n)_{n \geq 0}$ such that $x_1 = i$, whereas U_i consists of sequences with $x_0 = i$.

Next we have that

$$\Omega_{\geq 0} \text{[redacted]} = \coprod_{(i,j)\in\Gamma} \underbrace{U_i \cap V_j}_{\pi_{0,1}^{-1}\{i,j\}}$$

and the [redacted] main point is that

$$\sigma : U_i \cap V_j \xrightarrow{\sim} U_j$$

when $(i,j) \in \Gamma$. This is just the ~~statement~~ that [redacted] every sequence in U_j, that is, starting with j is the backward shift of a unique sequence ~~starting~~ with (i,j) provided $(i,j) \in \Gamma$. [redacted]

Let's now suppose that we have an invariant measure $d\mu$ on $\Omega_{\geq 0}$. This means

$$\mu(\sigma^{-1}A) = \mu(A)$$

for all cylinder sets A, and it implies that [redacted] $f \longmapsto f\sigma$ is an isometric embedding of $L^2(\Omega_{\geq 0}, d\mu)$ into itself.

[redacted]

April 18, 1988

Let $\Omega \subset I^{\mathbb{Z}}$ be the subspace of sequences (x_n) such that $\forall n$ $(x_n, x_{n+1}) \in \Gamma$ where $\Gamma \subset I \times I$ is a relation projecting surjectively onto each factor. We can view Γ as a $(0,1)$ matrix $p(x,y)$. In good cases the Frobenius thm. says p has unique (up to scalar factors) left and right eigenvectors ν, μ with positive entries, and these have the same eigenvalue λ which is the unique eigenvalue of maximum absolute value. Then we can define a measure on Ω as follows. On I^n we define the measure

$$\mu_n : (x_1, \dots, x_n) \longmapsto \lambda^{-n} \mu(x_1)\, p(x_1, x_2) \cdots p(x_{n-1}, x_n)\, \nu(x_n)$$

This is supported on $\Gamma \times_I \overset{n-1}{\cdots} \times_I \Gamma = \Omega_{[1,n]}$. Then because μ, ν are eigenvalues, it follows that μ_n on I^n pushes down to μ_{n-1} on I^{n-1} under the maps $I^n \rightrightarrows I^{n-1}$ omitting the first and last vertex. Thus we obtain a coherent family of measures on the finite quotients $\Omega_{[m,n]}$ of Ω, and hence a measure $d\mu$ on Ω. It is a probability measure provided μ, ν are normalized so that $\sum_x \mu(x)\nu(x) = 1$.

The Hilbert space $L^2(\Omega, d\mu)$ contains the filtration $L^2(\Omega_{\geq n})$. These subspaces are moved around by the shift σ.

~~$L^2(\Omega_{\geq n})$ is spanned by the functions $f\pi_k = (f\pi_0)\sigma^k$ where $\pi_k : \Omega \to I$ is the k-th projection and where~~

We have the projection $\Omega \longrightarrow \Omega_{\geq 0}$, call it ρ_0, which induces the embedding $\rho_0^* : L^2(\Omega_{\geq 0}) \hookrightarrow L^2(\Omega)$

Then we have

$$\begin{tikzcd} \Omega \arrow[d, "\rho_0"'] & \Omega \arrow[l, "\sigma^n"'] \arrow[d, "\rho_n"] \\ \Omega_{\geq 0} & \Omega_{\geq n} \arrow[l, "\sim"'] \end{tikzcd}$$

whence we see that

$$\sigma^n \rho_0^* L^2(\Omega_{\geq 0}) = \rho_n^* L^2(\Omega_{\geq n}).$$

We should think therefore of σ being a unitary autom. of $L^2(\Omega)$ carrying $L^2(\Omega_{\geq 0})$ inside ~~itself~~, and $\sigma^n L^2(\Omega_{\geq 0}) = L^2(\Omega_{\geq n})$.

It seems that $L^2(\Omega_{\geq 0})$ is an outgoing subspace in the sense of Lax-Phillips. Certainly $\bigcup \sigma^n L^2(\Omega_{\geq 0})$ is dense in $L^2(\Omega)$; it is not unreasonable to expect $\bigcap \sigma^n L^2(\Omega_{\geq 0})$ to be zero, although this might be technical to prove. Similarly we can expect $L^2(\Omega_{\leq 0})$ to be an incoming subspace

If all this works there should be a scattering operator which is describable in terms of the original $(0,1)$ matrix.

Notice that $L^2(\Omega_{\geq 0}) / \sigma L^2(\Omega_{\geq 0})$ is likely to be infinite dimensional. For example, if $\Gamma = I \times I$, then

$$L^2(\Omega_{\geq 0}) = V \otimes V \otimes V \otimes \cdots \qquad V = L^2(I)$$
$$\qquad\qquad\qquad\qquad \cup$$
$$\sigma L^2(\Omega_{\geq 0}) = \mathbb{C} \otimes V \otimes V \otimes \cdots$$

Let's return now to the Cuntz-Krieger algebra. I believe this is supposed to operate on $L^2(\Omega_{\geq 0})$.

Let's begin with the operators we have on $L^2(\Omega)$. We have the algebra $C(\Omega)$ of continuous functions and the automorphism σ. Thus we have the cross product C^*-algebra

$$C(\Omega) \rtimes \mathbb{Z}$$

acting on $L^2(\Omega)$.

Next we have the projector onto $L^2(\Omega_{\geq 0})$, call it e. A natural question is the relation between $e(C(\Omega)\times\mathbb{Z})e$ and the Cuntz-Krieger C^*-algebra.

First let's observe that $C(\Omega)\times\mathbb{Z}$ is generated by $C(I) \xrightarrow{\pi_0^*} C(\Omega)$, where $\pi_0 : \Omega \to I$ is the 0-th coordinate, and the autom. σ. For each $x \in I$ let P_x denote the projector corresponding to the subset of (x_n) with $x_0 = x$. Then $\sigma P_x \sigma^{-1}$ projects onto the subset of (x_n) with $x_1 = x$. Check:

$$(P_x f)(\vec{x}) = \chi_{\pi_0^{-1}\{x\}}(\vec{x})\, f(\vec{x})$$

$$\begin{aligned}(\sigma P_x \sigma^{-1} f)(\vec{x}) &= (P_x \sigma^{-1} f)(\sigma\vec{x}) \\ &= \underbrace{\chi_{\pi_0^{-1}\{x\}}(\sigma\vec{x})}_{\chi_{\pi_1^{-1}\{x\}}(\vec{x})}\ \underbrace{(\sigma^{-1}f)(\sigma\vec{x})}_{f(\vec{x})}\end{aligned}$$

We now consider the operators on $L^2(\Omega_{\geq 0})$, which we can think of as functions depending only on the coordinates x_n, $n \geq 0$. Clearly if f is such a function so is $f\sigma$, in fact $f\sigma \in L^2(\Omega_{\geq 1})$. Thus we have

$$\begin{tikzcd}
L^2(\Omega) \arrow[r, "\sim", "\sigma"'] & L^2(\Omega) \\
L^2(\Omega_{\geq 0}) \arrow[u, phantom, "\bigcup \rho^*"] \arrow[r, hook, "\sigma"] & L^2(\Omega_{\geq 0}) \arrow[u, phantom, "\bigcup \rho^*"]
\end{tikzcd}$$

So on our subspace $L^2(\Omega_{\geq 0})$ we have the isometry σ. From this we can construct the partial isometries $s_i = \sigma P_i$. These have domain projections

P_i = mult. by $\chi_{\pi_0^{-1}\{i\}}$

$$s_i^* s_i = P_i \sigma^{-1} \sigma P_i = P_i$$

and range projections

$$s_i s_i^* = \sigma P_i \sigma^{-1} = Q_i$$

where Q_i is multiplication by the characteristic function of the set $\pi_1^{-1}\{i\}$. It's worth noting that the multiplication by $\rho^*(f)$ operator on $L^2(\Omega)$, where f is a function on $\Omega_{\geq 0}$ preserves the subspace $\rho^* L^2(\Omega_{\geq 0})$, and in fact we have

$$\rho_*(\rho^*(f)\, g) = f \cdot \rho_*(g)$$

Finally we want to relate the P_i, Q_i.

It seems we may have the wrong isometries. Recall that we want the P_i's to give an orthogonal decomposition and the Q_j's to satisfy

$$Q_i = \sum_{(i,j) \in \Gamma} P_j$$

So take P_i to be the char. function of $\pi_0^{-1}\{i\}$ and Q_j to be the char. function of the set of sequences $(x_n)_{n\geq 0}$ such that $(j, x_0, x_1, \cdots) \in \Omega_{\geq 0}$

This gives projections with the required properties. What are the corresponding isometries?

Note that if $X \subset \Omega$, then on $L^2(\Omega)$ we have

$$(\sigma \chi_X \sigma^{-1} f)(\omega) = (\chi_X \sigma^{-1} f)(\sigma\omega)$$
$$= \chi_X(\sigma\omega)\ (\sigma^{-1} f)(\sigma\omega)$$
$$= \chi_{\sigma^{-1}X}(\omega)\ f(\omega)$$

so
$$\sigma \chi_X \sigma^{-1} = \chi_{\sigma^{-1}X}.$$

Somehow the point is that Q_i is bigger than P_i. Notice that σ on $\Omega_{\geq 0}$ is expanding, certainly it is many to one on $\Omega_{\geq 0}$. In fact Q_i = char function of the set of $(x_n)_{n\geq 0}$ such that $(i, x_0, x_1, \cdots) \in \Omega_{\geq 0}$.

???

Let's review. $\Omega_{\geq 0}$ = space of sequences $(x_n)_{n\geq 0}$ in $I^{\mathbb{N}}$ such that $(x_n, x_{n+1}) \in \Gamma$ for all $n \geq 0$. $X_i \subset \Omega_{\geq 0}$ the subset of sequences beginning with i; Y_i the subset of sequences beginning with a j such that $(i,j) \in \Gamma$. Thus

$$Y_i = \coprod_{\substack{j \ni \\ (i,j)\in\Gamma}} X_j$$

Notice that $\quad \sigma : X_i \xrightarrow{\sim} Y_i$

Now we want to translate this picture to $L^2(\Omega_{\geq 0})$. One attempt goes as follows. X_i is a subspace of $\Omega_{\geq 0}$ hence it inherits a measure and we have an embedding

$$L^2(X_i) \subset L^2(\Omega_{\geq 0}).$$

This subspace is the image of P_i = mult. by χ_{X_i}. Similarly we have

$$L^2(Y_i) \subset L^2(\Omega_{\geq 0})$$

as the image of Q_i = mult by χ_{Y_i}. But σ maps X_i one-one onto Y_i, so maybe it gives an isometry between $L^2(X_i)$ and $L^2(Y_i)$ somehow.

Lets go back to $\sigma : \Omega_{\geq 0} \twoheadrightarrow \Omega_{\geq 0}$ and try to understand its effect on functions. We have $\sigma^* : C(\Omega_{\geq 0}) \longrightarrow C(\Omega_{\geq 0})$, $\sigma^*(f) = f \circ \sigma$ and $\sigma_* : C(\Omega_{\geq 0}) \longrightarrow C(\Omega_{\geq 0})$ integration over the fibre. These should be related by

$$\int \sigma_*(f)\, g \, d\mu = \int f \, \sigma^*(g)\, d\mu$$

Let's take g to be the characteristic function of those sequences beginning with $(x_1, \dots, x_n)$, and take f to be a function of the first $n+1$ coords. The right side is

$$\sum_{x_0} f(x_0, x_1, \dots, x_n)\, \lambda^{-n-1} \mu(x_0) \cdot p(x_0, x_1) \cdots p(x_{n-1}, x_n)\, \nu(x_n)$$

If $\sigma_*(f)$ depends only on the first n-coordinates, the left side is

$$\sigma_*(f)(x_1,\dots,x_n)\lambda^{-n}\mu(x_1)p(x_1,x_2)\cdots p(x_{n-1},x_n)\nu(x_n)$$

which gives us the formula

$$\sigma_*(f)(x_1,\dots,x_n) = \sum_{x_0} f(x_0,x_1,\dots,x_n)\frac{\mu(x_0)p(x_0,x_1)}{\lambda\,\mu(x_1)}$$

This clearly ought to be valid in general.

Now let's see if $\sigma_* : L^2(\Omega_{\geq 0}) \longrightarrow L^2(\Omega_{\geq 0})$ induces an isomorphism of $L^2(X_i)$ onto $L^2(Y_i)$. Here $L^2(X_i)$ sits inside $L^2(\Omega_{\geq 0})$ as the space of functions supported in the set X_i of sequences with $x_0 = i$. Clearly the support of $\sigma_*(f)$ for $\mathrm{Supp}(f) \subset X_i$ is contained in $\sigma(X_i) = Y_i$. Suppose f has support in X_i, and depends only on $(x_0,\dots,x_n)$. Then

$$\sigma_*(f)(x_1,\dots,x_n) = f(i,x_1,\dots,x_n)\frac{\mu(i)\,p(i,x_1)}{\lambda\mu(x_1)}$$

Then

$$\|\sigma_*(f)\|^2 = \sum_{(x_1,\dots,x_n)} |f(i,x_1,\dots,x_n)|^2 \frac{\mu(i)^2p(i,x_1)^2}{\lambda^2\mu(x_1)^2}\mu(x_1)p(x_1,x_2)\cdots p(x_{n-1},x_n)\frac{\nu(x_n)}{\lambda^n}$$

whereas

$$\|f\|^2 = \sum_{(i,x_1,\dots,x_n)} |f(i,x_1,\dots,x_n)|^2\,\mu(i)\,p(i,x_1)\cdots p(x_{n-1},x_n)\frac{\nu(x_n)}{\lambda^{n+1}}$$

This doesn't work, but it would work if we used instead the operator

$$f \longmapsto f(i,x_1,\dots)\left(\frac{\mu(i)p(i,x_1)}{\lambda\,\mu(x_1)}\right)^{1/2}$$

It seems that the invariant measure might be subject to variation. This should be examined. We start with the finite set I and the relation $\Gamma \subset I \times I$. This gives rise to the space $\Omega \subset I^{\mathbb{Z}}$. What do we need to have an invariant measure? We should obtain one starting from any non-negative matrix $p(x,y)$, $x,y \in I$ having support Γ. We make the standard simplifying assumption that there is an $n \geq 1$ such that the map

$$\Gamma \times_I \overset{n}{\cdots} \times_I \Gamma \longrightarrow I \times I$$

is surjective. This implies that the matrix p^n has strictly positive entries and allows us to use the Frobenius thm. to construct the invariant measure.

On the other hand the backwards shift on $\Omega' = \Omega_{\geq 0}$ is an expanding map in some sense, so perhaps there is a canonical invariant measure, as in the case of expanding maps of compact manifolds. Recall the rough idea. Given a continuous function f one assigns a number to it as follows. One fixes a point ω_0 of Ω' and takes the average of the values of f at the points in $\sigma^{-n}\{\omega_0\}$, then takes the limit as $n \longrightarrow \infty$.

This seems to give the measure associated to the $(0,1)$ matrix given by Γ.

April 19, 1988

Let's go over what was learned yesterday. Let $\Gamma \subset I \times I$ project surjectively onto the two factors, and let $\Omega \subset I^{\mathbb{Z}}$, $\Omega' = \Omega_{\geq 0} \subset I^{\mathbb{N}}$ be the corresponding subshifts. Let's assume there is an N such that $\Gamma \times_I \overset{n-1}{\cdots} \times_I \Gamma \longrightarrow I \times I$ is surjective. Then for any non-negative matrix $p(x,y)$, $x,y \in I$ with support Γ, the Frobenius thm. implies the uniqueness up to scalar factors of (strictly) positive left and right eigenvectors $\mu(x)$, $\nu(x)$ for p. This allows us to define invariant measures on Ω, Ω' by taking the measures

$$\circledast \quad d\mu(x_0, \dots, x_n) = \lambda^{-n} \mu(x_0)\, p(x_0, x_1) \cdots p(x_{n-1}, x_n)\, \nu(x_n)$$

on $\Gamma \times_I \overset{n}{\cdots} \times_I \Gamma$ for different n.

We may normalize and suppose $\lambda = 1$ and $\sum_x \mu(x)\nu(x) = 1$.

We can also start with a Markov process, better Markov chain, with state space I such that Γ contains the allowed transitions. This gives a probability measure $k(x,y)$ on Γ and a probability measure $\mu(x)$ on I such that

$$\mu(x) = \sum_y k(x,y)$$
$$\mu(y) = \sum_x k(x,y)$$

(Assume $\mu(x) \neq 0$ for all $x \in I$ so Supp(k) projects onto both factors I.)

($k(x,y) = 0$ for $(x,y) \notin \Gamma$). The transition probability $p(x,y)$ is

$$p(x,y) = \frac{k(x,y)}{\mu(x)}$$

Then $\sum_x \mu(x)\, p(x,y) = \mu(y)$, $\quad \sum_y p(x,y) = 1$

The rest of the measure on Ω is constructed from the rule $\circledast$.

The conclusion is that the class of "Markov" measures on Ω are described exactly by matrices $k(x,y) \geq 0$ supported on Γ such that

$$\sum_y k(x,y) = \sum_y k(y,x) > 0$$

for each x, and also $\sum_{x,y} k(x,y) = 1$.

Let's now discuss the relation with the Cuntz-Krieger algebra. The way I would like to see this motivated is as follows. We work with the half-infinite sequence space $\Omega' = \Omega_{\geq 0}$ and the backwards shift σ. We have

$$\Omega' = \coprod_{i \in I} X_i$$

where X_i consists of sequences ~~starting~~ with i. Let $Y_i = \sigma(X_i)$; then Y_i consists of sequences $(x_n)_{n \geq 0}$ in Ω' such that $(i, x_0) \in \Gamma$; and

$$Y_i = \coprod_{\substack{j \ni \\ (i,j) \in \Gamma}} X_j$$

$$\sigma : X_i \xrightarrow{\sim} Y_i$$

Consider the Hilbert space $L^2(\Omega')$. Then we have

$$L^2(\Omega') = \bigoplus_{i \in I} L^2(X_i)$$

Actually why not first consider functions on Ω', say cylinder functions to begin with, all this alg. $A(\Omega')$. Then we have

$$A(\Omega') = \bigoplus_{i \in I} A(X_i)$$

$$A(Y_i) = \bigoplus_{\substack{j \ni \\ (i,j) \in \Gamma}} A(X_j)$$

and finally

$$\sigma^* : A(Y_i) \xrightarrow{\sim} A(X_i)$$

Let's consider the simplest case. This is where $I = \{1,2\}$ and $\Gamma = I \times I$ and we take the product measure. Then

$$\Omega' = X_1 \amalg X_2$$

$$\Omega' = Y_1 = Y_2$$

and the shift σ maps X_i bijectively onto $Y_i = \Omega'$.

What are the representations of the CK C^*-algebra in this example? One has a Hilbert space with a splitting

$$H = H_1 \oplus H_2$$

together with unitary isomorphisms

$$\theta_1 : H_1 \xrightarrow{\sim} H \qquad \theta_2 : H_2 \xrightarrow{\sim} H$$

From this we can construct finer decompositions

$$H = \underbrace{H_1}_{H_{11} \oplus H_{12}} \oplus \underbrace{H_2}_{H_{21} \oplus H_{22}}$$

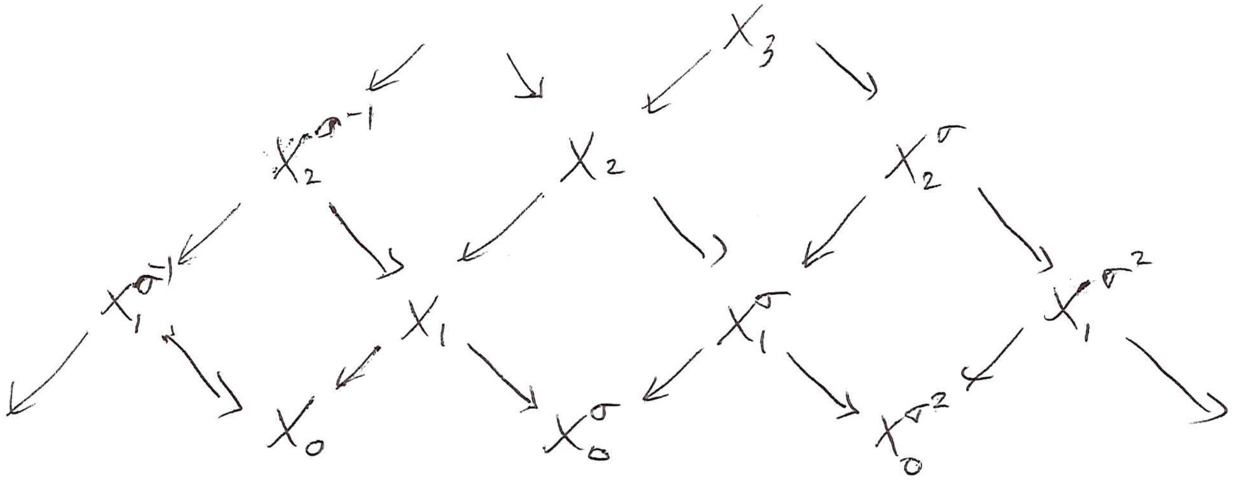
etc. We get an action of $C(\Omega')$ on H.

April 21, 1988

We learned above that the Cuntz-Krieger C^*-algebra is not likely to be well understood in terms of the Hilbert space $L^2(\Omega_{\geq 0})$ associated to a nice invariant measure (better: Markov measure). It seems that the CK algebra is associated to the "topological Markov chain" in a fashion discovered by Krieger.

[redacted] The real mystery appears to be the following. Suppose given (Ω, σ) a [redacted] subshift of finite type. Then there seems to be this CK C^*-algebra [redacted] associated to (Ω, σ), which, if not defined up to canonical isomorphism is at least defined up to Morita equivalence. (Wagoner claims $\mathrm{Aut}(\Omega_A, \sigma_A)$ acts on [redacted] the CK C^*-algebra associated to the $(0,1)$ matrix A, so it seems this algebra is defined up to canonical isomorphism.) Wagoner's claim has not appeared in print

Krieger works with non-negative integral matrices. Such a thing can be interpreted as a finite set X_0 together with a correspondence $X_1 \underset{p_2}{\overset{p_1}{\rightrightarrows}} X_0$. One can then construct the diagram of finite sequence spaces



in which the squares are cartesian, and define Ω to be the inverse limit of the diagram.

Rectangles: Let $\Omega \to X$ be a Markov partition. Then we have

$$\circledast \qquad \Omega \xrightarrow{\sim} \Omega_{\leq 0} \times_X \Omega_{\geq 0}$$

Given $\omega \in \Omega$ the "stable manifold" through ω is

$$W^s(\omega) = \{\vec{x} \in \Omega \mid x_n = \omega_n \text{ for } n \geq 0\}$$

by definition. If $p_+ : \Omega \to \Omega_{\geq 0}$ is the projection then

$$W^s(\omega) = p_+^{-1}\{p_+(\omega)\}$$

is the fibre of p_+ containing ω. (It's not clear why this should be viewed as the stable manifold. $\vec{x} \in W^s(\omega) \iff \vec{x}$ and ω [redacted] have the same coordinates for $n \geq 0$. Thus $\sigma^n(\vec{x})$ and $\sigma^n(\omega)$ become closer as $n \to +\infty$; in fact the whole set $\sigma^n W^s(\omega)$ has diameter $\to 0$ as $n \to +\infty$. The same however is [redacted] true of the sets $\sigma^k W^s(\omega)$ for different k.)

A rectangle R relative to $\circledast$ is a clopen subset of Ω which has the form

$$R = R' \times R''$$

with R' clopen in $\Omega_{\leq 0}$ and R'' clopen in $\Omega_{\leq 0}$ and both R' and R'' [redacted] lying over the some point of X.

April 22, 1988

At some point in the future it will be necessary for you to learn about the CK algebras. In the Cuntz-Krieger article (Inv. Vol. 56 (1980)) are the following points or ideas:

1) Study of the C^* algebra itself, denoted $\mathcal{O}_A$, A being the defining (0,1) matrix. $\exists$ nice basis of monomials, and a nice sequence of finite dimensional subalgebras whose union is the AF algebra corresponding to some kind of "dimension groups". Also there is a comm. subalgebra $\simeq C(\Omega_{\geq 0})$. Simplicity of $\mathcal{O}_A$ when A is irreducible (or aperiodic? One of these refers to $\exists n \; \forall i,j \;\; (A^n)_{ij} > 0$, the other to $\forall i,j \;\; \exists n \;\; (A^n)_{ij} > 0$.).

2) Identification of $\mathcal{K} \otimes \mathcal{O}_A$ with something related to Ω. Here one defines the stable "manifold" $W(x)$ through $x \in \Omega$ to consist of those points agreeing with x for large degree coordinates. $W(x)$ is σ-compact, and up to homeomorphism (maybe σ-homeom.) is independent of x. ~~[illegible]~~ Given $W(x)$ one can construct a cross-product of functions on $W(x)$, $C_0(W(x))$ I think, with a group of suitably defined finite-type ~~[illegible]~~ homeomorphisms of $W(x)$, and this cross product can be identified with $\mathcal{K} \otimes \mathcal{O}_A$.

April 23, 1988

Return to cyclic theory. We have the cochain algebra $C^\circ(A,B)$ and the 1-cochain $\rho : A \to B$ which is such that the "curvature" $d\rho + \rho^2 \in C^2(A,I)$. This is reminiscent of foliations, so it is worth asking whether the secondary classes constructed by Bott and others are relevant to cyclic theory. If I remember correctly a basic cochain algebra studied in connection with foliations, Haefliger classifying spaces, and Gelfand-Fuks cohomology is the following. One starts with the Weil algebra $W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$ for $\mathfrak{g} = \mathfrak{u}_n$ and divides out by the ideal generated by the universal Chern forms c_k for $k > n$.

For the cyclic theory one needs an appropriate stable analogue. It seems to me that all one can expect is to be found by considering the universal Chern-Simons algebra

$$R = \mathbb{C}\langle \alpha, d\alpha \rangle$$

with the J-adic filtration, where J is the ideal generated by the curvature $\alpha^2 + d\alpha$. This leads us then to the problem of finding the cohomology of $J^n/[J, J^{n-1}]$ and $R/[R,R] + J^n$.

Note that R is free as an algebra, as well as $R/J = \mathbb{C}\langle\alpha\rangle$. ~~free as an R-bimodule~~

April 24, 1988

The problem is to construct cyclic cocycles attached to Dirac operators, more abstractly, to unbounded p-summable Fredholm modules (A, H, D). Hopefully one can also treat the θ-summable case eventually.

Consider the odd or ungraded case, for example, $A = C^\infty(S^1)$, $H = L^2(S^1)$, $D = \frac{1}{i}(\partial_x + \alpha)$. It's clear that one wants to work in some algebra B of operators on H which contains an ideal I such that a trace is defined on some power. Moreover B should be generated by A, D in some sense. What's important about D is the involution modulo compacts it defines.

To fix the ideas let $B = \Psi^0(S^1)$, $I = \Psi^{-1}(S^1)$, or more generally we can consider the case of an odd manifold. Then to D we can associate $F = \frac{D}{\sqrt{1+D^2}}$ which is an involution modulo I. We have for $a \in A$

$$F^2 - 1 \, , \quad [F, a] \quad \in \quad I.$$

Then we can take $\rho : A \to B$ to be any lifting of the homomorphism $a \longmapsto ae \equiv ea \equiv eae \pmod{I}$, where $e = \frac{1}{2}(1+F)$. For example we can take $\rho(a) = ea$, ae, or eae. Each of these ρ's leads to odd cyclic cocycles and all belong to the same cyclic class.

To proceed further one probably has to find a good choice for ρ.

The question is whether I can understand the sort of infinite degree cyclic cocycle constructed by Connes, Jaffe + Lesnewski in terms of extensions. Thus the idea will be an even cocycle class on A ought to be given by an extension $B/I \longrightarrow A$ and a trace on B/I^∞. As yet I don't know what to expect B/I^∞ should be. One can take the inverse limit $\varprojlim B/I^n$, however there are lots of more interesting analytical possibilities.

To get some idea as to what to try, we might try assembling the different Chern-Simons forms. This brings up the problem of relating the different even forms by the S-operator.

Another idea is to try figure out how a "trace on B/I^∞" pairs with $K_0(A)$. In some way this should be a meaningful question in the case $\mathbb{C} \oplus \mathbb{C}e$ or perhaps Cuntz's algebra $\mathbb{C}[e] * \mathbb{C}[e]$. In the case of $\mathbb{C}[e] = \mathbb{C}[F] = \mathbb{C} \times \mathbb{C}$ the minimal choice for B is $\mathbb{C}[x]$ and the ideal I is $\mathbb{C}[x](x^2-1)$. Because B is commutative there is a unique choice for the lifting of F to an involution in B/I^n for all n. The lifting of e should be up to a constant factor

$$\int_{-1}^{x} (t+1)^{n-1}(1-t)^{n-1}\, dt$$

Now what I am looking for is a natural algebra B/I^∞ with a trace mapping onto

$\mathbb{C}[\boxed{x}]/I^n$ for all n ~~consistently~~ and such that e can be lifted into B/I^∞. This means that $\mathbb{C}[e] = A$ lifts back into B/I^∞. And I would like B/I^∞ to be some sort of Banach algebra.

It seems that there are not many choices at least if we want to keep it commutative, and in some sense generated by x. We have got to find an ~~idempotent~~ "function" of x, call it $e(x)$ such that $e(x)$ vanishes to infinite order at $x=-1$ and $=1$ to infinite order at $x=1$. Probably we want to use some convenient analytic function in the unit disk like

$$\text{const.} \int_{-1}^{x} e^{-\frac{1}{t+1} + \frac{1}{t-1}}\, dt$$

~~Next depending on our choice for the algebra~~

and then divide out by $e(x) - e(x)^2$

April 28, 1988

Cuntz–Krieger C^*-algebras. Connes explains them as examples of noncommutative spaces. Specifically let (Ω, σ) be a subshift of finite type. Then there is a natural equivalence relation on Ω which says that $x \sim y$ if $\sigma^n(x), \sigma^n(y)$ become arbitrarily close as $n \to \infty$, where this is to be interpreted in the sense of uniform structures, i.e. for any nbd. of the diagonal, $(\sigma^n x, \sigma^n y)$ belongs to this nbd for sufficiently large n. (Also there are various metrics one can use.) In terms of a choice of Markov partition, this means two sequences are equivalent iff $x_n = y_n$ for $n \gg 0$.

We get the same quotient space by taking the half shift space $\Omega_{\geq 0}$ (defined using a Markov partition) and saying $x \sim y \iff \sigma^n x = \sigma^n y$ for some $n \geq 0$. It's clear that any equivalence class is dense, so we indeed have a non-Hausdorff quotient space.

Connes claims that the Cuntz–Krieger C^*-algebra is the cross-product in a certain sense of the continuous functions on $\Omega_{\geq 0}$ by the above equivalence relation. What does this mean? If a group G acts on an algebra A, then the crossproduct $A \times G$ $(= A \otimes \mathbb{C}[G]$ essentially$)$ consists of $\sum_{g \in G} a_g g$, so if $A = C(X)$, then $A \times G$ looks like functions on $X \times G$. Thus it would seem that the CK-C^*-algebra should be some completion perhaps of functions on the graph of the equivalence relation.

So thus one should think of the CK algebra as essentially made from continuous functions $f(x,y)$

defined for $x, y \in \Omega_{\geq 0}$ with $x \sim y$. The topology on this graph is a kind of etale space topology. Thus for x fixed, the points y equivalent to x (i.e. agreeing a.e. with x) have the discrete topology.

Define the algebra structure by convolution

$$(f * g)(x, z) = \sum_y f(x,y)\, g(y,z)$$

where y runs over those sequences equivalent to x. In order for this to be well-defined we must suppose $f(x,y)$ has support proper over $\Omega_{\geq 0}$ relative to the first projection.

Now there is supposed to be a canonical trace on this algebra. This should be the form $f(x,y) \longmapsto \int f(x,x)\, d\mu(x)$ where $d\mu(x)$ is the (unique?) measure on $\Omega_{\geq 0}$ which is compatible with the equivalence relation. The problem is what this measure is. We want

$$\int \sum_y f(x,y)\, g(y,x)\; d\mu(x) = \int \sum_x f(x,y) g(y,x)\; d\mu(y)$$

which means that if we define ~~two~~ measures on the graph Γ as follows, then they coincide. The idea is that Γ is an etale space over $\Omega_{\geq 0}$ and so using local sections to push the measure on $\Omega_{\geq 0}$ into Γ, we get a measure on Γ such that

$$\int_\Gamma f(x,y)\, d\mu(x,y) = \int_{\Omega_{\geq 0}} \Big(\sum_{y \sim x} f(x,y) \Big)\, d\mu(x)$$

We therefore get two measures on Γ corresponding to the two projections of Γ onto $\Omega_{\geq 0}$.

Example: Suppose we take $\Omega_{\geq 0} = \prod_{\mathbb{N}} \{0,1\}$ better $\Omega_{\geq 0} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. The equivalence relation is generated by the translation action of $\bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. The unique invariant measure is the Haar measure on $\Omega_{\geq 0}$. The cross product algebra, if I recall correctly, is the CAR C^*-algebra.

The key point in the theory which I have yet to understand is the fact that the CK algebra is an AF algebra, i.e. its the inductive limit of finite dimensional C^*-algebras. Cuntz + Krieger show this by starting from the relations defining the algebra

April 30, 1988

Recall the defn of the CK C^*-algebra. It is generated by partial isometries s_i whose range projections $P_i = s_i s_i^*$ decompose 1:

$$1 = \sum_i P_i \qquad \text{hence } P_iP_j = 0 \text{ for } i \neq j.$$

and such that the domain projections are

$$Q_i = s_i^* s_i = \sum_{(i,j)\in\Gamma}' s_j s_j^*$$

Let's consider a composition $s_i s_j$. By assumption s_i is projection onto $\operatorname{Im} Q_i = \bigoplus_{(i,j)\in\Gamma} \operatorname{Im} P_j$ followed by an isomorphism of the latter with $\operatorname{Im} P_i$. If $(i,j) \in \Gamma$, then we have

$$\overbrace{H \xrightarrow[\text{proj}]{} Q_jH \xrightarrow{\sim} P_jH}^{s_j} \subset \overbrace{H \xrightarrow[\text{proj}]{} Q_iH \xrightarrow{\sim} P_iH}^{s_i}$$

so that $s_i s_j$ is ~~a~~ partial isometry with domain projection Q_j. (Check $s_j^* s_i^* s_i s_j = s_j^* Q_i s_j = s_j^* s_j = Q_j$)
On the other hand if $(i,j) \notin \Gamma$, then $P_jH \perp Q_iH$ so $s_i s_j = 0$.

Repeating this argument for $s_i s_j s_k$. Now $s_i s_j$ projects to Q_jH and embeds this in P_iH. If $(j,k) \notin \Gamma$, then $\operatorname{Im}(s_k)$ is "outside" Q_jH so $s_i s_j s_k = 0$. But if $(j,k) \in \Gamma$, then $\operatorname{Im}(s_k)$ is inside Q_jH, and so $s_i s_j s_k$ projects onto Q_kH and embeds this into P_iH.

The way to say this is that when (i, j, k, l, m) are such that consecutive pairs are in Γ, then s_m projects onto $Q_m H$ and isometrically embeds this onto $P_m H \subset Q_l H$, which is then isom. embedded by s_l onto $P_l H \subset \cdots$, etc. So we conclude that $s_{i_1} \cdots s_{i_p}$ is zero unless $(i_1, \dots, i_p)$ has consecutive pairs in Γ, and that in this case this product is a partial isometry with domain projection Q_{i_p}.

Let's try to obtain finite dimensional subalgebras. First note that we have

$$s_i^* s_j = 0 \qquad i \neq j$$

$$s_i^* s_i = \sum_{(i,j) \in \Gamma} s_j s_j^*$$

This implies that polynomials in the s_i, s_j^* can always put in normal ordered form with the s_j^* to the right. Let's use the notation $s_\mu = s_{i_1} \cdots s_{i_p}$ where $\mu = (i_1, \dots, i_p)$ is a sequence with consecutive pairs in Γ. We've seen s_μ is a partial isometry with domain projection Q_{i_p}. Thus s_μ^* is a partial isometry with range projection Q_{i_p}.

Let's consider $s_\mu s_\nu^*$ where $\mu = i_1, \dots, i_p$ and $\nu = j_1, \dots, j_q$. This will be zero unless Q_{j_q} and Q_{i_p} overlap. In this case one might as well write $s_\mu s_\nu^* = \sum' s_\mu P_i s_\nu^*$ where i runs over those indices with $P_i \le Q_{i_p}, Q_{j_p}$.

Put another way we see the algebra of polynomials in the s_i, s_j^* is spanned by the operator $s_\mu s_\nu^*$ where μ, ν are "admissible" sequences with common last index. These operators $s_\mu s_\nu^*$ are evidently partial isometries.

Let's now consider the monomials $s_i s_j^*$. We have

$$s_i s_j^* s_k s_\ell^* = 0 \quad \text{if } j \ne k$$

$$s_i s_j^* s_j s_\ell^* = \sum_{(j,k)\in\Gamma} s_i s_k s_k^* s_\ell^*.$$

This doesn't so, so instead consider $s_i P_k s_j^*$, where to get something non-zero we want $(i,k) \in \Gamma$ and $(j,k) \in \Gamma$. These monomials are closed under composition. In effect

$$s_i P_k s_j^* s_\ell P_n s_m^* = \begin{cases} 0 & j \ne \ell \\ s_i P_k Q_j P_n s_m^* & j = \ell \end{cases}$$

$$= \begin{cases} 0 & j \ne \ell \text{ or } k \ne n \\ s_i P_k s_m^* & j = \ell,\ k = n \end{cases}$$

$$\therefore \quad \boxed{s_i P_k s_j^* s_\ell P_n s_m^* = \delta_{j\ell}\,\delta_{kn}\, s_i P_k s_m^*}$$

Consider the simplest case where $a(i,j)=1$ for $i,j \in \mathbb{Z}/2\mathbb{Z}$. If H is a Hilbert space representation of the Cuntz-Krieger algebra, then $s_i : H \longrightarrow H$ are isometries embeddings such that one gets an isomorphism

$$(*) \qquad (s_0\ s_1) : \begin{matrix} H \\ \oplus \\ H \end{matrix} \xrightarrow{\sim} H$$

Thus we have

$$(s_0\ \ s_1)\begin{pmatrix} s_0^* \\ s_1^* \end{pmatrix} = s_0 s_0^* + s_1 s_1^* = 1$$

$$\begin{pmatrix} s_0^* \\ s_1^* \end{pmatrix}(s_0\ \ s_1) = \begin{pmatrix} s_0^* s_0 & s_0^* s_1 \\ s_1^* s_0 & s_1^* s_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We iterate the isomorphism $*$.

$$\begin{matrix} H^{\oplus 2} \\ \oplus \\ H^{\oplus 2} \end{matrix} \xrightarrow[\sim]{\begin{pmatrix} s_0 & s_1 & 0 & 0 \\ 0 & 0 & s_0 & s_1 \end{pmatrix}} \begin{matrix} H \\ \oplus \\ H \end{matrix} \xrightarrow[\sim]{(s_0\ \ s_1)} H$$

so

$$(s_0\ \ s_1)\begin{pmatrix} s_0 & s_1 & 0 & 0 \\ 0 & 0 & s_0 & s_1 \end{pmatrix} = \begin{pmatrix} s_0^2 & s_0 s_1 & 0 & 0 \\ 0 & 0 & s_1 s_0 & s_1^2 \end{pmatrix}$$

What I want to do next is to understand the relation of the above CK algebra with the CAR algebra.

Let's observe that to give H a C_1-module structure is ~~the same~~ as giving a grading

$$H = H_+ \oplus H_-$$

and to give a C_2-module structure is the same as giving such a grading together with an isomorphism of H_+ with H_-, so that we have

$$H = H_+ \otimes S_2$$

where $S_2 = \mathbb{C}^2$ is the module of spinors over C_2.

If we view C_{2n} as $C_2^{\otimes n}$, then to extend a C_{2n-2}-module structure

$$H = H' \otimes S_{2n-2}$$

to a C_{2n} structure is the same as writing $H' = H'' \otimes S_2 = H'' \oplus H''$.

Now let's look carefully at a representation of the CK algebra. Then for each n we get an orthogonal decomposition of H into 2^n pieces, namely the image of

$$s_\mu = s_{\mu_1} \cdots s_{\mu_n} : H \hookrightarrow H, \qquad \mu_i = 0 \text{ or } 1.$$

Moreover these pieces are canonically isomorphic to each other, in fact, to H. Thus it seems that H is naturally a C_{2n}-module for each n. In fact H should naturally be a representation of the CAR algebra; the best statement should identify the CAR algebra with a subalgebra of the CK algebra.

So the problem is to do this fairly explicitly. Now the viewpoint to adopt is to first look at the decomposition of H; this is to be identified with an action of the algebra of cont. functions on $\Omega_{\geq 0}$.

May 2, 1988

In connection with the subshift (Ω, σ) Connes mentioned two equivalence relations. The first which we discussed above says $x \sim y$ iff $d(\sigma^n(x), \sigma^n(y)) \to 0$ as $n \to \infty$, or equivalently if $x_n = y_n$ for $n \gg 0$. This leads to the "non-commutative space" being studied. The second equivalence relation "normalizes" the first and I think it is described simply by the action of σ.

Connes uses the words "horocycles" to describe the former, but I don't see his example.

Something to check when we have time is whether the CK algebra is the cross-product of the C^*-algebra describing the first quotient space (which should be an AF algebra) by the integers. There are some <u>analogies</u> worth investigating:

σ is an expanding map, hence it is analogous to Frobenius.

Loop groups: You've noticed that there are lots of invariant measures on Ω, namely, the Markov measures for different transition-probability-matrices. Yet there is only one CK-algebra. Similarly by looking at different measures on S^1 one gets lots of representations of the loop group, and yet there appears to be only one positive energy representation. There's an analogy between σ and the energy semi-group.

Thus Ω and the CK algebra are classical and quantum theories respectively.

Let's return to cyclic homology and extensions. We consider a unital algebra A and consider the universal extension

$$0 \longrightarrow I \longrightarrow B \underset{\xleftarrow{\ \rho\ }}{\longrightarrow} A \longrightarrow 0$$

with lifting ρ satisfying $\rho(1_A) = 1_B$. Thus

$$B = T(A) / (1 = \rho(1_A))$$

and B [redacted] is non-canon. isom. to $T(A/\mathbb{C})$. To simplify suppose A augmented

$$A = \mathbb{C} \oplus \mathfrak{a} \qquad B = T(\mathfrak{a}).$$

Our goal will be to derive exact sequences:

$$0 \to \overline{HC}_{2n+1}(A) \longrightarrow I^{n+1}/[I, I^n] \longrightarrow H_1(B, I^n) \longrightarrow \overline{HC}_{2n}(A) \to 0$$

$$0 \to \overline{HC}_{2n}(A) \longrightarrow HC_0(B/I^{n+1}) \longrightarrow H_1(B, B/I^n)_\sigma \longrightarrow HC_{2n-1}(A) \to 0$$

by proceeding directly on the level of formulas.

[redacted]

It seems that the way to proceed is to [redacted] try to produce the long exact sequence which results by splicing the above

$$(*) \quad \cdots \longrightarrow I^{n+1}/[I, I^n] \longrightarrow H_1(B, I^n) \longrightarrow \overline{HC}_0(B/I^{n+1}) \longrightarrow H_1(B, B/I^n)_\sigma \longrightarrow$$

My feeling is that everything should become clear once one really understands the isomorphism

$$\overline{HC}_0(B) \xrightarrow[\sim]{\delta} H_1(B, B) = \operatorname{Ker}\{\Omega^1_B \otimes_B \longrightarrow B\}$$

induced by $d: B \longrightarrow \Omega^1_B$

We can produce the exact sequence $\circledast$ as follows by putting together the exact sequences in Hochschild homology

$$\to \overline{HC}_0(B) \to \overline{HC}_0(B/I^{n+1}) \to 0$$

+

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H_1(B,I^{n+1}) \arrow[r] & H_1(B,B) \arrow[r] & H_1(B,B/I^{n+1}) \arrow[r] & I^{n+1}/[B,I^{n+1}] \arrow[r] \arrow[d, "\delta"] & \overline{HC}_0(B) \arrow[r] \arrow[d, "\delta"] & \overline{HC}_0(B/I^{n+1}) \arrow[r] \arrow[d, "\delta"] & 0 \\
 & & & 0 \arrow[r] & H_1(B,I^n) \arrow[r] & H_1(B,B) \arrow[r] & H_1(B,B/I^n) \arrow[r] & {}
\end{tikzcd}$$

By serpent lemma + diagram chasing this gives

$$\to H_1(B,B/I^{n+1}) \to I^{n+1}/[B,I^{n+1}] \xrightarrow{\delta} H_1(B,I^n) \to \overline{HC}_0(B/I^{n+1}) \to H_1(B,B/I^n) \to$$

Finally we've seen that we can divide out by the cyclic action on $H_1(B/I^{n+1})$ and $I^{n+1}/[B,I^{n+1}]$ and so obtain

$$\to H_1(B,B/I^{n+1})_\sigma \to I^{n+1}/[I,I^n] \to H_1(B,I^n) \to \overline{HC}_0(B/I^{n+1}) \to H_1(B,B/I^n)_\sigma$$

which is the exact sequence $\circledast$. Thus our problem now is to make the identifications

$$\overline{HC}_{2n}(A) = \mathrm{Im}\{H_1(B,I^n) \to \overline{HC}_0(B/I^{n+1})\}$$

$$\overline{HC}_{2n+1}(A) = \mathrm{Im}\{H_1(B,B/I^{n+1})_\sigma \to I^{n+1}/[I,I^n]\}$$

can be omitted

<u>May 3, 1988</u>

Let's consider $A = \mathbb{C} \oplus \mathfrak{a}$, $B = T(\mathfrak{a}) = \mathbb{C} \oplus \mathcal{B}$. Our goal is to derive directly the exact sequences for $HC_\cdot(\mathfrak{a}) = \overline{HC}_\cdot(A)$ in terms of traces connected with the extension

$$0 \longrightarrow I \longrightarrow B \longrightarrow A \longrightarrow 0.$$

The first exact sequence is

$$(*) \quad 0 \longrightarrow HC_1(\mathfrak{a}) \xrightarrow{\gamma} I/[B,I] \longrightarrow \mathcal{B}/[\mathcal{B},\mathcal{B}] \longrightarrow \mathfrak{a}/[\mathfrak{a},\mathfrak{a}] \to 0.$$

The question is how to derive this. One method, essentially the one in my paper, is to use the cyclic complex of $I \to B$ considered as a chain algebra. This yields a [redacted] 5 or 6 term exact sequence which when we use $HC_1(B) = 0$ gives the above sequence.

[redacted] The Connes homomorphism γ above [redacted] can be refined to [redacted] a map of complexes

$$\begin{tikzcd}
\mathfrak{a}_\lambda^{\otimes 3} \arrow[d, "b"] & 0 \arrow[d] \\
\mathfrak{a}_\lambda^{\otimes 2} \arrow[r] \arrow[d, "b"] & I/[B,I] \arrow[d] \\
\mathfrak{a} \arrow[r] & \mathcal{B}/[\mathcal{B},\mathcal{B}]
\end{tikzcd}$$

where the two horizontal arrows are the first Chern form $\mathrm{tr}(\rho^2 + d\rho)$ and its transgression form $\mathrm{tr}(\rho)$. The exactness of $(*)$ says that this map induces isomorphism on homology in degrees 0, 1. Notice that

$$\mathcal{B}/[\mathcal{B},\mathcal{B}] = \mathcal{B}/[\mathfrak{a},\mathcal{B}] = \bigoplus_{n>0} \mathfrak{a}_\sigma^{\otimes n}.$$

Recall [redacted] that

$$a_\lambda^{\otimes 2}/b\,a_\lambda^{\otimes 3} = \Omega_A^2/[A,\Omega_A^2]+[\Omega_A^1,\Omega_A^1]+d\Omega_A^1$$

and that $I/I^2 = \Omega_A^2$. This doesn't seem to help much.

Consider now the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & HC_1(a) \arrow[d] & \mathrm{Ker}(\alpha) \arrow[d] & & \\
 & a_\lambda^{\otimes 2}/b a_\lambda^{\otimes 3} \arrow[r] \arrow[d] & I/[B,I] \arrow[d, "\alpha"] & & \\
0 \arrow[r] & a \arrow[r, hook, "\rho"] \arrow[d] & B/[B,B] \arrow[r] \arrow[d] & \bigoplus_{n\geq 2} a_\sigma^{\otimes n} \arrow[r] & 0 \\
 & a/[a,a] \arrow[r, "\sim"] \arrow[d] & \mathrm{Cok}(\alpha) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

This shows that we must have an exact sequence

$$0 \to a_\lambda^{\otimes 2}/b a_\lambda^{\otimes 3} \longrightarrow I/[B,I] \longrightarrow \bigoplus_{n\geq 2} a_\sigma^{\otimes n} \longrightarrow 0$$

and conversely if this sequence is exact, then (*) holds.

The algebra B has a natural increasing filtration (in the general case where A is not assumed to be augmented). We consider the induced filtration on I. [redacted] This has the property ~~that~~ that it is compatible with the induced filtration on $A = B/I$.

Let's go over this carefully. Let A be supposed unital but not necessarily augmented. Define $B = T(A)/(1=\rho(1_A))$ and the filtration

$$F_pB = \rho(A)^p.$$

This is an increasing algebra filtration and $gr(B)$ is canonically isomorphic to $T(\bar{A})$. So far we haven't used that A is an algebra.

Now the algebra structure on A gives rise to a homomorphism $B \longrightarrow A$ of B onto A. Let I be the kernel.

In general if $\{F_pV\}$ is a filtration of a vector space V and if W is a subspace of V, then we define induced filtrations on W and V/W by

$$F_pW = W \cap F_pV$$

$$F_p(V/W) = (F_pV+W)/W \cong F_pV/F_pV\cap W$$

One then has exact sequences

$$0 \longrightarrow F_pW \longrightarrow F_pV \longrightarrow F_p(V/W) \longrightarrow 0$$

$$0 \longrightarrow gr(W) \longrightarrow gr(V) \longrightarrow gr(V/W) \longrightarrow 0$$

In our situation $F_0A = \mathbb{C}$, $F_pA = A$ for $p \geq 1$, hence we conclude

$$F_pI = 0 \qquad p = 0, 1$$

$$F_pI/F_{p-1}I = \bar{A}^{\otimes p} \qquad \text{for } p \geq 2.$$

But we ought to be able to describe F_pI

quite naturally. The initial filtration, that is, $F_p B$, is

$$F_p B = \rho(A)^p.$$

Now I is generated by the elements $\rho(a_1)\rho(a_2) - \rho(a_1 a_2) = K(a_1, a_2)$. So it should be clear that

$$\bar{A}^{\otimes 2} \longrightarrow F_2 I$$

$$(a_1, a_2) \longmapsto K(a_1, a_2)$$

is an isomorphism. Since we have an ideal it's clear that the elements

$$\rho(a_0)\, K(a_1, a_2)$$

are in $F_3 I$. It should be clear that

$$\bar{A}^{\otimes 2} \longrightarrow F_3 I / F_2 I$$

$$(a_0, a_1, a_2) \longmapsto \rho(a_0) K(a_1, a_2)$$

is an isomorphism. As a check we should relate $\rho(a_0) K(a_1, a_2)$ and $K(a_0, a_1)\rho(a_2)$. The difference should lie in $F_2 I$ because these two elements have the same image in $gr_3 B$.

$$\rho(a_0) K(a_1, a_2) - K(a_0, a_1)\rho(a_2)$$

$$= \cancel{\rho(a_0)\rho(a_1)\rho(a_2)} - \rho(a_0)\rho(a_1 a_2) - \cancel{\rho(a_0)\rho(a_1)\rho(a_2)} + \rho(a_0 a_1)\rho(a_2) + \rho(a_0 a_1 a_2) - \rho(a_0 a_1 a_2)$$

$$= K(a_0 a_1, a_2) - K(a_0, a_1 a_2)$$ which works.

Notice that the above formula is the Bianchi identity

$$dK + [\rho, K] = 0$$

Thus we should have fairly precise control over F_pI. Now the next step will be to understand $I/[I,a]$. The obvious thing to do is to look at the map of inductive systems

$$F_pI/[F_{p-1}I,\rho A] \longrightarrow F_pB/[F_{p-1}B,\rho A]$$

It seems we need a better language than filtered algebras. Thus I don't know that $F_pI \cap [I,\rho A] = [F_{p-1}I,\rho A]$. Perhaps the thing to do is to introduce modules

$$\bigoplus_{p\geq 0} h^p F_pI \qquad \text{over } \mathbb{C}[h]$$

In any case we want to study the map $I/[I,\rho A] \longrightarrow B/[B,\rho A]$ and get control of its kernel. We want to use the fact that $gr_pI = gr_pB$ for $p \geq 2$ in order to see that the kernel can be obtained from low stages in the filtration.

Let $x \in F_pI$ belong to $[B,\rho A]$, where $p>2$. Then the image of x in $gr_p(I) = gr_p(B) = \bar{A}^{\otimes p}$ lies in $(1-\sigma)\bar{A}^{\otimes p}$. I guess I am arguing that

$$B/[B,B] = B/[B,\rho A] \simeq \bigoplus \bar{A}^{\otimes p}_{\sigma}$$

and more precisely that $F_pB \cap [B,\rho A] = [F_{p-1}B,\rho A]$.?

Let's be more precise about what is needed. I let $x \in F_pI \cap [B,\rho A]$ and look at its leading term, i.e. its image in $F_pB/F_{p-1}B = \bar{A}^{\otimes p}$.

I want to conclude that there is a $y \in [F_{p-1}I, \rho A]$ with the same leading term as x. Thus it's enough to know $F_pB \cap [B, \rho A] = [F_{p-1}B, \rho A]$ and that $F_{p-1}I \twoheadrightarrow F_{p-1}B/F_{p-2}B$, which is OK for $p > 2$. Thus modulo $[I, \rho A]$ any element x of I vanishing in $B/[B,B]$ can be assumed to lie in $F_2 I$. The image of I in $gr_2 B = \bar{A}^{\otimes 2}$ lies in the skew-symmetric tensors.

So I guess we have shown that any element in

$$\mathrm{Ker}\{I/[I, \rho A] \longrightarrow \bar{B}/[B,B]\}$$

can be represented by an element in the image of

$$\bar{A}_\lambda^{\otimes 2} \longrightarrow I$$

$$(a_1, a_2) \longmapsto K(a_1, a_2) - K(a_2, a_1)$$

$$= [\rho(a_1), \rho(a_2)] - \rho([a_1, a_2])$$

Notice that the composition

$$\bar{A}_\lambda^{\otimes 2} \longrightarrow I \longrightarrow \bar{B}/[B,B]$$

is $(a_1, a_2) \longmapsto -\rho([a_1, a_2])$, which gives a surjection

$$\mathrm{Ker}\{\bar{A}_\lambda^{\otimes 2} \xrightarrow{b} \bar{A}\} \twoheadrightarrow \mathrm{Ker}\{I/[I, \rho A] \longrightarrow \bar{B}/[B,B]\}$$

Next when is an element of $\bar{A}_\lambda^{\otimes 2}$ such that its image in $F_2 I$ lies in $[I, \rho A]$?

We know that elements in $b\bar{A}_\lambda^{\otimes 3}$ have this property.

$$\begin{tikzcd}
\bar{A}_\lambda^{\otimes 3} \arrow[r] \arrow[d, "b"] & {[F_2 I, \rho A]} \arrow[d, phantom, "\cap"] \\
\bar{A}_\lambda^{\otimes 2} \arrow[r, hook] & F_2 I \subset F_3 I
\end{tikzcd}$$

$$(a_0, a_1, a_2)$$
$$\downarrow$$
$$(a_0a_1, a_2) - (a_0, a_1a_2) + (a_2a_0, a_1) \longmapsto \begin{aligned} &K(a_0a_1, a_2) - K(a_2, a_0a_1) \\ &- K(a_0, a_1a_2) + K(a_1a_2, a_0) \\ &+ K(a_2a_0, a_1) - K(a_1, a_2a_0) \end{aligned}$$

$$\|$$

$$[\rho A, F_2 I] \ni \left\{ \begin{aligned} &\rho(a_0)K(a_1, a_2) - K(a_0, a_1)\rho(a_2) \\ &\rho(a_2)K(a_0, a_1) - K(a_2, a_0)\rho(a_1) \\ &\rho(a_1)K(a_2, a_0) - K(a_1, a_2)\rho(a_0) \end{aligned} \right.$$

This is too complicated!

May 4, 1988

Let's see if we can use our Chern-Simons formula for the even Connes homomorphism to prove an index theorem.

Let us consider a real symplectic vector space V of dim $2n$ and let $\mathcal{A} = \mathcal{S}(V)$ the algebra of Schwartz functions. It ought to be possible to construct a family of smooth Weyl algebras depending on the parameter h which is a deformation of $\mathcal{A}$.

To begin we have to get notation straight. We have to start with the operator situation. To functions on V we want to assign operators such that Schwartz functions become trace class operators. To linear functions on V belong the operators satisfying the CCR.

To fix the ideas let $V = \{(x,\xi) \in \mathbb{R}^2\}$ and let q, p be the linear functions

$$q(x,\xi) = x$$
$$p(x,\xi) = p.$$

Then to q, p belong the operators on $L^2(\mathbb{R})$

$$q \longmapsto \text{multiplication by } x$$
$$p \longmapsto \frac{h}{i}\partial_x.$$

Following Weyl we assign to exponential fns. on V operators as follows

$$e^{i(aq+bp)} \longmapsto e^{i(ax + b\frac{h}{i}\partial_x)}$$

Then we extend this linearly to assign operators to any $f \in \mathcal{S}(V)$. This means that we take

$f(x,\xi) \in \mathcal{S}(V)$ expand it in exponentials

$$f(x,\xi) = \int e^{i(ax+b\xi)} \hat{f}(a,b)\, da\, db$$

and assign to $f(x,\xi)$ the operator

$$\int e^{iax + bh\partial_x} \hat{f}(a,b)\, da\, db$$

This leads to a deformed "composition" product on $\mathcal{S}(V)$ which we now work out.

It will probably be useful to abstract things a bit. Write v for a typical elt of V and λ for a typical element V^*. Let $W(\lambda)$ be the operator belonging to the function $v \mapsto e^{i\lambda v}$. Then we have

$$W(\lambda)W(\mu) = W(\lambda+\mu)\, e^{ihS(\lambda,\mu)}$$

where S is a skew pairing. If

$$f(v) = \int e^{i\lambda(v)} \hat{f}(\lambda)\, d^{2n}\lambda$$

Then
$$W(f) = \int W(\lambda)\, \hat{f}(\lambda)\, d^{2n}\lambda$$

so
$$W(f)\,W(g) = \int \underbrace{W(\lambda)\,W(\mu)}_{W(\lambda+\mu)\, e^{ihS(\lambda,\mu)}}\, \hat{f}(\lambda)\,\hat{g}(\mu)\, d^{2n}\lambda\, d^{2n}\mu$$

$$= \int W(\nu)\, e^{ihS(\lambda,\nu)} \hat{f}(\lambda)\,\hat{g}(\nu-\lambda)$$

$$= \int W(\nu) \left\{ \int e^{ihS(\lambda,\nu)} \hat{f}(\lambda)\,\hat{g}(\nu-\lambda)\, d\lambda \right\} d\nu$$

Let's continue on an abstract level.

We have the commutative algebra $\mathcal{A} = \mathcal{S}(V)$ and then we have a deformation of it $\mathcal{B}$ which additively is $\mathcal{A} + h\mathcal{A} + h^2\mathcal{A} + \cdots$ In this example there is a canonical lifting ρ of $\mathcal{A}$ into $\mathcal{B}$. To construct a cyclic $2n$-cocycle on $\mathcal{A}$, recall that we work in the algebra

$$C^{\cdot}(\mathcal{A}, \mathcal{B})$$

with the trace on it. We take the Chern-Simons or Chern transgression form which is an expression in the "connection" ρ and "curvature" $d\rho + \rho^2$.

Let's consider $\mathcal{A} = \mathcal{S}(S^1 \times \mathbb{R})$ and $\mathcal{B} = \mathcal{A} + h\mathcal{A} + h^2\mathcal{A} + \cdots$ with multiplication determined by the rule

$$e^{-ix} * p * e^{ix} = p + h$$

Consider the derivation $X = p\partial_p$ of $\mathcal{A}$. We want to extend it to $\mathcal{B}$; this requires

$$X(e^{-ix} * p * e^{ix}) = e^{-ix} * Xp * e^{ix} = e^{-ix} * p * e^{ix}$$

$$\| $$

$$X(p+h) \qquad = p + h$$

which means probably that on $\mathcal{B}$ we have

$$X = p\partial_p + h\partial_h$$

Next consider the trace on $\mathcal{B}$ given by

$$\tau(f) = \int \frac{dx\,dp}{2\pi h} f(h, x, p)$$

Then

$$\tau(Xf) = \int \frac{dx\,dp}{2\pi h} (p\partial_p + h\partial_h) f(h, x, p)$$

$$= \int \frac{dx\,dp}{2\pi h}\; p\,\partial_p f \;+\; \partial_h h \int \frac{dx\,dp}{2\pi h} f(h,x,p)$$

$$= -\,\tau(f) \;+\; \partial_h h\, \tau(f) \qquad \text{so}$$

$$\boxed{\tau(Xf) = h\partial_h \tau(f)}$$

Let $\tau_0(f)$ denote the coefficient of h^0 in $\tau(f)$. Thus if $f = f_0^{(x,p)} + h f_1^{(x,p)} + \cdots$ we have

$$\tau_0(f) = \int \frac{dx\,dp}{2\pi} f_1$$

Then from the above we have

$$\tau_0(Xf) = 0$$

Next we consider the cyclic 2-cocycle [redacted] on $\mathcal{A}$ associated to the extension $\mathcal{B}/h^2\mathcal{B}$, [redacted] the obvious lifting of $\mathcal{A}$ into $\mathcal{B}$, and the trace τ_0.

I reviewed earlier formulas. The transgression form is

$$\eta = tr\left(\tfrac{2}{3}A^3 + A\cdot dA\right) = tr\left(A\cdot F - \tfrac{1}{3}A^3\right)$$

and $d\eta = tr(F^2)$. When we calculate the transgression form using the trace given by τ_0 and cyclic averaging [redacted] the term $tr(A^3)$ is not zero. This leads me to suspect that the situation might be better in the Weyl case where g is canonical, or can be chosen in a canonical fashion.

May 5, 1988

Let's return to $\mathcal{A} = \mathcal{S}(V)$ where V is a real symplectic vector space of dimension $2n$. Elements of $\mathcal{S}(V)$ can be expanded in exponential functions $e^{ik(x)}$, where $k \in V^*$. We get a deformation algebra $\mathcal{B}$ of $\mathcal{A}$ which is generated by h and $\rho(k) = \rho(e^{ikx})$ satisfying the Weyl relations

$$\rho(k_1)\rho(k_2) = \rho(k_1+k_2)\, e^{ih S(k_1,k_2)}$$

where S is the skew form on V transported to V^*. (It might be better to suppose a skew form on V^* given to begin with.)

One has a trace on $\mathcal{B}$ with values in Laurent series in h given by

$$\tau\big(\rho(k)\big) = h^{-n}\delta(k)$$

Motivation. Suppose V polarized and V^* consists of the functions $k'q + k''p$ and that $\rho(k)$ is the operator

$$q_i = \text{mult. by } x_i \qquad p_j = \frac{h}{i}\partial_{x_j}$$

$$\begin{aligned}
\rho(k) &= e^{i(k'q + k''p)} \\
&= e^{ik'x + hk''\partial_x} \\
&= e^{ik'x}\, e^{hk''\partial_x}\, e^{-\frac{1}{2}[ik'x,\, hk''\partial_x]} \\
&= e^{\frac{1}{2}ihk'k''}\, e^{ik'x}\, e^{hk''\partial_x}
\end{aligned}$$

This operator has the Schwartz kernel

$$\langle x|\rho(k)|y\rangle = e^{\frac{1}{2}ihk'k''}\, e^{ik'x}\,\delta(x+hk''-y).$$

The trace is obtained by setting $x=y$ and ~~integrating~~

over x, so

$$\operatorname{tr}(\rho(k)) = e^{\frac{1}{2}ihk'k''}(2\pi)^n \delta(k')\,\delta(hk'')$$
$$= (2\pi)^n h^{-n}\,\delta(k)$$

Now let's calculate the cyclic 2-cocycle given by the Chern-Simons form

$$\operatorname{tr}(AF - \tfrac{1}{3}A^3) = \operatorname{tr}\left(\tfrac{2}{3}A^3 + AdA\right)$$

First we look at $\operatorname{tr}(A^3)$. This is the cochain

$$\underbrace{\tau(\rho(k_0)\rho(k_1)\rho(k_2))}\quad + \quad \text{cyc. perms.}$$

$$\rho(k_0)\,\rho(k_1+k_2)\,e^{ihS(k_1,k_2)} = \rho(k_0+k_1+k_2)\,e^{ih\left\{\begin{matrix}S(k_0,k_1+k_2)\\+S(k_1,k_2)\end{matrix}\right\}}$$

$$\therefore\ \tau(\rho(k_0)\rho(k_1)\rho(k_2)) = h^{-n}\underbrace{\delta(k_0+k_1+k_2)}_{\substack{\delta \text{ fn on}\\ \text{hypersurface}}}\ \underbrace{e^{ih\{S(k_0,k_1+k_2)+S(k_1,k_2)\}}}_{\substack{\text{smooth}\\ \text{function}}}$$

$$\boxed{\tau(\rho(k_0)\rho(k_1)\rho(k_2)) = h^{-n}\,\delta(k_0+k_1+k_2)\,e^{ih\,S(k_1,k_2)}}$$

So even in the canonical ~~situation~~ we have at hand we see that the trace of A^3 is not trivial.

$$\tfrac{2}{3}\operatorname{tr}(A^3) = \tfrac{2}{3}h^{-n}\delta(k_0+k_1+k_2)\left[e^{ihS(k_1,k_2)} + \text{cyc.}\right]$$

$$\operatorname{tr}(AdA) = \tau(\rho(k_0)(-\rho(k_1+k_2))) + \text{cyc.}$$

$$= -\tau\left(\rho(k_0+k_1+k_2)\,e^{ihS(k_0,k_1+k_2)}\right) + \text{cyc.}$$

$$= -h^{-n}\delta(k_0+k_1+k_2)\,e^{ihS(k_0,k_1+k_2)} + \text{cyc}$$

vanishes where $k_0+k_1+k_2=0$

Notice that when $k_0+k_1+k_2=0$ we have

$$S(k_2,k_0) = S(k_2,-k_1-k_2) = S(k_1,k_2)$$

and similarly $S(k_0,k_1) = S(k_1,k_2)$. Thus we conclude that

$$\boxed{\operatorname{tr}\left(\tfrac{2}{3}A^3 + A\,dA\right) = h^{-n}\,\delta(k_0+k_1+k_2)\left[2e^{ihS(k_1,k_2)} - 3\right]}$$

This is a cyclic 2-cochain on $\mathcal{A}$ whose boundary is $\operatorname{tr}(F^2)$

$$F^2 : \quad \rho(k_0+k_1)\left[e^{ihS(k_0,k_1)}-1\right]\rho(k_2+k_3)\left[e^{ihS(k_2,k_3)}-1\right]$$

$$= \rho(k_0+\cdots+k_3)\,\underbrace{e^{ihS(k_0+k_1,\,k_2+k_3)}\left[e^{ihS(k_0,k_1)}-1\right]\left[e^{ihS(k_2,k_3)}-1\right]}$$

$$\operatorname{tr}F^2 = h^{-n}\,\delta(k_0+\cdots+k_3)\Big(\qquad\qquad\qquad\Big)$$

$$+ \text{ cyclic perms.}$$

The leading term is

$$(i)^2 h^{2-n}\,\delta(k_0+\cdots+k_3)\left\{S(k_0,k_1)\,S(k_2,k_3) + \text{cyc. perms}\right\}$$

If $n=1$, ~~then~~ the coefficients ~~of~~ h^{-1}, h^0 of $\operatorname{tr}\left(\tfrac{2}{3}A^3 + A\,dA\right)$ are cyclic 2-cocycles. This gives the 2-cocycles

$$-\delta(k_0+k_1+k_2)$$

$$\delta(k_0+k_1+k_2)\;2iS(k_1,k_2)$$

Our problem is now to understand what is happening in general. This Weyl algebra deformation is a very good example. It shares some features with the universal case $\mathcal{B} = T(\mathfrak{a})$ in there is a nice grading on $\mathcal{B}/[\mathcal{B},\mathcal{B}]$, or at least for the trace $\tau : \mathcal{B} \to \mathbb{C}[h]h^n$ which is utilized.

I suspect that we have not yet found the S-operator's role in the Chern-Weil algebra.