January 16, 1988

I want to go carefully over the Goodwillie, Feigin, Tsygan result on the cyclic complex of a semi-direct product $R \oplus M$. This complex is $\mathbb{Z}$-graded by the degree in M. Thus for $p>0$

$$(1) \qquad CC_{q-1}(R\oplus M)(p) = \left((R\oplus M)^{\otimes q}/(1-t)\right)(p)$$
$$= \Big\{ \bigoplus_{i_0+\cdots+i_p+p=q} R^{\otimes i_0} \otimes M \otimes \cdots \otimes M \otimes R^{\otimes i_p} \Big\} / (1-t)$$

The result states that $\Sigma\, CC_*(R\oplus M)(p)$ is isomorphic to $\left(M[1] \otimes_R B(R) \otimes_R\right)^p_\sigma$. Now in degree q we have

$$(2) \qquad \left(M[1] \otimes_R B(R) \otimes_R\right)^p_q = \bigoplus_{i_1+\cdots+i_p+p=q} M \otimes R^{\otimes i_1} \otimes \cdots \otimes M \otimes R^{\otimes i_p}$$

Let us now define a map Φ from (2) to (1) in the obvious way, namely by identifying (2) with the part of the direct sum in braces in (1) with $i_0 = 0$. Let's check that Φ is a map of complexes. Take $p=2$ and consider $[m_1, x_1, \ldots, x_{i_1}, m_2, x_{i_1+1}, \ldots, x_{i_1+i_2}]$ in (2). Recall that the differential in $B(R)$ is an alternating sum (or commutator) of "face" operators each of which deletes a $\otimes$ sign. Thus in $M \otimes_R B(R) \otimes_R N$ the differential is

$$d[m, x_1, \ldots, x_p, n] = [mx_1, x_2, \ldots, x_p] + \sum_{i=1}^{p-1} (-1)^i [m, \ldots, x_i x_{i+1}, \ldots, n]$$
$$+ (-1)^p [m, x_1, \ldots, x_{p-1}, x_p n]$$

In $(M[1] \otimes_R B(R) \otimes_R)^p$ there will be extra signs due to the suspension of M occurring. Thus

$$d[m_1, x_1, \ldots, x_{i_1}, m_2, x_{i_1+1}, \ldots, x_{i_1+i_2}]$$

$$= -[m_1 x_1, x_2, \ldots] + \sum_{j=1}^{i_1-1} (-1)^{1+j} [\ , x_j x_{j+1}, \ ]$$

$$+ (-1)^{1+i_1} [\ \ , x_{i_1} m_2, \ \ ] + (-1)^{2+i_1} [\ldots, m_2 x_{i_1+1}, \ ]$$

$$+ \sum_{j=1}^{i_2-1} (-1)^{2+i_1+j} [\ldots, x_{i_1+j} x_{i_1+j+1}, \ldots]$$

$$+ (-1)^{2+i_1+i_2-1} [x_{i_1+i_2} m_1, \ldots\ldots]$$

(In this formula we are assuming that $i_1, i_2 > 0$)

How does this compare with the differential in the cyclic complex (1)? The sign of a face operator is the parity of the number of preceding faces. It's off by 1 from the number of elements preceding the comma. This also apply to the crossover term. Thus we conclude that Φ anti-commutes with d meaning that it is a map of complexes from the suspension of the cyclic complex.

Still should check what happens if either i_1, i_2 is zero.

$$\cancel{d[m_1, m_2, x_1, \ldots, x_k] = \underbrace{dm_1 \otimes [m_2, \ , x_k]\otimes}_{\in M \otimes_R B(R)} - m_1 \otimes d[m_2, \ldots, x_k]\otimes}$$

$$\cancel{= + m_1 \otimes m_2}$$

January 17, 1988

Recall that we have a map

$$\Phi : \left(M \overset{[1]}{\otimes}_R B(R) \otimes_R\right)^p \longrightarrow \left\{\sum CC_*(R \oplus M)\right\}(p)$$

and we are trying to check that it is a map of complexes. What is the differential on the left? $M \overset{[1]}{\underset{\wedge}{\otimes}}_R B(R)$ is a resolution of $M[1]$ by R-bimodules which are right free. One has

$$\left(M \overset{[1]}{\underset{\wedge}{\otimes}}_R B(R)\right)_{n+1} = M \otimes_R \left(R \otimes R^{\otimes n} \otimes R\right) = M \otimes R^{\otimes n} \otimes R$$

and

$$d[m, x_1, \dots, x_n, x_{n+1}] = \overset{=0}{dm} \otimes_R [1, x_1, \dots, x_{n+1}] - m \otimes_R d[1, x_1, \dots, x_{n+1}]$$

$$= -m \otimes_R \left\{[x_1, \dots, x_{n+1}] + \sum_{i=1}^{n} (-1)^i \,[1, \dots, x_i x_{i+1}, \dots]\right\}$$

$$= -\left([m x_1, \dots, x_{n+1}] + \sum_{i=1}^{n} (-1)^i [m, \dots, x_i x_{i+1}, \dots]\right)$$

Thus if we take an element of $\left(M[1] \otimes_R B(R) \otimes_R\right)^2$ say $[m_1, x_1, \dots, x_k, m_2, y_1, \dots, y_\ell]$ its differential is

$$d[m_1, x_1, \dots, x_k, m_2, y_1, \dots, y_\ell] =$$

$$d[m_1, \dots, x_k] \otimes_R [m_2, \dots, y_\ell] \otimes_R + (-1)^{k+1} [m_1, \dots, x_k] \otimes_R d[m_2, \dots, y_\ell] \otimes_R$$

$$= -[m_1 x_1, \dots] + \sum_{i=1}^{k-1} (-1)^{i+1} [m_1, \ , x_i x_{i+1}, \ ]$$

$$+ (-1)^{k+1} [m_1, x_1, \dots, x_k m_2, y_1, \dots, y_\ell]$$

$$+ (-1)^{k+2} [m_1, \dots, x_k, m_2 y_1, \dots, y_\ell] + \sum_{j=1}^{\ell-1} (-1)^{k+2+j} [m_1, \dots, m_2, \dots y_j y_{j+1}, \dots]$$

$$+ (-1)^{k+2+\ell} [y_\ell m_1, \dots, m_2, y_1, \dots, y_{\ell-1}]$$

On the other hand the right side in degree n is

$$\left\{\Sigma\, CC_*(R\oplus M)\right\}_n(p) = CC_{n-1}(R\oplus M)(p)$$

$$= \left\{(R\oplus M)^{\otimes n}/1-t\right\}(p)$$

$$= \left\{\bigoplus_{i_0+i_1+\cdots+i_p+p=n} R^{\otimes i_0}\otimes M\otimes\cdots\otimes M\otimes R^{\otimes i_p}\right\}/(1-t)$$

If we regard $[m_1, x_1, \ldots, x_k, m_2, y_1, \ldots, y_\ell]$ as an element of $CC_{n-1}(R\oplus M)(2)$ where $n = k+\ell+2$ then

$$b[m_1, x_1, \ldots, x_k, m_2, y_1, \ldots, y_\ell]$$

$$= [m_1x_1, \ldots] + \sum_{i=1}^{k-1}(-1)^i[m_1, x_ix_{i+1}, \ldots, m_2, \ldots]$$

$$+ (-1)^k[m_1, \ldots, x_km_2, \ldots] + (-1)^{k+1}[m_1, \ldots, m_2y_1, \ldots]$$

$$+ \sum_{j=1}^{\ell-1}(-1)^{k+1+j}[m_1, \ldots, m_2, \ldots, y_jy_{j+1}, \ldots]$$

$$+ (-1)^{k+1+\ell}[y_\ell m_1, \ldots, m_2, \ldots, y_{\ell-1}]$$

Thus d agrees with b on the suspension.

Check the case where $\ell = 0$.

$$d[m_1, x_1, \ldots, x_\ell, m_2] = d[m_1, \ldots, x_\ell]\otimes_R m_2\otimes_R + (-1)^{\ell+1}\cancel{[m_1, \ldots, x_\ell]\otimes_R dm_2\otimes_R}\ 0$$

$$= -[m_1x_1, \ldots, x_\ell, m_2] + \sum_{i=1}^{\ell-1}(-1)^{i-1}[m_1, \ldots, x_ix_{i+1}, \ldots, m_2]$$

$$+ (-1)^{\ell-1}[m_1, \ldots, x_\ell m_2]$$

$$b[m_1, x_1, \ldots, x_\ell, m_2] = [m_1x_1, \ldots, m_2] + \sum_{i=1}^{\ell-1}(-1)^i[m_1, \ldots, x_ix_{i+1}, \ldots, m_2]$$

$$+ (-1)^\ell[m_1, \ldots, x_\ell m_2]$$

So it's clear that Φ is a map of complexes. The next thing to be checked is the action of $\mathbb{Z}/p$. What we must check is that (case $p=2$)

$$[m_1, x_1, \ldots, x_k, m_2, y_1, \ldots, y_\ell]$$

$$(-1)^{(k+1)(\ell+1)} [m_2, y_1, \ldots, y_\ell, m_1, x_1, \ldots, x_k]$$

These are related by the flip in $\left(M \overset{[1]}{\underset{R}{\otimes}} B(R)\right) \otimes_R \left(M \overset{[1]}{\underset{R}{\otimes}} B(R)\right) \otimes_R$

have the same image in $CC_{n-1}(R \oplus M)(2)$ where $n = k+\ell+2$. But this is clear, because t is the cyclic permutation of degree n with the sign obtained by regarding the elements of $R \oplus M$ as odd.

At this point we are certain of the Goodwillie – Feigin – Tsygan isomorphism for $R \oplus M$. How do we see that Φ induces an isomorphism on the quotient by σ?

We can describe $CC_{n-1}(R \oplus M)(p)$ as follows. Let S be the set of cyclic embeddings of $[p]$ into $[n]$, where $[p] = \{1, \ldots, p\}$. Another way of describing an element of S is to give a card p subset with basepoint in $[n]$.

S is acted on by $\mathbb{Z}/p \times \mathbb{Z}/n$ and the quotient by the action of $\mathbb{Z}/p$ is the set of card p subsets of $[n]$. Given $s \in S$, let

$$V_s = R^{\otimes i_0} \otimes M \otimes \cdots \otimes M \otimes R^{\otimes i_p}$$

if the image of $s : [p] \longrightarrow [n]$ consists of the points

$$i_0 + 1, \; i_0 + i_1 + 2, \; \ldots, \; i_0 + i_1 + \cdots + i_{p-1} + p - 1$$

?

The naive to prove Φ is an isomorphism after dividing by $\mathbb{Z}/p$ would be to show it's surjective + injective. But Goodwillie's way is to show both sides are the $\mathbb{Z}/p$ quotient of the same thing. To each cyclic embedding $s : [p] \longrightarrow [n]$ we assign the tensor product

$$V_s = \bigotimes_{i=1}^{n} \begin{Bmatrix} R & \text{if } i \notin \operatorname{Im} s \\ M & \text{if } i \in \operatorname{Im} s \end{Bmatrix}$$

Alternatively

$$V_s = V_{s,1} \otimes V_{s,2} \otimes \cdots \otimes V_{s,n}$$

where

$$V_{s,i} = \begin{cases} R & i \notin \operatorname{Im} s \\ M & i \in \operatorname{Im} s \end{cases}$$

~~[illegible]~~ This depends only on the image of s, so perhaps I will write V_I for $I \subset [n]$.

In general one has

$$(R \oplus M)^{\otimes n} = \bigoplus_{I \subset [n]} V_I$$

and one has the cyclic skew-symmetric action of $\mathbb{Z}/n$. We have

$$CC_{n-1}(R \oplus M)(p) = \Big(\bigoplus_{\substack{I \subset [n] \\ \operatorname{card} I = p}} V_I \Big)_{\mathbb{Z}/n}$$

$$\|$$

$$\Big(\bigoplus_{\substack{s : [p] \hookrightarrow [n] \\ \text{cyclic}}} V_{\operatorname{Im}(s)} \Big)_{\mathbb{Z}/p \times \mathbb{Z}/n}$$

and

$$\left(\bigoplus_{\substack{s:[p]\hookrightarrow[n] \\ \text{cyclic}}} V_{\operatorname{Im}(s)} \right)_{\mathbb{Z}/n} = \bigoplus_{c_1+\cdots+c_p+p=n} M\otimes R^{\otimes c_1}\otimes\cdots\otimes M\otimes R^{\otimes c_p}$$

so we find that

$$CC_{n-1}(R\oplus M)(p) \cong \left\{ \left(M^{[1]}\otimes_R B(R)\otimes_R \right)^p_n \right\}_{\mathbb{Z}/p}$$

There are some intriguing things going on here. Basically I have the feeling that I have to do things like parallel transport, + time-ordered exponentials, but where instead of having functions (or fields) along the curve, I have modules. Thus tensor product of modules [redacted] occurs instead of composition (or product) of operators. [redacted]

Next I should see what happens with the signs when I the superalgebra $R\oplus I$ where $I = M$ but considered as having odd degree. The first difference is that in

$$\cancel{(R\oplus M)^{\otimes n} = \bigoplus_I}$$

$$(R\oplus I)^{\otimes n} = \bigoplus_{J\subset[n]} V_J$$

the action of $\mathbb{Z}/n$ is different. Here

$$V_J = \bigotimes_{i=1}^n \begin{cases} R & i\notin J \\ I & i\in J \end{cases}$$

and in determining the signs only the R factors are considered odd.

The idea now will be to try to get a map Φ from $(I \otimes_R B(R) \otimes_R)^p$ to a suitable shift of $CC_*(R \oplus I)(p)$. NO

In order to handle the case of the DGA $R \leftarrow I$ we have to be careful about the signs. What exactly does one mean by the cyclic complex of a DGA?

In general we obtain the Hochschild complex of an algebra A by taking tensor powers $A^{\otimes n}$ in degree $n-1$ and defining the boundary b to be the alternating sum of face operators.

$$b[a_1|\cdots|a_n] = \sum_{i=1}^{n-1} (-1)^{i-1} [\cdots|a_i a_{i+1}|\cdots] + (-1)^{n-1}[a_n a_1|\cdots]$$

The face operators are given by the multiplication map $A \otimes A \longrightarrow A$ applied to consecutive factors. The cross-over term is obtained by the flip isom.

$$(*) \quad (A^{\otimes(n-1)}) \otimes A \simeq A \otimes (A^{\otimes(n-1)})$$

followed by multiplying the initial two factors.

Now take A to be DGA. Then each tensor power $A^{\otimes n}$ is a complex, and the face operators are maps of complexes since the multiplication $A \otimes A \longrightarrow A$ is. Thus the **Hochschild** complex and the cyclic complex are complexes in the category of complexes, i.e. double complexes. (Thus the vertical + horizontal differentials commute and a sign must be introduced to define the total differential.)

Because the flip isomorphism ⊛ used for complexes is not the same as for vector spaces, the cross over term for the DGA setup (really I should say superalgebra setup) is not the same as for the underlying algebra.

For example, let's consider $A = R \twoheadleftarrow I$.

$$(**)\qquad (A^{\otimes n})(p) = (R \oplus I)^{\otimes n}(p)$$

$$= \bigoplus_{1 \le n_1 < \cdots < n_p \le n} R^{\otimes(n_1 - 1)} \otimes I \otimes R^{\otimes(n_2 - n_1 - 1)} \otimes I \otimes \cdots \otimes R^{\otimes(n - n_p)}$$

The cross-over term involves the flip of the last factor to the front together with the sign $(-1)^{n-1}$. If the last factor is R, i.e. $n_p < n$, then there is no other sign, but if the last factor is I then we have the additional sign $(-1)^{p-1}$ because we have moved I past $(p-1)$ other factors I. Note that after performing the flip we muliply, so if both first and last factors are I the crossover term gives zero.

Now we can define

$$\Phi : (I[1] \otimes_R B(R) \otimes_R)^p \longrightarrow \Sigma CC_{n-1}(R \oplus I)(p)$$

exactly as before. The calculation that Φ is a map of complexes is exactly the same except for the cross over term. We have defined Φ by mapping into the Hochschild group (**), actually into the part where the first factor is I (always $n_1 = 1$), and then projecting into the cyclic complex. The strange signs appear in the crossover term only when the last factor is I. However as the first factor is always I this means the crossover term has to be zero, and so the strange signs have no effect. Thus Φ is a map of complexes.

The next step is to show that Φ commutes with the skew-symmetric cyclic action on the left (i.e. on its source). Take $p=2$ and consider the elements

$$[m_1 | x_1 | \cdots | x_k | m_2 | y_1 | \cdots | y_\ell] \quad \in \left(I[1] \otimes_R \mathcal{B}(R) \otimes_R \right)^2$$

and

$$(-1)^{(k+1)(\ell+1)} [m_2 | y_1 | \cdots | y_\ell | m_1 | x_1 | \cdots | x_k].$$

This are related by the generator of $\mathbb{Z}/2$. When we apply Φ we obtain elements in $(R \leftarrow I)^{\otimes(k+\ell+2)}(2)$ which are denoted in the same way. The effect of $\mathbb{Z}/n\mathbb{Z}$ where $n = k+\ell+2$ introduces the sign

$$\underset{\text{moving } m_2 \text{ past } m_1}{(-1)} \; (-1)^{(n-1)(\ell+1)}$$

Now

$$\begin{aligned} (n-1)(\ell+1) &= (k+\ell+1)(\ell+1) \\ &\equiv k(\ell+1) + (\ell+1)^2 \qquad \text{mod } 2 \\ &\equiv (k+1)(\ell+1) \end{aligned}$$

showing that the Φ is consistent with the skew cyclic action. In the general case given

$$[\overbrace{m_1 \cdots\cdots m_2}^{\mu} \cdots\cdots \overbrace{m_p \cdots\cdots}^{\nu}] \in \left(I[1] \otimes_R \mathcal{B}(R) \otimes_R \right)^p$$

in order to move m_p around in front we acquire a normal sign $(-1)^{\mu\nu}$ $\left(= (-1)^{(n-1)\nu} \right.$ since $(n-1)\nu = (\mu+\nu-1)\nu \equiv \mu\nu \mod 2$

The same sign occurs in $(R \leftarrow I)^{\otimes n}(p)$ with an additional sign of $(-1)^{p-1}$ when one moves m_p around. Thus Φ is compatible with $(-1)^{p-1}\sigma = t_p$.

January 22, 1988

R unital algebra, consider the bifunctor $K \otimes_R L$, where K, L are complexes of R-modules (right and left respectively), with values in complexes. Assuming L_n flat for all n, does it follow that $? \otimes_R L$ preserves quasi-isomorphisms? It suffices by the mapping cone construction to show that $H_*(K) = 0 \implies H_*(K \otimes_R L) = 0$. If then K is exact and L is flat, the double complex $K_p \otimes_R L_q$ has exact rows. One can't conclude the total complex $K \otimes_R L$ is exact unless L is bounded below or K is bounded above. In the former case we can express L as the union of bounded flat complexes $L_{\leq n}$, and in the latter we can express K as the union of finite exact complexes:

$$\begin{tikzcd}
\cdots & K_{n-2} \arrow[l] & K_{n-1} \arrow[l] & K_n \arrow[l] & K_{n+1} \arrow[l] & K_{n+2} \arrow[l] \\
0 \arrow[u] & Z_{n-2} \arrow[l] \arrow[u, hook] & K_{n-1} \arrow[l] \arrow[u, equal] & K_n \arrow[l] \arrow[u, equal] & K_{n+1} \arrow[l] \arrow[u, equal] & \arrow[l] \\
0 \arrow[u] & 0 \arrow[l] \arrow[u] & Z_{n-1} \arrow[l] \arrow[u, hook] & K_n \arrow[l] \arrow[u, equal] & K_{n+1} \arrow[l] \arrow[u, equal] & \arrow[l]
\end{tikzcd}$$

Counterexample: $A = k[\varepsilon]/(\varepsilon^2)$ and take

$$K: \quad 0 \leftarrow \underset{0}{k} \leftarrow \underset{1}{A} \xleftarrow{\varepsilon} A \xleftarrow{\varepsilon} A \xleftarrow{\varepsilon} \cdots \qquad \text{exact}$$

$$L: \quad \xleftarrow{\varepsilon} A \xleftarrow{\varepsilon} A \xleftarrow{\varepsilon} A \xleftarrow{\varepsilon} A \xleftarrow{\varepsilon} A \xleftarrow{\varepsilon} \qquad \text{flat}$$

Then $K \otimes_A L$ is the double complex

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 & k \arrow[l] \arrow[d, "0"] & A \arrow[l] \arrow[d, "\varepsilon"] & A \arrow[l, "\varepsilon"'] \arrow[d] & {} \arrow[l] \\
0 & k \arrow[l] \arrow[d, "0"] & A \arrow[l] \arrow[d, "\varepsilon"] & A \arrow[l, "\varepsilon"'] \arrow[d, "\varepsilon"] & {} \arrow[l] \\
0 & k \arrow[l] \arrow[d, "0"] & A \arrow[l] \arrow[d, "\varepsilon"] & A \arrow[l, "\varepsilon"'] \arrow[d, "\varepsilon"] & {} \arrow[l] \\
0 & k \arrow[l] \arrow[d, "0"] & A \arrow[l] \arrow[d] & A \arrow[l, "\varepsilon"'] \arrow[d] & {} \arrow[l] \\
0 & {} \arrow[l] & {} \arrow[l] & &
\end{tikzcd}$$

This double complex is the union of the subcomplexes consisting of the first $p+1$ columns for different p. Since the p^{th} column for $p>0$ is exact, it follows the cohomology of the double complex is that of the 0th column, i.e. k in each degree. Thus $K \otimes_A L$ is not exact. QED.

<u>January 23, 1988</u>

Let $R/I = A$ with $R = T(V)$ to fix the ideas. We consider the semi-direct product $R \oplus \Omega^1_R$ which can be viewed as an extension of A with the ideal $J = I \oplus \Omega^1_R$. Then

$$J^{n+1} = I^{n+1} \oplus \sum_{k=0}^{n} I^k \Omega^1_R I^{n-k}$$

because $(\Omega^1_R)^2 = 0$. (More precisely any product $I^{i_0}\Omega^1_R I^{i_1}\Omega^1_R \cdots$ lies in $\Omega^1_R \cdot \Omega^1_R \cdots = 0$.)

We have two homomorphisms

$$R \rightrightarrows R \oplus \Omega^1_R \qquad \begin{array}{l} x \longmapsto x \\ \longmapsto x + dx \end{array}$$

are homomorphisms "over" A, hence carry I to J. Thus we get two homomorphisms

$$R/I^{n+1} \rightrightarrows R \oplus \Omega^1_R / J^{n+1} = R/I^{n+1} \oplus \Omega^1_R \Big/ \sum_{k=0}^{n} I^k \Omega^1_R I^{n-k}$$

of nilpotent extensions of A of order $n+1$.

These induce maps

$$HC_0(R/I^{n+1}) \rightrightarrows HC_0(R/I^{n+1} \oplus \Omega^1_R / \cdots)$$

which agree on the image of $HC_{2n}(A)$. The latter group is the direct sum of $HC_0(R/I^{n+1})$ and

$$\cancel{HC_0(R/I^{n+1}) \oplus} \quad \Omega^1_R \Big/ \Big(\sum_{k=0}^{n} I^k \Omega^1_R I^{n-k} + [R, \Omega^1_R]\Big)$$

However $I^k \Omega^1_R I^{n-k} \equiv I^n \Omega^1_R \mod [R, \Omega^1_R]$, so the last group is

$$\Omega^1_R / I^n \Omega^1_R + [R, \Omega^1_R] = R/I^n \otimes_R \Omega^1_R \otimes_R$$

Thus we have proved

<u>Prop.</u> The ~~maps~~ kernel of the map

$$(*) \qquad HC_0(R/I^{n+1}) = R/[R,R]+I^n \longrightarrow R/I^n \otimes_R \Omega^1_R \otimes_R$$

induced by the DR differential $d: R \to \Omega^1_R$. coincides with the ~~[illegible]~~ equalizer of two maps
$$HC_0(R/I^{n+1}) \rightrightarrows HC_0(R \oplus \Omega^1_R / J^{n+1})$$
induced by two maps of nilpotent extensions of A of order $n+1$. Hence the image of $HC_{2n}(A)$ is contained in the <u>kernel of $(*)$.</u>

Now take the case where $R = T(V)$, in which case we have established an exact sequence

$$0 \to HC_{2n}(A) \to HC_0(R/I^{n+1}) \xrightarrow{u} H_1(R, R/I^n)_\sigma \to HC_{2n-1}(A) \to$$

Our goal will be ~~[illegible]~~ to show the kernel of u coincides with that of $(*)$. Once this done we ~~[illegible]~~ see that $HC_{2n}(A)$ is the ~~[illegible]~~ subspace of $HC_0(R/I^{n+1})$ equalized by ~~two~~ maps from R/I^{n+1} to another order $(n+1)$ nilpotent extension of A. Thus we will have

$$HC_{2n}(A) = \varprojlim_{R \twoheadrightarrow A} HC_0(R/I^{n+1})$$

where the limit is taken over the category of extensions of A.

So what do we know about u above? We know it's natural ~~[illegible]~~ in the pair (R,I), ~~[illegible]~~ and this gives a commutative square by using the ~~[illegible]~~ map $(R,0) \subset (R,I)$:

$$\begin{tikzcd} HC_0(R) \arrow[r, "u'"] \arrow[d] & H_1(R,R)_\sigma \arrow[d] \\ HC_0(R/I^{n+1}) \arrow[r, "u"] & H_1(R, R/I^n)_\sigma \end{tikzcd}$$ irrelevant

I should have written down first

$$\begin{array}{ccc} HC_0(R/I^{n+1}) & \xrightarrow{\quad u \quad} & H_1(R, R/I^n) \\ \| & & \| \\ H_0\left((R\overset{!}{\otimes}_R)^{n+1}_\sigma/(I\overset{!}{\otimes}_R)^{n+1}_\sigma\right) & & H_1\left((R\overset{!}{\otimes}_R)^n_\sigma/(I\overset{!}{\otimes}_R)^n_\sigma\right) \\ \| & & \| \\ E^1_{n,n} & \xrightarrow{\quad d' \quad} & E^1_{n-1,n} \end{array}$$

and that this d' is induced by the ~~[illegible]~~ horizontal boundary map from the n-th column to the $(n-1)$th column of the double complex $CC(R \overset{!}{\leftarrow} R)/CC(R)$. This is a map

$$\sum (R\overset{!}{\otimes}_R)^{n+1}_\sigma \longrightarrow (R\overset{!}{\otimes}_R)^n_\sigma$$

which presumably has to be equivalent to Connes B-operator. In any case this would be pretty easy to calculate if required.

So we can assume (more or less) that our map

$$HC_0(R/I^{n+1}) \xrightarrow{u} H_1(R, R/I^n)_\sigma$$

is induced by

$$\begin{array}{cccc} B: & HC_0(R) & \longrightarrow & H_1(R,R) \\ & \| & & \cap \\ & R/[R,R] & \xrightarrow{d} & \Omega^1_R \otimes_R \end{array}$$

This will give a commutative diagram

$$\begin{tikzcd}
HC_0(R/I^{n+1}) \arrow[r] \arrow[rr, bend left, "d"] \arrow[dr, "u"'] & H_1(R,R/I^n) \arrow[d] & \subset R/I^n \otimes_R \Omega^1_R \otimes_R \\
 & H_1(R,R/I^n)_\sigma &
\end{tikzcd}$$

The real issue is whether the passage to the quotient by $\mathbb{Z}/n$ will lead to a larger kernel. However recall the exact sequence

$$\to H_1(R,R) \to H_1(R,R/I^n) \to I^n/[R,I^n] \to R/[R,R] \to 0$$

where $\mathbb{Z}/n$ acts with trivial action at the ends. Since we have

$$\begin{tikzcd}
HC_0(R) \arrow[r, "B"] \arrow[d] & H_1(R,R) \arrow[r, hook] \arrow[d] & \Omega^1_R \otimes_R \arrow[d] \\
HC_0(R/I^{n+1}) \arrow[r] & H_1(R,R/I^n) \arrow[r, hook] & R/I^n \otimes_R \Omega^1_R \otimes_R
\end{tikzcd}$$

it follows that the image of $HC_0(R/I^{n+1}) \to H_1(R,R/I^n)$ is contained in the invariant subspace for the cyclic group action. But the invariant subspace maps isomorphically onto the coinvariant space. QED.

Let's see if there is a map of complexes of R-bimodules

$$B(R) \otimes_R B(R) \longrightarrow B(R)$$

given by

$$[x_0|x_1|\cdots|x_{k+1}] \otimes_R [y_0|\cdots|y_{\ell+1}] \longrightarrow [x_0|\cdots|x_{k+1}y_0|\cdots|y_{\ell+1}]$$

One has

$$d([x_0|\cdots|x_{k+1}] \otimes_R [y_0|\cdots|y_{\ell+1}])$$

$$= d([x_0|\cdots|x_{k+1}]) \otimes_R [y_0|\cdots|y_{\ell+1}] + (-1)^k [x_0|\cdots|x_{k+1}] \otimes_R d[y_0|\cdots|y_{\ell+1}]$$

$$= \left\{ \sum_{i=0}^{k} (-1)^i [\cdots|x_i x_{i+1}|\cdots] \right\} \otimes_R [y_0|\cdots|y_{\ell+1}]$$

$$+ (-1)^k [x_0|\cdots|x_{k+1}] \otimes_R \left\{ \sum_{j=0}^{\ell} (-1)^j [\cdots|y_j y_{j+1}|\cdots] \right\}$$

$$= \sum_{i=0}^{k-1} (-1)^i [x_0|\cdots|x_i x_{i+1}|\cdots|x_{k+1} y_0|\cdots|y_{\ell+1}]$$

$$+ (-1)^k [x_0|\cdots|x_k x_{k+1} y_0|\cdots|y_{\ell+1}]$$

$$+ (-1)^k \Big\{ [x_0|\cdots|x_{k+1} y_0 y_1|\cdots|y_{\ell+1}] + \sum_{j=1}^{\ell} (-1)^j [x_0|\cdots|x_k y_0|\cdots|y_j y_{j+1}|\cdots] \Big\}$$

Check $k = 1$, $\ell = 0$

$$d\,([x_0|x_1|x_2] \otimes_R [y_0|y_1])$$

$$= ([x_0 x_1|x_2] - [x_0|x_1 x_2]) \otimes_R [y_0|y_1] + (-1)[x_0|x_1|x_2] \otimes ([y_0 y_1])$$

$$= [x_0 x_1|x_2 y_0|y_1] - [x_0|x_1 x_2 y_0|y_1] - [x_0|x_1|x_2 y_0 y_1]$$

$$d[x_0|x_1|x_2 y_0|y_1] = [x_0 x_1|x_2 y_0|y_1] - [x_0|x_1 x_2 y_0|y_1] + [x_0|x_1|x_2 y_0 y_1]$$

Thus doesn't work because of a sign. Actually it's even worse because I haven't even got the degrees straight. Thus

$$\mathcal{B}(R)_m \otimes_R \mathcal{B}(R)_n = R^{\otimes(m+2)} \otimes_R R^{\otimes(n+2)}$$
$$= R^{\otimes(m+n+3)}$$

e.g. (very embarrassing)

$$(R\otimes R)\otimes_R (R\otimes R) = R\otimes R\otimes R$$

On the other hand there is ~~[illegible]~~ an interesting map

$$\Sigma\left(I\overset{!}{\otimes}_R\right)^{p+1}_\sigma \longrightarrow \left(I\overset{!}{\otimes}_R\right)^{p}_\sigma$$

when I is an ideal in R. It is related to Connes B operator, and it involves replacing each of the I factors on the right with R. This suggests some sort of map from

$$\underbrace{\Sigma\, \mathcal{B}(R)\otimes_R R \otimes_R \mathcal{B}(R)} \longrightarrow \mathcal{B}(R)$$

in degree n this is

$$\bigoplus_{p+q=n-1} \mathcal{B}(R)_p \otimes_R \mathcal{B}(R)_q = \bigoplus_{p+q=n-1} R^{\otimes \overset{(n+2)}{\cancel{n+1}}(\cancel{p+q+3})}.$$

so it might work, however

January 24, 1988 (David is 24)

Let I be an ideal in R. We wish to define a map of complexes of R-bimodules

$$\circledast \qquad \mathcal{B}(R) \otimes_R \Sigma I \otimes_R \mathcal{B}(R) \longrightarrow \mathcal{B}(R)$$

NO

The left side in degree n is

$$\bigoplus_{k+1+l=n} (R \otimes R^{\otimes k} \otimes R) \otimes_R I \otimes_R (R \otimes R^{\otimes l} \otimes R)$$

$$= \bigoplus_{k+1+l=n} R \otimes R^{\otimes k} \otimes I \otimes R^{\otimes l} \otimes R$$

and the inclusion of $I \subset R$ will induce an obvious map to $\mathcal{B}(R)_n$. Let's now check compatibility with d. Let's consider a symbol

$$[x_0 | \cdots | x_k | z | y_1 | \cdots | y_{l+1}] \subset R \otimes R^{\otimes k} \otimes I \otimes R^{\otimes l} \otimes R$$

It's differential in the complex on the left side of $\circledast$ is

forgot that $d \neq b'$ in $B(R)_0$ see p. 498

$$d[x_0 | \cdots | x_k | 1] \otimes_R z \otimes_R [1 | y_1 | \cdots | y_{l+1}]$$
$$+ (-1)^k [x_0 | \cdots | x_k | 1] \otimes_R \cancel{dz}^{=0} \otimes_R [1 | \quad | y_{l+1}]$$
$$+ (-1)^{k+1} [x_0 | \cdots \quad | 1] \otimes_R z \otimes_R d[1 | \cdots | y_{l+1}]$$
$$= \sum_{i=0}^{k} (-1)^i [\cdots | x_i x_{i+1} | \cdots] \otimes_R [z | y_1 | \cdots | y_{l+1}] = [x_0 | \cdots | x_i x_{i+1} | \cdots | z | y_1 | \cdots | y_{l+1}]$$
$$+ \sum_{j=0}^{l} (-1)^{k+1+j} [x_0 | \cdots x_k | z] \otimes_R [1 | \cdots | y_j y_{j+1} | \cdots] = [x_0 | \cdots | x_k | z | \cdots | y_j y_{j+1} | \cdots]$$

provided we interpret $x_{k+1} = z$ $i = k$, $y_0 = 0$ $j = 0$

This is the same as $d[x_0 | \cdots | x_k | z | y_1 | \cdots | y_l]$ in $\mathcal{B}(R)$.

Thus ~~tensoring~~ $\otimes$ we obtain

$$\left(\sum' I \underset{R}{\otimes} \mathcal{B}(R)\right) \underset{R}{\otimes} \left(\sum' I \underset{R}{\otimes} \mathcal{B}(R)\right) \longrightarrow \sum' I \underset{R}{\otimes} \mathcal{B}(R)$$

a map of complexes, but this is not useful as the two I's on the left are treated differently.

The real problem here is to understand the maps

$$(**) \qquad \left(\sum' I \overset{!}{\otimes}_R\right)^{p+1}_\lambda \longrightarrow \left(\sum' I \overset{!}{\otimes}_R\right)^p_\lambda$$

which are given by the horizontal differentials of the double complex $CC(R \leftarrow I)$. It appears almost as if $\sum' I \otimes_R \mathcal{B}(R)$ is a DGA without unit with multiplier algebra R, and that we are taking its cyclic complex relative to R. But this doesn't seem to work.

Taking $I = R$ in $(**)$ should give something equivalent to Connes B operator.

January 26, 1988

Here's how to get one's hands on the differential in the spectral sequence which is needed for the exact sequences (R free)

$$0 \to \overline{H}C_{2n+1}(A) \to I^{n+1}/[I,I^n] \xrightarrow{d'} H_1(R,I^n) \to \overline{H}C_{2n}(A) \to 0$$

$$0 \to HC_{2n}(A) \to HC_0(R/I^{n+1}) \xrightarrow{d'} H_1(R,R/I^n)_\sigma \to HC_{2n-1}(A) \to 0$$

First note that for general R we have maps

$$\begin{tikzcd} H_0((I\overset{!}{\otimes}_R)^{p+1}_\sigma) \arrow[r, "d'"] \arrow[d, equal] \arrow[rrr, dashed, bend left] & H_1((I\overset{!}{\otimes}_R)^p_\sigma) \arrow[r, hook] & H_1((I\overset{!}{\otimes}_R)^p) \arrow[r] & H_1(R, I\otimes_R \overset{p}{\cdots} \otimes_R I) \arrow[d, phantom, "\cap"] \\ (I\otimes_R)^{p+1}_\sigma & & & I\otimes_R \overset{p}{\cdots} \otimes_R I \otimes_R \Omega^1_R \otimes \end{tikzcd}$$

which are functorial in R, I. By writing R as a quotient of a free algebra we see that the dotted arrow is determined by the case when R is free. Then one can take $I = R$ because the map

$$H_1(R, I\otimes_R \cdots \otimes_R I) \to H_1(R,R)$$

is injective. Thus we just need a formula for d' when $I = R$.

Similarly we have maps defined in general

$$\begin{tikzcd} H_0((R\overset{!}{\otimes}_R)^{p+1}_\sigma/(I\overset{!}{\otimes}_R)^{p+1}_\sigma) \arrow[r] \arrow[d, equal] & H_1((R\overset{!}{\otimes}_R)^p_\sigma/(I\overset{!}{\otimes}_R)^p_\sigma) \arrow[r, hook] & H_1((R\overset{!}{\otimes}_R)^p/(I\overset{!}{\otimes}_R)^p) \arrow[r] & H_1(R, R/I^p) \arrow[d, phantom, "\cap"] \\ HC_0(R/I^{p+1}) & & & R/I^p \otimes_R \Omega^1_R \otimes_R R \end{tikzcd}$$

and the composition is ~~[illegible]~~ determined using naturality by ~~[illegible]~~ the case when $I = 0$.

Thus we have to find

$$H_0((R\overset{!}{\otimes}_R)^{p+1}_\sigma) \xrightarrow{d'} H_1((R\overset{!}{\otimes}_R)^p_\sigma).$$

We did this on p388 even in the case of an ideal.

Remark: The double complex $CC(R \leftarrow I)/CC(R)$ has the form (up to a vertical shift by 1

$$\cdots \xrightarrow{\delta} \left(\Sigma I \overset{!}{\otimes}_R\right)^3_\lambda \xrightarrow{\delta} \left(\Sigma I \overset{!}{\otimes}_R\right)^2_\lambda \xrightarrow{\delta} \Sigma I \overset{!}{\otimes}_R$$

It would seem that one has the cyclic complex of $\Sigma I \otimes_R B(R)$ with multiplier algebra R. However I haven't found an algebra structure on $\Sigma I \otimes_R B(R)$.

I have to understand δ. Recall that it can be defined more generally for a map of R-bimodules $u: I \to R$ such that $u(x)y = xu(y)$ for this is what is needed to make $R \xleftarrow{u} I$ a DGA. We have

$$\begin{tikzcd}
\Sigma\, CC^s(R \oplus I)(p+1) \arrow[r] \arrow[d, equal] & \Sigma\, CC^s(R \oplus I)(p) \arrow[d, equal] \\
\left(\Sigma I \overset{!}{\otimes}_R\right)^{p+1}_\lambda \arrow[r, "\delta"] \arrow[d, "\text{standard lift}"'] & \left(\Sigma I \overset{!}{\otimes}_R\right)^{p}_\lambda \\
\left(\Sigma I \overset{!}{\otimes}_R\right)^{p+1} \arrow[r] & \left(\Sigma I \overset{!}{\otimes}_R\right)^{p} \arrow[u]
\end{tikzcd}$$

where the bottom arrow is induced by applying u to the last factor of I and using the map

$$B(R) \otimes_R \Sigma R \otimes_R B(R) \longrightarrow B(R)$$

I described a few days ago.

Next let's consider the problem of the S-operator. I want to relate Connes S-operator to the I-adic filtration, specifically I would like to prove commutativity of

$$\begin{tikzcd} HC_{2n+1}(A) \arrow[r] \arrow[d, "S"] & I^{n+1}/[I,I^n] \arrow[d] \\ HC_{2n-1}(A) \arrow[r] & I^n/[I,I^{n-1}] \end{tikzcd}$$

One might try to define a natural map

$$(I \overset{!}{\otimes}_R)^{p+1}_\sigma \longrightarrow (I \overset{!}{\otimes}_R)^p_\sigma$$

modelled on $I^{p+1}/[I,I^p] \longrightarrow I^p/[I,I^{p-1}]$. Note the factorization

$$\begin{tikzcd} I^{p+1}/[I,I^p] \arrow[rr] \arrow[dr] & & I^p/[I,I^{p-1}] \\ & I^p/[R,I^p] \arrow[ur] & \end{tikzcd}$$

so one might have a map

$$(I \overset{!}{\otimes}_R)^{p+1}_\sigma \longrightarrow (I \overset{!}{\otimes}_R)^p .$$

Let treat H_0 first, and define

$$(I \otimes_R)^{p+1}_\sigma \longrightarrow (I \otimes_R)^p$$

by

$$(x_0 | \cdots | x_p) \longmapsto (x_0 | \cdots | x_{p-1} u(x_p))$$

This is obviously compatible with multiplication by elements of R between consecutive elements. Moreover

$$(x_0 | \cdots | x_{p-1} u(x_p)) = (u(x_p) x_0 | x_1 | \cdots | x_{p-1})$$

$$= (x_p u(x_0) | x_1 | \cdots | x_{p-1}) = (x_p | u(x_0) x_1 | \cdots)$$

$$= (x_p | x_0 u(x_1) | x_2 | \cdots) = \cdots = (x_p | x_0 | \cdots | x_{p-2} u(x_{p-1}))$$

showing that it is compatible with the action of $\mathbb{Z}/(p+1)$.

<u>January 27, 1988</u>

I want to consider the various maps $(I \overset{!}{\otimes}_R)^{p+1} \longrightarrow (I \overset{!}{\otimes}_R)^{p}$ using the bimodule map $u : I \to R$ on one of the copies of I and the two maps

1)
$$X \overset{!}{\otimes}_R R \longrightarrow X \otimes_R R = X$$
$$R \overset{!}{\otimes}_R Y \longrightarrow R \otimes_R Y = Y.$$

This gives $2(p+1)$ possibilities initially, however it turns out that there are only $(p+1)$ of them, which can be viewed as applying the product

2)
$$I \overset{!}{\otimes}_R I \longrightarrow I \otimes_R I \longrightarrow I$$

at different points.

Recall first the product map

$$I \otimes_R I \longrightarrow I \qquad x \otimes y \longmapsto u(x)y = xu(y).$$

Then

$$\begin{tikzcd}
 & I \overset{!}{\otimes}_R I \arrow[dl, "u\otimes 1"'] \arrow[d] \arrow[dr, "1\otimes u"] & \\
R \overset{!}{\otimes}_R I \arrow[d] & I \otimes_R I \arrow[dl, "u\otimes 1"'] \arrow[dr, "1\otimes u"] & I \overset{!}{\otimes}_R R \arrow[d] \\
R \otimes_R I \arrow[dr, equal] & & I \otimes_R R \arrow[dl, equal] \\
 & I &
\end{tikzcd}$$

commutes, showing that the two maps $I \overset{!}{\otimes}_R I \to I$ that we obtain from u and 1) coincide with the product 2).

Next let's consider the two maps

3) $$I \overset{!}{\otimes}_R I \overset{!}{\otimes}_R I \rightrightarrows I \overset{!}{\otimes}_R I$$

(It might be useful to first consider the maps

$$I \otimes_R I \otimes_R I \longrightarrow I \otimes_R I$$

at first sight there are four, which turn out to be equal:

$$u(x)y \otimes z = x\, u(y) \otimes z = x \otimes u(y) z = x \otimes y\, u(z)$$

We've seen the first and last equality hold on the $\overset{!}{\otimes}$ level, so what we have to worry about is the analogue of the identity

$$xr \otimes y = x \otimes ry$$

on the $\overset{!}{\otimes}$ level.)

The two maps in 3) are

$$I \overset{!}{\otimes}_R I \overset{!}{\otimes}_R I \longrightarrow I \overset{!}{\otimes}_R R \overset{!}{\otimes}_R I \rightrightarrows I \overset{!}{\otimes}_R I$$

where the latter two result from the two maps

$$\mathcal{B}(R) \otimes_R \mathcal{B}(R) \underset{1 \otimes \varepsilon}{\overset{\varepsilon \otimes 1}{\rightrightarrows}} \mathcal{B}(R)$$

[redacted] Now we know these two maps are homotopic as maps of bimodule complexes. We can therefore conclude that the two maps 3) are homotopic as maps of $\overset{!}{\otimes}$ bimodule complexes.

Next I want to check the "cross-over term" that is I want to consider the maps

$$I \overset{!}{\otimes}_R I \overset{!}{\otimes}_R \longrightarrow I \overset{!}{\otimes}_R$$

Again there are four possibilities which coincide in pairs to yield two maps. Let's go over this

$$\begin{tikzcd}[column sep=tiny]
 & & I\otimes_R B(R)\otimes_R I\otimes_R B(R)\otimes_R \arrow[dl] \arrow[dr] & & \\
 & \cancel{R}\otimes_{\cancel{R}} B(R)\otimes_R I\otimes_R B(R)\otimes_R \arrow[dl] \arrow[d] & & I\otimes_R B(R)\otimes_{\cancel{R}} \cancel{R}\otimes_R B(R)\otimes_R \arrow[d] \arrow[dr] & \\
B(R)\otimes_R I\otimes_{\cancel{R}}\cancel{R}\otimes_R & \cancel{R}\otimes_{\cancel{R}} I\otimes_R B(R)\otimes_R \arrow[rr, phantom, "\simeq"] & & I\otimes_R \cancel{R}\otimes_R B(R)\otimes_R & I\otimes_R B(R)\otimes_{\cancel{R}}\cancel{R}\otimes_R
\end{tikzcd}$$

(commutes: product $I\overset{!}{\otimes}_R I\to I$)

The outer two arrows $\swarrow\ \searrow$ become the inner two $\swarrow\ \searrow$ if we apply the flip to $I\overset{!}{\otimes}_R I\overset{!}{\otimes}_R$.

What I learn is that the two maps $(I\overset{!}{\otimes}_R)^2\to I\overset{!}{\otimes}_R$ are related by the flip and they can be described by

$$I\overset{!}{\otimes}_R I\overset{!}{\otimes}_R \longrightarrow I\overset{!}{\otimes}_R R\overset{!}{\otimes}_R \rightrightarrows I\overset{!}{\otimes}_R$$

$$I\otimes_R B(R)\otimes_R B(R)\otimes_R \qquad\qquad I\otimes_R B(R)\otimes_R$$

where the latter two maps are again induced by $1\otimes\varepsilon,\ \varepsilon\otimes 1 : B(R)\otimes_R B(R)\longrightarrow B(R)$.

Now there ought to be a good way to keep track of which arrows coincide and which ones are homotopic. Look at the maps

$$I\otimes_R I\otimes_R \longrightarrow I\otimes_R$$

There are ~~potentially~~ four of them:

$$u(x)y \;(=)\; x\,u(y) \qquad\qquad u(y)x \;(=)\; y\,u(x)$$

On $\overset{!}{\otimes}_R$ level we have $\underbrace{\qquad\qquad\qquad\qquad}$ and we

homotopies $\quad x\,u(y)\sim u(y)x$ $\quad$??

$\qquad\qquad\quad\ \ y\,u(x)\sim u(x)y$

Better: Let's consider maps

$$\circledast \qquad (I\otimes_R)^{p+1} \longrightarrow (I\otimes_R)^p$$

There are $p+1$ face maps

$$(x_0, \dots, x_p) \longmapsto (x_0, \dots, x_{i-1}, x_i x_{i+1}, \dots, x_p) \qquad 0 \le i < p$$

$$(x_p x_0, x_1, \dots, x_{p-1})$$

The same maps are present with the derived $\otimes$ product. Now we saw that all the face maps on the tensor level $\otimes_R$ coincide and give the same map

$$(I\otimes_R)^{p+1}_\sigma \longrightarrow (I\otimes_R)^p$$

Let's review the problem. I have certain face maps

$$(I\overset{!}{\otimes}_R)^{p+1} \longrightarrow (I\overset{!}{\otimes}_R)^p$$

defined by using the "product"

$$I\overset{!}{\otimes}_R I \longrightarrow I\otimes_R I \longrightarrow I$$

to delete a $\overset{!}{\otimes}_R$ symbol. I would like to show these face maps are all homotopic, and at the same time I would like to understand the different homotopies between them. In particular I suspect that if I compose the homotopies between successive faces, then I get a self-homotopy of the first face, which should (or might) be the map

$$\Sigma(I\overset{!}{\otimes}_R)^{p+1} \longrightarrow (I\overset{!}{\otimes}_R)^p$$

occurring in the spectral sequence. (σ missing?)

Let's consider the two face maps

$$
\begin{array}{ccc}
(I \overset{!}{\otimes}_R)^2 & \longrightarrow & I \overset{!}{\otimes}_R \\
\| & & \\
I\otimes_R \mathcal{B}(R)\otimes_R I\otimes_R \mathcal{B}(R)\otimes_R & \longrightarrow & I\otimes_R \mathcal{B}(R)\otimes_R \\
x\otimes \xi\otimes y\otimes \eta\otimes & \longmapsto & x\varepsilon(\xi)y\otimes\eta\otimes \\
 & \longmapsto & y\varepsilon(\eta)x\otimes\xi\otimes
\end{array}
$$

These are certainly the effect of replacing the two $\overset{!}{\otimes}_R$ by $\otimes_R$. Notice that we get the same faces if we first replace one of the I factors by R and then apply ε to delete "!". For example

$$
\begin{tikzcd}
I\otimes_R \mathcal{B}(R)\otimes_R I\otimes_R \mathcal{B}(R)\otimes_R \\
R\otimes_R \mathcal{B}(R)\otimes_R I\otimes_R \mathcal{B}(R)\otimes_R \arrow[u, phantom, "\cap"] \arrow[d, shift right] \arrow[d, shift left] \\
I\otimes_R \mathcal{B}(R)\otimes_R
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & x\otimes\xi\otimes y\otimes\eta\otimes \arrow[d, mapsto] & \\
 & x\otimes\xi\otimes y\otimes\eta\otimes \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\varepsilon(x\xi)y\otimes\eta\otimes \arrow[d, equal] & & y\varepsilon(\eta)\otimes x\xi\otimes \arrow[d, equal] \\
x\varepsilon(\xi)y\otimes\eta\otimes & & y\varepsilon(\eta)x\otimes\xi\otimes
\end{tikzcd}
$$

or simpler

$$
\begin{tikzcd}
I\otimes_R \mathcal{B}(R)\otimes_R I\otimes_R \mathcal{B}(R)\otimes_R \\
I\otimes_R \mathcal{B}(R)\otimes_R R\otimes_R \mathcal{B}(R)\otimes_R \arrow[u, phantom, "\cap"] \arrow[d, shift right] \arrow[d, shift left] \\
I\otimes_R \mathcal{B}(R)\otimes_R
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=tiny]
 & x\otimes\xi\otimes y\otimes\eta\otimes \arrow[d, mapsto] & \\
 & x\otimes\xi y\otimes\eta \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
x\varepsilon(\xi y)\otimes\eta\otimes \arrow[d, equal] & & x\otimes\xi y\varepsilon(\eta)\otimes \arrow[d, equal] \\
x\varepsilon(\xi)y\otimes\eta\otimes & & y\varepsilon(\eta)x\otimes\xi\otimes
\end{tikzcd}
$$

Recall that I defined a map of complexes (p. 487)

$$\circledast \quad B(R) \otimes_R \Sigma R \otimes_R B(R) \longrightarrow B(R)$$

NOT compatible with d

which seems to be a basic building block in the horizontal boundary maps

$$\Sigma \left(I \overset{!}{\otimes}_R \right)^{p+1}_\sigma \xrightarrow{\ \delta\ } \left(I \overset{!}{\otimes}_R \right)^p_\sigma$$

However we note that $\circledast$ as a map of complexes of free $R \otimes R^0$-modules must be homotopic to zero, since $B(R) \to R$ is a quis and the left side doesn't begin until degree one.

There seems to be a paradox.

January 28, 1988

Resolution of the paradox: I made a mistake on p487 by not checking the cases $k=0$, $\ell=0$. The point is that $d=0$ on $\mathcal{B}(R)_0 = R\otimes R$, not $b' : R\otimes R \to R$. Instead of a map of complexes from $\Sigma(\mathcal{B}(R)\otimes_R \mathcal{B}(R)) \to \mathcal{B}(R)$ we seem to get a map

1) $$\mathcal{B}(R)\otimes_R R[\Delta(1)] \otimes_R \mathcal{B}(R) \longrightarrow \mathcal{B}(R)$$

which is a homotopy between $\varepsilon\otimes 1$ and $1\otimes\varepsilon$.

Let $\mathcal{A}(R)$ denote the acyclic Hochschild complex of R, so that $\mathcal{A}(R)_n = R^{\otimes(n+1)}$ and the differential is b'. We have an exact sequence

2) $$0 \longrightarrow R \longrightarrow \mathcal{A}(R) \longrightarrow \Sigma\mathcal{B}(R) \longrightarrow 0$$

$\mathcal{A}(R)$ is the cone on the map $\mathcal{B}(R) \xrightarrow{\varepsilon} R$.

The calculation on p487 shows that we have a map of complexes of R-bimodules

3) $$\mathcal{A}(R)\otimes_R \mathcal{A}(R) \longrightarrow \mathcal{A}(R)$$

given by
$$(x_0|\cdots|x_k)\otimes(y_0|\cdots|y_\ell) \longmapsto (x_0|\cdots|x_{k-1}|x_k y_0|y_1|\cdots|y_\ell)$$

This product is obviously associative and makes $\mathcal{A}(R)$ into a DGA containing R.

We have subcomplexes

$$\begin{tikzcd} & \mathcal{A}(R)\otimes_R\mathcal{A}(R) & \\ R\otimes_R\mathcal{A}(R) \arrow[ur, phantom, "\subset"] & \text{cart} & \mathcal{A}(R)\otimes_R R \arrow[ul, phantom, "\supset"] \\ & R\otimes_R R \arrow[ul, phantom, "\subset"] \arrow[ur, phantom, "\supset"] & \end{tikzcd}$$

which lead to an exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \Sigma B(R)^{\oplus 2} \arrow[r] & \mathcal{Q}(R)\otimes_R \mathcal{Q}(R)/R\otimes_R R \arrow[r] \arrow[d] & \Sigma B(R)\otimes_R \Sigma B(R) \arrow[r] & 0 \\
& & \mathcal{Q}(R)/R = \Sigma B(R) & &
\end{tikzcd}
$$

The vertical map is a homotopy equivalence, so it appears that $\Sigma^{-1}\{\mathcal{Q}(R)\otimes_R \mathcal{Q}(R)/R\otimes_R R\}$ is some sort of model for <u>the cylinder on $B(R)$.</u>

Now that we have straightened out the paradox, we want to see if we can ~~explain~~ understand the horizontal boundary operator δ. Consider the dotted arrow in

$$
\begin{tikzcd}
(\Sigma I \overset{!}{\otimes}_R)^{\otimes 2} \arrow[r] \arrow[drr, dashed, "\delta"'] & (\Sigma I \overset{!}{\otimes}_R)^{\otimes 2}_{\cancel{\lambda}} & = CC^s(R\oplus I)(2) \arrow[d, "\delta"] \\
& \Sigma I \overset{!}{\otimes}_R & = CC^s(R\oplus I)(1)
\end{tikzcd}
$$

This takes $(m_1|x_1|\cdots|x_k|m_2|y_1|\cdots|y_\ell)$ into

$$(u(m_1)|x_1|\cdots|m_2|y_1|\cdots) - (m_1|x_1|\cdots|u(m_2)|y_1|\cdots)$$

in $CC^s(R\oplus M)(1)$ which is equivalent to

$$(-1)^{(k+1)(\ell+1)}(m_2|y_1|\cdots|y_k|u(m_1)|x_1|\cdots|x_k)$$

$$-\ (m_1|x_1|\cdots|x_k|u(m_2)|y_1|\cdots|y_\ell)$$

(The minus sign is due to the fact that δ is obtained from the diffential in $(R\leftarrow I)^{\otimes n}$, where R is even and I is odd).

The natural question is whether the map

$$(m_1|\cdots|m_2|\cdots) \longmapsto -(m_1|\cdots|u(m_2)|\cdots)$$

$$\left(\sum I \overset{!}{\otimes}_R\right)^2 \longrightarrow \sum I \overset{!}{\otimes}_R$$

is a map of complexes. There's probably no trouble if $k, \ell > 0$. Suppose $k=0$.

$$\begin{aligned} d(m_1|m_2|y_1|\cdots|y_\ell) = {} & (m_1|m_2y_1|y_2|\cdots) \\ & + \sum_{i=1}^{\ell-1}(-1)^i(m_1|m_2|\cdots|y_iy_{i+1}|\cdots) \\ & + (-1)^\ell(y_\ell m_1|m_2|\cdots|y_{\ell-1}) \end{aligned}$$

$$\begin{aligned} d(m_1|u(m_2)|y_1|\cdots|y_\ell) = {} & -(m_1u(m_2)|y_1|\cdots|y_\ell) \\ & + (m_1|m_2y_1|y_2|\cdots) \\ & + \sum_{i=1}^{\ell-1}(-1)^i(m_1|m_2|\cdots|y_iy_{i+1}|\cdots) \\ & + (-1)^\ell(y_\ell m_1|m_2|\cdots) \end{aligned}$$

Thus we see that we don't have a map of complexes. On the other hand consider the other term in the formula for δ:

$$\begin{aligned} \delta(m_1|m_2|y_1|\cdots|y_\ell) = {} & (u(m_1)|m_2|y_1|\cdots|y_\ell) \\ & -(m_1|u(m_2)|y_1|\cdots|y_\ell) \end{aligned}$$

$$\equiv \underbrace{(-1)^{\ell+1}(m_2|y_1|\cdots|y_\ell|u(m_1))}_{} - (m_1|u(m_2)|y_1|\cdots|y_\ell)$$

which is λ. This comes from

$$\lambda(m_1|m_2|y_1|\cdots|y_\ell) = (-1)(-1)^{\ell+1}(m_2|y_1|\cdots|y_\ell|m_1)$$

by applying $-u$ to the right element in I. We have

$$d(-1)^{\ell}(m_2|y_1|\cdots|y_\ell|m_1) = (-1)^{\ell+1}\Big\{(m_2 y_1|y_2|\cdots|y_\ell|m_1) + \sum_{i=1}^{\ell-1}(-1)^i(m_2|\cdots|y_i y_{i+1}|\cdots|y_\ell|m_1) + (-1)^{\ell}(m_2|y_1|\cdots|y_\ell m_1)\Big\}$$

$$d(-1)^{\ell}(m_2|y_1|\cdots|y_\ell|u(m_1)) =$$
$$(-1)^{\ell+1}\Big\{(m_2 y_1|\cdots|y_\ell|u(m_1)) + \sum_{i=1}^{\ell-1}(-1)^i(m_2|\cdots|y_i y_{i+1}|\cdots|y_\ell|u(m_1)) + (-1)^{\ell}(m_2|\cdots|y_\ell\, u(m_1)) + (-1)^{\ell+1}(u(m_1)m_2|y_1|\cdots|y_\ell)\Big\}$$

Thus if we add the discrepancy for the two terms, we get

$$+(m_1 u(m_2)|y_1|\cdots|y_\ell) - (u(m_1)m_2|y_1|\cdots|y_\ell) = 0,$$

which checks the calculation.

The problem we have is to get control of the horizontal differentials in our double complex:

$(*)$

$$\to \left(\Sigma I \overset{!}{\otimes}_R\right)^3_\lambda \longrightarrow \left(\Sigma I \overset{!}{\otimes}_R\right)^2_\lambda \longrightarrow \Sigma I \overset{!}{\otimes}_R$$

By naturality the formulas will be determined by the case $I = R$.

One possible approach is as follows. Consider the DGA $a(R)$ and form its relative cyclic complex relative to R:

$$\cdots \xrightarrow{b} \left(a(R)\otimes_R\right)^2_\lambda \xrightarrow{b} a(R)\otimes_R .$$

Recall that $a(R)/R = \Sigma\mathcal{B}(R)$. It might be possible to show in ~~this~~ relative cyclic complex one can form a sort of reduced cyclic complex

$$\left(\overline{\mathcal{a}(R)}\otimes_R\right)^n_\lambda \qquad \text{where} \quad \overline{\mathcal{a}(R)} = \mathcal{a}(R)/R$$

If so, then this reduced cyclic complex ought to be the double complex ⊛.

Let's check ~~that~~ it works. We have to see that b preserves the "degenerate" subcomplex, so we take a symbol with a 1 in it. Suppose the 1 occurs strictly inside:

$$(x_0|\cdots|x_{i-1}|1|x_{i+1}|\cdots) \qquad 0<i<n$$

Then there are two faces outside the degenerate complex and these have opposite signs. Next suppose the 1 occurs at the left end:

$$b(1|x_1|\cdots|x_n) = (x_1|\cdots|x_n) + \text{deg stuff}$$
$$+(-1)^n \underbrace{(x_n|x_1|\cdots|x_{n-1})(-1)^{(\deg x_n)(\deg x_1+\cdots+\deg x_{n-1})}}_{=(-1)^{n-1}(x_1|\cdots|x_n)}$$

But the cyclic relations give $=(-1)^{n-1}(x_1|\cdots|x_n)$ and so it works. The case where $x_n=1$ is the same as the case $x_i=1$ for $0<i<n$.

So if everything above is correct we find a somewhat amazing isomorphism between the double complex $CC(R\xleftarrow{1}R)/CC(R)$

and the reduced relative cyclic complex of $\mathcal{a}(R)$ rel R. There's a shift by one both horizontally and vertically, which probably could be avoided if one reindexed the cyclic complex so that CC_n is the quotient of $(A\otimes)^n$.

At this point I probably have some understanding of the horizontal defferential in $CC(R \xleftarrow{1} R)$, however I don't see how to put the ideal I into the picture. There doesn't seem to be anyway to make an ideal in $\mathcal{Q}(R)$ which would be simple enough to handle.

So we could try returning to the original idea of a homotopy between $\varepsilon \otimes 1$ and $1 \otimes \varepsilon$.

<u>January 29, 1988</u>

The problem is to understand the horizontal differential

$$\delta : \Sigma \left(I \overset{!}{\otimes}_R\right)^{p+1}_\sigma \longrightarrow \left(I \overset{!}{\otimes}_R\right)^p_\sigma$$

in the spectral sequences. We can lift back to a map

$$\Sigma' \left(I \overset{!}{\otimes}_R\right)^{p+1} \longrightarrow \left(I \overset{!}{\otimes}_R\right)^p_\sigma .$$

How can we produce such a map? The idea is we have face maps

$$\delta_i : \left(I \overset{!}{\otimes}_R\right)^{p+1} \longrightarrow \left(I \overset{!}{\otimes}_R\right)^p \qquad i = 0, \ldots, p$$

defined using the product $I \overset{!}{\otimes}_R I \xrightarrow{\mu} I$ (and in the case of δ_p also the cyclic permutation σ_{p+1}). Precisely we have

$$\delta_i = \sigma_p^i \delta_0 \sigma_{p+1}^{-i} \qquad \text{for} \qquad i = 0, \ldots, p$$

e.g.

$$\begin{aligned} \delta_i(x_0 | \cdots | x_p) &= \sigma_p^i \delta_0 (x_i | \cdots | x_{i-1}) \\ &= \sigma_p^i (x_i x_{i+1} | \cdots | x_{i-1}) \\ &= (x_0 | \cdots | x_i x_{i+1} | \cdots | x_p) \qquad 0 \le i < p \end{aligned}$$

and

$$\begin{aligned} \delta_p(x_0 | \cdots | x_p) &= \delta_0 \sigma_{p+1}^{-p} (x_0 | \cdots | x_p) \\ &= \delta_0 (x_p | x_0 | \cdots | x_{p-1}) = (x_p x_0 | \cdots | x_p) \end{aligned}$$

In addition we have a homotopy between consecutive face maps, which comes from the canonical homotopy between $\varepsilon \otimes 1, 1 \otimes \varepsilon : B(R) \otimes_R B(R) \longrightarrow B(R)$.
More precisely we have a diagram

$$k' = h + \sigma_p h \sigma_{p+1}^{-1} + \cdots + \sigma_p^{p-1} h \sigma_{p+1}^{-p+1}$$

then

$$[d, k'] = \delta_p - \delta_0 = \delta_0 \sigma_{p+1} - \delta_0$$

and hence we also have a map of complexes

$$\boxed{\Sigma\left\{\left(I \overset{!}{\otimes}_R\right)^{p+1}\right\}^{\sigma} \xrightarrow{\tilde{k}'} \left(I \overset{!}{\otimes}_R\right)^{p}}$$

which might be useful.

Next if $\pi_p : \left(I \overset{!}{\otimes}_R\right)^p \twoheadrightarrow \left(I \overset{!}{\otimes}_R\right)^p_\sigma$ is the projection onto the covariants, then

$$\pi_p k = \pi_p\left(\sum_{i=0}^{p} \sigma_p^i h \sigma_{p+1}^{-i}\right) = \pi_p h \left(\sum_{i=0}^{p} \sigma_{p+1}^i\right)$$

so $\pi_p k = \bar{k}\, \pi_{p+1}$ for a map $\bar{k}$ on coinvariants

Similarly if $\iota_{p+1} : \left(I \overset{!}{\otimes}_R\right)^{p+1,\sigma} \hookrightarrow \left(I \overset{!}{\otimes}_R\right)^{p+1}$ is the inclusion of invariants, we have

$$k' \iota_{p+1} = \sum_{i=0}^{p-1} \sigma_p^i h \sigma_{p+1}^{-i} \iota_{p+1} = \left(\sum_{i=0}^{p-1} \sigma_p^i\right) h \,\iota_{p+1}$$

so that $k' \iota_{p+1} = \iota_p \overline{k'}$ for a map $\overline{k'}$ on invariants. Thus we have a diagram

$$\begin{tikzcd}
\Sigma\left(I \overset{!}{\otimes}_R\right)^{p+1,\sigma} \arrow[r, "\overline{k'}"] \arrow[d, hook, "\iota_{p+1}"'] \arrow[dr, "\tilde{k}'"] & \left(I \overset{!}{\otimes}_R\right)^{p,\sigma} \arrow[d, hook, "\iota_p"] \\
\Sigma\left(I \overset{!}{\otimes}_R\right)^{p+1} \arrow[r] \arrow[d, two heads, "\pi_{p+1}"'] \arrow[dr, "\tilde{k}"] & \left(I \overset{!}{\otimes}_R\right)^{p} \arrow[d, two heads, "\pi_p"] \\
\Sigma\left(I \overset{!}{\otimes}_R\right)^{p+1}_\sigma \arrow[r, "\bar{k}"] & \left(I \overset{!}{\otimes}_R\right)^{p}_\sigma
\end{tikzcd}$$

$$\begin{tikzcd}
I \overset{!}{\otimes}_R I \overset{!}{\otimes}_R I \arrow[dd, shift right, "\mu\otimes 1"'] \arrow[dd, shift left, "1\otimes\mu"] \arrow[dr] \arrow[rr, phantom, "="] & & I\otimes_R \mathcal{B}(R)\otimes_R I\otimes_R \mathcal{B}(R)\otimes_R I \arrow[d] \\
& I \overset{!}{\otimes}_R R \overset{!}{\otimes}_R I \arrow[dl, shift left] \arrow[dl, shift right] \arrow[r, phantom, "="] & I\otimes_R \mathcal{B}(R)\otimes_R \mathcal{B}(R)\otimes_R I \arrow[d, shift left] \arrow[d, shift right] \\
I \overset{!}{\otimes}_R I \arrow[rr, phantom, "="] & & I\otimes_R \mathcal{B}(R)\otimes_R I
\end{tikzcd}$$

Let h denote the resulting homotopy $\delta_0 \Rightarrow \delta_1$ so that

$$[d, h] = \delta_1 - \delta_0 = \sigma_p \delta_0 \sigma_{p+1}^{-1} - \delta_0$$

Thus

$$\delta_0 \xRightarrow{h} \sigma_p \delta_0 \sigma_{p+1}^{-1} \xRightarrow{\sigma_p h \sigma_{p+1}^{-1}} \sigma_p^2 \delta_0 \sigma_{p+1}^{-2} \Rightarrow \cdots \xRightarrow{\sigma_p^{p-1} h \sigma_{p+1}^{-p+1}} \underset{\substack{\| \\ \delta_p}}{\delta_0 \sigma_{p+1}} \xRightarrow{\cancel{\sigma_p^p} h \sigma_{p+1}} \sigma_p \delta_0$$

so

$$(*) \quad \boxed{[d, \underbrace{h + \sigma_p h \sigma_{p+1}^{-1} + \cdots + \sigma_p^p h \sigma_{p+1}^{-p}}] = \sigma_p \delta_0 - \delta_0}$$

(This has $p+1$ terms, one for each factor I in $(I \overset{!}{\otimes}_R)^{p+1}$ and there are also $p+1$ faces one for each $\overset{!}{\otimes}_R$ sign).

From $(*)$ we see that composing the degree one map

$$k = h + \sigma_p h \sigma_{p+1}^{-1} + \cdots + \sigma_p^p h \sigma_{p+1}^{-1}$$

from $(I \overset{!}{\otimes}_R)^{p+1}$ to $(I \overset{!}{\otimes}_R)^p$ with the projection onto $(I \overset{!}{\otimes}_R)^p_\sigma$ gives a map of complexes

$$\boxed{\Sigma (I \overset{!}{\otimes}_R)^{p+1} \xrightarrow{\tilde{k}} (I \overset{!}{\otimes}_R)^p_\sigma}$$

Notice also that if instead of k we omit the last term and put which commutes up to a scalar factor, i.e. the map

$$\pi_p k' \iota_{p+1} = p(\pi_p h \iota_{p+1})$$

$$\pi_p k \iota_{p+1} = (p+1)(\pi_p h \iota_{p+1})$$

Actually it appears we ought to use the map

$$\frac{1}{p(p+1)} N_p h N_{p+1} : (R \overset{!}{\otimes}_R)^{p+1} \longrightarrow (R \overset{!}{\otimes}_R)^p \quad (\text{degree } 1)$$

which is a map of complexes because

$$[d, N_p h N_{p+1}] = N_p(\sigma_p \partial_0 \sigma_{p+1}^{-1} - \partial_0) N_{p+1} = 0$$

and which agrees upon restriction to $(R \overset{!}{\otimes}_R)^{p+1, \sigma}$ with

$$\frac{1}{p} N_p h = \frac{1}{p} k'$$

and which agrees when projected to $(R \overset{!}{\otimes}_R)^p_\sigma$ with $\frac{1}{p+1} h N_{p+1} = \frac{1}{p+1} k$

Consider the case $p = 2$. Then we have

$$\begin{array}{c}
\Sigma\{\underbrace{R \overset{!}{\otimes}_R R \overset{!}{\otimes}_R}\} \xrightarrow{h + h\sigma} R \overset{!}{\otimes}_R \\
\mathcal{B}(R) \otimes_R \mathcal{B}(R) \otimes_R \\
\big\uparrow \, \Delta \otimes_R \\
\mathcal{B}(R) \otimes_R
\end{array}$$

where $\Delta : \mathcal{B}(R) \longrightarrow \mathcal{B}(R) \otimes_R \mathcal{B}(R)$ is the cup-product (which is a heq of bimodule complexes).

Recall that

$$\Delta(x_0|\cdots|x_{n+1}) = \sum_{\substack{[redacted] \\ 0\le k\le n}} (x_0|\cdots|x_k|1) \otimes_R (1|x_{k+1}|\cdots|x_{n+1})$$

and I think also that $h : B(R)\otimes_R B(R) \xrightarrow{+1} B(R)$ is given by

$$h(x_0|\cdots|x_{k+1})\otimes_R(y_0|\cdots|y_{\ell+1}) = (-1)^k (x_0|\cdots|x_{k+1}y_0|y_1|\cdots|y_{\ell+1})$$

~~Thus $\Delta\otimes_R$ on $B(R)\otimes_R$ is $(x_0|\cdots|x_k|1)\otimes_R(1|x_{k+1}|\cdots|x_n|1)\otimes_R$~~

$$\cancel{\Delta(x_0|\cdots|x_n) = \sum_{k=0}^{n} (x_0|\cdots|x_k|1|x_{k+1}|\cdots|x_n)}$$

$$\cancel{\sigma\Delta(x_0|\cdots|x_n) = \sum_{k=0}^{n} (1|x_{k+1}|\cdots|x_n|x_0|\cdots|x_k)}$$

Thus $\Delta\otimes_R$ on $B(R)\otimes_R$ is

$$(\Delta\otimes_R)(x_0|\cdots|x_n) = \sum_{k=0}^{n} (x_0|\cdots|x_k|1)\otimes_R(1|x_{k+1}|\cdots|x_n|1)\otimes_R$$

so

$$\sigma(\Delta\otimes_R)(x_0|\cdots|x_n) = \sum_{k=0}^{n} (-1)^{k(n-k)} (1|x_{k+1}|\cdots|x_n|1)\otimes_R(x_0|\cdots|x_k|1)\otimes_R$$

So $(h+h\sigma)(\Delta\otimes_R)(x_0|\cdots|x_n) =$

$$\sum_{k=0}^{n} (-1)^k (x_0|\cdots|x_k|1|x_{k+1}|\cdots|x_n)$$

$$+\sum_{k=0}^{n} (-1)^{k(n-k)+n-k} (1|x_{k+1}|\cdots|x_n|x_0|\cdots|x_k)$$

$$\equiv \sum_{k=0}^{n} (-1)^{(k+1)(n-k)} (1|x_{k+1}|\cdots|x_n|x_0|\cdots|x_k)$$

modulo degenerate elements.

January 30, 1988

The program: To find the S operator on the double complex with columns $(\Sigma I \overset{!}{\otimes}_R)^{p+1}_\lambda$

I think what I only have to do is to produce the operator S and show that it is compatible with the horizontal boundary maps, that is,

$$\begin{tikzcd}
\Sigma'(I\overset{!}{\otimes}_R)^{p+1}_\sigma \arrow[r, "\delta"] \arrow[d, "\Sigma(S)"'] & (I\overset{!}{\otimes}_R)^{p}_\sigma \arrow[d, "S"] \\
\Sigma(I\overset{!}{\otimes}_R)^{p}_\sigma \arrow[r, "\delta"] & (I\overset{!}{\otimes}_R)^{p-1}_\sigma
\end{tikzcd}$$

commutes. Once this is done the rest should follow by naturality. Why? We have maps of double complexes

$$CC(R\leftarrow I)/CC(R) \longrightarrow CC(R\leftarrow R)/CC(R) \longrightarrow CC(A\leftarrow A)/CC(A)$$

such that the left complex is quis to the fibre of the right map. By naturality S will act compatibly on all these complexes. Thus formula for S on $CC(R\leftarrow I)/CC(R)$ will be consist with S acting on $HC(A)$, provided I show the augmentation

$$CC(A\to A)/CC(A) \longrightarrow \Sigma' CC(A)$$

is compatible with S.

So what I really would like is a quasi-isomorphism of the bicomplex $CC(R\overset{!}{\leftarrow}R)/CC(R)$ with the Connes double complex (possibly normalized). The idea is to use the iterated S-maps

$$(R\overset{!}{\otimes}_R)^{p+1}_\sigma \xrightarrow{S^p} R\overset{!}{\otimes}_R$$

This raises the question of whether

$$\begin{tikzcd}
\left(R \overset{!}{\otimes}_R\right)^2_\sigma \arrow[r, "S"] \arrow[d, "\delta"'] & R \overset{!}{\otimes}_R \arrow[dl, "B"] \\
\Omega\left(R \overset{!}{\otimes}_R\right) &
\end{tikzcd}$$

commutes, but this is unlikely if S is to result from using $\varepsilon : B(R) \longrightarrow R$.

In any case the real issue to decide first is whether there is a nice S operator compatible with δ. S is to be a map of complexes

$$S : \left(I \overset{!}{\otimes}_R\right)^{p+1}_\sigma \longrightarrow \left(I \overset{!}{\otimes}_R\right)^p_\sigma$$

and in degree zero must be a map

$$I^{\otimes(p+1)}_\sigma \longrightarrow I^{\otimes p}_\sigma .$$

The only possibility it seems is

$$\frac{1}{p(p+1)} N_p \partial_0 N_{p+1}$$

where $\partial_0(x_0 \otimes \cdots \otimes x_p) = x_0 x_1 \otimes \cdots \otimes x_p$. The question is now whether this works - is it compatible with $\delta = \frac{1}{p(p+1)} N_p h N_{p+1}$?

There seem to be problems with this. We want to show that

$$N_{p-1} h N_p \partial_0 N_{p+1} = N_{p-1} \partial_0 N_p h N_{p+1}$$

$$\|$$

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p} N_{p-1} \underbrace{\sigma^i_{p-1} h \sigma^{-i}_p}_{h_i} \underbrace{\sigma^j_p \partial_0 \sigma^{-j}_{p+1}}_{\partial_j}$$

Roughly speaking ∂_j operating on $(I\overset{!}{\otimes}_R)^{p+1}$ deletes the j-th $\overset{!}{\otimes}_R$ sign whereas h_j deletes the j-th I. The way this identity could hold would be to identify in some order the terms

$$N_{p-1}\, h_i\, \partial_j \qquad \text{and} \qquad N_{p-1}\, \partial_i\, h_j$$

It's reasonable to expect that h_i, ∂_j commute (up to some shifts) provided i and j are not too close. So there might be a problem when the two indices are nearly equal. Let's take $I = R$ and look at the possibilities.

$$\begin{tikzcd}
\mathcal{B}(R)\otimes_R \mathcal{B}(R)\otimes_R \mathcal{B}(R) \arrow[r, "\substack{\varepsilon\otimes1\otimes1 \\ 1\otimes\varepsilon\otimes1}", "1\otimes1\otimes\varepsilon"'] \arrow[d, "h\otimes1,\ 1\otimes h"] & \mathcal{B}(R)\otimes_R \mathcal{B}(R) \arrow[d, "h"] \\
\mathcal{B}(R)\otimes_R \mathcal{B}(R) \arrow[r, "\substack{\varepsilon\otimes1 \\ 1\otimes\varepsilon}"'] & \mathcal{B}(R)
\end{tikzcd}$$

Note that $\varepsilon h = 0$, so that there are three maps $\hookrightarrow$ namely $h\otimes\varepsilon$, 0, $\varepsilon\otimes h$. We have

$$\varepsilon\otimes h = h(\varepsilon\otimes1\otimes1)$$
$$h\otimes\varepsilon = h(1\otimes1\otimes\varepsilon)$$

however $h(1\otimes\varepsilon\otimes1)$ is definitely <u>not</u> zero. In particular when I consider the maps involved for $p=1$:

$$\begin{tikzcd}
\overset{1}{\mathcal{B}(R)}\otimes_R \overset{2}{\mathcal{B}(R)}\otimes_R \overset{3}{\mathcal{B}(R)}\otimes_R \arrow[r, "\varepsilon\otimes 1\otimes 1", "{\substack{1\otimes\varepsilon\otimes 1\\ 1\otimes 1\otimes\varepsilon}}"'] \arrow[d, "{\substack{h\otimes 1,\ 1\otimes h,\\ (h\otimes 1)\sigma}}"'] & \mathcal{B}(R)\otimes_R \mathcal{B}(R)\otimes_R \arrow[d, "{h,\ h\sigma}"] \\
\mathcal{B}(R)\otimes_R \mathcal{B}(R)\otimes_R \arrow[r, "\varepsilon\otimes 1", "1\otimes\varepsilon"'] & \mathcal{B}(R)\otimes_R
\end{tikzcd}$$

Thus $\rightarrow\!\!\downarrow$ we have $\varepsilon\otimes h$, $(h\otimes\varepsilon)$, $(h\otimes\varepsilon)\sigma$, 0, 0, 0

and $\downarrow\!\!\rightarrow$ we have

$$h(\varepsilon\otimes 1\otimes 1) = \varepsilon\otimes h$$

$$h(1\otimes 1\otimes\varepsilon) = h\otimes\varepsilon \qquad h(1\otimes\varepsilon\otimes 1)$$

$$h\sigma(\varepsilon\otimes 1\otimes 1) = h(1\otimes\varepsilon\otimes 1)\sigma$$

$$h\sigma(1\otimes\varepsilon\otimes 1) = h(1\otimes 1\otimes\varepsilon)\sigma = (h\otimes\varepsilon)\sigma$$

$$h\sigma(1\otimes 1\otimes\varepsilon) = h(1\otimes\varepsilon\otimes 1)\sigma^2$$

So we seem to have 3 maps $h(1\otimes\varepsilon\otimes 1)$, $h(1\otimes\varepsilon\otimes 1)\sigma$, $h(1\otimes\varepsilon\otimes 1)\sigma^2$ left over.

Therefore it appears as if we can't find an S-operator along these lines.

February 2, 1988

New proof that the reduced cyclic complex $\overline{CC}(R)$ is quis $CC(A)/CC(\mathbb{C})$.

First we have a quis

$$\begin{array}{ccccc} \overline{CC}_{q-1}(R \leftarrow R) & = & (\overline{R} \leftarrow R)^{\otimes q}_{\lambda} & & \\ \uparrow & & \uparrow & & \\ \overline{C}C_{q-1}(\mathbb{C} \leftarrow \mathbb{C}) & = & (0 \leftarrow \mathbb{C})^{\otimes q}_{\lambda} & = & \mathbb{C}[q] \end{array}$$

for each q and so

$$\overline{CC}(R \leftarrow R) \quad \text{quis to} \quad \overline{CC}(\mathbb{C} \leftarrow \mathbb{C}) = \Sigma\, CC(\mathbb{C})$$

Now the key point is that for $p \geq 1$ the pth columns of the double complex $\overline{CC}(R \leftarrow I)$ ~~are~~ is isomorphic to $\Sigma^{-1}\left(\Sigma I \otimes_R \overline{B}(R) \otimes_R\right)^p_\lambda$, where $\overline{B}(R)$ is the normalized bar resolution. Thus one sees that there is a quis in

$$\begin{array}{ccccc} CC(R) & \longrightarrow & CC(R \leftarrow R) & \longrightarrow & CC(R \leftarrow R)/CC(R) \\ \alpha \downarrow & & \beta \downarrow & & \downarrow \text{quis} \\ \overline{CC}(R) & \longrightarrow & \overline{CC}(\overline{R} \leftarrow R) & \longrightarrow & \overline{CC}(R \leftarrow R)/\overline{CC}(R) \end{array}$$

Thus the fibre of α is quis the fibre of β, which by the calculation above is the same for $R = \mathbb{C}$, hence is $\Omega \Sigma\, CC(\mathbb{C}) = CC(\mathbb{C})$.

February 16, 1988

Goodwillie. Waldhausen's big theorem

$$\begin{cases} A(X) = Q(X_+) \times Wh(X) \\ \Omega^2 Wh(X) = \quad \mathcal{P}(X) \end{cases}$$

where $\mathcal{P}(X) = \varinjlim \underbrace{\mathrm{Diff}(M \times I \bmod M \times 0)}_{\text{pseudo-isotopies}}$ and M runs over thickenings of X.

$Wh(X)$ has $\pi_0 = \quad K_0(\mathbb{Z}[\pi_1 X])$, $\pi_1(Wh(X)) = Wh(\pi_1 X)$.

Alternative constructions of $A(X)$ as K-theory

1) Finite simplicial G-sets, G = simplicial group
2) finite CW extensions of X retracting onto X.
3) homotopy invertible matrices over $Q[(\Omega X)_+]$

where $Q = \varinjlim \Omega^n S^n$. These matrices are heq's of $X \cup \bigvee S^n$ over X.

Problem: There is a "trace" map analogous to Dennis trace,

$$A(X) \longrightarrow Q(\Lambda X_+)$$

ΛX = free loop space

The idea is to construct this and show it factors

$$\begin{tikzcd} A(X) \arrow[rr] \arrow[dr, dashed] & & Q(\Lambda X_+) \\ & Q(\Lambda X_+)^{hS^1} \arrow[ur] & \end{tikzcd}$$

homotopy fixpts.

Analogy with

$$\begin{tikzcd} K(R) \arrow[rr, "\text{Dennis}"] \arrow[dr, dashed] & & HH(R) \\ & HC^{-}(R) \arrow[ur] & \end{tikzcd}$$

Definition of Dennis map. $G = GL_n(R)$

$$\begin{tikzcd} BG \arrow[r] & (BG)^{\text{cyclic}} \arrow[r] & (M_n R)^{\text{cyc}} \arrow[r] & H(M_n R) \arrow[d, "\text{trace}"] \\ & & & H(R) \end{tikzcd}$$

<u>cyclic nerve</u> of a category

$$\begin{tikzcd} X_0 \arrow[r] & X_1 \arrow[d, dotted] \\ X_{k-1} \arrow[u] & \cdots \arrow[l] \end{tikzcd}$$

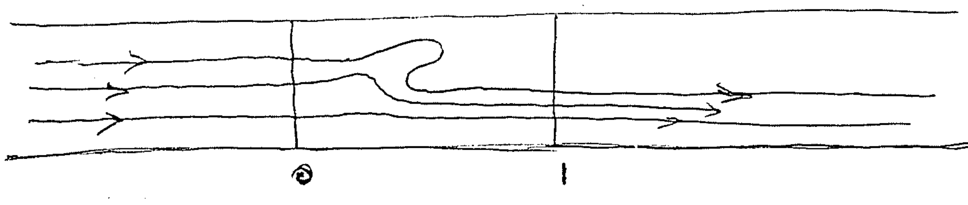
In the case of a groupoid $\mathcal{G}$, $B\mathcal{G} = \text{Nerve}(\mathcal{G})$ sits inside $\text{Nerve}^{\text{cyclic}}(\mathcal{G})$ as loops such that the product around the loop is 1.

In particular $B\mathcal{G}$ is a cyclic subset of $B\mathcal{G}^{\text{cyc}}$, hence there is a natural S^1-action on $B\mathcal{G}$, which <u>turns out to be trivial</u> up to homotop

Goodwillie's geometric trace map

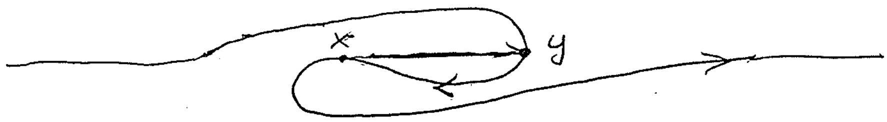
Think of a pseudo-isotopy as a scattering picture

$\mathbb{R}\times M$



The ψdoiso is a twistin of $\mathbb{R}\times M$ hol the left fixed

Now consider pairs of points (x,y), such that the normal flow $\longrightarrow$ carries x to y and the distorted flow carries y to x.



The set of these pairs, in good cases, is a compatifiable framed 2-manifold. It turns out this (+ general position) defines a map

$$\text{Pseudo isotopies}(M) \longrightarrow \Omega^2 Q(M_+)$$

This is a curious fixpoint construction. $\exists$ more general problem of periodic points.

February 16, 1988

Wodzicki's spectral sequence. Let I be an ideal in R non-unital. Then the filtration $R \supset I \supset 0$ of R as a vector space induces a filtration on the tensor powers $R^{\otimes n}$ and on their cyclic quotients.

$$\begin{tikzcd}
\cdot & R^{\otimes 4}_\lambda \arrow[l] \arrow[d] \arrow[r, phantom, "\supset"] & I\cdot R\cdot R\cdot R \arrow[dl, dashed] \arrow[r, phantom, "\supset"] & \begin{matrix} I\cdot I\cdot R\cdot R \\ +I\cdot R\cdot I\cdot R \end{matrix} \arrow[dl, dashed] \arrow[r, phantom, "\supset"] & I\cdot I\cdot I\cdot R \arrow[dl, dashed] \arrow[r, phantom, "\supset"] & I^{\otimes 4}_\lambda \arrow[dl, dashed] \\
(R/I)^{\otimes 3}_\lambda \arrow[u] \arrow[d] & R^{\otimes 3}_\lambda \arrow[l] \arrow[d] \arrow[r, phantom, "\supset"] & I\cdot R\cdot R \arrow[dl, dashed] \arrow[r, phantom, "\supset"] & I\cdot I\cdot R \arrow[dl, dashed] \arrow[r, phantom, "\supset"] & I^{\otimes 3}_\lambda \arrow[dl, dashed] & \\
\Lambda^2(R/I) \arrow[d] & \Lambda^2 R \arrow[l] \arrow[d] \arrow[r, phantom, "\supset"] & I\cdot R \arrow[dl, dashed] \arrow[r, phantom, "\supset"] & \Lambda^2 I \arrow[dl, dashed] & & \\
R/I & R \arrow[l] \arrow[r, phantom, "\supset"] & I & & &
\end{tikzcd}$$

Next consider what the b-operator does to a symbol $(x_0, \dots, x_n)$ having $\geq k$ elements in I. If two consecutive elements, say x_i, x_{i+1}, both are in I, then the symbol $(x_0, \dots, x_i x_{i+1}, \dots, x_n)$ has $k-1$ elements in I. Better, consider the face symbol $(\cdot\ , x_i x_{i+1}, \dots)$ in the formula for b; this is to be understood as the cross-over $(x_n x_0, \dots)$ when $i=n$. If both x_i, x_{i+1} are in the ideal I, then this face has $\geq k-1$ elements in I, and if at most one of x_i, x_{i+1} is in I, then the face has $\geq k$ elements. Thus we see the b operator relative to the filtration acts so as to decrease the number of I factors at most by 1, as indicated by the dotted arrows above.

When we pass to the associated graded complex, we obtain in degree $n-1$, the vector space

$$(I \oplus R/I)^{\otimes n}_{\lambda}$$

and the b-operator on the level of the associated graded complex involves only those faces where the two elements multiplied both belong to I. The other possibilities $R \otimes I \to I$, $I \otimes R \to I$, $R \otimes R \to R$ do not decrease the number of I-factors and so are zero on the associated graded level.

(~~Actually I restrict from $C(R)$ to the relative complex $K = \ker\{C(R) \to C(R/I)\}$ before taking the associated graded.~~ Also the filtration is defined so that

$$F_p K_n = \sum \operatorname{Im}\{\underbrace{R \otimes \cdots \otimes I \otimes \cdots}_{\substack{p \text{ factors } = R \\ n+1-p \;\;"\;\; = I}} \to R_{\lambda}^{\otimes(n+1)}\}$$

?

where $0 \leq p \leq n+1$)

It should be clear, from this description of the boundary operator b on the associated graded complex, that we have simply the cyclic complex of the semi-direct product algebra

$$I \oplus (R/I)$$

where $I \cdot R/I = R/I \cdot I = $
$(R/I)^2 = 0$

Thus the only thing that matters on the associated graded level is the multiplication in I.

Goodwillie's formula then tells us that gr_p up to some shift is

$$\left(\Sigma (R/I) \otimes_{I^+} B_N(I^+) \otimes_{I^+}\right)^p_\sigma \quad = \left\{ \left(\Sigma R/I\right)^{\otimes p} \otimes \left(\mathbb{C} \otimes_{I^+} B_N(I^+) \otimes_{I^+} \right. \right.$$

Here I want to use the non-unital version which describes $CC(A \oplus M)(p)$ as

$$\Sigma\, CC(A \oplus M)(p) = \left(\Sigma M \otimes_{A^+} B_N(A^+) \otimes_{A^+}\right)^p_\sigma$$

or the more general reduced version when A is unital

$$\Sigma\, \overline{C}C(A \oplus M)(p) = \left(\Sigma M \otimes_A B_N(A) \otimes_A\right)^p_\sigma$$

Then Wodzicki's spectral sequence becomes very clear. It involves the complex

$$\mathbb{C} \otimes_{I^+} B_N(I^+) \otimes_{I^+} \mathbb{C}$$

which gives $\mathrm{Tor}^{I^+}_*(\mathbb{C}, \mathbb{C})$.

He define I to be H-unital where this is trivial, i.e when

$$\mathrm{Tor}^{I^+}_*(\mathbb{C}, \mathbb{C}) = \begin{cases} \mathbb{C} & * = 0 \\ 0 & * > 0. \end{cases}$$

When this is the case we have

$$F_p / F_{p-1} \quad \text{is a resolution of} \quad \Sigma^{p-1}(R/I)^{\otimes p}_{\lambda}$$

$$F_0 = CC(I).$$

Actually I seem to be missing the following idea. We start the filtration of $CC(R)$ defined by

$$F_p CC_n(R) = \sum \operatorname{Im}\{\underbrace{R \otimes \cdots \otimes I \otimes \cdots}_{\substack{p \text{ factors of } R \\ n+1-p \ \text{———} \ I}} \to R_{\lambda}^{\otimes(n+1)}\}$$

Then we have

$$\operatorname{gr}_p CC(R) = CC(I \oplus R/I)(p)$$

$$\cong \begin{cases} CC(I) & p = 0 \\ \Sigma^{p-1}\{(R/I) \otimes \underbrace{\bar{B}(I)}_{\mathbb{C} \otimes_{I^+} B_N(I^+) \otimes_{I^+} \otimes \mathbb{C}}\}^{\otimes p}_{\lambda} & p \geq 1 \end{cases}$$

On the other in the complex F_p/F_{p-1} the lowest boundary operator

$$(R/I)^{\otimes p} \otimes I \xrightarrow{b} (R/I)^{\otimes p}_{\lambda}$$

is zero. This means that in the H-unital case, if we remove the 0-th group, the rest is acyclic. So if we cut down the $\operatorname{Ker}\{CC(R) \to CC(R/I)\}$ the relative cyclic complex, then F_p/F_{p-1} is acyclic for all $p \geq 1$, and so one gets excision.

<u>February 19, 1988</u>

Let's consider non-unital rings. First let's look at the reduced cyclic complex of $R \leftarrow I$, where R is unital and $\neq 0$. This is a double complex $\overline{CC}(R \leftarrow I)$ with

$$\overline{CC}_q(R \leftarrow I) = (\bar{R} \leftarrow I)_\lambda^{\otimes q+1}$$

and its seems that the p-th column for $p \geq 1$ is

$$\Sigma^{p-1}\left(I \otimes_R B^N(R) \otimes_R\right)^p_\sigma$$

where B^N is the normalized bar resolution, i.e. $B^N(R)_k = R \otimes \bar{R}^{\otimes k} \otimes R$. As a check we still have that the pth column for $p \geq 1$ is a model for $\Sigma^{p-1}\left(I \overset{!}{\otimes}_R\right)^p_\sigma$, hence

$$\overline{CC}(R \leftarrow I)/\overline{CC}(R) \longleftarrow CC(R \leftarrow I)/CC(R)$$

is a quis. The former resolves (when $I < R$), the relative reduced cyclic complex $\operatorname{Ker}\{\overline{CC}(R) \to \overline{CC}(A)\}$, and the latter resolves the relative cyclic complex $\operatorname{Ker}\{CC(R) \to CC$ and these we know are quis.

Now take a non-unital ring $\mathcal{a}$ and consider $R = \mathcal{a}^+$, $I = \mathcal{a}$. Then the rows of

$$\overline{CC}(\mathcal{a}^+ \leftarrow \mathcal{a})$$

are exact. The 0-th column is $CC(\mathcal{a}) = \overline{CC}(\mathcal{a}^+)$, and the p-th column is for $p \geq 1$

$$\Sigma^{p-1}\left(a \otimes_{a^+} B^N(a^+) \otimes_{a^+}\right)^p_\sigma$$

Notice that this doesn't for $p=1$ give the Hochschild homology. For $p=1$ in degree n it is

$$a \otimes_{a^+} (a^+ \otimes a^{\otimes n} \otimes a^+) \otimes_{a^+} = a^{\otimes(n+1)}$$

so one is not getting $a^+ \otimes a^{\otimes n}$, which gives the Hochschild homology.

Thus we conclude that the double complex $\overline{CC}(a^+ \leftarrow a) = CC(a \leftarrow a)$ gives an interesting resolution of $CC(a)$ which is not obvious. It certainly doesn't look anything ~~like~~ the Connes bicomplex up to column-wise quis.

Morita equivalence: Why are Hochschild and cyclic homology Morita invariant?

You have a derived category viewpoint for Hochschild homology. Is it possible to prove Morita invariance of Hochschild homology by derived functor methods?

What is a Morita equivalence? Suppose we have two algebras A, B. Then given a left B, right A bimodule ${}_BP_A$ we get a functor

$$P \otimes_A ? \; : \; Mod(A) \longrightarrow Mod(B)$$

and similarly a $A \otimes B^\circ$ module Q gives a functor the other way. Thus to get an equivalence of these module categories we need compatible isomorphisms

$$Q \otimes_B P = A \qquad\qquad P \otimes_A Q = B.$$

Once these hold we get an equivalence of the right module categories and also the bimodule categories:

$$X \longmapsto X \otimes_B P \qquad\qquad M \longmapsto P \otimes_A M \otimes_A Q$$

I guess by general nonsense it follows that P is f.g. projective over B and over A°, etc.

Natural problem: Characterize the functor $M \longmapsto M \otimes_A$ from bimodules to vector spaces intrinsically in categorically terms.

We can do $H^0(A, M) = \{m \mid am = ma \quad \forall a \in A\}$ as follows:

$$H^0(A,M) = \operatorname{Hom}_{A \otimes A^\circ}(A, M)$$

and we can identify bimodules with certain endofunctors of $\mathrm{Mod}(A)$. Thus an element of $H^0(A,M)$ should be a natural transf. from the identity functor $X \longmapsto A \otimes_A X$ to the functor $X \longmapsto M \otimes_A X$.

The real question seems to be why can $M\otimes_A$ be the "fixpoints" of the functor $M\otimes_A$? If one has ${}_BP_A$ and ${}_BQ_A$, then we have $Q\otimes_B P$ from $\mathrm{Mod}(A)$ to itself and also $P\otimes_A Q$ from $\mathrm{Mod}(B)$ to itself. Then the fact that there is an isomorphism

$$P\otimes_A Q\otimes_B \;=\; Q\otimes_B P\otimes_A$$

somehow expresses the well-known formulas

$$\mathrm{Fix}(fg) = \mathrm{Fix}(gf)$$

$$\mathrm{tr}(XY) = \mathrm{tr}(YX)$$

This seems to be a mystery.

February 24, 1988

Residues, duality, Lefschetz. It seems to be a good idea to review these: The reason is an analogy: If M is an R-bimodule then $M\otimes_R = M/[R,M]$ is some sort of fixpoint space. In the algebraic geometry situation, it is the restriction of M to the diagonal in the sense of quasi-coherent sheaves. Moreover the Cartier approach to residu which I would love to reconstruct uses a wrong-way map which I find reminiscent of $M\otimes_R$.

Example: Let A/k be smooth of dim n. Then there is a canonical residue isomorphism

$$\mathrm{Ext}^n_A(M,\Omega^n_{A/k}) = \mathrm{Hom}_k(M,k)$$

for M a A-module finite-diml over k. The problem is to construct this isomorphism. For example if $M = A/I$ where I is generated by a regular sequence we want a canonical map

$$\begin{array}{c}\mathrm{Ext}^n_A(A/I,\Omega^n_{A/k}) = \mathrm{Hom}_k(A/I,k) \xrightarrow{\text{eval on } 1} k \\ \| \\ \Lambda^n(I/I^2)^\vee \otimes_{A/I}\left(\Omega^n_{A/k}\otimes_A A/I\right)\end{array}$$

However the cotangent complex theory gives a complex

$$I/I^2 \longrightarrow \Omega^1_{A/k}\otimes_A A/I$$

This however gives an element of $\mathrm{Ext}^n_A(A/I,\Omega^n_{A/k})$ not a linear functional on it.

So it seems worthwhile to discuss the whole Lefschetz framework. Really I should say duality, because even over Riemann surfaces one needs the duality [redacted] given by the residues.

Where does one start? Duality (meaning Serre duality essentially) says that the dual of cohomology is cohomology of a dual sheaf. [redacted] In Grothendieck's form

$$\mathrm{Hom}_Y(f_! F, G) = \mathrm{Hom}_X(F, f^! G) \qquad \begin{array}{c} X^{F} \\ \downarrow f \\ Y^{G} \end{array}$$

where I have left out the derived functors. But more is involved:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

$$\begin{aligned} &\mathrm{Hom}_Y(f_! F, g_* G) \\ &= \mathrm{Hom}_{Y'}(g^* f_! F, G) \\ &= \mathrm{Hom}_{Y'}(f'_! g'^* F, G) \\ &= \mathrm{Hom}_{X'}(g'^* F, f'^! G) \end{aligned}$$

Thus there's a [redacted] Schwartz kernel theorem around. If we take $g = f$, then we have

$$\mathrm{Hom}_Y(f_! F, f_* F) = \mathrm{Hom}_{X \times_Y X}(pr_2^* F, pr_1^! F)$$

A natural question is [redacted] how to handle the trace! Let's proceed formally.

$$\begin{aligned} \mathrm{Hom}(f_! F, f_* F) &= (f_! F)^\vee \otimes (f_* F) \\ &= f_*(\check{F} \otimes \omega) \otimes (f_* F) \\ &= (f \times f)_*((\check{F} \otimes \omega) \boxtimes F) \qquad ? \end{aligned}$$

(not clear how to bring in Δ).

To gain insight we ought to look at DR cohomology and take a completely concrete viewpoint.

Let M be a smooth manifold of dimension n and $\Omega^\circ(M)$ its DR complex. Suppose M compact. Then one uses the basic pairing $C^\infty(M,E) \times C^\infty(M, E^\vee \otimes \omega) \longrightarrow \mathbf{C}^\infty(M,\omega) \xrightarrow{\int} \mathbb{C}$ where ω is the line bundle whose sections are smooth densities ~~measures~~ on M. Thus ω is $\Lambda^n T^*$ twisted by the orientation line bundle.

It seems then that to understand DR duality we must use this pairing, or really, we must use the integration of n-forms twisted by the orientation bundle. Maybe one can obtain this directly by looking at $\Delta : M \longrightarrow M \times M$ and setting up the Thom isomorphism. Thus one looks at the forms on $M \times M$ supported in a tubular nbd of the diagonal and one uses integration, Mayer-Vietoris to see this complex has the same cohomology as the twisted forms on M.

Does duality follow? Assume M oriented to simplify. One needs the Kunneth formula
$$H^*(M \times M) = H^*(M) \otimes H^*(M)$$
which follows by a Mayer-Vietoris argument. Then one takes the Kunneth components of the Thom class as in Bott's lectures, to prove Poincaré duality. Recall how this goes. One writes
$$\Delta_* 1 = \sum \alpha_i \, \beta_i$$
and uses
$$\alpha = pr_{1*}\left(\Delta_* 1 \cdot pr_2^* \alpha\right)$$
$$= pr_{1*}\left(\sum_i pr_1^*(\alpha_i)\, pr_2^*(\beta_i)\, pr_2^*(\alpha)\right)$$

$$= \sum_i (-1)^{|\alpha_i|} \alpha_i \int_M \beta_i \alpha$$

February 25, 1988

What I probably have to do is to learn the proof of the Atiyah-Bott-Lefschetz formula. One supposes given a complex of vector bundles and differential operators over a compact manifold which is elliptic. The ellipticity has the following consequence: We can consider the complex of smooth sections $C^\infty(M, E^\cdot) = \Gamma(E^\cdot)$ and also the complex of distributional sections, denote it $\tilde{\Gamma}(E^\cdot)$. The inclusion $\Gamma(E^\cdot) \hookrightarrow \tilde{\Gamma}(E^\cdot)$ is a quis

Next one uses the Schwartz kernel theorem to express operators from $\Gamma(E^\cdot)$ to $\tilde{\Gamma}(E^\cdot)$ as distributional sections over the product

$'$ = dual as LCTVS.

$$\mathrm{Hom}(\Gamma(E^\cdot), \tilde{\Gamma}(E^\cdot)) \underset{\text{NOT CLEAR}}{=} \Gamma(E^\cdot)' \otimes \tilde{\Gamma}(E^\cdot)$$

I need to know about $\tilde{\Gamma}$ for a vector bundle E. Let ω be the line bundle of densities on M. Then we have a pairing

$$\Gamma(E) \otimes \Gamma(E^\vee \otimes \omega) \longrightarrow \Gamma(\omega) \xrightarrow{\int_M} \mathbb{C}$$

and one defines

$$\tilde{\Gamma}(E) = \Gamma(E^\vee \otimes \omega)'.$$

Then

$$\mathrm{Hom}(\Gamma(E^\cdot), \tilde{\Gamma}(E^\cdot)) = \mathrm{Hom}(\Gamma(E^\cdot), \Gamma(E^{\cdot\vee} \otimes \omega)')$$

$$= \left(\Gamma(E^\cdot) \otimes \Gamma(E^{\cdot\vee} \otimes \omega)\right)'$$

$$= \Gamma(M \times M, pr_2^* E^\cdot \otimes pr_1^*(E^{\cdot\vee} \otimes \omega))'$$

This last space can be identified with

$$\tilde{\Gamma}(M \times M,\ pr_1^*(E^\cdot) \otimes pr_2^*(E^{\cdot\vee} \otimes \omega))$$

Now we also should have a similar description for the smooth kernel operators, so the kernel thm. gives the formulas

$$\boxed{\begin{aligned} \mathrm{Hom}\,(\Gamma(E^\cdot), \tilde{\Gamma}(E^\cdot)) &= \tilde{\Gamma}(M \times M,\ pr_1^*(E^\cdot) \otimes pr_2^*(E^{\cdot\vee} \otimes \omega)) \\ \mathrm{Hom}\,(\tilde{\Gamma}(E^\cdot), \Gamma(E^\cdot)) &= \Gamma(\text{———————}) \end{aligned}}$$

What about the trace? I think I have to go over the last part more carefully.

$$(*) \qquad \mathrm{Hom}\,(\tilde{\Gamma}(E^\cdot), \Gamma(E^\cdot)) = \Gamma(E^\cdot) \otimes \tilde{\Gamma}(E^\cdot)'$$

This is not obvious - in effect, if we had

$$\mathrm{Hom}\,(\Gamma(E), \Gamma(E)) = \Gamma(E) \otimes \Gamma(E)'$$

then traces would always be defined. However $(*)$ seems to be a consequence of nuclearity. In any case we have maps

$$\begin{array}{ccc} \mathrm{Hom}\,(\tilde{\Gamma}(E^\cdot), \Gamma(E^\cdot)) & \longleftarrow & \Gamma(E^\cdot) \otimes \tilde{\Gamma}(E^\cdot)' \\ & & \| \\ \Gamma(M \times M, pr_1^*(E^\cdot) \otimes pr_2^*(E^{\cdot\vee} \otimes \omega)) & = & \Gamma(E^\cdot) \otimes \Gamma(E^{\cdot\vee} \otimes \omega) \end{array}$$

So it's formally clear that the trace of a smooth kernel operator is given by restricting the kernel to the diagonal and integrating the little trace.

So far we haven't used the fact we have complexes or the ellipticity.

Let's [redacted] review. I have been considering [redacted] smooth vector bundles E over M. Then one should have

$$\mathrm{Hom}(\tilde{\Gamma}(E), \Gamma(E)) = \Gamma(E) \otimes \tilde{\Gamma}(E)'$$ [redacted]

$$= \Gamma(E) \otimes \Gamma(E^\vee \otimes \omega)$$

$$= \Gamma(M \times M, pr_1^*(E) \otimes pr_2^*(E^\vee \otimes \omega))$$

by Grothendieck's theory. From this one sees that the trace of a smooth kernel operator is the integral over M of the little trace. This is because [redacted] the basic pairing

$$\Gamma(E) \otimes \Gamma(E^\vee \otimes \omega) \longrightarrow \Gamma(\omega) \xrightarrow{\int_M} \mathbb{C}$$

is given by integration.

Now let's bring in the fact [redacted] that we have an elliptic complex. We can consider operators on $\Gamma(E^\cdot)$ or $\tilde{\Gamma}(E^\cdot)$ which are morphisms of complexes, and which are homotopic to zero. Hodge theory gives a homotopy

$$dh + hd = 1 - P$$

where P is projection on the harmonics. Now if we replace h by ρh, where ρ is 1 near ΔM and supported in a tubular nbd, then we get $d(\rho h) + (\rho h)d = 1 - U$, where U is a kind of Thom form. [redacted]

[redacted] U is a smooth kernel operator homotopic to the identity and supported in a tubular nbd of ΔM. One should obtain a Lefschetz formula for geometric endomorphisms transversal to Δ.

7

February 26, 1988

Case of a Riemann surface M. Let E be a holomorphic vector bundle on M. One has complexes

$$\Gamma(E) \xrightarrow{\bar\partial} \Gamma(E\otimes T^{0,1})$$

$$\Gamma(E^\vee\otimes T^{1,1}) \xleftarrow{\bar\partial} \Gamma(E^\vee\otimes T^{1,0})$$

pairing together. This gives rise to Serre duality.

To be more precise we have that the dual of the Dolbeault complex of E is the distributional Dolbeault complex of $E^\vee\otimes T^{1,0} = E^\vee\otimes K$. The latter is homotopy equivalent to the Dolbeault complex of $\check{E}\otimes K$ by ellipticity. Finally use the exactness of the "dual space" functor when the complexes ~~that the differentials have~~ are "split".

This raises the question of how to prove Poincaré duality for DR cohomology. I think the usual proof starts with a pairing, which is shown to be a duality first locally, then by Mayer-Vietoris. So what's the pairing?

$$\Gamma_c(U,\Lambda^pT^*) \times \Gamma(U,\Lambda^{n-p}T^*\otimes\omega) \longrightarrow \Gamma_c(U,\omega) \xrightarrow{\int} \mathbb{C}$$

This brings up another aspect of duality namely cohomology with compact support. Poincaré duality has a purely topological proof. The idea is to consider $U \longmapsto R\Gamma_c(U,\mathbb{C})'$ which is a sheaf. Actually it would be better to fix a flabby resolution of $\mathbb{C}$, e.g. the DR complex. Thus say consider $U \longmapsto \Gamma_c(U, \Lambda^\bullet T^*)' = \tilde{\Gamma}(U, \Lambda^\bullet T^* \otimes \omega)$; this is a complex of sheaves. By local calculations it is a resolution of ω, so the global sections give $H^\bullet(M, \omega)$, etc.

So far I have reviewed various ideas in Poincare duality, however I really want to get some ideas about the Riemann surface case. Here one looks at the $\bar\partial$ complex and one doesn't have partitions of unity

It seems that a sheaf theory treatment of Poincare duality doesn't use the product manifold $M \times M$. I think the product viewpoint is very important. It's probably better for me to view Poincare duality as a consequence of Künneth and the Thom isomorphism in the way I learned from Bott.

Recall the steps: The Thom isomorphism gives

$$\Delta_* 1 \in H^n(M \times M, M \times M - \Delta M).$$

One verifies that

$$\alpha = pr_{1*}\left(\Delta_* 1 \cdot pr_2^* \alpha\right)$$

(This uses integration over M. It would be nice if one obtained integration over M at the same time one proves the Thom isomorphism; the idea is that the integration is done locally using knowledge about balls.)

Suppose I don't know anything about integration of differential forms. Say that I discovered the DR complex in studying the cyclic homology of $C^\infty(M)$. Then how might I go about constructing integration over the manifold?

You need some sort of integration to prove the Poincare lemma. This involves integration over the unit interval. But let's suppose we have the Poincaré lemma, or more generally the homotopy axiom for DR cohomology. This gives the DR cohomology of $\mathbb{R}^n$.

What about the DR cohomology with compact supports on $\mathbb{R}^n$? Take $n=1$. Then $d: \Gamma_c(\mathbb{R}, \Lambda^0 T^*) \longrightarrow \Gamma_c(\mathbb{R}, \Lambda^1 T^*)$ is injective. Given a 1-form $g(x)dx$ with compact support, one can integrate it $\int_{-\infty}^{x} g(x)dx$ using our integration in dimension 1. We obtain a smooth function $f(x) = \int_{-\infty}^{x} g(x)dx$ with $df = g\,dx$, which has compact support iff $\int_{-\infty}^{\infty} g\,dx = 0$. Thus the F.T.C. computes the DR cohomology. We find $H^1_c(\mathbb{R}) = \mathbb{C}$ but the isomorphism depends on the orientation of $\mathbb{R}$.

If we assume Künneth theorems hold for products of manifolds, then we get the calculation of $H^*_c(\mathbb{R}^n)$. The next step might be to see if we have diffeomorphism invariance of the isomorphism $H^n_c(\mathbb{R}^n) \xrightarrow{\sim} \mathbb{C}$.

What I am trying to do is to understand the foundations of DR theory. Let us next look at the Thom isomorphism. We have an oriented vector bundle E/M and we want to compare the the DR cohomology of M with the DR cohomology of E ~~with~~ proper supports over M. Locally E is a trivial bundle, and if we use the Kunneth formula, then we can see that for small open $U \subset M$, we have

$$H^q_{proper/U}(E_U) \cong \begin{cases} 0 & q \neq n = \text{rank } E \\ \mathbb{C} & q = n \end{cases}$$

At the moment we just know that $H^n_{pr/U}(E_U)$ is 1-diml., but we have no canonical isomorphism. But this is necessary if I want obtain a global Thom class in $H^n_{pr/M}(E)$.

In fact I really ought to see what is involved in the construction of the Thom class. Thus I need to construct in each $H^n_{pr/U}(E/U)$ with U small a Thom class. Thus you would have to specify the Thom class as the unique class such that when restricted to a fibre and integrated with respect to an admissible coordinate system on the fibre gives 1. You check via homotopy considerations that admissible refers only to the orientation.

Actually you want an integration over the fibre map for <u>vector</u> bundles. What does this require? One can define it by trivializing the vector bundle, using Kunneth, but one has to check independence of the trivialization.

Thus it appears that there is some work involved in defining integration over the fibre for differential forms. The kind of work involved has something to do with diffeomorphism invariance of the integral of top-diml forms.

Another possibility would be that the integral over the fibre can be defined by global duality. This means for $f: X \to Y$, that

$$\int_Y f_*(\alpha)\beta = \int_X \alpha f^*(\beta)$$

which says that the transpose of $f^*: \Omega(Y) \to \Omega(X)$ induces $f_*: \Omega(X) \longrightarrow \Omega(Y)$. The smoothness of f implies that $(f^*)^t$ on currents induces f_* on forms.

I am also interested in whether knowing π_* for vector bundles might give it for manifolds. If I know the explicit integration for the normal bundle for $\Delta: M \hookrightarrow M \times M$, does this tell me about $\int_M$?

It seems reasonable to inquire whether there is a good way to define the integral of 1-forms on the circle. I really don't know if this is a reasonable question. Look at it this way. Differential forms are defined by functions on $M \times M$ which vanish on ΔM. What makes it possible to assign ~~[illegible]~~ a number to these?

February 28, 1988

I made some progress towarding reconstructing Cartier's residue construction. I start with the idea I remember from a lecture of Mumford on ~~[illegible]~~ the residue isomorphism. The point of this isomorphism is the following. One has two versions of the duality map $f^!$. If $f: X \to Y$ is finite, then one has

$$f^!(F) = R\underline{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, F)$$

whereas if f is smooth of relative dimension n, then one has

$$f^\#(F) = \Omega^n_{X/Y}[-n] \otimes_{\mathcal{O}_X} f^*(F)$$

Consider now a triangle

$$1) \qquad \begin{tikzcd} & X \arrow[d, "p"] \\ Z \arrow[ur, "j"] \arrow[r, "f"', "\text{finite}"] & Y \end{tikzcd} \quad p \text{ smooth} \qquad \therefore j \text{ finite}$$

Then ~~[illegible]~~ the residue isomorphism is a canonical isomorphism

$$2) \qquad j^! p^\# \simeq f^!$$

Mumford's idea is to reduce this to the case where f is the identity. Let $W = Z \times_Y X$. Then

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Z \arrow[r, "f"'] \arrow[ur, "j"] \arrow[u, bend left, "i"] & Y \end{tikzcd}$$

$$\begin{aligned} j^! p^\#(F) &= i^! f'^! p^\#(F) \\ &= i^! p'^\# f^!(F) \\ &= f^!(F) \end{aligned}$$

provided we have an isom $i^! p'^\# = id$.

Actually the residue isomorphism appears in other forms, e.g. if A/k is smooth where k is a field, then one wants an isomorphism

$$3) \qquad \mathrm{Ext}^n_A(M, \Omega^n_{A/k}) = M^*$$

for any A-module M which is finite dimensional over k. This is related to 1) 2) by letting $Z = \mathrm{Spec}$ of A/I where I kills M. In effect one considers $R\mathrm{Hom}_Z(M, ?)$ applied to 2).

However if I work with M built into the argument, then I don't need to assume Z is finite, and I can take $Z = X$ with $f = id_X$. ~~This gives the following isomorphism better for generalizing (I hope)~~

$$\cancel{R\mathrm{Hom}_A(M, \Omega^1_{A/k}) = R\mathrm{Hom}_{A\otimes A}(A, \mathrm{Hom}_k(M, \Omega^1_{A/k}))}$$

Pursuing this idea I was led to the following, which I could have stated immediately.

$$R\mathrm{Hom}_A(M, \Omega^n_{A/k}) = R\mathrm{Hom}_{A\otimes A}(A, \underbrace{\mathrm{Hom}_k(M, \Omega^n_{A/k})}_{= M^* \otimes_k \Omega^n_{A/k}})$$

when $\dim_k(M) < \infty$.

$$= R\mathrm{Hom}_{A\otimes A}(A, M^* \otimes_{(A\otimes k)} \underbrace{(A \otimes \Omega^n_{A/k})}_{\Omega^n_{A\otimes A/A\otimes k}})$$

Here we are directly reduced to the case of a smooth map with cross-section, specifically $A\otimes A$ over $A \otimes k = A$ with the diagonal $A\otimes A \longrightarrow A$.

Set $B = A \otimes A$, so that we have $A \to B \to A$

with the first map smooth. Let $J = \mathrm{Ker}\{B \to A\}$ so that for same reason J is locally generated by a regular sequence. Thus A is of finite Tor dim over B, and so for any B-module N we have

$$R\mathrm{Hom}_B(A, N) = R\mathrm{Hom}_B(A, B) \overset{L}{\otimes}_B N.$$

This can be seen explicitly using the Koszul resolution to compute the LHS. The Koszul resolution also gives

$$R\mathrm{Hom}_B(A, B) = \Lambda^n (J/J^2)^\vee [n]$$

where the dual and Λ^n are taken over $B/J = A$.

Now consider $N = M^* \otimes_A \Omega^n_{B/A}$. But B is flat over A, so that I get a flat B-resolution of N from a flat A-resolution of M^*. This means

$$R\mathrm{Hom}_B(A, B) \overset{L}{\otimes}_B N = R\mathrm{Hom}_B(A, B) \otimes_B \Omega^n_{B/A}[n] \overset{L}{\otimes}_A M^*$$

and the last L can be dropped since the homology groups of

$$R\mathrm{Hom}_B(A, B) \otimes_B \Omega^n_{B/A}[n] = R\mathrm{Hom}_B(A, \Omega^n_{B/A}[n])$$

are flat over A. In fact this is canonically A.

February 29, 1988

I think I am now in a position to reconstruct ~~[illegible]~~ how Tate found his 1-dimensional residue from Cartier's definition. I will have to first understand the A-module

$$A \otimes_{A\otimes A} \mathrm{Hom}_k(M, A) = \mathrm{Hom}_k(M,A)/[A, \mathrm{Hom}_k(M,A)]$$

where M is an A-module which is finite-dim'l over k. We have

$$A \otimes_{A\otimes A} \mathrm{Hom}_k(M,A) = A \otimes_{A\otimes A} (M^* \otimes_k A)$$

$$= A \otimes_A M^* = M^*.$$

so I have to understand how to go from a k-linear map $\lambda : M \longrightarrow A$ to a linear functional on M.

Let m_j be a basis for M over k and let $\mu_j \in M^*$ be the dual basis. Thus we have

$$m = \sum_j m_j \, \mu_j(m) \qquad \forall m \in M$$

(which would hold for projective modules). Then

$$\lambda(m) = \sum_j \lambda(m_j)\, \mu_j(m)$$

so

$$\lambda = \sum_j \mu_j \otimes \lambda(m_j) \in M^* \otimes A.$$

Now modulo $[A, M^* \otimes A]$ we have

$$\mu \otimes a = (\mu \otimes 1) a \equiv (a\mu \otimes 1)$$

where $(a\mu)(m) = \mu(am)$. Thus

$$\lambda \equiv \Big(\sum_j \lambda(m_j) \mu_j \Big) \otimes 1$$

where

$$\Big(\sum_j \lambda(m_j)\mu_j\Big)(m) = \sum \mu_j(\lambda(m_j)m)$$

This is the trace of the composition

$$M \xrightarrow{\lambda} A \xrightarrow{m} M$$

$\therefore$ We have proved

Lemma: If M is an A-module finite-diml over k, then

$$\mathrm{Hom}_k(M,A)/[A,\mathrm{Hom}_k(M,A)] \xrightarrow{\sim} M^*$$

$$\lambda \longmapsto (m \to \mathrm{tr}_M(m\lambda)).$$

~~Suppose now that f is a non-zero divisor such that A/fA is finite-dimensional over k.~~

Let's now construct the residue isomorphism

$$\mathrm{Ext}^1_A(M,A) = H^1\{\mathrm{Hom}_A(I\otimes_A M \longrightarrow (A\otimes A)\otimes_A M, A)\}$$

$I = \hat{\Omega}^1_A$

$$= H^1\{\mathrm{Hom}_{A\otimes A}(I \longrightarrow A\otimes A, \mathrm{Hom}_k(M,A))\}$$

$$= H^1\{\mathrm{Hom}_{A\otimes A}(I \longrightarrow A\otimes A, A\otimes A) \otimes_{A\otimes A} \mathrm{Hom}_k(M,A)\}$$

$$= \mathrm{Ext}^1_{A\otimes A}(A, A\otimes A) \otimes_{A\otimes A} \mathrm{Hom}_k(M,A)$$

$$= (\Omega^1_A)^\vee \otimes_{A\otimes A} \mathrm{Hom}_k(M,A)$$

$$= (\Omega^1_A)^\vee \otimes_A M^*$$

But it might be better to proceed as follows.

$$\operatorname{Ext}^1_A(M, A) = H^1\{\operatorname{Hom}_A(\mathcal{B}(A) \otimes_A M, A)\}$$

$$= H^1\{\operatorname{Hom}_{A\otimes A}(\mathcal{B}(A), \operatorname{Hom}_k(M, A))\}$$

$$= \operatorname{Der}(A, \underbrace{\operatorname{Hom}_k(M, A)}_{\text{bimodule}}) \Big/ \begin{matrix}\text{principal}\\ \text{derivations}\end{matrix}$$

$$\longrightarrow \operatorname{Der}(A, \underbrace{\operatorname{Hom}_k(M, A)/[A, \operatorname{Hom}_k(M, A)]}_{A\text{-module}})$$

$$= \operatorname{Der}(A, M^*)$$

Thus there is a canonical map

$$\operatorname{Ext}^1_A(M, A) \longrightarrow \operatorname{Hom}_A(\Omega^1_A, M^*)$$

when M is an A-module finite-dimensional over k, A any commutative ring.

In higher degrees we have

$$\operatorname{Ext}^n_A(M, A) = H^n\{\operatorname{Hom}_A(\mathcal{B}(A) \otimes_A M, A)\}$$

$$= H^n\{\operatorname{Hom}_{A\otimes A}(\mathcal{B}(A), \operatorname{Hom}_k(M, A))\}$$

$$= H^n(A, \operatorname{Hom}_k(M, A))$$

$$\xrightarrow[\text{map}]{\text{canon}} H^n(A, A \otimes_{(A\otimes A)} \operatorname{Hom}_k(M, A))$$

$$= H^n(A, M^*)$$

$$\xrightarrow[\text{map}]{\text{canon}} \operatorname{Hom}_A(\Omega^n_A, M^*)$$

Where does the last map come from?

$$\mathrm{Ext}^n_A(M, A^*) = H^n\{\mathrm{Hom}_{A\otimes A}(\mathcal{B}^N(A), M^*)\}$$

$$= H^n\{\mathrm{Hom}_A(A \otimes_{A\otimes A} \mathcal{B}^N(A), M^*)\}$$

Now $A \otimes_{A\otimes A} \mathcal{B}^N(A)$ is the Hochschild complex, and we know it is a commutative DG ring (under shuffle product. This means we have a map

$$\Lambda^n_A \underbrace{H_1(A,A)}_{=\,\Omega^1_A} \longrightarrow H_n(A,A)$$

and so we obtain canonical maps

$$\mathrm{Ext}^n_A(M, A^*) = H^n(A, \mathrm{Hom}_k(A, M))$$

$$\longrightarrow H^n(A, M^*)$$

$$\longrightarrow \mathrm{Hom}_A(H_n(A,A), M^*)$$

$$\longrightarrow \mathrm{Hom}_A(\Omega^n_A, M^*)$$

Furthermore I think ~~it~~ is probably fairly easy to be a bit more explicit by using the fact that $\mathrm{Coker}\{\mathcal{B}^N(A)_{n+2} \to \mathcal{B}^N_{n+1}(A)\}$ is the non-commutative $\hat{\Omega}^n_A$.

At this point it is more or less clear that there ought to be a simple approach to the residue map which uses almost nothing. One might start by representing an element of $\mathrm{Ext}^n_A(M, A)$ by an n-extension

$$0 \to A \to \cdots\cdots\cdots \to M \to 0$$

One splits this over k maybe.

Let's try to understand the case $n=1$. Start with an extension of A-modules

$$0 \to A \to E \xrightarrow{\pi} M \to 0$$

We want to [redacted] realize the class of this extension in $\mathrm{Ext}^1_A(M,A)$ by a map from $B(A)\otimes_A M$ to $(A \to E)$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat\Omega^1_A \otimes_A M & \longrightarrow & A\otimes M & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

The map α is given by

$$\alpha(a\otimes m) = a\,\rho(m)$$

where ρ is a k-linear cross-section of π.

We next take

$$\alpha' \in \mathrm{Hom}_A(\hat\Omega^1_A \otimes_A M, A) = \mathrm{Hom}_{A\otimes A}(\hat\Omega^1_A, \mathrm{Hom}_k(M,A))$$

and apply the map

$$\mathrm{Hom}_k(M,A) \longrightarrow M^*$$

which is given by the pairing

$$\mathrm{Hom}_k(M,A)\times M = \mathrm{Hom}_k(M,A)\times \mathrm{Hom}_A(A,M)$$

$$\xrightarrow{\ \circ\ } \mathrm{Hom}_k(M,M) \xrightarrow{\mathrm{tr}} k$$

Thus it seems we want to send an elt of $\hat\Omega^1_A$, say $x\,dy\,z$, into the linear fn on M given by

$$m \longmapsto \mathrm{tr}\,(m_1 \mapsto m\,\alpha'(x\,dy\,z\,m_1))$$

March 1, 1987

Let's consider A non-commutative unital and look at the isomorphism

$$\mathrm{Ext}^1_A(M,N) = H^1(A, \mathrm{Hom}_k(M,N))$$
$$= \mathrm{Der}(A, \mathrm{Hom}_k(M,N)) / \mathrm{Im}\ \mathrm{Hom}_k(M,N)$$

where M, N are A-modules. This can be made explicit as follows. Given an extension

$$(1) \qquad 0 \longrightarrow N \longrightarrow E \underset{\rho}{\overset{\pi}{\rightleftarrows}} M \longrightarrow 0$$

we choose a cross-section ρ of π, and define $f : A \longrightarrow \mathrm{Hom}_k(M,N)$ by

$$f(a)(m) = a\rho(m) - \rho(am)$$

Then f is a 1-cocycle or derivation whose class is independent of the choice of ρ.

~~Suppose now that $N = M$ and that M is finite-dimensional over k. Then from the derivation f we get a map~~

$$\cancel{\hat{\Omega}^1_A \xrightarrow{\tilde{f}} \mathrm{Hom}_k(M,M)}$$

We extend the derivation f to a bimodule morphism

$$\hat{\Omega}^1_A \xrightarrow{\tilde{f}} \mathrm{Hom}_k(M,N)$$

$$x(da)y \longmapsto x[a,\rho]y$$

Suppose now that $N = M$ is finite dim'l over k. Then we have the trace

$$\begin{tikzcd}
\hat{\Omega}^1_A \arrow[r, "\tilde{f}"] \arrow[d] & \mathrm{Hom}_k(M,M) \arrow[d] \arrow[rd, bend left, "tr"] & \\
\hat{\Omega}^1_A / [A, \hat{\Omega}^1_A] \arrow[r] & \mathrm{Hom}_k(M,M)/[A, \mathrm{Hom}_k(M,M)] \arrow[r, "tr"] & k
\end{tikzcd}$$

which means we have a Hochschild 1-cocycle

$$\varphi(a_0, a_1) = tr\left(a_0[a_1, \rho]\right)$$

Check: $$(b\varphi)(a_0, a_1, a_2) = tr\Big(\left(a_0 a_1 [a_2, \rho]\right) - \left(a_0[a_1 a_2, \rho]\right) + \left(a_2 a_0 [a_1, \rho]\right)\Big) = 0$$

This is not a cyclic 1-cocycle apparently, nor is it zero necessarily when ρ is replaced by an element $u \in Hom_k(M, M)$.

However when A is commutative, we have

$$\begin{aligned} tr\left(a_0[a_1, u]\right) &= tr\left(a_0 a_1 u - a_0 u a_1\right) \\ &= tr\left([a_0, a_1] u\right) = 0 \end{aligned}$$

[redacted] and $\hat{\Omega}^1_A / [A, \hat{\Omega}^1_A] = \hat{\Omega}^1_A \otimes_{A \otimes A} A = \Omega^1_A$ so we have a well-defined map

$$Ext^1_A(M, M) \longrightarrow Hom_k(\Omega^1_A, k)$$

which associates to the extension ① the map

$$x\,dy \longmapsto tr_M\left(x[y, \rho]\right)$$

I think this justifies ~~my~~ feeling that the Grothendieck residue symbol can be reduced to these commutator type calculations. More work is necessary to have a complete picture.

Let's go back to the non-commutative situation, where we have the isomorphism

$$Ext^1_A(M, N) = H^1(A, Hom_k(M, N))$$

We consider A non-commutative and suppose we have a resolution of left A-modules

$$\longrightarrow E_{n+1} \longrightarrow E_n \longrightarrow \cdots\cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0$$

and also a map $E_n / \operatorname{Im} E_{n+1} \longrightarrow N$. By Yoneda we have an element of

$$\operatorname{Ext}^n_A(M, N) = H^n\{\operatorname{Hom}_A(\mathcal{B}^N(A) \otimes_A M, N)\}$$

I claim we can represent this class by a map $\mathcal{B}^N(A) \otimes_A M \longrightarrow N[n]$ in a fairly concrete way. This is because $\mathcal{B}^N(A) \otimes_A M$ has a universal property among resolutions of M, namely it comes with a contracting homotopy s, which is k-linear, given by

$$s(a_0, a_1, \ldots, a_n, m) = (1, \bar{a}_0, a_1, \ldots, a_n, m)$$

and s satisfies $s^2 = 0$.

(In fact it seems that the same thing is true for $\mathcal{B}(A) \otimes_A M$ except that ~~the~~ condition $s^2 = 0$ is absent.)

Let's choose a contracting homotopy h of the above resolution complex such that $h^2 = 0$. Then we claim there is a unique morphism of A-module complexes which is compatible with the homotopies

$$\begin{tikzcd}
\cdots \arrow[r, "d"'] & A \otimes \bar{A} \otimes M \arrow[l, bend right, dashed, "s"'] \arrow[r, "d"'] \arrow[d, "f_1"] & A \otimes M \arrow[l, bend right, dashed, "s"'] \arrow[r, "d"'] \arrow[d, "f_0"] & M \arrow[l, bend right, dashed, "s"'] \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r, "d"'] & E_1 \arrow[l, bend right, dashed, "h"'] \arrow[r, "d"'] & E_0 \arrow[l, bend right, dashed, "h"'] \arrow[r, "d"'] & M \arrow[l, bend right, dashed, "h"'] \arrow[r] & 0
\end{tikzcd}$$

In effect

$$\mathcal{B}^N(A)_{n+1} \otimes_A M = A \otimes \bar{A}^{\otimes n+1} \otimes M$$

is the free A-module generated by the vector space $\mathcal{B}^N(A)_n \otimes_A M / s\mathcal{B}^N(A)_{n-1} \otimes_A M$.

So having defined f_n, f_{n-1} we define f_{n+1} to be the unique A-module map such that

$$f_{n+1} s = h f_n$$

We must check that f_{n+1} is well defined, i.e. that $h f_n s = 0$. But $h f_n s = h h f_{n-1} = 0$.

Next we must see that $\{f_n\}$ is a morphism of complexes. But

$$\begin{array}{ccc} d f_{n+1} s & \overset{?}{=} & f_n d s \\ \| & & \| \\ d h f_n & & f_n - \underbrace{f_n s d}_{\|} \\ \| & & \\ f_n - h d f_n & & h f_{n-1} d = h d f_n \end{array} \quad \text{induction}$$

so it works.

A nice thing to do is to note that we have

$$\begin{aligned} (a_0, a_1, \ldots, a_n, m) &= a_0 s(a_1, \ldots, a_n, m) \\ &= a_0 s(a_1 s(a_2, \ldots, m)) \\ &= a_0 s a_1 s a_2 s \cdots s m. \end{aligned}$$

Thus one has replaced the commas by s. Then the formula for the differential

$$d(a_0 s a_1 s a_2 s m) = a_0 a_1 s a_2 s m - a_0 s \underbrace{d(a_1 s a_2 s m)}_{a_1 a_2 s m \,-\, a_1 s \underbrace{d a_2 s m}_{a_2 m - \cancel{a_2 s d m}}}$$

or

$$d(a_0 s a_1 s a_2 s m) = a_0 a_1 s a_2 s m - a_0 s(a_1 a_2 s m - a_1 s a_2 m)$$

$$= a_0 a_1 s a_2 s m - a_0 s a_1 a_2 s m + a_0 s a_1 s a_2 m$$

Thus the formula for $f: \mathcal{B}^N(A) \otimes_A M \longrightarrow E_0$ is just

$$f(a_0, \ldots, a_n, m) = a_0 h a_1 h a_2 \cdots h a_n h m$$

<u>March 2, 1988</u>

Let's consider an A-module resolution

$$\begin{tikzcd} \cdots \arrow[r] & E_1 \arrow[l, dashed, bend right, "h"'] \arrow[r, "d"'] & E_0 \arrow[l, dashed, bend right, "h"'] \arrow[r, "d"'] & M \arrow[l, dashed, bend right, "h"'] \arrow[r] & 0 \end{tikzcd}$$

together with a splitting over k. Set $M_0 = M$ and $M_j = \mathrm{Im}(E_j \xrightarrow{d} E_{j-1})$. We have seen that there is a unique map of A-module resolutions

$$\mathcal{B}^N(A) \otimes_A M \xrightarrow{f} E$$

compatible with the splittings. f is given by

$$f_n(a_0, \ldots, a_n, m) = a_0 h a_1 \cdots h a_n h m$$

$$f_n : \mathcal{B}^N(A)_n \otimes_A M = A \otimes \bar{A}^{\otimes n} \otimes M \longrightarrow E_n$$

Let's consider the map to the truncated resolution

$$\begin{tikzcd} {} \arrow[r, "d"] \arrow[d] & A \otimes \bar{A}^{\otimes n} \otimes M \arrow[r, "d"] \arrow[d, "df_n"] & A \otimes \bar{A}^{\otimes n-1} \otimes M \arrow[r] \arrow[d, "f_{n-1}"] & {} \\ 0 \arrow[r] & M_n \arrow[r, hook] & E_{n-1} \arrow[r, "d"] & {} \end{tikzcd}$$

Then since $h^2 = 0$, we have

$$\begin{aligned} f_n(a_0, \ldots, a_n, m) &= a_0 h a_1 \cdots h a_n h m \\ &= a_0 h a_1 \cdots h [a_n, h] m \\ &\;\vdots \\ &= a_0 h [a_1, h] \cdots [a_n, h] m \end{aligned}$$

Moreover we've claim that $[a, h] : E_k \longrightarrow E_{k+1}$ carries M_{k-1} into M_k.

In effect $d[a,h] + [a,h]d = adh - dha + ahd - hda$
$= a - a = 0$

Thus we see that

$$d\big(f_n(a_0,\dots,a_n,m)\big) = a_0[a_1,h]\cdots[a_n,h]m \in M_n$$

Next I would like to interpret all this in terms of cup product of cocycles and non-commutative differential forms. I first identify Ω^n_A with the n-th image $b' B^N(A)_n$ in the normalized bar resolution.

Recall that we define Ω^n_A to be the bimodul

① $$\Omega^n_A = \Omega^1_A \otimes_A \overset{n}{\cdots} \otimes_A \Omega^1_A$$

where

$$\Omega^1_A = \mathrm{Coker}\ \{B^N(A)_2 \xrightarrow{b'} B^N(A)_1\}$$

$$= \mathrm{Im}\ \{B^N(A)_1 \xrightarrow{b'} B^N(A)_0\}$$

$$= \mathrm{Ker}\ \{A\otimes A \longrightarrow A\}$$

We define $d: A \longrightarrow \Omega^1_A$ by da = image of $(1,a,1) \in A\otimes\bar{A}\otimes A = B^N(A)_1$ or $da = a\otimes 1 - 1\otimes a$ in $A\otimes A$. From the splitting compatible with right mult.

② $$0 \longrightarrow \Omega^1_A \longrightarrow A\otimes A \underset{s}{\overset{\longrightarrow}{\longleftarrow}} A \longrightarrow 0$$

we get the right A-module isomorphism

③ $$\bar{A}\otimes A \xrightarrow{\sim} \Omega^1_A$$
$$(a_1,a_2) \longmapsto (da_1)a_2$$

From ①, ③ and an obvious induction we have a right A-module isomorphism

$$\bar{A}^{\otimes n}\otimes A \xrightarrow{\sim} \Omega^n_A$$
$$(a_1,\dots,a_n,a_{n+1}) \longmapsto da_1\cdots da_n\, a_{n+1}$$

Now let us tensor the exact sequence ② to get the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_A \otimes_A \Omega^n_A \arrow[r] \arrow[d, equal] & (A\otimes A)\otimes_A \Omega^n_A \arrow[r] \arrow[d, equal] & A \otimes_A \Omega^n_A \arrow[r] \arrow[d, equal] \arrow[l, dashed, bend right] & 0 \\
0 \arrow[r] & \Omega^{n+1}_A \arrow[r] & A \otimes \Omega^n_A \arrow[r] & \Omega^n_A \arrow[r] \arrow[l, dashed, bend right, "1\otimes"'] & 0
\end{tikzcd}$$

of A bimodules with splitting as right A-modules. Putting these together we get an exact sequence of A-bimodules

$$\begin{tikzcd}
{} \arrow[r] & A\otimes\Omega^2_A \arrow[r] \arrow[l, dashed, bend right] & A\otimes\Omega^1_A \arrow[r] \arrow[l, dashed, bend right] & A\otimes A \arrow[r] \arrow[l, dashed, bend right] & A \arrow[r] \arrow[l, dashed, bend right] & 0
\end{tikzcd}$$

Let's begin with the exact sequence

$$(dx)y \longleftarrow\!\!\shortmid\ x\otimes y \ \shortmid\!\!\longrightarrow xy$$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_A \arrow[r] & A\otimes A \arrow[r] \arrow[l, bend right] & A \arrow[r] \arrow[l, bend right, "1\otimes ?"'] & 0
\end{tikzcd}$$

$$(dx)y \longmapsto x\otimes y - 1\otimes xy$$

with splitting as right modules. Then we can tensor on the right and get

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_A\otimes_A\Omega^n_A \arrow[r] \arrow[d, equal] & (A\otimes A)\otimes_A\Omega^n_A \arrow[r] \arrow[d, equal] & \Omega^n_A \arrow[r] \arrow[d, equal] \arrow[l, bend right, "1\otimes"'] & 0 \\
& \Omega^{n+1}_A \arrow[r] & A\otimes\Omega^n_A \arrow[r] \arrow[l, bend right] & \Omega^n_A \arrow[l, bend right, "1\otimes"'] &
\end{tikzcd}$$

$$(dx)\omega \longleftarrow\!\!\shortmid\ x\otimes\omega \ \shortmid\!\!\longrightarrow x\omega$$

$$\longmapsto x\otimes\omega - 1\otimes x\omega$$

Now let's put these sequences together and we get a complex with splitting

$$\begin{tikzcd} \cdots \arrow[r] & A\otimes\Omega^2_A \arrow[r] \arrow[d, equal] & A\otimes\Omega^1_A \arrow[r] \arrow[l, bend right, "s"'] \arrow[d, equal] & A\otimes A \arrow[r] \arrow[l, bend right, "s"'] \arrow[d, equal] & A \arrow[r] \arrow[l, bend right, "s"'] & 0 \\ \arrow[r] & A\otimes\bar{A}^{\otimes 2}\otimes A \arrow[r] & A\otimes\bar{A}\otimes A \arrow[r] & A\otimes A \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Formally $s(a_0\otimes da_1 \cdots da_n a_{n+1}) = 1\otimes da_0 \cdots da_n a_{n+1}$ on the upper level, so on the lower level

$$s(a_0, a_1, \ldots, a_{n+1}) = (1, \bar{a}_0, a_1, \ldots, a_{n+1})$$

Notice that $(a_0, a_1, \ldots, a_{n+1}) = a_0\, s\, a_1 \cdots s\, a_{n+1}$ and ~~[illegible]~~ so the usual formula for b' follows. But I can also obtain it by

$$b'(a_0\otimes da_1 \cdots da_n a_{n+1}) = a_0 da_1 \cdots da_n a_{n+1} \quad \text{embedded in } A\otimes\Omega^{n-1}_A$$

~~$= d(a_0a_1)da_2 \cdots da_n a_{n+1} - da_0\, a_1\, da_2 \cdots da_n a_{n+1}$ $\longrightarrow$ $a_0a_1\otimes da_2 \cdots da_n a_{n+1}$ $-1\otimes a_0a_1 da_2 \cdots da_n a_{n+1}$ $- a_0\otimes a_1 da_2 \cdots$ $+ 1\otimes a_0a_1 da_2 \cdots$~~

~~$$= d(a_0a_1)da_2 \cdots da_n a_{n+1} - da_0\, d(a_1a_2) \cdots da_n a_{n+1} + \cdots\cdots + (-1)^{n-1} da_0 da_1 \cdots d(a_{n-1}a_n)a_{n+1} + (-1)^n da_0 da_1 \cdots da_{n-1} a_n a_{n+1}$$~~

~~which under the embedding of Ω^n_A in $A\otimes\Omega^{n-1}_A$ becomes~~

~~$= a$~~

$$= a_0 da_1 \otimes da_2 \cdots da_n a_{n+1}$$

$$= a_0(a_1\otimes 1 - 1\otimes a_1)\otimes_A da_2 \cdots da_n a_{n+1}$$

$$= a_0a_1\otimes da_2 \cdots da_n a_{n+1} - a_0\otimes a_1 da_2 \cdots da_n a_{n+1}$$

$$= a_0 a_1 \otimes da_1 \cdots da_n a_{n+1}$$

$$- a_0 \otimes \Big[d(a_1 a_2) \cdots\cdots\cdots$$

$$- da_1\, d(a_2 a_3) \cdots\cdots$$

$$+ \cdots\cdots\cdots$$

$$+ (-1)^{n-2} da_1\, da_2 \cdots d(a_{n-1} a_n)\, a_{n+1}$$

$$+ (-1)^{n-1} da_1 \cdots\cdots da_{n-1} (a_n a_{n+1}) \Big\}$$

which gives the formula for b'.

I think it would be nice to see directly why $\Omega^n_A = \cancel{\operatorname{Ker}\{\mathcal{B}^N(A)_n \xrightarrow{b'} \mathcal{B}^N(A)_{n-1}\}}$ $= \operatorname{Im}\{\mathcal{B}^N(A)_{n+1} \xrightarrow{b'} \mathcal{B}^N(A)_n\}$.

Remark: $\mathcal{B}^N(A)$ (really $\operatorname{Cone}\{\mathcal{B}^N(A) \to A\}$) also comes with the the left A-module contracting homotopy

$$s^L(a_0, \dots, a_n) = (a_0, \dots, a_n, 1)$$

Then $[b', s^L - s]_+ = 1 - 1 = 0$ so $s^L - s$ carries $\operatorname{Im}(b')$ into itself. This should be the DR d on $\Omega^{\cdot}_A$ up to sign.

March 3, 1988

Beilinson + Schechtman - Determinant bundles and Virasoro algebras.

Motivation: Let $\pi: X \longrightarrow S$ be a family of Riemann surfaces and let E be a vector bundle on X, everything being supposed holomorphic (or algebraic). Then one has a canonical line bundle on S

$$\lambda_E = \det(R\pi_*(E))$$

called the determinant line bundle of E. The idea will be to study this line bundle infinitesimally using an appropriate formalism which goes back to the way Atiyah constructed characteristic classes for vector bundles in alg. geometry.

Let E be a vector bundle over X (= any variety which is smooth). Let $\mathcal{A}_E$ be the sheaf of those differential operators $\partial: E \to E$ of ~~order~~ ≤ 1 whose symbol is given by a vector field ~~times~~ id_E. Thus $\sigma(\partial, \xi) = \langle v|\xi\rangle id_E$, $v \in \mathcal{T}_X =$ tangent bundle. Then we have an exact sequence of Lie algebras.

1) $\quad 0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathcal{A}_E \longrightarrow \mathcal{T}_X \longrightarrow 0$

A section of $\mathcal{A}_E$ is an infinitesimal automorphism of (M, E)

The extension 1) is also an extension of left $\mathcal{O}_X$-modules (and also right $\mathcal{O}_X$-modules). (Thus its an exact sequence of vector bundles). The exact sequence 1) determines an element in

$$H^1\{Hom_{\mathcal{O}_X}(\mathcal{T}_X, End(E))\} = H^1(X, \Omega^1_X \otimes_{\mathcal{O}_X} End(E))$$

which is the obstruction to the existence of a connection on E. Atiyah use this class to define characteristic classes for vector bundles with values in Hodge cohomology $H^p(X, \Omega^p)$.

When a splitting of 1) exists, i.e. a connection exists, then one obtains a curvature in $\Omega^2_X \otimes_{\mathcal{O}_X} End(E)$.

I guess we can think of the exact sequence 1) together with its various structure as being a kind of infinitesimal version of E.

The first theorem calculates the extension $\mathcal{A}_{\lambda_E}$, which because λ_E is a line bundle, is of the form

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{A}_{\lambda_E} \longrightarrow \mathcal{T}_S \longrightarrow 0$$

Preliminaries: $\pi: X \to S$ family of curves
One has the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_{X/S} & \longrightarrow & \mathcal{T}_X & \longrightarrow & \pi^*\mathcal{T}_S & \longrightarrow & 0 \\ & & \| & & \cup & \text{cart} & \cup & & \\ 0 & \longrightarrow & \mathcal{T}_{X/S} & \longrightarrow & \mathcal{T}_\pi & \longrightarrow & \pi^{-1}\mathcal{T}_S & \longrightarrow & 0 \end{array}$$

where $\mathcal{T}_\pi$ is defined so the square is cartesian. The bottom sequence is an exact sequence of Lie algebras. $\mathcal{T}_\pi$ is the sheaf of inf. autos.

of the map π. Let $\mathcal{T}^{\circ}_{\pi}$ be the DG Lie algebra

$$\longrightarrow 0 \longrightarrow \overset{(-1)}{\mathcal{T}_{X/S}} \longrightarrow \overset{(0)}{\mathcal{T}_{\pi}} \longrightarrow 0 \longrightarrow$$

Let $\Omega^{\circ}_{X/S} = (\mathcal{O}_X \longrightarrow \omega)$, $\omega = \Omega^1_{X/S}$ be the relative DR complex.

Notice that $\Omega^{\circ}_{X/S}$ is quis $\pi^{-1}\mathcal{O}_S$ and $\mathcal{T}^{\circ}_{\pi}$ is quis to $\pi^{-1}\mathcal{T}_S$. The following is then similar in spirit to the extension

$$0 \to \mathcal{O}_S \to \mathcal{A}_{\lambda_E} \longrightarrow \mathcal{T}_S \longrightarrow 0.$$

Def. A <u>π-algebra</u> $\mathcal{A}^{\circ}$ is a sheaf of DG Lie algebras on X with $\pi^{-1}\mathcal{O}_S$-module structure together with a 3 step filtration

$$0 = \mathcal{A}^{\circ}_{-3} \subset \mathcal{A}^{\circ}_{-2} \subset \mathcal{A}^{\circ}_{-1} \subset \mathcal{A}^{\circ}_0 = \mathcal{A}^{\circ}$$

$$\underbrace{\mathcal{A}^{\circ}_{-3} \subset \mathcal{A}^{\circ}_{-2}}_{\Omega^{\circ}_{X/S}[2] \cong} \quad \underbrace{\mathcal{A}^{\circ}_{-2} \subset \mathcal{A}^{\circ}_{-1}}_{\text{acyclic}} \quad \underbrace{\mathcal{A}^{\circ}_{-1} \subset \mathcal{A}^{\circ}_{0}}_{\cong \mathcal{T}^{\circ}_{\pi}}$$

satisfying various compatibility conditions on the Lie, filtration, $\mathcal{O}_S$-module structures.

Given a π-algebra $\mathcal{A}^{\circ}$ we can apply the functor $R^0\pi_* = H^0 \circ R\pi_*$ where π is proper to get an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow R^0\pi_*(\mathcal{A}^{\circ}) \longrightarrow \mathcal{T}_S \longrightarrow 0$$

(not clear in the holom. context, but perhaps OKAY in the algebra case : Why can't the other cohomology groups $R^{\circ}\pi_*(\pi^{-1}\mathcal{O}_S)$, $R^{\circ}\pi_*(\pi^{-1}\mathcal{T}_S)$ interfere?)

So the program is now ~~to~~ take the Atiyah extension $\mathcal{A}_E$ and refine it to a π-algebra, ~~[illegible]~~. Recall we have sheaves of Lie algebras and $\mathcal{O}_X$-bimodules

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow \mathcal{A}_E \xrightarrow[\varepsilon]{} \mathcal{T}_X \longrightarrow 0$$

on X. Set

$$\mathcal{A}_{E,\pi} = \varepsilon^{-1}(\mathcal{T}_\pi)$$

$$\mathcal{A}_{E/S} = \varepsilon^{-1}(\mathcal{T}_{X/S})$$

whence we have a DG Lie alg. $\mathcal{A}^{\bullet}_{\pi}$

$$\mathcal{A}_{E/S} \longrightarrow \mathcal{A}_{E,\pi}$$

with an [redacted] augmentation $\mathcal{A}^{\bullet}_{E,\pi} \longrightarrow \mathcal{T}^{\bullet}_{\pi}$ which is surjective with the acyclic kernel $\mathrm{Cone}(\mathrm{id}_{\mathrm{End}(E)})$.

Now we define an extension of $\mathcal{A}^{\bullet}_{E,\pi}$ by $\Omega^{\bullet}_{X/S}[2]$ called the <u>trace Ω-extension</u> of $\mathcal{A}_E$ and denoted ${}^{tr}\mathcal{A}^{\bullet}_E$.

$$\left({}^{tr}\mathcal{A}^{\bullet}_E\right)^0 = \mathcal{A}_{E,\pi}$$

$$\left({}^{tr}\mathcal{A}^{\bullet}_E\right)^{-2} = \mathcal{O}_X$$

In degree -1 we will have an extension

$$0 \longrightarrow \omega \longrightarrow \left({}^{tr}\mathcal{A}^{\bullet}_E\right)^{-1} \longrightarrow \mathcal{A}_{E/S} \longrightarrow 0$$

where the middle is defined subtly as a subquotient of jets $E \boxtimes E^{\circ}(2\Delta)/E \boxtimes E^{\circ}(-\Delta)$, $E^{\circ} = E^{*} \otimes \omega$.

The problem for me is now to understand this construction. It turns out to be interesting already in the case where $S = pt$, and it gives the Toledo – Tong proof of the 1-dimensional R-R. thm.

When $S = pt$ we have

$$\begin{tikzcd} \mathcal{T}_{X|S} \arrow[r, equal] \arrow[d, equal] & \mathcal{T}_X \arrow[r] \arrow[d, equal] & \pi^*\mathcal{T}_S = 0 \\ \mathcal{T}_{X|S} \arrow[r, equal] & \mathcal{T}_\pi & \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{End}(E) \arrow[r] & \mathcal{A}_E \arrow[r, "\varepsilon"] \arrow[d, equal] & \mathcal{T}_X \arrow[r] \arrow[d, equal] & 0 \\ & & \mathcal{A}_{E,\pi} \arrow[r] \arrow[d, equal] & \mathcal{T}_\pi \arrow[d, equal] & \\ & & \mathcal{A}_{E,S} \arrow[r] & \mathcal{T}_{X|S} & \end{tikzcd}$$

Thus we are looking for an extension

$$0 \longrightarrow \omega \longrightarrow \left(\mathrm{tr}\,\mathcal{A}^\circ_E\right)^{-1} \longrightarrow \mathcal{A}_E \longrightarrow 0$$

The Virasoro algebra of E is a central extension of the Lie algebra $\mathcal{A}_E$ by $\mathcal{H} = \omega/d\mathcal{O}$, where one works in the Zariski topology. This means that an open set U is of the form $U = X - \{x_1, \ldots, x_n\}$

We have the exact sequence of Zariski sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_X \xrightarrow{d} \omega \longrightarrow \mathcal{H} \longrightarrow 0$$

i.e. a triangle

$$\mathbb{C} \longrightarrow (\mathcal{O}_X \to \omega) \longrightarrow \mathcal{H}[-1] \longrightarrow 0$$

This gives

$$\begin{array}{ccccccc} H^1(U,\mathbb{C}) & \longrightarrow & H^1_{DR}(U) & \xrightarrow{\sim} & \mathcal{H}(U) & \longrightarrow & H^2(U,\mathbb{C}) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

since a constant sheaf is flask in the Zariski top. Here $U = X$. By Grothendieck's thm. $H^1_{DR}(U)$ is the usual $H^1(U,\mathbb{C})$, so that we have an exact sequence

$$0 \longrightarrow H^1_{DR}(X) \longrightarrow H^1_{DR}(U) \xrightarrow{(\mathrm{res}_{x_i})} \mathbb{C}^n \xrightarrow{\Sigma} \mathbb{C} \longrightarrow 0$$

To get further we have to bring in parametrices.

Let's try to reconstruct Toledo-Tong's proof of Riemann-Roch over a curve. What are the operators on $H^*(X,E)$? By Kunneth and Serre duality (for an n-diml proj. variety)

$$H^{\cdot}(X\times X, E \boxtimes E^*\otimes\omega[n]) = H^{\cdot}(X,E) \otimes \underbrace{H^{\cdot}(X, E^*\otimes\omega[n])}_{H^{\cdot}(X,E)^*}$$

Thus $\boxed{\mathrm{End}\left(H^{\cdot}(X,E)\right) = H^n(X\times X, E\boxtimes(E^*\otimes\omega))}$

Now $H^n_\Delta(X\times X, E\boxtimes(E^*\otimes\omega)) = \mathrm{Diff}(E)$. To see this one uses the local cohomology theory:

$$H^n_\Delta(X\times X, E\boxtimes(E^*\otimes\omega)) = H^0(X, \mathcal{H}^n_\Delta(X\times X, E\boxtimes(E^*\otimes\omega)))$$

and a further result

$$\mathcal{H}^n_\Delta(X\times X, E\boxtimes(E^*\otimes\omega)) = \varinjlim_i \mathcal{E}xt^n_{\mathcal{O}_{X\times X}}(\mathcal{O}_{X\times X}/\mathcal{J}^i_\Delta, E\boxtimes(E^*\otimes\omega))$$

+ further calculation.

Let's go back to curves. Then the sheaf of differential operators on E is

$$\mathcal{D}_E = \mathcal{H}^1_\Delta(X\times X, E \boxtimes E^\circ) \qquad E^\circ = E^* \otimes_{\mathcal{O}_X} \omega$$

$$= \varinjlim_i \mathcal{E}xt^1_{\mathcal{O}_{X\times X}}(\mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(-i\Delta), E\boxtimes E^\circ)$$

$$= \varinjlim_i \mathcal{O}_{X\times X}(i\Delta) \otimes_{\mathcal{O}_{X\times X}} E \boxtimes E^\circ \Big/ E \boxtimes E^\circ$$

$$= E \otimes_{\mathcal{O}_X} (\mathcal{O}_X \boxtimes \omega)(\infty\Delta) \otimes_{\mathcal{O}_X} E^* \Big/ E \boxtimes E^\circ$$

In the curve case Δ is a divisor in $X\times X$, so it's natural to consider all of the line bundles

$$\cancel{P_E = E \otimes_{\mathcal{O}_X} (\mathcal{O}_X \boxtimes \omega)(\infty\Delta) \otimes \mathcal{O}_X}$$

$\mathcal{O}_{X\times X}(n\Delta)$ and this leads to formal ψDO's.

So in the same way that the sheaf of differential operators on the trivial bundle is

$$\mathcal{D} = \mathcal{O}_X \boxtimes \omega(\infty\Delta)/\mathcal{O}_X \boxtimes \omega$$

the sheaf of formal ψDO is

$$P = \varprojlim_i \mathcal{O}_X \boxtimes \omega(\infty\Delta)/\mathcal{O}_X \boxtimes \omega(-i\Delta)$$

Set
$$P_{ab} = \mathcal{O}_X \boxtimes \omega((a+1)\Delta)/\mathcal{O}_X \boxtimes \omega((b+1)\Delta)$$

$$P_a = P_{a,-\infty} \qquad \text{etc.}$$

It seems that there are problems with identifying P with formal pseudo-differential operators in dimension 1 as I have been accustomed. Thus there is apparently no non-commutative residue defined on P. P consists of certain kernels

$$\frac{f(t_1,t_2)}{(t_1-t_2)^N}\,dt_2$$

and the log type singularity that occurs with ψDO's on the line, and which the non-commutative residue detects, is missing.

In any case we do have well-defined trace on P_{-1} namely the map

$$P_{-1} \longrightarrow P_{-1}/P_{-2} = \mathcal{O}_X \boxtimes \omega \quad / \mathcal{O}_X \boxtimes \omega(-\Delta) = \omega$$

March 4, 1988

To describe the Virasoro algebra $\hat{A}_E$ of (X,E) where X is a curve and E is a vector bundle on it. It is a central extension of the Atiyah algebra

$$0 \longrightarrow \omega/d\mathcal{O} \longrightarrow \hat{A}_E \longrightarrow A_E \longrightarrow 0$$

which comes from a central extension of Lie algebras

$$0 \longrightarrow \omega/d\mathcal{O} \longrightarrow \hat{\mathcal{D}}_E \longrightarrow \mathcal{D}_E \longrightarrow 0$$

There's some mystery here with the sheaf $\omega/d\mathcal{O}$. What in fact happens is that we have exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_{E,-1}/P_{E,-2} & \longrightarrow & P_E/P_{E,-2} & \longrightarrow & P_E/P_{E,-1} & \longrightarrow & 0 \\
 & & \| & & & & & & \\
 & & E\otimes E^*\otimes\omega & & \downarrow & & \| & & \\
 & & \downarrow{\scriptstyle tr} & & & & & & \\
0 & \longrightarrow & \omega & \longrightarrow & \text{"}\tilde{\mathcal{D}}_E\text{"} & \longrightarrow & \mathcal{D}_E & \longrightarrow & 0
\end{array}$$

but one doesn't get a well-defined bracket on $\tilde{\mathcal{D}}_E$ until one divides out by differentials of functions. In the algebraic case one doesn't lose anything, see p. 558.

Further structure on $\mathcal{P}$ – it's a bimodule over $\mathcal{D}_{\mathcal{O}_X}$ with formulas one expects for the way kernels compose, so that

$$\left(A(t)\partial_t + B(t)\right)\cdot \psi(t_1,t_2)\,dt_2 = \left(A(t_1)\partial_{t_1} + B(t_1)\right)\psi(t_1,t_2)\,dt_2$$

$$\psi(t_1,t_2)\,dt_2 \cdot \left(A(t)\partial_t + B(t)\right)$$
$$= \left[\psi(t_1,t_2)B(t_2) - \partial_{t_2}\left(\psi(t_1,t_2)A(t_2)\right)\right]dt_2$$

Now I ought to be able to compute the bracket in the Virasoro algebra $\hat{\mathcal{A}}(\mathcal{O}_X)$. Take two elements - each element is a triple

$$(\chi, B, \nu) \Longleftrightarrow \left(\frac{\chi(t_1)}{(t_2-t_1)^2} + \frac{B(t_1)}{(t_2-t_1)} + \nu\right)dt_2$$

The bracket is arranged so that

$$[a,b]_{\text{Virasoro}} = [\delta a, b] \quad \text{in } \hat{\mathcal{A}}(\mathcal{O}_X)$$

i.e. ad(a) is ad of the differential operator associated to $a \in \mathcal{P}$.

Take two elements (χ_i, B_i, ν_i) $i=1,2$ as above

$$\left(\chi_1(t_1)\partial_{t_1} + B_1(t_1)\right)\left(\frac{\chi_2(t_1)}{(t_2-t_1)^2} + \frac{B_2(t_1)}{t_2-t_1} + \nu_2(t_1)\right)dt_2$$

This calculation seems pointless at the present time. A more interesting question is why is P a $\mathcal{D}$-bi-module? Is it possible to define a composition on P, motivated by composition of operators? Maybe a better question would be whether there is an algebra of formal ψDO's on a Riemann surface, and if so what is the relation of P to it?

P is to be thought of as a sheaf of parametrices, probably that's what the letter means.

Let's consider formal ψDO's on a Riemann surface. In terms of a local coordinate t these would appear as formal series

$$\sum_{n=-N}^{\infty} a_n(t)\, \partial_t^{-n}$$

There's a composition law which is a suitable generalization of Leibniz's formula.

Another way to describe this algebra on X is to use the sheaf of formal micro-differential operators $\hat{\mathcal{E}}_X$ on T^*X. (See book of Schapira). We consider $\mathcal{E}_X(p)$ = sheaf of homogeneous functions on T^*X of degree p, and then use the Leibniz business to define a product on

$$\hat{\mathcal{E}}_X = \bigcup_n \prod_{p \le n} \mathcal{E}_X(p)$$

Then we restrict to $T^*X - X$ and apply π_* where $\pi : T^*X - X \longrightarrow X$. This gives an interesting algebra when $\dim(X) = 1$.

In any case it seems clear that we have nice sheaf of formal ψDO's on a Riemann surface (and perhaps something more if one takes advantage of "analytic convergence arguments" one uses to prove Cauchy-Kowalewski). Now the question is what's the relation between these formal ψDO's and P.

Recall that P consists of kernels

$$\left(\cdots + \frac{a_0(x)}{(y-x)} + a_0(x) + a_1(x)(y-x) + \cdots \right) dy$$

where $\dfrac{a(x)}{(y-x)^{n+1}}$ acts as

$$\operatorname{res}_{y=x} \left\{ \frac{a(x)}{(y-x)^{n+1}} f(y)\, dy \right\} = \frac{1}{n!} a(x)\, \partial_x^n f(x)$$

Let's try putting in a log term formula

$$\operatorname{res}_{y=x} \left\{ a(x) \log(y-x) f(y)\, dy \right\}$$

$$= \operatorname{res}_{y=x} \left\{ \frac{a(x)}{y-x} \left(-\int^y f \right) dy \right\} = -\int^x f = -\partial_x^{-1} f$$

Therefore formally it appears that the space P maps to the algebra of formal ψDO's, and the image is the differential operators. But then comes the non-commutative residue.

It's becoming clear that I really want to examine the local situation, especially

the Tate operator picture.

March 5, 1988:

Let's discuss the RR theorem on a complete n.s. curve X from the viewpoint of duality and intersections. Ultimately I want to look at the Toledo-Tong paper.

Let start with a differential operator $\partial : E \to E$, for example the identity. This induces a map on cohomology $H^{\cdot}(X,E)$ and we want to calculate the traces. Grothendieck's theory gives us the following way to proceed.

First of all there is the identification of the sheaf of differential operators $\mathcal{D}_E$ with local homology sheaves

$E^0 = E^* \otimes \omega$

1) $$\mathcal{H}^1_\Delta(E \boxtimes E^0) = \mathcal{D}_E$$

Next the passage from local to global

2) $$\Gamma(X, \mathcal{H}^1_\Delta(E \boxtimes E^0)) \longleftarrow H^1_\Delta(X \times X, E \boxtimes E^0)$$

Then we have the maps

3) $$H^1_\Delta(X \times X, E \boxtimes E^0) \longrightarrow H^1(X \times X, E \boxtimes E^0)$$

4) $$H^1(X \times X, E \boxtimes E^0) \underset{\text{Kunneth}}{\cong} \Big(H^{\cdot}(X,E) \otimes \underbrace{H^{\cdot}(X,E^0)}_{H^{\cdot}(X,E)^*}\Big)^1$$ Serre duality
$$= \mathrm{End}^0(H^{\cdot}(X,E)).$$

Combining these gives a map $\mathcal{D}_E(X) \to \mathrm{End}^0(H^{\cdot}(X,E))$ which has to be the induced map of a diff op.

on cohomology.

On the other hand we have the map

$$5)\quad H^1(X\times X, E\boxtimes E^\circ) \xrightarrow{\Delta^*} H^1(X, \mathrm{End}(E)\otimes\omega) \xrightarrow{tr} H^1(X,\omega) \xrightarrow{\int} \mathbb{C}$$

which gives the trace of an elt. of $\mathrm{End}^\circ(H^\cdot(X,E))$. This part is obvious from the nature of Kunneth and Serre duality.

Thus to compute the trace of ∂ on cohomology, we must represent it by an element of $H^1_\Delta(X\times X, E\boxtimes E^\circ)$, then apply 3), 5). So for everything holds in higher dimensions with suitable modifications.

Now we have

$$\mathcal{D}_E = \mathcal{H}^1_\Delta (E\boxtimes E^\circ) \xleftarrow{\sim} \varinjlim_n \underbrace{\mathrm{Ext}^1_{\mathcal{O}_{X\times X}}\left(\mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(-i\Delta), E\boxtimes E^\circ\right)}_{\cong\; E\boxtimes E^\circ(i\Delta)/E\boxtimes E^\circ}$$

which gives an exact sequence

$$6)\quad \boxed{0 \to E\boxtimes E^\circ \to E\boxtimes E^\circ(\infty\Delta) \xrightarrow{\delta} \mathcal{D}_E \to 0}$$

which I could have written down at the very beginning. The map δ uses the local residue.

This sequence immediately gives a map

$$\mathcal{D}_E(X) \longrightarrow H^1_\Delta(X\times X, E\boxtimes E^\circ)$$

which must be 2). It would be nice to see on an elementary level why this map composed with 3), 4) gives the action of differential operators on cohomology.

In any case we can now apply 3), 5) to get the trace. From 6) we get

$$\begin{tikzcd}
0 \arrow[r] & E\boxtimes E^\circ / E\boxtimes E^\circ(-\Delta) \arrow[r] \arrow[d, equal] & E\boxtimes E^\circ(\infty\Delta)/E\boxtimes E^\circ(-\Delta) \arrow[r] & \mathcal{D}_E \arrow[r] & 0 \\
& E\otimes E^\circ \arrow[d, "tr_E"] & & & \\
& \omega & & &
\end{tikzcd}$$

which defines an extension of $\mathcal{D}_E$ by ω. Following Beilinson + Schechtman, one can go further and form the induced extension

$$0 \longrightarrow \omega/d\mathcal{O} \longrightarrow \hat{\mathcal{D}}_E \longrightarrow \mathcal{D}_E \longrightarrow 0$$

which turns out to be a central extension of Lie algebras. We now see why their assertion that the boundary map

$$\mathcal{D}_E(X) \longrightarrow H^1(X, \omega/d\mathcal{O}) = H^2_{DR}(X) = \mathbb{C}$$

gives the trace of a diff'l operator on cohomology.

But actually we should check. We have the triangle

$$\mathbb{C} \longrightarrow (\mathcal{O} \xrightarrow{d} \omega) \longrightarrow \omega/d\mathcal{O}[-1]$$

so

$$H^1_{DR}(X) \xrightarrow{\sim} H^0(X, \omega/d\mathcal{O})$$
$$H^2_{DR}(X) \xrightarrow{\sim} H^1(X, \omega/d\mathcal{O})$$

(cohomology relative to Zariski topology).

March 8, 1988.

[redacted] Let M be an A-module which is finite-dimensional over k. Then we have constructed [redacted] maps

1) $$\mathrm{Ext}^n_A(M,M) \longrightarrow \mathrm{Hom}\,(H_n(A,A),k)$$

using the trace on $\mathrm{End}(M)$. [redacted] (These maps were used by Cartier to construct the Grothendieck residue symbol, I believe.)

I would like to generalize the construction to the case where M is finite projective over over an algebra B. Thus I want M to be an $A \otimes B^\circ$-module which is finite projective over B. Then we have a map

$$A \longrightarrow \mathrm{End}_{B^\circ}(M) \xrightarrow{tr_M} B/[B,B]$$

which vanishes on $[A,A]$, hence induces a k-linear map

$$A/[A,A] \longrightarrow B/[B,B]$$

But the generalization of 1) for $n=0$ is [redacted] the map

$$\begin{array}{c} \mathrm{Hom}_{A\otimes B^\circ}(M,M) \longrightarrow \mathrm{Hom}\left(A/[A,A]\;,\; B/[B,B]\right) \\ \| \\ \mathrm{Hom}_{B^\circ}(M,M)^A \end{array}$$

which sends $f \longmapsto (a \mapsto tr_M(af))$

Let's next consider the case $n=1$. Start with an $A\otimes B^\circ$-module extension:

$$0 \longrightarrow M \xrightarrow{i} E \underset{\leftarrow h}{\longrightarrow} M \longrightarrow 0$$

Since M is B-projective, we can choose a B-linear splitting h. Set

$$\varphi(a) = i^{-1}[a,h] \in \mathrm{End}_{B^\circ}(M)$$

Then φ is a derivation of A with values in $\mathrm{End}_{B^\circ}(M)$ considered as an A-bimodule in the obvious way. Changing $h \to h + iu$ where $u \in \mathrm{End}_{B^\circ}(M)$, changes $\varphi(a)$ to

$$i^{-1}[a, h+iu] = \underbrace{i^{-1}[a,h]}_{\varphi(a)} + \underbrace{i^{-1}[a, iu]}_{[a,u]} = (\delta u)(a).$$

Thus we get well-defined map

$$\mathrm{Ext}^1_{A\otimes B^\circ}(M,N) \longrightarrow H^1(A, \mathrm{Hom}_{B^\circ}(M,N))$$

when M is B°-projective. This map is an isomorphism, because $\mathcal{B}^N(A)\otimes_A M$ is a projective $A\otimes B^\circ$-module resolution of M. Thus we have more generally

$$\begin{aligned}\mathrm{Ext}^n_{A\otimes B^\circ}(M,N) &= H^n\{\mathrm{Hom}_{A\otimes B^\circ}(\mathcal{B}^N(A)\otimes_A M, N)\} \\ &= H^n\{\mathrm{Hom}_{A\otimes A^\circ}(\mathcal{B}^N(A), \mathrm{Hom}_{B^\circ}(M,N))\} \\ &= H^n(A, \mathrm{Hom}_{B^\circ}(M,N))\end{aligned}$$

Next suppose M finitely generated as well as projective over B°. Then we have the trace map

$$\mathrm{End}_{B^\circ}(M) \xrightarrow{\mathrm{tr}_M} B/[B,B]$$

which kills $[A, \mathrm{End}_{B^\circ}(M)]$. Thus we get a map

$$\mathrm{Ext}^n_{A\otimes B^o}(M,M) \xrightarrow{\sim} H^n\left(\mathrm{Hom}_{A\otimes A^o}(\mathcal{B}^N(A), \mathrm{End}_{B^o}(M))\right)$$

$$\longrightarrow H^n\left(\mathrm{Hom}_k(\mathcal{B}^N(A)\otimes_A , \mathrm{End}_{B^o}(M)\otimes_A)\right)$$

$$\longrightarrow H^n\{\mathrm{Hom}_k(\mathcal{B}^N(A)\otimes_A , B/[B,B])$$

$$= \mathrm{Hom}_k(H_n(A,A), B/[B,B]).$$

Notice that the formulas will be the same as before, namely if the element of Ext^n is represented by an exact sequence with splittings, then the cocycle is

$$\varphi(a_0,\dots,a_n) = \mathrm{tr}_M\left(a_0[a_1,h]\cdots[a_n,h]\right)$$

Next let's consider residues over an algebraic curve X. I want to understand how the residue construction yields differential operators from sections over $X\times X$ with poles along Δ. So we want the isomorphism

$$\mathcal{E}xt^1_{\mathcal{O}_{X\times X}}\left(\mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(-n\Delta), E\boxtimes F^o\right) \qquad F^o = F^*\otimes_{\mathcal{O}_X}\omega$$

$$= \mathcal{H}om_{\mathcal{O}_X}\left(J_{n-1}(F), E\right)$$

It should be enough to understand the case where $E=F=\mathcal{O}_X$. Thus we want an isomorphism

$$\mathcal{O}_X\boxtimes\omega(n\Delta)/\mathcal{O}_X\boxtimes\omega = \mathcal{H}om_{\mathcal{O}_X}\left(\mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(-n\Delta), \mathcal{O}_X\right)$$

i.e. a pairing

$$\left(\mathcal{O}_X \boxtimes \omega(n\Delta)/\mathcal{O}_X \boxtimes \omega\right) \times \left(\mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(-n\Delta)\right) \longrightarrow \mathcal{O}_X$$

Let's get the map straight in the case where $E = \mathcal{O}_X$, $F = \omega$ so that we want to associate to principal part of merom. function along Δ a differential operator $\partial : \omega \longrightarrow \mathcal{O}_X$. So we want a map

$$\mathcal{O}_{X\times X}(\infty\Delta)/\mathcal{O}_{X\times X} \longrightarrow \mathrm{Diff}(\omega, \mathcal{O}_X)$$

This map will be given by sending [redacted] $\dfrac{f(x,y)}{(y-x)^n}$ and $g(y)\,dy$ to $\operatorname{Res}_{y=x}\left(\dfrac{f(x,y)g(y)\,dy}{(y-x)^n}\right)$.

To define the residue we proceed as follows. First we identify

$$\mathcal{O}_{X\times X}(n\Delta)/\mathcal{O}_{X\times X} = \mathrm{Ext}^1_{\mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(-n\Delta)}\left(\mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(-n\Delta), \mathcal{O}_{X\times X}\right)$$

We have the canonical extension

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{X\times X}(-n\Delta)/\mathcal{O}_{X\times X}(-2n\Delta) \arrow[r] \arrow[d, "\xi"] & \mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(-2n\Delta) \arrow[r] & \mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(-n\Delta) \arrow[r] \arrow[l, bend right, dashed, "h"'] & 0 \\ & \mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(n\Delta) & & & \end{tikzcd}$$

and there's a map induced by $\xi \in \mathcal{O}_{X\times X}(n\Delta)/\mathcal{O}_{X\times X}$.

Choose the splitting h to be linear over $\mathcal{O}_X \otimes \mathbb{C}$. Then for any $g \in \mathcal{O}_{X\times X}$, we consider the operator

$$\xi\,[g, h] \quad \text{on} \quad \mathcal{O}_{X\times X}/\mathcal{O}_{X\times X}(-n\Delta)$$

and we take its trace over $\mathcal{O}_X \otimes \mathbb{C}$. This trace is the $\operatorname{Res}_{y=x}(\xi\, d_y g)$. Note that it depends on $dg \in \Omega^1_{X\times X/X\times pt} = \mathcal{O}_X \boxtimes \omega$

This is all a bit confusing and it's not immediately clear how to link it with the situation on p 568. This is the problem with Cartier's procedure, which perhaps improves with Tate's versions. My interest in Tate's version is because it gives rise to a cyclic cocycle $\mathrm{res}(f\,dg)$ unlike the Hochschild cocycles obtained so far.

I have to go back to the cyclic 1-cocycle occurring for Toeplitz operators on the circle. Thus suppose we have an extension

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & R \arrow[r] & A \arrow[r] \arrow[l, bend right, dashed, "\rho"'] & 0 \end{tikzcd}$$

and a linear functional $\tau : I/[R,I] \longrightarrow \mathbb{C}$. If we choose a linear lifting ρ, then we obtain a cyclic 1-cocycle (normalized) on A given by

$$\varphi(a_0,a_1) = \tau\big([\rho(a_0),\rho(a_1)] - \rho[a_0,a_1]\big)$$

In the circle case, we have $A = C^\infty(S^1)$ acting on $L^2(S^1) = H^+ \oplus H^-$, and R is an algebra of operators on H^+ containing the Toeplitz operators $T_f = PfP$ where $P =$ orthogonal projection on H^+. The lifting ρ is then $\rho(f) = PfP$, and the cyclic cocycle is

$$\varphi(f,g) = \mathrm{tr}_{H_+}\Big([PfP, PgP] - P\overbrace{[f,g]}^{\text{can be } \neq 0 \text{ for matrix functions}}P\Big)$$

Now
$$\begin{aligned}[PfP, PgP] &= PfPgP - PgPfP \\ &= Pf\big(gP + [P,g]\big)P - P\big(Pg - [P,g]\big)fP\end{aligned}$$

Thus

$$[PfP, P_gP] = Pf[P,g]P + P[P,g]fP + P[f,g]P$$

Now

$$\operatorname{tr}(P[P,g]fP) = \operatorname{tr}([P,g](1-P)fP)$$
$$= \operatorname{tr}((1-P)fP[P,g]) = \operatorname{tr}((1-P)f[P,g](1-P))$$

simpler is

$$\operatorname{tr}(P[P,g]fP) = \operatorname{tr}(P[P,g]f) = \operatorname{tr}([P,g](1-P)f)$$
$$= \operatorname{tr}((1-P)f[P,g]).$$

So we get

$$\varphi(f,g) = \operatorname{tr}(Pf[P,g]) + \operatorname{tr}((1-P)f[P,g])$$
$$= \operatorname{tr}(f[P,g])$$

~~Let's discuss the Tate method as explained by Beilinson–Schechtman. Let V = ℂ((t))ⁿ, = E(ů), E(U) = ℂ[[t]]ⁿ~~

Let's go over the Tate method as presented by Beilinson–Schechtman. $U = \operatorname{Spec} \mathbb{C}[[t]]$, $\mathring{U} = \operatorname{Sp}(\mathbb{C}((t)))$, E vector bundle on U, $F = E(\mathring{U}) \cong \mathbb{C}((t))^n$ $\supset E(U) \cong \mathbb{C}[[t]]^n$. Call a $\mathbb{C}$-linear subspace V bounded if $V \subset t^{-N}E(U)$ some N and open if $V \supset t^N E(U)$ for some N; lattice = bounded + open subspace. Let $R \subset \operatorname{End}_{\mathbb{C}}(F)$ be the alg. of continuous operators ($T \in R \iff$ (V open $\Rightarrow T^{-1}(V)$ open)), let $I_0 = \{A \in R \mid \operatorname{Im} A \text{ bounded}\}$, $I_1 = \{A \in R \mid \operatorname{Ker} A \text{ open}\}$. Then I_0, I_1 are ideals in R such that $I_0 + I_1 = R$. Also a trace is defined on $I_0 \cap I_1$ such that

$$\operatorname{tr}(ab) = \operatorname{tr}(ba) \qquad a \in I_0, \quad b \in I_1$$

This apparently is not what I want although it suggests an approach, namely one has an R-bimodule extension

$$0 \longrightarrow I_0 \cap I_1 \longrightarrow I_0 \oplus I_1 \longrightarrow R \longrightarrow 0$$

and a trace on the kernel. This leads to a linear functional on $H_1(R,R)$, in fact to a cyclic 1-cocycle class, which in turn gives a central extension $\hat{R}$ of R by $\mathbb{C}$. B+S show this leads to the Virasoro extension $\hat{\mathcal{D}}_E(\dot{U})$ of $\mathcal{D}_E(\dot{U})$ by pulling back via the embedding $\mathcal{D}_E(\dot{U}) \subset R$.

So let's take up this idea that the residue is the linear functional on $\hat{\Omega}^1_A$ associated to an A-bimodule extension

$$0 \longrightarrow K \longrightarrow Q \xrightarrow{\pi} A \longrightarrow 0$$

together with a trace τ on $K \otimes_A$. Such an extension with trace ought to be easy to find. Since we want to obtain a 1-cocycle from this we want to have a bimodule map $A \otimes A^\circ \longrightarrow Q$ which means an element of Q.

It seems that we want to take Q to consist of operators which are linear combinations of $a_1 P a_2$ for $a_1, a_2 \in A$ and smooth kernel operators. The map π takes the symbol of this ψDO and restricts to $\xi > 0$, so it gives $a_1 P a_2 \longrightarrow a_1 a_2$.

Let's compute the cocycle by mapping $B^N(A)$ into the above resolution.

$$\begin{tikzcd}
0 \arrow[r] & \hat{\Omega}^1_A \arrow[r] \arrow[d, "f_1"'] & A\otimes A \arrow[r] \arrow[d, "f_0"', "\substack{(a_0,a_1)\\ \downarrow \\ a_0 P a_1}"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r] & Q \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

$$f_1(da) = f_0(a\otimes 1 - 1\otimes a) = aP - Pa = [a,P]$$

and so the [redacted] linear functional on $\hat{\Omega}^1_A$ is
$$a_0\, da_1\, a_2 \longmapsto tr\left(a_0[a_1,P]a_2\right)$$
which means that we have the Hochschild cocycle
$$\varphi(a_0,a_1) = tr\left(a_0[a_1,P]\right)$$

Relation with the Beilinson-Schechtman version. Notice that $A \subset R$ where $A \cong M_n(\mathbb{C}((t)))$, i.e. we are looking at $End(E)(\mathring{U})$. In fact $\mathcal{D}_E(\mathring{U}) \subset R$. Next note that P = the projection on $E(U) = \mathbb{C}[[t]]^n$ is in I_0, whereas $1-P$ is in I_1. So to obtain the cocycle for their extension [redacted] we can use the lifting

$$\begin{tikzcd}
0 \arrow[r] & \hat{\Omega}^1_R \arrow[r] \arrow[d, "f_1"] & R\otimes R \arrow[r] \arrow[d, "f_0"] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_0\cap I_1 \arrow[r] & I_0\oplus I_1 \arrow[r] & R \arrow[r] & 0
\end{tikzcd}$$

with $f_0(r_0,r_1) = r_0\begin{pmatrix} P \\ 1-P\end{pmatrix} r_1$. Then

$$f_1(dr) = i^{-1}\left[r, \begin{pmatrix} P \\ 1-P\end{pmatrix}\right] = i^{-1}\begin{pmatrix} [r,P] \\ -[r,P]\end{pmatrix} = [r,P]$$

so we [redacted] get the same cocycle. [redacted]

Now lots of questions arise. It seems that we have lots of extensions of A that all give rise to the same cocycle. First of all there is the Toeplitz extension. This is an extension of algebras and it is only by manipulation that we can get from the formula $\mathrm{tr}\left([\rho(f),\rho(g)] - \rho[f,g]\right)$ to the formula $\mathrm{tr}\left(f[g,P]\right)$. This transition seems to be important, and I suspect a proper understanding would link cyclic cocycles to non-commutative differential forms as Connes has done. It might be related to the Stinespring construction.

But there are also two bimodule extensions which lead to the same cocycle. The bimodule Q is even an algebra, but it's non-unital.

March 10, 1988

Back to Beilinson + Schechtman. Let X be a complete non-singular curve over $\mathbb{C}$, E a vector bundle on X. $\mathcal{D}_E = \mathcal{D}iff(E,E)$ sheaf of differential operators on E. Atiyah extension:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{D}iff_0(E,E) & \longrightarrow & \mathcal{D}iff_{\leq 1}(E,E) & \longrightarrow & \mathcal{H}om_{\mathcal{O}_X}(\Omega^1 \otimes_{\mathcal{O}_X} E, E) & \longrightarrow 0 \\
 & & & & & \shortparallel & \\
 & \| & & \cup & & \mathcal{E}nd(E) \otimes \mathcal{T}_X & \\
 & & & & & \cup \; id_E \otimes ? & \\
0 \longrightarrow & \mathcal{E}nd(E) & \longrightarrow & \mathcal{A}_E & \longrightarrow & \mathcal{T}_X & \longrightarrow 0
\end{array}$$

$\mathcal{A}_E$ = sheaf of infinitesimal autos. of (X, E).

$E^\circ = \underbrace{E^*}_{\mathcal{H}om_{\mathcal{O}_X}(E,\mathcal{O}_X)} \otimes \underbrace{\omega}_{\Omega^1_X}$. Have canonical isom

$$\boxed{E \boxtimes E^\circ((k+1)\Delta) / E \boxtimes E^\circ = \mathcal{D}iff_{\leq k}(E,E)}$$

given by residue: $E \simeq \mathcal{O}_X^r$, t local coord on X

$x = t_1,\ y = t_2$

$$\psi(x,y)\,dy = \sum_{n=0}^{k} \frac{a_n(x)}{(y-x)^{n+1}}\,dy$$

$$f(y) \longmapsto \operatorname*{Res}_{y=x} \psi(x,y) f(y)\,dy$$

Virasoro algebra of E is a sheaf of Lie algebras $\hat{\mathcal{A}}_E$ which is a central extension

$$0 \longrightarrow \omega/d\mathcal{O} \longrightarrow \hat{\mathcal{A}}_E \longrightarrow \mathcal{A}_E \longrightarrow 0$$

Actually there is a central extension of $\mathcal{D}_E$ as a Lie algebra. In fact it's better to think of there being a cyclic 1-cocycle class on $\mathcal{D}_E$ as an algebra with values in $\omega/d\mathcal{O}$. Already this should be interesting over the circle.

Let's review the construction of this central extension. Let

$$\mathcal{P}_E = \varinjlim_i E \boxtimes E^\circ(\infty\Delta)/E \boxtimes E^\circ(-i\Delta)$$

$$\mathcal{P}_{E,n} = \varinjlim_i E \boxtimes E^\circ((n+1)\Delta)/E \boxtimes E^\circ(-i\Delta)$$

Then $\mathcal{P}_E$ is a $\mathcal{D}_E$-bimodules, and we have a bimodule extension

$$0 \longrightarrow \mathcal{P}_{E,-1} \longrightarrow \mathcal{P}_E \longrightarrow \mathcal{D}_E \longrightarrow 0$$

On the other hand we have a map

$$\mathcal{P}_{E,-1} \longrightarrow \mathcal{P}_{E,-1}/\mathcal{P}_{E,-2} = E \boxtimes E^\circ/E \boxtimes E^\circ(-\Delta)$$

$$= E \otimes E^\circ \xrightarrow{tr_E} \omega \longrightarrow \omega/d\mathcal{O}$$

I think that $E \otimes E^\circ$ is a $\mathcal{D}_E$-bimodule, so that we have a $\mathcal{D}_E$-bimodule extension

$$0 \longrightarrow \underset{\displaystyle E\otimes E^\circ}{\mathcal{P}_{E,-1}/\mathcal{P}_{E,-2}} \longrightarrow \mathcal{P}_E \longrightarrow \mathcal{D}_E \longrightarrow 0$$

Now certainly

$$E \otimes E^\circ \xrightarrow{tr_E} \omega \longrightarrow \omega/d\mathcal{O}$$

should kill brackets. In fact the way $\mathcal{D}_E$

acts on E° is by assigning to ∂ in $\mathcal{D}_E$ the adjoint operator ${}^t\partial$ on E°. Then the vanishing of brackets is the formula

$$\langle \partial\xi, \eta\rangle - \langle \xi, {}^t\partial\eta\rangle = d(\text{something})$$

At this stage we automatically get a Hochschild cocycle class

$$[P_E/P_{E,-2}] \in \mathrm{Ext}^1_{\mathcal{D}_E\otimes\mathcal{D}_E^{\circ}}(\mathcal{D}_E, E\otimes E^{\circ}) \longrightarrow H^1\{\mathrm{Hom}(\mathcal{B}(\mathcal{D}_E)\otimes_{\mathcal{D}_E}, \omega/d\mathcal{O})\}$$

and one wants to see why it kills exact 1-forms, so that it's cyclic.

It seems to me that everything ought to work globally on the circle, or with analytic functions on an annulus. In this case I should be able to compare P_E with ψDO's on the circle.

An interesting question at this point is how one should think about this Virasoro extension with the kernel $\omega/d\mathcal{O}$. Recall we have

$$\mathbb{C} \longrightarrow (\mathcal{O} \xrightarrow{d} \omega) \longrightarrow \omega/d\mathcal{O}[-1]$$

so in the Zariski topology on our curve, we have

$$0 \longrightarrow H^1_{DR}(U) \xrightarrow{\sim} H^0(U, \omega/d\mathcal{O}) \longrightarrow 0,$$

$$\hookrightarrow H^2_{DR}(U) \longrightarrow H^1(U, \omega/d\mathcal{O}) \longrightarrow 0$$

Question: What is the quantization on the sheaf level?

On an algebraic curve we have these central extensions of sheaves of Lie algebras, which generalize the Heisenberg Lie algebra giving rise to CCRs. For example we can take the central extension of $\mathcal{O}_X$ defined by the cocycle $f,g \longmapsto fdg$. I should study this first algebraically and try to obtain adelic type representations. Then I should try to make sense out of the picture holomorphically.

Let's now try to understand the B+S theorem about $\mathcal{A}_{\lambda_E}$. One has X a family of curves over S. One wants to construct something on X such that applying π_* gives $\mathcal{A}_{\lambda_E}$.

B+S manage to describe $\mathcal{A}_{\pi_* E}$ in the case $R^1\pi_*(E) = 0$. First of all one has

$$
\begin{aligned}
R(\pi\times\pi)_*(E \boxtimes E^0) &= R\pi_*(E) \otimes R\pi_*(E^0) \\
&= R\pi_*(E) \otimes R\pi_*(E)^\vee = End(R\pi_*(E)) \\
&= End(\pi_* E)
\end{aligned}
$$

One defines B_E by

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E\boxtimes E^0 & \longrightarrow & E\boxtimes E^0(2\Delta) & \longrightarrow & Diff_{\leq 1}(E,E) & \longrightarrow & 0 \\
 & & \| & & \cup & \text{cart} & \cup & & \\
0 & \longrightarrow & E\boxtimes E^0 & \longrightarrow & B_E & \longrightarrow & \mathcal{A}_{E/S} & \longrightarrow & 0
\end{array}
$$

March 11, 1988

Here's the general B+S picture. Suppose X/S smooth proper map of rel dim n, $\omega = \Omega^n_{X/S}$. Grothendieck - Sato isomorphism:

$$\Delta \subset X \times X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$$

$$\mathcal{D}_{\mathcal{O}_X/S} = \mathcal{H}^n_\Delta(p_2^*\omega)$$

View this as an isomorphism in the derived cat (of sheaves on $X \times_S X$)

$$\mathcal{D}_{\mathcal{O}_X/S} = R\mathcal{H}^0_\Delta(p_2^*\omega[n])$$

Then one has a canonical map $\mathcal{D}_{\mathcal{O}_X/S} \to p_2^*\omega[n]$ which one can restrict to $\mathcal{A}_{\mathcal{O}_X/S}$, and then $\mathcal{B}$ so as to have a triangle which is the first row in the following

$$\begin{tikzcd}
\mathcal{B} \arrow[r] \arrow[d, equal] & \mathcal{A}_{\mathcal{O}_X/S} \arrow[r] \arrow[d] & p_2^*\omega[n] \arrow[d] \\
\mathcal{B} \arrow[r] & \mathcal{A}_{\mathcal{O}_X,\pi} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
 & \pi^{-1}\mathcal{T}_S \arrow[r, equal] & \pi^{-1}\mathcal{T}_S
\end{tikzcd}$$

$\mathcal{C}$ is defined so that the second row is a triangle. Recall

$$\mathcal{A}^E_{|S} = \text{inf. autos of } (X,E) \text{ over } S$$

$$\mathcal{A}_{E,\pi} = \text{inf. autos of } (\pi, E)$$

$$\mathcal{T}_\pi = \text{inf autos of } \pi$$

so that the second column is a triangle. It follows the the last column is a triangle.

Applying $R(\pi\times\pi)_*$ we get the triangle

$$\begin{array}{ccccc} R(\pi\times\pi)_*(p_2^*\omega[n]) & \longrightarrow & R(\pi\times\pi)_*\mathcal{C} & \longrightarrow & R(\pi\times\pi)_*\left(\overset{\pi^{-1}\mathcal{J}_S}{\mathcal{J}/\pi}\right) \\ \| & & & & \| \\ R\mathcal{E}nd\,(R\pi_*\mathcal{O}_X) & & & & \mathcal{J}_S \end{array}$$

The last step I guess uses the fact $\pi^{-1}\mathcal{J}_S$ is supported on Δ so that it should have been written $\Delta_*\left(\overset{\pi^{-1}\mathcal{J}_S}{\mathcal{J}/\pi}\right)$, and then

$$R(\pi\times\pi)_*\left(\Delta_*\left(\overset{\pi^{-1}\mathcal{J}_S}{\mathcal{J}/\pi}\right)\right) = R\pi_*(\mathcal{J}_\pi)$$

But then $R\pi_*\left(\overset{\pi^{-1}\mathcal{J}_S}{\mathcal{J}/\pi}\right) = \mathcal{J}_S$ I guess because (assumes fibres connected) and you have a constant sheaf on the fibres (?).

In any case they claim

$$R(\pi\times\pi)_*(\mathcal{C}) = \mathcal{A}_{R\pi_*(\mathcal{O}_X)}$$

So let's look at this in rel dimension 1.

First it might be better to observe that one is taking a pushout

$$\begin{tikzcd} \mathcal{A}_{E/S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cocart}"] & E\boxtimes E^\circ[n] \arrow[d] \\ \mathcal{A}_{E,\pi} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \pi^{-1}\mathcal{J}_S \arrow[r, equal] & \pi^{-1}\mathcal{J}_S \end{tikzcd}$$

Recall

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_{X|S} \arrow[r] \arrow[d, phantom, "\parallel"] & \mathcal{T}_X \arrow[r] \arrow[d, phantom, "\cup"] & \pi^*\mathcal{T}_S \arrow[r] \arrow[d, phantom, "\cup"] & 0 \\
0 \arrow[r] & \mathcal{T}_{X/S} \arrow[r] & \mathcal{T}_\pi \arrow[r] & \pi^{-1}\mathcal{T}_S \arrow[r] & 0
\end{tikzcd}$$

$\pi^*\mathcal{T}_S$ not a Lie alg.

ext. of Lie algs

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}nd\, E \arrow[r] \arrow[d, phantom, "\parallel"] & \mathcal{A}_E \arrow[r] \arrow[d, phantom, "\cup"] & \mathcal{T}_X \arrow[r] \arrow[d, phantom, "\cup"] & 0 \\
0 \arrow[r] & \mathcal{E}nd\, E \arrow[r] \arrow[d, phantom, "\parallel"] & \mathcal{A}_{E,\pi} \arrow[r] \arrow[d, phantom, "\cup"] & \mathcal{T}_\pi \arrow[r] \arrow[d, phantom, "\cup"] & 0 \\
0 \arrow[r] & \mathcal{E}nd\, E \arrow[r] & \mathcal{A}_{E/S} \arrow[r] & \mathcal{T}_{X|S} \arrow[r] & 0
\end{tikzcd}$$

$$\therefore \quad \mathcal{A}_{E,\pi}/\mathcal{A}_{E/S} = \mathcal{T}_\pi/\mathcal{T}_{X|S} = \pi^{-1}\mathcal{T}_S$$

Now define $\mathcal{B}_E$ by

$$\begin{tikzcd}
0 \arrow[r] & E \boxtimes E^\circ \arrow[r] \arrow[d, phantom, "\parallel"] & E \boxtimes E^\circ(2\Delta) \arrow[r] \arrow[d, phantom, "\cup"] & \mathcal{D}_{E/S,\le 1} \arrow[r] \arrow[d, phantom, "\cup"] & 0 \\
0 \arrow[r] & E \boxtimes E^\circ \arrow[r] & \mathcal{B}_E \arrow[r] & \mathcal{A}_{E/S} \arrow[r] & 0
\end{tikzcd}$$

Finally consider

$$\mathrm{Cone}\,(\mathcal{B}_E \longrightarrow \mathcal{A}_{E/S} \longrightarrow \mathcal{A}_{E,\pi})$$

so that we have a triangle

$$E \boxtimes E^\circ[1] \longrightarrow \mathrm{Cone(above)} \longrightarrow \pi^{-1}\mathcal{T}_S$$

Now let us assume that $R^1\pi_*(E) = 0$, whence $R\pi_*(E \boxtimes E^\circ[1]) = \mathcal{E}nd\; R\pi_*(E) = \mathcal{E}nd(\pi_* E)$ ~~[illegible]~~ is concentrated in degree zero. Thus we get an exact sequence by applying $R^0(\pi\times\pi)_*$

$$0 \longrightarrow \mathcal{E}nd(\pi_* E) \longrightarrow R^0(\pi\times\pi)_*(\mathrm{Cone}) \longrightarrow \mathcal{T}_S \longrightarrow 0$$

The claim is that this is the Atiyah extension of $\pi_* E$. The proof (which is pretty sketchy) calculates $R^0(\pi\times\pi_*)(\mathrm{Cone})$ using a relative divisor: $T \subset X$ finite over S (which exists locally). Thus the cohomology is represented by certain kinds of "singular part" data which can then be interpreted by residues as operators on $\pi_* E$.

Construction of ${}^{tr}\mathcal{A}^{\bullet}_{E,\pi}$ <u>sheaf of DG Lie algebras</u>

$$\begin{tikzcd}
\left({}^{tr}\mathcal{A}_{E,\pi}\right)^{-2} \arrow[r] \arrow[d, equal] & \left({}^{tr}\mathcal{A}_{E,\pi}\right)^{-1} \arrow[r] \arrow[d, equal] & \left({}^{tr}\mathcal{A}_{E,\pi}\right)^{0} \arrow[d, equal] \\
\mathcal{O}_X & \mathcal{B}_E/\mathrm{Ker}\,\varphi & \mathcal{A}_{E,\pi}
\end{tikzcd}$$

where

$$\begin{tikzcd}
0 \arrow[r] & E\boxtimes E^{\circ} \arrow[r] & E\boxtimes E^{\circ}(2\Delta) \arrow[r] & \mathcal{D}_{E/S,\leq 1} \arrow[r] & 0 \\
0 \arrow[r] & E\boxtimes E^{\circ} \arrow[r] \arrow[u, equal] \arrow[d, "\Delta^*"] \arrow[dd, bend left, "\varphi"] & \mathcal{B}_E \arrow[r] \arrow[u, phantom, "\cup"] \arrow[dd] & \mathcal{A}_{E/S} \arrow[r] \arrow[u, phantom, "\cup"] \arrow[dd, equal] & 0 \\
 & E\otimes E^{\circ} \arrow[d, "tr"] & & & \\
0 \arrow[r] & \omega \arrow[r] & \mathcal{B}_E/\mathrm{Ker}\,\varphi \arrow[r] & \mathcal{A}_{E/S} \arrow[r] & 0
\end{tikzcd}$$

Basic triangle

$$
\begin{array}{c}
\left((\mathcal{O}_X \longrightarrow \omega)[-2] \right) \\
\downarrow \\
\left(\mathcal{O}_X \longrightarrow \mathcal{B}_E / \operatorname{Ker}\varphi \longrightarrow \mathcal{A}_{E,\pi} \right) \\
\downarrow \\
\left(\mathcal{A}_{E/S} \longrightarrow \mathcal{A}_{E,\pi} \right) \overset{\text{quis}}{\sim} \mathcal{T}_{E/S} \to \mathcal{T}_\pi \overset{\text{quis}}{\sim} \pi^{-1}\mathcal{T}_S
\end{array}
$$

It's a sheaf of DG Lie algebras over X so when we apply $R^0\pi_*$ it gives a sheaf of Lie algebras on S. In fact one gets an exact sequence of Lie algebras

$$
0 \longrightarrow \underset{\substack{\| \\ \mathcal{O}_S}}{R^2\pi_*(\mathcal{O}_X \to \omega)} \longrightarrow R^0\pi_*({}^{tr}\mathcal{A}^{\bullet}_{E,\pi}) \longrightarrow \mathcal{T}_S \longrightarrow 0
$$

and their theorem asserts this is the Atiyah extension $\mathcal{A}_{\lambda_E}$.