548 - 625 April 9 - May 21

Connes homology April 29, 594

Atiyah Bott on Duistermaat Heckman: 568 - 577

April 9, 1982

Today I want to finish the computation of the heat kernel terms, using power series expansions where necessary. Recall

$$\Delta = \frac{1}{g}\underbrace{\left(\partial_z - \alpha^*\right)}_{\tilde{D}}\underbrace{\left(\partial_{\bar z} + \alpha\right)}_{D}$$

where α is a matrix of C^∞ functions of z, and g is the function giving the metric: $ds^2 = g\, dz d\bar z$, or $\left|\frac{\partial}{\partial z}\right|^2 = g$.

$$\langle z|e^{t\Delta}|0\rangle = \underbrace{\frac{1}{2\pi t}e^{-\frac{u}{t}}}_{\phi} bA(t,z) \qquad \begin{array}{l} A \sim A_0^{(z)} + tA_1^{(z)} + \cdots \\ b = b(z). \end{array}$$

$$\phi^{-1}(\partial_t - \Delta)\phi = \partial_t - \frac{1}{t} + \frac{u}{t^2} - \frac{1}{g}\left(\tilde{D} - \frac{1}{t}\partial_z u\right)\left(D - \frac{1}{t}\partial_{\bar z} u\right)$$

$$= \frac{1}{t^2}\left[u - \frac{1}{g}|\partial_z u|^2\right] + \frac{1}{t}\left[-1 + \frac{1}{g}\partial^2_{z\bar z}u + \frac{1}{g}\left(\partial_z u\, D + \partial_{\bar z} u\, \tilde{D}\right)\right] + \partial_t - \frac{1}{g}\tilde{D}D.$$

Gives $\frac{1}{g}|\partial_z u|^2 = u \implies u = \frac{1}{2}r^2$. Finally we also want to introduce $b(z)$ satisfying

$$\begin{cases} \left[-1 + \frac{1}{g}\partial^2_{z\bar z}(u)\right]b + \underbrace{\frac{1}{g}\partial_z u\, \partial_{\bar z} b + \frac{1}{g}\partial_{\bar z} u\, \partial_z b}_{r\frac{\partial}{\partial r}b} = 0. \\ b(0) = 1. \end{cases}$$

and then we have

$$\nabla_{r\frac{d}{dr}} A_0 = \frac{1}{g}\left(\partial_z u\, D + \partial_{\bar z} u\, \tilde{D}\right)A_0 = 0 \qquad A_0(0) = 1.$$

$$\left(1 + \nabla_{r\frac{d}{dr}}\right)A_1 = \frac{1}{gb}\tilde{D}D\, bA_0$$

$$\left(n + \nabla_{r\frac{d}{dr}}\right)A_n = \frac{1}{gb}\tilde{D}D\, bA_{n-1}$$

I am interested in the exact value of bA_1 at 0.

$$(bA_1)(0) = A_1(0) = \left(\frac{1}{gb}\tilde{D}D\, bA_0\right)(0)$$

hence we have to compute b, A_0 to the 2nd order.

First case is where $\alpha = 0$. Then $A_0 \equiv 1$ and so we need $\frac{1}{g}\partial^2_{z\bar z} b$ at 0. Let's work this out for the metric

$$g(z) = \frac{1}{(1-\varepsilon|z|^2)^2}$$

which I have seen has constant curvature $R = -\varepsilon$.

$$r(z) = \int_0^{|z|} \sqrt{2g}\, d|z| = \sqrt{2}\int_0^{|z|} (1+\varepsilon|z|^2 + \varepsilon^2|z|^4 + ..)\, d|z|$$

$$= \sqrt{2}|z|\left(1 + \frac{\varepsilon}{3}|z|^2 + \varepsilon^2\frac{|z|^4}{5} + ..\right) = \frac{\sqrt{2}|z|}{2\varepsilon}\log\frac{1+\varepsilon|z|}{1-\varepsilon|z|}$$

$$r(z) = \sqrt{2}\left(|z| + \frac{\varepsilon}{3}|z|^3 + \cdots\right)$$

$$\frac{r^2}{2} = |z|^2 + \frac{2\varepsilon}{3}|z|^4 + .. \qquad \partial_z\partial_{\bar z}\, \bar z^2 z^2 = \partial_z\, 2\bar z z^2 = 4|z|^2$$

$$\partial^2_{z\bar z}\left(\frac{r^2}{2}\right) = 1 + \frac{8\varepsilon}{3}|z|^2 + ..$$

$$\frac{1}{g}\partial^2_{z\bar z}\left(\frac{r^2}{2}\right) = \left(1 + \frac{8\varepsilon}{3}|z|^2 + ..\right)\left(1-\varepsilon|z|^2\right)^2 = 1 + \underbrace{\left(\frac{8}{3}-2\right)}_{2/3}\varepsilon|z|^2 + ..$$

$$-1 + \frac{1}{g}\partial^2_{z\bar z}\left(\frac{r^2}{2}\right) = \frac{2}{3}\varepsilon|z|^2 + ..$$

$$\shortparallel$$

$$-r\frac{d}{dr}\log(b) = -r\frac{d}{d|z|}\log b \Big/ \frac{dr}{d|z|} = -|z|\frac{d\log b}{d|z|}\Big/ \frac{|z|}{r}\sqrt{2g}$$

$$\therefore\quad |z|\frac{d}{d|z|}\log b = \left(-\frac{2}{3}\varepsilon|z|^2 + \right)\frac{|z|}{r}\sqrt{2g} = -\frac{2}{3}\varepsilon|z|^2 + \cdots$$

$$\log b = -\frac{1}{3}\varepsilon|z|^2 + \cdots$$

$$b = 1 - \frac{1}{3}\varepsilon|z|^2 + \cdots$$

So if I take the trivial line bundle over a constant curvature Riemann surface, then

$$A_1(0) = \left(\frac{1}{gb}\partial^2_{z\bar z} b\right)(0) = -\frac{1}{3}\varepsilon$$

so
$$\langle 0|e^{t\Delta}|0\rangle \sim \frac{1}{2\pi t}\left[1 - \frac{1}{3}\varepsilon t + O(t^2)\right]$$

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{tr}(e^{-tA} - P_0)\, t^s \frac{dt}{t}$$

hence $\zeta_A(0) =$ constant term in the asymptotic expansion for $\operatorname{tr}(e^{-tA} - P_0)$. So

$$\zeta_A(0) = \int_M a_1 - \dim(\operatorname{Ker} A).$$

In the case of the trivial line bundle I have found

$$\int_M a_1 = \frac{1}{2\pi}\left(-\frac{1}{3}\varepsilon\right) \operatorname{vol}(M) = \frac{1}{6\pi}\int_M R \cdot \operatorname{vol} = \frac{1}{6\pi}(2-2g)\pi$$

$$= \frac{1}{3}(1-g)$$

and hence in this case $\zeta(0) = \frac{1}{3}(1-g) - 1$.

Next we want the case where α is present. The equation for A_0 is $\nabla_{r\frac{d}{dr}} A_0 = 0$, $A_0 = 1$ at $z=0$. Now $\nabla_{r\frac{d}{dr}}$ is a multiple of $\bar{z}\tilde{\mathcal{D}} + z\mathcal{D}$, and hence we can use our old formula for A_0: (p. 502)

$$A_0(z) = 1 + (z\alpha_0^* - \bar{z}\alpha_0) + \frac{1}{2}(z\alpha_0^* - \bar{z}\alpha_0)^2$$

$$+ \frac{1}{2}\left(z\left(\partial_z\alpha^*|_0 z + \partial_{\bar{z}}\alpha^*|_0 \bar{z}\right) - \bar{z}\left(\partial_z\alpha|_0 z + \partial_{\bar{z}}\alpha|_0 \bar{z}\right)\right)$$

So now I want to use the formula

$$A_1(0) = \left(\frac{1}{gb}\tilde{\mathcal{D}}\mathcal{D} b A_0\right)(0) \qquad \left\{\begin{array}{l} b = 1 - \frac{1}{3}\varepsilon|z|^2 \\ g(0) = 1 + O(|z|^2). \end{array}\right.$$

$$= (\tilde{\mathcal{D}}\mathcal{D}A_0)(0) + (\tilde{\mathcal{D}}\mathcal{D}(b-1)A_0)(0)$$

have to diff. $|z|^2$ twice

from p 502 $\rightarrow$
$$= \frac{1}{2}\left(\frac{\partial\alpha^*}{\partial\bar{z}} + \frac{\partial\alpha}{\partial z} + \alpha\alpha^* - \alpha^*\alpha\right)\Big|_0 - \frac{1}{3}\varepsilon$$

curvature form for the connection is (p. 517)

$$D^2 = [\tilde{\mathcal{D}}\mathcal{D} - \mathcal{D}\tilde{\mathcal{D}}]\,dz\,d\bar{z} = \left[\frac{\partial\alpha^*}{\partial\bar{z}} + \frac{\partial\alpha}{\partial z} + \alpha\alpha^* - \alpha^*\alpha\right] dz\,d\bar{z}$$

So the local contribution at $z=0$ to be integrated is

$$\frac{1}{2\pi}\left(\frac{1}{2}\operatorname{tr}\left(\frac{\partial\alpha^*}{\partial\bar z}+\frac{\partial\alpha}{\partial z}\right)-\frac{\overset{\text{rank }E}{(1)}}{3}\varepsilon\right)\Bigg|_0 \, i\,dz\,d\bar z$$

and so it seems we get

$$\boxed{\zeta_E(0) = \frac{1}{2}(\deg E) + \frac{(\text{rank }E)}{3}(1-g) - \dim H^0(E)}$$

(The $\frac{1}{3}$ needs checking; it might be better if it were really $\frac{1}{2}$ for then $\zeta(0)=0$, in the case of slope 0 and vanishing cohomology)

For R.R. one doesn't need to know the exact nature of the $\frac{1}{3}$-term. In effect "Serre" duality tells us that under the isomorphisms (conjugate linear)

$$E \;\sim\; E^*\otimes\Omega^{1,1} \qquad E\otimes\Omega^{0,1} \simeq E^*\otimes\Omega^{1,0}$$

obtained from the metrics, that $\bar\partial_E^*$ corresponds to $\bar\partial_{E^*\otimes K}$ where $K=\Omega^{1,0}$. Hence $\zeta_E(s)=\zeta_{E^*\otimes K}(s)$ and so

$$\zeta_E(0) = \frac{1}{2}(\deg E) + \frac{1}{3}r(1-g) - h^0(E)$$

$$\parallel$$

$$\zeta_{E^*\otimes K}(0) = \frac{1}{2}(\deg E^*\otimes K) + \quad '' \quad - h^0(E^*\otimes K)$$

gives
$$h^0(E) - h^0(E^*\otimes K) = \frac{1}{2}\left(\deg E - \deg(E^*\otimes K)\right)$$
which is R.R.

Compute the other Laplacean ζ value.

$$-\bar\partial\bar\partial^* = \mathcal{D}\frac{1}{g}\tilde{\mathcal{D}} = \frac{1}{g}\mathcal{D}^g\tilde{\mathcal{D}}$$

where $\mathcal{D}^g = g\,\mathcal{D}\frac{1}{g} = \partial_{\bar z} + \alpha - \partial_{\bar z}\log g$. Since I have never used that $\alpha^* =$ adjoint of α I can use the above formulas. Will get different term

$$\int \frac{1}{2\pi}\,\frac{1}{2}\operatorname{tr}\left(\frac{\partial\alpha^*}{\partial\bar z}+\frac{\partial\alpha}{\partial z} - \partial^2_{z\bar z}\log g\right) i\,dz\,d\bar z = \frac{1}{2}(\deg E - \text{rank}.\deg T)$$

$$= \frac{1}{2} \deg (E^* \otimes K)$$ T = tangent bundle since $g = \left|\frac{\partial}{\partial z}\right|^2$.

so again we get R.R. out but no way to check the suspicious $\frac{1}{3}$.

I can check the calculation over S^2 using eigenvalues of the Laplacean from spherical harmonics. I want to take the line bundle $O(-1)$ over S^2 and evaluate its ζ function at $s = 0$. One takes the Yang-Mills minimum model, i.e. the curvature form is constant and gives degree -1. Unfortunately I have to extrapolate since I don't know the group theory. If I consider the trivial bundle, then I can relate the Laplacean on S^2 to the Laplacean on $\mathbb{R}^3$

$$\Delta_{\mathbb{R}^3} = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{\phi,\theta}$$

Harmonic functions on space can be decomposed into homogeneous components. If u is homogeneous of degree l and harmonic, then

$$0 = r^2 \Delta u = l(l+1) u + \Delta_{\phi,\theta} u$$

and so the Laplacean on S^2 has the eigenvalues $l(l+1)$, $l = 0, 1, 2, \ldots$. One knows the multiplicities are $2l+1$. Hence the ζ-function is

$$\sum_{l=1}^{\infty} (2l+1) \frac{1}{[l(l+1)]^s}$$ $l(l+1) = (l+\frac{1}{2})^2 - \frac{1}{4}$

My guess for the line bundle $O(-1)$ is that one should use the same formulas for the multiplicity and eigenvalues but take $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ This gives

$$\sum_{n=1}^{\infty} 2n \frac{1}{(n^2 - \frac{1}{4})^s}$$ wrong should be n^2

I compute the value at 0 via the asymptotic expansion as follows:

$$\sum_{n=1}^{\infty} 2n \frac{1}{\left(n^2-\frac{1}{4}\right)^s} = \frac{1}{\Gamma(s)} \int_0^\infty \sum_1^\infty 2n\, e^{-tn^2} e^{t\frac{1}{4}}\, t^s \frac{dt}{t}$$

But $\displaystyle \frac{1}{\Gamma(s)} \int_0^\infty \sum_1^\infty 2n e^{-tn^2} t^s \frac{dt}{t} = \sum_1^\infty 2n \frac{1}{n^{2s}} = 2\zeta_R(2s-1)$

$$(*) \quad \sim \begin{cases} 2\,\frac{1}{2s-1-1} = \frac{1}{s-1} & \text{as } s \to 1 \\ 2\,\zeta_R(-1) = 2\cdot\left(-\frac{1}{12}\right) = -\frac{1}{6} & \text{as } s \to 0 \end{cases}$$

Check the value of $\zeta_R(-1)$.

$$\frac{1}{e^t-1} = \frac{1}{t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + O(t^3)$$

$$\begin{array}{r} 1 \\ \underline{1+\frac{t}{2}+\frac{t^2}{6}+\frac{t^3}{24}} \\ -\frac{t}{2}-\frac{t^2}{6}-\frac{t^3}{24} \\ \underline{-\frac{t}{2}-\frac{t^2}{4}-\frac{t^3}{12}} \\ \frac{t^2}{12}+\frac{t^3}{24} \\ \underline{\frac{t^2}{12}+\frac{t^2}{24}} \\ 0 \end{array}$$

$\displaystyle \zeta_R(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{e^t-1}\, t^s \frac{dt}{t}$ The singularities are obtained from $\displaystyle \frac{1}{\Gamma(s)} \int_0^1 t^{n+s} \frac{dt}{t} = \frac{1}{\Gamma(s)(n+s)} = \frac{s(s+1)\cdots(s+n-1)}{\Gamma(s+n)(n+s)} \to (-1)^n n!$ as $s \to -n$

Thus the term $\frac{t}{12}$ gives $\boxed{\zeta_R(s) = -\frac{1}{12} \text{ at } s=-1}$.

So from $(*)$ we conclude

$$\sum_1^\infty 2ne^{-tn^2} \sim \frac{1}{t} - \frac{1}{6} + O(t) \quad \text{as } t \downarrow 0$$

$$\left(\sum_1^\infty 2ne^{-tn^2}\right) e^{t/4} \sim \left(\frac{1}{t} - \frac{1}{6} + \ \right)\left(1+\frac{t}{4}+\cdots\right)$$

$$\sim \frac{1}{t} + \left(-\frac{1}{6}+\frac{1}{4}\right) + \cdots = \frac{1}{t} + \frac{1}{12} + \cdots$$

and therefore

$$\zeta(s) = \sum_{n=1}^{\infty} 2n \frac{1}{\left(n^2-\frac{1}{4}\right)^s} \xrightarrow[s\to 0]{\text{as}} \frac{1}{12}$$

which isn't anywhere consistent with

$$\zeta(0) = \tfrac{1}{2}\deg + \tfrac{1}{3}(1-g)r - h^0 \qquad = -\tfrac{1}{2} + \tfrac{1}{3}$$

Let's try the same method on the ζ for the Laplacean on functions

$$\zeta(s) = \sum_{\ell=1}^{\infty} (2\ell+1) \frac{1}{\ell(\ell+1)^s}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{\ell=1}^{\infty} (2\ell+1)\, e^{-t(\ell+\frac{1}{2})^2}\, e^{t/4}\, t^s \frac{dt}{t}$$

Note the sum starts at $\ell = 1$. Now

$$\frac{1}{\Gamma(s)}\int_0^\infty \sum_{\ell=0}^{\infty} (2\ell+1) e^{-t(\ell+\frac{1}{2})^2} t^s \frac{dt}{t} = \sum_0^\infty (2\ell+1)\frac{1}{(\ell+\frac{1}{2})^{2s}} = 2^{2s}\sum_0^\infty \frac{1}{(2\ell+1)^{2s-1}}$$

$$= 2^{2s}\left(1-\frac{1}{2^{2s-1}}\right)\zeta_R(2s-1)$$

$$\sim \begin{cases} 2^2\left(1-\frac{1}{2}\right)\frac{1}{2s-2} = \frac{1}{s-1} & \text{as } s\to 1 \\ (1-2)\left(-\frac{1}{12}\right) = \frac{1}{12} & \text{as } s \to 0 \end{cases}$$

Thus

$$\left[\sum_{\ell=0}^{\infty} (2\ell+1) e^{-t(\ell+\frac{1}{2})^2}\right] e^{t/4} \sim \left(\frac{1}{t}+\frac{1}{12}+\cdots\right)\left(1+\frac{t}{4}+\cdots\right)$$

$$= \frac{1}{t} + \frac{1}{3} + \cdots$$

so

$$\left[\sum_{\ell=1}^{\infty} (2\ell+1) e^{-t(\ell+\frac{1}{2})^2}\right] e^{t/4} = \frac{1}{t} - \frac{2}{3} + \cdots$$

and hence $\zeta(0) = -\frac{2}{3}$ which does agree with the formula

$$\boxed{\zeta_E(0) = \frac{1}{2}\deg E + \frac{1}{3}(1-g)\operatorname{rank} E - h^0(E)}$$

$$0 \qquad\qquad \frac{1}{3} \qquad\qquad -1$$

April 11, 1982

Consider the 2 sphere as $SU_2/\text{max. torus}$ and the bundle $\mathcal{O}(k)$ as an induced bundle. I know that the space of C^∞ sections of $\mathcal{O}(k)$ is then an induced representation of SU_2 whose irreducible pieces can be computed using Frobenius ~~████~~ reciprocity. Lets parameterize the irreducible reps. of SU_2 by spin no: $0, \frac{1}{2}, 1, \dots$ so that the representation with highest weight n has dimension $2n+1$ and Casimir eigenvalue $(n+\frac{1}{2})^2 - \frac{1}{4}$.

By Frobenius reciprocity the induced repn. from the weight $k \in \mathbb{Z}\cdot\frac{1}{2}$ contains the repns. with heighest weights $|k|, |k|+1, \dots$ each of multiplicity 1. The Laplacean on sections of the bundle should be the Casimir operator up to a constant depending on k, the idea being that the repns. induced from k and $-k$ are the same, but one has holomorphic sections and the other doesn't. So its clear we want to subtract $(k+\frac{1}{2})^2 - \frac{1}{4}$ to get the Lap:

$$-\Delta = \text{Casimir} - \left[(k+\tfrac{1}{2})^2 - \tfrac{1}{4}\right] \qquad \text{on } \mathcal{O}(2k)$$

Thus the eigenvalues of $-\Delta$ will be $(n+\frac{1}{2})^2 - (k+\frac{1}{2})^2$ for $n = |k|,$ ~~████~~ $|k|+1, |k|+2, \dots\dots$.

Take $k = -\frac{1}{2}$ so that we get the bundle $\mathcal{O}(-1)$.
~~█~~ $n = \frac{1}{2}, \frac{3}{2}, \dots\dots = m - \frac{1}{2}$ where $m = 1, 2, \dots$; $\dim = 2m$, and eigenvalues of $-\Delta = m^2$, hence

$$\zeta = \sum_{m=1}^{\infty} 2m \frac{1}{m^{2s}} = 2\zeta_R(2s-1) = 2\left(-\tfrac{1}{12}\right) = -\tfrac{1}{6} \quad \text{at } s=0$$

which agrees with

$$\begin{array}{ccccc} \frac{1}{2}\deg E & + & \frac{1}{3}(1-g)r & - & h^0(E) \\ -\frac{1}{2} & + & \frac{1}{3} & = & -\frac{1}{6} \end{array}$$

$$\boxed{\zeta_E(0) = \frac{1}{2}\deg(E) + \frac{1}{3}(1-g)\,\mathrm{rank}(E) - h^0(E)}$$

Let's go back to the general problem of putting a metric on the line bundle L via some version of analytic torsion. I recall that have many ways to construct holomorphic sections of the line bundle L. Presumably L is trivial, hence these sections give me functions which will turn out to be some kind of determinants. Variation will give some sort of trace involving the inverse of the $\bar\partial$ operator.

Guess: Return to abstract situation of a Fredholm operator $V_1 \xrightarrow{T} V_0$ of index p. Select a finite dimensional subspace $F \subset V_0$ and a finite-codim subspace W of V_1 such that the map induced by T:

$$W \xrightarrow{\bar{T}} V_0/F$$

has index 0. Then $\operatorname{tr}(\bar{T}^{-1}\delta\bar{T})$ is formally going to give me the variation of the determinant. So on the surface it looks like I can handle the difficulties by passing from T to $\bar{T}$.

$\mathcal{L}$ = dual cohomology determinant line bundle over the space of holom. structures on the C^∞ bundle E over the Riemann surface M. I have produced lots of holomorphic sections of $\mathcal{L}$ and now the goal is to define $|s|^2$ for these sections using a suitable variant of analytic torsion. The main problem is as follows: Given two of my sections s_1, s_2 then at points where they don't vanish we have $s_2 = f s_1$ with f holomorphic and non-vanishing. Thus $\log|s_2|^2 = \log|s_1|^2 + \log|f|^2$, and hence from the ζ fn approach I am going to have to see the term $\log|f|^2$ analytically. So it is first necessary to see exactly what f is like.

First look at index 0 and take s_1 to be the canonical section of $\mathcal{L}$. Use the notation $T: V_1 \longrightarrow V_0$ for the $\bar\partial$ operator assoc. to a holom. structure, so $V_1 = C^\infty(E)$, $V_0 = C^\infty(E \otimes \Omega^{0,1})$. Fix a finite-dimensional subspace F of V_0. Then for T which are transversal to F, we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(T) \arrow[r] \arrow[d, equal] & T^{-1}(F) \arrow[r] \arrow[d, hook] & F \arrow[r] \arrow[d, hook] & \mathrm{Cok}(T) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ker}(T) \arrow[r] & V_1 \arrow[r, "T"] & V_0 \arrow[r] & \mathrm{Cok}(T) \arrow[r] & 0
\end{tikzcd}$$

and on this open set of T we have

$$L_T = \lambda(F) \otimes \lambda(T^{-1}F)^* = \mathrm{Hom}(\lambda(T^{-1}F), \lambda(F)).$$

The canonical section assigns to T the map induced by $T: T^{-1}F \to F$. To get another section s_2 choose a surjection $V_1 \xrightarrow{\pi} F$ and assign to T the effect of λ of the map

$$T^{-1}(F) \subset V_1 \xrightarrow{\pi} F.$$

The s_2 is non-vanishing where this maps is an isom., i.e. where T induces an isomorphism of $\mathrm{Ker}(\pi)$ with V_0/F.

Next compute the ratio $f = s_2/s_1$ on the open set where T is an isom. We get that f is the effect on $\lambda(F)$ of the composition:

$$F \xrightarrow{i} V_0 \xrightarrow{T^{-1}} V_1 \xrightarrow{\pi} F$$

and so
$$f = \det(\pi T^{-1} i).$$

So we reach the following problem: Over the open set where T is an isomorphism we have $|s_1|^2$ by analytic torsion, and we know $f = \det(\pi T^{-1} i)$, hence we know what $|s_2|^2 = |f|^2 |s_1|^2$ is. The problem is to interpret $|s_2|^2$ as an analytic torsion in a way that extends to the set where s_2 is non-vanishing, i.e. where $\overline{T} : \mathrm{Ker}(\pi) \longrightarrow \mathrm{Coker}(i)$ is an isom.

We have an obvious candidate for the torsion of $\overline{T}$, namely to use the Hilbert space structures on $\mathrm{Ker}(\pi)$, $\mathrm{Cok}(i)$ to define $\overline{T}^*$, then form $\zeta(s)$, etc.

Consider line bundles of degree of degree 1 over $M = \mathbb{C}/\Gamma$ or of degree 0 over $M = S^2$. In this case $h^0 = 1$ and so $\zeta(0)$ is constant, and we could hope to get a formula for $-\delta\zeta'(0)$.

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left[\mathrm{tr}\left(e^{-tA}\right) - h^0 \right] t^s \frac{dt}{t}$$

$$-\delta\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{tr}\left(e^{-tA}\,\delta A\right) t^s \, dt$$

Now you want to integrate by parts to obtain

$$\frac{-\delta_A \zeta_A(s)}{s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{tr}\left(\frac{e^{-tA} - Q}{A}\,\delta A\right) t^s \frac{dt}{t}$$

where has to be the identity on the 0-eigenstates at least. Obvious choice for Q is projection P_0 on $\mathrm{Ker} A$. Now because $\zeta_A(0)$ is constant, we have $\frac{-\delta\zeta_A(s)}{s} \longrightarrow -\delta\zeta_A'(0)$ as $s \to 0$

So you get the general formula

$$-\delta\zeta_A'(0) = \text{constant term in } \operatorname{tr}\left(\frac{e^{-tA}-P_0}{A}\,\delta A\right) \quad \text{as } t\downarrow 0.$$

However this result is perhaps meaningless for me because I won't be using $-\zeta'(0)$ to define the norm of a section, but probably a modified ζ-fn depending on the section. The important thing about this formula is it shows that we should cut our space down so that A^{-1} becomes defined

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April 13, 1982

Let's go back to the case of index 0. We choose maps $V_1 \xrightarrow[\pi]{} F \overset{i}{\hookrightarrow} V_0$. Over the open set of $T: V_1 \to V_0$ which are transversal to F we have

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ker} & \to & V_1 & \xrightarrow{T} & V_0 & \to & \mathrm{Cok} & \to 0 \\ & \wr\!\! \mid & & \cup & & \cup & & \wr\!\!\mid & \\ 0 \to & \mathrm{Ker} & \to & T^{-1}F & \to & F & \to & \mathrm{Cok} & \to 0 \end{array}$$

and

$$\mathcal{L}_T = \lambda(F) \otimes \lambda(T^{-1}F)^* = \mathrm{Hom}\left(\lambda(T^{-1}F), \lambda(F)\right).$$

The canonical section s_1 is the map induced by $T: T^{-1}F \to F$. The section s_2 is the map induced by $T^{-1}F \subset V_1 \xrightarrow{\pi} F$. Hence $f = s_2/s_1$ is the determinant of the map

$$F \xrightarrow{T^{-1}} T^{-1}F \subset V_1 \xrightarrow{\pi} F$$

or

$$F \overset{i}{\hookrightarrow} V_0 \xrightarrow{T^{-1}} V_1 \xrightarrow{\pi} F \qquad f = \det(\pi T^{-1} i).$$

Question: Does s_2 depend in an obvious way, or natural way, on the finite rank map $i\pi: V_1 \to V_0$? In some sense s_2 depends only on the subspaces $\mathrm{Ker}\,\pi$, $\mathrm{Im}\, i$ and an isomorphism of $\lambda(V_1/\mathrm{Ker}\,\pi) \simeq \lambda(F)$. In effect

$$T^{-1}F \subset V_1 \to V_1/\mathrm{Ker}\,\pi$$

induces

$$\lambda(T^{-1}F) \longrightarrow \lambda(V_1/\mathrm{Ker}\,\pi)$$

which then maps to $\lambda(F)$. This procedure works when the index $\neq 0$. So I conclude that thinking of $i\pi$ as an arbitrary finite rank map may not be useful.

Correct way to think:

$$\mathcal{L}_T^* = \lambda(T^{-1}F) \otimes \lambda(F) \subset Hom(\Lambda V_0, \Lambda V_1)$$

and we produce a simple family of linear fnls. on the latter space by maps of rank 1:

$$\Lambda V_1 \xrightarrow[\substack{0 \text{ off degree} \\ = \dim(V_1/\mathrm{Ker}\,\pi)}]{} \lambda(V_1/\mathrm{Ker}\,\pi) \simeq \lambda(F) \hookrightarrow \Lambda V_0$$

So we now have these sections s_1, s_2 and a definition of $|s_1|^2$ using analytic torsion. I am trying to find a definition of $|s_2|^2$ using analytic torsion which will be compatible with $|s_2|^2 = |f|^2 |s_1|^2$.

First we should understand what happens algebraically. Suppose that we have a procedure for defining $\det(T)$, i.e. say that the spaces are finite-dimensional and one gives an isom $\lambda(V_0) \simeq \lambda(V_1)$. This trivializes $\mathcal{L}_T$ so that s_1 becomes the function

$$s_1(T) = \det(T)$$

So we can ask what is $s_2(T)$? This is a simple question:

$$\begin{tikzcd} V_1 \arrow[r, "T"] \arrow[d, "\pi"'] & V_0 \\ F_1 & F_0 \arrow[u, hook, "i"'] \end{tikzcd}$$

and we give $\lambda(V_1) \simeq \lambda(V_0)$, $\lambda(F_1) \simeq \lambda(F_0)$. Then we have

$$\lambda(V_1) = \lambda(F_1) \otimes \lambda(\mathrm{Ker}\,\pi)$$
$$\lambda(V_0) = \lambda(F_0) \otimes \lambda(\mathrm{Cok}\, i)$$

and so associated to T is a map $\lambda(\mathrm{Ker}\,\pi) \to \lambda(\mathrm{Cok}\, i)$.

?

Better version:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, two heads, "\pi"] & V \\ F & F \arrow[u, hook, "i"] \end{tikzcd}$$

$$s_2(T) = \det(\pi T^{-1} i)\, \underbrace{s_1(T)}_{\det(T)}$$

What is $s_2(T)$? We claim it is defined for all T, not just the invertible ones. The obvious candidate is $\det(\bar{T})$ where $\bar{T}$ is the induced map from $\operatorname{Ker}\pi$ to $\operatorname{Cok} i$ and we use the obvious isom $\lambda(\operatorname{Ker}\pi) \cong \lambda(F)^* \otimes \lambda(V) \simeq \lambda(\operatorname{Cok} i)$.

<u>Lemma</u>: Let V be a finite diml vector space and let $\pi: V \twoheadrightarrow F$, $i: F \hookrightarrow V$ be surjective and injective homomorphisms respectively. Then for any invertible $T: V \longrightarrow V$ we have

$$\det(\bar{T}) = \det(\pi T^{-1} i)\det(T)$$

where $\bar{T}: \operatorname{Ker}\pi \longrightarrow \operatorname{Cok} i$ is the map induced by T and its determinant is defined using the canonical isoms

$$\lambda(\operatorname{Ker}\pi) = \lambda(F)^* \otimes \lambda(V) = \lambda(\operatorname{Cok} i)$$

<u>Proof</u>: Both sides are alg. functions of T, so we can suppose $\pi T^{-1} i$ is invertible. This implies that $T^{-1} i(F)$ is a complement to $\operatorname{Ker}\pi$. Hence

$$\begin{tikzcd} 0 \arrow[r] & T^{-1}i(F) \arrow[r] \arrow[d, "T"] & V \arrow[r] \arrow[d, "T"] & \operatorname{Ker}\pi \arrow[r] \arrow[d, "\bar{T}"] & 0 \\ 0 \arrow[r] & F \arrow[r] & V \arrow[r] & \operatorname{Cok} i \arrow[r] & 0 \end{tikzcd}$$

so that $\bar{T}$ is invertible, hence $T(\operatorname{Ker}\pi)$ is complementary to $\operatorname{Im} i$. So T becomes a direct sum of

$$T^{-1} i(F) \xrightarrow{T} iF \qquad \text{and} \qquad \operatorname{Ker}\pi \xrightarrow{T} T(\operatorname{Ker}\pi)$$

and the rest follows by looking at the volume elements.

April 14, 1982

It might be possible to compute the ζ fns. explicitly for line bundles over $M = \mathbb{C}/\Gamma$ with constant curvature form. If such a line bundle is pulled back to $\mathbb{C}$ it should be a Gaussian line bundle.

Let's start with the trivial line bundle over $\mathbb{C}$ with metric $|1|^2 = g$. The curvature form is

$$\bar\partial \partial \log g = -\partial^2_{z\bar z} \log g \; dz\, d\bar z$$

So if $g = e^{-|z|^2}$, the curvature is $dz\,d\bar z$, and the first Chern form is $\frac{i}{2\pi} dz\, d\bar z = \frac{dx\,dy}{\pi}$. This same form makes sense over $M = \mathbb{C}/\Gamma$, but must be an integral class to come from a line bundle. Hence we want

$$\int_M \frac{dx\,dy}{\pi} = \frac{\operatorname{Im} \tau}{\pi} = 1 \qquad \text{for line bundles of degree 1}$$

So let L denote a line bundle over M with curvature form $dz\,d\bar z$, and $\pi : \mathbb{C} \longrightarrow M$ the covering map. Then $\pi^*(L)$ is a line bundle over $\mathbb{C}$ with curvature $dz\,d\bar z$. If we trivialize $\pi^*(L)$ by a non-vanishing holom. section s, we get a positive fn. $g = |s|^2$ such that $\bar\partial \partial \log g = dz\,d\bar z$, so we get a $(1,0)$ form $\theta = \partial \log g$ with $\bar\partial \theta = dz\,d\bar z$.

Let's do this more abstractly. Given ω a closed purely imaginary 2 form representing an integral cohomology class, we consider hermitian line bundles with curvature ω. ~~[illegible]~~ Two such line bundles differ by a flat line bundle, hence over $\mathbb{C}$ there is an ~~[illegible]~~ isomorphism of them unique up to multiplication by an elt. of S^1. If I choose a $(1,0)$ form θ with $d\theta = \omega$, then I can think of ~~[illegible]~~ $\theta - \bar\theta$ as a connection form for a connection on the trivial line bundle

$$D = d + A = (\partial + \theta) + (\bar\partial - \bar\theta) \qquad ?$$

Correct method. $D = d + \frac{1}{2}(\theta - \bar{\theta})$ preserves the standard metric $|f|^2$ on $\mathcal{O}$. A holom. section for the associated cx. structure is a fn. s with $(\bar{\partial} - \frac{1}{2}\bar{\theta})s = 0$. Now if I compute the connection form using s as a basis I get

$$\frac{Ds}{s} = \frac{1}{s}(\partial + \tfrac{1}{2}\theta)s$$

$\frac{1}{2}\theta$ since $(\partial - \frac{1}{2}\theta)\bar{s} = 0$

But $$\partial \log |s|^2 = \frac{1}{s}\partial s + \frac{1}{\bar{s}}\partial\bar{s} = \frac{1}{s}\partial s + \frac{1}{2}\theta, \text{ hence}$$

$$\frac{Ds}{s} = \partial \log |s|^2$$

as it should be. However notice that the connection form defined this way does not coincide with θ. So the mistake to avoid seems to be that if $D = d + A = d + (\alpha - \bar{\alpha})$ is a hermitian connection on $\mathcal{O}$, then $\alpha \neq \partial \log|s|^2$ for a holomorphic section.

Repeat. Suppose given ω a closed purely imaginary 2 form, in fact suppose ω is exact: $\omega = d(\frac{1}{2}(\eta - \bar{\eta}))$ where η is of type (1,0). I will suppose the base is simply-connected, so then I know there is a unique hermitian line bundle with this connection. I can realize this hermitian line bundle as the trivial hermitian line bundle with the connection

$$D = d + \tfrac{1}{2}(\eta - \bar{\eta})$$

and a different choice of η can be undone by a gauge transformation.

Notice that this line bundle need not be holomorphically trivial in any obvious way, because one has to solve $$(\bar{\partial} - \tfrac{1}{2}\bar{\eta})s = 0$$
globally with s non-vanishing.

Even when the bundle is holomorphically trivial it is not clear that there is a good trivialization of it.

So now let's apply this to $\omega = dz d\bar{z}$ over $\mathbb{C}$. Pick η: $d\,\frac{1}{2}(\eta - \bar{\eta}) = \omega$ or $\bar{\partial}\eta = \omega$. The simplest choice seems to be $\eta = -\bar{z}dz$, hence $A = \frac{1}{2}(z d\bar{z} - \bar{z} dz)$

Check: $A = \frac{1}{2}(z\,d\bar{z} - \bar{z}\,dz) = i(y\,dx - x\,dy)$, $dA = -2i\,dx\,dy = dz\,d\bar{z}$

Next we want to consider the translation $T_\alpha : z \to z+\alpha$. First denote by L the trivial line bundle with standard inner product, but equipped with the connection $D = d + A$, A as above. Since the form $dz\,d\bar{z}$ is invariant under T_α it follows that we have a unique isomorphism $T_\alpha^*(L) \simeq L$ up to a scalar. Thus given a section of L, i.e. a function $f(z)$, we can translate it to $f(z+\alpha)$ which is a section of $T_\alpha^*(L)$ and then associate to it a section of L: $g(z)f(z+\alpha)$, with this process compatible with D:

$$g(z)\big((d+A)f\big)(z+\alpha) = (d+A)\big(g(z)f(z+\alpha)\big)$$

need:
$$g\,A(z+\alpha) = dg + A(z)g$$

or
$$A(z+\alpha) - A(z) = d\log g$$
$$\frac{1}{2}(\alpha\,d\bar{z} - \bar{\alpha}\,dz) = d\log g$$

or
$$g(z) = e^{\frac{1}{2}(\alpha\bar{z} - \bar{\alpha}z)}$$

so
$$\boxed{(T_\alpha^* f)(z) = e^{\frac{1}{2}(\alpha\bar{z} - \bar{\alpha}z)} f(z+\alpha)}$$

Another way to see this formula is as follows. Holomorphic sections of L are given by solutions of

$$\left(\bar{\partial} - \tfrac{1}{2}\eta\right)f = 0 \quad \text{or} \quad \left(\bar{\partial}_z + \tfrac{1}{2}z\right)f = 0$$

and hence a holomorphic section of L is of the form

$$f(z) = e^{-\frac{|z|^2}{2}} h(z), \qquad \bar{\partial}_z h = 0$$

Moreover
$$\|f\|^2_{H^2(L)} = \int e^{-|z|^2}|h|^2 \frac{dx\,dy}{\pi}$$
$$= \int e^{-|z+\alpha|^2}|h(z+\alpha)|^2 \quad ''$$
$$= \int e^{-|z|^2}\left|e^{-|\alpha|^2/2 - \bar{\alpha}z} h(z+\alpha)\right|^2 \quad ''$$

and we know from past experience that on the holomorphic fns. repn we have

$$T_\alpha(h) = e^{-|\alpha|^2/2 - \bar{\alpha} z} h(z+\alpha).$$

Hence on sections of L we want

$$T_\alpha : f \longmapsto e^{-|z|^2/2} T_\alpha(\underbrace{e^{|z|^2/2} f}_{h})$$

$$= e^{-|z|^2/2} e^{-|\alpha|^2/2 - \bar{\alpha} z} e^{|z+\alpha|^2/2} f(z+\alpha)$$

$$= e^{\frac{1}{2}(\alpha\bar{z} - \bar{\alpha} z)} f(z+\alpha).$$

Start again. We begin with a hermitian line bundle $\tilde{L}$ over $M = \mathbb{C}/\Gamma$ with the curvature form $dz d\bar{z} = \tilde{\omega}$. Then we pull it back and fix an isomorphism $\pi^*(\tilde{L}) \simeq L$ where L denotes the trivial hermitian line bundle over $\mathbb{C}$ with connection $D = d + \frac{1}{2}(z d\bar{z} - \bar{z} dz)$. Given a period $\gamma \in \Gamma$ it gives an autom. of $\pi^*(\tilde{L})$, which we transport to L, and hence must be of the form

$$f \longmapsto c_\gamma T_\gamma(f) = c_\gamma e^{\frac{1}{2}(\gamma\bar{z} - \bar{\gamma} z)} f(z+\gamma)$$

with $|c_\gamma| = 1$. Now

$$T_\beta(T_\gamma f) = e^{\frac{1}{2}(\beta\bar{z} - \bar{\beta} z)} (T_\gamma f)(z+\beta)$$

$$= e^{\frac{1}{2}(\beta\bar{z} - \bar{\beta} z)} e^{\frac{1}{2}(\gamma(\bar{z}+\bar{\beta}) - \bar{\gamma}(z+\beta))} f(z+\beta+\gamma)$$

$$= e^{\frac{1}{2}(\gamma\bar{\beta} - \bar{\gamma}\beta)} T_{\beta+\gamma} f$$

hence the c_γ satisfy

$$c_\beta c_\gamma e^{\frac{1}{2}(\gamma\bar{\beta} - \bar{\gamma}\beta)} = c_{\beta+\gamma}.$$

This implies that $e^{\gamma\bar{\beta} - \bar{\gamma}\beta} = 1$, i.e. that $\gamma\bar{\beta} - \bar{\gamma}\beta \in 2\pi i\mathbb{Z}$ for all $\beta, \gamma \in \Gamma$, hence taking $\beta = 1, \gamma = \tau$

$$2i\,\mathrm{Im}(\tau) \in 2\pi i\mathbb{Z} \quad \text{or} \quad \frac{\mathrm{Im}\,\tau}{\pi} \in \mathbb{Z}.$$

This checks out the fact the form $\tilde{\omega}$ has to be integral.

Notice also that different choices for the $\{c_\gamma\}$ differ by homomorphisms $\Gamma \to S^1$, so that once a convenient c_γ is chosen, the others can be simply described just like flat line bundles.

At this point we have a nice description of our line bdle $\tilde{L}$ over M, namely sections are smooth functions over $\mathbb{C}$ which are invariant under the action of the $\{c_\gamma T_\gamma\}$. The problem now arises as to whether we can find eigenfns. for the $\bar\partial$ operator. Do first over $\mathbb{C}$, and use $d\bar z$ to identify $\mathcal{O} \xrightarrow{\sim} \Omega^{0,1}$, so that $\bar\partial$ becomes an operator on $\tilde{L}$. The connection on L is $D = d + A$, $A = \frac{1}{2}(z\,d\bar z - \bar z\,dz)$, hence holomorphic sections are defined by $(\bar\partial + \frac{1}{2} z\,d\bar z)f = 0$. Hence we think of the $\bar\partial$ operator on $\tilde{L}$ as lifting to the operator $\partial_{\bar z} + \frac{1}{2}z$ on L. Look for eigenfns. upstairs:

$$\left(\partial_{\bar z} + \tfrac{1}{2}z\right) f = \lambda f$$

$$e^{-\frac{1}{2}|z|^2}\,\partial_{\bar z}\, e^{\frac{1}{2}|z|^2} f = \lambda f$$

hence
$$f = e^{-\frac{1}{2}|z|^2 + \lambda \bar z}\, h(z)$$

where $h(z)$ is holomorphic. So now we try to choose h so as to satisfy the periodicity condition.

Let's see what we get for $\lambda = 0$, using the fact that there is one holomorphic section of $\tilde L$ with a single zero on M. This tells me that there is a unique (up to a multiplicative constant) holomorphic function $h(z)$ such that

$$c_\gamma\, e^{-\frac{1}{2}|\gamma|^2 - \bar\gamma z}\, h(z+\gamma) = h(z).$$

Thus
$$\frac{d}{dz}\log h(z+\gamma) - \frac{d}{dz}\log h(z) = \bar\gamma .$$
On the other hand
$$\frac{d}{dz}\log \sigma(z+\gamma) - \frac{d}{dz}\log\sigma(z) = \ell\gamma + m\bar\gamma \qquad m = \frac{\pi}{\operatorname{Im}\tau} = 1$$

hence

$$\frac{d}{dz}\log\frac{h(z+\gamma)}{\sigma(z+\gamma)} - \frac{d}{dz}\log\frac{h(z)}{\sigma(z)} = -\ell\gamma$$

so $$\frac{d}{dz}\log\left(\frac{h(z)}{\sigma(z)}\, e^{+\ell z^2/2}\right) = \frac{d}{dz}\left(\log\frac{h(z)}{\sigma(z)}\right) + \ell z$$

is doubly-periodic, hence constant. Thus

$$h(z) = e^{-\ell\frac{z^2}{2} + cz}\,\sigma(z)$$

and different values of a will correspond to different $\{c_\gamma\}$. Recall

$$\frac{\sigma(z+\gamma)}{\sigma(z)} = e^{(\ell\gamma+\bar\gamma)z + b(\gamma)}$$

Now $$\frac{\sigma(z+\beta+\gamma)}{\sigma(z+\beta)}\,\frac{\sigma(z+\beta)}{\sigma(z)} = e^{(\ell\gamma+\bar\gamma)(z+\beta) + b(\gamma) + (\ell\beta+\bar\beta)z + b(\beta)}$$

showing $$b(\beta+\gamma) = b(\beta) + b(\gamma) + (\ell\gamma+\bar\gamma)\beta \qquad \bmod 2\pi i\mathbb{Z}.$$

Since $\bar\gamma\beta - \gamma\bar\beta \in 2\pi i\mathbb{Z}$, we conclude that

$$\begin{aligned} b(\beta+\gamma) - \frac{\ell}{2}(\beta+\gamma)^2 - \frac{1}{2}|\beta+\gamma|^2 = {} & b(\beta) + b(\gamma) + \cancel{\ell\gamma\beta} + \bar\gamma\beta \\ & - \frac{\ell}{2}\beta^2 - \frac{\ell}{2}\gamma^2 - \cancel{\ell\gamma\beta} \\ & - \frac{1}{2}|\beta|^2 - \frac{1}{2}|\gamma|^2 - \frac{1}{2}(\beta\bar\gamma + \bar\beta\gamma) \end{aligned}$$

should be matched by $\{c_\gamma\}$.

April 16, 1982

Consider the following: M compact symplectic manifold, $f : M \to \mathbb{R}$ such that X_f is periodic, i.e. gives an S^1-action. To understand all the good theorems that hold in this situation.

First look at equivariant cohomology. Recall

$$\cdots \longrightarrow H_S(M^S) \xrightarrow{i_*} H_S(M) \longrightarrow H_S(M - M^S) \longrightarrow \cdots$$

$$\begin{array}{c} H_S(M) \\ \big\downarrow i^* \\ H_S(M^S) \end{array}$$

where $i^* i_*$ is multiplication by the equivariant Euler class of the normal bundle ν. The normal bundle ν is a real bundle over M^S with S^1-action, hence decomposes w.r.t. the irreducible non-trivial real repns. of S^1. Hence it has a unique complex structure such that only the positive characters of S^1 occur.

We would like to know when

$$e(\nu) = i^* i_* : \begin{array}{c} H_S(M^S) \\ \| \\ H(M^S) \otimes \underbrace{H_S(pt)}_{\mathbb{Z}[u]} \end{array} \longrightarrow H_S(M^S)$$

is injective. Now $\nu = \bigoplus_{n>0} \nu_n$ where S^1 acts on ν_n by the character z^n, so

$$e(\nu) = \prod e(\nu_n)$$

Assume for simplicity that a single character χ occurs. Then

$$e_{equiv}(\nu) = \xi^d + c_1(\nu)\xi^{d-1} + \cdots + c_d(\nu)$$

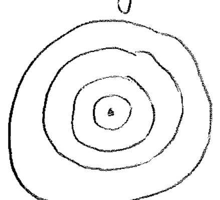
where $c_i(\nu) \in H(M^S)$ are the ordinary Chern classes, and $\xi = c_1(\chi) \in H^2(BS)$. It is now clear that $i^* i_*$ is always injective in rational cohomology, but not [redacted] in general. One must assume either that $H(M^S)$ is torsion free, or that only the character [redacted] $z \mapsto z$ occurs in the normal bundle ν, or something similar.

What does the symplectic structure tell us? At a point of M^S, the tangent space has a symplectic form, and we have this S^1 action. The infinitesimal generator X_f has got [redacted] eigenvalues of the form $\pm ni$, $n \in \mathbb{Z}$, if the period is 2π. We conclude M^S is a symplectic submanifold, and ν is a symplectic vector bundle. We can also [redacted] conclude that f is non-degenerate in the normal direction, because the 2nd order terms in f give the effect of X_f on the <u>tangent</u> space. What about the sign of f?

Let V be a real symplectic vector space, f a quadratic function on V such that X_f is periodic of period 2π. Then the eigenvalues of X_f on V are $\pm ni$ $n \in \mathbb{Z}$ and occur in conjugate pairs. It's clear that different $|n|$ eigenspaces are orthogonal, so assume X_f has eigenvalues $\pm ni$ for a single n. Then $\frac{1}{|n|} X_f = J$ gives a complex structure on V. [redacted] This gives me the [redacted] positive frequency [redacted] complex structure. However the sign of f perhaps specifies a complex structure?

The point is that the S^1 action determines a complex structure by the positive frequency rule, however the symplectic 2 form won't have positive sign relative to this complex structure. In the plane here's what happens.

Start with V real, 2diml and $\Lambda^2 V \xrightarrow{\omega} \mathbb{R}$. Then given an S^1 action we get a unique family of invariant circles



and so we know what $\pm 90°$ is. But ω gives us an orientation hence V has a complex structure. However the S^1 action can rotate clockwise or counterclockwise depending on f.

Correct way to think: The maximal compact subgroup of $Sp_{2n}(\mathbb{R})$ is U_n, hence a symplectic manifold always has an almost complex structure. Similarly a compact group leaving the symplectic structure invariant with (by averaging) leave an almost complex structure invariant.

Again let $S = S^1$ act on M, supposed compact & oriented. Then for the map $\pi : M \to pt$ we have π_* on equivariant cohomology:

$$\begin{tikzcd} H_S(M) \arrow[d, "\pi_*"] & H_S(M^S) \arrow[l, "i_*"'] \arrow[dl, "\pi'_*"] \\ \mathbb{Z}[u] = H_S(pt) & \end{tikzcd}$$

If we invert u, and work rationally (or else assume S acts freely on $M - M^S$, so that $H_S(M-M^S)$ is finite-diml, then i_* becomes an isomorphism. Hence

$$\pi_*(\alpha) = \pi_* i_*(\beta) = \pi'_*(\beta)$$

where $i_*(\beta) = \alpha \implies i^*(\alpha) = i^*(i_*(\beta)) = (i^* i_* 1)\beta$, hence we get

$$\boxed{\pi_*(\alpha) = \pi'_*\left(\frac{i^*\alpha}{i^* i_* 1}\right)}$$

Let's try to understand this in the case of isolated fixpoints. At a fixpt, the torus acts on the tangent space which decomposes into non-trivial irred repns. If I assume a complex structure, then each of these is given by a character, so we get a sequence of integers $\lambda_1, \dots, \lambda_n \neq 0$ such that

$$i^* i_* 1 = \left(\prod_{j=1}^{n} \lambda_j \right) u^n .$$

So next let's look at the action of S^1 on $\mathbb{P}(V)$ obtained from an action ρ on $V = \mathbb{C}^n$ given by

$$\rho(z) = \begin{pmatrix} z^{a_1} & & \\ & \ddots & \\ & & z^{a_n} \end{pmatrix} .$$

I'll assume the a_i are distinct so that the fixpoints are isolated. Now the equivariant cohomology is given by the projective bundle thm.

$$H_S(\mathbb{P}V) = H_S(pt)[\xi] \big/ \left(\xi^n + c_1(V)\xi^{n-1} + \cdots + c_n(V) \right) .$$

~~One has $\pi^* L_i \subset (L_1 \oplus \cdots \oplus L_n = V) \to \mathcal{O}(1)$, hence $c_1(\mathcal{O}(1) \otimes L_i^\vee)$ belongs to the hyperplane~~

We have

$$\mathcal{O}(-1) \subset \pi^* V \xrightarrow{pr_i} \pi^* L_i$$

whose vanishing describes the hyperplane of lines projecting trivially into L_i. This hyperplane has coh. class

$$c_1(\mathcal{O}(1) \otimes \pi^* L_i) = \xi + c_1(L_i)$$

hence the relation is $\prod (\xi + c_1(L_i)) = 0$. Also the normal bundle to the point L_i is

$$\begin{aligned} \nu &= \mathrm{Hom}\left(\mathcal{O}(-1), V/\mathcal{O}(-1)\right) \quad \text{at } \mathcal{O}(-1) = L_i \\ &= \mathrm{Hom}(L_i, V/L_i) \\ &= \bigoplus_{j \neq i} \underbrace{L_j \otimes L_i^*}_{\text{character } z \mapsto z^{a_j - a_i}} \end{aligned}$$

and so

$$i^* i_* 1 \text{ at } L_i = \left[\prod_{j \neq i} (a_j - a_i) \right] u^{n-1}$$

Now $\mathcal{O}(1)$ restricted to the point L_i is L_i^*, so $i^* \xi = -a_i u$. So

$$\pi_*(\xi^k) = \sum_{i=1}^{n} \frac{(-a_i)^k}{\prod_{j \neq i}(a_j - a_i)} u^{k-(n-1)}$$

which we can check for $k \le n-1$ using the Lagrange interpolation formula.

Now the idea I learned from Bott is to use differential forms. Realize elements of $H_S(M)$ as differential forms on the fibre space $P_S \overset{S}{\times} M \longrightarrow B_S$, or rather the finite-diml approximation to B_S given by $\mathbb{C}P_N$. The map π_* will be realized by actual integration over the fibre.

So our problem will be to actually construct differential forms in $S^{2N+1} \times^S M$, and then physically integrate them over M to get forms on $\mathbb{C}P_N$. Actually I am mainly interested in the 2 diml cohomology class ξ which should arise as follows. We start with the symplectic form ω on M which I will think of as a curvature form of a line bundle L. Assuming L were S^1-equivariant then we would get a line bundle over $S^{2N+1} \times^S M$, and ξ would be the class of its curvature. Work out the formulas on the differential form level.

April 18, 1982

Recall the stationary phase approximation. Suppose $f : M \longrightarrow \mathbb{R}$ has non-degenerate critical points, M compact, dV a volume form on M, then as $t \longrightarrow +\infty$

$$\int_M e^{itf}\, g\, dV \approx \sum_{P \text{ critical}} \frac{e^{itf(P)}}{(t)^{n/2}}\, g(P)\, c(P)\left(1 + O\left(\tfrac{1}{t}\right)\right)$$

where the constants $c(P)$ are computed as follows from the Hessian of f. Choose Morse lemma coordinates at P

$$f(x) - f(P) = \tfrac{1}{2}\sum \lambda_i x_i^2$$

and suppose $dV = d^n x$ at P. Then

$$\int_{\mathbb{R}^n} e^{it\frac{1}{2}\sum \lambda_i x_i^2}\, d^n x = \frac{(2\pi)^{n/2}}{t^{n/2}\prod |\lambda_i|^{1/2}}\, e^{i\frac{\pi}{4}(\text{signature})}$$

because

$$\int e^{ia\frac{x^2}{2}}\, dx = \cancel{\frac{(2\pi)^{1/2}}{|a|^{1/2}}}\ \frac{(2\pi)^{1/2}}{(-ia)^{1/2}}$$

where $(-ia)^{1/2}$ denotes the square root in the RHP. So

$$\int e^{ia\frac{x^2}{2}}\, dx = \frac{(2\pi)^{1/2}}{|a|^{1/2}}\begin{cases} e^{i\frac{\pi}{4}} & a > 0 \\ e^{-i\frac{\pi}{4}} & a < 0 \end{cases}$$

Thus
$$c(P) = \frac{(2\pi)^{n/2}}{\prod_i |\lambda_i|^{1/2}}\, e^{i\frac{\pi}{4}(\text{signature})} = \frac{(2\pi)^{n/2}}{\det\left(-i\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^{1/2}}$$

where the latter is suitably interpreted.

The remarkable thm is that this approximation is exact in the following case: M symplectic, X_f periodic, and $g\, dV = \omega^n$, where ω is the canonical 2-form and $\dim M = 2n$. I want now to go over the proof of this which I learned from Bott.

Since X_f is periodic we get an $S = S^1$ action on M and hence can consider equivariant cohomology $H_S(M)$. We use real coefficients so that the Gysin map

$$i_* : H_S(M^S) \longrightarrow H_S(M)$$

and restriction $\quad i^* : H_S(M) \longrightarrow H_S(M^S)$

are isomorphisms when localized. Precisely:

$$\to H_S(M^S) \xrightarrow{i_*} H_S(M) \longrightarrow H_S(M - M^S) \longrightarrow \cdots$$

$$\longrightarrow H_S(M, M^S) \to H_S(M) \xrightarrow{i^*} H_S(M^S) \longrightarrow$$

and the circled groups are finite since S acts with only finite isotropy groups, and we use real cohomology.

If $\pi : M \to pt$, then we have integration over the fibre $\quad \pi_* : H_S(M) \longrightarrow H_S(pt) = \mathbb{R}[u]$, $\quad \deg u = 2$, and

$$\begin{tikzcd} H_S(M^S) \arrow[r, "i_*"] \arrow[d, equal] & H_S(M) \arrow[d, "\pi_*"] \\ \bigoplus_P H_S(pt) \arrow[r, "\sum_P"'] & H_S(pt) \end{tikzcd}$$

Use formulas

$$i_*(i^*\alpha) = i_*1 \cdot \alpha$$

$$i^*(i_*\beta) = (i^*i_*1)\beta$$

$$i^*i_*1 = e(\nu) \qquad \nu = \text{normal bundle of } M^S \subset M$$

At a critical point P the circle S acts on the tangent space and one decomposes this repn.

Interesting point: Because U_n is the maximal compact subgroup of Sp_{2n}, any symplectic manifold has an almost complex structure unique up to homotopy and one can make this invariant under the action of a compact group. So at the fixpt P one gets characters of S which one can identify with non-zero integers $\lambda_1, \dots, \lambda_n$ and then

$$e(\nu)\Big|_P = \prod_1^n (\lambda_i u)$$

where $u \in H_S^2(pt)$ corresponds to the identity character.

<u>Formula</u>: If $\beta = \frac{i^*(\alpha)}{e(\nu)}$, then $i_*\beta = \alpha$, so

$$\pi_*(\alpha) = \sum_P \frac{i^*(\alpha)}{e(\nu)}(P)$$

So now we apply this to various classes $\alpha \in H_S(M)$. The idea is to take the 2-form ω and to extend it to an equivariant class.

$$H_S(M) = \varinjlim_N H^*(S^{2N-1} \times^S M)$$

We will work with differential forms on $S^{2N-1} \times M$, and use the fact that if X is the vector field describing the S^1-action, then a form on $S^{2N-1} \times^S M$ can be identified with a form on $S^{2N-1} \times M$ killed by $\theta(X)$, $i(X)$.

Formulas over $S^{2N-1} = \{ (z_i)_{1 \le i \le N} \mid \sum |z_i|^2 = 1 \}$. Think of this as the unit circle bundle in $\mathcal{O}(-1)$ over $\mathbb{CP}_{N-1}$. This line bundle carries a canonical connection such that at any vector $v \in \mathcal{O}(-1) = S^{2N-1}$, the flat directions are the ones perpendicular to the line $\mathbb{C}v$. Maybe better is to say that a tangent vector to S^{2N-1} at v is a vector δv such that $\langle v | \delta v \rangle + \langle \delta v | v \rangle = 0$; the tangent space to S^{2N-1} at v can be identified with $(\mathbb{R}v)^\perp = \mathbb{R}iv \oplus (\mathbb{C}v)^\perp$. Only the $(\mathbb{C}v)^\perp$ represents flat directions for the canonical connections. Hence a flat direction is a vector δv such that $\langle v | \delta v \rangle = 0$, and so a 1-form vanishing on flat directions is

$$\sum \bar{z}_j \, dz_j = \frac{1}{2} \sum (\bar{z}_j dz_j - z_j d\bar{z}_j) \quad \text{on } S^{2N-1}$$

This is the connection form for the canonical connection on $\mathcal{O}(-1)$. Obviously invariant under multiplication by $e^{i\theta}$. Also if we consider the embedding $e^{i\theta} \longmapsto e^{i\theta} v$ of a fibre and pull-back we get

$$\langle e^{i\theta} v \mid i e^{i\theta} d\theta \, v \rangle = i \, d\theta$$

which is $\frac{dz}{z}$ if $z = e^{i\theta}$.

Let's put $\eta = \frac{1}{2i} \sum (\bar{z}_j dz_j - z_j d\bar{z}_j)$ so that η restricts to $d\theta$ on each fibre, i.e.

$$i\left(\frac{d}{d\theta}\right) \eta = 1.$$

Then $d\eta$ is killed both by $\theta\left(\frac{d}{d\theta}\right)$ and $i\left(\frac{d}{d\theta}\right)$ and so represents a 2-dim'l class on $\mathbb{C}P_{N-1}$. Which one? The curvature form for $\mathcal{O}(-1)$ is

$$d \sum \bar{z}_j \, dz_j = \sum d\bar{z}_j \, dz_j = - \sum dz_j \, d\bar{z}_j$$

and we know $\frac{i}{2\pi} \sum dz_j \, d\bar{z}_j$ represents $c_1(\mathcal{O}(1)) = u$. Hence $\frac{1}{2\pi} d\eta$ represents the cohomology class u, which checks checks up to sign because it restricts to $\frac{d\theta}{2\pi}$ on the fibre.

Let X denote the vector field on $S^{2N-1} \times M$ given by $\frac{d}{d\theta}$ on S^{2N-1} and X_f on M. Recall that X_f is defined by

$$i(X_f)\,\omega = df.$$

Then let $\xi = d(f\eta) + \omega$. This is a closed 2-form on $S^{2N-1} \times M$ invariant under S because f, η, ω are. Also

$$i(X)\left[d(f\eta) + \omega \right] = \left[\theta(X) - d\, i(X)\right](f\eta) + df$$

$$= -df + df = 0$$

since $i(X)\eta = 1$. Thus ξ represents a 2-dim'l cohomology class of $H_S(M)$.

apply
$$\pi_*(\alpha) = \sum \frac{i^*(\alpha)}{c(\nu)}(P)$$
to the class represented by ξ^{n+k}. Then $\pi_*(\alpha)$ is rep. by
$$\int_M (f\,d\eta + df\,\eta + \omega)^{n+k} = \int_M \binom{n+k}{n} (f\,d\eta)^k \omega^n$$
using $(df)^2 = 0$ and the fact that $df.\omega^j$ has odd degree. This is
$$(d\eta)^k (n+k)! \int_M \frac{f^k}{k!} \frac{\omega^n}{n!}$$
which by the fixpt formula is
$$\sum_P \frac{(f(P)\,d\eta)^{n+k}}{\prod \lambda_i(P)\,u} = (d\eta)^k \sum_P \frac{f(P)^{n+k}}{\prod \lambda_i(P)} (2\pi)^n$$
since $d\eta = 2\pi u$. Hence one gets
$$\int_M \frac{f^k}{k!} \frac{\omega^n}{n!} = \sum_P \frac{f(P)^{n+k}}{(n+k)!} \frac{(2\pi)^n}{\prod \lambda_i(P)}$$
or taking the generating function
$$\int_M e^{itf} \frac{\omega^n}{n!} = \sum_P \frac{e^{itf(P)}}{(it)^n} \frac{(2\pi)^n}{\prod \lambda_i(P)}$$

April 19, 1982

Parametrics for the $\bar\partial$ operator over a Riemann surface. Let's fix a holomorphic vector bundle over a Riemann surface M, and consider the $\bar\partial$ operator

$$\bar\partial : E \longrightarrow E \otimes \Omega^{0,1}.$$

Suppose this operator is invertible. It's inverse - how can it be described?

Recall that over $\mathbb{C}$ a fundamental solution for $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$ is $\frac{1}{\pi z}$, hence solutions of $\partial_{\bar z} f = \delta(z-z')$ are of the form $\frac{1}{\pi(z-z')} + h(z)$, where $h(z)$ is holomorphic. When $\bar\partial : E \to E \otimes \Omega^{0,1}$ is invertible its inverse will be given by a Schwartz kernel $K(z,z')dz'$ which is a distribution on $M \times M$ with values in the bundle $pr_1^* E \otimes pr_2^*(\check{E} \otimes \Omega^{1,0})$ which is holomorphic off the diagonal, and which looks like $\frac{1}{\pi(z-z')} dz'$ + holom. near the diagonal.

Hence it is clear the inverse of $\bar\partial$ is constructed in a rather straightforward way by choosing global sections of E with simple poles at the different points z' of M.

Let's now go over the construction of a parametrix for $\bar\partial$, that is, an inverse modulo smoothing operators. This time we look for a $K(z,z')dz'$ smooth off the diagonal such that it looks like $\frac{1}{\pi(z-z')}dz'$ near the diagonal.

April 20, 1982:

To understand how to construct a parametrix for $\bar\partial : E \longrightarrow E \otimes \Omega^{0,1}$ where E is a line bundle. This is a map on C^∞ sections going in the other direction given by a kernel:

$$(Kg)(z) = \int K(z,z')\,g(z') \qquad g \in C^\infty(E \otimes \Omega^{0,1}).$$

So for fixed z, $K(z,z')$ has got to be a $1,0$ form, because then $K(z,z')\,g(z')$ will be a $1,1$ form in z' which can be integrated. Thus we see that K will be a distribution section of the bundle

$$Hom\,(pr_2^* E,\, pr_1^* E) \otimes pr_2^* \Omega^{1,0}$$

over $M \times M$.

Next we try to get at $K(z,z')$ by taking g to be a δ-function at z'. Suppose to simplify that $\bar\partial$ is an isomorphism and K is its inverse. Then we find that on taking g to be a δ-function at z', that $K(z,z')$ is a holomorphic section of E with a singularity at $z=z'$. In fact from our local knowledge we know that $K(z,z')$ has a simple pole at $z=z'$. Now we can't talk about the residue of a function with a simple pole, only the residue of a differential. So for example if $E = \Omega^{1,0}$, then we have a simple meaning for $K(z,z')$. This is a meromorphic differential form $\frac{dz}{\pi(z-z')}$ with simple pole at z' having residue $\frac{1}{\pi}$. Similarly g = the δ-function at z' = the distributional section of $\Omega^{1,0} \otimes \Omega^{0,1} = \Omega^{1,1}$ supported at z' with integral $f(z')$ against a fn. f.

So for the moment I conclude that it [redacted] would be very nice to take $E = \Omega^{1,0}$. In this case [redacted] $\bar{\partial} : \Omega^{1,0} \to \Omega^{1,1}$ is not invertible, that is we can't construct [redacted] meromorphic diff'ls with only a simple pole.

What we can do is this. Given a point P' of M choose a coordinate function [redacted] near this point and consider $\dfrac{dz}{\pi(z-z')}$ which is a differential form over the coordinate nbd. U defined for z' in this coordinate nbd. Now [redacted] Choose $\psi \in C_0^\infty(U)$ with $\psi \equiv 1$ near P' and put

$$K(z,z') = \psi(z)\frac{dz}{\pi(z-z')}$$

This is a section of $\Omega^{1,0}$ depending on $z' \in U$. It is smooth unless $z = z'$ and $\psi(z) \neq 0$. If z' remains in the region where $\psi \equiv 1$, then $\bar{\partial}_z K(z,z')$ is smooth for $z \neq z'$ and as z comes near z' it comes into the region with $\psi \equiv 1$, hense $\partial K(z,z') =$ [redacted] $\delta(z,z')$ for z, z' [redacted] in the $\psi \equiv 1$ region.

~~[illegible] So for~~

Therefore we get a covering $\{U\}$ of M and for each U a $K_U(z,z')$ which is a globally defined $(1,0)$ form in z depending on $z' \in U$ such that

$$\bar{\partial}_U K(z,z') = \delta(z,z') \qquad \text{for } z, z' \in U.$$

So now put

$$K(z,z') = \sum_U{}' K_U(z,z')\,\varphi_U(z')$$

where $\sum \varphi_U = 1$ is a partition of 1 with $\operatorname{Supp} \varphi_U \subset U$. Then

$$\partial K(z,z') = \sum_U{}' \partial K_U(z,z')\,\varphi_U(z')$$

smooth for $z \neq z'$, and for z close to z' it is

$$\sum_{u,\ \varphi_u(z')\neq 0} \delta(z,z')\,\varphi_u(z') = \delta(z,z')$$

Next I want to use this parametrix to prove Riemann-Roch. I am going to use P for the parametrix instead of K which is want to use for smooth kernels. Now we have constructed P satisfying

$$PD = I - K_0 \quad \text{on } C^\infty(E) \qquad\qquad D = \bar\partial$$
$$DP = I - K_1 \quad \text{on } C^\infty(E\otimes\Omega^{0,1})$$

hence the identity map on the $\bar\partial$ complex is homotopy to the map given by K_0, K_1; this is the standard proof that the cohomology is finite-dimensional. Also

$$\begin{aligned} \text{index} &= \operatorname{tr} \text{Identity on cohomology} \\ &= \operatorname{tr} \ K^\cdot \text{ on cohomology} \\ &= \operatorname{tr}(K^\cdot) = \operatorname{tr}(K_0) - \operatorname{tr}(K_1) = \operatorname{tr}\,[D,P] \end{aligned}$$

Let's compute the commutator. If we use the construction $P = \sum \psi_u P_u \varphi_u$ where $\sum \varphi_u = 1$ is a partition and $\psi_u \equiv 1$ on $\operatorname{Supp}\varphi_u$, then we have $\bar\partial P(z,z') = \delta(z,z')$ near the diagonal so that $\operatorname{tr}(K_1) = 0$, and the index will come from the diagonal behavior of $P\bar\partial$. Hence a better choice for P is $P = \sum \varphi_u P_u \psi_u$ which should satisfy $PD = \delta$ near Δ, and reduce the index to computing the diagonal part of DP.

$$DP = \bar\partial \sum \varphi_u(z) \frac{dz_u}{\pi(z_u - z'_u)} \psi_u(z')$$

$$= \sum \bar\partial \varphi_u(z) \frac{dz_u}{\pi(z_u - z'_u)} \psi_u(z') + \underbrace{\sum \varphi_u(z)\,\delta(z,z')\,\psi_u(z')}_{\delta(z,z')}$$

April 23, 1982

To under the L^2 index thm. of Atiyah. One has a discrete group Γ acting freely on X, an elliptic operator $D: E \longrightarrow F$ over X invariant under Γ. One defines

$$\dim_\Gamma(\mathrm{Ker}(D)) = \int_{X/\Gamma} \mathrm{tr}\, \langle x | \mathbf{E} | x \rangle \qquad \text{taken in } L^2(E)$$

where $\mathbf{E}$ is the orthogonal projection on $\mathrm{Ker}\, D$, and similarly $\dim_\Gamma(\mathrm{Cok}(D))$. Then the thm reads

$$\mathrm{ind}_\Gamma(D) = \mathrm{ind}(\bar{D} \text{ over } \bar{X} = \Gamma \backslash X)$$

The idea behind the proof is to use some sort of parametrix P for the operator D which corresponds to a parametrix $\bar{P}$ for $\bar{D}$, and to use the formula

$$\mathrm{ind}(\bar{D}) = \mathrm{tr}([\bar{D}, \bar{P}])$$

suitably interpreted. (Recall that if we have $\Gamma(E) \underset{D}{\overset{P}{\rightleftarrows}} \Gamma(F)$ with $PD = I - K_0$, $DP = I - K_1$, then P is a homotopy between I and $K = (K_0, K_1)$, so that

$$\begin{aligned}
\mathrm{ind}(D) &= \mathrm{tr}(\mathrm{Id\ on\ } H^*) = \mathrm{tr}(K \text{ on } H^*) \\
&= \mathrm{tr}(K_0) - \mathrm{tr}(K_1) \\
&= \mathrm{tr}(I - PD) - \mathrm{tr}(I - DP) \overset{\text{"}}{=}\!\!\text{"}\ \mathrm{tr}([D, P]).
\end{aligned}$$
)

What I need to do is find exactly what P should be. Presumably a natural candidate is to take P to be the inverse of the unboundeded operator $D: L^2(E) \ominus \mathrm{Ker}\, D \longrightarrow \overline{\mathrm{Im}\, D} \subset L^2(F)$. Then we have

$$\begin{aligned}
PD &= I - E_0 \qquad & E_0 &= \text{orth proj. on } \mathrm{Ker}(D) \\
DP &= I - E_1 \qquad & E_1 &= \text{―――} \ \overline{\mathrm{Im}}(D).
\end{aligned}$$

and so $\mathrm{ind}_\Gamma(D) = \mathrm{tr}_\Gamma([D, P])$, and our problem is to identify the latter with $\mathrm{tr}[\bar{D}, \bar{P}]$.

A natural question is whether the P defined in this way induces the analogously defined $\bar{P}$.

Let's look at the case of the operator $D = \partial_{\bar{z}} : 1 \to 1$ over $\mathbb{C}$ with $\Gamma =$ a lattice subgroup of translations so that $\bar{X} = \mathbb{C}/\Gamma$ is an elliptic curve. We can understand D on $L^2(\mathbb{C})$ by means of the Fourier transform:

$$f(z) = \int e^{\mu\bar{z} - \bar{\mu}z} \, \hat{f}(\mu) \, \frac{d^2\mu}{\text{const.}}$$

$\partial_{\bar{z}} f$ corresponds to $\mu \hat{f}(\mu)$. So $\mathrm{Ker}(D) = 0$, $\mathrm{Cok}(D) = 0$ but the image of D on L^2 is not closed.

Now the parametrix which corresponds to multiplying by $\frac{1}{\mu}$ on the transform side is given by the kernel $\frac{1}{\pi(z-z')}$. This gives the L^2-inverse of D.

Pass to $\bar{D} = \partial_{\bar{z}}$ over $\mathbb{C}/\Gamma$. Here the corresponding $\bar{P}$ is given by the kernel $Q(z-z')$, where

$$Q(z) = \sum_{\substack{\text{dual} \\ \text{lattice}}}{}' \frac{e^{\mu\bar{z} - \bar{\mu}z}}{\mu} \frac{1}{\mathrm{vol}(\mathbb{C}/\Gamma)}$$

which is a version of the Weierstrass ζ-fn adjusted by a linear term in $\bar{z}$ so as to be doubly-periodic.

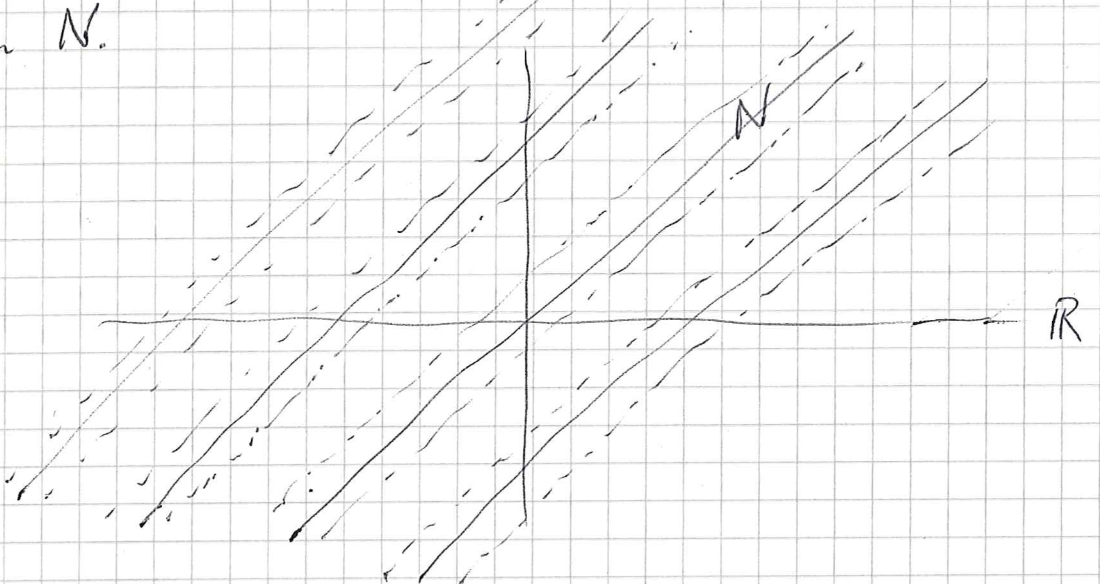
$$Q(z) = \frac{1}{\pi} \sum_{\gamma \in \Gamma}{}' \left(\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} \right) + c_1 \bar{z} + c_2$$

and to have integral zero.

This seems to show that the P defined as above doesn't induce the corresponding $\bar{P}$. Perhaps a better choice is to use a parametrix of the form

$$\frac{\rho(z-z')}{\pi(z-z')}$$

where $\rho \in C_0^\infty(\mathbb{C})$ satisfies $\rho \equiv 1$ near $z = 0$, and has small enough support that the sets $\gamma + \operatorname{Supp} \rho$, $\gamma \in \Gamma$ are disjoint. So what we are doing is to take a $\bar{P}$ over $\mathbb{C}/\Gamma$ whose kernel is supported in a small nbd. $\bar{N}$ of the diagonal. Then when lifted back to $\mathbb{C}$ its support is a disjoint union $\coprod_{\gamma \in \Gamma} (\gamma, 0) + N$ and then we take P to be the part of $\bar{P}$ supported in N.



So this gives me what sort of operator on the transform level? $f(z) = \frac{\rho(z)}{\pi z}$ satisfies $\partial_{\bar{z}}\left(\frac{\rho(z)}{\pi z}\right) = \delta(z) + C_0^\infty$, hence its transform when multiplied by μ is $1 +$ rapidly decreasing fn.. Hence basically one takes $\frac{1}{\mu}$ and modifies it near 0 so to be smooth. Thus

$$\hat{f}(\mu) = \frac{g(\mu)}{\mu} \qquad g \to 1 \quad \text{as } \mu \to \infty$$

and where g is divisible by μ. Then $\bar{P}$ on the Fourier transform level is given by $\hat{f}(\mu)$ restricted to the dual lattice points.

In general then we pick a parametrix $\bar{P}$ for $\bar{D}$ whose kernel is supported in a small tubular nbd. of the diagonal in $(\bar{X})^2 = (\Gamma \backslash X)^2$. Then I lift $\bar{P}$ back to X, and I define P to be the part of it supported in the corresponding nbd of the diagonal in $(X)^2$. It's then clear that because D is a differential operator, we obtain the kernel for $[D,P]$ by the same process from $[\tilde{D},\tilde{P}]$. Consequently we get

$$\mathrm{ind}(\tilde{D}) = \int_{\Gamma\backslash X} tr \langle x | [D,P] | x \rangle \overset{\text{defn.}}{=} Tr_{\Gamma}([D,P])$$

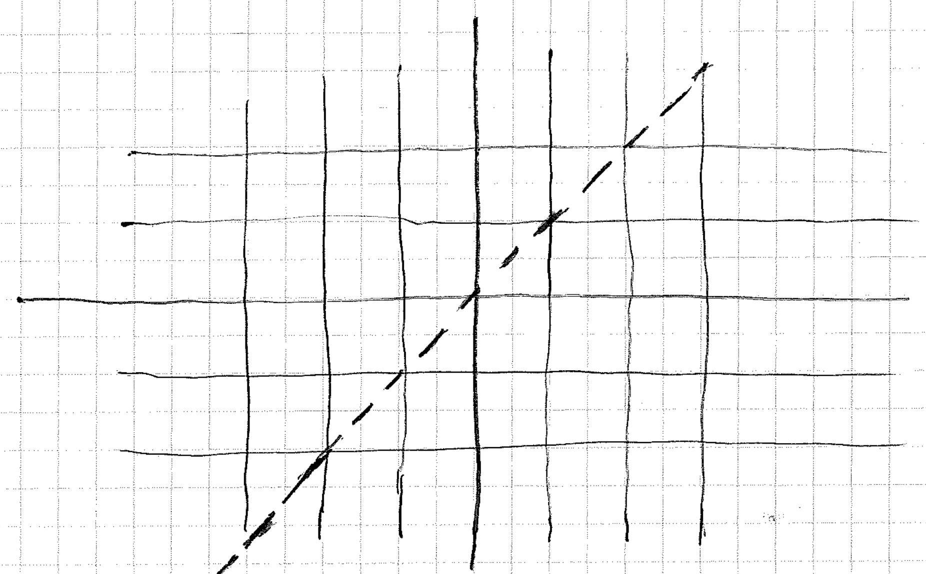
Therefore we see that the real problem is to get from the L^2-index to this Tr_Γ, and this is ultimately a trace calculation in a certain algebra of operators. The operators work in $L^2(X,E)$ and $L^2(X,F)$. To simplify assume $E=F=$ trivial bundle. So I have operators on $L^2(X)$ which commute with the action of Γ. Now if we use a measure space isomorphism $X \xrightarrow{\sim} (\Gamma\backslash X)\times\Gamma$, then we have

$$L^2(X) = L^2(\Gamma\backslash X)\,\hat{\otimes}\,\ell^2(\Gamma)$$

For example

$$L^2(\mathbb{R}) = \widehat{\bigoplus_{n\in\mathbb{Z}}} L^2([n,n+1]) = L^2([0,1])\,\hat{\otimes}\,\ell^2(\mathbb{Z}).$$

I can [redacted] picture the operators by their kernels



Now Γ invariance says that the blocks related by translation in the direction $\nearrow$ are the same. From this viewpoint the operators I am working with are in the ring:

$$\mathrm{End}(L^2(X/\Gamma)) \otimes \text{group ring of } \Gamma$$

and the trace I am interested is the normal trace on the first factor (hence End here should be interpreted as trace class operators) and the evaluation at the identity element on the group ring.

The problem goes as follows. Working on the Hilbert space $L^2(X)$ we the quasi-inverse Q for D defined using projections, so that

$$\begin{cases} QD = I - p_0 & p_0 = \text{proj on } \mathrm{Ker}\, D \\ DQ = I - p_1 & p_1 = \text{——— } \mathrm{Cok}\, D \end{cases}$$

and on the other hand with our parametrix P we have

$$\begin{cases} PD = I - K_0 \\ DP = I - K_1 \end{cases}$$

Now the point is that to prove $tr(p_0) - tr(p_1) = tr\,[D,Q]$ is the same as $tr(K_0) - tr(K_1) = tr\,[D,P]$, we are going to need

$$tr\,[D, P-Q] = 0.$$

In all this should appear tr_Γ and we are working in an operator algebra and we are working in an algebra (= commutant of Γ in $\mathrm{End}(L^2(X))$) where the trace class operators are different.

So for example look at the commutant of Γ in the operators on $\ell^2(\Gamma)$. Such operators are given by right translation by elements of Γ and linear combinations of these. When Γ is abelian, use $\ell^2(\Gamma) \simeq L^2(\Gamma^\vee)$ and then we want the commutant of multiplication by characters in $\Gamma^\vee$, and hence the commutant is $L^\infty(\Gamma^\vee)$. The trace is then the integral over $\Gamma^\vee$ relative to Haar measure, and its defined on every element of $L^\infty(\Gamma^\vee)$.

Key calculation:

$$PDQD = (I-K_0)(I-p_0) = I - K_0 - p_0 + K_0 p_0$$
$$QDPD = (I-p_0)(I-K_0) = I - p_0 - K_0 + p_0 K_0$$

have difference with trace 0, because $\mathrm{tr}\,[K_0, p_0] = 0$ since both are of trace class. Now

$$\begin{aligned} PDQD &= P(I-p_1)D = PD - Pp_1 D \\ &= I - K_0 - p_1 DP + [p_1 D, P] \\ &= I - K_0 - p_1 + p_1 K_1 + [p_1 D, P] \\ QDPD &= Q(I-K_1)D \\ &= I - p_0 - K_1 + K_1 p_1 + [K_1 D, Q] \end{aligned}$$

The latter two commutators have 0 trace because $p_1 D$ $(K_1 D)$ should be ~~bounded~~ trace class and P (Q) should be bounded. So we get

$$\mathrm{tr}(K_0 + \underline{p_1 - p_0} - K_1) = 0$$

Better proof:

$$\begin{aligned} (PD)Q &= (I-K_0)Q = Q - K_0 Q \\ P(DQ) &= P(I-p_1) = P - Pp_1 \\ \therefore\ P - Q &= Pp_1 - K_0 Q \end{aligned}$$

$$[D, P-Q] = (DPp_1 - Pp_1 D) - (DK_0 Q - K_0 Q D)$$

But $\mathrm{tr}(DPp_1) = \mathrm{tr}(p_1 DP) = \mathrm{tr}(Pp_1 D)$ again using DP, P bounded, $p_1, p_1 D$ bounded + trace class.
Similarly for the other term so

$$\mathrm{tr}\,[D, P-Q] = 0$$

April 25, 1982: Goal – to understand Alain Connes.

It is necessary to understand Atiyah's work carefully:

Idea 1: Atiyah's operator version of K-homology classes.

Take a compact smooth manifold X. Symbols of [redacted] elliptic pseudo-diffl operators over X give elements of $K_c(T_X^*)$. On the other hand T_X^* is a naturally a symplectic manifold, and hence has an almost complex structure unique up to homotopy. In particular it is orientable for K-theory, so one has a Gysin map $\pi_! : K_c(T_X^*) \longrightarrow K(pt) = \mathbb{Z}$. The 'index thm. says that

$$\text{Index}(\psi) = \pi_!(\sigma_\psi).$$

[redacted] Presumably $K_c(T_X^*)$ can be identified with the K-homology of X. In any case using the natural module structures of $K(X)$ on $K_c(T_X^*)$, we see any element of $K_c(T_X^*)$ defines a map $K(X) \to K(pt) = \mathbb{Z}$.

Let's do this last step analytically. Start with a degree 0 pseudo-differential operator $P: E \to F$ over X. [redacted] Think of P as a bounded operator from $L^2(E) \longrightarrow L^2(F)$. Then for any $f \in C(X)$ one knows that $[P,f]$ is a compact operator. (The reason is that this commutator is a ψDO of order -1, hence bounded from $L^2(E) =$ [redacted] $H_0^2(E)$ to $H_1^2(E)$, and because $H_1^2(E) \hookrightarrow H_0^2(E)$ is compact.)

Now we must see how to [redacted] take an element in [redacted] $K(X)$ and get an integer. The simplest case is [redacted] for $1 \in K(X)$ in which case we have to define the index of P. Assuming this can be done lets take a vector bundle N over X, and suppose N is the image of a projector $e \in M_n(C(X))$. Then $L^2(N \otimes E) = \operatorname{Im}(e)$ on $L^2(E)^{\oplus n}$, and so the effect of tensoring with N is just taking the image of e on the n-fold direct sum of the Hilbert spaces. Of course we don't have that

$$P^{\oplus n} : L^2(E)^{\oplus n} \longrightarrow L^2(F)^{\oplus n}$$

commutes with e, however, because e is made of continuous functions we have that $[e, P^{\oplus n}]$ is compact. So if we think of $P^{\oplus n}$ as a block matrix corresponding to the e-decomposition

$$P^{\oplus n} = \left(\begin{array}{c|c} eP^{\oplus n}e & eP^{\oplus n}(1-e) \\ \hline (1-e)P^{\oplus n}e & (1-e)P^{\oplus n}(1-e) \end{array}\right)$$

then the off-diagonal blocks are compact, so the diagonal blocks will be Fredholm. So we get a Fredholm op.

$$L^2(N \otimes E) \xrightarrow{eP^{\oplus n}e} L^2(N \otimes F)$$

whose index is what we want.

Atiyah's idea is now to abstract this. A K-homology class over X should be given by the following: Two Hilbert spaces H_i which are modules over $C(X)$ and a Fredholm operator $P : H_1 \longrightarrow H_2$ such that $[P, f]$ is compact for all $f \in C(X)$.

These triples (H_1, H_2, P) obviously push forward under a map of spaces. Using the fact that complex bordism maps onto (?) K-homology, one sees that any K-homology class can be realized by such triples.

[redacted] Let's review. We have [redacted] a Fredholm operator $P: H_1 \longrightarrow H_2$, where H_i are $A = C(X)$-modules, and where $[P,f]$ is compact for all $f \in A$. Then [redacted] can define a map [redacted] $K(A) \longrightarrow \mathbb{Z}$ by sending a projector $e \in M_n(A)$ into the index of $eP^{\oplus n}e$.

So now let us [redacted] try to get at the index via a trace. Hence I will assume the compact operators appearing are of trace class.

The basic formula is that if $Q: H_2 \to H_1$ is an inverse of P modulo trace class operators, then

$$\operatorname{Index}(P) = \operatorname{tr}[P,Q] \qquad (\text{strictly } \operatorname{tr}(PQ-I) - \operatorname{tr}(QP-I))$$

Replace A by $M_n(C(X))$ and $P^{\oplus n}, Q^{\oplus n}$ by simply P, Q. Then as e ranges over idempotents in [redacted] A, [redacted] I want

$$\operatorname{Ind}(ePe) = \operatorname{tr}[ePe, eQe].$$

[redacted]

Now I should bring in Tate's approach to the residue: If $\pi: L^2(S^1) \longrightarrow H^2(S^1)$ is the projection, then one has

$$\operatorname{Res} f\,dg = \operatorname{tr}[\pi f, \pi g]$$

$$\operatorname{Res} f\,df^{-1} = \operatorname{tr}[\pi f, \pi f^{-1}] = \operatorname{Ind}(\pi f)$$

(e.g. if $f = z^n$, then $\operatorname{Ind}(\pi f) = -n$). This sort of thing shows that you want to regard the [redacted] index as a special case of the traces which is a bilinear forms.

So the idea is that by understanding traces in A we get a kind of cohomology classes (ultimately Hochschild cohomology). Then the index will be computed in terms of the curvature of the idempotent.

Something to think about: To define the map $K_0(A) \to \mathbb{Z}$ defined by a P one uses

$$\mathrm{Ind}(ePe) = \mathrm{tr}([eP, eQ]e)$$

and keeps P fixed while varying e. On the other has the Toeplitz operator viewpoint has a fixed projection e and the invertible operator P varies, so you get a map from a K_1 to $\mathbb{Z}$.

Hochschild cohomology: Given an algebra A one considers the category of A-bimodules M and derives the functor

$$H^0(A, M) = \mathrm{Hom}_{A\otimes A^{op}}(A, M)$$

$$= \{m \mid am = ma \quad \forall a\}$$

namely a map $\varphi: A \to M$ such that $\varphi(axb) = a\varphi(x)b$ is determined by $\varphi(1) = m$, which must then satisfy $\varphi(x) = \varphi(x \cdot 1) = x\varphi(1) = xm$, as well as $\varphi(x) = \varphi(1 \cdot x) = mx$.

We want to take $M = A^*$ in which case

$$(af)(x) = f(xa) \qquad \langle x | af \rangle = \langle xa | f \rangle$$

$$(fa)(x) = f(ax) \qquad \langle x | fa \rangle = \langle ax | f \rangle$$

Hence

$$H^0(A, A^*) = \text{maps } f: A \to \mathbb{C} \text{ such that } f(xa) = f(ax)$$

gives all traces on the algebra A.

Higher Hochschild cohomology uses the standard triple resolution

$$\cdots \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; A\otimes A\otimes A \; \substack{\longrightarrow\\ \longrightarrow} \; A\otimes A \longrightarrow A$$

which leads to the cochain complex

$$C^0(A,M) \longrightarrow C^1(A,M) \rightrightarrows C^2(A,M) \quad \cdots$$

An element $\varphi \in C^1(A,M) = \mathrm{Hom}_{A\otimes A^{op}}(A\otimes A\otimes A, M)$ can be viewed as a function $\varphi(x,y,z)$ such that

$$\varphi(ax,y,zb) = a\,\varphi(x,y,z)\,b.$$

and $$(\delta\varphi)(x_0,x_1,x_2,x_3) = \varphi(x_0x_1,x_2,x_3) - \varphi(x_0,x_1x_2,x_3) + \varphi(x_0,x_1,x_2x_3)$$

But it's simpler to write

$$\varphi(x_0,x_1,x_2) = x_0\,\psi(x_1)\,x_2$$

and the the right-side for $\delta\varphi$ is

$$x_0x_1\,\varphi(x_2)x_3 - x_0\varphi(x_1x_2)x_3 + x_0\varphi(x_1)x_2x_3$$

or simpler

$$(\delta\psi)(x_1,x_2) = x_1\psi(x_2) - \psi(x_1x_2) + \psi(x_1)x_2.$$

So the general pattern is clear.

Next take $M = A^*$. Then we we should associate to $\psi : A \to A^*$, the form

$$\varphi(x_0,x_1) = \langle x_0, \psi(x_1)\rangle$$

whence to $\delta\psi$ belongs

$$\langle x_0, \delta\psi(x_1,x_2)\rangle = \langle x_0x_1, \psi(x_2)\rangle - \langle x_0, \psi(x_1x_2)\rangle + \langle x_2x_0, \psi(x_1)\rangle$$

or finally:

$$(\delta\varphi)(x_0,x_1,x_2) = \varphi(x_0x_1,x_2) - \varphi(x_0,x_1x_2) + \varphi(x_2x_0,x_1)$$

April 28, 1982

The question is whether Connes cohomology might be related to Lie algebra cohomology.

Let's first describe Connes cohomology. Let $C^{\cdot}(A;A^*)$ be the Hockschild cochain complex with values in the dual of A. Then an element of $C^q(A;A^*)$ can be identified with a map $\varphi : A^{\otimes(q+1)} \longrightarrow \mathbb{C}$, $x_0 \otimes \cdots \otimes x_q \longmapsto \varphi(x_0, \ldots, x_q)$. We think of this as $\varphi(x_0, \ldots, x_q) = \langle x_0 \,|\, \tilde{\varphi}(x_1, \ldots, x_q)\rangle$ where $\tilde{\varphi}$ is an actual Hochschild cochain. Then in the $\sim$ version

$$(\delta\tilde{\varphi})(x_1, x_2, x_3) = x_1\tilde{\varphi}(x_2, x_3) - \tilde{\varphi}(x_1x_2, x_3) + \tilde{\varphi}(x_1, x_2x_3) - \tilde{\varphi}(x_1, x_2)x_3$$

so
$$\begin{aligned}(\delta\varphi)(x_0, x_1, x_2, x_3) &= \langle x_0 \,|\, \delta\tilde{\varphi}(x_1, x_2, x_3)\rangle \\ &= \langle x_0x_1 \,|\, \tilde{\varphi}(x_2, x_3)\rangle - \cdots - \langle x_3x_0 \,|\, \tilde{\varphi}(x_1, x_2)\rangle \\ &= \varphi(x_0x_1, x_2, x_3) - \varphi(x_0, x_1x_2, x_3) + \varphi(x_0, x_1, x_2x_3) \\ &\quad - \varphi(x_3x_0, x_1, x_2).\end{aligned}$$

Among the elements of $C^q(A;A^*)$ are those $\varphi(x_0, \ldots, x_q)$ which are sign-symmetric for cyclic permutations, i.e.

$$\varphi(x,y) = -\varphi(y,x) \qquad q = 1$$
$$\varphi(x,y,z) = \varphi(y,z,x) \qquad q = 2$$

etc. These form a subcomplex of the Hochschild cohomology. For example

$q=0$ $\qquad \delta\varphi(x_0, x_1) = \varphi(x_0x_1) - \varphi(x_1x_0)$ is skew-symm.

$q=1$ $\qquad \begin{cases} \delta\varphi(x_0, x_1, x_2) = \varphi(x_0x_1, x_2) - \varphi(x_0, x_1x_2) + \varphi(x_2x_0, x_1) \\ \delta\varphi(x_1, x_2, x_0) = \varphi(x_1x_2, x_0) - \varphi(x_1, x_2x_0) + \varphi(x_0x_1, x_2) \end{cases}$

these two are equal if φ is skew-symm.

Connes cohomology $H^*(A)$ is defined to be the cohomology of this subcomplex.

<u>April 29, 1982</u>

Connes exact sequence

$$0 \longrightarrow H^0(A) \xrightarrow{\sim} H^0(A;A^*) \longrightarrow \underset{\overset{\shortparallel}{0}}{H^{-1}(A)} \longrightarrow H^1(A)$$
$$\longrightarrow H^1(A;A^*) \longrightarrow H^0(A) \longrightarrow H^2(A)$$
$$\longrightarrow H^2(A;A^*) \longrightarrow H^1(A) \longrightarrow H^3(A) \longrightarrow$$

seems to result from the following. In the Hochschild cochain complex $C^\bullet(A;A^*)$ sits the complex $C_c^\circ(A)$ of Connes cochains. There seems to exist a quasi-isom. of

$$C^\circ(A;A^*)/C_c^\circ(A) \sim \Sigma C_c^\circ(A)$$

which then leads to the above sequence.

Here's how we can use the sequence to do computations. Take $A = \mathbb{C}$ or more generally the simple algebra $M_n(\mathbb{C})$. Then the Hochschild cohomology

$$H^*(A;A^*) = \mathrm{Ext}^*_{A\otimes A^{op}}(A, A^*)$$

vanishes in positive dimensions because $A\otimes A^{op}$ is semi-simple. Also $H^0(A) = H^0(A;A^*) = \mathbb{C}$ because there is a unique trace. So what we find is the beautiful periodicity

$$H^0 = H^2 = H^4 = \cdots\cdots = \mathbb{C}$$
$$H^1 = H^3 = H^5 = \cdots\cdots = 0.$$

Follow Loday and work with homology instead. Hochschild homology is derived functors of

$$H_0(A;M) = M/[A,M] \qquad \therefore \quad H_q(A;M) = \mathrm{Tor}_q^{A\otimes A^{op}}(A,M)$$

and is computed by the standard complex

$$\cdots A\otimes A\otimes A\otimes A \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; A\otimes A\otimes A \rightrightarrows A\otimes A$$

where the faces use the product in A. Taking $M = A$ gives the complex

$$A\otimes A\otimes A \longrightarrow A\otimes A \longrightarrow A$$

$$a\otimes b \longmapsto ab - ba$$

$$a\otimes b\otimes c \longmapsto ab\otimes c - a\otimes bc + ca\otimes b \qquad \text{etc.}$$

and Connes complex is a quotient by the cyclic group. So we get the [redacted] basic sequence

$$\to H_n^H(A) \to H_n^C(A) \longrightarrow H_{n-2}^C(A) \to H_{n-1}^H(A) \to \cdots\cdots$$

Loday's [redacted] conjecture (maybe thm.) is that

$$H_{n-1}^C(A) = \text{Prim}\{H_n(gl(A))\}$$

so the interesting thing here is the periodicity map

$$H_n^C(A) \longrightarrow H_{n-2}^C(A)$$

which goes <u>down in degree.</u>

<u>Remark</u>: For $k = \mathbb{C}$ the calculation of Connes cohomology is trivial since A is 1-dimensional, so the symmetry condition makes all the odd degree cochains $= 0$.

Suppose A is a smooth f.t. k-algebra. Then the Hochschild homology of A is

$$\text{Tor}_p^{A\otimes A}(A, A) = \Omega^p_{A/k}$$

hence vanishes above the dimension of A. Hence the <u>Connes homology is periodic in large dimensions.</u>

Question: Is Connes homology the same for A and $M_n(A)$?

April 30, 1982

Let's use Connes exact sequence to compute the Connes homology for a smooth k-algebra in terms of the de Rham complex. We know that

$$H_p^H(A) = \Omega^p_{A/k}$$

and

$$H_0^C = H_0^H = \Omega^0 .$$

Next we have

$$\begin{array}{ccccccccc} H_2^C & \longrightarrow & H_0^C & \xrightarrow{\alpha} & H_1^H & \longrightarrow & H_1^C & \longrightarrow & 0 \\ & & \Omega^0 & & \Omega^1 & & & & \end{array}$$

and let's assume the map α can be identified with d. Then we have

$$H_1^C = \Omega^1/d\Omega^0 .$$

$$\begin{array}{ccccccccccc} H_3^C & \longrightarrow & H_1^C & \xrightarrow{\alpha} & H_2^H & \longrightarrow & H_2^C & \longrightarrow & H^0_{DR} & \longrightarrow & 0 \\ & & \| & & \| & & & & & & \\ & & \Omega^1/d\Omega^0 & & \Omega^2 & & & & & & \end{array}$$

so if we identify α again with d we get

$$0 \longrightarrow \Omega^2/d\Omega^1 \longrightarrow H_2^C \longrightarrow H^0_{DR} \longrightarrow 0$$

$$\begin{array}{ccccccccccc} H_4^C & \longrightarrow & H_2^C & \xrightarrow{\alpha} & H_3^H & \longrightarrow & H_3^C & \longrightarrow & H^1_{DR} & \longrightarrow & 0 \\ & & & & \| & & & & & & \\ & & & & \Omega^3 & & & & & & \end{array}$$

Assuming the upper sequence splits and α again can be identified with d on Ω^2 and 0 on H^0_{DR}, we get

$$0 \longrightarrow \Omega^3/d\Omega^2 \longrightarrow H_3^C \longrightarrow H^1_{DR} \longrightarrow 0$$

$$\begin{array}{ccccccccc} H_5^C & \longrightarrow & H_3^C & \longrightarrow & H_4^H & \longrightarrow & H_4^C & \longrightarrow & H^2_{DR} \oplus H^0_{DR} \longrightarrow 0 \\ & & \Omega^3/d\Omega^2 \oplus H^1_{DR} & & \Omega^4 & & & & \end{array}$$

So we conclude the following formulas should hold.

$$H_0^C = \Omega^0$$
$$H_1^C = \Omega^1/d\Omega^0$$
$$H_2^C = \Omega^2/d\Omega^1 \oplus H^0_{DR}$$
$$H_3^C = \Omega^3/d\Omega^2 \oplus H^1_{DR}$$
$$H_4^C = \Omega^4/d\Omega^3 \oplus H^2_{DR} \oplus H^0_{DR}$$

evident periodicity c map $H_n^C \to H_{n-2}$

$$H_n^C = \Omega^n/d\Omega^{n-1} \oplus H^{n-2}_{DR} \oplus H^{n-4}_{DR} \oplus \cdots\cdots$$

Let's take the [redacted] viewpoint which we get from Loday that Connes [redacted] cohomology is some sort of Lie K-theory closely connected with ordinary K-theory for C^*-algebras.

~~Let's take a C algebra A of the stable type so that~~

Take a Banach algebra A. Associated to it is a K-theory made up [redacted] from projection operators e in $M_n(A)$ for different n and a K_1-theory made up of automorphisms in $M_n(A)$ for different n.

We need to understand the significance of Connes constructions. One key idea is that his cocycles give characteristic numbers to elements of $K_0(A)$ via the curvature mechanism. Let's understand this in low dimensions.

Let's start with [redacted] an algebra with 1 A. Then one forms an algebra of non-commutative differential forms ΩA which is

$$A \longrightarrow A\otimes\bar{A} \longrightarrow A\otimes\bar{A}\otimes\bar{A} \longrightarrow \cdots\cdots \qquad \bar{A} = A/\mathbb{C}1$$
$$a \longmapsto 1\otimes\bar{a},\ a\otimes\bar{b} \mapsto 1\otimes\bar{a}\otimes\bar{b}$$

It is obtained from the descent cosimplicial ring

$$A \rightrightarrows A\otimes A \ \substack{\to\\\to\\\to}\ A\otimes A\otimes A \cdots$$

by a process of normalization. No see below Connes notation is

$$a\,dx_1\cdots dx_n \quad \text{for} \quad a\otimes\bar{x}_1\otimes\cdots\otimes\bar{x}_n$$

Now given a right A-module M a connection is a map

$$D: M \longrightarrow M\otimes\bar{A}$$

such that $D(ma) = m\otimes\bar{a} + (Dm)a$. For example, suppose $M = eA$ where e is an idempotent. Then we can define

$$D(ea) = e\otimes\bar{a} + (e\otimes\bar{e})a$$

where the product on the right is computed as follows

$$(e\otimes\bar{e})a \text{ “=” } e\,de\,a = e\,d(ea) - e(e\,da)$$
$$= e\otimes\overline{ea} - e\otimes\bar{a}$$

Thus

$$\boxed{D(ea) = e\otimes\overline{ea}}$$

is the definition. Now check the identity

$$D((ea)b) = e\otimes\overline{eab} = e\,d(eab)$$
$$= e\left[d(ea)\,b + (ea)\,db\right]$$
$$= D(ea)b + (ea)\,db$$

so it works modulo the identity

$$d(xy) = x\,dy + dx\,y$$

which is more or less the definition. Actually to be sure this works one must take the Alexander–Whitney product on $A \quad A\otimes A \quad A\otimes A\otimes A$ and check that it descends to the normalized complex.

This doesn't seem to work. Thus it is not true that the [redacted] DG ring

$$A \longrightarrow A\otimes A \longrightarrow A\otimes A\otimes A \longrightarrow \cdots$$

$$a \longmapsto 1\otimes a - a\otimes 1$$

$$a\otimes b \longmapsto 1\otimes a\otimes b - a\otimes 1\otimes b + a\otimes b\otimes 1$$

with the Alexander-Whitney cup product

$$(a_0\otimes\cdots\otimes a_p)(b_0\otimes\cdots\otimes b_q) = a_0\otimes\cdots\otimes(a_p b_0)\otimes b_1\otimes\cdots\otimes b_q$$

admits $A \longrightarrow A\otimes\bar{A} \longrightarrow A\otimes\bar{A}\otimes\bar{A} \quad \cdots$

as a quotient. In effect the kernel of the [redacted] evident surjection is not an ideal:

$$a\otimes 1 \in \operatorname{Ker}(A\otimes A \longrightarrow A\otimes\bar{A})$$

but $\quad (a\otimes 1)\cdot b = a\otimes b \quad$ needn't be in the kernel

Thus it seems to be necessary to directly define the right multiplication in order to make Connes ΩA into a ring. Let's assume this has been done, and suppose we have a connection on a right A[redacted]-module M. Then we can extend D to $M\otimes_A \Omega A$ ~~[illegible]~~ by

$$D(m\omega) = Dm\cdot\omega + m\cdot d\omega.$$

Let's check

$$D((ma)\omega) = D(ma)\,\omega + (ma)d\omega$$
$$= ((Dm)\,a + m\,da)\,\omega + m\,a\,d\omega$$
$$D(m(a\omega)) = Dm\,(a\omega) + m\,d(a\omega).$$

Then

$$D^2(m\omega) = D(Dm\,\omega + m\,d\omega)$$
$$= (D^2 m)\,\omega - \cancel{Dm\,d\omega} + \cancel{Dm\,d\omega} + \cancel{m\,d^2\omega}$$

where we must check

$$D((m\omega_1)\omega_2) = D(m(\omega_1\omega_2)) = Dm\,(\omega_1\omega_2) + m\,d(\omega_1\omega_2)$$
$$= (Dm)\,\omega_1\omega_2 + m\,d\omega_1\cdot\omega_2 + (-1)^{\omega_1} m\cdot\omega_1\,d\omega_2$$
$$= D(m\omega_1)\,\omega_2 + (-1)^{\omega_1} m\omega_1\,d\omega_2$$

So for the connection $De = ede$ on eA we have the curvature $D^2e = D(ede) = ededе$. Suppose consider higher powers

$$D^3 e = D(e(de)^2) = De\,(de)^2 = e(de)^3$$

and so forth.

Now that one has this algebra of non-commutative differential forms, one wants cycles to "integrate" the forms over which in particular will give us a characteristic number attached to elements of K_0.

0-cycles: These have to be linear functions on A which will allow us to assign numbers to equivalent idempotents. Two idempotents e, e' are [redacted] equivalent when $\exists\, f, g \in A$ with $e = fg$ and $e' = gf$. Hence we see that a 0-cycle is just a trace on A.

[redacted]

In general [redacted] Connes shows that any [redacted] element of $H_C^{ev}(A)$ determines by the connection-curvature construction a homomorphism of $K_0(A)$ to k. Thus one has a map

$$(*) \qquad H_C^{ev}(A) \longrightarrow \mathrm{Hom}(K_0(A), k)$$

or better (probably) a map

$$K_0(A) \longrightarrow H^{C}_{ev}(A)$$

e.g.

$$K_0(A) \longrightarrow H_0^C(A) = A/[A,A]$$

which sends a projector to its [redacted] universal trace.

So for a smooth algebra A we expect maps

$$(**) \qquad K_0(A) \longrightarrow H^C_{2k}(A) = \Omega^{2k}/d\Omega^{2k-1} \oplus H^{2k-2}_{DR} \oplus H^{2k-4}_{DR} \oplus \cdots$$

but Connes construction is periodic, i.e., the map $(*)$ above factors thru $\beta = 1$, where β = periodicity. Hence one can conclude that the maps $(**)$ are probably nothing but the Chern character map

$$ch : K_0(A) \longrightarrow H^{ev}_{DR}(A)$$

Consider on the K_1-level the simplest case. We have the map

$$g \longmapsto g^{-1}dg \qquad\qquad A^* \longrightarrow A\otimes\bar{A}$$

and to get a characteristic number we need a linear map $\tau : A\otimes\bar{A} \longrightarrow k$ which gives a homomorphism $A^* \longrightarrow k$. But

$$g_1g_2 \longmapsto (g_1g_2)^{-1}d(g_1g_2) = g_2^{-1}(g_1^{-1}dg_1)g_2 + g_2^{-1}dg_2$$

and hence we want τ to be invariant under inner autos. on $A\otimes\bar{A}$. Thus I want

$$\tau(a(xdy)-(xdy)a) = 0$$

So now think of τ as a map $A\otimes A \longrightarrow k$ such that

$$\tau(xdy) = \tau(x,y)$$

Then we have

$$\tau(x,1) = 0$$

$$\tau(ax,y) - \tau(x,ya) + \tau(xy,a) = 0. \qquad \text{(from)}$$

If one puts $x=1$ in the latter

$$\tau(a,y) - \tau(1,ya) + \tau(y,a) = 0$$

so that τ skew-symmetric $\Longleftrightarrow$ $\tau(1,x)=0$, i.e. $\tau(dx)=0$.

The condition $\tau(dx)=0$ is natural so that paths of units have constant charac. no. In effect

$$\frac{d}{dt}\tau(g^{-1}dg) = \tau(-g^{-1}\dot{g}g^{-1}dg) + \underbrace{\tau(g^{-1}d\dot{g})}$$

$$\tau(d(g^{-1}\dot{g})) - \underbrace{\tau(dg^{-1}\dot{g})}_{\tau(g^{-1}dgg^{-1}\dot{g})}$$

$$\begin{aligned} & - \tau(g^{-1}\dot{g}g^{-1}dg) \\ &= \tau(d(g^{-1}\dot{g})) + \tau(g^{-1}dg\,g^{-1}\dot{g}) \\ &= 0 \end{aligned}$$

if τ is both central + vanishes on dx. Therefore we see how $\quad g \longmapsto \tau(g^{-1}, g) = \tau(g^{-1}dg) \quad$ τ Connes 1-cocycle

gives a homomorphism $A^* \longrightarrow k$ constant on components of A^*, hence also for $Gl_n(A)$, and hence a map

$$K_1(A) \longrightarrow k.$$

As for a Connes 1-coboundary $\quad \tau(x,y) = \varphi(xy) - \varphi(yx)$

$$\varphi(g^{-1} \cdot g) - \varphi(g \cdot g^{-1}) = 0.$$

May 10, 1982

Review all we know about the determinant line bundle over the space $\mathcal{A}$ of holomorphic structures on a C^∞ vector bundle E over a closed Riemann surface M. Each such holomorphic structure can be thought of as a $\bar\partial$-operator

$$D: E \longrightarrow E \otimes \Omega^{0,1}$$

i.e. a first order operator whose symbol is the map $E \otimes T^* \longrightarrow E \otimes \Omega^{0,1}$ given by id $\otimes$ projection. Thus $\mathcal{A}$ is an affine space for $\Gamma(\underline{Hom}(E, E\otimes\Omega^{0,1}))$, that is, we can write $D = D_0 + \alpha$, where D_0 is a fixed point of $\mathcal{A}$ and $\alpha \in \Gamma(\mathrm{End}(E)\otimes\Omega^{0,1})$.

Since M is compact the kernel + cokernel of D are finite-dimensional. In fact these are just the cohomology groups $H^0(M,\mathcal{E})$, $H^1(M,\mathcal{E})$ of the sheaf of holomorphic sections of E for the complex structure. Riemann-Roch gives the index

$$h^0 - h^1 = \deg E + (\operatorname{rank} E)(1-g).$$

A first project is to understand this result analytically.

One chooses a parametrix $P: E\otimes\Omega^{0,1} \longrightarrow E$. If chosen sufficiently well, then one has

$$PD = 1 - K^0$$

$$DP = 1 - K^1$$

where K^0, K^1 are trace class operators. Then if we think of P as a homotopy operator on the complex

$$C^* : \quad \Gamma(E) \xrightarrow{D} \Gamma(E\otimes\Omega^{0,1})$$

we see the identity is homotopic to the map $K = (K^0, K^1)$, hence $h^0 - h^1 = tr(\text{id on } H^*) = tr(K \text{ on } H^*) = tr(K \text{ on } C^*)$, where the last equality comes from the additivity of the trace.

Now we must understand what a parametrix looks like. It is given by a Schwarz kernel $P(z,z')\,d^2z'$, where $P(z,z')$ is a distribution with "values" $P(z,z') \in \mathrm{Hom}(E_{z'} \otimes \Omega^{0,1}_{z'}, E_z)$, and $d^2z' \in \Omega^{1,1}_{z'}$ is a volume element. We can write this instead as

$$P(z,z')\,dz'$$

where $P(z,z')$ has values in $\mathrm{Hom}(E_{z'}, E_z)$ and $dz' \in \Omega^{1,0}_{z'}$ is local $(1,0)$ form. In other given a section f of $E \otimes \Omega^{0,1}$, locally written $f(z)\,d\bar{z}$, then Pf is the section of E given by

$$(Pf)(z) = \int P(z,z') f(z')\,dz'\,d\bar{z}'$$

In order to recover the kernel of P, take f to approach a δ fn. in a suitable sense.

How can I make this clear? Let's first suppose D is invertible and that P is the inverse of D. Then fix a point $y \in M$ and $e \in E_y$ and choose a coord. fn. z near y. Then I take $f \in \Gamma(E \otimes \Omega^{0,1})$ to be

$$z' \longmapsto e\,\delta_y(z')\,d\bar{z}'$$

Then Pf will be a section of E holomorphic away from y. At y it has a simple pole with residue

$$\lim_{x \to y} \pi(z(x) - z(y))\,Pf(z(x)) = (e\,dz)_y$$

Here I use the fact that $\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\pi(z-y)}\right) = \delta_y(z)$ in a local coordinate patch.

Repeat: a parametrix is given by a kernel $P(z,z')\,dz'$ which is a distribution section of the bundle $\underline{\mathrm{Hom}}(pr_1^* E, pr_2^* E \otimes \Omega^{1,0})$ over $M \times M$, which is smooth off ΔM and such that the diagonal behavior is such that in a local coordinate patch

$$\circledast \qquad \lim_{z \to z'} \pi(z-z')\, P(z,z')\, dz' = id_{E_{z'}} \cdot dz'$$

(Is this the same as $P(z,z') = \dfrac{id}{\pi(z-z')} + \text{smooth}$?
Actually this latter doesn't maybe make sense, since $P(z,z') \in \mathrm{Hom}(E_{z'}, E_z)$.)

So the point seems to involve what precisely a parametrix is. The condition $\circledast$ is obviously independent of the choice of complex structure on E, and presumably is the kind of weak parametrix you get just from the symbol.

Let us then begin with the parametrix $P = \dfrac{1}{\pi(z-z')}$ on the trivial bundle over a coordinate patch. Then you consider a different holomorphic structure $D = \partial_{\bar z} + \alpha$. Then can I still use P to compute the index? Better: suppose P is a very good parametrix for D, say $P = D^{-1}$, then how good a parametrix is P for $D_1 = D + \alpha$?

$$PD_1 = P(D+\alpha) = I + P\alpha$$
$$D_1P = (D+\alpha)P = I + \alpha P$$

Now α is a multiplication operator and P blows up along the diagonal, so $P\alpha$, αP are not of trace class.

Let's now go over how we must construct a better parametrix for D_1. Put $P_1 = P + \beta$. We want $D_1P_1 = (D+\alpha)(P+\beta) = I + \alpha P + D\beta + \alpha\beta$ close to I, hence can define β so that $D\beta + \alpha P = 0$ i.e. $\beta = -P\alpha P$. Then $D_1P_1 = I - \alpha P \alpha P$, $P_1 D_1 = I - P\alpha P\alpha$.

Let's review this calculation in general. Suppose $PD = I - K^0$, $DP = I - K'$, but that K^0, K' are not of trace class. Put $P_1 = P + \alpha$. We would like

$$P_1 D = PD + \alpha D = I - K^0 + \alpha D$$

to be I, i.e. $\alpha = K^0 D^{-1}$. Take $\alpha = K^0 P$. Then

$$P_1 D = I - K^0 + K^0 \underbrace{PD}_{I-K^0} = I - (K^0)^2$$

$$\begin{aligned} DP_1 &= D(P + K^0 P) = D(P + (I - PD)P) \\ &= 2(DP) - (DP)^2 = 2(I-K') - (I-K')^2 \\ &= I - (K')^2 \qquad [\text{Simpler } DK^0 = K'D \text{ and as above}] \end{aligned}$$

Similar formulas in general hold. To derive them think of P as a homotopy operator between I and K, and one wants the homotopy operator between I and $(K)^n$.

$$\begin{aligned} I - (K)^n &= I - (I - (I-K))^n \\ &= n(I-K) - \binom{n}{2}(I-K)^2 + \cdots + (-1)^{n-1}(I-K)^n \end{aligned}$$

$$I - (K')^n = D\left(nP - \binom{n}{2}P(DP) + \cdots \right)$$

$$I - (K^0)^n = \left(nP - \binom{n}{2}(PD)P + \cdots \right) D$$

so

$$P_n = nP - \binom{n}{2} PDP + \binom{n}{3} PDPDP - \cdots$$

In the case of a Riemann surface, one must go to $\mathrm{tr}(K)^3$ before one gets something of trace class.

Next let's go over the cohomology determinant line bundle and it's holomorphic sections. So where am I to start? Let's think of our $\bar\partial$ operator as a Fredholm operator

$$V_1 \xrightarrow{T} V_0$$

between suitable Sobolev spaces. Then one [redacted] knows how to define a K-class on the space of Fredholm operators as follows. We choose a finite dimensional subspace F of V_0 and consider all T transversal to F, i.e. $\operatorname{Im} T + F = V_0$. Then we have [redacted] exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} T \arrow[r] \arrow[d, equal] & V_1 \arrow[r, "T"] & V_0 \arrow[r] & \operatorname{Cok} T \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Ker} T \arrow[r] & T^{-1}(F) \arrow[r, "T"] \arrow[u, phantom, "\cup" description] & F \arrow[r] \arrow[u, phantom, "\cup" description] & \operatorname{Cok} T \arrow[r] & 0
\end{tikzcd}$$

[redacted] Over the open set $\mathcal{U}_F$ where T is transversal to F we have a vector bundle with fibre $T^{-1}(F)$ and the constant bundle with fibre F. The difference $-T^{-1}(F)+F$ is a canonical K-element on $\mathcal{U}_F$. Evident compatibility as F increases, etc.

We will be interested in the line bundle $\mathcal{L}$ [redacted] obtained by taking the determinant of this K-elt. Hence the fibre of $\mathcal{L}$ at T is

$$\mathcal{L}_T = \lambda(F) \otimes \lambda(T^{-1}F)^* = \operatorname{Hom}(\lambda(T^{-1}F), \lambda(F))$$

when $T \in \mathcal{U}_F$. Notice that we have a canonical isom

$$\mathcal{L}_T = \lambda(F) \otimes \lambda(T^{-1}F)^* = \lambda(\operatorname{Cok} T) \otimes \lambda(\operatorname{Ker} T)^*$$

How do we get sections of $\mathcal{L}$? Think of the case where $\operatorname{Ker} T = 0$. Then $\operatorname{Cok} T$ is like the quotient bundle on the Grassmannian, so $\mathcal{L}$ is its highest exterior power, and we get sections of $\mathcal{L}$ from elements of $\Lambda^p V_0$ where $p = \dim(\operatorname{Cok} T)$, in particular from a p-dimensional subspace of V_0 equipped with a volume. Similarly in the case where $\operatorname{Cok} T = 0$ we have the highest exterior power of the [redacted] dual of the subbundle and so [redacted] get sections from $\Lambda^q V_1^*$, $q = \dim \operatorname{Cok} T$.

What's the general case? Let's fix the index of our Fredholm operators, call ~~it~~ d and choose a finite-dimensional subspace F of dim p in V_0 and a finite-diml. ~~quotient-space~~ $V_1 \twoheadrightarrow W$ of dim q of V_1 ~~with~~ $p+d=q$. This isn't clear.

What I want to do is to see sections of the line bundle $\mathcal{L}$ which are obviously global holom. sections, and I want to see when they vanish. I recall that one has an embedding

$$** \qquad \mathcal{L}_T^* \subset \operatorname{Hom}^{(d)}(\Lambda V_0, \Lambda V_1)$$

which is canonical. Namely one has canonical maps

$$\lambda(\operatorname{Ker} T) \otimes \Lambda(\operatorname{Im} T) \hookrightarrow \Lambda V_1$$

$$\Lambda V_0 \twoheadrightarrow \lambda(\operatorname{Cok} T) \otimes \Lambda(\operatorname{Im} T)$$

hence a canonical map

$$\lambda(\operatorname{Cok} T)^* \otimes \Lambda V_0 \twoheadrightarrow \Lambda(\operatorname{Im} T) \hookrightarrow \lambda(\operatorname{Ker} T)^* \otimes \Lambda V_1$$

etc. The first map we can think of as intersection with $\operatorname{Im} T$, the second with pulling back into V_1. So the total effect is T^{-1} on subspaces.

So if we have $**$ above, then we can define a section of $\mathcal{L}$, i.e. a homomorphism $\mathcal{L}_T^* \longrightarrow \mathbb{C}$, by an arbitrary linear fn. on $\operatorname{Hom}^{(d)}(\Lambda V_0, \Lambda V_1)$

$$\operatorname{Hom}^{(d)}(\Lambda V_0, \Lambda V_1) = \prod_{p \geq 0} \operatorname{Hom}(\Lambda^p V_0, \Lambda^{p+d} V_1)$$

Such things in particular are given by elements of

$$\Lambda^p V_0 \otimes (\Lambda^{p+d} V_1^*)$$

and in particular by a f.d. subspace $F \subset V_0$ and q. space $V_1 \twoheadrightarrow W$ equipped with volumes

May 11, 1982

The problem is to trivialize the cohomology determinant line bundle $\mathcal{L}$. Suppose that using analytic torsion I succeed in putting a hermitian metric on $\mathcal{L}$ such that the curvature is the Kähler form on the space $\mathcal{A}$ of connections. Because $\mathcal{A}$ is simply-connected two hermitian line bundles with the same curvature forms are isomorphic, hence it suffices to put a metric on the trivial line bundle over $\mathcal{A}$ with the curvature given by the Kähler form.

Let the Kahler form be ω, e.g. $\omega = \sum dz_i d\bar{z}_i$ over $\mathbb{C}^n$. A metric on the trivial line bundle is a function $g > 0$ giving the norm^2 of the section 1, and the curvature form is

$$\bar\partial \partial \log g$$

Put $h = \log g$, so that we want to solve

$$\bar\partial \partial h = \omega$$

with h real valued. A solution, recall, puts a suitable metric on the trivial bundle with $|1|^2 = e^h$. Another solution h_1 gives an isomorphic bundle, i.e. there exists an invertible holom. f with

$$|f|^2 e^h = e^{h_1}$$

To see this: $\bar\partial\partial(h_1 - h) = d\,\partial(h_1 - h) = 0$, and since $H^1 = 0$ one has a φ with $d\varphi = \partial(h_1 - h)$, hence $\bar\partial\varphi = 0$ and φ is holom. Moreover

$$d(\varphi + \bar\varphi) = \partial(h_1 - h) + \bar\partial(h_1 - h) = d(h_1 - h)$$

so that $\varphi + \bar\varphi + h = h_1 + \text{const}$ so $|f|^2 e^h = e^{h_1}$ where $f = e^\varphi$.

This argument shows the uniqueness when $H^1 = 0$, but we still need the existence of h starting from ω. Again the point is that

$$\bar\partial(\partial h) = d(\partial h)$$

[redacted] and because $H^2 = 0$, we can solve $d\eta = \omega$ with η purely imaginary. Then the $(1,0)$ component α of η times 2 satisfies $\bar\partial$ [redacted] $\alpha = \omega$, ∂ [redacted] $\alpha = 0$. Hence

$$\eta = \tfrac{1}{2}(\alpha - \bar\alpha) \qquad d\eta = \tfrac{1}{2}(\cancel{\partial\alpha} + \overset{\omega}{\bar\partial\alpha} - \overset{-\bar\omega}{\partial\bar\alpha} - \cancel{\bar\partial\bar\alpha}) = \omega$$

whereas

$$d\left(\tfrac{1}{2}(\alpha + \bar\alpha)\right) = \tfrac{1}{2}(\cancel{\partial\alpha} + \underset{\omega}{\bar\partial\alpha} + \underset{+\bar\omega}{\partial\bar\alpha} + \cancel{\bar\partial\bar\alpha}) = 0.$$

Hence as $H^1 = 0$ we have $\alpha + \bar\alpha = dh$, whence $\alpha = \partial h$ and so we have solved $\bar\partial\partial h = \omega$.

Let's put it all together. Let me assume that analytic torsion gives me a metric on $\mathcal{L}$ such that $\bar\partial\partial \log|s|^2 = \omega$ for all of my [redacted] sections s. Suppose also that I have a real-valued fn. h on $\mathcal{A}$ such that $\bar\partial\partial h = \omega$. Then I want to see how to construct an everywhere non-vanishing holom. section of $\mathcal{L}$. But by the above analysis of uniqueness (where we described solns. of $\bar\partial\partial(h_1 - h) = 0$) we know there is a unique-up-to-multiplicative-constant holom. function f with

$$|fs|^2 = |f|^2 |s|^2 = e^h \text{ [redacted]},$$

[redacted] f defined where s is non-vanishing over 1-connected open sets. So now it's clear: You pick h and then locally up-to-multiplying by an elt. of S^1, there is a unique holom. section s with $|s|^2 = h$, then you trivialize the resulting flat S^1-bundle to obtain a global such [redacted] section.

Let's now consider the index 0 case.
Here the coh. det. bundle $\mathcal{L}$ has a canonical section s.
In finite dims. we have for $T: V_1 \longrightarrow V_0$

$$\mathcal{L}_T = \lambda(\operatorname{Ker} T)^* \otimes \lambda(\operatorname{Cok} T) \underset{\text{canon}}{\cong} \lambda(V_1)^* \otimes \lambda(V_0) = \operatorname{Hom}(\lambda(V_1), \lambda(V_0))$$

Hence $\mathcal{L}$ is the trivial bundle with fibre $\operatorname{Hom}(\lambda(V_1), \lambda(V_0))$.
The canonical section assigns to T the map $\lambda(T)$, giving det(T). Thus we have

$$\begin{tikzcd} \mathcal{O} \arrow[r, "s"', "\text{canon. section}"] \arrow[rr, bend left, "\det"] & \mathcal{L} \arrow[r, "\sim"', "\text{trivialization}"] & \mathcal{O} \end{tikzcd}.$$

Now the idea is to define a metric in $\mathcal{L}$ by

$$|s_T|^2 = |\det T|^2$$

but in this finite-diml. case there is no curvature.

In the Riemann surface case the proposal is to define a metric at least over the fat cell. by

$$|s_D|^2 = -\zeta'_{D^*D}(0).$$

This metric will have the Kähler form for curvature, hence if we pick an h with $\bar{\partial}\partial h = \omega$, then there should exist an inv. holom. fn. f with

$$|fs|^2 = |f|^2 |s|^2 = e^h$$

Then we have

$$\begin{tikzcd} \mathcal{O} \arrow[r, "s", "\text{canon. section}"'] \arrow[rr, bend left, "f^{-1}"] & \mathcal{L} & \mathcal{O} \arrow[l, "fs"', "\sim"] \end{tikzcd}$$

This is an isom. of hermitian bundles where $\mathcal{O}$ has $|1|^2 = e^h$

and so f^{-1} is an entire function on $\mathcal{A}$ which I can think of as det(D). (renormalized).

Let's go over this a bit more clearly. I am going to (try to) equip $\mathcal{L}$ with analytic-torsion-defined metric, which should have curvature ω. If I choose an h with $\bar\partial\partial h = \omega$, then I know $\mathcal{L}$ becomes isomorphic as hermitian line bundle to $\mathcal{O}$ with metric $|1|^2 = e^h$. Hence given one of my ~~sections~~ determinant-type sections of $\mathcal{L}$ the trivialization gives me an actual determinant function over $\mathcal{a}$

$$\begin{tikzcd} \mathcal{O} \arrow[r, "s"] \arrow[rr, bend left, "\det"] & \mathcal{L} & \mathcal{O} \arrow[l, "\sim"', "\text{trivialization}"] \end{tikzcd}$$

This is a canonical process [redacted] which depends only on the choice of h. So therefore [redacted] what is the simplest choice for h?

Over $\mathbb{C}$ we have $\omega = dz\,d\bar z = \bar\partial\partial(-|z|^2)$. [redacted]
Two h's differ by $\varphi + \bar\varphi$ where φ is holom., so we see the natural choices for h are $-|z|^2 + a\bar z + \bar a z + b$, and this family is closed under translation. Presumably the same holds for the space of connections.

Let's now fix an h of the form $-\|\alpha\|^2 + \text{linear}$ where we work around a basepoint. Next let's go back to [redacted] the problem of defining $\det(D)$ by the variational formula

$$\begin{aligned}\delta \log\det(D) &= \log\det(1 + D^{-1}\delta D) \\ &= \operatorname{tr}\log(1 + D^{-1}\delta D) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \operatorname{tr}(D^{-1}\delta D)^n \end{aligned}$$

where the first two terms are infinite and have to be renormalized. Recall also the analytic torsion formula

$$\delta(-\zeta'(0)) = \lim_{s\to 0} \operatorname{tr}\left((D^*D)^{-s} D^{-1}\delta D\right) + \text{c.c.}$$

Let's put these ideas together

$$\begin{tikzcd} \mathcal{O} \arrow[r, "s"', "\text{canon.}"] \arrow[rr, bend left, "\det"] & \mathcal{L} & \mathcal{O} \arrow[l, "\sim"', "\frac{s}{\det}"] \end{tikzcd}$$

$$e^{-\zeta'(0)} = |s|^2 \qquad \left|\frac{s}{\det}\right|^2 = e^h$$

$$-\zeta'(0) = h + \log|\det|^2$$

$$\partial(-\zeta'(0)) = \partial h + \partial \log \det$$

$$\alpha \mapsto \operatorname{tr}_{reg}(D^{-1}\alpha) = \lim_{s\to 0} \operatorname{tr}\left((D^*D)^{-s} D^{-1}\alpha\right)$$

Now we know from our calculations that this regularized trace is supposed to consist of two parts, a non-holom. part which is exactly canceled by ∂h and a holom. part which maybe we can use to define $\partial \log \det$.

So therefore if everything is to work there has to be some process by which we can define

$$\delta \log \det D = \operatorname{tr}(D^{-1}\delta D) \qquad \text{first order}$$

holomorphic in $D, \delta D$.

May 12, 1982

In the index 0 case over the open set of $\mathcal{a}$ where $h^0 = h^1 = 0$ we have a metric on $\mathcal{L}$ defined by the analytic torsion, namely

$$|s|^2 = e^{-\zeta'(0)}$$

where s is the canonical section. The curvature of this metric is the Kähler form ω over $\mathcal{a}$, so if I choose an h real-valued with $\bar\partial\partial h = \omega$, then $\mathcal{L}$ + metric has the same curvature as $\mathcal{O}$ with metric $|1|^2 = e^h$. So modulo a 1-connectedness problem which we can forget about locally, we get a trivialization of $\mathcal{L}$

$$\mathcal{O} \xrightarrow[\substack{\text{canon.}\\ \text{section}}]{s} \mathcal{L} \xrightarrow[\substack{\text{isom.}\\ \text{preserving}\\ \text{metric}}]{\sim} \mathcal{O}$$

and the composition is a kind of determinant function. So therefore once h has been chosen there is a regularized determinant function defined. This regularized determinant is what I want to understand directly.

Recall that $-h$ is the distance squared in $\mathcal{a}$ from some basepoint. So let us fix a basepoint $D_0 : E \longrightarrow E \otimes \Omega^{0,1}$ in $\mathcal{a}$ which is invertible. Then h is determined, so that we should have defined a regularized $\det(D)$, where $D = D_0 - \alpha$, α is small. It might be true that the way to work in the basepoint is to calculate D^{-1} approximately from D_0^{-1}. We want

$$\delta \log \det(D) = -\operatorname{tr}(D^{-1}\delta\alpha)$$
$$= -\operatorname{tr}\left((D_0-\alpha)^{-1}\delta\alpha\right) \qquad \text{formally.}$$

[redacted] and also

$$\log\det(D) = \log\det(D_0^{-1}(D_0-\alpha)) = +\operatorname{tr}\log(1-D_0^{-1}\alpha)$$
$$= -\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{tr}(D_0^{-1}\alpha)^n$$

so

$$\delta\log\det(D) = -\operatorname{tr}\left(D_0^{-1}\delta\alpha + D_0^{-1}\alpha D_0^{-1}\delta\alpha + \cdots\right).$$

It looks like the obvious candidate is

$$\text{reg. tr.}\left((D_0-\alpha)^{-1}\delta\alpha\right) = \operatorname{tr}\left(\left((D_0-\alpha)^{-1}-D_0^{-1}-D_0^{-1}\alpha D_0^{-1}\right)\delta\alpha\right)$$

Let's review March 11, p. 441. The idea is to define $\det_{(2)}(D;D_0)$ locally around D_0 by the series

$$-\log\det_{(2)}(D;D_0) = \sum_{n\geq 3}\frac{1}{n}\operatorname{tr}(D_0^{-1}$$ [redacted] $$\alpha)^n$$

where $\alpha = D_0 - D$. I then show that if D_0 is changed then this expression varies by a quadratic function of α. Specifically let D_0 be a fn. of t

$$\frac{d}{dt}\left[-\log\det_{(2)}(D;D_0)\right] = \sum_{n\geq 3}\operatorname{tr}\left((D_0^{-1})^{\circ}\alpha(D_0^{-1}\alpha)^{n-1}\right)$$
$$+\operatorname{tr}\left(D_0^{-1}\dot\alpha(D_0^{-1}\alpha)^{n-1}\right)$$

Now $\dot\alpha = +\dot D_0$ since D is fixed, and $(D_0^{-1})^{\circ} = -D_0^{-1}\dot D_0 D_0^{-1}$, so we get

$$\sum_{n\geq 3} -\operatorname{tr}\left(D_0^{-1}\dot D_0(D_0^{-1}\alpha)^n\right) + \operatorname{tr}\left(D_0^{-1}\dot D_0(D_0^{-1}\alpha)^{n-1}\right)$$

and the series telescopes to give

$$\frac{d}{dt}\left[-\log\det_{(2)}(D;D_0)\right] = \operatorname{tr}\left(D_0^{-1}\dot D_0(D_0^{-1}\alpha)^2\right)$$

which is evidently a quadratic fn. of α.

An obvious thing to do is to see that this procedure for regularizing the trace is consistent with the

ζ function or heat equation method, where one defines

$$\text{reg. tr.}(D^{-1}\delta\alpha) = \lim_{s\to 0} \operatorname{tr}\left((D^*D)^{-s}D^{-1}\delta\alpha\right)$$

via analytic continuation. Now we know this has a part which is not holomorphic in α. What this is is $i(\delta\alpha)\partial(-\zeta'(0))$, and the anti-holom. part should be $i(\delta\alpha)\partial h$. So if $h = -\|\alpha\|^2$, then $i(\delta\alpha)\partial h = -(\alpha|\delta\alpha)$. A different h will have linear terms.

Recall

$$\begin{tikzcd} \mathcal{O} \arrow[r, "s"', "\text{canon section}"'] \arrow[rr, bend left, "\det"] & \mathcal{L} \arrow[r, "\sim", "\text{isom obtained from } h"'] & \mathcal{O} \end{tikzcd}$$

$$\frac{s}{\det} \longleftarrow\!\shortmid\; 1$$

$$\left|\frac{s}{\det}\right|^2 = |1|^2 = e^h$$

$$e^{-\zeta'(0)} = e^h|\det|^2 \quad\text{or}\quad -\zeta'(0) = h + \log|\det|^2$$

Thus
$$\partial(-\zeta'(0)) = \operatorname{tr}\left((D^*D)^{-s}D^{-1}\delta\alpha\right)\Big|_{s=0}$$
$$= \partial h + \partial\log\det.$$

If $h = -(\alpha|\alpha)$, then $\partial h = -(\alpha|\delta\alpha)$ is a linear form on the tangent space which vanishes at the basepoint $\alpha = 0$. On the other hand if we ~~tried~~

$$(*)\qquad \partial\log\det = \operatorname{tr}\left((D^{-1} - D_0^{-1} - D_0^{-1}\alpha D_0^{-1})\delta\alpha\right)$$

then this also vanishes at $\alpha = 0$. Since the analytic regularization is $\neq 0$ at $\alpha = 0$, we see that $(*)$ is not consistent with $h = -(\alpha|\alpha)$.

I have to understand the analytical regularization better. First I need to understand D^{-1}.

Recall $D: E \longrightarrow E \otimes T^{0,1}$. Its inverse is given by a Schwarz kernel

$$(D^{-1}f)(x) = \int \langle x | D^{-1} | y \rangle f(y).$$

Hence $f(y) \in E_y \otimes T^{0,1}_y$ and $\langle x | D^{-1} | y \rangle \in E_x \otimes E_y^* \otimes T^{1,0}_y$.

Hence if φ is a local coordinate near y we can write

$$\langle x | D^{-1} | y \rangle = G(x,y)\, d\varphi_y \qquad G(x,y) \in E_x \otimes E_y^*$$

Now we know that $(\varphi(x) - \varphi(y))\, G(x,y)$ is smooth. (Recall the argument in a different notation with $\varphi(y) = 0$. We have

$$(\partial_{\bar z} - \alpha)\ G(z) = \delta(z)$$

hence

$$(\partial_{\bar z} - \alpha)\ z G(z) = z\,\delta(z) = 0,$$ hence $zG(z)$ is smooth.) Therefore we see that the good way to write things is

$$\langle x | D^{-1} | y \rangle = \frac{F(x,y)}{\pi[\varphi(x) - \varphi(y)]}\, d\varphi_y$$

where $F(x,y) \in E_x \otimes E_y^*$ is smooth and $=$ id when $x = y$.

In the simpler notation where $\varphi(y) = 0$ and $\varphi =$ the fn. z on $\mathbb{C}$, we have

$$G(z) = \frac{F(z)}{\pi z} = \frac{1 + az + b\bar z + O(z^2)}{\pi z}$$

and the regularized value is

$$\frac{\partial}{\partial z}\big(z G(z)\big)\Big|_{z=0} = \frac{a}{\pi}.$$

In other words

$$G(z) = \underbrace{\frac{1}{\pi z}}_{\text{set}=0} + \frac{a}{\pi} + \underbrace{\frac{b\bar z}{\pi z}}_{\text{integrates normally to } 0} + O(z)$$

~~What is missing~~ However

$$\left(\frac{\partial}{\partial \bar{z}} - \alpha\right) \pi \cancel{z} G(z) = 0$$

so that
$$\frac{\partial}{\partial \bar{z}}\left(\pi z G(z)\right) = \frac{\partial}{\partial \bar{z}}\left(1 + az + b\bar{z} + \dots\right) = b \quad \text{at } z=0$$
$$\shortparallel$$
$$\alpha\left(\pi \cancel{z} G(z)\right) = \pi \alpha(0) \quad \text{at } z = 0.$$

What is missing is this. I know this regularization process which gives $\frac{a}{\pi}$ depends on the local coordinate z. Hence there must be in the heat equation regularization a compensating term

May 21, 1982

Today I want to compute carefully the variation of the analytic torsion in the general case. We compute the heat kernel locally around a point. So we identify a nbd. with an open subset of $\mathbb{C}$ containing 0, and we trivialize the bundle E by an orthonormal frame, whence

$$D = (\partial_{\bar z} + \alpha)\, d\bar z$$

where α is a matrix of smooth functions. Suppose the metric on M given by

$$ds^2 = \rho(dx^2 + dy^2)$$

which means that $\sqrt{\rho}\, dx$, $\sqrt{\rho}\, dy$ is an orthonormal frame in T^*, hence

$$|d\bar z|^2 = |dx|^2 + |dy|^2 = \frac{2}{\rho} = \frac{1}{g}$$

(where g is defined to be $\rho/2$). Also the volume elt is

$$\rho\, dx dy = \frac{\rho}{2}\, i\, dz d\bar z = g\, i\, dz d\bar z.$$

Now we can compute D^* as follows:

$$\begin{aligned}(Df \mid h\, d\bar z) &= \int ((\partial_{\bar z} + \alpha) f)^* h\, \underbrace{|d\bar z|^2\, g\, i\, dz d\bar z}_{i\, dz d\bar z} \\ &= \int f^* (-\partial_z + \alpha^*) h\ i\, dz d\bar z \\ &= (f \mid g^{-1}(-\partial_z + \alpha^*) h)\end{aligned}$$

Thus

$$\boxed{D^*(h\, d\bar z) = -\, g^{-1}(\partial_z - \alpha^*) h}$$

and so

$$-D^* D = g^{-1}\underbrace{(\partial_z - \alpha^*)}_{\tilde{\mathcal{D}}}\underbrace{(\partial_{\bar z} + \alpha)}_{\mathcal{D}}$$

is the Laplacean. Put

Next we want the heat kernel. Put

$$\langle z | e^{-tD^*D} | 0 \rangle = \underbrace{\frac{1}{t} e^{-\frac{u}{t}}}_{\phi} \; U$$

Then
$$\phi^{-1}(\partial_t + D^*D)\phi = \phi^{-1}(\partial_t - g^{-1}\tilde{\mathcal{D}}\mathcal{D})$$
$$= \partial_t - \frac{1}{t} + \frac{u}{t^2} - g^{-1}\left(\tilde{\mathcal{D}} - \frac{1}{t}\partial_z u\right)\left(\mathcal{D} - \frac{1}{t}\partial_{\bar z} u\right)$$
$$= \frac{1}{t^2}\left(u - g^{-1}|\partial_z u|^2\right) + \frac{1}{t}\left(-1 + g^{-1}\partial^2_{z\bar z} u + g^{-1}(\partial_z u\,\mathcal{D} + \partial_{\bar z} u\,\tilde{\mathcal{D}})\right)$$
$$+ \partial_t - g^{-1}\tilde{\mathcal{D}}\mathcal{D}$$

I claim that if $u = \frac{r^2}{2}$ where $r(z)$ = distance from z to 0, then $|\partial_z u|^2 = g u$. As
$$du = \left(\frac{1}{\sqrt{g}}\partial_z u\right)\sqrt{g}\,dz + \left(\frac{1}{\sqrt{g}}\partial_{\bar z} u\right)\sqrt{g}\,d\bar z$$
and $\sqrt{g}\,dz$, $\sqrt{g}\,d\bar z$ are an orthonormal frame for $T^*\otimes\mathbb{C}$, we have
$$\nabla u = \frac{1}{\sqrt{g}}\partial_z u \,\frac{1}{\sqrt{g}}\partial_{\bar z} + \frac{1}{\sqrt{g}}\partial_{\bar z} u\,\frac{1}{\sqrt{g}}\partial_z$$
$$= g^{-1}\left(\partial_z u\,\partial_{\bar z} + \partial_{\bar z} u\,\partial_z\right)$$
so
$$|\nabla u|^2 = g^{-1}\left(|\partial_z u|^2 + |\partial_{\bar z} u|^2\right) = g^{-1}2\,|\partial_z u|^2$$
Thus $g^{-1}|\partial_z u|^2 = u$ becomes $|\nabla u|^2 = 2u$. Since $|\nabla r| = 1$, we have $\left|\nabla \frac{r^2}{2}\right|^2 = |r\nabla r|^2 = r^2 = 2\frac{r^2}{2}$.

This also shows that ∇u is the vector field $r\frac{d}{dr}$, and hence that
$$(g^{-1}\partial_{\bar z} u)\tilde{\mathcal{D}} + (g^{-1}\partial_z u)\mathcal{D} = \nabla_{r\frac{d}{dr}}$$
the latter denoting covariant differentiation in the v.b. E for the connection associated to D.

Now it might be useful to have Taylor series expansions of these quantities around $z=0$. Suppose
$$g = \sum g_{\alpha\bar\beta}\frac{z^\alpha}{\alpha!}\frac{\bar z^\beta}{\beta!} = g_0 + g_1 z + g_{\bar 1}\bar z + g_2\frac{z^2}{2} + g_{1\bar 1} z\bar z + g_{\bar 2}\frac{\bar z^2}{2} + \cdots$$

$$u = |z|^2\left(u_0 + u_1 z + u_{\bar 1}\bar z + u_2 \text{[redacted]} \frac{z^2}{2} + u_{1\bar 1} z\bar z + u_{\bar 2}\frac{\bar z^2}{2} + \cdots\right)$$

$$\partial_z u = \bar z\left(u_0 + u_1 z + u_{\bar 1}\bar z + u_2\frac{z^2}{2} + u_{1\bar 1}\bar z z + u_{\bar 2}\frac{\bar z^2}{2}\right.$$
$$\left. + z\bar z\left(u_1 + u_2 z + u_{1\bar 1}\bar z + \cdots\right)\right)$$

$$= u_0\bar z + (2u_1) z\bar z + u_{\bar 1}\bar z^2 + \text{[redacted]}$$
$$\left(\frac{u_2}{2} + u_2\right) z^2\bar z + \left(u_{1\bar 1} + u_{1\bar 1}\right) z\bar z^2 + u_{\bar 2}\frac{\bar z^3}{2}$$

$$= \bar z\left[u_0 + \left(2u_1 z + u_{\bar 1}\bar z\right) + \left(\frac{3u_2}{2} z^2 + 2u_{1\bar 1} z\bar z + u_{\bar 2}\frac{\bar z^2}{2}\right) + \cdots\right]$$

$$|\partial_z u|^2 = |z|^2\left[u_0^2 + u_0\left(3u_1 z + 3u_{\bar 1}\bar z\right) + u_0\left(2(u_1^2 + u_2) z^2 + (5u_1u_{\bar 1} + 4u_{1\bar 1}) z\bar z\right.\right.$$
$$\left.\left. + 2(u_{\bar 1}^2 + u_{\bar 2})\bar z^2\right) + \cdots\right]$$

$$gu = |z|^2\left(g_0 + g_1 z + g_{\bar 1}\bar z + \cdots\right)\left(u_0 + u_1 z + u_{\bar 1}\bar z + \cdots\right)$$
$$= |z|^2\left(g_0u_0 + \left(u_0(g_1 z + g_{\bar 1}\bar z) + g_0(u_1 z + u_{\bar 1}\bar z)\right) + \cdots\right)$$

giving the equations

$$u_0^2 = g_0u_0 \implies u_0 = g_0$$

$$g_1 z + g_{\bar 1}\bar z + \cancel{u_1 z} + \cancel{u_{\bar 1}\bar z} = 3u_1 z + 3u_{\bar 1}\bar z$$

$$u_1 = \tfrac{1}{2}g_1 \qquad u_{\bar 1} = \tfrac{1}{2}g_{\bar 1}$$

So we have

$$u = |z|^2\left(g_0 + \tfrac{1}{2}g_1 z + \tfrac{1}{2}g_{\bar 1}\bar z + \text{quad. terms}\right)$$

$$\partial_z u = \bar z\left(g_0 + g_1 z + \tfrac{1}{2}g_{\bar 1}\bar z + \cdots\right)$$

$$g^{-1}\partial_z u = \bar z\left(1 - \frac{\tfrac{1}{2}g_{\bar 1}\bar z + \text{quad}}{g_0 + g_1 z + \cdots}\right)$$

$$= \bar z\left(1 - \tfrac{1}{2}g_0^{-1}g_{\bar 1}\bar z + \text{quad. terms}\right)$$

The next point is to [redacted] make $\phi = \frac{1}{t}e^{-\frac{u}{t}}$ even better by constructing a function $\xi(z)$ with the property that

$$\left(-1+g^{-1}\partial^2_{z\bar z}u\right)\xi + g^{-1}\left(\partial_z u\,\partial_{\bar z}+\partial_{\bar z}u\,\partial_z\right)\xi = 0$$

Then $\langle z|e^{-tD^*D}|0\rangle = \frac{1}{\pi t}e^{-\frac{u}{t}}\xi\ \ (\xi^{-1}U)$

and

$$(\phi\xi)^{-1}\left(\partial_t - g^{-1}\tilde{D}D\right)\phi\xi = \frac{1}{t}\nabla_{r\frac{d}{dr}} + \partial_t - \xi^{-1}g\tilde{D}D\xi .$$

Next I need to understand $\xi^{-1}U=F$. The idea is that it is a smooth fn. of t,z with an asymptotic expansion

$$F(t,z) \sim F_0(z) + tF_1(z) + \cdots$$

as $t\downarrow 0$, and hence

$$0 = \nabla_{r\frac{d}{dr}}F_0 = \left[(g^{-1}\partial_z u)(\partial_{\bar z}-\alpha^*) + (g^{-1}\partial_{\bar z}u)(\partial_z+\alpha)\right]F_0$$

So try $\quad F_0 = I + az + b\bar z + \cdots$

$$\bar z(1+O(z))(b-\alpha_0^*+O(z)) + z(1+O(z))(a+\alpha_0+O(z)) = 0$$

$$\bar z(b-\alpha_0^*) + z(a+\alpha_0) + O(z^2) = 0$$

$$\therefore\ b=\alpha_0^*,\quad a=-\alpha_0 .$$

$$\mathrm{tr}\left((D^*D)^{-s}D^{-1}\delta\alpha\right) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{tr}\left(e^{-tD^*D}D^{-1}\delta\alpha\right)t^s\frac{dt}{t}$$

$$\mathrm{tr}\left(e^{-tD^*D}D^{-1}\delta\alpha\right) = \int d^2y\,\underbrace{\int d^2x\,\mathrm{tr}\langle y|e^{-tD^*D}|x\rangle\langle x|D^{-1}|y\rangle\,\delta\alpha(y)}_{\text{this expression to be understood as } t\to 0}$$

Next point is that the heat kernel as $t\downarrow 0$ decays exponentially except along the diagonal so that one only has to worry about x near y. So we supposed $x=z$, $y=0$

in $\mathbb{C}$. Then we have the integral

$$\int \underbrace{i g\, dz d\bar{z}\ \frac{1}{2\pi t} e^{-\frac{r(z)^2}{2t}}}_{\text{approaches } \delta(0)} \xi(z)\ F(t,z)\ \underbrace{\langle z | D^{-1} | 0 \rangle}_{\frac{F_1(z)}{\pi z}}$$

Now we have $F(t,z) - F_0(z)$ is divisible by t, and since $\frac{1}{\pi z}$ is integrable, we can replace F by F_0. What will be the argument? We expand

$$F_0(z) F_1(z) = 1 + az + b\bar{z} + O(z^2)$$

and expect the answer $\frac{a}{\pi}$. But this depends on knowing that

$$\int i g \frac{dz d\bar{z}}{2\pi} \frac{1}{t} e^{-\frac{r(z)^2}{2t}} \xi(z) \begin{cases} \frac{1}{z} \\ \frac{\bar{z}}{z} \end{cases} = 0$$

which is the case if there is invariance under S^1.

Substitute

$$\int \frac{i}{2\pi} \overbrace{dz d\bar{z}}\, g(\sqrt{t}z)\ e^{-\frac{r(\sqrt{t}z)^2}{2t}} \xi(\sqrt{t}z) \begin{cases} \frac{1}{\sqrt{t}z} \\ \frac{\bar{z}}{z} \end{cases}$$

$$r(\sqrt{t}z) = t|z|^2 \left(g_0 + \tfrac{1}{2} g_1 \sqrt{t} z + \tfrac{1}{2} g_{\bar{1}} \sqrt{t} \bar{z} + O(t)\right)$$

$$\int \frac{i\, dz d\bar{z}}{2\pi}\ e^{-|z|^2 (g_0 + \frac{1}{2} g_1 \sqrt{t} z + \frac{1}{2} g_{\bar{1}} \sqrt{t} \bar{z} + O(t))} \cdot g(\sqrt{t}z)\, \xi(\sqrt{t}z) \cdot \frac{1}{\sqrt{t}z}$$

So we see the $\frac{\bar{z}}{z}$ won't contribute, but that we have to be careful about the leading term.

Look invariantly at the sort of situation we have. Fix $y \in M$. We know the holomorphic bundle E has an r-diml space ($r = \text{rank } E$) of meromorphic sections with at most simple pole at y, and that such sections s are determined by their residue at y, which is an elt of $E_y \otimes (T_y^{1,0})^*$, namely, $\lim_{x \to y} [\varphi(x) - \varphi(y)]\, s(x) \otimes (d\varphi_y)^\vee$. $\varphi = $ coord

So if I choose a generator $d\varphi_y$ for $\Omega^{1,0}_y$, then for each $e \in E_y$, there is a unique section s of E holom. off y with behavior $\frac{e}{\varphi - \varphi(y)}$ + holom. near y. Hence if I choose a basis for E_y and let $s_1, \ldots, s_r$ be the corresponding meromorphic sections, then the s_i will trivialize E over $U - \{y\}$, where U is some nbd. of y. This is a fairly canonical trivialization, as it depends only on the generator of $\Omega^{1,0}_y$. Hence in particular a definite way of transporting elements of E near y, however, we can't quite identify E_x with E_y for x near but $\neq y$. Wait: Given $x \in U - \{y\}$ and an elt e of E_x, there is a unique section s with $s_x = e$ an s having only a simple pole at y; then take $\lim\, [\varphi - \varphi(y)]\, s$. This identifies E_x + E_y but not continuously in x.

Next because of the connection on E we have a way to transport elts of E along curves, hence given the metric on M we have a way to identify E_x with E_y by transporting along the geodesic from y to x. Now we are trying to define by heat kernel regularization a finite part of $G(x,y) = \langle x | D^{-1} | y \rangle$ as $x \to y$. Now $G(x,y) \in E_x \otimes E^*_y \otimes \Omega^{1,0}_y$ so the finite part will belong to $E_y \otimes E^*_y \otimes \Omega^{1,0}_y$.

Here is a possible thing to do. Suppose given an element of $E_y \otimes (\Omega^{1,0}_y)^*$. Hence we have a corresp. section s of E. For each $x \in U$ transport s_x by the connection radially back to an element of E_y. Then take the average of the resulting elements of E_y using the Gaussian measures converging to the δ-fn. measure at y.

It is completely clear that what I have described corresponds to the regularization process

$$\lim_{t\downarrow 0} \int \langle y | e^{-tD^*D} | x \rangle \, d^2x \, \langle x | D^{-1} | y \rangle$$

because

$$\langle y | e^{-tD^*D} | x \rangle \, d^2x = \underbrace{\frac{1}{2\pi t} \, e^{-\frac{r^2}{2t}(y,x)} (?)(y,x)}_{\text{Gaussian measure on } M \text{ centered at } x} \; F(t,y,x)$$

and

$$F(t,y,x) \longrightarrow \text{radial transport isomorphism of } E_x \text{ to } E_y$$

as $t \downarrow 0$.

The basic problem I have is to decide whether to work with a holomorphic coordinate around y or with normal coordinates. The former simplifies the singularity of $\langle x|D^{-1}|y\rangle$, the latter simplifies the Gaussian factors.