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<u>February 28, 1982</u>

Poisson Summation Formula: V real vector space with volume dx, Γ lattice in V, $\Gamma^* \subset V^*$ the dual lattice: $\Gamma^* = \{\mu \in V^* \mid \mu(\Gamma) \subset 2\pi\mathbb{Z}\}$, f a fn on V, then

$$\sum_{\gamma\in\Gamma} f(x+\gamma) = \frac{1}{\mathrm{vol}(\Gamma\backslash V)} \sum_{\mu\in\Gamma^*} \hat{f}(\mu)\, e^{i\mu x}$$

where $$\hat{f}(\mu) = \int_V f(x) e^{-i\mu x}\, dx$$

<u>Example</u>: $\Gamma = \mathbb{Z} \subset \mathbb{R}$, $\Gamma^* = 2\pi\mathbb{Z}$, $f(x) = e^{-\frac{t}{2}x^2}$,

$\hat{f}(x) = \frac{\sqrt{2\pi}}{\sqrt{t}} e^{-\frac{x^2}{2t}}$ so

$$\sum_{\mathbb{Z}} e^{-\frac{t}{2}(x-n)^2} = \sum_{\mathbb{Z}} \frac{\sqrt{2\pi}}{\sqrt{t}} e^{-\frac{1}{2t}(2\pi n)^2} e^{i2\pi n x}$$

Put $t \mapsto 2\pi t$ and this simplifies to

$$\sum e^{-\pi t(x-n)^2} = \frac{1}{\sqrt{t}} \sum e^{-\frac{\pi}{t}n^2} e^{2\pi i n x}.$$

Let's apply this to compute the analytic torsion of a flat line bundle over S^1, i.e. the complex

$$\mathcal{O} \xrightarrow{\frac{d}{d\theta} - \alpha} \mathcal{O}.$$

The eigenvalues are $n - \alpha$ as n runs over $\mathbb{Z}$, so

$$\zeta(s,x) = \sum{}' \frac{1}{|x-n|^{2s}} \qquad \text{converges for } \mathrm{Re}(s) > \tfrac{1}{2}$$

$$\zeta(s,x)\Gamma(s) = \sum{}' \frac{1}{|x-n|^{2s}} \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

$$\pi^{-s}\zeta(s,x)\Gamma(s) = \int_0^\infty \Big(\sum_{n\in\mathbb{Z}} e^{-\pi t|x-n|^2}\Big) t^s \frac{dt}{t}$$

So now I want to do the analytic continuation. Because I suppose $x \notin \mathbb{Z}$, the Θ function decays as $t \to +\infty$.

As $t \to 0$, the functional equation shows it behaves like $\frac{1}{\sqrt{t}}$ which should contribute a simple pole at $s = 1/2$. Introduce

$$\theta(t,x) = \sum e^{-\pi t (x-n)^2}$$

$$\theta_1(t,x) = \sum e^{-\pi t n^2} e^{2\pi i n x}$$

so that $$\theta(t,x) = \frac{1}{\sqrt{t}}\, \theta_1\left(\frac{1}{t}, x\right)$$

and $$\int_0^\infty [\theta_1(t,x) - 1]\, t^s \frac{dt}{t} = \sum_{n\neq 0} e^{2\pi i n x} \int_0^\infty e^{-\pi t n^2} t^s \frac{dt}{t}$$

$$= \sum_{n \neq 0} \frac{e^{2\pi i n x}}{(\pi n^2)^s}\, \Gamma(s)$$

$$= \pi^{-s} \underbrace{\left(\sum_{n\neq 0} \frac{e^{2\pi i n x}}{(n^2)^s} \right)}_{\zeta_1(s,x)} \Gamma(s)$$

It looks like this converges for $Re(s) > 0$.

Do the analytic continuation:

$$\pi^{-s} \zeta(s;x)\Gamma(s) = \int_0^\infty \theta(t,x)\, t^s \frac{dt}{t} = \int_0^1 + \underbrace{\int_1^\infty}_{\text{entire}}$$

$$= \int_0^1 \frac{1}{\sqrt{t}}\, \theta_1(1/t, x)\, t^s \frac{dt}{t} + \int_1^\infty$$

$$= \int_1^\infty \theta_1(t,x)\, t^{1/2 - s} \frac{dt}{t} + \int_1^\infty$$

$$= \underbrace{\int_1^\infty t^{1/2-s} \frac{dt}{t}}_{\left.\frac{t^{1/2-s}}{1/2 - s}\right]_1^\infty = \frac{1}{s - 1/2}} + \underbrace{\int_1^\infty [\theta_1(t,x) - 1]\, t^{1/2-s} \frac{dt}{t} + \int_1^\infty \theta(t,x)\, t^s \frac{dt}{t}}_{\text{entire}}$$

So therefore I see that the function

$$\zeta(s,x)\Gamma(s)$$

is entire except for a simple pole at $s=1/2$. Since $\Gamma(s) \sim \frac{1}{s}$ as $s \to 0$, it follows that $\zeta(s,x)=0$ at $s=0$, and hence the torsion is defined. But we have the functional equation

$$\pi^{-s}\zeta(s,x)\Gamma(s) = \pi^{s-1/2}\zeta_1(1/2-s)\Gamma(1/2-s)$$

Notice that because $\Gamma(s) \sim 1/s$ as $s \to 0$, it follows that

$$\frac{d}{ds}\zeta(s,x)\Big|_{s=0} = \lim_{s\to 0} \pi^{-s}\zeta(s,x)\Gamma(s)$$

$$= \pi^{-1/2}\zeta_1(1/2)\underbrace{\Gamma(1/2)}_{\sqrt{\pi}} = \zeta_1(1/2)$$

But

$$\zeta_1(1/2,x) = \sum_{n\neq 0} \frac{e^{2\pi i n x}}{|n|}$$

can be calculated using the logarithm.

$$-\log(1-z) = \sum_1^\infty \frac{z^n}{n}$$

$$\zeta_1(1/2,x) = -\log\left(1-e^{2\pi i x}\right) - \log\left(1-e^{-2\pi i x}\right)$$

$$= -\log\left(1-e^{2\pi i x}-e^{-2\pi i x}+1\right)$$

$$= -\log\left(2-2\cos(2\pi x)\right)$$

But recall $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$. Thus

$$\zeta'(0,x) = \zeta_1(1/2) = -\log\left(4\sin^2(\pi x)\right)$$

$$\parallel$$

$$-\log\det\Delta = -2\log\text{ torsion}$$

Hence

$$\text{torsion} = |2\sin(\pi x)|$$

We should compare this with the naive computation of the relative determinant of

$$C^\infty(S^1) \xrightarrow{\frac{d}{d\theta} - x} C^\infty(S^1)$$

The eigenvalues are $n - x$ and so as a first try

$$\frac{\det\left(\frac{d}{d\theta} - x\right)}{\det\left(\frac{d}{d\theta}\right)} = \prod \left(\frac{n-x}{n}\right)$$

which up to a constant is

$$x \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) = \frac{\sin(\pi x)}{\pi}$$

Next consider the case where x is complex say $x + iy$.

$$\zeta(s, x+iy) = \sum_{\mathbb{Z}} \frac{1}{|x+iy-n|^{2s}}$$

$$\pi^{-s}\Gamma(s)\zeta(s, x+iy) = \int_0^\infty \sum_n e^{-\pi t|x+iy-n|^2} \; t^s \frac{dt}{t}$$

$$\sum_n e^{-\pi t[(x-n)^2+y^2]} = \Bigg(\underbrace{\sum e^{-\pi t(x-n)^2}}_{\theta(t,x) = \frac{1}{\sqrt{t}}\sum e^{-\frac{\pi n^2}{t}} e^{2\pi i n x}}\Bigg) e^{-\pi t y^2}$$

We know $\theta(t,x) \sim \frac{1}{\sqrt{t}}$ exponentially as $t \searrow 0$.
So therefore we have the asymptotic expansion:

$$\theta(t,x)e^{-\pi t y^2} \sim \frac{1}{\sqrt{t}}\left(1 - (\pi y^2)t + \frac{(\pi y^2)^2}{2!}t^2 + \cdots\right)$$

and hence by a general result we know that

$\pi^{-s}\Gamma(s)\zeta(s, x+iy)$ is meromorphic with simple poles at $s = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots\ldots$. In particular one sees for $y \neq 0$ that there is no functional equation.

Here's how to evalutete $\zeta'(0, x+iy)$. We have

$$\pi^{-s}\Gamma(s)\zeta(s, x+iy) = \int_0^\infty \frac{1}{\sqrt{t}} \sum e^{-\frac{\pi n^2}{t} + 2\pi i n x - \pi t y^2} t^s \frac{dt}{t}$$

$$= \underbrace{\int_0^\infty \sum_{n \neq 0} e^{-\frac{\pi n^2}{t} + 2\pi i n x - \pi t y^2} t^{s-\frac{1}{2}} \frac{dt}{t}}_{\text{entire fn}} + \underbrace{\int_0^\infty e^{-\pi t y^2} t^{s-\frac{1}{2}} \frac{dt}{t}}_{\frac{\Gamma(s-\frac{1}{2})}{(\pi y^2)^{s-\frac{1}{2}}}}$$

I need the Bessel $K_{-1/2}$ fn.

$$K_{-1/2}(r) = \int_0^\infty e^{-\frac{r}{2}(t+t^{-1})} t^{-1/2} \frac{dt}{t}$$

which one can evaluate exactly as follows. Put

$$t + t^{-1} = 2 + u^2, \quad u^2 = t - 2 + \frac{1}{t} = (t^{1/2} - t^{-1/2})^2 \quad \text{or}$$

$$u = t^{1/2} - t^{-1/2} \quad \text{so that } 0 < t < \infty \text{ is } -\infty < u < \infty.$$

$$\frac{dt}{t^{3/2}} = -2d(t^{-1/2}) \qquad t^{-1/2}u = 1 - (t^{-1/2})^2$$

$$(t^{-1/2})^2 + u(t^{-1/2}) - 1 = 0 \implies t^{-1/2} = \frac{-u + \sqrt{u^2+4}}{2}$$

$$\implies \frac{dt}{t^{3/2}} = -2d(t^{-1/2}) = -2\,\frac{1}{2}\left[-1 + \frac{1}{2}(u^2+4)^{-1/2}2u\right]du$$

$$= \left(1 - \frac{u}{\sqrt{u^2+4}}\right)du. \quad \text{Thus}$$

$$K_{-1/2}(r) = \int_{-\infty}^\infty e^{-\frac{r}{2}(2+u^2)}\left(1 - \frac{u}{\sqrt{u^2+4}}\right)du$$

odd fn. so Gauss. is 0.

$$= e^{-r}\int_{-\infty}^\infty e^{-\frac{r}{2}u^2}du = \boxed{e^{-r}\frac{\sqrt{2\pi}}{\sqrt{r}} = K_{-1/2}(r)}$$

I want $\int_0^\infty e^{-at-b/t}\frac{dt}{t^{3/2}}$ $\qquad t \longmapsto \sqrt{\frac{b}{a}}\,t$

$$= \int_0^\infty e^{-\sqrt{ab}\,t-\sqrt{ab}/t}\left(\sqrt{\frac{b}{a}}\right)^{-1/2}\frac{dt}{t^{3/2}} = e^{-2\sqrt{ab}}\frac{\sqrt{2\pi}}{\sqrt{2\sqrt{ab}}}\left(\frac{a}{b}\right)^{1/4}$$

$$\boxed{\int_0^\infty e^{-at-b/t}\,t^{-1/2}\frac{dt}{t} = \sqrt{\frac{\pi}{b}}\,e^{-2\sqrt{ab}}}$$

So

$$\zeta'(0,x+iy) = \lim_{s\to 0}\pi^{-s}\Gamma(s)\zeta(s,x+iy) \qquad \int \frac{\Gamma(-\frac{1}{2}+1)}{(-1/2)} = -2\sqrt{\pi}$$

$$= \sum_{n\neq 0} e^{2\pi i n x}\sqrt{\frac{\pi}{\pi n^2}}\,e^{-2\sqrt{\pi y^2\pi n^2}} + \frac{\Gamma(-1/2)}{(\pi y^2)^{-1/2}}$$

$$= \sum_{n\neq 0}\frac{e^{2\pi i n x - 2\pi|ny|}}{|n|} - 2\pi|y|$$

$$= -\log\left(1-e^{2\pi i x}e^{-2\pi|y|}\right) - \log\left(1-e^{-2\pi i x}e^{-2\pi|y|}\right) \underbrace{}_{-2\pi|y|}$$

But $\zeta'(0) = -\log\det\Delta = -2\log\text{torsion}$. Hence

$$\text{torsion} = \left|1-e^{2\pi i x}e^{-2\pi|y|}\right|e^{\pi|y|}$$

$$= \left|e^{\pi|y|-\pi i x} - e^{-\pi|y|+\pi i x}\right| \qquad \text{doesn't change if } x\mapsto -x$$

$$= \left|e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}\right|$$

$$= 2\left|\sin\pi(x+iy)\right|$$

which agrees with the naive result

$$\text{torsion} = |\det| = \left|\frac{\sin\pi(x+iy)}{\pi}\right|$$

General remarks: For a positive operator A

$$\zeta_A(s) = \mathrm{tr}(A^{-s})$$ [redacted]

$$\Gamma(s)\zeta_A(s) = \mathrm{tr}\int_0^\infty e^{-tA}\, t^s \frac{dt}{t} = \int_0^\infty \mathrm{tr}(e^{-tA})\, t^s \frac{dt}{t}$$

I assume that A has no zero eigenvalues; then the only problems with convergence come from $t \searrow 0$. Next one has for Laplaceans at least an asymptotic expansion

$$\mathrm{tr}(e^{-tA}) = t^{-d/2}(a_0 + a_1 t + a_2 t^2 + \cdots) \qquad \text{as } t \searrow 0$$

where the a_i are integrals over the manifold. So by analytic continuation one sees that $\Gamma(s)\zeta_A(s)$ is meromorphic with simple poles having the behavior

$$\int_0^1 a_n t^{n-d/2}\, t^s \frac{dt}{t} = a_n \left. \frac{t^{n-\frac{d}{2}+s}}{n-\frac{d}{2}+s} \right]_0^1 = \frac{a_n}{s-(\frac{d}{2}-n)}$$

So for odd manifolds $\zeta_A(s)$ will have [redacted] simple zeroes at [redacted] integers $n \le 0$ to cancel the poles of $\Gamma(s)$, and can be expected to have simple poles at half integer points $\frac{d}{2}-n$, $n \ge 0$. For even manifolds one has that $\zeta_A(s)$ has simple poles at $\frac{d}{2}, \frac{d}{2}-1, \ldots, 1$ and is holomorphic elsewhere.

Now let's return to the [redacted] elliptic curve case and the case of the operator

$$\mathcal{O} \xrightarrow{\frac{\partial}{\partial \bar u} - z} \mathcal{O}$$

Let set this up carefully: $X = \Gamma \backslash \mathbb{C}$ usual volume, and $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$ so that $\mathrm{vol}(X) = (\mathrm{Im}\,\tau)$. The eigenfns. are [redacted] $e^{\mu\bar u - \bar\mu u}$ where $\mu \in \frac{\pi}{\mathrm{Im}\,\tau}\Gamma$ and hence the [redacted] eigenvalues are $\mu - z$, hence the Θ function is

$$\sum_\mu{}' e^{-\pi t|z-\mu|^2}$$

This is the usual θ-function of a lattice in $\mathbb{C}$ and hence one has the Poisson formula:

$$\sum_{\mu\in\frac{\pi}{\mathrm{Im}\,\tau}\Gamma} e^{-\pi t|z-\mu|^2} = \sum_{\gamma\in\Gamma} \frac{\int_{\mathbb{C}} e^{-\pi t|z|^2} e^{-\gamma\bar{z}+\bar{\gamma}z}\,dx\,dy}{\mathrm{vol}\left(\frac{\pi}{\mathrm{Im}\,\tau}\Gamma \,\middle|\, \mathbb{C}\right)}\, e^{\gamma\bar{z}-\bar{\gamma}z}$$

$$\int_{\mathbb{C}} e^{-\pi t|z|^2-\gamma\bar{z}+\bar{\gamma}z}\,dx\,dy = \frac{e^{-\frac{|\gamma|^2}{\pi t}}}{t} \qquad \underset{\mu\text{-lattice}}{\mathrm{vol}} = \frac{\pi^2}{\mathrm{Im}\,\tau}$$

So it seems that

$$\sum_{\mu\in\frac{\pi}{\mathrm{Im}\,\tau}\Gamma} e^{-\pi t|z-\mu|^2} = \frac{1}{t}\sum_{\gamma\in\Gamma} e^{-\frac{|\gamma|^2}{\pi t}}\, e^{\gamma\bar{z}-\bar{\gamma}z}\left(\frac{\mathrm{Im}\,\tau}{\pi^2}\right)$$

When we do the analytic continuation this time we will drop the term $\gamma=0$ and get

$$\pi^{-s}\Gamma(s)\,\zeta(s,z) = \sum{}' e^{\gamma\bar{z}-\bar{\gamma}z}\underbrace{\int_0^\infty e^{-\frac{|\gamma|^2}{\pi t}}\, t^{s-1}\frac{dt}{t}}_{\frac{\Gamma(1-s)}{(|\gamma|^2/\pi)^{1-s}}}\left(\frac{\mathrm{Im}\,\tau}{\pi^2}\right)$$

and so we end up with

$$\zeta'(0,z) = \left(\sum_{\Gamma}{}' \frac{e^{\gamma\bar{z}-\bar{\gamma}z}}{|\gamma|^2}\right)\left(\frac{\mathrm{Im}\,\tau}{\pi}\right)$$

Notice that we have

$$\frac{\partial^2}{\partial z\,\partial\bar{z}}\sum{}'\frac{e^{\gamma\bar{z}-\bar{\gamma}z}}{|\gamma|^2} = -\sum{}' e^{\gamma\bar{z}-\bar{\gamma}z}$$

$$= -\underset{\mu\text{-lattice}}{\mathrm{vol}}\sum_{\mu}\delta(z-\mu) + 1$$

hence

$$\frac{\partial^2}{\partial z\,\partial\bar{z}}\zeta'(0,z) = -\pi\sum_{\mu}\delta(z-\mu) + \frac{\mathrm{Im}\,\tau}{\pi}$$

which means that $\zeta'(0,z)$ is going to be a solution of Laplace's equation with singularities of type $\log |z|$.

Now in fact we have $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$ and hence

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

On the other hand $\Delta \log r = 2\pi \delta(r)$, hence we conclude that $\zeta'(0,z)$ is a renormalized version of

$$-2\sum_{\mu} \log |z-\mu|.$$

If we recall $\zeta'(0,z) = -2\log(\text{torsion})$, things check out as they should.

References:

Ray + Singer (torsion for d op) Adv. in Math 1971

" (torsion for $\bar{\partial}$ op) Annals of Math 1973

The above torsion is computed completely including the constant factor which depends on τ. Also torsion is related to a Selberg ζ function for genus $g > 1$.

March 1, 1982

Let X be a Riemann surface (compact) and E a C^∞-vector bundle over X, and $\mathcal{A}$ the space of holom. structures on E. A point of $\mathcal{A}$ gives us an elliptic complex

$$E \xrightarrow{\bar\partial_A} E \otimes \Omega^{0,1} \qquad A \in \mathcal{A}$$

from which we compute the cohomology $H^*(X,E)$ for the given holom. structure. This gives a family of elliptic operators on X parameterized by $\mathcal{A}$, hence a perfect complex on $\mathcal{A}$ which has a determinant line bundle. Thus we get a line bundle over $\mathcal{A}$ which should be holomorphic. Call this line bundle $\lambda(R\pi_*\mathcal{E})$

Here's how we can produce sections in certain cases of the dual line bundles. Let's suppose the degree d and rank r of E are such that generically the $H^1=0$. Then over an open set of $\mathcal{A}$, $\pi_*\mathcal{E}$: $E \mapsto H^0(X,E)$ is a vector bundle of a given rank $\rho = d + r(1-g)$, and hence we have a map

$$\lambda(R\pi_*\mathcal{E}) = \lambda(\pi_*\mathcal{E}) \subset \Lambda^\rho(C_\infty(E))$$

Hence associated to any element of the dual of $\Lambda^\rho(C_\infty(E))$ is a section of $\lambda(R\pi_*\mathcal{E})^*$, over the open set where $H^1=0$.

In general as $\bar\partial_A$ varies we have a family of Fredholm operators $\Gamma(\bar\partial_A): \Gamma(E) \longrightarrow \Gamma(E\otimes T^{0,1})$, and we use the construction of the index of this family. At a point A_0 one chooses a finite-diml subspace $F \subset \Gamma(E\otimes T^{0,1})$ which maps onto $\operatorname{Coker}\Gamma(\bar\partial_{A_0})$

$$\begin{array}{ccccccccc}
0 \to & H^0(E_A) & \longrightarrow & \Gamma(E) & \xrightarrow{\Gamma(\bar\partial_A)} & \Gamma(E\otimes T^{0,1}) & \longrightarrow & H^1(E_A) & \to 0 \\
 & \| & & \cup & \text{transv.} & \cup & & \| & \\
0 \to & H^0(E_A) & \longrightarrow & \Gamma(\bar\partial_A)^{-1}F & \longrightarrow & F & \longrightarrow & H^1(E_A) & \to 0
\end{array}$$

Then for A near A_0, $F + \mathrm{Im}\,\Gamma(\bar\partial_A) = \Gamma(E \otimes T^{0,1})$ and so $\Gamma(\bar\partial_A)^{-1}(F)$ is a sub bundle of $\Gamma(E)$. On this open set we have that $R\pi_*(\mathcal{E})$ is quis the complex $\Gamma(\bar\partial_A)^{-1}F \longrightarrow F$, and so

$$\lambda(R\pi_*(\mathcal{E}))^* = \lambda(F) \otimes \lambda(\Gamma(\bar\partial_A^{-1})F)^*$$

Let $p = \text{index} = \underbrace{\dim(\Gamma(\bar\partial_A)^{-1}F)}_{p+q} - \underbrace{\dim F}_{q}$ and suppose we are given an element $\alpha \in \Lambda^p(\Gamma(E))^*$. I want to show how this defines a section of $\lambda(R\pi_*(\mathcal{E}))^*$.
Put $H = \Gamma(E)$, $H' = \Gamma(E \otimes T^{0,1})$, $T = \Gamma(\bar\partial_A)$. By transversality

$$\begin{tikzcd}
Q \arrow[r] & Q' \\
H \arrow[u, two heads] \arrow[r, "T"] & H' \arrow[u, two heads] \\
T^{-1}F \arrow[u, phantom, "\cup"] \arrow[r] & F \arrow[u, phantom, "\cup"] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

T induces an isom $Q \xrightarrow{\sim} Q'$. We have canonical map

$$\Lambda^q(F) \otimes \Lambda^s(Q') \subset \Lambda^{q+s}(H')$$

$$\Lambda^{p+q}(T^{-1}F) \otimes \Lambda^s(Q) \subset \Lambda^{p+q+s}(H)$$

hence $$\Lambda^{p+q+s}(H)^* \otimes \Lambda^{q+s}(H') \longrightarrow \Lambda^{p+q}(T^{-1}F)^* \otimes \Lambda^s(Q)^* \otimes \Lambda^s(Q') \otimes \Lambda^q(F)$$

so if we use the pairing $\Lambda(Q)^* \otimes \Lambda(Q) \longrightarrow \mathbb{C}$ together with $\Lambda^s(Q') \simeq \Lambda^s(Q)$ furnished by T, we therefore get a map

$$\Lambda^{p+q+s}(H)^* \otimes \Lambda^{q+s}(H') \longrightarrow \text{sections of } \lambda(R\pi_*(\mathcal{E}))^* \text{ over } T \text{ transversal to } F.$$

depending on F. ?

Better approach: Take $\alpha \in \Lambda^p(H)^*$ pull-back to $\Lambda^p(T^{-1}F)^*$. On the other hand given $\beta \in \Lambda^q(F)^*$, one can pull it back to $\Lambda^q(T^{-1}F)^*$, then multiply with the inverse image of α to land in $\Lambda^{p+q}(T^{-1}F)^*$. Thus we get a map

$$\begin{array}{ccccc} \Lambda^p(H)^* & \longrightarrow & \Lambda^p(T^{-1}F)^* & = & \Lambda^q(T^{-1}F) \otimes \lambda(T^{-1}F)^* \\ & & & & \downarrow \\ & & \lambda(F) \otimes \lambda(T^{-1}F)^* & = & \Lambda^q(F) \otimes \lambda(T^{-1}F)^* \end{array}$$

as desired which is clearly 0 unless $T^{-1}F \to F$ is onto. This shows that the sections of $\lambda(R\pi_*(\mathcal{E}))^*$ produced from elts of $(\Lambda^p\Gamma(\mathcal{E}))^*$ in the cases where $H' = 0$ generically vanish on the rest of the bundles.

More generally to define

$$(*) \quad \Lambda^s(H') \otimes \Lambda^{p+s}(H)^* \longrightarrow \Lambda^q(F) \otimes \Lambda^{p+q}(T^{-1}F)^*$$

it should be enough to define a map

$$\Lambda^s(H') \otimes \Lambda^{p+q}(T^{-1}F) \longrightarrow \Lambda^{p+s}(H) \otimes \Lambda^q(F)$$

Now by transversality

$$0 \longrightarrow T^{-1}F \longrightarrow H \oplus F \longrightarrow H' \longrightarrow 0$$

so that one has canonical maps

$$\Lambda^s(H') \otimes \lambda(T^{-1}F) \hookrightarrow \Lambda^{p+q+s}(H \oplus F) \twoheadrightarrow \Lambda^{p+s}(H) \otimes \lambda(F)$$

Not yet clear how the maps $(*)$ fit together for different s.

March 2, 1982

Yesterday I reached the following situation: Consider two vector spaces V_1, V_0 and the space of all maps $T : V_1 \to V_0$ which are Fredholm, i.e. the kernel + cokernel are finite-dimensional. Over this space I have a canonical line bundle whose fibre at T can be canonically identified with

(1) $$\lambda(\text{Coker}\, T) \otimes \lambda(\text{Ker}\, T)^*.$$

A finite-dimensional subspace F of V_0 which spans $\text{Coker}(T_0)$ defines an open neighborhood of T_0 in the space over which the line bundle can be identified with

$$\lambda(F) \otimes \lambda(T^{-1}F)^*$$

Problem: Understand the sections of this line bundle. Fix the index of T, call it p.

Example 1: Suppose T is onto. Then we have a map

$$T \longmapsto \text{Ker}\, T \subset \text{Grass}_q(V_1) \qquad q = -p$$

and the line bundle on the set of onto T is the pull-back of the bundle $\lambda(S)^*$, where S is the subbundle on the Grassmannian. Thus we get over the space of onto T sections of the line bundle (1) given by

$$\Gamma(\text{Grass}_q(V_1), \lambda(S)^*) = (\Lambda^q V_1)^*$$

These sections are constant along the fibres of $T \mapsto \text{Ker}\, T$. The fibre over $K \in \text{Grass}_q(V_1)$ is the set of isomorphisms of $V_1/K \xrightarrow{\sim} V_0$ and hence is a general linear group, so its an affine variety. This means that our line bundle over the open set of T which are onto, has many more sections than come from $(\Lambda^q V_1)^*$.

~~However the group of autos. of the pair (V_1, V_0) acts on the set of T and equivariantly on the line bundle, and it's clear that $(\Lambda^q V_1)^*$ should be the equivariant sections.~~ Meaningless

Example 2: Take the open set where T is injective whence we get a map $T \longmapsto \operatorname{Im}(T) \in \mathrm{Grass}_{\mathrm{cod}(p)}(V_0)$ and the line bundle is the pull-back of the bundle $\lambda(Q)$, $Q =$ quotient bundle on the grassmannian. So we get sections over the space of injec T given by

$$\Gamma\left(\mathrm{Grass}_{\mathrm{cod}\, p}(V_0), \lambda(Q)\right) = \Lambda^p(V_0).$$

~~and these are the invariant sections.~~

Example 3: Look at the stratum where $\dim \mathrm{Cok}(T) = p+q$ and $\dim(\operatorname{Ker} T) = q$. Then we get sections given by elements of

$$\Lambda^{p+q}(V_0) \otimes \Lambda^q(V_1)^*$$

It seems that these sections should be essentially minors of the operator T. Look at the finite-diml case: $V_1^d \xrightarrow{T} V_0^{d+p}$, gives

$q = d-k \Rightarrow p+q = p+d-k$

$$\Lambda^k(T) : \Lambda^k(V_1^d) \longrightarrow \Lambda^k(V_0^{p+d})$$

If in addition I am given an elt. of $\Lambda^{p+d-k} V_0 \otimes \Lambda^{d-k}(V_1)^*$ then we can multiply by $\Lambda^k(T) \in \Lambda^k(V_0) \otimes \Lambda^k(V_1)^*$ to land in $\lambda(V_0) \otimes \lambda(V_1)^*$. So what I seem to be getting is a subspace

$$\bigoplus_q \Lambda^{p+q}(V_0) \otimes \Lambda^q(V_1)^* \subset \Gamma\left(T \longmapsto \lambda(\operatorname{Coker} T) \otimes \lambda(\operatorname{Ker} T)^*\right)$$

For example, if the index $p = 0$, then we are associating to a matrix its minors of various sizes.

In the infinite-diml case we take a limit over all finite-diml subspaces F inside V_0. For F large enough T is transversal to F so we get

$$0 \longrightarrow T^{-1}F \longrightarrow V_1 \oplus F \longrightarrow V_0 \longrightarrow 0$$

so maps

$$\lambda(T^{-1}F) \otimes \Lambda(V_0) \hookleftarrow \Lambda(V_1 \oplus F) \twoheadrightarrow \Lambda(V_1) \otimes \lambda(F)$$

and hence a map

$$(*) \qquad \Lambda(V_0) \longrightarrow \Lambda(V_1) \otimes \Big(\underbrace{\lambda(F) \otimes \lambda(T^{-1}F)^*}_{\text{line bundle at } T} \Big)$$

associated to T which is independent of the choice of F. This will make elements of

$$\mathrm{Hom}\big(\Lambda(V_0), \Lambda(V_1)\big)^*$$

to give sections of the line bundle, which means there is a certain line attached to T in

$$\mathrm{Hom}\big(\Lambda(V_0), \Lambda(V_1)\big)$$

which is undoubtedly the thing defined by $(*)$. If index (T) = dim Ker − dim Cok $= p$, then the line of maps $\Lambda(V_0) \longrightarrow \Lambda(V_1)$ raises degree by p. This map should be the inverse image.

Review: Given $T: V_1 \longrightarrow V_0$ of index p it induces a map

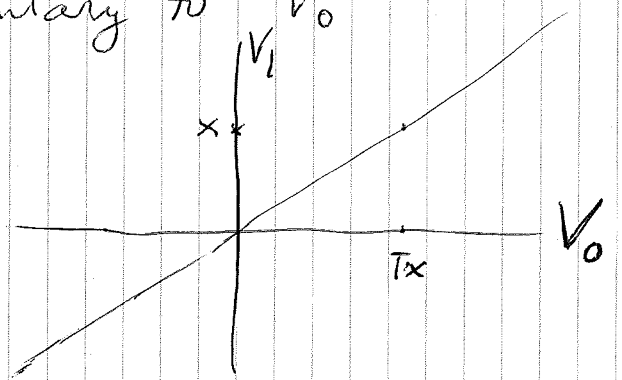
$$\Lambda(V_0) \longrightarrow \Lambda(V_1)$$

defined up to a scalar raising degrees by p. (When $V_1 = V_0$ one gets a projective repn. of the monoid of Fredholm operators; and actually since only T^{-1} is being used one might be able to extend to T's defined on a subspace of V_1.

Next point: I am thinking of these T as points in a kind of Grassmannian, hence I would like to associate to T a line in a wedge-space. So

$$\mathrm{Hom}(\Lambda(V_0), \Lambda(V_1)) \supset \Lambda(V_0^*) \otimes \Lambda(V_1) = \Lambda(V_0 \oplus V_1 ; V_0)$$

and the latter contains lines for subspaces of $V_0 \oplus V_1$ commensurable with V_0. Now the graph of T is complementary to V_0



so in infinite dimensions is never commensurable with V_0. However we are interested in the ~~graph~~ correspondence T^{-1} which will be commensurable with V_0 "provided most of the eigenvalues of T are infinite". So if the eigenvalues of T go off to ∞ fast enough, the Hilbert space version of the wedge space might contain the required line.

Suppose then that V_1, V_0 are Hilbert spaces and $T: V_1 \to V_0$ is a Fredholm operator. When does the graph of T determine an element is the L^2-Fock space $\Lambda(V_0 \oplus V_1 ; V_0)$? Using unitary transformations one can suppose T diagonal. I can reduce to the situation where T is an isomorphism, precisely its kernel + cokernel are 0. Then if $Te_n = \lambda_n f_n$ where $\{e_n\}$ is an orth. base for V_1, $\{f_n\}$ an orth. base for V_0, one has

$f_1 \wedge f_2 \wedge \cdots\cdots = |0\rangle$ generates the line belong. to V_0

$(f_1 \wedge \frac{1}{\lambda_1} e_1) \wedge (f_2 \wedge \frac{1}{\lambda_2} e_2) \wedge \cdots$ generates the line belong. to graph(T^{-1})

and the graph of T^{-1} gives an ℓ^2 element in the Fock space when
$$\prod\left(1+\frac{1}{|\lambda_n|^2}\right) < \infty$$
i.e. when $\sum \frac{1}{|\lambda_n|^2} < \infty$.

This is the just the sort of thing that fails for $\bar\partial$ on a Riemann surface. Thus for the elliptic curve and $\bar\partial - z$ we saw the eigenvalues were $\mu - z$ with μ running over the dual lattice, and I know that $\sum \frac{1}{|\mu - z|^2} = \infty$.

The thing I want to understand now is why a τ-function is defined for line bundles over a Riemann surface represented by clutching functions over a small S^1, and why a τ-function can't apparently be defined by the $\bar\partial$ operator.

So review the τ-fn. situation. We have a point ∞ on a Riemann surface X and a small circle S^1 around ∞. We have a line bundle L_0 over X with $h^0 = h^1 = 0$ trivialized over S^1 and its interior by a holom. section. Then inside $V = L^2(S^1)$ we have the subspace H_- of sections extending holomorphically inside S^1, and the subspace W_0 of sections extending holom. outside S^1. By assumption that $h^0 = h^1 = 0$ we have

$$V = H_- \oplus W_0$$

Now suppose that we have a holomorphic fn. f on S^1 with values in $\mathbb{C}^*$, a typical example being

$$f = e^{x_1 z + x_2 z^2 + \cdots + x_n z^n}$$

where z denotes a coordinate on the disk. Specifically we suppose that if D_- = interior of S^1, then $z : D_- \xrightarrow{\sim} \{z \in \mathbb{C} \mid |z| > 1\}$. Then we can use f as a clutching function to define a new line bundle over X, call it L_f with

$$\Gamma(U_-, L_f) = H_-$$
$$\Gamma(U_+, L_f) = f W_0$$

and hence the cohomology of L_f is given by the complex

$$H_- \oplus W_0 \xrightarrow{(1, f)} V$$

[redacted] Now consider the Fock space $\mathcal{F}$ belonging to V with ground state $|0\rangle$ corresponding to $H_+ = (H_-)^\perp$. [redacted] Let u_W [redacted] generate the line in $\mathcal{F}$ corresponding to W, which should be ℓ^2-commensurable with H_+. Lift f to $\tilde{f}$ on the Fock space and then you can define

$$\tau = \langle 0|\tilde{f}|u_W\rangle / \langle 0|u_W\rangle .$$

Presumably $\langle 0|u_W\rangle \neq 0$ when W is complementary to H_- and u_W is defined. (Clear: W is the graph of $T: H_+ \longrightarrow H_-$ and we can choose orthonormal bases so that $Te_n = \lambda_n f_n$. Then $\left\{ \frac{e_n + \lambda_n f_n}{\sqrt{1+|\lambda_n|^2}} \right\}$ is an orth. basis for W, so we can take

$$u_W = \frac{e_1 + \lambda_1 f_1}{\sqrt{1+|\lambda_1|^2}} \wedge \cdots\cdots\cdots$$

[redacted] whence $\langle 0|u_W\rangle = \prod (1+|\lambda_n|^2)^{-1/2}$.

We see from the above that the only indeterminacy in the τ function comes from lifting f to $\tilde{f}$. In the situation of interest the f's that we consider form an abelian Lie group, [redacted] isom. to $\mathbb{C}^n$, [redacted] and the central extension formed of the $\tilde{f}$ is abelian. The τ function does not depend just on f and if we try to define $\tau(f)$, then two choices differ by a character in the variable f.

<u>Conjecture:</u> Fix a C^∞ vector bundle E over a Riemann surface X. Assume $\deg(E) = \operatorname{rank}(E)(g-1)$, so that $h^0 = h^1 = 0$ for most holom. structures on E. Fix such a structure $\bar\partial : E \longrightarrow E \otimes \Omega^{0,1}$, so that the others are $\bar\partial - \omega : E \to E \otimes \Omega^{0,1}$ as $\omega \in \Gamma(X, \operatorname{End}(E) \otimes \Omega^{0,1})$. Then it should be possible

to define as relative determinant $\det\left((\bar\partial)^{-1}(\bar\partial-\omega)\right) = \tau(\omega)$ analytic in ω and we have

$$|\tau(\omega)| = \frac{\text{analytic torsion of } \bar\partial-\omega}{\text{torsion of } \bar\partial}$$

provided the structure $\bar\partial_\omega = \bar\partial - \omega$ has $h^0 = h^1 = 0$, and zero otherwise.

(This conjecture is probably slightly off by the exponential of a Kähler type metric.)

Example: Consider the family of all rank 2, degree (-2) vector bundles over $\mathbb{P}^1$ obtained from maps $f: S^1 \to GL_2$ of degree 0. The cohomology is computed via the cx

$$z^{-1}H_-^2 \oplus H_+^2 \xrightarrow{(1,f)} V^2$$

and one has the τ-function

$$\tau(\tilde f) = \langle 0|\tilde f|0\rangle.$$

which is defined on the covering group. In order to get a τ function defined on the set of f we have to lift $f \mapsto \tilde f$. I can restrict f to send 1 to 1 and to be unitary, in which case, it is equivalent to the lattice fH_+^2, and hence we have a complex manifold structure on the set of f, and the possible $\tilde f$ maybe give a holom. line bundle. The condition that $\tilde f$ be unitary gives a hermitian structure on this line bundle. Finally

$$\frac{|\langle 0|\tilde f|0\rangle|}{\|\tilde f|0\rangle\|} = |\cos\theta| \qquad \theta \text{ angle between } |0\rangle, \tilde f|0\rangle.$$

is independent of the choice of $\tilde f$. Thus if we have a holomorphic way to define $\tau(f) \mapsto \tau(\tilde f)$, then

$$|\tau(f)| = \|\tilde f|0\rangle\| \cdot \frac{|\langle 0|\tilde f|0\rangle|}{\|\tilde f|0\rangle\|}$$

and the second factor could be analogous to the analytic torsion.

<u>Conjecture</u>. Fix a C^∞ hermitian vector bundle E over X. Then the canonical determinant line bundle over the space of holom. structures on E has a canonical hermitian metric + hence a canonical connection. When the index is 0, the norm of the (section of this) canonical line bundle is the analytic torsion.

Toward defining the determinant

$$d \log \det(\bar\partial_0 - \omega) = -\operatorname{tr}\left[(\bar\partial_0-\omega)^{-1} d\omega\right]$$

$$= - \operatorname{tr}\left\{ \bar\partial_0^{-1} d\omega + \bar\partial_0^{-1}\omega\,\bar\partial_0^{-1} d\omega + \cdots\right\}.$$

Consider the elliptic curve case with $\omega = z$. Then

$$-\operatorname{tr}\left[(\bar\partial_0-\omega)^{-1} d\omega\right] = \left(\sum_\mu \frac{1}{z-\mu}\right) dz$$

which makes no sense. To get convergence you must perform two subtractions

$$\sum_\mu \left\{\frac{1}{z-\mu} + \frac{z}{\mu} + \frac{z^2}{\mu^2}\right\}.$$

Therefore it seems that what we want is

$$\tau(\omega) = \det{}^{(2)}\left(1 - \bar\partial_0^{-1}\omega\right).$$

It's likely that this is well-defined, and will give an entire function on the space of holom. structures

March 4, 1982

I have the feeling that it makes more sense algebraically to replace a Fredholm operator $T: V_1 \longrightarrow V_0$ by a Fredholm correspondence, suitably defined. Let's try the definition of a subspace $W \subset V_1 \oplus V_0$ such that the projections $W \xrightarrow{p_1} V_1$, $W \xrightarrow{p_0} V_0$ are Fredholm. More generally take a pair of such Fredholm maps. Actually we seem to need only that p_0 be Fredholm.

Simple problem. In finite dimensions suppose we have two correspondences

$$\begin{tikzcd} & W' \arrow[r] \arrow[d] & V_0 \\ W \arrow[r] \arrow[d] & V_1 & \\ V_0 & & \end{tikzcd}$$

Then we have

$$\lambda(W) \otimes \lambda(V_0)^* \subset \mathrm{Hom}(\Lambda V_0, \Lambda V_1)$$

$$\lambda(W') \otimes \lambda(V_1)^* \subset \mathrm{Hom}(\Lambda V_1, \Lambda V_0)$$

and so

$$\begin{tikzcd} \lambda(W)\otimes\lambda(V_0)^*\otimes\lambda(W')\otimes\lambda(V_1)^* \subset \mathrm{Hom}(\Lambda V_0, \Lambda V_1) \otimes \mathrm{Hom}(\Lambda V_1, \Lambda V_0) \arrow[d, "\mathrm{tr}"] \\ \mathbb{C} \end{tikzcd}$$

What is this map? Example: If W is the graph of an homomorphism A from V_0 to V_1, $\lambda(W) \otimes \lambda(V_0)^* = \mathbb{C}$ and the map from ΛV_0 to ΛV_1 we get is $\Lambda(A)$. So if W' is the graph of $B: V_1 \to V_0$ we are getting

$$\mathrm{tr}_{\Lambda V_0}(\Lambda B \, \Lambda A) = \mathrm{tr}_{\Lambda V_0}(\Lambda(BA)) = \det(1 + BA)$$

Question: Can this thing be interesting when W

has a non-zero index?

Actually I really yet don't understand composition of correspondences. It's not always defined, but I think the induced map on the wedge spaces should be zero in this case.

Perhaps it is possible to understand the effect of correspondences on ΛV using the Clifford generators. So let us take a correspondence

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"'] & W \\ V & \end{tikzcd}$$

and try to realize its effect on $\Lambda(V)$.

$$\operatorname{Ker} p \rightarrow W \twoheadrightarrow \operatorname{Im} p \subset V.$$

Start with $F \subset V$, then intersect with $\operatorname{Im} p$ which corresponds to multiplying by $i(\lambda_1)\cdots i(\lambda_k)$, where $\lambda_1, \ldots, \lambda_k$ is a basis for $(\operatorname{Im} p)^\perp \subset V^*$. Then we have to pull back to W. This can be realized by choosing a section of $W \twoheadrightarrow \operatorname{Im} p$, call it s, then using $\Lambda(s) : \Lambda(\operatorname{Im} p) \rightarrow \Lambda(W)$ and then multiplying by $e(\omega_1) \wedge \cdots \wedge e(\omega_\ell)$, where $\omega_1, \ldots, \omega_\ell$ is a basis for $\operatorname{Ker} p$. Extend s to a map $s : V \rightarrow W$. Then we get the following formula for the effect on ΛV:

$$\Lambda(q)\, e(\omega_1) \wedge \cdots e(\omega_\ell)\, \Lambda(s) \;\; i(\lambda_1)\cdots i(\lambda_k).$$

Hence in general the operator we get belonging to a correspondence is a normal product sort of thing

$$e(\omega_1' \wedge \cdots \wedge \omega_\ell')\; \Lambda(s')\; i(\lambda_1 \wedge \cdots \wedge \lambda_k)$$

where $\omega_1, \ldots, \omega_\ell \in V$, $\lambda_1 \cdots \lambda_k \in V^*$, $s' : V \rightarrow V$.

Clearly if $q: W \to V$ is not injective on the kernel of p, then we get the zero map. Hence W must embed in $V \times V$, for this to be non-zero on ΛV. What is the trace of the map on ΛV when the index is zero? Seems to be $\det(1+s')$, where s' is what the correspondence does as a map from $\operatorname{Im} p$ to $V/q(\operatorname{Ker} p)$??

Effectively we are thinking of our correspondence in the form

$$\begin{tikzcd} & V \arrow[d, two heads] \\ I \arrow[r, "s"] \arrow[d, hook] & C \\ V & \end{tikzcd}$$

where I and C have the same dimension. The map on ΛV is

$$e(\omega_1 \wedge \cdots \wedge \omega_p)\, \Lambda(s)\, i(\lambda_1 \wedge \cdots \wedge \lambda_p)$$

where $\{\omega_i\}$ is a basis for $\operatorname{Ker} V \twoheadrightarrow C$, and $\{\lambda_i\}$ is a basis for $I^\vee \subset V^*$. The trace is the same as that of

$$\Lambda(s)\left(i(\lambda_1 \wedge \cdots \wedge \lambda_p)\, e(\omega_1 \wedge \cdots \wedge \omega_p)\right)$$

where the second factor is a map

$$\Lambda^k C \xrightarrow{e(\omega_1 \wedge \cdots \wedge \omega_p)} \Lambda^{k+p} V \xrightarrow{i(\lambda_1 \wedge \cdots \wedge \lambda_p)} \Lambda^k I$$

Better viewpoint: We are thinking of correspondences of V with itself of index 0 as subspaces W of $V \times V$ of the same dimension as V. On this Grassmannian we have the dense open set of correspondences which are graphs of homomorphisms $T: V \to V$, and we have the map $\det(1+T) = \operatorname{tr}(\Lambda T)$. We are trying to extend this function to the Grassmannian, which is impossible, as it acquires poles. So one thing we can say is that we have canonical maps

$$\lambda(W) \otimes \lambda(V)^* \subset \mathrm{Hom}(\Lambda V, \Lambda V) \xrightarrow{tr} \mathbb{C}$$

And another thing we can say is that we are trying to make sense of

$$\begin{array}{l} W \xrightarrow{B} V \\ \downarrow A \\ V \end{array} \qquad \det(1 + A^{-1}B) = \frac{\det(A+B)}{\det(A)}$$

as A becomes singular.

So let us take an endomorphism T and allow it to become infinite and let us see if we can control what's going on.

Positive result: It seems that the natural generalization of the Fredholm map Grassmannian is the set of correspondences $W \subset V_0 \times V_1$ such that $W \xrightarrow{pr_1} V_0$ is Fredholm. Attached to each such W is a line in

$$\mathrm{Hom}(\Lambda V_0, \Lambda V_1)$$

so one gets a line bundle over this Grassmannian whose dual has lots of sections. Restricting to W such that $W \xrightarrow{\sim} V_1$, we get the set of Fredholm maps $T : V_1 \longrightarrow V_0$, as before. But now if $V_1 = V_0$, the trace of the element of $\mathrm{Hom}(\Lambda V_0, \Lambda V_0)$ in the case of index 0 is some nice version of

$$\frac{\det(1 + T^{-1})}{\det(T^{-1})}$$

which explains why I was interested in the case where the eigenvalues of T grow very fast.

Notice that this type of Grassmannian fits nicely with differential operators $D : V_1 \to V_0$ which are not everywhere defined.

The next thing will be to put a hermitian structure and connection on this line bundle. Begin with the finite-dimensional case. We have been looking at subspaces W of $V_0 \times V_1$. To such a W we have associated a line

$$\lambda(W) \subset \Lambda(V_0 \times V_1) = \Lambda V_0 \otimes \Lambda V_1$$
$$= (\Lambda V_0)^* \otimes \lambda(V_0) \otimes \Lambda V_1$$

or a line $\lambda(W) \otimes \lambda(V_0)^* \subset (\Lambda V_0)^* \otimes (\Lambda V_1) = \mathrm{Hom}(\Lambda V_0, \Lambda V_1)$.
Thus the line bundle over the Grassmannian is the highest exterior power of the subbundle. Given inner products on V_0 and V_1 one gets an inner product on $(\Lambda V_0), (\Lambda V_1)$ and thus an inner product on $\lambda(W) \otimes \lambda(V_0)^*$.

In infinite dimensions

$$\Lambda(V_0^*) \otimes \Lambda(V_1) = \Lambda(V_0 \oplus V_1 ; V_0)$$

contains a line for any subspace $W \subset V_0 \oplus V_1$ commensurable with V_0, which in the case, where one has

$$\begin{array}{ccc} W & \xrightarrow{B} & V_1 , \\ {\scriptstyle A}\downarrow\cong & & \\ V_0 & & \end{array}$$

means that $A^{-1}B$ has finite rank.

In the L^2-theory we want that $A^{-1}B$ be Hilbert-Schmidt.

Example: Look at the correspondence in $\mathbb{C}/\Gamma$ case:

$$\begin{array}{ccc} & H^2_{(1)} & \subset H^2_{(k)} \\ \frac{\partial}{\partial \bar{u}} - z & \cong\downarrow & \\ & H^2_{(0)} & \end{array}$$

Then for the orth. basis e_μ of $H^2_{(0)}$ we have

$$e_\mu \longmapsto \frac{1}{\mu - z} e_\mu$$

and $\left\|\frac{1}{\mu-z}e_\mu\right\|^2_{(k)} = \frac{|\mu|^{2k}}{|\mu-z|^2}$. Thus we do get a Hilbert-Schmidt operator provided $k<0$.

The next project will be to understand the metric and connection on the canonical line bundle over the Grassmannian.

March 5, 1982

Let's compute the canonical connection on $\mathcal{O}(-1)$ over $\mathbb{P}(V)$ and its curvature, when V has an inner product. A non-zero element of $\mathcal{O}(-1)$ is the same thing as a non-zero $\psi \in V$. Suppose $\|\psi\| = 1$, so that we have a point in the unit circle bundle of $\mathcal{O}(-1)$. If $L = \mathbb{C}\psi$, then ψ lies over $L \in \mathbb{P}(V)$. The tangent space to [redacted] $\mathbb{P}(V)$ at L can be identified with $\mathrm{Hom}(L, V/L)$, and [redacted] because of the [redacted] inner product on V, we have $V/L \cong L^\perp$. But also $\mathrm{Hom}(L, L^\perp) \hookrightarrow \mathbb{P}(V)$ by associating to a map A [redacted] its graph, [redacted] $\Gamma(A) = \{\psi + A\psi \mid \psi \in L\}$. Therefore we can lift A to $\mathcal{O}(-1)$ by associating to $\psi \in L$ the vector going from ψ to $\psi + A\psi$. Here I am thinking of $\mathcal{O}(-1) - (0\text{-section})$ [redacted] $= V - \{0\}$, so that a tangent vector in $\mathcal{O}(-1)$ will be given by a vector in V.

Notice that

$$\begin{tikzcd}
T_{\mathcal{O}(-1)} \text{ at } \psi = V \arrow[d] \arrow[drr] & & \\
T_{\mathbb{P}V} \text{ at } \mathbb{C}\psi = \mathrm{Hom}(\mathbb{C}\psi, V/\mathbb{C}\psi) & \cong & V/\mathbb{C}\psi
\end{tikzcd}$$

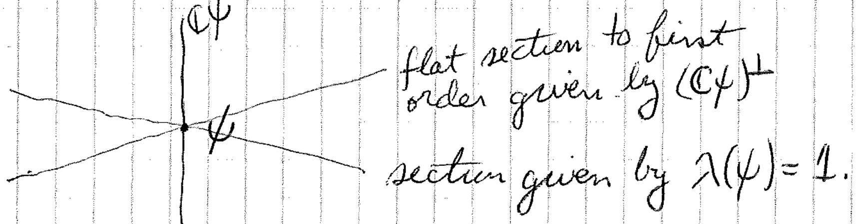
and the connection just defined uses the [redacted] inner product to construct a section of $V \to V/\mathbb{C}\psi$. This tells us that a curve [redacted] $t \mapsto \psi_t$ in $\mathcal{O}(-1)^{-0\,\text{section}}$ is flat relative to the connection when $\dot{\psi} \perp \psi$. In particular

$$\frac{d}{dt}\|\psi\|^2 = \langle \dot{\psi} | \psi \rangle + \langle \psi | \dot{\psi} \rangle = 0.$$

so the connection preserves the metric.

Next we want to show that if s is a local holomorphic section of $\mathcal{O}(-1)$, then it covariant derivative Ds

relative to this connection is a form of type $1,0$. Take $\lambda : V \to \mathbb{C}$; on the open set of L complementary to $\operatorname{Ker}\lambda$ we get a section s of $\mathcal{O}(-1)$ associating to L the unique vector $s(L) \in L$ with $\lambda s(L) = 1$. So the ~~actual section s consists of a~~ graph of the section s consists of all vectors ψ with $\lambda(\psi) = 1$, so it is an affine hyperplane. The connection we have defined says that a section flat to first order at ψ is given by the affine hyperplane thru ψ perpendicular to ψ.



The covariant derivative is the "difference" of these two hyperplanes, viewed as a map from the tangent space

$$\operatorname{Hom}(\mathbb{C}\psi, V/\mathbb{C}\psi)$$

to the fibre $\mathbb{C}\psi$. It's a complex linear map so therefore is a section of $\Omega^{1,0}$.

More precisely given λ I want the section Ds of $\mathcal{O}(-1) \otimes T^{1,0}$ over the open set of L with $\lambda(L) = \mathbb{C}$. The value of Ds at L is an element of

$$L \otimes (T^{1,0} \text{ at } L) = L \otimes \operatorname{Hom}(L, V/L)^*.$$

But $\operatorname{Ker}\lambda$, $L^\perp$ are two complements to L, hence $\operatorname{Ker}\lambda$ is the graph of an element of $\operatorname{Hom}(L^\perp, L)$. Thus it seems that

$$Ds = s \otimes \theta$$

where θ at L is the linear form on the tangent space $\operatorname{Hom}(L, L^\perp)$ obtained from the element of $\operatorname{Hom}(L^\perp, L)$ measuring the difference between $\operatorname{Ker}(\lambda)$ and $L^\perp$ as complements to L.

<u>Point</u>: When one has a holomorphic line bundle L with inner product, then the canonical connection assigns to a local holomorphic section s the connection form θ (hence $Ds = s \otimes \theta$ by definition) given by

$$\theta = \partial \log |s|^2$$

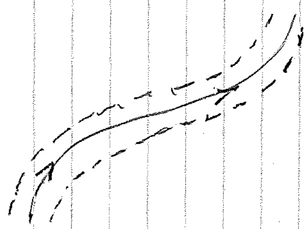
The curvature is then

$$d\theta = \bar{\partial} \partial \log |s|^2 .$$

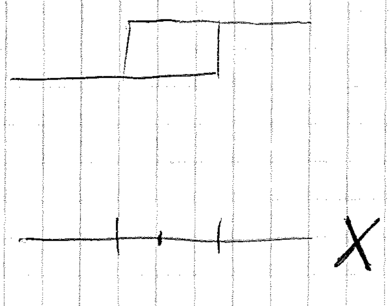
The point is that changing the metric by a scale factor doesn't affect either of these, so they might make sense over a Riemann surface.

March 6, 1982.

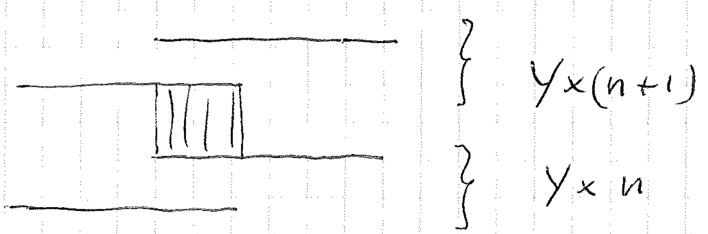
We have two ways to produce families of v.b. over a Riemann surface: 1) Bott-Atiyah method of holomorphic structures on a fixed C^∞-vector bundle. 2) Clutching function. Take an embedding $S^1 \subset X$ and a given vector bundle E over X. Orient the curve S^1 hence in a tubular neighborhood of the curve we can talk about the left and the right of the curve



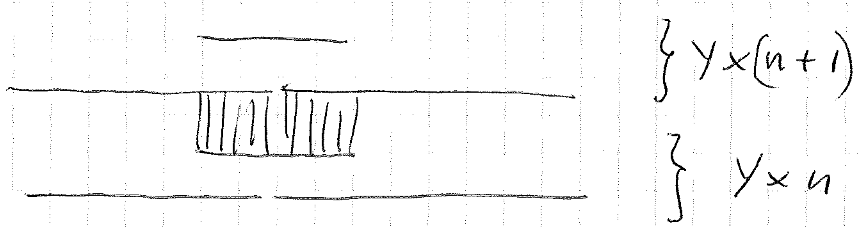
Now over the S^1 take an automorphism g of the bundle E, and suppose it is analytic, hence it extends over a tubular nbd of the curve. Now I can define a new holomorphic vector bundle as follows. It's enough to give the sections over small open sets, and any small open set is divided into a left and right by the curve so you have the usual construction. Put another way we can construct by the clutching construction a twisted version of E in the strip which is isomorphic to E on either side. Best approach is to cut the surface along the curve and thicken the edges. Then we get an open surface Y such that X is obtained by identifying the two copies of the annulus.



The clutching function is just descent data for the map $Y \to X$. Actually maybe we can construct an infinite cyclic covering space of X in this way [redacted]. [redacted] Namely take $Y \times \mathbb{Z}$ and glue the right [redacted] strip of $Y \times n$ to the left [redacted] strip of $Y \times (n+1)$.



So it should be possible to obtain from $S^1 \subset X$ an element of $H^1(X, \mathbb{Z})$; yes, [redacted] from the Gysin homomorphism $H^0(S^1, \mathbb{Z}) \longrightarrow H^1(X, \mathbb{Z})$. Note that in the case [redacted] where the S^1 divides the surface in 2 parts we get a trivial covering



Let's consider the clutching function construction [redacted] in the case of a small S^1 around a point. V = space of sections of E over S^1, H_- = sections holom. inside S^1, W = sections holom. outside S^1. Then

$$H_- \oplus W \xrightarrow{(1,g)} V$$

is the Cech complex for computing the cohomology of the bundle E_g. This is Fredholm, so one gets a [redacted] line bundle over the space of g.

The way to think of this line bundle is as follows: [redacted] The subspaces gW are all commensurable (in the l^2 sense) and hence determine lines in the wedge space

$$\Lambda(H_- \oplus W; W)$$

Hence the fibre of the line bundle at g is the line L_{gW} belonging to gW in the wedge space.

Next suppose we are in the situation of index 0, and [redacted] $h^0(E) = h^1(E) = 0$, whence W is complementary to H_-. Then there should be [redacted] a linear functional $\langle 0|$ on the wedge space which one gets from the decomposition $V = H_- \oplus W$. Hence we get a canonical map

$$L_{gW} \xrightarrow{\langle 0|} \mathbb{C}$$

which is non-zero for gW which are complementary to H_-.

[redacted]

Next if [redacted] one supposes E has a hermitian structure and a [redacted] volume is put on S^1, then V becomes a Hilbert space, hence so does the wedge space, and so the line L_{gW} acquires an inner product.

What you should really check out in the present situation is why the subspaces W [redacted] are Hilbert-Schmidt with respect to $H_+ = (H_-)^{\perp}$.

Let's go back to the case of holomorphic structures on a smooth vector bundle. This leads to a family of Fredholm operators

$$V_1 \xrightarrow{T} V_0$$

to which we can associate a line

$$\lambda(T) \in \mathrm{Hom}(\Lambda V_0, \Lambda V_1)$$

of degree = index of T. Now V_0 and V_1 are Hilbert spaces, [redacted] and from the construction of $\lambda(T)$, I can probably make it into a map between Hilbert wedge spaces. For example the most interesting case is where $T^{-1} : V_0 \longrightarrow V_1$ is a bounded operator in which case $\lambda(T)$ is the line spanned by $\Lambda(T^{-1})$. Unfortunately T^{-1} is not Hilbert-Schmidt in the cases I am interested in.

Recall that a bounded operator $A : V \longrightarrow W$ is Hilbert-Schmidt when it is in the image of

$$\underbrace{V^* \hat{\otimes} W}_{\substack{\text{Hilbert-space} \\ \text{completion}}} \longrightarrow \mathrm{Hom}(V, W)$$

or equivalently if $\mathrm{tr}(A^*A) < \infty$. So if I have this map $A: V \to W$ which is Hilbert-Schmidt, I know its graph in $V \times W$ will determine a line in the Fock space

$$\Lambda(V \oplus W; V) = \Lambda(V^*) \otimes \Lambda(W) \qquad (L^2 \text{ version})$$

which means the operator $\Lambda(A): \Lambda V \to \Lambda W$ is Hilbert-Schmidt. One can also see this from the fact that if A has eigenvalues λ_n (i.e. $Av_n = \lambda_n w_n$ for orth. bases of V, W) then $\Lambda(A)$ has eigenvalues $\lambda_{i_1} \cdots \lambda_{i_n}$ or

$$\mathrm{tr}(\Lambda(A)^* \Lambda(A)) = \mathrm{tr}\, \Lambda(A^*A) = \prod (1 + \lambda_i^2)$$

which converges when $\sum \lambda_i^2$ does.)

Even when A is not Hilbert-Schmidt I can hope for the following. Let us take a path in the space of holomorphic structures, then we get a family of operators A_t (assume we stay in the open set where $h^0 = h^1 = 0$) and hence a family of lines

$$L_t = \mathbb{C}\Lambda(A_t) \subset \mathrm{Hom}(\Lambda V_0, \Lambda V_1) \quad L^2 \text{ version}$$

Given $\psi_0 \in L_0$ I can hope to show that there is a unique path $\psi_t \in L_t$ such that

$$\psi_t \perp \dot{\psi}_t$$

in the sense that $\dot{\psi}_t$ is sufficiently bounded that it's trace with ψ_t is defined, and then this trace is 0.

Let's now calculate in the elliptic curve case. Here $V_0 = V_1 = C^\infty$ functions on $\mathbb{C}/\Gamma$ and the operator $A: V_0 \to V_0$ is $\frac{1}{\bar\partial - \omega}$. Using the natural basis e_μ one has

$$A : e_\mu \longmapsto \frac{1}{\mu - \omega} e_\mu$$

Suppose now that $\omega = \omega(t)$. Set

$$\psi_t = c_t \Lambda(A_t) \in \mathrm{Hom}(\Lambda V_0, \Lambda V_0)$$

$$\mathring{\psi}_t = c_t \frac{d}{dt} \Lambda(A_t) + \mathring{c}_t \Lambda(A_t)$$

$$\mathrm{tr}(\psi^* \mathring{\psi}) = |c|^2 \,\mathrm{tr}\left(\Lambda(A)^* \frac{d}{dt}\Lambda(A)\right) + \bar{c}\mathring{c}\ \mathrm{tr}\left(\Lambda(A)^*\Lambda(A)\right)$$

Now

$$\mathrm{tr}\left(\Lambda(A_u)^* \Lambda(A_t)\right) = \prod_\mu \left(1 + \overline{\frac{1}{\mu - \omega_u}}\ \frac{1}{\mu - \omega_t}\right)$$

so

$$\frac{\partial}{\partial t} \log \mathrm{tr}\left(\Lambda(A_u)^* \Lambda(A_t)\right) = \mathrm{tr}\left(\Lambda(A_u)^* \frac{d}{dt}\Lambda(A_t)\right) \Big/ \mathrm{tr}\left(\Lambda A_u^* \Lambda A_t\right)$$

$$= \sum_\mu \frac{1}{1 + \overline{\frac{1}{\mu - \omega_u}} \frac{1}{\mu - \omega_t}}\ \overline{\frac{1}{\mu - \omega_u}}\ \frac{1}{(\mu - \omega_t)^2}\ \mathring{\omega}_t$$

So put $u = t$ and you find

$$\frac{\mathrm{tr}\left(\Lambda(A_t)^* \frac{d}{dt}\Lambda(A_t)\right)}{\mathrm{tr}\left(\Lambda(A_t)^* \Lambda(A_t)\right)} = \sum_\mu \frac{1}{|\mu - \omega_t|^2 + 1}\left(\frac{\mathring{\omega}_t}{\mu - \omega_t}\right)$$

which is a finite quantity. So hence I can define c_t so that

$$\frac{\mathrm{tr}(\psi_t^* \mathring{\psi}_t)}{\mathrm{tr}(\psi_t^* \psi_t)} = 0$$

March 8, 1982

Problem: We have an operator $D = \bar\partial + A$ and we are trying to make sense of its determinant. Something similar occurs in Schwinger's paper V. Let's review the setup.

Schwinger looks at the Dirac equation on 4-space. With no EM field it takes the form

$$i\frac{\partial\psi}{\partial t} = \left(\sum \alpha_j \frac{1}{i}\frac{\partial}{\partial x_j} + \alpha_0 m\right)\psi$$

and with EM field given by the gauge field A_μ one replaces $\partial/\partial x_\mu$ by $\partial/\partial x_\mu + A_\mu$. In any case one gets an equation of the form

$$i\partial_t \psi = H\psi$$

where H is a self-adjoint operator (possibly depending on t) working on vector functions in the x-variables.

He considers the situation where A_μ is supported in a time interval $[-T, T]$. Then one gets a standard scattering setup on the space of solutions of the free Dirac equation. Call this space V; it is a Hilbert space with the self-adjoint operator H_0. The scattering operator on V takes a ψ at time 0 propagates backward to time $-T$, then forward ~~to~~ time T with the field present, then freely back to time 0. This gives a nice unitary operator S on V.

When one quantizes, one forms the Fock space of V relative to the negative energy subspace for the operator H_0. The quantum problem is to lift the operator S to the Fock space. There is no problem probably constructing such a lifting, namely, one can use the formula

$$\tilde S = \langle 0|\tilde S|0\rangle : e^{R} :$$

where R is some sort of scattering matrix of S relative to the splitting: $V = V_+ \oplus V_-$.

However what seems to be interesting is that Schwinger has some way to make sense of the scalar $\langle 0|\tilde{S}|0\rangle$. In general we have

$$i\partial_t\psi = (H_0 + H')\psi$$

where $H'(t)$ depends on t. The S-matrix is given by

$$S = T\left\{e^{-i\int_{-\infty}^{\infty} e^{iH_0t}H'(t)e^{-iH_0t}\,dt}\right\}.$$

Modulo the problem of integrating an ordinary D.E. in Fock space, the real problem seems to be to make sense of $H'(t)$ as an operator on Fock space. One has to make sense out of $\mathrm{tr}(P^- H'(t))$, where P^- is the projection on the negative energy space. Possibly something special is going on in the case of the Dirac equation.

<u>Question</u>: Take a hermitian vector bundle E over a Riemann surface X and consider a curve $S^1 \xrightarrow{\alpha} X$. Does a holomorphic structure on E give one a way of going between $L^2(E, \alpha(S^1))$ for different curves α?

Presumably given another curve β, one starts with a section over α extends it to a holomorphic section and then restricts to β. Of course, this sort of Cauchy problem is poorly posed for the $\bar{\partial}$ operator, which suggests we should look for an imaginary time version of the Schwinger theory.

<u>Question</u>: According to Coleman notes, fermion integration of the action $\int \bar{\psi} D \psi$ leads to

the determinant of the operator D. Presumably one can also define various Green's functions [redacted] from this fermion integration. What are these?

Recall the formula

$$\frac{\int e^{-\tilde{\psi} A \psi}\, \psi_i \tilde{\psi}_j}{\int e^{-\tilde{\psi} A \psi}} = (A^{-1})_{ij}$$

More generally if we introduce independent anti-commuting variables [redacted] $\tilde{J}_i$, J_i then $\left((\tilde{\psi} - \tilde{J} A^{-1}) A (\psi - A^{-1} J) \right)$

$$\frac{\int e^{-\tilde{\psi} A \psi + \tilde{J}\psi + J\tilde{\psi}}}{\int e^{-\tilde{\psi} A \psi}} = e^{\tilde{J} A^{-1} J}.$$

Hence in a natural way the Green's functions are matrix elements of $\Lambda(A^{-1})$. On the other hand Green's functions traditionally have an interpretation as vacuum expectation values,

March 10, 1982

Facts about ζ functions of diffl. ops.

Let B be an elliptic ψDO of positive order m on a compact manifold of dim. n. Assume B self-adjoint + positive so that all eigenvalues are >0.

$$\zeta_B(s) = \operatorname{Tr} B^{-s} = \sum \mu^{-s}$$

Then: This converges for $Re(s) > n/m$ + is analytic there

Meromorphic continuation: for all integers $N \geq -n$

$$\zeta_B(s) = \sum_{\substack{k=-n \\ k\neq 0}}^{N} \frac{a_k}{s+k/m} + \underbrace{\phi_N(s)}_{\text{Anal for } Re(s) > -N/m}$$

a_k are given by local integral fmlas: $a_k = \int \alpha_k$

No pole at $s=0$. One knows $\zeta_B(0)$ is also given by a[redacted] local integral formula.

If B is a differential operator, then [redacted] there are no poles at $s=0,-1,-2,\dots$ and the ζ_B values are given by local integral formulas.

Proposition (2.9 of Atiyah, Patodi, Singer: Spectral asym. + Riem. geom. III, Math. Proc. Camb. 79 (1976)): Let $u \mapsto B_u$ be a C^∞ family of pos. s.s. elliptic ops. of pos. order m

$$\frac{d}{du} \operatorname{tr}\left(B_u^{-s}\right) = -s \operatorname{tr}\left(\left(\tfrac{d}{du}B_u\right) B_u^{-s-1}\right) \qquad Re\, s \gg 0.$$

Interesting point: Let $K_s(x,y)$ be the Schwartz kernel of A^{-s}. Then for $x\neq y$, $K_s(x,y)$ is an entire fn. of s which vanishes for $s=0$ (and when A is a differential operator for $s=0,-1,-2,\dots$ because then A^{-s} is a local operator). Now $K_0(x,y) = \delta(x,y)$ because $A^0 = \text{Identity}$. However the surprising point is that $K_s(x,x)$, when analytically continued from $Re(s)$ large, is not infinite at $s=0$. This is what leads to

$$\zeta(s) = tr(A^{-s}) = \int K_s(x,x)\,dx$$

being finite at $s=0$.

For example let us take the operator $\Delta = D^*D$ where $D = \frac{\partial}{\partial \bar{u}} - z$ on functions over the elliptic curve $\mathbb{C}/\Gamma$. Then we have the eigenfns. $e^{\bar{\mu}u - \mu\bar{u}}$ with eigenvalues $|z-\mu|^2$, and hence

$$K_s(x,y) = \langle x \,|\, \Delta^{-s} \,|\, y \rangle = \left(\sum \frac{e^{\bar{\mu}(x-y) - \mu\overline{(x-y)}}}{|z-\mu|^{2s}} \right) \frac{1}{vol(\mathbb{C}/\Gamma)}.$$

This $K_s(x,y)$ is a distribution and we have just computed its Fourier series. Notice that as $s \to 0$ we get the δ-fn distribution as we should.

However we can fix x,y and analytically continue $K_s(x,y)$ from $Re(s) > 1$. We can do the analytic continuation using $u = x-y$

$$\pi^{-s}\Gamma(s) K_s(x,y) = \sum \frac{\bar{\mu}u - \mu\bar{u}}{|z-\mu|^{2s}} \int_0^\infty e^{-\pi t} t^s \frac{dt}{t}$$

$$= \int_0^\infty \Big(\underbrace{\sum_\mu e^{-\pi t|z-\mu|^2 + \bar{\mu}u - \mu\bar{u}}} \Big) t^s \frac{dt}{t}$$

Now the functional equation for the θ function will say this equals $\frac{1}{t}$ another θ function at $\frac{1}{t}$. As $t \searrow 0$, assuming $u \notin \Gamma$ the other θ fn. won't contain the term 1, since θ approaches the δ fn. in u as $t \searrow 0$. Thus we see that the integrand θ fn. decays both as $t \searrow 0$ and as $t \to +\infty$, so $\pi^{-s}\Gamma(s) K_s(x,y)$ is entire for $x-y \notin \Gamma$. Thus we see that $K_s(x,y)$ vanishes for $s = 0, -1, -2, \ldots$ as $x-y \notin \Gamma$.

March 11, 1982:

Let's consider the space $\mathcal{a}$ of holomorphic structures on an E with $\deg = rg(g-1)$. Fix a structure with vanishing cohomology: $\bar\partial : E \longrightarrow E \otimes \Omega^{0,1}$. Then other structures are given by $\bar\partial - A$, where $A : E \to E \otimes \Omega^{0,1}$ is a linear map. Now $\bar\partial$ is invertible, hence

$$\bar\partial^{-1}(\bar\partial - A) = 1 - \underbrace{\bar\partial^{-1} A}_{K}$$

I conjecture that it should be possible to define

$$f(A) = \det_{(2)}(1-K)$$

which should be an entire function of A. Moreover this function f should have the same zeroes as the canonical section of the determinant line bundle $\mathcal{L}$ over $\mathcal{a}$, so that it is equivalent to a trivialization of the line bundle $\mathcal{L}$.

Obvious questions: How does this $\det_{(2)}$ vary with respect to the original choice of $\bar\partial$? Can one prove these conjectures for line bundles using the gauge gp?

Mathematical problems connected with $\det_{(2)}$:

What I really have is an affine space $\mathcal{a}$ over which I want to **define** a determinant function. Thus the goal is to construct $D \longmapsto \det(D)$ which should be an entire function of D. First attempt is to use the formula

$$\delta \log \det(D) = tr\left(D^{-1}\delta D\right)$$

to define $\log\det(D)$ up to an additive constant. The problem with this is that the trace isn't defined. In the case at hand $D + \delta D = D+A$ where

A is a 0-th order operator, and we know that $K = D^{-1}A$ satisfies $\mathrm{tr}(K^3) < \infty$, but $\mathrm{tr}(K^2) = \infty$.

Let's work in a neighborhood of D_0 and try to define $\det(D_0 - \lambda A)$ for small λ, using the formula

$$\frac{d}{d\lambda} \log\det(D_0 - \lambda A) = -\,\mathrm{tr}\,(D_0 - \lambda A)^{-1}A$$
$$= -\,\mathrm{tr}\,(1 - \lambda \underbrace{D_0^{-1}A}_{K})^{-1} D_0^{-1}A$$
$$= -\,\mathrm{tr}\left(K + \lambda K^2 + \lambda^2 K^3 + \cdots\right)$$
$$= -\left\{(\mathrm{tr}K) + \lambda(\mathrm{tr}K^2) + \lambda^2(\mathrm{tr}K^3) + \cdots\cdots\right\}.$$

This shows that

$$\frac{d^3}{d\lambda^3} \log\det(D_0 - \lambda A) = -\left\{2{\cdot}1\,\mathrm{tr}(K^3) + 3{\cdot}2\lambda\,\mathrm{tr}(K^4) + \cdots\right\}$$

is well-defined, and fixes whatever definition of $\log\det(D_0 - \lambda A)$ we use up to [redacted] a quadratic function of λ.

Local obstruction problem. Let us define

$$-\log\det_{(2)}(D; D_0) = +\,\mathrm{tr}\left(\tfrac{1}{3}K^3 + \tfrac{1}{4}K^4 + \cdots\right)$$
$$= \sum_{n \geq 3} \frac{1}{n}\,\mathrm{tr}(K^n)$$

where $1 - K = D_0^{-1}(D) = D_0^{-1}(D_0 - A) = 1 - D_0^{-1}A \Rightarrow K = D_0^{-1}A$.

This should be well-defined for D in a neighborhood of D_0. Now pick a point D_1 in this neighborhood. Then the question is whether

$$\log\det_{(2)}(D; D_1) - \log\det_{(2)}(D; D_0)$$

is a quadratic function of D.

This is a simpler version of the global obstruction problem, namely to define a $\log\det(D)$ such that

$$\log\det D - \log\det_{(2)}(D; D_0)$$

is a quadratic function of D for any D_0.

Put $D_t = (1-t)D_0 + tD_1 = D_0 + t\underbrace{(D_1 - D_0)}_{B}$

and compute $\frac{d}{dt} \log \det_{(2)}(D, D_t)$ as a function of D.

$$-\log \det_{(2)}(D, D_t) = \sum_{n \geq 3} \frac{1}{n} \operatorname{tr}(K^n)$$

where $\quad K = D_t^{-1}(D_t - D)$ $\quad$ (so that $D_t^{-1}D = 1 - K$).

Then $\quad \dot{K} = -D_t^{-1}B\,D_t^{-1}(D_t - D) + D_t^{-1}(B)$

$$= D_t^{-1}B\,(1-K)$$

and

$$\frac{d}{dt}\left(-\log\det_{(2)}(D, D_t)\right) = \sum_{n \geq 3} \operatorname{tr}(\dot{K}K^{n-1})$$

$$= \sum_{n \geq 3} \operatorname{tr}\left(D_t^{-1}B\,(I-K)\,K^{n-1}\right)$$

So provided you know that $\operatorname{tr}(D_t^{-1}B\,K^2) < \infty$ (i.e. $\operatorname{tr}\left(D_t^{-1}B\,(D_t^{-1}A)^2\right) < \infty$), then the sum will telescope to give

$$\frac{d}{dt}\left(-\log\det_{(2)}(D, D_t)\right) = \operatorname{tr}\left(D_t^{-1}B\,K^2\right)$$

where $\quad K = D_t^{-1}\underbrace{(D_t - D)}_{A}$ $\quad$ so that $D = D_t - A$.

Thus we get $\quad \operatorname{tr}\left(D_t^{-1}B\,(D_t^{-1}A)^2\right)$

which is obviously a quadratic function of A, hence a quadratic function of D.

Therefore I conclude the local obstruction to constructing a $\log\det(D)$ is zero.

Let E be a vector bundle over a Riemann surface and $\alpha : S^1 \hookrightarrow X$ an embedded oriented curve. Then given g an auto. of $\alpha^*(E)$ we can form the clutched bundle E_g. So we get a family of vector bundles parameterized by g, hence a line bundle over the space of g and a canonical section of the dual line bundle in the index 0 case.

Pick a point $\infty \in X - \alpha$. Then replacing E by $E(\infty)$ leads to an exact sequence

$$0 \longrightarrow E_g \longrightarrow E_g(\infty) \longrightarrow E(\infty)/E \longrightarrow 0$$

hence $\lambda(H^*(E_g)) \otimes \lambda(E(\infty)/E) \xrightarrow{\sim} \lambda(H^*(E_g(\infty)))$, so the line bundles for the family $\{E_g\}$ and $\{E(\infty)_g\}$ are isomorphic. Now

$$0 \to H^0(X, E_g) \longrightarrow H^0(X-\alpha, E) \xrightarrow{(j_-) - g j_+} H^0(\alpha, E) \longrightarrow H^1(X, E_g) \to 0$$

and suppose we restrict to $H^1(X, E_g) = 0$. Then

$$\begin{array}{ccccc}
? & \lambda(H^*(E_g)) = \lambda H^0(E_g) & \subset & \Lambda^p H^0(X-\alpha, E) & ? \\
 & \otimes \lambda(E(\infty)/E) & & \otimes \lambda(E(\infty)/E) & \\
 & \downarrow \wr & & \downarrow \wr & \\
 & \lambda((H^0(E(\infty)_g)) & \subset & \Lambda^{p+r} H^0(X-\alpha, E(\infty)) &
\end{array}$$

$p = \dim H^0$

so we end up embedding our line bundle in an ~~inductive~~ inverse limit of wedge-spaces

$$\varprojlim_n \Lambda^{p+nr}(H^0(X-\alpha, E(n\infty))) \otimes \lambda(E(n\infty)/E)^* .$$

This looks completely different from the previous construction of wedge-spaces. Here's the explanation

Suppose one has an exact sequence

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow V_1/V_0 \longrightarrow 0$$

with V_1/V_0 finite dimensional. Then I claim we have a map

$n = \dim(V_1/V_0)$

$$\Lambda^q V_1 \otimes \lambda(V_1/V_0)^* \longrightarrow \Lambda^{q-n}(V_0)$$

which corresponds to intersecting a subspace of V_1 with V_0. More generally given a Fredholm map $V_0 \xrightarrow{T} V_1$ we have a map

$$\Lambda V_1 \otimes \lambda(T) \longrightarrow \Lambda V_0$$

raising degree by the index in T.

<u>March 12, 1982</u>

$\alpha : S^1 \hookrightarrow X$, E vector bundle over X. Then by the clutching construction we can twist E by any auto. g of $\alpha^*(E)$ to get a family of vector bundles E_g. We get the "determinant-of-cohomology" line bundle $\mathcal{L}$ over the space $\mathcal{O}$ of g. We saw that we have an embedding ($p =$ index of E_g).

$$\mathcal{L}_g \subset \varprojlim_n \Lambda^{p+rn}\left(H^0(X-\alpha, E(n\infty))\right) \otimes \lambda\left(E(\infty)/E\right)^*$$

where ∞ is a fixed point off α. This came from the exact sequence

$$0 \longrightarrow H^0(E_g) \longrightarrow H^0(X-\alpha, E) \xrightarrow{j_1 - g j_2} H^0(\alpha, E) \longrightarrow H^1(E_g) \to 0$$

In the case where α is a small circle around ∞, then we have

$$H^0(X-\alpha, E(n\infty)) = z^n H_- \oplus H_+$$

and

$$H^0(E_g^{(n\infty)}) = z^n H_- \cap g H_+ \subset H^0(\alpha, E)$$

hence we get a <u>more efficient embedding</u>

$$\mathcal{L}_g \subset \varprojlim_n \Lambda^{p+rn}(z^n H_-) \otimes \lambda(z^n H_-/H_-)^*$$

(Actually I should be more careful: $H^0(X-\alpha, E)$ means holom. sections which extend analytically across α from either side, and also g should be analytic on α).

Question: Why did we get a projective representation of the loop group?

There we had the situation of the loop group G acting on the set $\mathcal{O}$ of outgoing subspaces W. The vector bundle is constructed from H_- and W. So we have the situation

$$\begin{tikzcd} & \mathcal{L} \arrow[d] \\ G \times \Theta \arrow[r] & \Theta \end{tikzcd}$$

and for some reason there is a canonical isomorphism $g^*\mathcal{L} \simeq \mathcal{L}$ <u>up to a scalar, for each</u> $g \in G$.

The key question: Why do we get a projective repn. of the loop group? It appears in the split case ($X - \alpha$ has two components) that the cohomology-determinant-line is a line in a wedge-space representation.

March 13, 1982:

Interesting problem: Why can you get a projective action of the loop group on the line bundle, when the curve separates the Riemann surface? The reason seems to be that the line bundle embeds in a representation.

Suppose $(\cdot\infty)$ and let $V =$ sections over the punctured disk, $W_0 =$ sections over $X - \infty$, and $H_- =$ sections over disk, all for the initial bundle E_0. Then for the twisted bundle by the auto g of V, we compute cohomology by

$$0 \longrightarrow H^0(E) \longrightarrow H_- \oplus W_0 \xrightarrow{(1,-g)} V \longrightarrow H^1(E) \longrightarrow 0.$$

Assume $\dim H^0 = p$, $\dim H^1 = 0$, then $H^0(E) = H_- \cap gW_0$ embeds in H_-, so as before we get an embedding

$$\mathcal{L} \subset \varprojlim_n \Lambda^{p+nr}(z^n H_-) \otimes \lambda(z^n H_-/H_-)^*$$

Notice: Assume $H_- \oplus W_0 = V$, then

$$\Lambda^{p+nr}(z^n H_-) = \Lambda^{p+nr}(H_- \oplus (z^n H_- \cap W_0)) = \sum_g \Lambda^g(H_-) \otimes \Lambda^{p-g+nr}(z^n H_- \cap W_0)$$

so
$$\Lambda^{p+nr}(z^n H_-) \otimes \lambda(z^n H_-/H_-)^* = \sum_g \Lambda^g(H_-) \otimes \Lambda^{g-p}(z^n H_- \cap W_0)^*$$

So we see that

$$\varprojlim_n \Lambda(z^n H_-) \otimes \lambda(z^n H_-/H_-)^* = \varprojlim_n \Lambda(H_-) \otimes \Lambda(z^n H_- \cap W_0)^*$$

This looks like a wedge space. Assume that $\bigcup z^n H_- = V$, so that $\bigcup (z^n H_- \cap W_0) = W_0$. Then what we have is

$$\Lambda H_- \mathbin{\hat{\otimes}} \hat{\Lambda} W_0^*$$

a topological version of the wedge space

$$\Lambda(V; W_0) = \Lambda H_- \otimes \Lambda(W_0^*)$$

In fact what we are getting is the

space $\mathrm{Hom}(\Lambda W_0, \Lambda H_-)$, so that we have lines attached to subspaces of $H_- \oplus W_0$ which project Fredholmly on W_0.

March 14, 1982

Consider the clutching situation: an $\overset{\text{analytic}}{\text{curve}}$ $\alpha : S^1 \hookrightarrow X$, a vector bundle E over X, so that as g varies over $\mathcal{a} = \mathrm{Aut}(\alpha^* E)$, we get a family E_g of vector bundles over X. Then we get a cohomology-determinant line bundle [redacted] $\mathcal{L}$ over $\mathcal{a}$. Problem: Is $g^*\mathcal{L}$ naturally isom. to $\mathcal{L}$ up to a scalar?

More general situation: Suppose $\mathcal{a}$ parameterizes a family of vector bundles over X and $\mathcal{G}$ is a group acting on $\mathcal{a}$. [redacted] Suppose a central extension $\tilde{\mathcal{G}}$ acts on $\mathcal{L}$ over the $\mathcal{G}$ action on $\mathcal{a}$. Then $\tilde{\mathcal{G}}$ acts on $V = \Gamma(\mathcal{a}, \mathcal{L}^*)$, hence if $\mathcal{L}^*$ is generated by its sections we get a surjection

$$V_{\mathcal{a}} \longrightarrow \mathcal{L}^*$$

$V_{\mathcal{a}}$ = trivial bundle $\mathcal{a} \times V \to \mathcal{a}$

hence an injection

$$\mathcal{L} \hookrightarrow V_{\mathcal{a}}^* .$$

In other words we get an equivariant-under-$\tilde{\mathcal{G}}$ way of embedding $\mathcal{L}_A$ in V for each $A \in \mathcal{a}$. The converse is clear: Namely, if we find a proj. repn. W of $\mathcal{G}$ and an equivariant-under-the-associated-$\tilde{\mathcal{G}}$ way of embedding $\mathcal{L}_A$ in W. Then clearly $\tilde{\mathcal{G}}$ acts on $\mathcal{L}$. Moreover we have

$$\mathcal{L} \subset W_{\mathcal{a}}^* \iff W_{\mathcal{a}} \twoheadrightarrow \mathcal{L}^*$$

i.e. the sections of $\mathcal{L}^*$ coming from W span $\mathcal{L}^*$

[redacted] If $\mathcal{G}$ acts transitively on $\mathcal{a}$, then $V = \Gamma(\mathcal{a}, \mathcal{L}^*)$ is an induced representation of $\tilde{\mathcal{G}}$, and then any subrepresentation W of V will give a surjection

$W_a \twoheadrightarrow \mathcal{L}^*$ by invariance.

Summary: $\mathcal{G}$ acts projectively on $\mathcal{L}$ iff it is possible to equivariantly imbed $\mathcal{L}$ in a projective repn. W of $\mathcal{G}$. [redacted] When $\mathcal{G}$ acts transitively on $\mathcal{A}$, the possible W^* are simply repns. of $\tilde{\mathcal{G}}$ which map non-trivially to $V = \Gamma(\mathcal{A}, \mathcal{L}^*)$. (modulo duality problems)

So now let us consider clutched vector bundles. Here the cohomology is calculated by the sequence

$$0 \longrightarrow H^0(E_g) \longrightarrow W \xrightarrow{\ j_1 - g j_2\ } V \longrightarrow H^1(E_g) \longrightarrow 0$$

where $W_0 =$ sections of E over the Reimann surface $\overline{X - \alpha}$ with boundary

$V =$ sections of $\alpha^* E$ over S^1.

A better sequence for my purpose is probably

$$0 \longrightarrow H^0(E_g) \longrightarrow W_0 \oplus V \xrightarrow{(j_1, g j_2) - \Delta} V \times V \longrightarrow H^1(E_g) \longrightarrow 0$$

because this is completely analogous to the old map

$$W_0 \oplus H_- \xrightarrow{\ (g, 1)\ } V.$$

Now we have to review [redacted] why in the last situation it is possible to identify $\mathcal{L}$ with the line corresponding to W in a certain wedge space.

We have

$$0 \longrightarrow H_- \cap W \longrightarrow H_- \oplus W \longrightarrow V \longrightarrow V/H_- + W \longrightarrow 0$$

and the corresponding determinant line is $\mathcal{L} = \lambda(H_- \cap W) \otimes \lambda(V/H_+ + W)^*$. Pick an $H > H_-$ with H/H_- f.d. and $H + W = V$. Then we have an exact sequence

$$0 \longrightarrow H_- \cap W \longrightarrow H \cap W \longrightarrow H/H_- \longrightarrow V/H_- + W \longrightarrow 0$$

so $\mathcal{L} = \lambda(H_- \cap W) \otimes \lambda(V/H_- + W)^* = \lambda(H \cap W) \otimes \lambda(H/H_-)^*$

embeds in $\Lambda(H) \otimes \lambda(H/H_-)^*$. In this way the determinant line $\mathcal{L}_W$ attached to any subspace W complementary to H_- modulo finite-diml. subspaces gets embedded in the wedge space

$$\mathcal{L}_W \subset \varprojlim_H \Lambda(H) \otimes \lambda(H/H_-)^*$$

where the limit is taken over all H containing H_- with H/H_- f.d.

Now I want to look at this in the case of $V \times V$ with H_- being the diagonal, and I want the wedge space to be a representation of the group of $(1,g)$ where g runs over a group of autos. of V. In order to act projectively on the wedge space I seem to need that $(1,g)$ preserve ΔV up to commensurability, which is possible only if $g = 1 +$ finite rank. This clearly doesn't work for our clutching functions, but one might be able to do something in the topological situation.

Let's take $V = L^2(S^1)$ and consider Fock space of $V \times V$ with respect to the subspace ΔV. Then for what maps $g: V \longrightarrow V$ is Γ_g going to give rise to a line in this Fock space? If we write Γ_g as the graph of a map T from ΔV to $(\Delta V)^\perp$, then the condition is that T be Hilbert–Schmidt i.e. $\mathrm{tr}(T^*T) < \infty$.

$$\Delta V = \{ (v,v) \mid v \in V \} \qquad pr_{\Delta V}(v_1, v_2) = \tfrac{1}{2}(v_1+v_2, v_1+v_2)$$

$$(\Delta V)^\perp = \{ (v,-v) \mid v \in V \} \qquad pr_{(\Delta V)^\perp}(v_1, v_2) = \tfrac{1}{2}(v_1-v_2, v_2-v_1)$$

Hence $\quad (v, gv) = \frac{1}{2}(v+gv, v+gv) + \frac{1}{2}(v-gv, -v+gv)$

and so $\quad T: \frac{1}{2}(v+gv, v+gv) \longmapsto \frac{1}{2}(v-gv, -v+gv)$

so that $\quad T: (w,w) \longmapsto \left((1-g)(1+g)^{-1}w, -(1-g)(1+g)^{-1}w\right).$

Thus if we identify ΔV with V via $\frac{1}{\sqrt{2}}(w,w) \longleftrightarrow w$ and similarly $(\Delta V)^{\perp}$ we see that T is the operator

$$Tw = (1-g)(1+g)^{-1}w$$

If $1-g$ is Hilbert Schmidt, then $1+g = 2-(1-g) = 2+K$, K compact, is Fredholm, so that provided g has no -1 eigenvector $1+g$ is invertible and so T is Hilbert-Schmidt. So we can conclude that operators $(1,g)$ where $g = 1+H.S.$ act projectively on the Fock space $\hat{\Lambda}(V\times V; \Delta V)$.

Consider elliptic curve case: $X = \mathbb{C}/\Gamma \xrightarrow{\sim} \mathbb{C}^*/q^{\mathbb{Z}}$

$$u \longmapsto e^{2\pi i u} = t$$
$$\tau \longmapsto e^{2\pi i \tau} = q.$$

Here $V = L^2(S^1)$ and the basic intersection situation is where $W =$ analytic functions on annulus $|q| < |t| < 1$ with L^2 boundary behavior $= \{ \sum a_n t^n \mid \sum |a_n|^2 < \infty, \sum |q^n a_n|^2 < \infty \}$. Thus $W \subset V \times V$ is the closed space spanned by $(t^n, q^n t^n)$, i.e. it is the graph of the operator $t^n \longmapsto q^n t^n$.

Clearly $W \cap \Delta V = \mathbb{C}$. Also $W + \Delta V$ is of codim 1 in $V\times V$. In effect look at $W \longrightarrow V\times V/\Delta V = V$ which is essentially the map $t^n \longmapsto (1-q^n)t^n$. given $\sum c_n t^n \in V$ we try to solve

$$\sum a_n (1-q^n) t^n = \sum c_n t^n$$

which we can do with $a_n = \dfrac{c_n}{1-q^n}$ if $c_0 = 0$.

Then

$$\sum_{n\neq 0} |a_n|^2 = \sum_{n \neq 0} \frac{|c_n|^2}{|1-q^n|^2} \leq C \sum |c_n|^2$$

since $\dfrac{1}{|1-q^n|^2}$ $\begin{array}{l} \longrightarrow 1 \text{ as } n \to \infty \\ \longrightarrow 0 \text{ as } n \to -\infty \end{array}$ and hence is bounded.

Similarly $\quad \sum |q^n a_n|^2 = \sum_{n\neq 0} \left|\frac{q^n}{1-q^n}\right|^2 |c_n|^2 \leq C \sum |c_n|^2$.

The above calculation checks our feeling that even with these L^2 boundary values we can compute cohomology, and hence should have a good vector bundle.

Unfortunately neither W nor $W^\perp$ are going to give lines in the Fock space. In effect W is the graph of $g : z^n \mapsto q^n z^n$, so $T = (1-g)(1+g)^{-1}$ has the eigenvalues

$$\frac{1-q^n}{1+q^n} \quad \begin{matrix} \longrightarrow 1 & \text{as } n \to \infty \\ \longrightarrow -1 & \text{as } n \to -\infty \end{matrix}$$

and so T is not Hilbert-Schmidt. Similarly for $W^\perp$ which is the graph of $z^n \mapsto -q^n z^n$.

Problem: Suppose $\alpha : S^1 \hookrightarrow X$ is a small circle about the point ∞, E is a given vector bundle, and V, H_-, W are defined as usual. Make V into a Hilbert space. How do I know that W differs from $H_+ = (H_-)^\perp$ by a H.S. operator? It seems that I need this in order to write the solution of KdV using vacuum expectation values.

Let's assume E is a line bundle L, that ∞ is a Weierstrass point on a hyperelliptic surface, that z^2 is a function with double pole at ∞ and regular elsewhere and that $\alpha(S^1)$ is the inverse image of a circle $|z^2| = R$ around ∞ under the map $z^2 : X \longrightarrow \mathbb{P}^1$. Then $(z^2)_* L$ is a 2-dim'l vector bundle E over $\mathbb{P}^1$ and V, H_-, W are respectively L^2 sections over the circle $|z^2| = R$, those that extend to the disk containing infinity, and those extending to the other disk.

Let's start over carefully. It is necessary to understand in the small circle case precisely why the line bundle $\mathcal{L}$ can be embedded in a representation. Fix the ideas and start with a line bundle L over $\mathbb{P}^1$, and let's restrict the clutching function g to be algebraic over S^1. First of all we have a decomposition $H_- \oplus W = V$ assume L has degree -1. Then we want the g's to operate on V, so let's suppose $V =$ **all** meromorphic sections of L without poles over S^1. Then $W =$ meromorphic sections of L without poles in the disk inside S^1, $H_- =$ merom. sections regular outside S^1. Then all the subspaces gW are commensurable, so we get an orbit of lines in $\Lambda(V;W)$. So the question is why we can identify the cohomology-determinant $\mathcal{L}$ with this sub line bundle of $\Lambda(V;W)$.

Let's assume $H_- \oplus W = V$. The clutched bundle E_g has cohomology given by

$$0 \to H^0(E_g) \to H_- \oplus gW \to V \to H^1(E_g) \to 0$$

Choose a subspace W_1 of finite codim in both W and gW whence we get a subbundle E_1 of E_g belonging to H_-, W_1 and

$$\begin{array}{ccccccccc} 0 \to & H^0(E_1) & \to & H^0(E_g) & \to & \Gamma(E_g/E_1) & \to & H^1(E_1) & \to H^1(E_g) \to 0 \\ & \| & & & & \| & & \| & \\ & 0 & & & & gW/W_1 & & V/H_- + W_1 = W/W_1 & \end{array}$$

so we get

$$\mathcal{L} = \lambda(H^0(E_g)) \otimes \lambda(H^1(E_g)^*) = \lambda(gW/W_1) \otimes \lambda(W/W_1)^*$$

sitting inside of $\underline{\Lambda(V;W) = \varinjlim \Lambda(V/W_1) \otimes \lambda(W/W_1)^*}$.

Proposition: Fix a decomposition $H_- \oplus W_0 = V$. Over the set of subspaces W commensurable with W_0 is the line bundle assigning to W the line in $\Lambda(V; W_0)$. This line bundle is isomorphic to the one assigning to W the line $\lambda(H_- \cap W) \otimes \lambda(V/H_- + W)^*$.

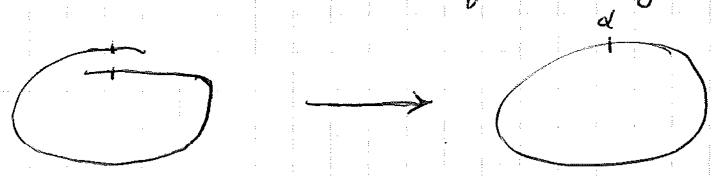
The proof is as above. The point maybe is that the latter line bundle is essentially independent of H_-.

March 15, 1982:

Take the clutching fn. construction in the case of line bundles. In this case g is a function given over the curve, and hence is independent of the initial line bundle. Thus we have a map $L \mapsto L_g$. I claim

(1) $$L_g = \mathcal{O}_g \otimes L$$

To see this recall that we [redacted] define L_g by lifting



L to Y and then using g to identify L over the two copies of the strip around the curve α. The obvious map

$$\mathcal{O} \otimes L \xrightarrow{\sim} L \qquad f \otimes s \longmapsto fs$$

over Y commutes with the identification:

$$\begin{tikzcd} f \otimes s \arrow[r, mapsto] \arrow[d, mapsto] & fs \arrow[d, mapsto] \\ gf \otimes s \arrow[r, mapsto] & gfs \end{tikzcd}$$

and hence we get by descent an isomorphism $\mathcal{O}_g \otimes L \to L_g$.

From (1) we see that to understand the clutching construction on line bundles, all we have to do is to understand the map into the Jacobian: $(\alpha, g) \mapsto \mathcal{O}_{(\alpha, g)}$. For fixed α this is a homomorphism, i.e. we have

(2) $$\mathcal{O}_{g_1} \otimes \mathcal{O}_{g_2} \simeq \mathcal{O}_{g_1 g_2} .$$

Obvious Questions: (i) Relate $\deg \mathcal{O}_g$ to $\deg(g)$.

(ii) What is the tangent map at $g = id$? This will be a linear map from $\Gamma(\alpha, \mathcal{O}) \longrightarrow H^1(X, \mathcal{O})$.

(i) is a question of topology and can be done as follows. Map X to $S^1 \times [-1,1] / S^1 \times \{-1,1\}$ by sending the outside of the strip to a point. One then has maps

$$X \longrightarrow S^1 \times [-1,1] / S^1 \times \{-1,1\} \longleftarrow S^2$$

which induce isos. on H^2, and a compatible line bundles over each of these space. This reduces one to the case of the clutching construction over S^2 where one knows that $\deg \mathcal{O}_g = \pm \deg g$, the sign depending on one's conventions.

(ii) should be a special case of a map

$$\Gamma(\alpha, \mathrm{End}(E)) \longrightarrow H^1(X, \mathrm{End}(E))$$

which is the deformation version of the clutching construction, i.e. I take an infinitesimal clutching function and ask for the resulting deformation. The way this map results probably via the covering situation

$$N \rightrightarrows Y \longrightarrow X$$

where N is the tubular neighborhood of α. This gives

$$0 \to \Gamma(X,E) \to \Gamma(Y,E) \rightrightarrows \Gamma(N,E) \longrightarrow H^1(X,E) \to H^1(Y,E)$$

and $H^1(Y,E)$ should be zero because Y is an open Riemann surface.

This should be interesting already in the case where α is a small circle around a point ∞, $Y = (X-\infty) \cup U$ where U is a disk and $N = U - \infty$. Then we get

$$0 \to \Gamma(X,E) \to \Gamma(X-\infty, E) \longrightarrow \Gamma(U-\infty, E) / \Gamma(U,E) \longrightarrow H^1(X,E) \to 0$$

In fact we even know this sequence holds in the algebraic situation, i.e. that

$$0 \longrightarrow H^0(X, E) \longrightarrow H^0_{alg}(X-\infty, E) \longrightarrow E\otimes F / E\otimes \mathcal{O}_\infty \longrightarrow H^1(X, E) \longrightarrow 0$$

Here are some general conclusions which can be drawn about the clutching function process. The first is that the clutching-process is a map

$$\Gamma(\alpha, \underline{Aut}(E)) \longrightarrow H^1(X, \underline{Aut}(E))$$

which is <u>onto</u> in the holomorphic context because holomorphic vector bundles are trivial over open Riemann surfaces. Thus any vector bundle over X can be obtained from E by a suitable clutching function over the fixed curve α.

March 15, 1982 (continued). (compare Feb 22)

Let's review the explicit formulas of the Japanese. Suppose we have two closed subspaces H, W of the Hilbert space V such that $V = H + W$, & $H \cap W$ is 1-dim'l. I suppose that $H^\perp$ is L^2-close enough to W, so that W determines a line in the Fock space of V centered at $H^\perp$. Let $|0\rangle$ correspond to H^-. Now let's consider

$$\lambda \longmapsto \langle 0 | a_\lambda u_W \rangle$$

where $\lambda \in V^*$ and u_W spans the line belonging to W. Here a_λ is interior product. I claim that this linear functional on V^* is represented by a non-zero element of $H \cap W$. To see this, note that you get 0 if $\lambda(W) = 0$, and because $\langle 0|$ is the wedge of all linear fnls. vanishing on H, you get $\langle 0 | a_\lambda = 0$ if $\lambda(H) = 0$. But $(H \cap W)^\perp = H^\perp + W^\perp$, so that the linear fnl. vanishes if $\lambda(H \cap W) = 0$, and hence is given by an elt. of $H \cap W$.

Better: If $V = H + W$, $H \cap W$ 1-dim'l, then let v span $H \cap W$, and put $W = \langle v \rangle + W_1$ so that $V = H \oplus W_1$. Then $u_W = a_v^* u_{W_1}$, so that

$$\langle 0 | a_\lambda | u_W \rangle = \langle 0 | a_\lambda a_v^* | u_{W_1} \rangle$$

$$= \langle \lambda | v \rangle \langle 0 | u_{W_1} \rangle - \langle 0 | a_v^* a_\lambda | u_{W_1} \rangle$$

and the last term is zero, because $\langle 0|$ is the wedge of all (ind.) linear fnl. vanishing on H and $v \in H$, so $\langle 0 | a_v^* = 0$.

Better: Work in $V = L^2(S^1)$ with $H = zH_- = \text{span } 1, z^{-1}, z^{-2}, \dots$ and W L^2-commensurable with $H_+ = \text{span of } 1, z, z^2, \dots$.

Then let $v = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots \in (1+H_-)\cap W$. Then

$$a_v|0\rangle = a_v(z^0\wedge z^1\wedge\cdots)$$
$$= z^1\wedge z^2\wedge\cdots = a_0|0\rangle$$

represents the subspace $H^\perp$. Then

$$\langle 0|a_0^* a_\lambda|u_W\rangle = \langle 0|a_v^* a_\lambda|u_W\rangle$$
$$= -\langle 0|a_\lambda \underbrace{a_v^*|u_W\rangle}_{0 \text{ as } v\in W} + \langle\lambda|v\rangle\langle 0|u_W\rangle$$

and so you get the formula

$$\langle\lambda|v\rangle = \frac{\langle 0|a_0^* a_\lambda|u_W\rangle}{\langle 0|u_W\rangle}$$

for the B.A. function v. In particular, taking $\lambda = \delta_z$ gives

$$v(z) = \frac{\langle 0|a_0^* \psi_z|u_W\rangle}{\langle 0|u_W\rangle}.$$

The next point is to write ψ_z out as the vertex operator.

March 16, 1982

Yesterday we noticed, as a consequence of the fact that holomorphic vector bundles over an open Riemann surface are trivial, that for any n-diml v.b. E_0 and curve α the clutching construction map

$$\Gamma(\alpha, \underline{\mathrm{Aut}}(E_0)) \longrightarrow H^1(X, \underline{\mathrm{Aut}}(E_0))$$
$$= \text{set of vector bundles of rank } n$$

is surjective. More specifically take $E_0 = \mathcal{O}^n$, whence $\underline{\mathrm{Aut}}(E_0) = GL_n(\mathcal{O})$, and fix a single point ∞ on X. Then given another rank n bundle E we can trivialize it over $X - \{\infty\}$ and over a disk neighborhood U of ∞. The two trivializations produce a holom. map $g : U - \{\infty\} \longrightarrow GL_n(\mathbb{C})$ and E = result of clutching the trivial bundle E_0 using g.

Next look at this algebraically. We want to restrict the clutching function to be meromorphic at ∞. What this means is that we associate to g the lattice, $g(\mathcal{O}_\infty^n)$, at ∞ it gives rise to, and then E_g is the bundle given by $E_0 = \mathcal{O}^n$ on $X - \infty$ and the lattice $g(\mathcal{O}_\infty^n)$ at ∞. So I get a map

$$GL_n(F)/GL_n(\mathcal{O}_\infty) \longrightarrow H^1(X, \underline{GL}_n)$$

and now I can ask what its image is. It's not onto for $n = 1$. What we get is the subgroup of $\mathrm{Pic}(X)$ generated by $\mathcal{O}(\infty)$. In general if I were to take E_0 = a given line bundle L, then clutching at ∞ by meromorphic functions gives only the line bundles $L(n\infty)$, $n \in \mathbb{Z}$.

On the other hand from the viewpoint of deformations the map

$$\mathfrak{gl}_n(F)/\mathfrak{gl}_n(\mathcal{O}_\infty) \longrightarrow H^1(X, \mathfrak{gl}_n(\mathcal{O}))$$

is onto, in fact for any quasi-coherent sheaf $\mathcal{F}$ we have

$$0 \to \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X-\infty, \mathcal{F}) \longrightarrow \mathcal{F}_\infty \otimes F / \mathcal{F}_\infty \longrightarrow H^1(X, \mathcal{F}) \to 0.$$

Now recall the theorem that over a Dedekind domain A a v.b. is of the form $A^{n-1} \oplus L$ with L invertible. This means that two vector bundles E_0, E over X of the same rank, and with $\lambda(E_0) \equiv \lambda(E)$ modulo the subgp of $\mathrm{Pic}(X)$ generated by $\mathcal{O}(\infty)$, become isomorphic over $X - \{\infty\}$ as algebraic v.b. [redacted] Choosing such an isomorphism, we can identify E with E_0 [redacted] equipped with a different lattice at the point ∞. [redacted] This shows that modulo the problem with the determinant line bundle, the clutching function map at a point is surjective.

Basic difference:

Atiyah-Bott: $\mathcal{a} \longrightarrow H^1(X, \underline{GL_n})$ has image all vector bundles of a given degree.

Clutching (algebraic): $GL_n(\hat{F}_\infty) \longrightarrow H^1(X, \underline{GL_n})$ has image all vector bundles whose determinant is [redacted] trivial in [redacted] $\mathrm{Pic}(X-\infty)$.

March 16, 1982 (continued)

Time to understand symplectic structure on a family of vector bundles in the various contexts.

Let's review the formalism in the case of loop groups. Let's start with a connected Lie group G and a principal G-bundle over S^1 which I will suppose is trivial

$$S^1 \times G \longrightarrow G$$

The gauge group $\mathcal{G}$ is the space of maps $S^1 \to G$. It operates on the space $\mathcal{A}$ of connections. If we choose a coordinate t on S^1, then connections are operators

$$D = \frac{d}{dt} - A \qquad\qquad A : S^1 \longrightarrow \mathrm{Lie}(G) = \mathfrak{g}$$

and then

$$gDg^{-1} = g\left(\frac{d}{dt} - A\right)g^{-1} = \frac{d}{dt} + g\left(-g^{-1}\frac{dg}{dt}g^{-1}\right) - gAg^{-1}$$
$$= \frac{d}{dt} - \left[gAg^{-1} + \frac{dg}{dt}g^{-1}\right].$$

Hence if we identify the affine space $\mathcal{A}$ with the space of $\mathfrak{g}$-valued 1-forms $A\,dt$, then the action is given by

$$g * A = gAg^{-1} + \frac{dg}{dt}g^{-1}$$

and the infinitesimal action by

$$X * A = [X, A] + \frac{dX}{dt}.$$

~~In the Riemann surface case the same sort of formulas can be expected to hold because~~

Next we want to show that the orbits of $\mathcal{G}$ on $\mathcal{A}$ have symplectic structure. Assume there is an invariant inner product on $\mathfrak{g}$, denote it (X,Y). Given a point $D = \frac{d}{dt} - A$ consider two tangent vectors to the orbit thru D defined by maps $X, Y : S^1 \longrightarrow \mathfrak{g}$. Thus the tangent vectors are the

1-forms $\quad (X * A)^{dt} = \left([X, A] + \frac{dX}{dt}\right) dt \quad$ and similarly for Y.

Define

$$\langle X, Y \rangle_A = \int (X * A, Y)\, dt$$

$$= \int \left\{ ([X, A], Y) + \left(\frac{dX}{dt}, Y\right) \right\} dt$$

It's clear this is skew-symmetric in X, Y, that it vanishes if $X * A = 0$, so that it defines a skew-form on the tangent space to the orbit, and finally that if $\langle X, Y \rangle_A = 0$ for all Y, then $X * A = 0$ so that this skew-form is non-degenerate.

Consider the function on $\mathfrak{a}$ given by

$$f_X(A) = \int (X, A)\, dt$$

Then applying the vector field Y to this function gives

$$(Y f_X)(A) = \int (X, Y * A)\, dt = \langle Y, X \rangle_A$$

In other words we have for the 2 form Ω defined on any orbit of $\mathcal{G}$ in $\mathfrak{a}$ defined above

$$Y f_X = \Omega(\cancel{X}, X) = i(Y) i(X) \Omega$$

whence $\quad df_X = i(X)\Omega$. $\quad$ Hence

$$i(X)\, d\Omega = \underbrace{\theta(X)\Omega}_{0 \text{ by invariance of } \Omega} - d \underbrace{i(X)\Omega}_{df_X} = 0$$

for all X, whence $d\Omega = 0$. So we see the orbits of $\mathcal{G}$ on $\mathfrak{a}$ have symplectic structure.

But now when we have a symplectic manifold on which $\mathcal{G}$ operates, we can look for the moment map to $\mathrm{Lie}(\mathcal{G})^* \cong \mathrm{Lie}(\mathcal{G})$

March 17, 1982

Yesterday I went over the symplectic structure on the loop group ΩG, where G is a connected Lie group whose Lie algebra has an invariant non-degenerate quadratic form, e.g. $\mathfrak{gl}_n$ with the form $X, Y \longmapsto \mathrm{tr}(XY)$. I want to check that this all works when I consider S^1 to be replaced by an annulus, e.g. the punctured disk and I take analytic maps from the annulus to G.

Let the annulus be $S = \{ z \in \mathbb{C} \mid r_1 < |z| < r_2 \}$, where $r_1 < 1 < r_2$. The interesting case is where $r_1 = 0$, so that S is a punctured disk, and I would like to keep track of things so as not to choose a specific coordinate z. Start with an analytic principal G-bundle over S. Since S is an open Riemann surface, there should be no loss in assuming the bundle is trivial. The gauge group G then consists of holom. maps $S \longrightarrow G$, and it acts on the set of connections $D = d - \theta$ in the usual way

$$gDg^{-1} = d - (g\theta g^{-1} + dg \cdot g^{-1})$$

So now introduce the skew-form on the gauge algebra (consisting of analytic maps $S \rightarrow \mathfrak{g}$) by

$$\langle X, Y \rangle_\theta = \int_{S^1} (X * \theta, Y) = \int_{S^1} ([X,\theta], Y) + (dX, Y)$$

where, of course, S^1 is any circle in the annulus. (Better notation: $\oint$). This obviously is a skew-form defined on the tangent space to the G-orbit thru θ.

In particular taking $\theta = 0$, the orbit is $\mathcal{G}/G$ which we can identify with analytic maps $S \to G$ sending a fixed basepoint 1 to the identity elt. of G.

An interesting point is that the symplectic structure is still defined, where $\mathcal{G}$, $\mathcal{a}$ are replaced by germs of analytic functions around the point.

Next classify the different $\mathcal{G}$ orbits in $\mathcal{a}$. Given a connection, parallel translation defines a **monodromy** transformation, whose conjugacy class in $\mathcal{G}$ is well-defined. So, for example, we take the connection $D = d - \theta$ put $\theta = A\,dz$; then parallel translation is obtained by solving $dY = \theta Y$ or

$$\frac{dY}{dz} = A \cdot Y \qquad Y : S \to G$$

Y is multi-valued. The solutions are unique up to right multiplication by a constant matrix. Thus parallel translation around the center point of a solution Y gives new Y = old $Y \cdot C$. On the other hand taking another $Y' = Y \cdot B$ we get new Y' = old $Y' \cdot C'$

$$(\text{new } Y) \cdot B = (\text{old } Y \cdot C) B = (\text{old } Y') B^{-1} C B$$

hence $C' = B^{-1} C B$.

~~Next suppose we have two connections with the~~

Actually one should fix a basepoint in S and one above in the principal bundle. This way the monodromy becomes a map $\pi_1(S, *) \to G$. Given a connection you consider parallel translation

$$\frac{dY}{dz} = AY$$

starting with $Y = I$ at $*$. Then monodromy is value of Y

If 2 connections have the same monodromy, say $\frac{dY}{dz} = AY$, $\frac{dZ}{dz} = BZ$, then ZY^{-1} is a single-valued function, hence is a map $g : S \to G$, so is in the gauge gp. Thus $gY = Z$, so

$$B = \frac{dZ}{dz} Z^{-1} = \frac{d(gY)}{dz}(gY)^{-1} = \frac{dg}{dz} g^{-1} + g^{-1}Ag$$

are in the same orbit under $\mathcal{G}$ with $g(*) = 1$. If we allow the value of g at $*$ to be arbitrary we get the full conjugacy class of the monodromy as the unique invariant of the orbit.

<u>Summary</u>: The $\mathcal{G}$ orbits on $\mathcal{A}$ are in 1-1 correspondence with conjugacy classes in G, the correspondence being given by the monodromy.

Next point is that one can make A constant by a gauge transformation iff the monodromy is in the image of the exponential map $\mathfrak{g} \to G$. The stabilizer of a connection in the gauge gp. is essentially the centralizer of the monodromy.

Situation: I have produced on various orbits of the gauge group symplectic structures. The gauge group $\mathcal{G}$ is now going to be interpreted as the set of clutching functions, and the problem will be whether the symplectic structure has anything to do with vector bundles.

Finite-dimensional-case: Consider the action of G on $\mathfrak{g}^* \cong \mathfrak{g}$ when there is an invariant form. Then we get a symplectic structure on each orbit

Let's digress and try to understand the Kyoto τ function connected with the Riemann-Hilbert problem. The RH problem here means taking a fn. $g: S^1 \longrightarrow GL_n$ and factoring it into analytic pieces for the inside + outside of S^1 on $\mathbb{P}^1$. Also in the Jap case g is constant except at a finite set of points where it jumps and then one has to specify carefully what the factorization should look like.

If g is of degree 0 and gives rise to the bundle $\mathcal{O}^n$ on $\mathbb{P}^1$, then the factorization exists and is essentially unique, and there is a VEV formula for it whose denominator is $\langle 0|\tilde{g}|0\rangle$, where $\tilde{g}$ is a lift of g to a wedge-space operator.

This quantity $\langle 0|\tilde{g}|0\rangle$ is the τ fn. It isn't well-defined unless one specifies what $\tilde{g}$ is, and in the singular case only its logarithmic variation:

$$\delta \log \langle 0|\tilde{g}|0\rangle = \frac{\langle 0|\widetilde{\delta g . g^{-1}}\, \tilde{g}|0\rangle}{\langle 0|\tilde{g}|0\rangle}$$

is defined. Here $\delta g . g^{-1}$ is an element of the Lie algebra of $\mathcal{G}$, i.e. a map $S^1 \longrightarrow \mathfrak{gl}_n$.

In practice what happens I guess is that one is given a family of g's

March 18, 1982

We have to get the basic structure of the orbits of the Bott-Atiyah situation straight. Fix a C^∞ bundle with hermitian structure E over the Riemann surface M. Then we have

$$\mathcal{A} = \text{space of unitary connections on } E$$
$$\mathcal{G} = \text{group of autos. of the hermitian bundle } E$$
$$\mathcal{G}^c = \text{group of autos. of the } C^\infty \text{ bundle } E.$$

There is an obvious action of $\mathcal{G}$ on $\mathcal{A}$. Because $\mathcal{A}$ can be identified with the space of holomorphic structures on E we get an action of $\mathcal{G}^c$ on $\mathcal{A}$.

The analogy is the following: $\mathcal{A}$ is a complex manifold on which the complex gp. $\mathcal{G}^c$ acts, and $\mathcal{G}$ is the maximal compact subgroup of $\mathcal{G}$. For example we can consider $T = (S^1)^n \subset U_n$ acting on $\mathbb{P}(\mathbb{C}^n)$ and the extension to the complex torus T^c. Given a line L generated by a unit vector $v = (z_1, \dots, z_n)$, the T^c orbit of L is described by the subset of $z_j \neq 0$, hence you get one orbit for each non-empty subset of $\{1, \dots, n\}$. The T orbit is described by the point of the simplex $\sum_{i=1}^n |z_i|^2 = 1$ with these vertices.

Let's take the trivial line bundle over M. Then $\mathcal{A}$ can be identified with the space of 1-forms θ on X which are purely imaginary: $\theta = \theta' + \theta''$ where $\theta'' = -\overline{\theta'}$, or equivalently with the space of forms θ'' of type $(0,1)$. The gp. $\mathcal{G}^c = \text{Maps}(M, \mathbb{C}^*)$ acts by

$$g D'' g^{-1} = g(d'' - \theta'')g^{-1} = d'' - (\theta'' + d''g \cdot g^{-1})$$

so the $\mathcal{G}^c$ orbits on $\mathcal{A}$ will be described by the cokernel of

$$\mathcal{G}^c \xrightarrow{d'' \log} \Gamma\Omega^{0,1}.$$

Exponential sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "2\pi i"] \arrow[d] & \Gamma\Omega^{0,0} \arrow[r, "\exp"] \arrow[d, equal] & \mathcal{G}^c \arrow[r] \arrow[d, "d'' \log"] & H^1(M,\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \Gamma\Omega^{0,0} \arrow[r, "d''"] & \Gamma\Omega^{0,1} \arrow[r] & H^1(M,\underline{\mathcal{O}}) \arrow[r] & 0
\end{tikzcd}$$

This shows that $\quad \operatorname{Coker}\{\mathcal{G}^c \to \Gamma\Omega^{0,1}\} = H^1(M,\mathcal{O}) / H^1(M, 2\pi i\mathbb{Z})$

is the Jacobian variety, i.e. the different isomorphisms classes of line bundles of degree 0.

Next we want to compute the gauge gp. orbits. $\mathcal{G} = \mathrm{Maps}(M, S^1)$. They are the cokernel of

$$\mathcal{G} \xrightarrow{d\log} \text{purely imaginary 1-forms}$$

$$\mathcal{G} \xrightarrow{d\log} i\Gamma\Omega^1_r \qquad r = \text{real-valued}$$

But we have the exponential sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "2\pi i"] & i\Omega^0_r \arrow[r, "\exp"] \arrow[d, equal] & \mathcal{G} \arrow[r] \arrow[d, "d\log"] & H^1(M,\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \Gamma i\Omega^0_r \arrow[r, "d"] & \Gamma i\Omega^1_r \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Now we have

$$0 \to H^1(M,\mathbb{R}) \longrightarrow C \longrightarrow \Gamma\Omega^2_r \longrightarrow H^2(M,\mathbb{R}) \longrightarrow 0$$

so that we conclude the set of gauge orbits is an extension.

$$0 \to \underbrace{H^1(M, i\mathbb{R}) / H^1(M, 2\pi i\mathbb{Z})}_{H^1(M,S^1)} \longrightarrow \operatorname{Coker}\{\mathcal{G} \xrightarrow{d\log} i\Omega^1_r\} \xrightarrow{d} \left\{\begin{matrix}\text{exact forms} \\ \text{in } i\Omega^2_r\end{matrix}\right\} \to 0$$

In other words

$$\boxed{0 \longrightarrow \underset{\substack{\text{flat line} \\ \text{bundles (unitary)}}}{H^1(M,S^1)} \longrightarrow \mathcal{G}\backslash\mathcal{A} \xrightarrow[\text{curvature}]{F} \left\{\begin{matrix}\text{exact purely imag.} \\ \text{2-forms}\end{matrix}\right\} \to 0}$$

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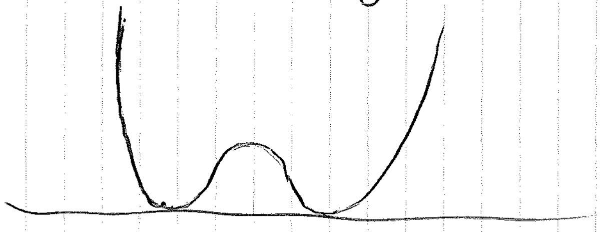
Instantons: Back to Coleman.

Consider motion in a potential on the line:

$$H = \frac{p^2}{2m} + U(x) \qquad \text{where} \quad p = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

and $U(x)$ is something like



Quantities of interest are the ground energy $E_0(\hbar)$ and the projection operator P_0 on the ground state $\langle x|0\rangle\langle 0|x'\rangle$, as functions of $\hbar$. We use formulas

$$E_0 = \lim_{\beta\to\infty} -\frac{1}{\beta}\log\left(\operatorname{tr} e^{-\beta H}\right)$$

$$P_0 = \lim_{\beta\to\infty} \frac{e^{-\beta H}}{\operatorname{tr}(e^{-\beta H})}$$

and we calculate these using path integrals.

$$\langle x|e^{-\beta H}|x'\rangle = \text{integral over paths } x:[0,\beta]\to\mathbb{R} \text{ with } x(0)=x',\ x(1)=x.$$

Need

$$\langle x|e^{-\Delta\beta H}|x'\rangle = \underbrace{\langle x|e^{-\Delta\beta\frac{p^2}{2m}}|x'\rangle}\, e^{-\Delta\beta U(x')}$$

$$\int \frac{dp}{2\pi\hbar}\, e^{i(p/\hbar)\Delta x - \Delta\beta\frac{p^2}{2m}} = \int \frac{d\xi}{2\pi}\, e^{i\xi\Delta x - \Delta\beta\frac{\hbar^2\xi^2}{2m}}$$

$$= e^{-\frac{m}{\hbar^2\Delta\beta}\frac{(\Delta x)^2}{2}}\ \frac{1}{\sqrt{2\pi\hbar^2\Delta\beta/m}}$$

Thus

$$\langle x|e^{-\beta H}|x'\rangle = \int e^{-\frac{1}{\hbar^2}\int_0^\beta \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 d\tau - \int_0^\beta U(x)\,d\tau}\,[x].$$

But now put $\tau = \beta t$ so that $t\in[0,1]$, and we end up with the formula

$$\langle x | e^{-\beta H} | x' \rangle = \int_{\substack{x(0)=x' \\ x(1)=x}} e^{-\frac{1}{\beta\hbar^2} \underbrace{\int_0^1 \frac{m}{2}\left(\frac{dx}{dt}\right)^2 dt}_{K.E.} - \beta \underbrace{\int_0^1 U(x)\,dt}_{P.E.}}$$

and similar the partition fn. $tr(e^{-\beta H})$ is expressed as an integral over closed paths.

So what we want to look at is an integral of the form:

$$\circledast \qquad \int e^{-\frac{1}{\beta\hbar^2}(K.E.) - \beta(P.E.)}$$

We are interested in the situation where first $\beta \to \infty$. Compare this with the classical limit, which is to understand

$$e^{-\frac{i}{\hbar}Ht} \quad \text{as } \hbar \to 0. \qquad \beta = \frac{i}{\hbar}t$$

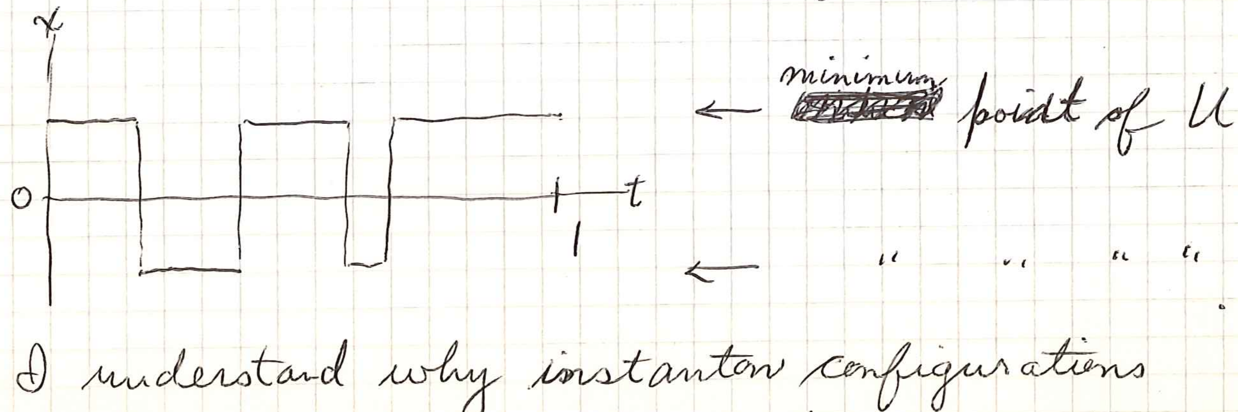
Then

$$\langle x | e^{-\frac{i}{\hbar}Ht} | x' \rangle = \int e^{\frac{i}{\hbar}\underbrace{\left[\frac{1}{t}(K.E.) - t(P.E.)\right]}_{S}}.$$

t occurs because paths are over $[0,1]$

Here t is fixed, and $\hbar \to 0$, so that what is important is the critical points of the action.

So now look at $\circledast$ as $\beta \to \infty$. Then the potential energy dominates, hence the important configurations in the limit are paths which stay (locally constant) at the minimum values of U, i.e. the instanton configurations.



So now I understand why instanton configurations occur, but I don't yet understand how to compute their contribution