January 1, 1981

Let's recall the oscillator Hamiltonian

$$(1) \qquad H = \omega_0 b^* b + \sum_k \left(\omega_k a_k^* a_k + b^* \bar{\gamma}_k a_k + a_k^* \gamma_k b \right)$$

encountered in laser physics (Glauber). It turns out that this Hamiltonian occurs also in the Lee model.

Following Barton (Intro to Adv. Field Theory) instead of the more complicated treatment in Schweber one considers the V, N particles to exist only in a single mode and that they are pinned down at the origin. What this means is that V, N states are described by the exterior algebra on a 2 dim'l vector space with annihilator operators ψ_V, ψ_N. The basic interactions

$$V \rightleftarrows N + \Theta$$

are described by terms $\qquad \psi_V^* \psi_N a_k + a_k^* \psi_N^* \psi_V$.

The operators $\psi_V^* \psi_N$ and $\psi_N^* \psi_V$ flip between the states $\psi_N^* |0\rangle$, $\psi_V^* |0\rangle$, and are zero on the other states. Thus we have essentially a 2-level atom coupled to a family of simple harmonic oscillators. All the interesting analysis occurs on the "sectors" with 1 V-particle or 1 NΘ-particle, and this is described by the 1-particle states of the oscillator (1). ω_0 is the "mass" of the V-particle.

Let's first review the behavior of the 1-particle states of (1). We have

$$e^{-iHt}\theta(t) = \int_{-\infty}^{\infty} \frac{dW}{2\pi} \, i \, \frac{e^{-iWt}}{W - H + i0^+}$$

and
$$\frac{1}{W-H} \doteq \frac{1}{W-H_0} + \frac{1}{W-H_0} H_{int} \frac{1}{W-H_0} + \cdots$$

Thus if $\Phi = b^*|0\rangle$ we have

$$\frac{1}{W-\mathbb{H}}\Phi = \frac{1}{W-\omega_0}\Phi + \frac{1}{W-H_0}|\gamma\rangle\frac{1}{W-\omega_0} + \frac{1}{W-\omega_0}\langle\gamma|\frac{1}{W-H_0}|\gamma\rangle\frac{1}{W-\omega_0} + \cdot$$

$$= \frac{1}{W-\omega_0-g(W)}\Phi + \frac{1}{W-H_0}|\gamma\rangle\frac{1}{W-\omega_0-g(W)}$$

where

$$\mathbb{H} = \left(\begin{array}{c|c} \omega_0 & \langle\gamma| \\ \hline |\gamma\rangle & H_0 \end{array}\right) \qquad H_0 = \sum \omega_k a_k^* a_k$$

and

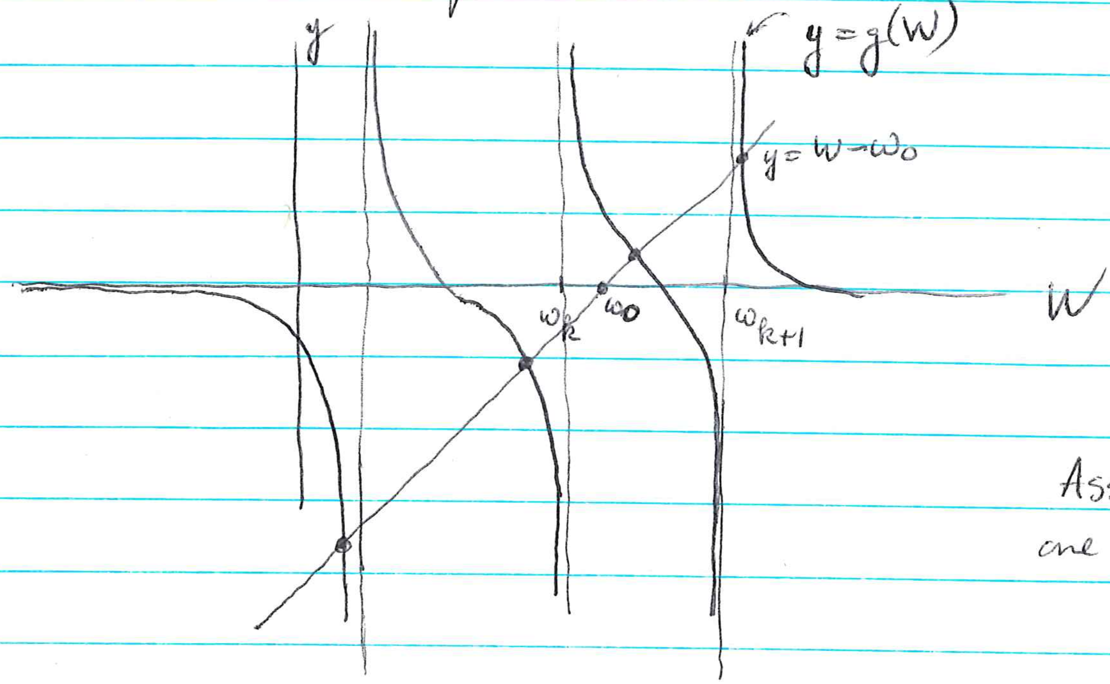
$$g(W) = \langle\gamma|\frac{1}{W-H_0}|\gamma\rangle = \sum_k{}' \frac{|\gamma_k|^2}{W-\omega_k}$$

Check:

$$(W-\mathbb{H})\left[\Phi + \frac{1}{W-H_0}|\gamma\rangle\right] = (W-\omega_0)\Phi - |\gamma\rangle$$
$$+ (W-H_0)\frac{1}{W-H_0}|\gamma\rangle - \langle\gamma|\frac{1}{W-H_0}|\gamma\rangle\Phi$$
$$= \left[W-\omega_0-g(W)\right]\Phi$$

$$\therefore \quad \frac{1}{W-\mathbb{H}}\Phi = \frac{1}{W-\omega_0-g(W)}\left(\Phi + \frac{1}{W-H_0}|\gamma\rangle\right)$$

Let's consider the case where H_0 has discrete spectrum. One can plot



Assume ω_0 not one of the ω_k.

Where the line $y = W - \omega_0$ intersects the graph of g we get the solutions of

$$W - \omega_0 - g(W) = 0.$$

These are the eigenvalues of $\mathbb{H}$; call them $\tilde{\omega}_k$. The corresponding eigenfunctions are clearly

$$\left(\Phi + \frac{1}{\tilde{\omega}_k - H_0} |\gamma\rangle \right)$$

because this is what one gets from $\frac{1}{2\pi i} \oint dW \frac{1}{W - H} \Phi$.
In more detail, the contour integral around $\tilde{\omega}_k$ gives the projection onto the $\tilde{\omega}_k$ eigenspace

$$|\Psi_k\rangle\langle\Psi_k|\Phi\rangle = \frac{1}{2\pi i} \oint \frac{dW}{W - \mathbb{H}} \Phi$$

$$= \frac{1}{1 - g'(\tilde{\omega}_k)} \left(\Phi + \frac{1}{\tilde{\omega}_k - H_0} |\gamma\rangle \right)$$

hence
$$|\langle \Phi | \Psi_k \rangle|^2 = \frac{1}{1 - g'(\tilde{\omega}_k)}$$

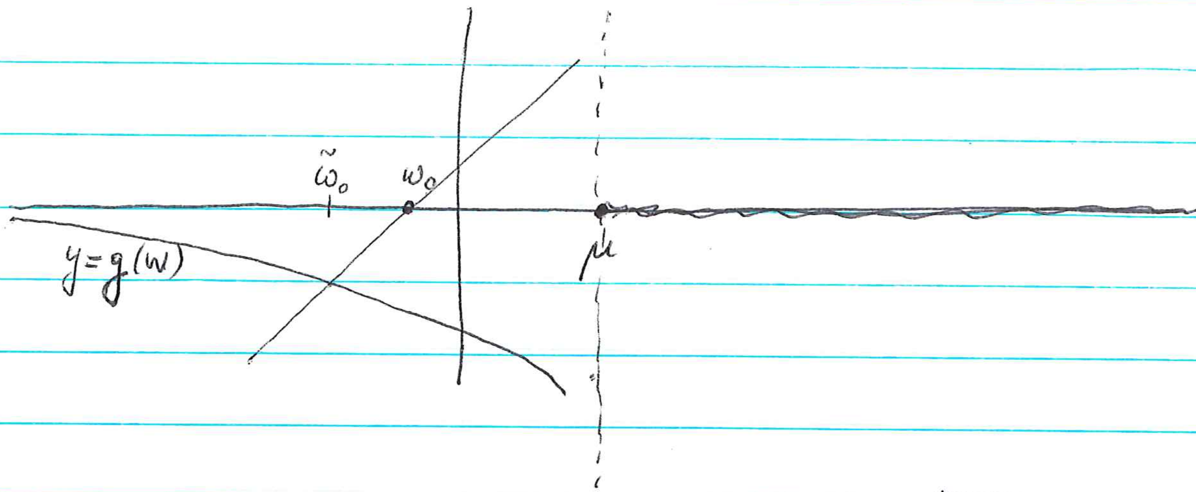
and so
$$|\Psi_k\rangle = \frac{1}{\sqrt{1 - g'(\tilde{\omega}_k)}} \left(\Phi + \frac{1}{\tilde{\omega}_k - H_0} |\gamma\rangle \right)$$

In the Lee model one assumes that the spectrum of H_0 is $[\mu, \infty)$, because the a_k correspond to bosons of mass μ and momentum k, so that $\omega_k = \sqrt{k^2 + \mu^2}$. Also the V particle is assumed to be stable, which means that its mass is $< \mu$.

In this situation

$$g(W) = \int_{\mu}^{\infty} \frac{\rho(\omega)\,d\omega}{W-\omega} \qquad \rho > 0$$

and we are interested in the case $W < \mu$, where g is real and < 0.



Put $\tilde{\omega}_0 = \omega_0 + \delta\omega$.

$\tilde{\omega}_0$: actual V-mass; ω_0: bare V-mass; $\delta\omega$: level shift

$$\boxed{\delta\omega = \tilde{\omega}_0 - \omega_0 = g(\tilde{\omega}_0)}$$

Only $\tilde{\omega}_0$ has a physical meaning. Instead of starting ω_0 and solving for $\tilde{\omega}_0$, one might start

January 3, 1981

Lee model: If one restrict to the sector with 1 V-particle or 1 $N\theta$-particle, then we have a situation described by

$$i\frac{\partial\psi}{\partial t} = \mathbb{H}\psi \qquad \mathbb{H} = \left(\begin{array}{c|c} \omega_0 & \langle\gamma| \\ \hline |\gamma\rangle & H_0 \end{array}\right)$$

The Hilbert space is the direct sum: [redacted]

$$\mathcal{H} = \mathbb{C}\,\Phi \oplus \mathcal{H}_0$$

where $\mathcal{H}_0$ and H_0 describe the θ states. In the Lee model the θ-particle is described by a scalar field

$$\phi(x,t) = [redacted] \sum_k \frac{1}{\sqrt{2\omega_{[redacted]} V}}\left(a_k e^{+i(kx-\omega t)} + a_k^* e^{-i(kx-\omega t)}\right).$$

Here $k \in \left(\frac{2\pi}{L}\mathbb{Z}\right)^3$, $\omega = \sqrt{k^2+\mu^2}$, $V = L^3$. The actual interaction is

$$\sum_k \gamma_k (b^* a_k + a_k^* b) = -g_0 \sum_k \frac{u(\omega)}{\sqrt{2\omega V}}(b^* a_k + a_k^* b)$$

where u is a cut-off function. Here $b^* a_k$ is to be interpreted as $|\Phi\rangle\langle k|$.

Recall that

$$\frac{1}{W-H}\Phi = \frac{1}{W-\omega_0 - g(W)}\left(\Phi + \frac{1}{W-H_0}|\gamma\rangle\right)$$

where

$$g(W) = \langle\gamma|\frac{1}{W-H_0}|\gamma\rangle$$

$$= \sum_k g_0^2 \frac{u(\omega)^2}{2\omega V}\frac{1}{W-\omega}$$

Now $\sum_k \sim V\int \frac{d^3k}{(2\pi)^3}$ so in the $V\to\infty$ limit

$$g(W) = \int \frac{\overbrace{d^3k}^{4\pi|k|^2 d|k|}}{(2\pi)^3}\, g_0^2 \frac{u(\omega)^2}{2\omega}\frac{1}{(W-\omega)}$$

$$= \frac{g_0^2}{4\pi^2}\int_0^\infty \frac{d|k|\; |k|^2 u^2}{\omega(W-\omega)}$$

$\omega^2 = |k|^2+\mu^2$

$\omega\, d\omega = |k|\, d|k|$

$$= \frac{g_0^2}{4\pi^2}\int_\mu^\infty \frac{d\omega\; |k|\, u(\omega)^2}{W-\omega}$$

Thus the spectral measure involved for H_0 and the vector γ is

$$d\omega\, |k|\, u^2 = d\omega\left(\sqrt{\omega^2-\mu^2}\right)u^2 \qquad \text{on } [\mu, \infty).$$

Now one wants to let the cut-off fn. $u(\omega)$ approach 1. Then one gets the measure

$$d\omega \cdot |k| = d\omega\left(\sqrt{\omega^2-\mu^2}\right)$$

for which the Riesz-Herglotz integral

$$\int_\mu^\infty \frac{d\omega\,\sqrt{\omega^2-\mu^2}}{1+\omega^2} = \infty$$

is logarithmically divergent.

Next we want to compute the scattering. Let φ_k be an eigenfunction for H_0. Better, start with $\varphi \in \mathcal{H}_0$. Then

$$\frac{1}{W-\mathbb{H}}\varphi = \left(\frac{1}{W-\mathbb{H}_0} + \frac{1}{W-\mathbb{H}} H_{int}\frac{1}{W-H_0}\right)\varphi$$

better $\frac{1}{W-\mathbb{H}}(W-H_0)\varphi = \varphi + \frac{1}{W-\mathbb{H}} \overbrace{H_{int}\,\varphi}^{\Phi\langle\gamma|\varphi\rangle}$

so
$$\frac{1}{W-H}(W-H_0)\varphi = \varphi_k + \frac{1}{W-\omega_0-g(W)}\left(\Phi + \frac{1}{W-H_0}|\gamma\rangle\right)\langle\gamma|\varphi_k\rangle$$

Now take $\varphi = \varphi_k$ whence we have

$$(W-\omega_k)\varphi_k = (W-H)\left[\varphi_k + \text{-------------}\right]$$

Therefore we get eigenfunctions for H as $W \to \omega_k$. In particular, letting $W = \omega_k + i\varepsilon$, we get

$$\boxed{\psi_k^+ = \varphi_k + \frac{1}{\omega_k-\omega_0-g(\omega_k+i\varepsilon)}\left(\Phi + \frac{1}{\omega_k+i\varepsilon-H_0}|\gamma\rangle\right)\gamma_k}$$

(Here's a better way to view the above: In general

$$\frac{1}{W-H} = \frac{1}{W-H_0} + \frac{1}{W-H_0}T(W)\frac{1}{W-H_0}$$

where
$$T(W) = H_{int} + H_{int}\frac{1}{W-H_0}H_{int} + \cdots\cdots$$

In the present case, between states in $\mathcal{H}_0$ we have

$$\boxed{T(W) = |\gamma\rangle\frac{1}{W-\omega-g(W)}\langle\gamma|}$$

so
$$\psi_k^+ = \varphi_k + \frac{1}{\omega_k+i\varepsilon-H_0}T(\omega_k+i\varepsilon)\varphi_k \qquad \text{(in general)}$$

$$= \varphi_k + \left(\Phi + \frac{1}{\omega_k+i\varepsilon-H_0}|\gamma\rangle\right)\frac{\gamma_k}{\omega_k-\omega_0-g(\omega_k+i\varepsilon)}$$

Similarly
$$\psi_k^- = \varphi_k + \left(\Phi + \frac{1}{\omega_k-i\varepsilon-H_0}|\gamma\rangle\right)\frac{\gamma_k}{\omega_k-\omega_0-g(\omega_k-i\varepsilon)}$$

(not very clear))

Let's assume now that we are the situation of a radial Schroedinger equation, so that the spectrum of H_0 is of multiplicity one. Then ψ_k^+, ψ_k^- should lie in a 1-dimensional space and so $\psi_k^+ = S_k \psi_k^-$. Clearly

$$\boxed{S_k = \frac{\omega_k - \omega_0 - g(\omega_k + i\varepsilon)}{\omega_k - \omega_0 - g(\omega_k - i\varepsilon)}}$$

in order that the coefficients of Φ work out. We can check this works:

$$\left[\omega_k - \omega_0 - g(\omega_k + i\varepsilon)\right]\psi_k^+ \overset{?}{=} \left[\omega_k - \omega_0 - g(\omega_k - i\varepsilon)\right]\psi_k^-$$

$$\shortparallel$$

$$(\omega_k - \omega_0 - g(\omega_k + i\varepsilon))\,\varphi_k + \left(\Phi + \frac{1}{\omega_k + i\varepsilon - H_0}|\gamma\rangle\right)\bar{\gamma}_k$$

$$\left(\frac{1}{\omega_k + i\varepsilon - H_0}|\gamma\rangle - \frac{1}{\omega_k - i\varepsilon - H_0}|\gamma\rangle\right)\bar{\gamma}_k \overset{?}{=} \varphi_k\left(g(\omega_k + i\varepsilon) - g(\omega_k - i\varepsilon)\right)$$

Now

so
$$\frac{1}{\omega_k + i\varepsilon - H_0} - \frac{1}{\omega_k - i\varepsilon - H_0} = -2\pi i\,\delta(\omega_k - H_0)$$

hence the above is

$$\left(-2\pi i \underbrace{\delta(\omega_k - H_0)\gamma}_{\text{multiple of } \varphi_k}\right)\bar{\gamma}_k \overset{?}{=} \varphi_k\left(-2\pi i \langle\gamma|\delta(\omega_k - H_0)|\gamma\rangle\right)$$

so it works because $\bar{\gamma}_k = \langle\gamma|\varphi_k\rangle$.

In the actual Lee model one works in 3 dimensions. The scattered wave

$$\frac{1}{\omega_k - \omega_0 - g(\omega_k + i\varepsilon)}\left(\frac{1}{\omega_k + i\varepsilon - H_0}|\gamma\rangle\right)\langle\gamma|\varphi_k\rangle$$

is spherically symmetric, because γ is. Hence there is only S-wave scattering. Thus if we average over the rotation group the equation

$$\psi_k^+ = \varphi_k + \text{scattered wave} + \text{mult. of } \Phi$$

we see that the scattering is multiplication by S_k (as given on the preceding page) on the S-wave components and the identity on higher components.

Hence

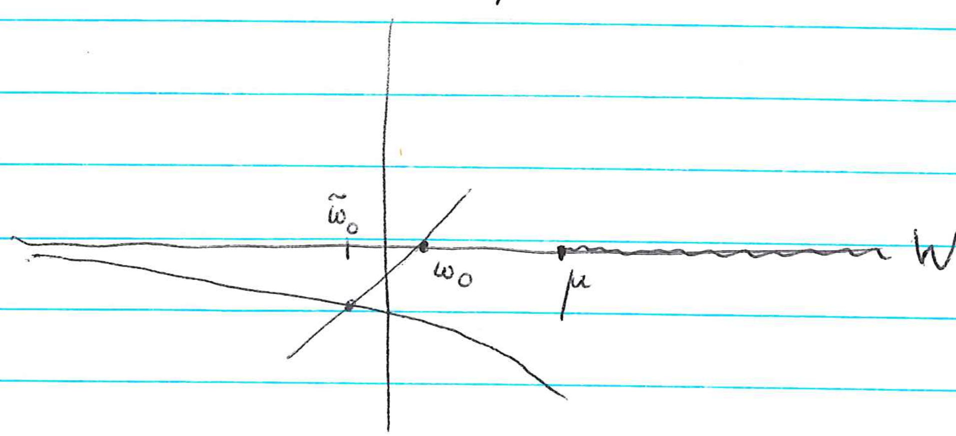
$$S_\omega = \frac{\omega - \omega_0 - g(\omega + i\varepsilon)}{\omega - \omega_0 - g(\omega - i\varepsilon)}$$

is the S-matrix.

Now we want to rewrite this so as to understand how to remove the cutoff. The main idea is that the V-particle mass $\tilde{\omega}_0$ and the S-matrix are "physical", i.e. determined by experiment.

$$g(W) = \frac{g_0^2}{4\pi^2} \int_0^\infty \frac{d|k|\, |k|^2\, u(\omega)^2}{\omega\,(W - \omega)} \qquad \omega = \sqrt{k^2 + \mu^2}$$

$$= \frac{g_0^2}{4\pi^2} \int_\mu^\infty \frac{d\omega\, |k|\, u(\omega)^2}{W - \omega} \qquad |k| = \sqrt{\omega^2 - \mu^2}$$



The equation defining $\tilde{\omega}_0$ is $\quad \tilde{\omega}_0 - \omega_0 - g(\tilde{\omega}_0) = 0$

but since $\tilde{\omega}_0$ is measurable, we should use this to define ω_0:

$$\omega_0 = \tilde{\omega}_0 - g(\tilde{\omega}_0)$$

Then the scattering becomes

$$S_\omega = \frac{\omega - (\tilde{\omega}_0 - g(\tilde{\omega}_0)) - g(\omega + i\varepsilon)}{\omega - (\tilde{\omega}_0 - g(\tilde{\omega}_0)) - g(\omega - i\varepsilon)} .$$

This involves the difference of two g values and hence is better convergent:

$$g(\tilde{\omega}_0) - g(W) = \frac{g_0^2}{4\pi} \int_\mu^\infty d\omega\, |k|\, u^2 \left[\frac{1}{\tilde{\omega}_0 - \omega} - \frac{1}{W - \omega} \right]$$

$$\frac{W - \tilde{\omega}_0}{(\tilde{\omega}_0 - \omega)(W - \omega)}$$

$$\frac{g(\tilde{\omega}_0) - g(W)}{\tilde{\omega}_0 - W} = -\frac{g_0^2}{4\pi^2} \int_\mu^\infty \frac{d\omega\, |k|\, u^2}{(\omega - \tilde{\omega}_0)(\omega - W)}$$

hence we get

$$S_\omega = \frac{1 + f(\omega + i\varepsilon)}{1 + f(\omega - i\varepsilon)}$$

where

$$f(W) = \frac{g_0^2}{4\pi^2} \int_\mu^\infty \frac{d\omega\, |k|\, u^2}{(\omega - \tilde{\omega}_0)(\omega - W)}$$

Now the other quantity of interest is the so-called renormalization constant. The ground state is proportional to

$$\Phi + \frac{1}{\tilde{\omega}_0 - H_0} |\gamma\rangle$$

$$\left\| \Phi + \frac{1}{\tilde{\omega}_0 - H_0} |\gamma\rangle \right\|^2 = 1 + \frac{g_0^2}{4\pi^2} \int_\mu^\infty \frac{d\omega\, |k|\, u(\omega)^2}{(\tilde{\omega}_0 - \omega)^2} = 1 + f(\tilde{\omega}_0)$$

There are various levels of divergence in this problem. Suppose we first look at dimension 1. Then

$$g(W) = \text{const} \int_0^\infty \frac{d|k|\; u(\omega)^2}{\omega\,(W-\omega)} \qquad d|k| = \frac{\omega\, d\omega}{|k|}$$

$$= \text{const} \int_\mu^\infty \frac{d\omega \quad u(\omega)^2}{|k|\,(W-\omega)}$$

remains convergent as the cutoff is removed: $u(\omega) \to 1$. This means that the bare mass

$$\omega_0 = \tilde{\omega}_0 - g(\tilde{\omega}_0)$$

is finite. Thus we have a nice standard perturbation problem, with a finite ground energy shift for the discrete state and scattering for the continuum states.

Next look at dimension 2 where

$$g(W) = \text{const} \int_\mu^\infty \frac{d\omega \cdot u(\omega)^2}{W - \omega}$$

Here as $u \to 1$, this integral is logarithmically divergent. Thus the bare mass of the V-particle is infinite. However the rest of the problem is finite, by which I mean that the states

$$\Phi + \frac{1}{\tilde{\omega}_0 - H_0} |\gamma\rangle$$

$$\psi_k^+ = \varphi_k + \frac{1}{\omega_k - \omega_0 - g(\omega_k + i\varepsilon)} \left(\Phi + \frac{1}{\tilde{\omega}_0 - H_0} |\gamma\rangle \right) \gamma_k$$

are well-defined.

January 5, 1981.

In the Lee model

$$\frac{1}{W-H}\Phi = \frac{1}{W-\omega_0-g(W)}\left(\Phi + \frac{1}{W-H_0}|\gamma\rangle\right)$$

where

$$g(W) = \frac{g_0^2}{4\pi^2}\int_\mu^\infty \frac{d\text{[redacted]}\,|k|\,u^2(\omega)}{W-\omega}$$

The physical energy of the V-particle $\tilde{\omega}_0$ is the unique root of

$$\tilde{\omega}_0 - \omega_0 - g(\tilde{\omega}_0) = 0$$

with $\tilde{\omega}_0 < \mu$. This we use to eliminate the base mass ω_0 in favor of $\tilde{\omega}_0$. Then

$$W-\omega_0-g(W) = W-\tilde{\omega}_0 + g(\tilde{\omega}_0) - g(W)$$

$$g(\tilde{\omega}_0) - g(W) = \frac{g_0^2}{4\pi^2}\int_\mu^\infty \frac{d\omega\,|k|\,u^2}{1}\left(\frac{1}{\tilde{\omega}_0-\omega} - \frac{1}{W-\omega}\right)$$

$$= (W-\tilde{\omega}_0)\, g_0^2 \underbrace{\frac{1}{4\pi^2}\int_\mu^\infty \frac{d\omega\,|k|\,u^2}{(\omega-\tilde{\omega}_0)(\omega-W)}}_{\text{define this to be } f(W)}$$

Consequently

$$\left\langle\Phi\right|\frac{1}{W-H}\left|\Phi\right\rangle = \frac{1}{W-\omega_0-g(W)} = \frac{1}{(W-\tilde{\omega}_0)(1+g_0^2 f(W))}$$

Recall that the S matrix is [redacted] multiplication [redacted] on S-wave components by

$$S_\omega = \frac{\omega-\omega_0-g(\omega\text{[redacted]}+i\varepsilon)}{\omega-\omega_0-g(\omega-i\varepsilon)} = \frac{1+g_0^2 f(\omega+i\varepsilon)}{1+g_0^2 f(\omega-i\varepsilon)}$$

As the cutoff is removed: $u \to \infty$, the integral defining $f(W)$ diverges logarithmically. However

$$\hat{f}(W) = f(W) - f(\tilde{\omega}_0) = \frac{1}{4\pi^2}\int_{\mu}^{\infty} \frac{d\omega\,|k|u^2}{(\omega-\tilde{\omega}_0)}\left[\frac{1}{\omega - W} - \frac{1}{\omega-\tilde{\omega}_0}\right]$$

$$= (W-\tilde{\omega}_0)\frac{1}{4\pi^2}\int_{\mu}^{\infty}\frac{d\omega\;|k|u^2}{(\omega-\tilde{\omega}_0)^2(\omega - W)}$$

converges. The idea is to let the coupling constant g_0 change with the cutoff, so as to obtain a good limit. Thus

$$S_\omega = \frac{1+g_0^2 f(\omega+i\varepsilon)}{\cdots} = \frac{1+g_0^2 f(\tilde{\omega}_0) + g_0^2\hat{f}(\omega+i\varepsilon)}{\cdots}$$

$$= \frac{1+g^2\hat{f}(\omega+i\varepsilon)}{1+g^2\hat{f}(\omega-i\varepsilon)}$$

where $$g^2 = \frac{g_0^2}{1+g_0^2 f(\tilde{\omega}_0)}$$ is the "dressed" coupling constant, which could be determined experimentally. However the problem is that for $g_0^2 > 0$ one has

$$g^2 < \frac{1}{f(\tilde{\omega}_0)}$$

and $f(\tilde{\omega}_0)\to\infty$, so it seems one is forced into having g_0 imaginary.

Consider now

$$\langle\Phi|\frac{1}{W-H}|\Phi\rangle = \frac{1}{(W-\tilde{\omega}_0)(1+g_0^2 f(W))}$$

and its Fourier transform (upper half W-plane)

$$\int\frac{dW}{2\pi}\,e^{-iWt}\langle\Phi|\frac{1}{W+i\varepsilon-H}|\Phi\rangle = -i\,\Theta(t)\langle\Phi|e^{-iHt}|\Phi\rangle$$

~~In the Lee model this gadget is the V-particle propagator~~

because $\Phi = \psi_V^* |0\rangle$. so

$$\left\langle \Phi \right| \frac{1}{W-H} \left| \Phi \right\rangle = \frac{1}{(W-\tilde{\omega}_0)(1+g_0^2 f(\tilde{\omega}_0))(1+g^2\hat{f}(W))}$$

is called the unrenormalized V-particle propagator. Since

$$\left\| \Phi + \frac{1}{\tilde{\omega}_0 - H_0} |\gamma\rangle \right\|^2 = 1 + g_0^2 f(\tilde{\omega}_0)$$

the renormalized propagator is obtained by making the residue $=1$ at $W = \tilde{\omega}_0$, so it is

$$\boxed{\frac{1}{(W-\tilde{\omega}_0)(1+g^2\hat{f}(W))} \quad \text{renormalized V-particle propagator}}$$

If this were a Herglotz function in the no cutoff limit, then it's clear we would have a 1-parameter unitary group with cyclic vector, etc.

$$1+g^2\hat{f}(W) = 1 + g^2(W-\tilde{\omega}_0)\frac{1}{4\pi^2}\int_\mu^\infty \frac{d\omega\,|k|\,u^2}{(\omega-\tilde{\omega}_0)^2(\omega-W)}$$

better

$$1+g^2\hat{f}(W) = \frac{1}{1+g_0^2 f(\tilde{\omega}_0)}\left(1+g_0^2 f(W)\right)$$

Now

$$f(W) = \frac{1}{4\pi^2}\int_\mu^\infty \frac{d\omega\,|k|\,u^2}{(\omega-\tilde{\omega}_0)(\omega-W)}$$ is >0 and increasing for $W<\mu$

If $g_0^2<0$, then $1+g_0^2 f(W)$, as W goes from $-\infty$ to $\tilde{\omega}_0$, goes from 1 to $1+g_0^2 f(\tilde{\omega}_0) = +\frac{g_0^2}{g^2} < 0$. Hence $1+g^2\hat{f}(W)$ has a root $\omega_1 < \tilde{\omega}_0$. The residue is $\omega_1 - \tilde{\omega}_0 < 0$, so one has a negative norm state. This is the "ghost" state.

~~We can evaluate $\hat{f}(W)$ by residues. Take $\tilde{\omega}_0 = 0$~~

Perhaps the simplest example of one of these Lee-type models has the V-particle propagator

$$\langle \Phi | \frac{1}{W - \mathbb{H}} | \Phi \rangle = \frac{1}{W - \omega' + ci\sqrt{W}} \qquad \text{with } c > 0.$$

A residue calculation gives

$$\int_0^\infty \sqrt{x}\, dx \left(\frac{1}{W - x} + \frac{1}{x} \right) = -\pi i \sqrt{W}$$

where $\sqrt{W}$ is defined off $\mathbb{R}_{>0}$ by requiring $\operatorname{Im}(\sqrt{W}) > 0$

ω' is not the bare mass ω_0, because we have already had to adjust for the fact that

$$g(W) = \int_0^\infty \frac{\sqrt{x}\, dx}{W - x} \qquad \text{diverges.}$$

January 8, 1981

Project: Consider a Riemann surface (closed) X. Atiyah & Bott have shown what gauge theory over X is like. One fixes a rank n, and degree d, and then gets a gauge theory [redacted] whose configurations are all connections A (or holomorphic structures) on a C^∞ hermitian vector bundle. [redacted] Morse theory allows them to compute the cohomology of the different strata of bundles of [redacted] rank n and degree d.

On the other hand, physics treats gauge theory as a form of statistical mechanics. One wants to compute various averages over all configurations. For example a correlation function $\langle A(x_1) A(x_2) \cdots A(x_n) \rangle$ or something like Wilson's loop integral which is the average of $A \longmapsto \mathrm{tr}$(parallel transport of A around γ). (Notice that in the case of a lattice gauge theory, γ is a bunch of edges $\vec{e}_i$ and the connection gives unitary matrices $U(\vec{e}_i)$ for each edge, so one is trying to compute the integral of

$$\mathrm{tr}\left(U(\vec{e}_1) \cdots\cdots U(\vec{e}_n) \right)$$

over all connections?)

So a basic problem seems to be to relate what Atiyah + Bott do to what is done in physics. Probably one needs to embed everything in representation theory of the gauge group, but perhaps it is possible to find simple examples. [redacted] The first example should deal with $\Omega^1 U_n$ or vector bundles over $\mathbb{C}P^1$.

Let's consider the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. I recall that one good way to describe vector bundles with hermitian inner product over S^2 is to use clutching functions. I should recall what I know about bundles over S^2 and $\Omega U(n)$. My problem is to connect what I know about Laurent polynomials with the connection A of gauge fields.

Let's begin where the physicists do by trivializing the bundle over $\mathbb{C}$. So we consider the trivial $U(n)$ bundles over $\mathbb{C}$ and all connections on it. A connection is a $\mathfrak{g}$-valued 1-form $A_\mu dx^\mu$ where $\mathfrak{g}$ = Lie alg of $U(n)$ = skew-hermitian $n\times n$ matrices. The covariant derivative is

$$D_\mu = \partial_\mu + A_\mu$$

which means that if ψ is a section of our vector bundle, then

$$D\psi = (\partial_\mu\psi + A_\mu\psi)\, dx^\mu$$

The curvature is the operator D^2

$$(\partial_\mu + A_\mu)(\partial_\nu + A_\nu)\ dx^\mu dx^\nu \qquad ?$$

(Do for $n=1$. Then ψ is a function

$$\begin{aligned} D^2\psi &= (d+A)(d+A)\psi \\ &= (d+A)(d\psi + A\psi) \\ &= dA\psi + Ad\psi + \underbrace{AA\psi}_{0 \text{ if } n=1} \\ &= (dA + AA)\psi \ .) \end{aligned}$$

Better: think of D as an operator on vector-valued forms. Then A takes a vector form into a vector form with one higher degree. Also

$$dA = d(A_\mu dx^\mu) = \partial_\mu A_\nu\ dx^\mu dx^\nu$$

$$AA = A_\mu dx^\mu A_\nu dx^\nu = A_\mu A_\nu\ dx^\mu dx^\nu$$

and
$$dA + AA = \tfrac{1}{2} F_{\mu\nu}\, dx^\mu dx^\nu \quad \text{where}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$
is the so-called field-strength tensor.

According to Bott given a [redacted] C^∞ vector bundle E over a Riemann surface with hermitian structure, there is a 1-1 correspondence between connections [redacted] on the bundle preserving the hermitian structure and holomorphic structures on E. Given the connection

$$D : \underline{E} \longrightarrow \underline{E \otimes T^*} \qquad T^* = \overset{dz}{\widehat{T^{10}}} \oplus \overset{d\bar z}{\widehat{T^{01}}}$$

one gets the holomorphic structure by saying $s \in \Gamma(E)$ is holomorphic $\Longleftrightarrow$ $Ds \in \Gamma(E \otimes T^{10})$. The differential operator

$$\bar\partial : \underline{E} \longrightarrow E \otimes T^* \big/ E \otimes T^{10} \simeq E \otimes T^{01}$$

is determined of first order, so one gets holomorphic sections through each point of E.

Conversely if one is given the holomorphic structure and the hermitian structure $\langle \,|\, \rangle$, one gets the connection as follows. Let s_i be a frame of holomorphic sections. Then

$$d\, [redacted] \langle s_i | s_j \rangle = \langle D s_i | s_j \rangle + \langle s_i | D s_j \rangle$$

if D is to preserve $\langle \,|\, \rangle$, so if $D s_i =$ [redacted] $s_j \theta_{ji}$, $\theta_{ji} \in \Omega^{10}$ we have

$$d \langle s_i | s_j \rangle = [redacted]$$
$$= \langle s_k | s_j \rangle \bar\theta_{ki} + \langle s_i | s_k \rangle \theta_{kj}$$

hence $\langle s_i | s_k \rangle \theta_{kj} = \partial \langle s_i | s_j \rangle$ determines θ_{kj}.

Now suppose we start out with a vector bundle E with hermitian structure over S^2. We can then trivialize it over $\mathbb{C}$ and describe connections as gauge forms $A = A_\mu dx^\mu$. A natural question is when such a form given over $\mathbb{C}$ defines a connection on E over S^2.

January 9, 1980

Let's go over the Atiyah-Bott theory of Yang-Mills over a Riemann surface X. Fix a C^∞ vector bundle E with hermitian structure [redacted] The space of connections on E preserving the hermitian product is an affine space for the vector [redacted] space of sections of $Hom(E,E) \otimes T^*$ which are [redacted] skew-hermitian, [redacted] Better: the connections on E form a bundle $\underset{\sim}{C}(E)$ over X which is a torsor for [redacted] the vector bundle $Hom^{sk}(E,E) \otimes T^*$, where $Hom^{sk}(E,E)$ denotes skew-hermitian endomorphisms. Put $C = \Gamma(X, \underset{\sim}{C}(E))$

The gauge group is $\mathcal{G} = Aut(E) = \Gamma(X, \underset{\sim}{Aut}(E))$ and its Lie algebra is $\Gamma(Hom^{sk}(E,E))$.

Let's work out some formulas. First let us take an open covering [redacted] $\{U_{[redacted]}\}$ and trivialize the bundle over the open sets in this covering

$$E_U \underset{s_U}{\overset{\sim}{\longleftarrow}} U \times \mathbb{C}^n$$

Then
$$s_U = s_V g_{VU} \qquad \text{over } U \cap V$$
where g_{VU} [redacted] $: U \cap V \longrightarrow U(n)$ is a cocycle:
$$g_{WU} = g_{WV} g_{VU} \qquad \text{over } U \cap V \cap W.$$
A connection D on E is given by
$$D(s_U \psi) = s_U (d + A_U)\psi$$
where A_U is a $\mathfrak{g}$-valued form over U, $\mathfrak{g} = Lie(U(n))$. We have

$$\cancel{d + A_U = s_U \, D \, s_U^{-1} = s_V g_{VU} D g_{UV} s_V^{-1}}$$

$$d + A_u = s_u^{-1} D s_u$$
$$= g_{uv} s_v^{-1} D s_v g_{vu} = g_{uv}(d + A_v) g_{vu}$$

so that
$$\boxed{A_u = g_{uv} A_v g_{vu} + g_{uv}\, d g_{vu}}$$

Let g be a gauge transformation, that ~~is~~, an automorphism of E. Then g is given by maps $g_u : U \longrightarrow U(n)$ defined by
$$g_u = s_u^{-1} g s_u .$$

[redacted] g sends D to $g D g^{-1}$. Hence
$$s_u^{-1}(g D g^{-1}) s_u = g_u (d + A_u) g_u^{-1}$$
$$= d + [g_u A_u g_u^{-1} + g_u d g_u^{-1}]$$

Therefore $\boxed{\text{if } D = \{A_u\}, \text{ then } g D g^{-1} = \{g_u A_u g_u^{-1} + g_u d g_u^{-1}\}.}$

Consider an infinitesimal gauge transf. $g^{-1} = 1 + h$ where $h \in \Gamma(Hom^{sk}(E,E))$. Then dropping U subscript
$$\delta A = (1-h)A(1+h) + (1-h)d(1+h) - A$$
$$= dh + [A, h]$$

What this means is that the tangent space to the orbit thru A ~~for~~ the gauge group can be identified with the space of $\tilde{\mathfrak{g}}$-valued 1-forms of the form $\{dh + [A,h]\} = Dh - hD$, where h is a $\tilde{\mathfrak{g}}$-valued 0-form. Here $\tilde{\mathfrak{g}} = Hom^{sk}(E,E) = P \times^{U(n)} \mathfrak{g}$, P = principal bdle.

The bundle $\tilde{\mathfrak{g}} = P \times^{U(n)} (\mathrm{Lie}(U(n)))$ is important. Its sections are elements of the Lie algebra of the gauge group. Differences between connections are sections of $\tilde{\mathfrak{g}} \otimes T^*$; curvature D^2 is a section of $\tilde{\mathfrak{g}} \otimes \Lambda^2 T^*$. A connection on E induces one on $\tilde{\mathfrak{g}}$, leading to a sequence of differential operators

$$\begin{array}{ccccccc} \tilde{\mathfrak{g}} & \longrightarrow & \tilde{\mathfrak{g}} \otimes T^* & \longrightarrow & \tilde{\mathfrak{g}} \otimes \Lambda^2 T^* & \longrightarrow & \cdots \\ h & \longmapsto & [D,h] = Dh - hD & & & & \\ & & B & \longmapsto & [D,B] = DB + BD & & \end{array}$$

The composite is bracket with the curvature D^2 since

$$D(Dh - hD) + (Dh - hD)D = [D^2, h].$$

The Yang-Mills functional is as follows: Given D its curvature is

$$D^2 = \{ (d+A)^2 = dA + AA \} \in \Gamma(\tilde{\mathfrak{g}} \otimes \Lambda^2 T^*)$$

On a Riemann surface $\Lambda^2 T^*$ is 1-dimensional, and the metric + orientation on the Riemann surface gives a trivialization + metric for $\Lambda^2 T^*$. Combining with an invariant metric on $\tilde{\mathfrak{g}}$ allows us to define $|D^2|$ and

$$\|D^2\|^2 = \int |D^2|^2$$

This is the Yang-Mills functional. It is a quartic functional on the space of connections which we have seen is an affine space belonging to $\Gamma(\tilde{\mathfrak{g}} \otimes T^*)$.

Let's digress to look at line bundles. In this case $\tilde{\mathfrak{g}}$ is a trivial bundle, and $AA = 0$ so that the curvature $D^2 = \{dA\}$ is linear on the space of connections. Thus the Yang-Mills functional is quadratic so the Yang-Mills connections form a vector space once a center is fixed. A form $D = \{A\}$ is stationary when

$$\langle dA | dB \rangle = 0 \qquad \text{for all} \quad B \in C_0^\infty(T^*)$$

$$\iff \quad d^* dA = 0.$$

Thus dA is a harmonic form on X of degree 2. It must be a multiple of the volume form, and since it represents a Chern class $\int dA = \pm \frac{i}{2\pi} c_1(E)$.

If we fix such an $\{A_u\}$ then the other Yang-Mills fields are of the form $\{A_u + B\}$, where B is a closed 1-form. The orbit thru A of the gauge group consists of forms $\{A_u + \frac{df}{f}\}$ where $f: X \longrightarrow S^1$. Now any exact 1-form du is in the form $\frac{df}{f} = \frac{de^u}{e^u}$. Clearly

$$\frac{\{\frac{df}{f} \in \Gamma(T^*)\}}{\{du \in \Gamma(T^*)\}} = [X, S^1] = H^1(X, \mathbb{Z})$$

Hence

$$\frac{\text{Yang-Mills } A}{\text{gauge group}} = \frac{\text{closed 1-forms}}{\{\frac{df}{f}\}} = \frac{H^1(X, \mathbb{R})}{H^1(X, \mathbb{Z})}$$

is the Picard group.

So next consider higher rank. Given D its curvature is a section of $\tilde{\mathfrak{g}} \otimes \Lambda^2 T^*$. Since we are provided with a section of $\Lambda^2 T^*$ given by the volume we get an endomorphism of E. Now I want to show that when D is a Yang-Mills field, this endomorphism commutes with D, hence with parallel transport defined by D. Thus we will get a endomorphism of the associated holomorphic bundle.

Look at a variation of D; it is of the form $D+B$ where $B \in \Gamma(\tilde{\mathfrak{g}} \otimes T^*)$. Then $(D+B)^2 = D^2 + [D,B] + B^2$. For D to be Yang-Mills means that

$$\langle (D+B)^2 | (D+B)^2 \rangle$$

vanishes to the first order in B, i.e.

$$\langle [D,B] \mid D^2 \rangle = 0.$$

Therefore in the sequence

$$\tilde{\mathfrak{g}} \otimes T^* \xrightarrow{\tilde{D} = [D,\ ]} \tilde{\mathfrak{g}} \otimes \Lambda^2 T^*$$

the curvature D^2 is orthogonal to the image. Now compute locally. Choose coordinates x, y so that $dx\,dy$ is the canonical volume and $D = d + A$ where $A = A_x\,dx + A_y\,dy$ and A_x, A_y are skew-hermitian matrix functions. A section of $\tilde{\mathfrak{g}} \otimes T^*$ is of the form $B_x\,dx + B_y\,dy$ and

$$\tilde{D}(B_x\,dx + B_y\,dy) = [d + A_x\,dx + A_y\,dy,\ B_x\,dx + B_y\,dy]$$

$$= \left(\partial_x B_y + [A_x, B_y] - \partial_y B_x - [A_y, B_x]\right) dx\,dy$$

Thus $F\,dx\,dy$ is orthogonal to the image of $\tilde{D}$ when F is orthogonal to the image of $\partial_x + [A_x,\ ]$, and $\partial_y + [A_y,\ ]$. However, recall we are working with matrices $B \in C_0^\infty$:

$$\langle \partial_x B_y + [A_x, B_y], F \rangle = \langle B_y, -\partial_x F - [A_x, F] \rangle.$$

Thus $D_x = \partial_x + [A_x,\ ]$ and $D_y = \partial_y + [A_y,\ ]$ kill F, which establishes that F is invariant under the connection.

(Now if our volume is normalized properly we know that the trace of D^2 (which is a constant function × volume) must be the degree of E, since $\frac{i}{2\pi}\int tr(D^2) = \deg(E)$.)

<u>Summary</u>: If D is a Yang-Mills connection on E then if we write $D^2 = F\cdot\omega$, where ω is our volume form and $F \in \Gamma(\tilde{\mathfrak{g}})$, then F is an endomorphism of E preserving both metric and connection. The eigenvalues of F are purely imaginary and the eigenbundles are orthogonal. Thus our bundle will be an orthogonal direct sum of Yang-Mills bundles where the endo F is a constant, where the constant is related to the degree of the bundle.

So to understand Yang-Mills bundles we can assume the curvature form is a constant multiple of the volume form ω. If the degree is zero, the curvature is zero, so the bundle comes from a unitary representation of π_1, i.e. it's a semi-stable bundle of degree 0.

Let's now understand a Yang-Mills connection D such that F is a multiple of the identity.

$$\langle (D+B)^2 \mid (D+B)^2\rangle = \langle D^2 + [D,B] + B^2 \mid D^2 + [D,B] + B^2\rangle$$

$$= \langle D^2 | D^2\rangle + 2\underbrace{\langle D^2 \mid [D,B]\rangle}_{0} + \underbrace{2\langle D^2 | B^2\rangle + \|[D,B]\|^2}_{\text{2nd variation}}$$

Now if $B = B_x\,dx + B_y\,dy$, then

$$B^2 = [B_x, B_y]\,dx\,dy$$

with be orthogonal to $D^2 = F\,dx\,dy$, because F is a multiple of the identity matrix. Also we have that

$$\tilde{\mathfrak{g}} \xrightarrow{\tilde{D}} \tilde{\mathfrak{g}}\otimes T^* \xrightarrow{\tilde{D}} \tilde{\mathfrak{g}}\otimes \Lambda^2 T^*$$

satisfies $\qquad \tilde{D}^2(h) = [D^2, h] = 0.$

So we see that the 2nd variation is ≥ 0 for such a D. The null space is given by $B \in \Gamma(\tilde{\mathfrak{g}} \otimes T^*)$ with $[D, B] = 0$.

January 10, 1980:

Given a vector bundle with hermitian ~~structure~~ E over X choose local coordinates x^μ on X such that $dx^1 \cdots dx^n$ is the volume form. Given a connection $D = \{d + A_\mu\}$ on E its curvature is

$$\begin{aligned} D^2 = (d+A)^2 &= dA + AA \\ &= (\partial_\mu A_\nu + A_\mu A_\nu)\, dx^\mu dx^\nu \\ &= \tfrac{1}{2}\big(\underbrace{\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]}_{F_{\mu\nu}}\big)\, dx^\mu dx^\nu \end{aligned}$$

Provided $dx^\mu dx^\nu$ is a orthonormal frame for $\Lambda^2 T^*$ the action is

$$\frac{1}{4}\int \sum_{\mu\nu} |F_{\mu\nu}|^2 \, dx^1 \cdots dx^n$$

and D is a YM connection when for all $B \in \Gamma(\tilde{\mathfrak{g}} \otimes T^*)$

$$\sum_{\mu\nu} \big\langle \underbrace{\partial_\mu B_\nu - \partial_\nu B_\mu + [A_\mu, B_\nu] + [B_\mu, A_\nu]}_{\tilde{D}_\mu(B_\nu) - \tilde{D}_\nu(B_\mu)} \,\big|\, F_{\mu\nu} \big\rangle = 0$$

$$\tilde{D}_\mu = \partial_\mu + [A_\mu, \ ].$$

i.e when
$$\sum_{\mu\nu} \langle \tilde{D}_\mu(B_\nu) \,|\, F_{\mu\nu} \rangle = 0.$$
$$\parallel$$
$$\langle B_\nu \,|\, -\tilde{D}_\mu F_{\mu\nu} \rangle$$

Thus D is a YM connection when

$$\sum_\mu \tilde{D}_\mu(F_{\mu\nu}) = 0 \qquad \text{all } \nu$$

For a Riemann surface this says

$$\tilde{D}_1(F_{12}) = \tilde{D}_2(F_{21}) = 0$$

and hence the endom. F of E given by the curvature D^2 + volume form commutes with parallel translation. Therefore a YM connection on E is obtained by splitting E

into an orthogonal direct sum of bundles, and then putting on each summand a connection whose curvature is a scalar endomorphism times the volume element.

Simplest case is a bundle E with a flat connection $F=0$. Such a thing defines a n-diml. unitary representation of π_1. Also we have a de Rham complex

$$* \qquad E \xrightarrow{D} E\otimes T^* \xrightarrow{D} E\otimes \Lambda^2 T^*$$

Now any global holomorphic section of E is constant. (Proof: For such a section $|s|^2$ has to have a maximum. Say it occurs at $z=0$ and we trivialize the bundle near 0 by a flat orthonormal frame. Then s can be identified with a vector $f_i(z)$ of holom. fns. near 0 such that $\varphi(z) = \sum |f_i(z)|^2$ has a max. at 0. But $\partial_{\bar z}\partial_z \varphi = \sum |f_i'(z)|^2 \geq 0$, so φ is subharmonic, so having a max at 0 means it is constant $\Rightarrow \sum |f_i'(z)|^2 = 0$ so s is constant.) [redacted] This is a special case of the fact that the Hodge decomposition holds for the de Rham complex above; The cohomology is represented by harmonic forms which have a type decomposition.

Now suppose one has an E with D such that F is a scalar. Then $\tilde D$ on $Hom(E,E)$ is flat, because we've seen [redacted] before that for

$$Hom(E,E) \xrightarrow{\tilde D} Hom(E,E)\otimes T^* \xrightarrow{\tilde D} Hom(E,E)\otimes \Lambda^2 T^*$$

we have $\tilde D(h) = [D^2, h]$. This will be zero if F is a scalar. So I can conclude that the holomorphic endos. of E are the same as the flat endos and that

$$H^0(Hom(E,E)) = [redacted] \left(Hom_{[redacted]}(E_\xi, E_\xi)\right)^{\pi_1(X)} \qquad \xi = \text{basept.}$$

(It appears that when F is a scalar, one has a projective representation of $\pi_1(X)$?)

January 11, 1980

Let E have YM connection D such that F is a scalar. Then we've seen that $Hom(E,E)$ is flat with global holomorphic sections $Hom(E_\xi,E_\xi)^{\pi_1(X)}$. This is a C^*-algebra so that it's semi-simple (there can't be a 2-sided nilpotent ideal because x^*x is non-nilpotent in $Hom(E_\xi,E_\xi)$). Thus it is a product of matrix algebras, which means that the bundle E decomposes as an orthogonal direct sum of YM bundles with the same curvature.

If further E is indecomposable, then

$$h^0(Hom(E,E)) = 1$$

so by RR

$$1 - h^1(Hom(E,E)) = \overset{0}{\overset{\shortparallel}{\deg}} + \overset{n^2}{\overset{\shortparallel}{\text{rank}}}(1-g)$$

or

$$h^1(Hom(E,E)) = n^2(g-1) - 1$$

From standard deformation theory one knows that deformations of vector bundles over curves are formally unobstructed, and that the tangent space for a universal family is $H^1(X, \underline{Hom}(E,E))$. We've also seen that in the sequence

$$\tilde{\mathfrak{g}} \xrightarrow{\tilde{D}} \tilde{\mathfrak{g}} \otimes T^* \xrightarrow{\tilde{D}} \tilde{\mathfrak{g}} \otimes \Lambda^2 T^*$$

the H^1 gives the tangent space to D in the space of YM connections modulo the tangent space to the gauge orbit. Since $\tilde{\mathfrak{g}}$ = skew-hermitian part of $Hom(E,E)$, it's clear that

$$\begin{matrix}\text{normal space to} \\ \text{gauge orbit of } D \\ \text{in YM connections}\end{matrix} \cong H^1(Hom(E,E)) \quad \begin{matrix}\text{tangent space to} \\ E \text{ in moduli} \\ \text{space}\end{matrix}$$

In the case where $X = \mathbb{CP}^1$, $\pi_1(X) = 0$ so that $H^0(\mathrm{End}(E)) = \mathrm{End}(E_\xi)$ is a matrix ring for a YM bundle with scalar F. Thus the indecomposable YM bundles are line bundles with connection having curvature = multiple of volume form.

$\therefore$ Over the projective line a YM bundle is an orthogonal direct sum of 1-dimensional YM bundles, and ~~[illegible] YM bundle is [illegible] bundle with connection~~ there is only one YM line bundle of each degree.

I still don't understand how to analyze the normal structure to a YM bundle having an F which is not scalar. According to Bott the second variation should have negative eigenvalues, and this has to be used to understand the cohomology of the moduli spaces.

Let's go back to trying to understand the vacuum structure of a 2 space time diml gauge theory (after Coleman's notes).

Coleman considers gauge fields A over $\mathbb{R}^2$ and defines the action by

$$\int |F|^2 dx\,dy$$

where $dA + AA = F\,dx\,dy$. Thus he uses the volume $dx\,dy$ over $\mathbb{R}^2$ which does not extend to a volume on S^2.

Find standard volume on S^2. The bundle $\mathcal{O}(1)$ has the basic orthonormal frame consisting of $s_0 = 1$, $s_\infty = z$, and so the metric is

$$\|s_0\|^2 = \frac{1}{1+|z|^2}$$

If A_0 is the connection form: $\quad D s_0 = s_0 A_0 \qquad A_0 \in \Gamma(T^{1,0})$

$$d\|s_0\|^2 = \langle s_0 | s_0 A_0 \rangle + \langle s_0 A_0 | s_0 \rangle = \|s_0\|^2 (A_0 + \bar{A}_0)$$

So $\quad A_0 = \partial \log \|s_0\|^2 = \partial \log \frac{1}{1+|z|^2} = -\frac{\bar{z}\,dz}{1+|z|^2}$

Check: $\quad A_\infty = \partial \log \|s_\infty\|^2 = \partial \log \frac{|z|^2}{1+|z|^2} = \frac{dz}{z} - \frac{\bar{z}\,dz}{1+|z|^2}$

Thus $\quad D^2 = dA_0 = \left(-\frac{1}{1+\bar{z}} + \bar{z}\frac{1}{(1+|z|^2)^2} z\right) d\bar{z}\,dz$

$$= \frac{dz\,d\bar{z}}{(1+|z|^2)^2} = \frac{-2i\,dx\,dy}{(1+|z|^2)^2}$$

Check $\quad \int \frac{i}{2\pi} D^2 = \frac{1}{\pi}\iint \frac{r\,dr\,d\theta}{(1+r^2)^2} = \frac{2\pi}{\pi}\int_0^\infty \frac{r\,dr}{(1+r^2)^2} = \left.\frac{-1}{1+r^2}\right|_0^\infty = 1$

which is the degree of $O(1)$.

So the standard volume on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ is

$$\frac{1}{\pi}\frac{dx\,dy}{(1+r^2)^2} \qquad \left(\begin{array}{c}\text{this is}\\ \text{normalized so}\\ \text{that } \int = 1\end{array}\right)$$

Look: We know that a YM field A can be gauge transformed until F is a constant diagonal matrix. Thus if we define the action using the volume $dx\,dy$ we won't get any finite action YM fields.

January 12, 1981

Think of S^4 as the projective line over the quaternions and cover it by

$$U: \quad x \neq \infty$$
$$V: \quad x \neq 0.$$

We take the canonical line bundle $O(1)$ with

$$s_U = 1 \qquad s_V = x$$

and transition function $g_{UV} = x$. (Think by analogy with $S^2 = \mathbb{C} \cup \{\infty\}$.) Now in general given connections

$$D_U^\circ = s_U (d + A_U^\circ) s_U^{-1} \quad \text{on } U$$

and similarly on V we can piece them via a partition of unity $\rho_U + \rho_V = 1$ to get a connection

$$\rho_U D_U^\circ + \rho_V D_V^\circ \underset{\text{over } U}{=} s_U \Big(\underbrace{\rho_U (d + A_U^\circ) + g_{UV} \rho_V (d + A_V^\circ) g_{VU}}_{d + A_U} \Big) s_U^{-1}$$

hence

$$A_U = \rho_U A_U^\circ + \rho_V (g_{UV} A_V^\circ g_{VU} + g_{UV}\, dg_{VU})$$

So if we start with the trivial connections over U, V we get

$$\begin{cases} A_U = \rho_V\, g_{UV}\, dg_{VU} \\ A_V = \rho_U\, g_{VU}\, dg_{UV} \end{cases}$$

Thus in our case

$$A_U = \rho_V\, x\, dx^{-1}$$

$$A_V = \rho_U\, x^{-1} dx$$

The simplest choice for ρ is $\quad \rho_U = \dfrac{1}{1+r^2} \quad \rho_V = \dfrac{r^2}{1+r^2}$

where $r^2 = x^* x$. (Recall a quaternion x is $a + bj$ where $a, b \in \mathbb{C}$ and that $x^* = \bar{a} - j\bar{b} = \bar{a} - bj$ so that $x^* x = (\bar{a} - bj)(a + bj) = |a|^2 + \bar{a}bj - \underbrace{bja}_{\bar{a}j} - \underbrace{bjbj}_{\bar{b}j} = |a|^2 + |b|^2$)

$x x^* = (a + bj)(\bar{a} - bj) = |a|^2 + |b|^2$

Thus

$$A_u = \frac{r^2}{1 + r^2}\, x\, dx^{-1} = x(-x^{-1} dx\, x^{-1})\, x x^* \frac{1}{1+r^2}$$

$$A_u = -dx\, x^* \frac{1}{1+r^2} \qquad \text{regular at } x = 0$$

$$A_V = \frac{1}{1+r^2}\, x^{-1} dx$$

Now compute the curvature:

$$\begin{array}{r} x\,dx^* + dx\,x^* \\ \| \end{array}$$

$$dA_u = dx\, dx^* \frac{1}{1+r^2} + dx\, x^* \left(-\frac{1}{(1+r^2)^2}\right) d(r^2)$$

$$A_u A_u = dx\, x^* dx\, \cancel{x}^* \frac{1}{(1+r^2)^2}$$

$$\therefore dA_u + A_u A_u = dx\, dx^* \left[\frac{1}{1+r^2} - \frac{r^2}{(1+r^2)^2}\right]$$

$$= \frac{dx\, dx^*}{(1+r^2)^2}$$

Now $dx\, dx^* = (da + db\, j)(d\bar{a} - j\, d\bar{b}) = da\, d\bar{a} + db\, d\bar{b}$

$$\cancel{db\, j\, d\bar{a}} - \cancel{da\, d\bar{b}\, j}$$

hence this curvature form is [redacted] scalar-valued. It doesn't seem to be invariant under parallel translation. In effect $D_\nu = \partial_\nu + [A_\nu, \cdot\,]$, so since the form is scalar valued the bracket term in $D_\mu F_{\mu\nu}$ is zero. ~~[illegible]~~. Now

$$dx\, dx^* = da\, d\bar{a} + db\, d\bar{b} = -2i(dx^0 dx^1 + dx^2 dx^3)$$

so only $F_{\mu\nu} \neq 0$ for $\{\mu, \nu\} = \{0, 1\}$ or $\{2, 3\}$; so $\partial_1 F_{10} = 0$ which is false.

However I have been computing the curvature of a connection that might not preserve the inner product. I ought to see if the form A_u has values in the correct Lie algebra, which is the Lie algebra of $S^3 \simeq SU(2)$. I am thinking of quaternions as 2×2 complex matrices, so that

$$a + bj = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$$

Think of $\mathbb{H}$ as a right $\mathbb{C}$ vector space with basis $1, j$ then multiplication by j is the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$j(1 \;\; j) = (j \;\; -1) = (1 \;\; j)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

[redacted]

$$\boxed{\cancel{(a+bj)^* = \bar{a} - bj = \begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix}}}$$

$$a(1 \;\; j) = (a \;\; aj) = (a \;\; j\bar{a}) = (1 \;\; j)\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$$

~~so~~ $a + bj$ corresponds to

$$\begin{pmatrix} a & \\ & \bar{a} \end{pmatrix} + \begin{pmatrix} b & \\ & \bar{b} \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} a & -b \\ \bar{b} & \bar{a} \end{pmatrix}$$

as above. [redacted] Thus

$$x^0 + x^1 i + x^2 j + x^3 k = \begin{pmatrix} x^0 + x^1 i & -x^2 - x^3 i \\ x^2 - x^3 i & x^0 - x^1 i \end{pmatrix}$$

$$= x^0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x^1\begin{pmatrix} i & \\ & -i \end{pmatrix} + x^2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + x^3\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

Unfortunately this is not the preferred notation:

$$x^4 + i\vec{x}\cdot\vec{\sigma} = x^4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x^1\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + x^2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + x^3\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$