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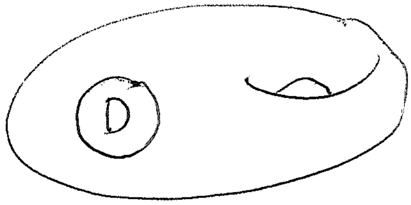
May 26, 1987

Back to bosonization. Here's Graeme's ideas on how to do QFT for the holomorphic functions. Take the usual picture. On the circle we have the space $C^\infty(S)$ of smooth functions with the skew form $\int f\,dg$. Look at the subspace $\mathcal{O}(\Sigma - D)$ of boundary values of holomorphic functions.



This is isotropic. We have

$$\mathcal{O}(\Sigma-D) \subset \left\{ f \in C^\infty(S) \;\middle|\; \int f\omega = 0 \quad \forall \omega \in \Omega^1(\Sigma-D) \right\} = V_1$$

$$\subset \left\{ f \in C^\infty(S) \;\middle|\; \int f\,dg = 0 \quad \forall g \in \Omega^0(\Sigma-D) \right\} = V_0$$

Then V_0 is the annihilator of $\mathcal{O}(\Sigma-D)$ with respect to the skew form. Note that

$$V_0/V_1 = \text{dual of } \Omega^1(\Sigma-D)/d\,\Omega^0(\Sigma-D)$$

But the holomorphic DR complex over $\Sigma - D$ has the cohomology groups $\mathbb{C}$, $H^1(\Sigma-D,\mathbb{C}) = H^1(\Sigma,\mathbb{C})$.

This shows that $\mathcal{O}(\Sigma-D)$ is not maximal isotropic, and also suggests what to do. The space $H^1(\Sigma,\mathbb{C})$ has a symplectic form given by the pairing to $H^2(\Sigma,\mathbb{C})$, and ~~the~~ subspace $H^{1,0}$, $H^{0,1}$ are maximal isotropic. Thus we can define something intermediate between V_1 and V_0.

Unfortunately this Hodge decomposition on $H^1(\Sigma-D,\mathbb{C})$ that comes from $H^1(\Sigma-D,\mathbb{C}) \xleftarrow{\sim} H^1(\Sigma,\mathbb{C})$ might depends on the way the disk is filled in. You would like to define intrinsically a subspace of functions on the circle associated to just the surface $\Sigma - D$.

Thus can you define a space of differential forms on S containing $d\,\mathcal{O}(\Sigma-D)$ and contained in $\Omega^1(\Sigma-D)$. Two candidates are suggested: 1) $\Omega^1(\Sigma)$. 2) Since

$$\left(\Omega^1(\Sigma-D)/d\mathcal{O}(\Sigma-D)\right)^\vee = \left(H^1(\Sigma-D)\right)^\vee$$
$$= H_1(\Sigma-D)$$

one can take an isotropic subspace ~~of the latter~~, i.e. the a or b cycles.

Let's see if we can set this up more cleanly. Can I show that $\mathcal{O}(\Sigma-D) = V_1 = \{f \in C^\infty(S) \mid \int f\omega = 0$ for all $\omega \in \Omega^1(\Sigma-D)\}$?

Let's try to construct a QFT associated to holomorphic functions on a Riemann surface. We begin with the circle. We consider the smooth functions on the circle with the skew-symmetric form

$$\frac{1}{2\pi i}\int f\,dg$$

We form a Heisenberg algebra whose generators are $\rho(f)$ satisfying

$$\rho(f)^* = \rho(\bar{f})$$
$$[\rho(f),\rho(g)] = \frac{1}{2\pi i}\int f\,dg$$

There is a class of irreducible representations of this Weyl algebra. Each has a vacuum state which is an eigenvector for the $\rho(f)$ with f holom. in the disk. The representations may have different characters on the center which is spanned by $\rho(1)$.

Now we can obviously do the same thing for a finite number of circles.

Let's consider a connected Riemann surface Σ with boundary $\partial\Sigma = \coprod_{i=1}^{n} S_i$. We take an irreducible repn. $\mathcal{H}$ of the Heisenberg algebra with generators $\rho(f)$, $f \in C^\infty(\partial\Sigma)$. The problem is to assign a line in this Hilbert space $\mathcal{H}$ which is associated to the holomorphic functions $\mathcal{O}(\Sigma)$. We want this line to be killed by the operators $\rho(f)$ with f the boundary value of an element of $\mathcal{O}(\Sigma)$. Actually we want ~~this~~ line to be spanned by an eigenvector for these operators.

~~To fix the ideas suppose $\partial\Sigma = S$ is a single circle.~~ $\mathcal{O}(\Sigma)$ is an isotropic subspace of $C^\infty(\partial\Sigma)$, but it is not a maximal isotropic subspace. We wish to extend it to a maximal isotropic subspace. Let V be its annihilator relative to the skew pairing on $C^\infty(\partial\Sigma)$. V consists of f such that

$$\int_{\partial\Sigma} f\,dg = 0 \qquad \text{for all } g \in \mathcal{O}(\Sigma)$$

First note that V contains $\mathcal{O}(\Sigma)$ because of Stoke's thm. and the fact that holom. 1-forms are closed. Also note that V contains the locally constant functions on $\partial\Sigma$, denote this

$$H^0(\partial\Sigma, \mathbb{C})$$

and also the subspace

$$V_1 = \left\{ f \in C^\infty(\partial\Sigma) \,\middle|\, \int_{\partial\Sigma} f\omega = 0 \quad \forall \omega \in \Omega^1(\Sigma) \right\}$$

where $\Omega^1(\Sigma)$ is the space of holom. 1-forms on Σ.
We have the inclusions

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, phantom, "\subset"] \arrow[dr, phantom, "\subset"] & \mathcal{O}(\Sigma) \arrow[r, phantom, "\subset"] & V_1 \arrow[r, phantom, "\subset"] & V \\
& H^0(\partial\Sigma, \mathbb{C}) \arrow[urr, phantom, "\subset"] & &
\end{tikzcd}
$$

Now we also know that the cohomology $H^1_{DR}(\Sigma)$ is given by the holom. DR complex

$$
0 \longrightarrow \underset{\substack{\parallel \\ \mathbb{C}}}{H^0_{DR}(\Sigma)} \longrightarrow \mathcal{O}(\Sigma) \xrightarrow{d} \Omega^1(\Sigma) \longrightarrow \underset{\substack{\parallel \\ \mathbb{C}^{2g+r-1}}}{H^1_{DR}(\Sigma)} \longrightarrow 0
$$

By duality V/V_1 is dual to $\Omega^1(\Sigma)/d\mathcal{O}(\Sigma)$, hence

$$V/V_1 = H_1(\Sigma)\otimes\mathbb{C}$$

$$\dim(V/V_1) = 2g+r-1.$$

But now if we fill in the disks we find that the complex

$$\mathcal{O}(\Sigma) \oplus \prod_{i=1}^{r}\mathcal{O}(D_i) \longrightarrow \prod_{i=1}^{r} C^\infty(S_i)$$

gives the cohomology $H^i(\Sigma \cup \bigcup D_i, \mathcal{O})$. So the kernel + cokernel have dimension $1, g$ respectively. Thus we expect a max isotropic subspace of $C^\infty(\partial\Sigma)$ containing $\mathcal{O}(\Sigma)$ to have extra dimension g. ??

$H^0(\partial\Sigma, \mathbb{C})$ is the kernel of the skew pairing so it must be contained in any maximal isotropic subspace. Thus a max. isotropic W

which contains $\Theta(\Sigma)$ contains also $\Theta(\Sigma) + H^0(\partial\Sigma, \mathbb{C})$. The conjectured picture is

$$H_1(\Sigma,\mathbb{C}) \left\{ \begin{array}{c} V \\ \cup \\ W \\ \cup \\ \Theta(\Sigma) \oplus H^0(\partial\Sigma,\mathbb{C}) \\ \cup \\ \Theta(\hat{\Sigma}) = V_1 \end{array} \right. \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} H_1(\hat{\Sigma},\mathbb{C}) \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} H_1(\partial\Sigma,\mathbb{C})/\mathbb{C} \end{array}$$

In any case it should be clear that $H_1(\Sigma,\mathbb{C})$ after killing the image of $H_1(\partial\Sigma,\mathbb{C})$ should have a natural symplectic ~~structure~~. W should then correspond to a maximal isotropic subspace.

Thus to get a state in the irreducible representation of the Heisenberg algebra gen. by $C^\infty(\partial\Sigma)$ with a given character on $H^0(\partial\Sigma,\mathbb{C})$ we must have choosen a maximal isotropic subspace of $H_1(\Sigma,\mathbb{C})$.

The problem now is to really understand the new functions on $\partial\Sigma$ which must be added to the holomorphic functions to get a maximal isotropic subspace of $C^\infty(\partial\Sigma)$. In the disconnected case we add locally constant functions. How about the integrals of holomorphic 1-forms?

Take $r=1$. I pick ~~an isotrop~~ an element γ of $H_1(\Sigma,\mathbb{C})$ and corresponding to γ should be a new function on the boundary, call it f such that

$$\int_{\partial\Sigma} f\omega = 0$$

for all $\omega \in \Omega^1(\Sigma)$ such that $\int_\gamma \omega = 0$.

May 28, 1987

Log-harmonic functions. These are functions $\gamma : \Sigma \longrightarrow \mathbb{C}^\times$ such that $\log \gamma$ is a harmonic function on the Riemann surface Σ. (Notice: It is not meaningful to speak of a harmonic map between Riemann surfaces unless metrics are given. ~~It's~~ not true that f holom, u harmonic $\Longrightarrow$ $f \circ u$ harmonic, e.g. $z^2 \circ x = x^2$.)

We have to start on the circle. Configurations are smooth maps $\gamma : S^1 \longrightarrow \mathbb{T}$. Think of S^1 as $\mathbb{R}/2\pi\mathbb{Z}$, then we can write

$$\gamma(x) = e^{if(x)}$$

where f is a real function on $\mathbb{R}$ unique up to an additive constant in $2\pi\mathbb{Z}$ such that

$$f(x+2\pi) - f(x) \in 2\pi\mathbb{Z}.$$

Then $\frac{1}{2\pi}\left[f(x+2\pi) - f(x)\right] = \text{degree } \gamma.$

Before we looked at or we took the configurations to be real fns. $f : S^1 \longrightarrow \mathbb{R}$. The difference now is that we replace the zero mode, which gives the constant term $\frac{1}{2\pi}\int f(x)dx \in \mathbb{R}$, by a pair in $\mathbb{R}/2\pi \times \mathbb{Z}$. I mean by this that when we look at the Fourier series expansion of f, then what is killed by $-\partial_x^2$ is now a linear function

$$ax + b$$

where $a \in \mathbb{Z}$, and b is well-defined only mod $2\pi\mathbb{Z}$.

It's clear that the main difference between the harmonic theory and the log harmonic theory is going to be the presence of this zero mode.

Let's try then to understand the difference between free motion $H = \frac{p^2}{2}$ of a particle on the line on one hand and on the circle on the other. Better, let's study the latter. When we do the quantum mechanics, the momentum is quantized, but the quantization is not unique. It can be twisted by an arbitrary flat line bundle. What is the best way to describe this structure mathematically?

One forms the Heisenberg group which is a central extension of $S^1 \times \mathbb{Z}$.

$$1 \longrightarrow \mathbb{T} \longrightarrow \Gamma \longrightarrow S^1 \times \mathbb{Z} \longrightarrow 1$$

We can think of Γ as acting on $L^2(S^1)$; it consists of operators which are products of translation operator multiplication by characters of S^1, and scalars of abs. value 1. As usual a point of momentum space, i.e. $n \in \mathbb{Z}$, gives rise to the position-like operator e^{inx}, and a point of configuration space $a \in \mathbb{R}/2\pi\mathbb{Z}$ gives rise to a momentum-like operator $a\partial_x$.

I guess I want to think of a quantization as consisting of a given definite choice for the position operator corresponding to the function e^{ix} and a definite choice of momentum operator

$$p = \frac{1}{i}\partial_x + c$$

The different choices for c modulo $\mathbb{Z}$ will lead to different quantizations.

The only natural vectors in the Hilbert space are the eigenvectors for p, i.e. the exponential functions e^{inx}, $n \in \mathbb{Z}$. The eigenvectors for e^{ix} are δ-functions and so not in the Hilbert space. However under the imaginary time evolution $e^{-tH} = e^{-t\frac{p^2}{2}}$, which is the heat operator, these become "normalizable" states.

Maximal abelian subgroups of Γ correspond to maximal isotropic subgps of $\mathbb{T} \times \mathbb{Z}$. These are $\mathbb{T} = S^1$ and cyclic groups with generators $(\zeta, 1)$ for $\zeta \in \mathbb{T}$. Only the eigenvectors for the inverse image of $\mathbb{T}$ in Γ are normalizable.

So far we have discussed S^1 as a configuration space, whereas we actually have $S^1 \times \mathbb{Z}$ as zero mode configurations. This makes things appear doubly difficult.

It seems that the first thing to check carefully is that when we view the circle as the boundary of the disk there is a distinguished line in the Fock space

May 29, 1987

We have a few days to find the details on bosonization, and thereby atone for the sins of yesterday's talk. Here's the program:

We start with the basic representation of the loop group $L\mathbb{T}$ or more precisely of its central extension $\tilde{L}\mathbb{T}$. This is constructed once and for all over the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. We then take a Riemann surface X with boundary whose boundary is identified with S^1. Thus we can now take invertible functions on X and identify their boundary values with elements of $L\mathbb{C}^*$. In particular we can look at the subgroup which is the image of

$$\mathcal{O}^\times(X) \longrightarrow L\mathbb{C}^\times$$

(If X is connected this is an injection.)

Next we recall the structure of $\mathcal{O}^\times(X)$:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}(X) \longrightarrow \mathcal{O}^\times(X) \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow 0$$

We restrict the central extension $\tilde{L}\mathbb{C}^\times$ over $\mathcal{O}^\times(X)$ where it splits. However the splittings form an affine space over the group of characters of $\mathcal{O}^\times(X)$.

We propose to fix one lifting and then look at vectors in the Fock space which are eigenvectors under the lifted subgroup. It is the same as varying the lifting and looking for fixed vectors under the different lifted subgroups. There should be at most one eigenvector for a given character of $\mathcal{O}^\times(X)$.

What is confusing is the fact that there are so many charaters of $\mathcal{O}(X)/\mathbb{Z}$ which seem unrelated to the desired answer. We want the different lines in Fock space that we obtain to correspond to the different line bundles on $X \cup D$.

In other words when we construct the Fock space initially [redacted] by fermion means, we [redacted] start with a line bundle on the circle, namely the trivial line bundle. Fock space is then the renormalized exterior tensor space of the space of sections of this line bundle on the circle. Then given a holom. line bundle ξ on $X \cup D$ with a trivialization over D, we get the subspace W of holom. sections of ξ over X. The line in Fock space is then characterized by being killed by the [redacted] $c(s)$, $b(t)$ operators for $s \in \Gamma(X, \xi)$, [redacted] $t \in \Gamma(X, \xi^{-1} \otimes K)$. ($c(s)$ is exterior mult. by s and $b(t)$ is interior mult. by t.)

We have to understand what goes on in the case where $X \cup D = S^2$. Notice that in the simple harmonic oscillator situation, if one asks for eigenvectors for the annihilation operator a, then one gets the coherent states $e^{\lambda a^*}|0\rangle$. This means that if we look for the different eigenvectors of $\mathcal{O}^\times(X)$, then we are getting "translates" of the states we want.

[redacted] Now the problem appears to be involved with the actual basic loop group representation. It's not enough to say you have a representation of the central extension $\tilde{L}\mathbb{T}$ since that doesn't assign operators to specific loops or ~~functions~~. When one works with the CCR one asks for specific operators to be attached to elements of the symplectic vector space. Thus you give more than the central extension, you give actual coset representatives, i.e. a lifting of $L\mathbb{T}$ into $\widetilde{L\mathbb{T}}$. Notice that there is no topological obstruction since $L\mathbb{T} \sim \mathbb{Z} \times S^1$, so circle bundles are trivial.

May 31, 1987

Discussion of bosonization. We start with the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. From the complex Hilbert space $L^2(S^1)$ we construct a Clifford algebra with generators $\psi^*(f)$, $\psi(f)$ satisfying the CAR. There is a canonical irreducible $*$ repn. $\mathcal{F}$ of this Clifford algebra. $\mathcal{F}$ is a renormalized version of $\Lambda L^2(S^1)$. The loop group $L\mathbb{T}$ acts on the Clifford algebra and this leads to an action of a central extension $\tilde{L}\mathbb{T}$ called the extended loop group, on $\mathcal{F}$.

It will be necessary soon to have precise control over elements of $\tilde{L}\mathbb{T}$. Notice that at the moment we do not have a CCR setup, i.e. we do not have operators on $\mathcal{F}$ attached to elements of $L\mathbb{T}$. Such a thing amounts to a cross-section of the map $\tilde{L}\mathbb{T} \to L\mathbb{T}$.

Next we consider a conn. Riemann surface X with boundary ∂X which is a single circle. We choose a parametrization $\partial X \xrightarrow{\sim} S^1$; this is needed if I want to continue working in the Hilbert space framework. Then functions on ∂X determine fermion operators on $\mathcal{F}$.

Now let ξ be a holomorphic line bundle on X extending the trivial line bundle on ∂X. Better, let ξ be a holomorphic line bundle on X ~~[illegible]~~ equipped with a trivialization over ∂X. Then the space $\Gamma(X,\xi)$ of holomorphic sections of ξ determines a line in $\mathcal{F}$. It is the unique line killed by $\psi^*(f)$, $f \in \Gamma(X,\xi)$ and $\psi(g)$, $g \in \Gamma(X,\xi)^\perp$. Denote this line by $\mathbb{C}|\xi\rangle$.

The natural question is ~~[illegible]~~ how to describe these lines. First note that any holomorphic line bundle on X is trivial. Hence we are considering the different trivializations of the trivial line bundle over ∂X, i.e. elements of the complex loop group $L\mathbb{C}^\times$. Up to isomorphism, ξ is just an element of $L\mathbb{C}^\times/\mathbb{C}^\times$.

I don't believe that $L\mathbb{C}^\times$ acts (projectively) on $\mathcal{F}$, but it does act on certain vectors. For example $e^{c_1 a^* + c_2 a}|0\rangle$ is well-defined. So I guess we can say the set of lines $\mathbb{C}|\xi\rangle$ is an orbit of $L\mathbb{C}^\times$ in $\mathbb{P}(\mathcal{F})$.

Let's take another viewpoint. Another way to describe a ξ is by means of a holomorphic structure on the trivial C^∞ line bundle, that is, by a $\bar\partial$-operator $\bar\partial + \alpha$ where $\alpha \in \Omega^{0,1}(X)$. Isomorphisms are given by complex gauge transformations which are 1 on ∂X. This leads to the same result.

Next we consider the boson picture. I presume that there is a central extension $\tilde{L}\mathbb{C}^\times$ of $L\mathbb{C}^\times$ by $\mathbb{C}^\times$, whose commutator pairing is given by Deligne's dilog construction. Then this extension becomes trivial over $\Gamma(X, \mathcal{O}^\times)$. The possible ways of lifting $\Gamma(X,\mathcal{O}^\times)$ into $\tilde{L}\mathbb{C}^\times$ form a torsor over $\mathrm{Hom}(\Gamma(X,\mathcal{O}^\times), \mathbb{C}^\times)$.

Consider the s.h.o. The possible ways of lifting the isotropic subspace $\mathbb{C}a \subset \mathbb{C}p + \mathbb{C}q$ into the Heisenberg algebra $\mathbb{C}p + \mathbb{C}q + \mathbb{C}$ are of the form

$$za \longmapsto za + c_1 z + c_2 \bar z$$

We want complex linear liftings $za \longmapsto z(a-\lambda)$. Similarly in the loop group setup we want holomorphic characters $\mathrm{Hom}_{hol}(\Gamma(X,\mathcal{O}^*), \mathbb{C}^*)$.

A better approach will be to choose (or construct) a lifting of $\Gamma(X,\mathcal{O}^\times)$ into $\tilde{L}\mathbb{C}^\times$, and then look at the different eigenvectors for this group. Then the eigenvectors should be the different lines $\mathbb{C}|\xi\rangle$. Invariantly we can say that the inverse image of $\Gamma(X,\mathcal{O}^\times)$ in $\widetilde{L}\mathbb{C}^\times$ is abelian and the lines $\mathbb{C}|\xi\rangle$ are its eigenvectors in the same way that the coherent states are the eigenvectors of a.

Look at the cohomology sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma(X,\mathcal{O}) \longrightarrow \Gamma(X,\mathcal{O}^\times) \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow 0$$

$$0 \longrightarrow H^1(X,\mathcal{O}^\times) \longrightarrow \underset{\overset{\|}{0}}{H^2(X,\mathbb{Z})} \longrightarrow 0$$

Thus

$$0 \longrightarrow \Gamma(X,\mathcal{O})/\mathbb{Z} \longrightarrow \Gamma(X,\mathcal{O}^\times) \longrightarrow H^1(X,\mathbb{Z}) \to 0$$

$$0 \longrightarrow \mathbb{C}^\times \longrightarrow \Gamma(X,\mathcal{O})/\mathbb{Z} \longrightarrow \Gamma(X,\mathcal{O})/\mathbb{C} \longrightarrow 0$$

Notice that because $\Gamma(X,\mathcal{O}^\times)$ preserves the subspace $\Gamma(X,\xi)$, the state $|\xi\rangle$ is an eigenvector for the inverse image $\widetilde{\Gamma}(X,\mathcal{O}^\times)$. Hence we get a lifting of $\Gamma(X,\mathcal{O}^\times)$ into $\widetilde{L}\mathbb{C}^\times$ by considering the subgroup of $\widetilde{\Gamma}(X,\mathcal{O}^\times)$ fixing $|\xi\rangle$. This is how we can proceed to define the explicit loop group operators g_n.

Let's review. We start with the Fock space on the circle which we can define as a certain irreducible representation of the CCR. We then get a representation of a central extension $\widetilde{L}\mathbb{C}^\times$ of the loop group $L\mathbb{C}^\times$. This central extension splits over $\Gamma(X,\mathcal{O}^\times)$. Actually it is more accurate to say that the inverse image $\widetilde{\Gamma(X,\mathcal{O}^\times)}$ of $\Gamma(X,\mathcal{O}^\times)$

in $\tilde{L}\mathbb{C}^\times$ is abelian. In fact we should consider the extension

$$0 \longrightarrow \mathbb{C}^\times \longrightarrow \widetilde{\Gamma(X,\mathcal{O}^\times)} \longrightarrow \Gamma(X,\mathcal{O}^\times) \longrightarrow 0$$

carefully. Certainly this splits because there is no topological obstruction. And any of the eigenvectors $|\xi\rangle$ determines a splitting.

Question: Is $\widetilde{\Gamma(X,\mathcal{O}^\times)}$ maximal abelian in $\tilde{L}\mathbb{C}^\times$, or equivalently is $\Gamma(X,\mathcal{O}^\times)$ its own annihilator in $L\mathbb{C}^\times$?

Any $|\xi\rangle$ determines a lifting of $\Gamma(X,\mathcal{O}^\times)$, in fact any eigenvector of $\tilde{\Gamma}$ does. Question: Are the eigenvectors $|\xi\rangle$ all the eigenvectors of $\tilde{\Gamma}$?

Notice that $L\mathbb{C}^\times$ acts on $\tilde{\Gamma}$ and hence on the eigenvectors. Irreducibility and the fact that $\tilde{\Gamma}$ is maximal abelian would indicate that a single orbit of characters of $\tilde{\Gamma}$ should occur. This is all heuristic as $\tilde{L}\mathbb{C}^\times$ doesn't act on the Hilbert space.

Repeat: We have a well understood fermionic business which we want to translate entirely in bosonic terms. As a first attempt one can proceed as in Graeme's lecture Wednesday. One has a real symplectic vector space $V = C^\infty(S^1,\mathbb{R})/\mathbb{R}$ with $\int f\,dg$, and one can form the canonical representation of the CCR. Then $\Gamma(X,\mathcal{O})/\mathbb{C}$ is an isotropic subspace of V, which is maximal when the genus is zero. Because of the way the Fock space has been defined, we are given a lifting of $\Gamma(X,\mathcal{O})/\mathbb{C}$ into operators. So in genus zero we have a unique state killed by the operators belonging to $\Gamma(X,\mathcal{O})/\mathbb{C}$ and the other eigenvectors for $\widetilde{\Gamma(X,\mathcal{O})/\mathbb{C}}$ are coherent states. In higher genus $W = \Gamma(X,\mathcal{O})/\mathbb{C}$ is not maximal isotropic, rather

$$W^\circ/W = H^1(X,\mathbb{C}) \cong \mathbb{C}^{2g}$$

It is possible that there is a problem with real structures. Let's check this. To each $f \in C^\infty(S^1)/\mathbb{C}$ we have an operator $\phi(f)$ attached which satisfies $\phi(f)^* = \phi(\bar{f})$, and we have

$$[\phi(f), \phi(g)] = \frac{i}{2\pi i}\int f\,dg$$

If $f, g \in \Gamma(X,\mathcal{O})$, then

$$[\phi(f), \phi(g)] = \frac{i}{2\pi i}\int f\,dg = \frac{1}{2\pi i}\iint df\,dg = 0$$

$$[\phi(f), \phi(\bar{f})] = \frac{i}{2\pi i}\iint df\,d\bar{f} > 0 \qquad (f \notin \mathbb{C})$$

Thus $\phi(\Gamma(X,\mathcal{O})/\mathbb{C})$ should be extendable to a ~~space~~ of annihilation operators, maybe I should say, maximal isotropic positive subspace. Consider

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \oplus \bar{\mathcal{O}} \longrightarrow \mathcal{H} \longrightarrow 0$$

This gives

$$0 \longrightarrow \mathbb{C} \longrightarrow \Gamma(X,\mathcal{O}) \oplus \Gamma(X,\bar{\mathcal{O}}) \longrightarrow \Gamma(X,\mathcal{H}) \to H^1(X,\mathbb{C}) \to 0$$

It follows that the orthogonal, or rather the annihilator to the subspace $\left(\Gamma(X,\mathcal{O})/\mathbb{C} + \overline{\Gamma(X,\mathcal{O})/\mathbb{C}}\right)_{real}$ in $C^\infty(S^1)_{real}/\mathbb{R}$ is a real symplectic space of dim $2g$. I forgot to mention that $\Gamma(X,\mathcal{H}) = C^\infty(S^1)$ by the solution to the Dirichlet problem, so the above gives

$$0 \longrightarrow \Gamma(X,\mathcal{O})/\mathbb{C} \oplus \Gamma(X,\bar{\mathcal{O}})/\mathbb{C} \longrightarrow C^\infty(S^1)/\mathbb{C} \to H^1(X,\mathbb{C}) \to 0$$

Here is how to describe the map $\Gamma(X,\mathcal{H}) \to H^1(X,\mathbb{C})$. We have exact sequences of sheaves

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{O}\oplus\bar{\mathcal{O}} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{O} \arrow[r] & K \arrow[r] & 0 \end{tikzcd}$$

which yield

$$\begin{tikzcd} & \Gamma(X,\mathcal{H}) \arrow[r] \arrow[d, "\partial"] & H^1(X,\mathbb{C}) \arrow[d, equal] & \\ \Gamma(X,\mathcal{O}) \arrow[r, "d"] & \Gamma(X,K) \arrow[r] & H^1(X,\mathbb{C}) \arrow[r] & 0 \end{tikzcd}$$

So the map from $\Gamma(X,\mathcal{H})$ to $H^1(X,\mathbb{C})$ takes a harmonic function f to the holom. 1-form ∂f which then is a closed 1-form on X. Note that because of

$$0 \longrightarrow \bar{\mathcal{O}} \longrightarrow \mathcal{H} \xrightarrow{\partial} K \longrightarrow 0$$

we have
$$0 \to \Gamma(X,\bar{\mathcal{O}}) \to \Gamma(X,\mathcal{H}) \xrightarrow{\partial} \Gamma(X,K) \to 0$$

A natural question now is what is the induced "real" symplectic structure on $H^1(X,\mathbb{C})$ which comes from the fact that it is quotient of $C^\infty(S^1)/\mathbb{C}$ by a "real" subspace on which the skew form is non-degenerate. Since we have the real subsheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow (\mathcal{O}\oplus\bar{\mathcal{O}})_{real} \longrightarrow \mathcal{H}_{real} \longrightarrow 0$$

it is clear that the "real" structure on $H^1(X,\mathbb{C})$ is given by the real subspace $H^1(X,\mathbb{R})$.

What must be happening is this. We take a real harmonic function f and form $*df$ which is also a real form. It is closed because f is harmonic, so it determines an element of $H^1(X,\mathbb{R})$. If f is holomorphic or anti-holom, then $*df = \pm i\,df$ and the cohomology class is zero.

Review: Starting from $C^\infty(S^1)/\mathbb{C}$ with its real symplectic structure we get a Weyl algebra acting on a Fock space. The subspace $\Gamma(X,\mathcal{O})/\mathbb{C}$ is isotropic and positive, so we have an orthogonal decomposition

$$\left(\Gamma(X,\mathcal{O})/\mathbb{C} \oplus \Gamma(X,\bar{\mathcal{O}})/\mathbb{C}\right) \oplus B = C^\infty(S^1)/\mathbb{C}$$

where B is a space with real symplectic structure which is isomorphic to $H^1(X,\mathbb{C})$ at least as a space with the real structure $H^1(X,\mathbb{R})$.

Let us defer the problem of the symplectic form. This subspace B_{real} is canonical and canonically isom. to $H^1(X,\mathbb{R})$, so there shouldn't be too much problem.

So we reach Graeme's problem where we are left with the Weyl algebra of $B \simeq H^1(X,\mathbb{R})$ acting irreducibly on the subspace of $\mathcal{F}$ annihilated by the $\phi(f)$, $f \in \Gamma(X,\mathcal{O})/\mathbb{C}$. As there is nothing canonical in $H^1(X,\mathbb{R})$ except $H^1(X,\mathbb{Z})$, it's not clear how to obtain normalizable states in $\mathcal{F}$.

Now we must look at the loop group situation and try to find out why this problem disappears. I think the basic difficulty in the loop group case is that we can't write down canonically the central extension. I can't say that Fock space is the canonical irreducible representation of a set of CCRs.

I think we have to return to the action of the extended loop group on fermion Fock space. want to first calculate the commutator pairing for the central extension.

Let's recall we have the operators associated to the Lie algebra of $L\mathbb{C}^\times$, i.e. functions on S^1. ρ_n = effect of multiplying by z^n, so $\rho_n = :\sum \psi^*_{k+n}\psi_k:$. Recall the normal ordering only affects ρ_0. Recall also that we have the shift operator τ such that $\tau e_s = e_{1+s}$. Then we have the relations

$$\tau^* = \tau^{-1}, \qquad \rho_n^* = \rho_{-n}$$

$$[\rho_{-n}, \rho_m] = m\,\delta_{n,m}$$

$$\tau^{-1}\rho_0\tau = \rho_0 + 1. \qquad \tau^{-1}\rho_n\tau = \rho_n$$

Now given a loop $f: S^1 \to \mathbb{T}$ we factor it

$$f(z) = z^m e^{c_0} e^{\sum_{n\neq 0} c_n z^n}$$

and define

$$\phi(f) = \tau^m e^{c_0\rho_0} e^{\sum_{n\neq 0} c_n \rho_n}$$

Notice this is well defined only if ρ_0 has integer eigenvalues, since c_0 is only defined up to $2\pi i\mathbb{Z}$.

Now we can calculate the commutator pairing.

$$\phi(f)\,\phi(g) = \underbrace{\tau^m e^{c_0\rho_0}\tau^{m'} e^{c_0'\rho_0}}_{\tau^{m+m'} e^{(c_0+c_0')\rho_0}\, e^{c_0 m'}} e^{\sum_{n\neq 0} c_n\rho_n} e^{\sum_{n\neq 0} c_n'\rho_n}$$

Now $$\Big[\sum_{n\neq 0} c_n\rho_n, \sum_{n\neq 0} c_n'\rho_n'\Big] = \sum_n n\, c_{-n} c_n'$$

and to understand this we can suppose $m = c_0 = m' = c_0' = 0$.

Then $$\frac{1}{2\pi i}\int \log f \,\frac{dg}{g} = \frac{1}{2\pi i}\int \sum c_n z^n \sum n c_n' z^{n-1}\,dz$$

$$= \sum n\, c_{-n} c_n'$$

Thus on the subgroup where $m=m'=c_0=c_0'=0$ the commutator pairing is

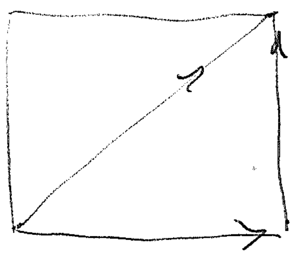
$$e^{\frac{1}{2\pi i}\int \log f \,\frac{dg}{g}}$$

and on the subgroup generated by z and constants the pairing is

$$\underbrace{z^m e^{c_0}}_{=f},\ \underbrace{z^{m'} e^{c_0'}}_{=g} \longmapsto e^{c_0 m' - c_0' m} = \frac{f^{\deg g}}{g^{\deg f}}(0)$$

As far as I can see it is not the tame symbol.

Note that Deligne's method gives for $f=g=z$ the monodromy of the canonical line bundle over $S^1\times S^1$ over the diagonal.



Since the axes have monodromy 1, applying Stokes's to the triangle gives a monodromy of $e^{i\pi}=-1$.

Alternate approach: Take the central extension of $L\mathbb{T}$ acting on fermion Fock space and restrict the action to the inverse image of the constant loops. This inverse image has to be abelian as the commutator pairing is necessarily trivial (the circle being "topologically cyclic"). So this inverse image is $S^1\times S^1$ and the characters occurring on the 2nd factor have to be integral.

Another point is that the commutator pairing has to vanish on cyclic groups.

So the conclusion is that the dilog pairing is not the commutator pairing. It seems to be the cocycle describing the central extension of the maximal torus of $L(SU(2))$ which acts on the basic representation.

Our next project will be to see if we can find the lines $\mathbb{C}|\xi\rangle$ in the fermion Fock space $\mathcal{F}$ in the boson picture. Recall we have the central extension $\tilde{L}\mathbb{T}$ acting on the fermion Fock space. The constant loops can be lifted to define a $\mathbb{Z}$ grading of $\mathcal{F}$. Fix the charge 0 sector and cut down ~~to~~ loops of degree 0. We then get an irreducible representation of a central extension of $C^\infty(S^1,\mathbb{R})/\mathbb{R}$ ~~as~~ Lie algebra.

I think I can the section $\mathcal{F}_0$ as being a representation of operator ρ_n $n \neq 0$ satisfying the standard relations. Recall we are trying to find a line in $\mathcal{F}_0$ which comes from a holomorphic line bundle ξ on X trivialized along S^1. We know this line has to be an eigenline for the inverse image of $\Gamma(X,\mathcal{O}^\times)$. Recall this consists of degree 0 loops.

June 1, 1987

<u>Problem</u>: Let us consider the trivial holomorphic line bundle over a Riemann surface X with parametrized boundary $\partial X = S^1$. In the fermion Fock space $\mathcal{F} = \Lambda^{\text{ren}} L^2(S^1)$ is a line $\mathbb{C}|\phi\rangle$ corresponding to the subspace of functions which extend holomorphically to X. There is a central extension $\tilde{L}\mathbb{C}^\times$ which "acts" on $\mathcal{F}$, and there is a unique lifting of $\Gamma(X, \mathcal{O}^\times)$ into $\tilde{L}\mathbb{C}^\times$ such that the lifted group fixes the vector $|\phi\rangle$, better, is such that $|\phi\rangle$ is fixed under the lifted group.

The problem is to construct the lifting of $\Gamma(X, \mathcal{O}^\times)$ into $\tilde{L}\mathbb{C}^\times$ directly. ~~Hopefully~~ it will involve some sort of operator like the operator T encountered in the harmonic theory. Recall this takes a function on the circle, extends it to a harmonic function, then takes the normal derivative on the boundary. We saw in the case of a real analytic boundary and parametrization that the operator T differs by a smooth kernel operator from the similar operator on the disk.

I expect to describe $\tilde{L}\mathbb{C}^\times$ in disk terms, i.e. relative to the natural exponential fns. determined by the parametrization. In particular I will have explicit operators attached to elements of $L\mathbb{C}^\times$, i.e. a cross section of $\tilde{L}\mathbb{C}^\times \longrightarrow L\mathbb{C}^\times$. I think I will be able to work on the Lie algebra level. First I can restrict to degree 0 loops, then to the sector which contains $|\phi\rangle$. This fixes the lifting over the constant loops.

So it seems that one has an 'irreducible' representation of a central extension of the Lie algebra of functions on S^1 mod constants - $C^\infty(S^1)/\mathbb{C}$. This is perfectly explicit using the operators ρ_n on F. Now we have to look at $\Gamma(X, \mathcal{O}^\times)/\mathbb{C}^\times \subset L^{\deg 0}\mathbb{C}^\times/\mathbb{C}^\times$ and construct a lifting to operators fixing $|\phi\rangle$.

On the Lie alg. level we have the operator ρ_n corresponding to the function z^n on the circle. Given a $f \in \Gamma(X, \mathcal{O})/\mathbb{C}$ we must find the eigenvalue of $\rho(f) = \sum_{n\neq 0} c_n \rho_n$ $\qquad f = \sum_{n\in\mathbb{Z}} c_n z^n$ acting on $|\phi\rangle$. Some sort of trace. I can find this eigenvalue as

$$\lambda(f) = \frac{\langle 0 | \rho(f) | \phi \rangle}{\langle 0 | \phi \rangle}$$

This must be pretty close to the τ function, whatever this is. This is approximately

$$\lambda(f) = \int f(z) \underbrace{\langle \psi^*(z)\psi(z) \rangle}_{\text{Green's function}}$$

June 2, 1987

Given a circle S we wish to see if we can produce a central extension of $C^\infty(S, \mathbb{C}^\times)$ which is unique up to canonical isomorphism. First choose a line bundle L over the circle. Then sections of L and sections of $L^{-1} \otimes \omega$ where ω is the bundle of densities can be paired, allowing us to form the Clifford algebra of $\Gamma(S, L) \oplus \Gamma(S, L^{-1} \otimes \omega)$. Secondly we choose an irreducible representation $\mathcal{F}$ of this Clifford algebra lying in the appropriate Shale class. Now given a loop $\gamma \in C^\infty(S, \mathbb{C}^\times)$ it acts on the Clifford algebra and there exist invertible operators T on $\mathcal{F}$ such that conjugation by T gives the action on the Clifford algebra. The central extension consists of those invertible operators on $\mathcal{F}$ such that conjugation by T coincides with the actions of an elt of $C^\infty(S, \mathbb{C}^\times)$.

Now we have made two choices and we want to see their effect. Any two line bundles are isomorphic. Let's take two line bundles L, L' and suppose we have Fock spaces $\mathcal{F}, \mathcal{F}'$. We choose an isom. $\alpha : L \to L'$, whence there exists a compatible isomorphism $\mathcal{F} \to \mathcal{F}'$, call it $\tilde{\alpha}$ which is unique up to a scalar. Then $\tilde{\alpha}$ induces an isomorphism $T \mapsto \tilde{\alpha} T \tilde{\alpha}^{-1}$ between the two central extensions of $L\mathbb{C}^\times$. If we choose another isom. $\beta : L \to L'$, then $\beta = \alpha\gamma$, where $\gamma \in L\mathbb{C}^\times$. We want to see that $\tilde{\beta} T \tilde{\beta}^{-1} = \tilde{\alpha} T \tilde{\alpha}^{-1}$ for T in the central extension given by $\mathcal{F}$. Now $\tilde{\beta}^{-1}\tilde{\alpha}$ belongs to this central extension and lies over γ. We want to see that $T = (\tilde{\beta}^{-1}\tilde{\alpha}) T (\tilde{\beta}^{-1}\tilde{\alpha})^{-1}$ and this is false.

So we learn that the central extension of $L\mathbb{C}^\times$ depends on the line bundle L.

Also I forgot to mention that one needs the orientation of the circle to specify the Shatten class. One uses a ψDO on the circle whose symbol is $\mathrm{sgn}(\xi)$, and this requires an orientation to be defined.

Therefore once I fix a line bundle on the circle and the orientation of the circle I can speak of "the" central extension of $L(\mathbb{C}^\times)$. I probably want to give a description of this extension in Segal-Wilson terms.

Now let's consider the situation where the circle is the boundary of X a R.S. since I must give the line bundle on the circle it is natural to give the extension to the surface. For the moment let us just concentrate on the trivial line bundle.

Take the trivial line bundle over the circle. This fixes a central extension of the loop group. Now there is a canonical lifting of $\Gamma(X, \mathcal{O}^\times)$ into this central extension which we have to describe.

The problem seems to be the following. We have a fixed central extension of $L(\mathbb{C}^\times)$. We have various descriptions of it, but for the moment, the most reliable is as a certain group of operators. I think this must be closely related to giving it via a cocycle.

Let's discuss the cocycle viewpoint. Given a cocycle on $L\mathbb{T}$ one writes down a group law on $\mathbb{T} \times L\mathbb{T}$ and gets a central extension. The cocycle also gives a set of commutation relations which have an essentially unique irred repn. The problem seems to be the following. Your cocycle seems to involve a param. of the circle, but the central extension you want comes from just the choice of the trivial line bundle on the circle.

Here's the crux of the bosenization problem. Start with a circle S and a complex line bundle ξ over it. Let $V = \Gamma(S, \xi)$. There should then be a Fock space $\mathcal{F}$ with operators $\psi^*(v) = v \wedge$ and $\psi(\lambda) = \lambda \lrcorner$ for all $v \in V = \Gamma(S, \xi)$ and all $\lambda \in \Gamma(S, \xi^{-1} \otimes T_S^*)$. Whenever we have a splitting $V = W_+ \oplus W_-$ defined by a ψDO F of order 0 $\ni F^2 = 1$ and $\sigma(F, \xi) = \operatorname{sign}(\xi)$ we have a line in $\mathcal{F}$ corresp. to W_- and a dual line corresp. to W_+.

Now let's look at multiplication by $f \in C^\infty(S)$ on V. Then there is an operator $\rho(f)$ on $\mathcal{F}$, unique up to an additive constant such that

$$[\rho(f), \psi^*(v)] = \psi^*(fv)$$
$$[\rho(f), \psi(\lambda)] = \psi(-f\lambda)$$

Thus we get a central extension $\tilde{L}\mathbb{C}$ of $L\mathbb{C}$ as Lie algebras.

Next let us fix a sector in $\mathcal{F}$, i.e. an eigenspace for $\rho(1)$. Then we can redefine $\rho(1)$ to make it 0 on this sector. This gives a lifting α of constant loops

$$\begin{tikzcd} & \tilde{L}\mathbb{C} \arrow[r, twoheadrightarrow] \arrow[d] & \tilde{L}\mathbb{C}/\alpha\mathbb{C} \arrow[d] \\ \mathbb{C} \arrow[r] \arrow[ur, dotted, "\alpha"] & L\mathbb{C} \arrow[r, twoheadrightarrow] & L\mathbb{C}/\mathbb{C} \end{tikzcd}$$

into the central extension. The image of α is normal because commutator pairing any elt of $L\mathbb{C}$ with constant loops is trivial. Thus we actually obtain a central extension $\tilde{L}\mathbb{C}/\alpha\mathbb{C}$ of $L\mathbb{C}/\mathbb{C}$ which acts on the sector, call it $\mathcal{F}^0$, of our Fock space $\mathcal{F}$.

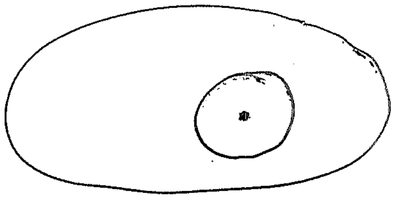
Now starting from $L\mathbb{C}/\mathbb{C}$ with its symplectic form we can construct a central extension with cross-section and an irreducible representation of the central extension. This gives the standard construction of the boson Fock space associated to $L\mathbb{C}/\mathbb{C}$.

<u>The crux of the problem</u> is that there is no natural identification between these central extensions, and hence no identification (up to scalars) of the space $\mathcal{F}^0$ with the standard boson Fock space. What is missing is a cross-section of the extension

$$\tilde{L}\mathbb{C}/\alpha\mathbb{C} \longrightarrow L\mathbb{C}/\mathbb{C}$$

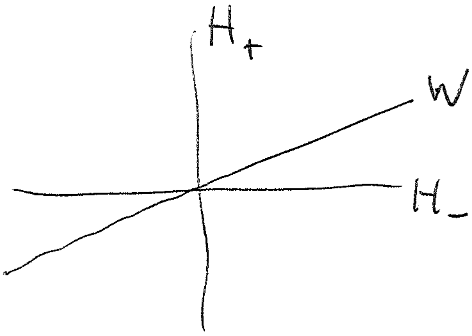
such that the **cocycle associated** is $\frac{1}{2}\frac{1}{2\pi i}\int f dg$

Let us look at genus zero. Take a closed Riemann surface Σ of genus 0 and a local coordinate z centered at p and let $D: |z| < 1$. We form $\mathcal{F} =$ fermionic Fock space belonging to $L^2(S^1)$. Then



$$L^2(S^1) = \underbrace{H_+}_{\substack{\text{spanned} \\ \text{by } z, z^2, \ldots}} \oplus \underbrace{H_-}_{\substack{\text{spanned} \\ \text{by } 1, z^{-1}, z^{-2}, \ldots}}$$

Let $W = \Gamma(\Sigma - D, \mathcal{O})$. Then W is complementary to H_+ and is close to H_-



In $\mathcal{F}^0$ we have $|0\rangle$ corresp. to H_- and $|\phi\rangle$ corresp. to W. whereas $\langle 0|$ corresp. to H_+.

Next we want to look at $\mathcal{F}^0$ as a boson Fock space, i.e. as a [redacted] repn. of [redacted] the CAR associated to $C^\infty(S^1)/\mathbb{C}$ with the symplectic form $\int f dg$. For this purpose I need a cross-section of the central extension of $C^\infty(S^1)/\mathbb{C}$ which acts on $\mathcal{F}^0$. There's the standard choice of [redacted] cross-section

$$f \text{ holom on } |z|<1 \implies \langle 0|\rho(f) = 0$$
$$f \text{ holom on } |z|>1 \implies \rho(f)|0\rangle = 0.$$

Thus $\langle 0|\rho_n = \rho_{-n}|0\rangle = 0$ for $n>0$ which is consistent with earlier formulas.

Now there's another choice in the present ~~situation~~ based on the [redacted] exactness of

$$0 \to \mathbb{C} \to \Gamma(\Sigma - D, \mathcal{O}) \oplus \Gamma(\bar{D}, \mathcal{O}) \longrightarrow C^\infty(S) \longrightarrow \underset{0}{\overset{}{H^1(\Sigma,\mathbb{C})}}$$

$$\implies \left(\Gamma(\Sigma - D, \mathcal{O})/\mathbb{C}\right) \oplus \Gamma(\bar{D},\mathcal{O})/\mathbb{C} = C^\infty(S)/\mathbb{C}$$

[redacted] ~~gives another cross-section~~

Thus we can define another lifting ~~to~~ operators $f \mapsto \tilde{\rho}(f)$ such that

$$\langle 0|\tilde{\rho}(f) = 0 \qquad \text{if} \quad f \in \Gamma(\bar{D},\mathcal{O})/\mathbb{C}$$
$$\tilde{\rho}(f)|\phi\rangle = 0 \qquad \text{if} \quad f \in \Gamma(\Sigma - D, \mathcal{O})/\mathbb{C}$$

Thus $\tilde{\rho}(f) = \rho(f) + \lambda(f)$ where $\lambda(f) = 0$ $f \in \Gamma(\bar{D},\mathcal{O})/\mathbb{C}$

June 3, 1987 (married 26 years)

Consider X a Riemann surface (compact connected) with ∂X a circle S. Let ξ be a holomorphic line bundle on X.

I first should describe carefully what happens over the circle. One starts with ξ just a line bundle on S. I will suppose that it is possible to construct some sort of fermion Fock space $\mathcal{F}$ which [redacted] is a renormalized version of ΛV, where $V =$ [redacted] $\Gamma(S, \xi)$. Thus $\mathcal{F}$ has operators $\psi^*(v) =$ "$v \wedge$" and $\psi(\lambda) =$ "$\lambda \lrcorner$" for $v \in V$ and $\lambda \in \Gamma(S, \xi^{-1} \otimes T^*)$. Moreover $\mathcal{F}$ should contain a line for each $W \subset V$ which is the image of an idempotent ψDO of order 0, [redacted] having the symbol $\begin{cases} 1 & \xi < 0 \\ 0 & \xi > 0 \end{cases}$. (Standard convention is $\psi_n^* |0\rangle = 0 \quad n < 0$ i.e. $|0\rangle$ corresponds to H_-, whereas H_+ corresponds to $\langle 0|$.)

Graeme's approach gives [redacted] various candidates for $\mathcal{F}$ as spaces of holomorphic sections of the determinant line bundle over the Grassmannian. Let's defer these problems.

Now the loop group $L\mathbb{C}^\times = C^\infty(S, \mathbb{C}^\times)$ acts naturally on $V = \Gamma(S, \xi)$, and there should be a central extension $\tilde{L}\mathbb{C}^\times$ which acts on $\mathcal{F}$. Using the Segal-Wilson paper it should be possible to define $\tilde{L}\mathbb{C}^\times$ independently of $\mathcal{F}$. We also should [redacted] obtain the commutator pairing for this central extension as

$$(f, g) \longmapsto (-1)^{(\deg f)(\deg g)} \cdot \text{Deligne's } \langle f, g \rangle$$

So assuming the circle theory is set up nicely, let's now discuss the surfaces X. There is a line $\mathbb{C}|\phi\rangle$ in $\mathcal{F}$ associated to $\Gamma(X,\xi)$ = space of holom. sections. Let $\Gamma(X,\mathcal{O}^\times)$ be the invertible holom. fns. on X; as X is connected restriction to S is injective:

$$\Gamma(X,\mathcal{O}^\times) \subset L\mathbb{C}^\times.$$

Let $\widetilde{\Gamma(X,\mathcal{O}^\times)}$ be the inverse image of $\tilde{L}\mathbb{C}^\times$ over this subgroup. It's abelian, say because it is abelian mod scalars and it leaves the line $\mathbb{C}|\phi\rangle$ invariant. One can also calculate ~~it~~ the commutator pairing. There is a unique lifting of $\Gamma(X,\mathcal{O}^\times)$ into $\widetilde{\Gamma(X,\mathcal{O}^\times)}$ which acts as the identity on the line $\mathbb{C}|\phi\rangle$.

Other liftings differ from this lifting by characters $\chi : \Gamma(X,\mathcal{O}^\times) \longrightarrow \mathbb{C}^\times$. I would like to show that for each lifting there is a unique line fixed (elementwise) under the lifted group. I want to first ~~XXX~~ investigate ~~characterize~~ the existence.

We can get a lot of characters of $\Gamma(X,\mathcal{O}^\times)$ from the commutator pairing with elements of $L\mathbb{C}^\times$. This gives a natural map

$$L\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times) \longrightarrow \Gamma(X,\mathcal{O}^\times)^\wedge$$

and it is clear that the character χ_f belonging to $f \in L\mathbb{C}^\times$ is such that $\rho(f)|\phi\rangle$ is an eigenvector for the lifted $\Gamma(X,\mathcal{O}^\times)$ with character χ_f. Here $\rho(f)$ denotes any elt of $\tilde{L}\mathbb{C}^\times$ lying over f. If $g \in \Gamma(X,\mathcal{O}^\times)$, let $\rho(g)$ be the lifting fixing $|\phi\rangle$, so

$$\rho(g)\,|\phi\rangle = |\phi\rangle \qquad g \in \Gamma(X, \mathcal{O}^\times)$$

In symbols.

$$\rho(g)\rho(f)\,|\phi\rangle = \langle g,f\rangle\, \rho(f)\rho(g)\,|\phi\rangle$$
$$= \underbrace{\langle g,f\rangle}_{\chi_f(g)}\, \rho(f)\,|\phi\rangle.$$

There is another way to view the line $\mathbb{C}\cdot\rho(f)|\phi\rangle$ for $f \in L\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times)$ as follows. This line corresponds to the subspace $f\,\Gamma(X, \xi)$ in $\Gamma(S, \xi)$. I'd like somehow to say that ξ on S has been extended to a holom. line bundle on X differently. But perhaps one should bring in the fact that all holomorphic line bundles on X are trivial, which means that $\Gamma(X, \xi) = \Gamma(X, \mathcal{O})v$, for some v unique up to multiplying by elts. of $\Gamma(X, \mathcal{O}^\times)$. v is also non-vanishing.

Let's return now to the map

$$L\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times) \longrightarrow \Gamma(X,\mathcal{O}^\times)^\wedge$$

defined by the commutator pairing. We would like this to be an isomorphism, or at least injective with dense image. This somehow means $\Gamma(X,\mathcal{O}^\times)$ is maximal isotropic.

Recall the structure of $\Gamma(X,\mathcal{O}^\times)$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma(X,\mathcal{O}) \longrightarrow \Gamma(X,\mathcal{O}^\times) \longrightarrow H^1(X,\mathbb{Z}) \to 0$$

Thus $\Gamma(X, \mathcal{O}^\times)$ has a filtration with quotients (in increasing order)

$$\mathbb{C}^\times, \quad \Gamma(X,\mathcal{O})/\mathbb{C}, \quad H^1(X,\mathbb{Z})$$

So $\Gamma(X,\mathcal{O}^\times)^\wedge$ should have a filtration with quotients

$$H^1(X,\mathbb{Z})^\wedge, \quad \left(\Gamma(X,\mathcal{O})/\mathbb{C}\right)^\wedge, \quad \mathbb{Z}$$

We'd like to identify these as coming from $L\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times)$. Clearly the degree gives a surjective homomorphism of this gp to $\mathbb{Z}$ hence

$$0 \longrightarrow L^0\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times) \longrightarrow L\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times) \longrightarrow \mathbb{Z} \longrightarrow 0$$

The next thing we want to do is to identify $H^1(X,\mathbb{Z})^\wedge$ with a subgroup of $L^0\mathbb{C}/\Gamma(X,\mathcal{O}^\times)$. Now $H^1(X,\mathbb{Z})^\wedge$ is the group of flat line bundles on X. ? Any such line bundle is trivial and so upon choosing a trivialization the connection form is a holomorphic 1-form ω. What we are saying is

$$0 \to \mathbb{C}^\times \longrightarrow \Gamma(X,\mathcal{O}^\times) \xrightarrow{d\log} \Gamma(X,\Omega^1) \longrightarrow H^1(X,\mathbb{C}^\times) \longrightarrow 0$$

Wait: The group of flat line bundles is $H_1(X,\mathbb{Z})^\wedge$ not $H^1(X,\mathbb{Z})^\wedge$, so at this point we must be using the symplectic form on $H_1(X,\mathbb{Z})$.

Here is how to embed $H^1(X,\mathbb{C}^\times)$ in $L^0\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times)$:

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\Gamma(X,\mathcal{O}^*)/\mathbb{C}^* \arrow[d, "d\log"] \arrow[rr, equal] & & \Gamma(X,\mathcal{O}^*)/\mathbb{C}^* \arrow[d] \\
\Gamma(X,\Omega^1) \arrow[r, hook, "\int"] \arrow[d] & C^\infty(S)/\mathbb{C} \arrow[r, "\exp", "\sim"'] & L^0\mathbb{C}^*/\mathbb{C}^* \arrow[d] \\
H^1(X,\mathbb{C}^*) \arrow[rr, hook] \arrow[d] & & L^0\mathbb{C}^*/\Gamma(X,\mathcal{O}^*) \arrow[d] \\
0 & & 0
\end{tikzcd}$$

This shows the cokernel is

$$\operatorname{Coker}\left\{H^1(X,\mathbb{C}^*) \hookrightarrow L^0\mathbb{C}^*/\Gamma(X,\mathcal{O}^*)\right\} = C^\infty(S)\Big/\textstyle\int\Gamma(X,\Omega^1)$$

$$\xrightarrow{\sim} \{\omega\in\Omega^1(S) \mid \textstyle\int\omega = 0\}\Big/\Gamma(X,\Omega^1)$$

Thus we come back to showing that the pairing

$$\overset{g}{C^\infty(S)}\Big/\textstyle\int\Gamma(X,\Omega^1) \times \overset{f}{\Gamma(X,\mathcal{O})}/\mathbb{C} \longmapsto \int f\,dg$$

or

$$\Omega^1(S)_{\int=0}\Big/\Gamma(X,\Omega^1) \times \Gamma(X,\mathcal{O})/\mathbb{C} \ni (\omega,f) \longmapsto \int f\omega$$

is non-degenerate.

June 4, 1987

Let's review the latest version of bosonization. Given a circle S and a line bundle L over it we consider the Clifford algebra associated to $\Gamma(S,L) \oplus \Gamma(S,L^{-1}\otimes T^*)$. Consider also splittings $\Gamma(S,L) = W^+ \oplus W^-$ defined by involutions which are ψDO's of order zero with symbol $\operatorname{sign}(\xi)$. I would like to claim that there is a representation $\mathcal{F}$ of the Clifford algebra such that each splitting as above gives rise to a unique line $\mathbb{C}|W^-\rangle$ in $\mathcal{F}$ and a line of linear functionals $\mathbb{C}\langle W^+|$ on $\mathcal{F}$. These lines are characterized by being the unique states killed by appropriate subspaces in $\Gamma(S,L)\oplus\Gamma(S,L^{-1}\otimes T^*)$.

It seems that I ought to be able to adapt Graeme's machinery to construct this Fock space $\mathcal{F}$ in a non L^2 setting. Basically one has a Grassmannian consisting of the W^+ which occur, and a determinant line bundle over this Grassmannian, and elements of $\mathcal{F}$ should be be given by certain sections of the dual of this line bundle.

But let's not worry about the details for the moment, and let's just assume this $\mathcal{F}$ exists and is intrinsically associated to (S,L). In particular if one has an isomorphism $(S,L)\simeq(S',L')$, then there will be an isomorphism $\mathcal{F}\simeq\mathcal{F}'$ unique up to a scalar factor. This gives a central extension of the loop group $L\mathbb{C}^\times = C^\infty(S,\mathbb{C}^\times)$; denote the central extension $\widetilde{L\mathbb{C}}^\times$.

We know this central extension depends on L, that although $L\mathbb{C}^\times$ doesn't depend on L, this extension does. In effect an isomorphism of L

that is, an elt. of the loop group lifts to an inner autom. of $\tilde{L}\mathbb{C}^\times$ which is non-trivial. It seems that once we have this central extension we can recover $\mathcal{F}$ up to isomorphism, canonical up to scalar factors, as a certain kind of irreducible representation of this central extension.

(This perhaps explains the problems with square integrability. In effect one feels happy with a Hilbert space version of $\mathcal{F}$ associated to a central extension of $L\mathbb{T}$ by $\mathbb{T}$. But inside the $\tilde{L}\mathbb{C}^\times$ obtained from L I don't see how to find an extension of $L\mathbb{T}$ by $\mathbb{T}$, and any choice I make would not be invariant under autos. of L.

One way to get a completely intrinsic L^2-situation would be to start with L a spin structure on the circle. Then we know that $\Gamma(S,L)$ has a canonical hermitian inner product. so we get an $\mathcal{F}$ which is a Hilbert space, and a central extension $\tilde{L}\mathbb{T}$ of $L\mathbb{T}$ by $\mathbb{T}$, which acts by unitary operators on $\mathcal{F}$.)

<u>Assertion</u>: Associated to a pair (S,L) consisting of a line bundle over a circle is a central extension $\tilde{L}\mathbb{C}^\times$ of $L\mathbb{C}^\times = C^\infty(S,\mathbb{C}^\times)$ by $\mathbb{C}^\times$. There (probably) is a canonical irreducible repn. of this central extension.

Now that one has an understanding of the circle one wants to bring in a Riemann surface. On the fermion side, an extension of L to a holomorphic line bundle on X determines a line in $\mathcal{F}$. This is the line associated to the subspace $W = \Gamma(X, L)$ of holomorphic sections of L.

Since holomorphic line bundles on X are trivial, this extension of L on S is completely described by an orbit of $\Gamma(X, \mathcal{O}^\times)$ on the non-vanishing sections of L over S. Thus the possible holomorphic extensions of L/S to X is a torsor under $L\mathbb{C}^\times / \Gamma(X, \mathcal{O}^\times)$.

Recall Wodzicki's observation that the algebra of classical ψDO symbols (complete symbols) has a non-trivial center. The non-trivial center contains the identity and the Hilbert transform. This gives then a class of splittings of $\Gamma(S, L)$, namely involutions which are ψDO's of order 0 and which have the Hilbert transform for their complete symbol. Thus you are looking at operators F on $\Gamma(S, L)$ such that conjugation with $e^{it\varphi}$ has a certain asymptotic expansion:

$$e^{it\varphi} F e^{-it\varphi} u = \operatorname{sign}(d\varphi) \cdot u + O(t^{-N}) \quad \text{all } N$$

for $d\varphi \neq 0$ on $\operatorname{Supp}(u)$.

What is the goal of this search for the proper foundations? I want a fermion Fock space, i.e. a representation of the Clifford algebra of $\Gamma(S, L) \oplus \Gamma(S, L^{-1} \otimes T^*)$ containing unique lines and unique dual lines for each kinds of subspace. Once this is obtained I get a representation of a central extension of $C^\infty(S, \mathbb{C}^\times)$ on $\mathcal{F}$. I would like to be able to say that this central extension has a unique irreducible representation of a certain type and that $\mathcal{F}$ is this irreducible repn.

Let's return to the setup where we have X with boundary S. Then we have identified the possible ways of extending $L|S$ to a holom. line bundle on X. These are the orbits of $\Gamma(X, \mathcal{O}^\times)$ on the nowhere vanishing sections of L. These orbits form a torsor under

$$L\mathbb{C}^\times / \Gamma(X, \mathcal{O}^\times).$$

To each extension of L on S to a holomorphic line bundle on X we have a subspace $\Gamma(X, L) \subset \Gamma(S, L)$. In fact $\Gamma(X,L) = \Gamma(X, \mathcal{O})s$, where s is the corresponding nowhere-vanishing section of L. This subspace determines a line in Fock space and the line determines the subspaces. Now $\Gamma(X, \mathcal{O})s = \Gamma(X, \mathcal{O})s' \iff s, s'$ belong to the same orbit. Thus we conclude that

$$L\mathbb{C}^\times / \Gamma(X, \mathcal{O}^\times) = \tilde{L}\mathbb{C}^\times / \widetilde{\Gamma(X, \mathcal{O}^\times)} \hookrightarrow \mathbb{P}(\mathcal{F})$$

provided a basepoint orbit is chosen.

Thus $\widetilde{\Gamma(X, \mathcal{O}^\times)}$ is the stabilizer of each line in $\mathcal{F}$ that we are interested in, and each of these line is an eigenvector of $\overline{\Gamma(X,\mathcal{O}^\times)}$. The natural question is whether these are all the eigenvectors. This question depends on our model for $\mathcal{F}$.

Let's now fix the sector of $\mathcal{F}$ we are interested in and restrict to degree 0 loops. This means we fix the degree of the line bundle that would be obtained by filling X in with a disk and extending L to this disk. And it means that we can replace $L\mathbb{C}^\times$ by the Lie algebra $L\mathbb{C}/\mathbb{C}$.

~~Thus I have a family of lines in $\mathcal{F}^0$ which is a torsor under~~

~~$L\mathbb{C}/\Gamma(X,\mathcal{O}) \simeq L^0(\mathbb{C})/\Gamma$~~

Thus I have now a central extension as Lie algebra $\tilde{L}\mathbb{C}/\mathbb{C}$ of $L\mathbb{C}/\mathbb{C}$ which acts on $\mathcal{F}^0$. This central extension depends on L over S. What I've done is to choose the sector $\mathcal{F}^0$, cut $L\mathbb{C}^\times$ to $L^0\mathbb{C}^\times$ and to lift the constant loops $\mathbb{C}^\times$ into $\tilde{L}^0\mathbb{C}^\times$ so as to act trivially on $\mathcal{F}^0$. Then $L\mathbb{C}^\times$ is replaced by $\tilde{L}^0\mathbb{C}^\times/\mathbb{C}^\times \xleftarrow[\exp]{\sim} \tilde{L}\mathbb{C}/\mathbb{C}$.

Standard construction of boson Fock space starts from the symplectic space $\tilde{L}\mathbb{C}/\mathbb{C}$, and it leads to a representation of a central extension equipped with a cross-section. Thus to each $f \in L\mathbb{C}/\mathbb{C}$ should belong an operator $\rho(f)$ depending linearly on f such that $[\rho(f), \rho(g)] = \frac{1}{2\pi i}\int f\,dg$.

So now we need to return to

$$L\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times) \xhookrightarrow{\text{uses basept } |\phi\rangle} \mathbb{P}(\mathcal{F}).$$

Cutting down gives

$$L^\circ\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times) \hookrightarrow \mathbb{P}(\mathcal{F}_0).$$

I recall from yesterday that we have

$$\begin{tikzcd}
\Gamma(X,\mathcal{O}^\times) \arrow[rrr, equal] \arrow[d, "\int d\log"] & & & \Gamma(X,\mathcal{O}^\times) \arrow[d, "\int"] \\
\Gamma(X,\Omega^1) \arrow[r, hook] \arrow[d] & \{\omega\in\Omega^1(S) \mid \int\omega=0\} \arrow[r, "\int^\times", "\sim"'] & C^\infty(S^1)/\mathbb{C} \arrow[r, "\exp", "\sim"'] & L^\circ\mathbb{C}^\times/\mathbb{C}^\times \arrow[d] \\
H^1(X,\mathbb{C}^\times) & & & L^\circ\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times)
\end{tikzcd}$$

So we have a canonical embedding

$$H^1(X,\mathbb{C}^\times) \hookrightarrow L^\circ\mathbb{C}^\times/\Gamma(X,\mathcal{O}^\times) \xhookrightarrow{|\phi\rangle} \mathbb{P}(\mathcal{F}_0)$$

Let's now try to begin from the bosonic end. We start with $LT = C^\infty(S,\mathbb{T})$, and propose to construct a central extension and a unitary representation of this central extension. We consider the real symplectic space $C^\infty(S,\mathbb{R})/\mathbb{R}$ and find the representation of the CCR associated to this space with fixed vectors associated to a class of polarizations determined by the smooth structure. Call this representation $\mathcal{B}^\circ$; it is a unitary repn.

of a central extension $\tilde{L}^\circ\mathbb{T}$ with cross-section of $L^\circ\mathbb{T}$, and we can then induce this repn. up to $L\mathbb{T}$. NO, we have to choose a central extension $\tilde{L}\mathbb{T}$ of $L\mathbb{T}$ restricting to $\tilde{L}^\circ\mathbb{T}$. This seems to mean choosing a loop of degree 1.

In any case we have B° canonically constructed. Now consider the surface X with boundary S, and the embedding

$$\Gamma(X,\mathcal{O}^\times)/\mathbb{C}^\times \hookrightarrow L\mathbb{C}^\times/\mathbb{C}^\times \xleftarrow{\sim} C^\infty(S)/\mathbb{C}.$$

In other words we consider the logarithms of holomorphic functions on X restricted to the circle S. These are well-defined functions modulo constants on the circle. Thus we have

$$0 \subset \underbrace{\Gamma(X,\mathcal{O})/\mathbb{C} \subset \log\Gamma(X,\mathcal{O}^\times)/\mathbb{C}}_{H^1(X,\mathbb{Z})} \underbrace{\subset \int\Gamma(X,\Omega^1)/\mathbb{C}}_{H^1(X,\mathbb{C}^\times)} \subset C^\infty(S)/\mathbb{C}$$

Here is a [redacted] situation that I ought to analyze relative to the real structures. Identify $C^\infty(S)/\mathbb{C}$ with $\Gamma(X,\mathcal{H})/\mathbb{C}$ where $\mathcal{H}$ is the sheaf of harmonic functions. Then from

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}\oplus\bar{\mathcal{O}} \longrightarrow \mathcal{H} \longrightarrow 0$$

we get

$$0 \longrightarrow \Gamma(X,\mathcal{O})/\mathbb{C} \oplus \Gamma(X,\bar{\mathcal{O}})/\mathbb{C} \to \Gamma(X,\mathcal{H})/\mathbb{C} \to H^1(X,\mathbb{C}) \to 0$$

[redacted]

Basic question: When we project $\log \Gamma(X,\mathcal{O}^\times)/\mathbb{C}$ onto $H^1(X,\mathbb{C})$ do we get real classes?

The basic is this. We have a central extension of $L^0\mathbb{C}^\times/\mathbb{C}^\times$ which acts on a fermion Fock space in such a way that $\Gamma(X,\mathcal{O}^\times)/\mathbb{C}^\times$ has eigenvectors. I can ask the same question for the particular central extension I construct. And I can ask a stronger question as to whether there are eigenvectors for $\Gamma(X,\mathcal{O}^\times)/\mathbb{C}^\times$ which are killed under the specific operators $\rho(f)$ attached to elements $f \in \Gamma(X,\mathcal{O})$.

Let's explain concretely

$$\begin{array}{ccccccccc} & & & & & & C^\infty(S) & & \\ & & & & & & \| & & \\ 0 \longrightarrow & \mathbb{C} & \longrightarrow & \Gamma(X,\mathcal{O}) \oplus \Gamma(X,\bar{\mathcal{O}}) & \longrightarrow & & \Gamma(X,\mathcal{H}) & \longrightarrow & H^1(X,\mathbb{C}) \longrightarrow 0 \\ & & & & & & f & \longmapsto & [*df] \end{array}$$

If f harmonic, then $*df$ is closed so represents a cohomology class. If f is holom. (or anti-holom), then $*df = i\,df$ since $df \in \Gamma(X,T^{1,0})$. Then $*df$ is exact, so the coh. class is trivial. Conversely if $[*df]=0$ i.e. $*df = dg$, then

$$*\partial f = -i\partial f = \partial g \implies \partial(g+if) = 0$$
$$*\bar\partial f = i\bar\partial f = \bar\partial g \implies \bar\partial(g-if) = 0$$

so f is the sum of a holomorphic and anti-holom. fn.

Note $dx \wedge *dx$ is supposed to be >0, so that

$$*(dx) = dy \qquad *dy = -dx$$

Thus $*dz = dy - i\,dx = -i\,dz \qquad \therefore * = -i \text{ on } T^{1,0}$

Now the key question becomes what is the image of the map

$$\log \Gamma(X, \mathcal{O}^{\times}) + \mathbb{C} \subset C^{\infty}(S^1)/\mathbb{C} \longrightarrow H^1(X, \mathbb{C})$$

You take a holom. invertible fn. f; its logarithm $\log f$ is well-defined on S up to an additive constant. You then extend $\log f$ to a harmonic function u and look at the cohomology class of $*du$.

Next take a holomorphic 1-form ω and use the exact sequence

$$0 \longrightarrow \Gamma(X, \bar{\mathcal{O}}) \longrightarrow \Gamma(X, \mathcal{H}) \longrightarrow \Gamma(X, \Omega^1) \longrightarrow 0$$

to write $\omega = \partial v$ with v harmonic?

Consider f on S such that $\int_S f\omega = 0 \quad \forall \omega \in \Gamma(X, \Omega^1)$

Then extend f to a harmonic function. Then $\bar{f}$ is also harmonic, so $\partial \bar{f}$ is a holomorphic 1-form, so

$$\int_S f \partial \bar{f} = \int_X df\, \partial \bar{f} = \int_X \bar{\partial} f\, \partial \bar{f} = 0$$

But $\bar{\partial} f\, \partial \bar{f} = (\partial_{\bar{z}} f)(\partial_z \bar{f})\, d\bar{z}\, dz = |\partial_{\bar{z}} f|^2\, 2i\, dx dy$.

Thus $\bar{\partial} f = 0$ and the harmonic extension of f is holomorphic. This means that the annihilator of the functions on S which are the integrals of holom. 1-forms is the bdry values of holomorphic functions. If duality works, then the annihilator of $\Gamma(X, \mathcal{O})$ is $\int \Gamma(X, \Omega^1)$

June 5, 1987

How to show the annihilator of $\Gamma(X,\mathcal{O})$ is $\int\Gamma(X,\Omega^1)$, as Kronheimer showed me. One considers the 1-forms on X which are closed + coclosed:

$$d\omega = d(*\omega) = 0.$$

Write $\omega = \omega' + \omega'' \in \Omega^{1,0}(X) \oplus \Omega^{0,1}(X)$, then

$$*(\omega) = -i\omega' + i\omega''$$

so this means $d\omega' = \bar\partial\omega' = 0$ and $d\omega'' = \partial\omega'' = 0$.

Thus we have

$$Z = \{\omega \mid d\omega = d(*\omega) = 0\} = \Gamma(X,\Omega^1) \oplus \Gamma(X,\overline{\Omega}^1)$$

Now on the space of 1-forms on X we have a skew form $\int_X \omega\omega'$ and this sets up a non-degenerate pairing between $\Omega^{1,0}(X)$ and $\Omega^{0,1}(X)$. In effect given $f(z,\bar z)\,dz$, then $i\int f\,dz \cdot \bar f\,d\bar z > 0$ for $f \neq 0$. Thus this skew form restricted to Z is such that $\Gamma(X,\Omega^1)$ and $\Gamma(X,\overline{\Omega}^1)$ are maximal isotropic.

But the natural line to follow is this. First observe that f harmonic $\Rightarrow$ $df \in Z$. Then ask what $Z/d\Gamma(X,\mathcal{H})$ is. Use sheaf theory:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{H} \xrightarrow{\ d\ } \Omega^1 \oplus \overline{\Omega}^1 \longrightarrow 0$$

Yields.

$$0 \longrightarrow \mathbb{C} \longrightarrow \Gamma(X,\mathcal{H}) \xrightarrow{\ d\ } \Gamma(X,\Omega^1) \oplus \Gamma(X,\overline{\Omega}^1) \longrightarrow H^1(X,\mathbb{C}) \to 0$$

Another idea is to use the Hodge theorem. This says that one can represent elements of $H^1(X,\mathbb{C})$ uniquely by harmonic 1-forms on X vanishing on ∂X. Here harmonic means lying in Z. (Thus we learn that harmonic has a meaning for a function as a 2nd order DE condition, and for 1-forms as a 1st order DE condition.)

To prove this Hodge thm. take an element of

$H^1(X,\mathbb{C})$ and represent it by $\omega \in \Gamma(X,\Omega^1)$. Restrict ω to $\partial X = S$ where it becomes df then extend f to a harmonic fn. Then $\omega - df$ is harmonic and represents the same cohomology closed and it vanishes on S. Uniqueness: Suppose ω harmonic, vanishes on S, and is exact. Then $\omega = df$, $d * df = 0 \implies f$ harmonic. Then $df = 0$ on S, so f is constant on S, so by uniqueness in the Dirichlet problem f is constant and $\omega = 0$.

So now let's look at the annihilator question. We take a function f on S such that $\int_S f\,dg = 0$ for all $g \in \Gamma(X,\mathcal{O})$. Extend f harmonically. Then

$$\int_S f\,dg = \int_X df\,dg$$

and so $df \in Z$ and $df \in$ annihilator $d\Gamma(X,\mathcal{O})$. Now we know within Z that

$$\Gamma(X,\Omega^1)^\circ = \Gamma(X,\overline{\Omega^1})$$

whereas we want $[d\Gamma(X,\mathcal{O})]^\circ$. Claim

$$[d\Gamma(X,\mathcal{O})]^\circ = \Gamma(X,\Omega^1) + \text{harmonic forms vanishing on } S$$

Check $\supset$. $\int_X df\,\omega = \int_S f\omega = 0$ if $\omega|S = 0$.

The rest should be a dimension count: $d\Gamma(X,\mathcal{O})^\circ / \Gamma(X,\Omega^1)^\circ$ can be at most have the dimension of $\Gamma(X,\Omega^1)/d\Gamma(X,\mathcal{O}) = H^1(X,\mathbb{C})$ and we know the harmonic forms vanishing on S have this dimension.

So finally take $df \in [d\Gamma(X,\mathcal{O})]^\circ$ and write it $df = \omega + \eta$ with $\omega \in \Gamma(X,\Omega^1)$ and η vanishing on S. When we restrict to S we see $df = \eta$, so $f \in \int \Gamma(X,\Omega^1)$ as claimed.

Let's see if these ideas can be used to solve Graeme's problem having to do with the fact that the Θ fn. is not in the Hilbert space. The program he started was to take $C^\infty(S)/\mathbb{C}$ with its real symplectic structure, take the associated bosonic Fock space, then find somehow a line inside due to the holomorphic structure on X. The first requirement is that the line be killed by $\Gamma(X,\mathcal{O})/\mathbb{C}$ lifted to operators on $\mathcal{F}$ in the given lifting. Better, elements of $C^\infty(S)/\mathbb{C}$ are represented as operators on $\mathcal{F}$.

So look at the subspace killed by $\Gamma(X,\mathcal{O})/\mathbb{C}$ it is going to be the irreducible representation for the finite-dimensional real symplectic space which is the orthogonal of

$$\Gamma(X,\mathcal{O})/_{\mathbb{C}} \oplus \Gamma(X,\bar{\mathcal{O}})/_{\mathbb{C}} \subset C^\infty(S)/\mathbb{C}$$

and we know this orthogonal space is isomorphic to $H^1(X,\mathbb{C})$.

Another approach: When the circle^S appears as the boundary of X, the real symplectic space $C^\infty(S)/\mathbb{C}$ is naturally embedded in a larger real symplectic space, namely, the space of harmonic one-forms. Moreover the latter $\Gamma(X,\Omega^1) \oplus \overline{\Gamma(X,\Omega^1)}$ is naturally polarized, so there is an obvious boson Fock space with operators attached to harmonic 1-forms and vacuum vector killed by holomorphic 1-forms. So now what we try to do is to project in some way.

I can restrict to the operators from $C^\infty(S)/\mathbb{C}$ i.e. to exact harmonic 1-forms. This is many copies of the irreducible representation of $C^\infty(S)/\mathbb{C}$.

Let $A = \Gamma(X,\Omega^1) \oplus \overline{\Gamma(X,\Omega^1)}$ be the space of harmonic 1-forms, let B be the subspace of forms vanishing on S so that

$$A = dH \oplus B$$

where $H = \Gamma(X,\mathcal{H})/\mathbb{C}$. This decomposition is orthogonal for the symplectic form, since

$$\int_X df\,\omega = \int_S f\omega = 0 \qquad \text{if} \quad \omega/S = 0$$

Now $dH = \Gamma(X,\mathcal{H})/\mathbb{C} \simeq C^\infty(S)/\mathbb{C}$ is the symplectic space whose canonical representation is supposed to contain a line. Better, we want to construct a line in this canonical repn.

Corresponding to $A = dH \oplus B$ is an isomorphism

$$\mathcal{F}_A = \mathcal{F}_H \otimes \mathcal{F}_B$$

So far we have a canonical vector in $\mathcal{F}_A$. We want to produce a line in $\mathcal{F}_H$, hence ~~line~~ we need a linear map $\mathcal{F}_B \longrightarrow \mathbb{C}$, or densely defined linear map such that we can apply it to $|0\rangle \in \mathcal{F}_A$.

This approach has to be compared with the following. Let B' be the annihilator of $C = \left(\Gamma(X,\mathcal{O}) + \Gamma(X,\overline{\mathcal{O}})\right)/\mathbb{C}$ in H. We have

$$\mathcal{F}_H = \mathcal{F}_C \otimes \mathcal{F}_{B'}$$

where $\mathcal{F}_C$ has a canonical ground state killed by $\Gamma(X,\mathcal{O})/\mathbb{C}$. Now $B' \oplus B$ is the annihilator of C in A.

My feeling is that inside $\mathcal{F}_{B'} \otimes \mathcal{F}_B$ is a vacuum $|0\rangle$ coming from the image of $\Gamma(X, \Omega^1)$ in $B' \oplus B$. We still need to product a linear fnl $\mathcal{F}_B \longrightarrow \mathbb{C}$. However it is conceivable that this linear fnl is densely-defined yet still makes sense when applied to $|0\rangle$.

If this works we have attached something on $H^1(X,\mathbb{C})$ linked to its integral structure and the complex structure of X.

Note: A harmonic form ω determines two cohomology classes because both ω and $*\omega$ are closed.

$$\begin{tikzcd}[column sep=small]
 & & \Gamma(X,\Omega^1)\oplus\Gamma(X,\bar{\Omega}^1) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
 & \Gamma(X,\Omega^1)\oplus d\Gamma(X,\bar{\mathcal{O}}) \arrow[dl, dash] \arrow[dr, dash] & d\Gamma(X,\mathcal{H}) \arrow[d, dash] & d\Gamma(X,\mathcal{O})\oplus\Gamma(X,\bar{\Omega}^1) \arrow[dl, dash] \arrow[dr, dash] & \\
\Gamma(X,\Omega^1) \arrow[dr, dash] & & d\Gamma(X,\mathcal{O})\oplus d\Gamma(X,\bar{\mathcal{O}}) \arrow[dl, dash] \arrow[dr, dash] & & \Gamma(X,\bar{\Omega}^1) \arrow[dl, dash] \\
 & d\Gamma(X,\mathcal{O}) \arrow[dr, dash] & & d\Gamma(X,\bar{\mathcal{O}}) \arrow[dl, dash] & \\
 & & 0 & &
\end{tikzcd}$$

Here's the situation to understand.

Let V be a real symplectic vector space, let Γ be a lattice in V which coincides with its dual with respect to the symplectic form. For example $V = H^1(X,\mathbb{R})$, $\Gamma = H^1(\cancel{X,\mathbb{Z}})$. Now take $V \oplus V$ and take a polarization. I probably want the symplectic structure to be $-\omega \oplus \omega$ on $V \oplus V$. The picture is to form the double of X, that is, the union of X and its conjugate. Then the polarization on $H^1(X \cup \bar{X}, \mathbb{C})$ is given by the Hodge decomposition.

Now the polarization $W \subset V_c \oplus V_c$ should project onto the second factor, and probably both factors.

Next we take the QM representation $\mathcal{F} \otimes \mathcal{F}$ of $V \oplus V$ with the ground state $|0\rangle$ killed by W. Now $0 \oplus \Gamma \subset V \oplus V$ when exponentiated gives a free abelian group Γ acting on $\mathcal{F}$. The idea is that there should be a fixed vector α under Γ in a larger Hilbert space and inner product with α gives a map $\mathcal{F} \otimes \mathcal{F}' \longrightarrow \mathcal{F}$ which can be applied to $|0\rangle$.

Think of V as having the basis x, p_x and $V \oplus V$ as having the basis x, p_x, y, p_y. The model for $\mathcal{F} \otimes \mathcal{F}$ is to be L^2 fns in x, y with $p_x = \frac{1}{i}\partial_x$ etc. Then $|0\rangle$ has to be a Gaussian

$$e^{-\frac{1}{2}(\alpha x^2 + 2\beta xy + \gamma y^2)}$$

where $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ is a symmetric matrix with positive definite real part.

June 6, 1987

Yesterday I seemed to show that associated to a Riemann surface X with boundary the circle S there is an associated line in the Fock space of $C^\infty(S)/\mathbb{C}$. In order to see this I need to write $V = H^1(X,\mathbb{R})$ as a sum of complementary maximal isotropic subspaces. Thus to do the calculations one wants to use a,b cycles on X.

Let's check some details. Let $V = \mathbb{R}^2$, $\Gamma = \mathbb{Z}^2$. Identify the CCR representation of V with $L^2(\mathbb{R})$ with $V = \mathbb{R}q + \mathbb{R}p$ acting as $q = x$, $p = \frac{1}{i}\partial_x$. I want Γ to be integral, so as to act on $L^2(\mathbb{R})$. Let the two generators of Γ act as $e^{2\pi i x}$, e^{∂_x}. What are the eigenfunctions of Γ? The eigenfunction with eigenvalue 1 is

$$\sum_{n\in\mathbb{Z}} \delta(x-n)$$

This is not is $L^2(\mathbb{R})$, but it probably lives in the Sobolev space H_{-1} where H_1 consists of f such that $\partial_x f$ and xf lie in $H_0 = L^2$. I can check this by seeing that its pairing with such an f is well-defined.

$$\left\langle \sum_n \delta(x-n), f(x) \right\rangle = \sum_n f(n)$$

Since $\partial_x f \in L^2$, f is continuous, so $f(x)$ is well-defined for any x. Then by Cauchy-Schwartz

$$\left|\sum_n f(n)\right|^2 \leq \left(\sum \frac{1}{n^2}\right)\left(\sum (n f(n))^2\right)$$

showing the sum $\sum f(n)$ is always finite.

The other eigenfunctions are

$$v_{a,b} = \sum_n e^{2\pi i n a} \delta(x-b-n)$$

One has
$$e^{2\pi i x} v_{ab} = e^{2\pi i b} v_{ab}$$
$$e^{\partial_x} v_{ab} = e^{2\pi i a} v_{ab}$$

and
$$\langle v_{ab}, f \rangle = \sum_n e^{2\pi i n a} f(b+n)$$

So if $f = e^{-\frac{1}{2}x^2}$ is the harmonic oscillator ground state this gives

$$\sum e^{-\frac{1}{2}(b+n)^2 + 2\pi i n a} = e^{-\frac{1}{2}b^2} \sum_n e^{-\frac{1}{2}n^2 + (2\pi i a - b)n}$$

which makes sense for $a, b \in \mathbb{C}$. This last fact shows that the construction I want to use ought to produce vectors for any character $\Gamma \to \mathbb{C}^\times$.

The next step will be to use a, b cycles in earnest. We have the "real" symplectic space $C^\infty(S)/\mathbb{C}$ and the subspace $\mathcal{O}_K \oplus \bar{\mathcal{O}}_K$ on which the skew form is non degenerate. This defines a complementary space $B_i \subset C^\infty(S)/\mathbb{C}$ which is isomorphic to $H^1(X, \mathbb{C})$. Let's work with real spaces:

$$C^\infty(S, \mathbb{R})/\mathbb{R} = \mathrm{Re}(\mathcal{O}) \oplus B$$

and the natural identification $B \to H^1(X, \mathbb{R})$ takes f

to its harmonic extension and then the class of $*df$. (Recall: If $*df = dg$, then $d*dg = -d^2f = 0$ so g is harmonic. Also $-i\partial f = \partial g$, $i\bar\partial f = \bar\partial g$ so $f+ig \in \mathcal{O}$, $f-ig \in \bar{\mathcal{O}}$ and $f \in Re(\mathcal{O})$.)

When we quantize we want to use a maximal isotropic subspace of B, call it B_0. It consists of forms whose a-periods are arbitrary and whose b-periods are zero. The Hilbert space will be a tensor product of holomorphic functions on $\mathcal{O}$ with L^2 functions on B_0. Our task is to find a line in this space that somehow associated to $\Gamma(X, \mathcal{O}^\times)$. The first condition is that it should be killed by the operators in $\mathcal{O}$. Thus we are after some function on B_0.

How to construct a function on B_0? We have to use the integral structure, and something involving the holomorphic structure on X. From p.832 we expect a Θ-type function on B_0.

We have to somehow describe B explicitly as a space of functions on the circle.

Let's try to get a picture of the different subspaces of harmonic forms. First of all we are ultimately interested in the CCR representation $\mathcal{F}$ associated to $C^\infty(S)/\mathbb{C} \cong \Gamma(X, \mathcal{H})/\mathbb{C} \xrightarrow{\sim} d\Gamma(X, \mathcal{H})$. We want to produce a vector in $\mathcal{F}$ in a canonical way. It should be annihilated by $d\Gamma(X, \mathcal{O})$, hence really we are concerned with the annihilator B of $d\Gamma(X, \mathcal{O}) + d\Gamma(X, \bar{\mathcal{O}})$ in $d\Gamma(X, \mathcal{H})$, and its CCR representation.

Now I don't seem to be able to describe B very well, and from what Kronheimer showed me it seems very natural to work in the larger space of harmonic 1-forms.

Set $\mathcal{H}^1 = \Gamma(X, \Omega^1) \oplus \Gamma(X, \overline{\Omega}^1)$ for the space of harmonic 1-forms. Let $K \subset \mathcal{H}^1$ be the harmonic 1-forms vanishing on S. This we know is complementary to $d\Gamma(\mathcal{H})$ and is isomorphic to $H^1(X, \mathbb{C})$. Let $L \subset \Gamma(X, \Omega^1)$ be the space of holom. form components of elements of K. Then clearly $K + *K = L \oplus \overline{L} \subset \mathcal{H}^1$.

We saw in Kronheimer's proof that

$$d\Gamma(X,\mathcal{O})^\circ = \Gamma(X,\Omega^1) + K = \Gamma(X,\Omega^1) \oplus \overline{L}$$

$$d\Gamma(X,\overline{\mathcal{O}})^\circ = L \oplus \Gamma(X,\overline{\Omega}^1)$$

whence $\left(d\Gamma(X,\mathcal{O}) + d\Gamma(X,\overline{\mathcal{O}})\right)^\circ = L \oplus \overline{L} = K + *K$.

Thus from K we construct the symplectic complement to $d\Gamma(X,\mathcal{O}) + d\Gamma(X,\overline{\mathcal{O}})$, and it is $K + *K = L \oplus \overline{L}$

Note L is a complement to $d\Gamma(X,\mathcal{O})$ in $\Gamma(X,\Omega^1)$.

Now the space $B \subset d\Gamma(X,\mathcal{H})$ is

$$B = d\Gamma(X,\mathcal{H}) \cap (K + *K)$$

Thus $*K$ and B are both complements to K, but apparently they are not equal.

Let us consider the double of X namely $X \underset{S}{\cup} \overline{X}$. Then $H^1(X \underset{S}{\cup} \overline{X}) = H^1(X) \oplus H^1(X)$ and any cohomology class is represented by a unique harmonic form by Hodge theory. Also we have the Hodge decomposition into holomorphic and anti-holomorphic forms; eigenspaces of $*$.

There is an obvious orientation-reversing diffeomorphism of the double; call it reflection and denote it by τ. τ and $*$ anti-commute and generate $\mathbb{Z}/2 \ltimes \mathbb{Z}/4$. The effect of τ on the harmonic forms is to divide them into even and odd under τ. The odd forms vanish on S, and since τ has to flip the two factors, one sees half the harmonic forms are even and half are odd, and the former are the ones vanishing on the boundary and the latter ones vanishing normal to the boundary. $*$ interchanges the two.

So let us summarize. There's a way to identify the space $K + *K$ of harmonic 1-forms on X with the space of harmonic forms on the double. K is identified with the odd harmonic forms on the double and $*K$ with the even forms. The best way to say it is that the restriction map from harmonic 1-forms on the double to harmonic forms on X is injective and has image $K + *K$. This restriction map is compatible with $*$, so that the spaces $L, \bar{L}$ are the restrictions of the holomorphic and anti-holomorphic 1-forms on the double. Now

So we have a simple interpretation of the space $K + *K = L \oplus \bar{L}$ of harmonic 1-forms in which we are working. Now we would the rest of the structure interpreted in the double. We want the symplectic form on K and the complementary space B which consists of differentials

Of certain harmonic functions

B is easy to describe. It is the subspace of harmonic forms on $X \cup \bar{X}$ which become exact on X.

We seem to have then the following structure on $H^1(X \cup \bar{X}, \mathbb{C})$. Involution τ, symplectic form reversed by τ, $*$ operator reversed by τ, and a decomposition $H^1(X \cup \bar{X}, \mathbb{C}) = H^1(X, \mathbb{C}) \oplus H^1(X, \mathbb{C})$ flipped by τ. It seems then that I ought to be able to recover everything.

So let's see if I understand the picture well enough to translate everything in terms of the cohomology of the double. On the cohomology $H^1(X \cup \bar{X}, \mathbb{R})$ I have the following structure

symplectic form $\quad \langle \omega, \omega' \rangle = \int \omega\omega'$

involution $\tau \quad \ni \quad \langle \tau\omega, \tau\omega' \rangle = -\langle \omega, \omega' \rangle$

decomposition $\quad H^1(X \cup \bar{X}, \mathbb{R}) = B \oplus \tau B$

(here $B = \operatorname{Ker}\{H^1(X \cup \bar{X}, \mathbb{R}) \longrightarrow H^1(X, \mathbb{R})\}$)

complex structure $* \quad \ni \quad *^2 = -1$

$$\tau * = - * \tau$$

$$\int \omega \cdot * \omega > 0$$

Let's represent the cohomology $H^1(X \cup \bar{X}, \mathbb{R})$ by the space of harmonic 1-forms on $X \cup \bar{X}$ and consider the restriction map to harmonic 1-forms on X. This is injective. The harmonic forms on $X \cup \bar{X}$ break into odd and even under τ, and these two types are interchanged by $*$. We can also spot them whether they vanish on S or whether their normal derivative does. So K the harm. 1-forms on X vanishing on the boundary is the image of the $\tau = -1$ eigenspace.

The problem with this is that the symplectic forms do not agree. Thus if ω, ω' are harmonic forms on $X \cup X'$ which are odd under τ, i.e. vanish tangentially to the boundary, then

$$\int \omega\omega' = -\int \tau(\omega)\tau(\omega') = -\int \omega\omega'$$

vanishes.

So let's return to our initial setup where we let K = harmonic 1-forms on X vanishing on ∂X and work with $K + *K = L \oplus \bar{L}$, and we give inside this space B = the harmonic 1-forms in $K + *K$ which are exact. This is sort of a mess - let's try to get it more specific. Remember that B, K are non-degenerate and orthogonal for the symplectic form.

So the pattern is this: You are given a real symplectic space V decomposed into $B \oplus K$ where B, K are non-degenerate and orthogonal. Next you are given a complex structure $*$ on V, and a self-dual lattice in K. Think of the complex structure as some sort of correspondence from K to B and the lattice as defining a state in $\mathcal{F}_K$. You then want the image of this state in $\mathcal{F}_B$ under this correspondence.

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Let V be a real symplectic vector space and let Γ be a lattice which is self-dual. Then Γ operates on the canonical representation $\mathcal{F}_V$ of V, and the representation is of multiplicity one. One can realize the representation as the functions on $\hat{\Gamma}$. Therefore there is a unique eigenvector fixed under Γ except that it is a distribution, the δ fn. at $0 \in \hat{\Gamma}$.

Now let Γ become complex, i.e. we consider a lattice $\Gamma \subset V_c$, which is self dual. This means that $\operatorname{rank}(\Gamma) = \dim V$, and that $\Gamma = \{ v \in V_c \mid \frac{1}{i}[v,\gamma] \in 2\pi\mathbb{Z}$ for all $\gamma \in \Gamma \}$

It seems that if Γ satisfies a positivity condition then there is a unique vector in $\mathcal{F}_V$ fixed under the operators $e^{i\gamma}$.

I want to do some calculations where V is 2 dimensional. One thing we can probably do in general is to choose a maximal isotropic subgroup of Γ; this is the choice of a, b cycles. Then the complex subspace W spanned by this isotropic subgroup is maximal isotropic in V_c. If we assume $[\gamma, \gamma^*] > 0$ for $0 \neq \gamma \in \Gamma$, then this gives a polarization of V.

Let's take $V_c = W \oplus \overline{W}$, $W = \mathbb{C}a$ as usual and let $i\Gamma$ be spanned by

$$(2\pi i)a \quad \text{and} \quad \lambda a + a^*$$

We look for analytic functions $f(z)$ fixed under the operators

$$(e^{2\pi i a} f)(z) = f(z + 2\pi i)$$

$$(e^{\lambda a + a^*} f)(z) = e^{\frac{1}{2}\lambda} e^{z} f(z+\lambda)$$

In effect $\quad e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$ so

$$e^{a^*+\lambda a} = e^{-\frac{1}{2}[a^*,\lambda a]}\, e^{a^*} e^{\lambda a} = e^{\frac{1}{2}\lambda} e^{z} e^{\lambda \partial_z}$$

The equation $f(z+2\pi i) = f(z)$ says $f(z) = \sum c_n (e^z)^n$ and the second gives

$$f(z) = e^{\frac{1}{2}\lambda} e^{z} f(z+\lambda)$$

$$\sum c_n e^{nz} = e^{\frac{1}{2}\lambda} \sum c_n e^{z+n(z+\lambda)}$$

$$= e^{\frac{1}{2}\lambda} \sum c_{n-1} e^{(n-1)\lambda} e^{nz}$$

$$c_n = e^{\frac{1}{2}\lambda + (n-1)\lambda} c_{n-1}$$

$$c_n = e^{\frac{n^2}{2}\lambda} c_0$$

Thus we have a unique entire function (up to scalar factor) fixed under $e^{i\Gamma}$ which is

$$f(z) = \sum_{n\in\mathbb{Z}} e^{\frac{n^2}{2}\lambda + nz}$$

We have to assume that $\mathrm{Re}(\lambda) < 0$ in order that this series converges.

The next question is whether it is in the Hilbert space, i.e. $\|f\|^2 = \int e^{-|z|^2} |f(z)|^2 \frac{d^2z}{\pi} < \infty$.

Recall

$$\langle e^{c'z}, e^{cz} \rangle = e^{\bar{c}'c}$$

Hence

$$\|f\|^2 = \sum_{m,n} e^{\frac{m^2}{2}\bar{\lambda} + \frac{n^2}{2}\lambda + mn}$$

The real part of the quadratic form is

$$-\frac{1}{2}\left(\ell m^2 + \ell n^2 - 2mn\right) \qquad \ell = -\mathrm{Re}(\lambda)$$

and this is negative definite when

$$\begin{vmatrix} \ell & -1 \\ -1 & \ell \end{vmatrix} = \ell^2 - 1 > 0.$$

There the condition is that $Re(\gamma) < -1$. However this is just the positivity condition on the lattice. Thus suppose $\gamma = m(2\pi i)a + n(\lambda a + a^*)$ where $m, n \in \mathbb{Z}$, and we want to test $[\gamma, \gamma^*] \geqslant 0$. Can divide by n + ask for

$$\left[ixa + \lambda a + a^*, \ (\overline{ix+\lambda})a^* + a \right] \geqslant 0$$

$$|ix + \lambda|^2 - 1 \geqslant 0$$

for all $x = 2\pi \frac{m}{n} \in 2\pi\mathbb{Q}$. This gives $|Re\,\lambda|^2 > 1$, but not the condition that $Re\,\lambda < 0$.

Next let's set up the general case. We suppose V real symplectic of dimension $2g$ and $\Gamma \subset V_{\mathbb{C}}$ a lattice of rank $2g$ such that

$$\Gamma = \check{\Gamma} \overset{\text{def}}{=} \{ v \in V_{\mathbb{C}} \mid [v, \gamma] \in 2\pi i \mathbb{Z} \quad \forall \gamma \in \Gamma \}$$

and such that $[\gamma, \gamma^*] > 0$ for all $0 \neq \gamma \in \Gamma$. Then we choose a symplectic basis for Γ; apparently Γ always contains complementary maximal isotropic subgroups. To see that choose γ so that $\mathbb{Z}\gamma \subset \Gamma$ is a direct summand (i.e. γ generates $\mathbb{Q}\gamma \cap \Gamma$), then $\exists\ \lambda : \Gamma \longrightarrow \mathbb{Z}$ with $\lambda(\gamma) = 1$; λ is represented by $\gamma' \in \Gamma$ so γ, γ' generate a $\mathbb{Z}^2 \subset \Gamma$ on which the symplectic form is a perfect pairing; this splits off a symplectic $\mathbb{Z}^2$ and one proceeds inductively.

Choose then a symplectic basis for Γ call it $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ where the only non-zero pairings are $[\alpha_i, \beta_i] = 2\pi i$

Now we polarize V by taking W to be spanned by $\alpha_1, \dots, \alpha_g$.

W inherits a hermitian inner product $[w, w^*]$ and the Fock space is given by holomorphic functions on W square integrable with respect to the Gaussian measure $e^{-\|w\|^2} \frac{d^{2g}w}{\pi^g}$. [redacted]

Actually before I proceed I should go do the $g=1$ case more carefully. I should have taken $\iota\Gamma$ to have the generators

$$2\pi i \varepsilon^{-1} a$$

$$\lambda a + \varepsilon a^*$$

where I can suppose $\varepsilon > 0$ [redacted] by rotation in W.

Then

$$f = e^{2\pi i \varepsilon^{-1} a} f \implies f(z) = f(z + 2\pi i \varepsilon^{-1}) \implies f = \sum c_n e^{n\varepsilon z}$$

$$f = e^{\frac{1}{2}\lambda\varepsilon} e^{\varepsilon z} e^{\lambda \partial_z} f \implies \sum c_n e^{n\varepsilon z} = e^{\frac{1}{2}\lambda\varepsilon} e^{\cancel{\varepsilon z}} \sum_n c_{n-1} e^{n\varepsilon z + n\varepsilon\lambda - \varepsilon\lambda}$$

$$\implies c_n = e^{(n-\frac{1}{2})\varepsilon\lambda} c_{n-1} \implies c_n = e^{\frac{n^2}{2}\varepsilon\lambda}$$

So $f(z) = \sum e^{\frac{n^2}{2}\varepsilon\lambda + n\varepsilon z}$ and

$$\|f\|^2 = \sum e^{\frac{m^2}{2}\overline{\varepsilon\lambda} + \frac{n^2}{2}\varepsilon\lambda + mn|\varepsilon|^2}$$

This time we need to have $\mathrm{Re}(\varepsilon\lambda) < 0$ and

$$\begin{pmatrix} \mathrm{Re}(\varepsilon\lambda) & |\varepsilon|^2 \\ |\varepsilon|^2 & \mathrm{Re}(\varepsilon\lambda) \end{pmatrix} < 0$$

and [redacted] the second condition includes the first.

This calculation shows there is a single positivity condition, but it is not just $[\gamma, \gamma^*] > 0$ for $0 \neq \gamma \in \Gamma$.

Next let us do the general case. We have the orthonormal basis for W given by the annihilation ops $a_1, \dots, a_g$ and another basis $\alpha_1, \dots, \alpha_g$ in $i\Gamma \cap W$. We want to use the representation of the commutation relations on the holomorphic functions $f(z)$, where $z = (z_1, \dots, z_g)$ and $a_i = \partial_{z_i}$, $a_i^* = z_i$. Thus e^{α_j} will be the translation operator

$$e^{\alpha_j} f(z) = f(z + \alpha_j)$$

and an f invariant under these translations is a linear combination of exponentials

$$e^{\lambda^t z}$$

where $\lambda^t \alpha_j \in 2\pi i \mathbb{Z}$ for $j = 1, \dots, g$. To keep with previous notation, let us write $\lambda^t z = n^t \varepsilon z$ with $n = (n_1, \dots, n_g) \in \mathbb{Z}^g$. Then we want $\varepsilon \alpha_j \in 2\pi i \mathbb{Z}^g$ which means the simplest choice for the α_j is the columns of the matrix $2\pi i \varepsilon^{-1}$.

Let us then write

$$\alpha_j = \sum_i \alpha_{ij} a_i \qquad \alpha \in GL_n(\mathbb{C})$$

where $\alpha = 2\pi i \varepsilon^{-1}$. Then

$$e^{\sum_i \alpha_{ij} \partial_{z_i}} e^{n^t \varepsilon(z)} = e^{n_k \varepsilon_{k\ell} z_\ell} e^{[\alpha_{ij} \partial_{z_i}, n_k \varepsilon_{k\ell} z_\ell]}$$

$$= e^{n_k \varepsilon_{k\ell} z_\ell + \underbrace{n_k \varepsilon_{ki} \alpha_{ij}}_{2\pi i \delta_{kj}}} = e^{n^t \varepsilon z}$$

Next we need the other generators for Γ. These should be

$$\beta = \lambda a + \varepsilon a^*$$

and more precisely

$$\beta_j = \lambda_{kj} a_k + \varepsilon_{jk} a_k^*$$

Check: $[\alpha_j, \beta_q] = [2\pi i\, \alpha_{pj} a_p,\ \lambda_{kq} a_k + \varepsilon_{qk} a_k^*]$

$$= 2\pi i\ \underbrace{\alpha_{pj}\,\delta_{pk}\,\varepsilon_{qk}}\ = 2\pi i\ \varepsilon_{qk}\alpha_{kj} = 2\pi i \delta_{jq}$$

Thus the formulas ~~___~~ are

$$\begin{cases} \alpha = a^t (2\pi i)\varepsilon^{-1} \\ \beta = a^t \lambda + \varepsilon \cdot a^* \end{cases}$$

So now we have $f(z) = \sum c_n e^{n^t \varepsilon z}$ invariant under the translation coming from the α_j, and we want it to be invariant under e^β. Thus

$$f(z) = e^{\beta_j} f(z) = e^{\lambda_{kj} a_k + \varepsilon_{jk} a_k^*} f(z)$$

$$= e^{\frac{1}{2}\varepsilon_{jk}\lambda_{kj}}\ e^{\varepsilon_{j\ell} z_\ell} f(z_k + \lambda_{kj})$$

$$= e^{\frac{1}{2}\varepsilon_{jk}\lambda_{kj}}\ e^{\varepsilon_j z} \sum c_n e^{n^t \varepsilon (z + \lambda_j)}$$

$$= e^{\frac{1}{2}\varepsilon_{jk}\lambda_{kj}} \sum c_n e^{(n + \hat{e}_j)^t \varepsilon z + n^t \varepsilon \lambda \hat{e}_j}$$

$$= e^{\frac{1}{2}\hat{e}_j \varepsilon\lambda \hat{e}_j} \sum c_{n-\hat{e}_j}\, e^{n^t \varepsilon z + (n - \hat{e}_j)^t \varepsilon \lambda \hat{e}_j}$$

$$\boxed{c_n = e^{(n - \frac{1}{2}\hat{e}_j)^t \varepsilon\lambda \hat{e}_j}\ c_{n-\hat{e}_j}}$$ recursion relations.

Here $\hat{e}_j$ is the unit vector in the jth direction. Now $\varepsilon\lambda$ should be symmetric:

$$0 = [\beta_i, \beta_j] = [\lambda_{ki} a_k + \varepsilon_{ik} a_k^*, \lambda_{\ell j} a_\ell + \varepsilon_{j\ell} a_\ell^*]$$

$$= \lambda_{ki}\varepsilon_{jk} - \varepsilon_{ik}\lambda_{kj} = (\varepsilon\lambda)_{ji} - (\varepsilon\lambda)_{ij}$$

and so the solution to the recursion relation is the same

$$c_n = e^{\frac{1}{2} n^t \varepsilon\lambda n} c_0$$

Thus we have that the unique fixed vector is

$$f(z) = \sum_n e^{\frac{1}{2} n^t \varepsilon\lambda n + n^t \varepsilon z}.$$

$$\overline{m_k \varepsilon_{k\ell} z_\ell}\ \underbrace{n_i \varepsilon_{ij} z_j} = m_k \overline{\varepsilon_{k\ell}}\, \varepsilon_{i\ell} n_i$$

Then

$$\|f\|^2 = \sum_{m,n} e^{\frac{1}{2} m^t \overline{\varepsilon\lambda} m + \frac{1}{2} n^t \varepsilon\lambda n + n^t \varepsilon\varepsilon^* m}$$

which will be finite provided the matrix

$$\mathrm{Re}\begin{pmatrix} \overline{\varepsilon\lambda} & \varepsilon\varepsilon^* \\ (\varepsilon\varepsilon^*)^t & \varepsilon\lambda \end{pmatrix}$$

is negative definite.

Problem: Interpret this positivity geometrically e.g without <u>choosing half of the lattice Γ.</u>

Idea: The real subspaces V and $\mathbb{R}\Gamma$ in V_c have conjugations attached. In the case of V the conjugation is adjoint: $x \longrightarrow x^*$. The symplectic form is $\frac{1}{i}[x,y]$ and this is preserved up to conjugating the values

$$\overline{\left(\frac{1}{i}[x,y]\right)} = -\frac{1}{i}[x,y]^* = \frac{1}{i}[x^*,y^*]$$

similarly $\mathbb{R}\Gamma$ has an involution σ attach which is $+1$ on $\mathbb{R}\Gamma$ and -1 on $i\mathbb{R}\Gamma$, and the skew form is real on $\mathbb{R}\Gamma$. Thus we have

$$\frac{1}{i}[\sigma x, \sigma y] = \overline{\frac{1}{i}[x,y]}$$

since this is true for the four cases $x \in \mathbb{R}\Gamma$ or $i\mathbb{R}\Gamma$ and similarly for y; or both sides are conjugate linear in x, y.

Then $x \mapsto \sigma(x^*)$ is complex linear and symplectic:

$$\frac{1}{i}[x,y] = \frac{1}{i}\overline{[x^*,y^*]} = \frac{1}{i}[\sigma(x^*), \sigma(y^*)].$$

Finally there is a natural positivity condition for a complex symplectic transformation, namely that the graph should be a polarization.

Let's calculate $x \mapsto \sigma(x^*)$ or its inverse $x \mapsto \sigma(x)^*$ in the example where

Γ:	$\frac{2\pi i}{\varepsilon} a$	$\left(\frac{\varepsilon}{2\pi i}\right)\Gamma$:	a
	$\lambda a + \varepsilon a^*$		$\frac{\lambda \varepsilon a + \varepsilon^2 a^*}{2\pi i}$

This supposes ε real, say $\varepsilon > 0$.

Thus $\boxed{\sigma(a) = -a}$ and supposing $\varepsilon > 0$

$$\lambda a + \varepsilon a^* = \sigma(\lambda a + \varepsilon a^*) = -\bar{\lambda} a + \varepsilon \sigma(a^*) \quad \text{or}$$

$$\boxed{\sigma(a^*) = \frac{1}{\varepsilon}(\lambda + \bar{\lambda})\, a + a^*}$$

I want to take now kernels

$$K = \sum |u_i\rangle \otimes \langle v_i| \quad \in \mathcal{F} \otimes \overline{\mathcal{F}}$$

which are to be interpreted as operators on $\mathcal{F}$ in the usual way:

$$K|w\rangle = \sum |u_i\rangle \otimes \langle v_i | w\rangle$$

I would like K to have the property that it transforms a to $a' = \lambda a + \mu a^*$ in the sense that

$$a'K = Ka \qquad \text{i.e.} \quad a' = KaK^{-1}$$

Thus we want $K \in \mathcal{F} \otimes \overline{\mathcal{F}}$ to be killed by $a' \otimes 1 - 1 \otimes a^*$, i.e.

$$\sum_i |a'u_i\rangle \otimes \langle v_i| = \sum_i |u_i\rangle \otimes \langle a^* v_i|$$

Similarly if $(a^*)' = K(a^*)K^{-1}$, then K should be killed by $(a^*)' \otimes 1 - 1 \otimes a$. It seems convenient to change notation $a \otimes 1 \to a$, $1 \otimes a \to b$ etc. Then our kernel K is killed by $a' - b^*$ and $(a^*)' - b$ where

$$\begin{pmatrix} a \\ a^* \end{pmatrix}' = K \begin{pmatrix} a \\ a^* \end{pmatrix} K^{-1} = \begin{pmatrix} \lambda a + \mu a^* \\ \nu a + \pi a^* \end{pmatrix}$$

Then we must have

$$1 = [a', (a^*)'] = [\lambda a + \mu a^*, \nu a + \pi a^*] = \lambda\pi - \mu\nu.$$

which checks with

$$[a' - b^*, (a^*)' - b] = [a', (a^*)'] + \overset{-1}{[b^*, b]} = 0.$$

We also need positivity, that for any $\xi \in \mathbb{C}(a' - b^*) + \mathbb{C}(a^{*\prime} - b)$ one has $[\xi, \xi^*] \geq 0$.

Let's take a slightly more general approach

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~~and suppose that~~ and work with the real symplectic space V. Let the symplectic transf be $v \longmapsto v'$. Then we want $K \in \mathcal{F} \otimes \mathcal{F}$ to be killed by the operators $v' \otimes 1 - 1 \otimes v$ for $v \in V$. Thus we need

$$[v' \otimes 1 - 1 \otimes v,\ w' \otimes 1 - 1 \otimes w] = [v', w'] + [v, w]$$

$$0 = [v', w'] - [v, w]$$

showing $v \longmapsto v'$ has to be symplectic. Positivity means

$$[v' \otimes 1 - 1 \otimes v,\ (v')^* \otimes 1 - 1 \otimes v] > 0 \qquad \text{if } 0 \neq v \in V$$

i.e. $\qquad [v', (v')^*] > 0$

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Let V be a real symplectic vector space of dim $2g$, and let $\mathcal{F}$ be its Heisenberg representation. If $v \to T(v)$ is a symplectic transformation of V, one knows there is a unitary operator K on $\mathcal{F}$, unique up to scalars, such that

$$T(v) = K v K^{-1}$$

More generally we can analytically continue and associate operators to complex symplectic transformations satisfying a certain positivity condition. In this case K is given by a kernel, i.e., a vector in $\mathcal{F} \otimes \mathcal{F}^*$.

Let's write $\rho(v)$, or simply v, for the operator on $\mathcal{F}$ corresponding to v. Then $\rho(v) = \rho(v)^\dagger$ and

$$\frac{1}{i}[\rho(v), \rho(v')] = \text{symplectic form } \Omega(\omega, \omega')$$

We define $\check{\rho}(v)$ on $\mathcal{F}^*$ by ~~$\check{\rho}(v)(\lambda) = -\lambda\rho(v)$~~

~~Then $\mathcal{F}^* \otimes \mathcal{F} \longrightarrow \mathbb{C}$, $(\lambda, a) \to \lambda(a)$ satisfies~~

$$\check{\rho}(v)\langle\xi| = -\langle\xi|\rho(v)$$

Then the pairing

$$\mathcal{F}^* \otimes \mathcal{F} \longrightarrow \mathbb{C}$$

$$\langle\xi| \otimes |\eta\rangle \longmapsto \langle\xi|\eta\rangle$$

satisfies

$$\left(\check{\rho}(v)\otimes 1 + 1 \otimes \rho(v)\right)\left(\langle\xi| \otimes |\eta\rangle\right) = -\langle\xi|\rho(v) \otimes |\eta\rangle + \langle\xi| \otimes \rho(v)|\eta\rangle$$

$$\longmapsto -\langle\xi|\rho(v)|\eta\rangle + \langle\xi|\rho(v)|\eta\rangle = 0$$

Also

$$\check{\rho}(v_1)\check{\rho}(v_2)\langle\xi| = -\check{\rho}(v_1)\langle\xi|\rho(v_2) = \langle\xi|\rho(v_2)\rho(v_1)$$

so $\quad [\check{\rho}(v_1), \check{\rho}(v_2)]\langle\xi| = \langle\xi|\,[\rho(v_2), \rho(v_1)]$

$$= -\left([v_1, v_2]\right)\langle\xi|$$

Thus for the contragredient repn we have

$$\boxed{[\check{\rho}(v_1), \check{\rho}(v_2)] = -[v_1, v_2]}$$

that is the opposite sign for the symplectic form.

Now let $\quad K = \sum_i |\xi_i\rangle \otimes \langle\eta_i| \quad \in \mathcal{F}\otimes\mathcal{F}^*$ represent an operator consistent with a transformation $v \longmapsto v'$ from V_c to V_c. Thus

$$v'K = Kv \qquad \forall v \in V$$

or

$$(\rho(v')\otimes 1 + 1\otimes\check{\rho}(v))\,K = 0.$$

It follows that the scalar

$$[\rho(v_1')\otimes 1 + 1\otimes\check{\rho}(v_1),\ \rho(v_2')\otimes 1 + 1\otimes\check{\rho}(v_2)]$$

$$= [v_1', v_2'] - [v_1, v_2]$$

is zero, i.e. that $v \mapsto v'$ is symplectic or equivalently that the graph of (v', v) in $V_c \oplus V_c$ is isotropic, where the second factor has the opposite sign for the symplectic form. Also the scalar

$$[\rho(v')\otimes 1 + 1\otimes\check{\rho}(v),\ \rho(v')^{+}\otimes 1 + 1\otimes\check{\rho}(v)^{+}]$$

$$= [v', v'^{+}] - [v, v^{+}] \geq 0$$

because we know its matrix element

$$\langle K|?|K\rangle \quad \text{is} \quad \|(\rho(\sigma')^{+}\otimes 1 + 1\otimes \check{\rho}(\sigma)^{+})|K\rangle\|^2$$

Summaring the conditions we want on our transformation $\sigma \longmapsto \sigma'$ are

$$[\sigma_1', \sigma_2'] = [\sigma_1, \sigma_2] \qquad \text{(symplectic)}$$

$$[\sigma', \sigma'^{+}] \geqslant [\sigma, \sigma^{+}] \qquad \text{(positivity)}$$

and we want the latter to be $>$ for $\sigma \neq 0$.

Let's check carefully that this positivity condition coincides with what we found before for the lattice Γ generated by

$$2\pi i \varepsilon^{-1} a$$

$$\lambda a + \varepsilon a^*$$

Γ is the image of $(2\pi\mathbb{Z})q + \mathbb{Z}p$ under the symplectic transformation T such that

$$T(q) = i\varepsilon^{-1} a$$

$$T(p) = \lambda a + \varepsilon a^*$$

We now wish to check

$$[T(zq+\omega p), T(zq+\omega p)^{+}] \overset{?}{\geqslant} [zq+\omega p, \bar{z}q+\bar{\omega}p]$$

We can suppose $\omega = i$, whence

$$[zi\varepsilon^{-1}a + i(\lambda a + \varepsilon a^*), (\qquad)^{+}] \overset{?}{\geqslant} z+\bar{z}$$

$$|z\varepsilon^{-1}+\lambda|^2 - |\varepsilon|^2 \overset{?}{\geqslant} z+\bar{z}$$

Now I've seen before that I can suppose $\varepsilon > 0$. We thus need to see when

$$|z+\lambda|^2 - \varepsilon^2 \overset{?}{\geqslant} \varepsilon\, 2\mathrm{Re}(z)$$

Put in $z = x + iy$ and then by taking $y = -\operatorname{Im}\lambda$ you minimize the LHS without changing the right side. If $r = \operatorname{Re}(\lambda)$ we want

$$(x+r)^2 - \varepsilon^2 \overset{?}{\geqslant} 2\varepsilon x$$

$$x^2 + 2(r-\varepsilon)x + r^2 - \varepsilon^2 \overset{?}{\geqslant} 0$$

$$(r-\varepsilon)^2 \overset{?}{\leqslant} r^2 - \varepsilon^2$$

$$0 \leqslant 2r\varepsilon - 2\varepsilon^2 \quad \text{or finally} \quad r \geqslant \varepsilon.$$

This isn't quite the right ~~sign~~, but it does include a condition on the ~~sign~~ of $\operatorname{Re}(\lambda)$.

I can probably explain the sign. It is probably due to calculating $\check{\rho}(\sigma)^\dagger$ incorrectly. Do this carefully.

A linear fnl on $\mathcal{F}$ is of the form $\langle \xi |$ with $\xi \in \mathcal{F}$. We have defined

$$\check{\rho}(\sigma) \langle \xi | = - \langle \xi | \rho(\sigma) = - \langle \rho(\sigma)^\dagger \xi |$$

Now the inner product of this and $\langle \eta |$ is

$$\big\langle \langle \eta | \,\big|\, \check{\rho}(\sigma) \langle \xi | \big\rangle \overset{\text{defn}}{=} \big(\check{\rho}(\sigma)\langle \xi |\big) | \eta \rangle = - \langle \rho(\sigma)^\dagger \xi | \eta \rangle$$

$$= \langle \xi | -\rho(\sigma)\eta \rangle = \big\langle \langle -\rho(\sigma)\eta | \,\big|\, \langle \xi | \big\rangle$$

Thus

$$\check{\rho}(\sigma)^\dagger \langle \eta | = \langle -\rho(\sigma)\eta | = - \langle \eta | \rho(\sigma)^\dagger$$

and

$$\check{\rho}(\sigma)\, \check{\rho}(\sigma)^\dagger \langle \eta | = \check{\rho}(\sigma) \langle \eta | -\rho(\sigma)^\dagger$$

$$= \langle \eta | \rho(\sigma)^\dagger \rho(\sigma)$$

$$\check{\rho}(\sigma)^\dagger \check{\rho}(\sigma) \langle \eta | = \check{\rho}(\sigma)^\dagger \big(\langle \eta | -\rho(\sigma)\big) = \langle \eta | \rho(\sigma) \rho(\sigma)^\dagger$$

so

$$[\check{\rho}(v), \check{\rho}(v)^\dagger] = [\rho(v)^\dagger, \rho(v)] = -[v, v^\dagger]$$

Well this is what I used at the bottom of p. 851, so the sign has not been explained.

The preceding is not very clear. Let us start again.

I have a transformation $v \longmapsto v'$ on V_c and I would like to find an operator K such that

$$\rho(v')K = K\rho(v) \qquad \forall v \in V_c$$

Moreover I want $K = \sum u_i \otimes \lambda_i \in \mathcal{F} \otimes \mathcal{F}^*$.

Thus

$$\rho(v')Ku = \sum \rho(v')u_i \lambda_i(u)$$

$$= K\rho(v)u = \sum u_i \lambda_i(\rho(v)u)$$

i.e. K is killed by $\rho(v') \otimes 1 - 1 \otimes \rho(v)^t$. Then

$$0 = [\rho(v_1') \otimes 1 - 1 \otimes \rho(v_1)^t, \rho(v_2') \otimes 1 - 1 \otimes \rho(v_2)^t]$$

$$= [\rho(v_1'), \rho(v_2')] + \underbrace{[\rho(v_1)^t, \rho(v_2)^t]}_{-[\rho(v_1), \rho(v_2)]^t}$$

$$= [v_1', v_2'] - [v_1, v_2]$$

since the transpose of a scalar is the same scalar.

Next we want to calculate

$$[\rho(v')\otimes 1 - 1 \otimes \rho(v)^t, (\rho(v') \otimes 1 - 1 \otimes \rho(v)^t)^\dagger]$$

$$= [\rho(v'), \rho(v')^\dagger] + [\rho(v)^t, (\rho(v)^t)^\dagger]$$

I claim that

$$\left(\rho(\sigma)^t\right)^\dagger = \left(\rho(\sigma)^\dagger\right)^t.$$

To see this take two elements $\lambda, \mu \in \mathcal{F}^*$ and represent them as $\langle x|$, $\langle y|$ with $x, y \in \mathcal{F}$. Then by definition of the inner product on $\mathcal{F}^*$ we have

$$\langle \lambda | \mu \rangle = \langle y | x \rangle .$$

Now

$$\left(\rho(\sigma)^t \lambda\right)(u) = \lambda\left(\rho(\sigma) u\right) = \langle x | \rho(\sigma) u \rangle$$

$$= \langle \rho(\sigma)^\dagger x | u \rangle$$

so $\rho(\sigma)^t \lambda$ is rep. by $\langle \rho(\sigma)^\dagger x |$, and so

$$\langle \rho(\sigma)^t \lambda | \mu \rangle = \langle y | \rho(\sigma)^\dagger x \rangle$$

$$\langle \lambda | \left(\rho(\sigma)^t\right)^\dagger \mu \rangle \quad = \langle \rho(\sigma) y | x \rangle$$

so

$$\left(\rho(\sigma)^t\right)^\dagger \mu = \langle \rho(\sigma) y | = \langle y | \rho(\sigma)^\dagger$$

$$= \left(\rho(\sigma)^\dagger\right)^t \mu .$$

as claimed.

Thus

$$\left[\rho(\sigma)^t, \left(\rho(\sigma)^t\right)^\dagger\right] = \left[\rho(\sigma)^t, \left(\rho(\sigma)^\dagger\right)^t\right]$$

$$= - \left[\rho(\sigma), \rho(\sigma)^\dagger\right]$$

and again we get the positivity condition

$$\left[\rho(\sigma'), \rho(\sigma')^\dagger\right] - \left[\rho(\sigma), \rho(\sigma)^\dagger\right] \geq 0 .$$

As a check consider $K = e^{-\omega a^* a}$.

Then $K a K^{-1} = e^{-\omega a^* a}\, a\, e^{\omega a^* a} = e^{\omega} a$

$K a^* K^{-1} = e^{-\omega} a^*$

and $\left[z(e^{\omega} a) + w(e^{-\omega} a^*), (\quad)^{\dagger} \right] = e^{2\omega}|z|^2 - e^{-2\omega}|w|^2$

$\left[za + wa^*, (\quad)^{\dagger} \right] = |z|^2 - |w|^2$

and $e^{2\omega}|z|^2 - e^{-2\omega}|w|^2 \overset{?}{\geqslant} |z|^2 - |w|^2$

$\underbrace{(e^{2\omega} - 1)}_{>0}|z|^2 \geqslant \underbrace{(e^{-2\omega} - 1)}_{<0}|w|^2$ OK.

Now I want to consider the complex symplectic transformation T such that

$q' = T(q) = i\varepsilon^{-1} a$

$p' = T(p) = \lambda a + \varepsilon a^*$

Then if [redacted] K exists with $q'K = Kq$, we have $e^{2\pi q'}K = K e^{2\pi q}$, and we now find the error as we want $e^{2\pi i q}$. Thus we want

$(iq)' = i\varepsilon^{-1} a$ $\qquad q' = \varepsilon^{-1} a$

$(-ip)' = \lambda a + \varepsilon a^*$ $\qquad p' = i(\lambda a + \varepsilon a^*)$

which means we test

$$\Big[\underbrace{(-iz\varepsilon)(iq)' + (-ip)'}_{(z+\lambda)a + \varepsilon a^*}, (\quad)^*\Big] \overset{?}{\geqslant} \Big[\underbrace{(-iz\varepsilon)(iq) - ip}_{z\varepsilon q - ip}, \underbrace{(\quad)^*}_{\bar{z}\varepsilon q + ip}\Big]$$

$$\underline{|z+\lambda|^2 - \varepsilon^2 \overset{?}{\geqslant} -z\varepsilon - \bar{z}\varepsilon}$$ It works!

Here is a simple version of the positivity condition. Recall we have V, $R\Gamma$ two real subspaces of V_c on which $[\ ,\ ]$ is purely imaginary. We get a complex symplectic transformation $v \mapsto v'$ of V_c by choosing an isomorphism of V with $R\Gamma$. Then we want to check $[v', v'^*] \geq [v, v^*]$. But

$$[v', (v')^*] = [v, ((v')^*)'^{-1}]$$

better, let $v' = T(v)$, and then we have

$$[v, T^{-1}((Tv)^*)] \geq [v, v^*]$$

But what is the transformation $T^{-1} * T$? It's the conjugation associated to $T^{-1}(V)$.

Better, we want to check

$$[Tv, (Tv)^*] \geq [v, v^*]$$

or

$$[v, v^*] \geq [T^{-1}v, (T^{-1}v)^*]$$

$$\qquad\qquad\qquad\qquad \| $$

$$[v, T(*(T^{-1}v))]$$

and $T * T^{-1}$ is the involution σ associated to the real subspace $R\Gamma$. Thus the positivity condition is

$$[v, v^*] \geq [v, \sigma(v)]$$

where $v \mapsto v^*$ is the conjugation defined by V and σ is the conjugation defined by $R\Gamma$.

Let V be a real symplectic vector space of dimension $2g$, and let $\mathcal{F}_V$ be the associated Heisenberg representation. To each $v \in V$ we have an hermitian operator $\{\longmapsto v\}$ on $\mathcal{F}_V$ linear in v such that

$$[v, v'] = i\,\omega(v, v')$$

where ω is the symplectic form. We extend $\mathbb{C}$-linearly to $V_c = V \otimes \mathbb{C}$, so that the operator v^* adjoint to v is the operator associated to $\bar{v}$.

Suppose we have a linear transformation $T: V_c \longrightarrow V_c$ such that there exists an operator K satisfying

$$T(v)\,K = K v. \qquad v \in V_c$$

Furthermore suppose $K = \sum_i u_i \otimes \lambda_i \in \mathcal{F} \otimes \mathcal{F}^*$ (dual space). Then K is killed by $T(v) \otimes 1 - 1 \otimes v^t$ for all $v \in V_c$, where v^t denotes the operator on the dual

$$v^t \lambda = \lambda v.$$

We have (assuming $K \neq 0$)

$$[T(v_1) \otimes 1 - 1 \otimes v_1^t,\ T(v_2) \otimes 1 - 1 \otimes v_2^t]$$

$$= [Tv_1, Tv_2] \otimes 1 + 1 \otimes \underbrace{[v_1^t, v_2^t]}_{-[v_1, v_2]^t}$$

$$= [Tv_1, Tv_2] - [v_1, v_2] \qquad \text{(a scalar operator)}$$

$$= 0 \qquad \text{(since it kills } K\text{)}.$$

Thus T is a complex symplectic transformation.

Next we regard $\mathcal{F}^*$ as the Hilbert space conjugate to $\mathcal{F}$, by identifying $u \in \mathcal{F}$ with the linear functional $\langle u|$. Then if $\lambda = \langle x|$ $\mu = \langle y|$, we have $\langle \lambda | \mu \rangle = \langle y | x \rangle$.

We claim that

$$\boxed{(v^t)^* = (v^*)^t}$$

In effect if $\lambda(z) = \langle x | z \rangle$, then $(v^t\lambda)(z) = \langle x | v z \rangle = \langle v^* x | z \rangle$, hence

$$\langle v^t \lambda | \mu \rangle = \langle y | v^* x \rangle$$

$$\langle \lambda | \underbrace{(v^*)^t \mu}_{\text{rep by } v^{**}y} \rangle = \langle v y | x \rangle$$

proving $(v^t)^* = (v^*)^t$.

Thus

$$[T v \otimes 1 - 1 \otimes v^t, (T v \otimes 1 - 1 \otimes v^t)^*]$$

$$= [T v, (T v)^*] \otimes 1 + 1 \otimes [v^t, (v^t)^*]$$

$$= [T v, (T v)^*] - [v, v^*] \qquad \text{(a scalar operator)}$$

≥ 0 because its matrix coefficient $\langle K| \ |K\rangle$ is ≥ 0

In fact more is true, because if this matrix coefficient $= 0$ for some $v \neq 0$ in V_c, then K is killed by both $T v \otimes 1 - 1 \otimes v^t$ and its adjoint, and hence by their sum which is a nonzero hermitian operator in the representation of $V \oplus V$ on $\mathcal{F} \otimes \mathcal{F}^*$. But one knows the spectrum of such an operator is continuous, so it can't have a (normalizable) eigenvector. This shows the hermitian form $[T v, (T v)^*] - [v, v^*]$ is positive definite on V_c

Conversely if $T: V_c \to V_c$ is symplectic, then $W = \{T(v)\otimes 1 - 1 \otimes v^t \mid v \in V_c\}$ is a maximal isotropic subspace of $V_c \oplus V_c$, on which the hermitian form $[w, w^*]$ is positive-definite. Thus W is a polarization and there is a unique line in $\mathcal{F}\otimes\mathcal{F}^*$ killed by W. This proves

<u>Prop.</u> Given a [redacted] transformation $T: V_c \longrightarrow V_c$, there is a non-zero $K \in \mathcal{F}\otimes\mathcal{F}^*$ such that

$$T(v)K = Kv \qquad \forall v \in V_c$$

iff i) $[T(v_1), T(v_2)] = [v_1, v_2] \qquad \forall v_1, v_2 \in V_c$

ii) $[T(v), T(v)^*] - [v, v^*] \geq 0 \qquad \forall v \in V_c$ with equality iff $v = 0$.

Now suppose $\Gamma \subset V_c$ is a lattice of rank $2g$ which is self-dual in the sense that

$$\Gamma = \Gamma^0 \stackrel{\text{def}}{=} \{v \in V_c \mid [\gamma, v] \in 2\pi i \mathbb{Z}, \ \forall \gamma \in \Gamma\}$$

Then the real subspace [redacted] $\mathbb{R}\Gamma$ of V_c is such that the symplectic form restricted to $\mathbb{R}\Gamma$ is real. Hence there is a symplectic transformation T of V_c carrying V onto $\mathbb{R}\Gamma$.

Let σ be the conjugation on V_c with fixed subspace $\mathbb{R}\Gamma$. We shall reformulate[redacted] the positivity condition ii) in terms of σ. [redacted] Substituting $T^{-1}(v)$ for v in ii) yields

$$[v, v^*] \geq [T^{-1}v, (T^{-1}v)^*]$$
$$= [v, (T^* T^{-1})(v)] = [v, \sigma v]$$

June 9, 1987

Let Γ be a lattice in V_c such that it satisfies the conditions discussed above which ensure the existence of a unique fixed line $\mathbb{C}\phi$ under $e^{i\Gamma}$. I would like to get a more direct construction of $\mathbb{C}\phi$. The idea is to consider the function

$$F(v) = \langle \phi | e^{v} \phi \rangle \qquad v \in V_c$$

which for $v \in iV$ is just the positive definite function on the Heisenberg group associated to the cyclic vector ϕ.

Then F is an entire function on V_c $\ni$

$$\begin{aligned} F(v+i\gamma) &= \langle \phi | e^{v+i\gamma} \phi \rangle \\ &= \langle \phi | e^{-\frac{1}{2}[v,i\gamma]} e^{v} e^{i\gamma} \phi \rangle \\ &= e^{-\frac{1}{2}[v,i\gamma]} F(v) \end{aligned}$$

$$\begin{aligned} F(v-i\gamma^*) &= \langle \phi | e^{-i\gamma^* + v} \phi \rangle \\ &= \langle \phi | e^{-\frac{1}{2}[-i\gamma^*,v]} e^{-i\gamma^*} e^{v} \phi \rangle \\ &= e^{+\frac{1}{2}[i\gamma^*,v]} F(v) \end{aligned}$$

This gives quasi-periodicity of $F(v)$ with respect to the lattice $i\Gamma + i\Gamma^*$, which has rank $4g$ inside of $V_c \simeq \mathbb{C}^{2g}$. Now the integrality + positivity conditions on Γ should show there is a unique such F satisfying $F(0) = 1$.

To simplify put $F(v) = \langle \phi | e^{iv} | \phi \rangle$ so that

$$F(v) = \langle \phi | e^{iv} e^{i\gamma} \phi \rangle = \langle \phi | e^{\frac{1}{2}[iv, i\gamma]} e^{iv + i\gamma} | \phi \rangle$$

$$= e^{\frac{1}{2}[\gamma, v]} F(v + \gamma)$$

$$F(v) = \langle \phi | e^{+i\gamma^*} e^{iv} | \phi \rangle \qquad e^{-i\gamma}\phi = \phi \text{ as } -\gamma \in \Gamma$$

$$= \langle \phi | e^{\frac{1}{2}[i\gamma^*, iv]} e^{i\gamma^* + iv} | \phi \rangle$$

$$= e^{\frac{1}{2}[v, \gamma^*]} F(v + \gamma^*)$$

Thus we are asking that F be fixed under the operators

$$e^{\frac{1}{2}[\gamma, a^*]} e^{\gamma a} = e^{\frac{1}{2}[\gamma, a^*] + \gamma a}$$

$$e^{\frac{1}{2}[a^*, \gamma^*]} e^{\gamma^* a} = e^{\frac{1}{2}[a^*, \gamma^*] + \gamma^* a}$$

These commute because

$$\left[\frac{1}{2}[\gamma_1, a^*] + \gamma_1 a, \ \frac{1}{2}[a^*, \gamma_2^*] + \gamma_2^* a \right]$$

$$= \frac{1}{2}[\gamma_1, \gamma_2^*] - \frac{1}{2}[\gamma_1, \gamma_2^*] = 0$$

$$\left[\frac{1}{2}[\gamma_1, a^*] + \gamma_1 a, \ \frac{1}{2}[\gamma_2, a^*] + \gamma_2 a \right]$$

$$= \frac{1}{2}[\gamma_2, \gamma_1] - \frac{1}{2}[\gamma_1, \gamma_2] = [\gamma_1, \gamma_2] \in 2\pi i \mathbb{Z}$$

Now because the operators $e^{\frac{1}{2}[\gamma, a^*] + \gamma a}$ don't commute we can't find a multiplier $f(v)$ such that $f(v) F(v)$ is Γ-periodic. Thus we apparently can't use Fourier series to represent $F(v)$.

To get a little insight let us restrict σ to lie in $\mathbb{R}\Gamma$. Then a function $F(\sigma)$, $\sigma \in \mathbb{R}\Gamma$ fixed under the operators $e^{\frac{1}{2}[\gamma,\sigma]+\gamma\partial_\sigma}$ is the same as a section of a line bundle over the torus $\mathbb{R}\Gamma/\Gamma$. These we can represent by Schwartz functions on half of $\mathbb{R}\Gamma$

June 10, 1987

Recall we are trying to find $F(\sigma) = \langle \phi | e^{i\sigma}\phi \rangle$ by means of the relations

$$F(\sigma) = e^{\frac{1}{2}[\gamma,\sigma]} F(\sigma+\gamma) = e^{\frac{1}{2}[\sigma,\gamma^*]} F(\sigma+\gamma^*).$$

The problem is that entire functions satisfying the first relation for all $\gamma \in \Gamma$ are not given by Fourier series. The best one can do is to represent them by functions in half the variables, i.e. to divide Γ into "a, b cycles".

But more generally one can consider a lattice Λ of maximal rank in a complex V_c and look for entire $F(\sigma)$ satisfying relations

$$\circledast \qquad F(\sigma) = e^{\frac{1}{2}g(\lambda)} e^{\varphi_\lambda(\sigma)} F(\sigma+\lambda) \qquad \forall \lambda \in \Lambda.$$

where $\varphi_\lambda \in V_c^*$. Consistency requires

$$e^{\frac{1}{2}g(\lambda) + \varphi_\lambda(\sigma) + \frac{1}{2}g(\lambda') + \varphi_{\lambda'}(\sigma+\lambda)} F(\sigma+\lambda+\lambda')$$
$$= e^{\frac{1}{2}g(\lambda+\lambda') + \varphi_{\lambda+\lambda'}(\sigma)} F(\sigma+\lambda+\lambda')$$

so that

$$\boxed{\begin{aligned} &\varphi_{\lambda+\lambda'} = \varphi_\lambda + \varphi_{\lambda'} \\ &\tfrac{1}{2}\big(g(\lambda) + g(\lambda')\big) + \varphi_{\lambda'}(\lambda) \equiv \tfrac{1}{2}g(\lambda+\lambda') \mod 2\pi i\mathbb{Z} \end{aligned}}$$

This consistency condition amounts to an action of Λ on the trivial line bundle over V_c covering the translation action. Thus we are looking for holomorphic sections of a holom. line bundle L over the complex torus V_c/Λ. Thus the space of F satisfying ⊛ is finite-dimensional.

Observe that $\lambda \mapsto e^{\frac{1}{2}q(\lambda)}$, $\Lambda \longrightarrow \mathbb{C}^\times$ is a quadratic character and that $e^{\varphi_{\lambda'}(\lambda)}$ is the associated ~~[illegible]~~ pairing $\Lambda\times\Lambda \longrightarrow \mathbb{C}^\times$ which is necessarily symmetric. Thus $\varphi_{\lambda'}(\lambda)-\varphi_{\lambda}(\lambda')$ is a skew-symmetric form on Λ with values in $2\pi i\mathbb{Z}$.

Next observe that $\tilde{q}(\lambda) = q(\lambda) - \varphi_\lambda(\lambda)$ satisfies

$$\frac{1}{2}\tilde{q}(\lambda) + \frac{1}{2}\tilde{q}(\lambda') + \frac{1}{2}\left[\varphi_{\lambda'}(\lambda) - \varphi_{\lambda}(\lambda')\right] \equiv \frac{1}{2}\tilde{q}(\lambda+\lambda') \mod (2\pi i\mathbb{Z})$$

which means that $\left(e^{\frac{1}{2}\tilde{q}(\lambda)}\right)^2 = e^{\tilde{q}(\lambda)}$ is a character $\Lambda \longrightarrow \mathbb{C}^\times$.

Thus up to dividing-by-2-problems one has $q(\lambda) = \varphi_\lambda(\lambda) + \beta(\lambda)$ with β linear in λ.

June 15, 1987

Let X be a Riemann surface with boundary having g handles and r boundary circles: $\partial X = \bigcup_{i=1}^{r} S_i$. Fill in the circles with disks to obtain $\tilde{X} = X \cup \bigcup D_i$

X

$g=1,\ r=3$

Then we have the Mayer-Vietoris sequence

$$0 \leftarrow \overset{\mathbb{Z}}{H_0(\tilde{X})} \leftarrow \overset{\mathbb{Z}}{H_0(X)} \oplus \overset{\mathbb{Z}^r}{H_0(\cup D_i)} \leftarrow \overset{\mathbb{Z}^r}{H_0(\cup S_i)}$$

$$\leftarrow H_1(\tilde{X}) \leftarrow H_1(X) \oplus \cancel{H_1(\cup D_i)} \leftarrow \overset{\mathbb{Z}^r}{H_1(\cup S_i)}$$

$$\leftarrow \underset{\mathbb{Z}}{\underset{\shortparallel}{H_2(\tilde{X})}} \leftarrow 0$$

Note that $\mathbb{Z} = H_2(\tilde{X}) \longrightarrow H_1(\cup S_i) = \mathbb{Z}^r$ is essentially the diagonal because if we consider $[X]$ as a 2-chain then $\partial[X] = \sum_{i=1}^{r} [S_i]$. Thus $\sum_{i=1}^{r} [S_i] \in H_1(\cup S_i)$ goes to zero in $H_1(X)$.

From the exact sequence we get

$$0 \longrightarrow \mathbb{Z}^r/\Delta\mathbb{Z} \longrightarrow H_1(X) \longrightarrow \underset{\mathbb{Z}^{2g}}{\underset{\shortparallel}{H_1(\tilde{X})}} \longrightarrow 0$$

hence $\quad H_1(X) \cong \mathbb{Z}^{2g+r-1}$.

Dually we have

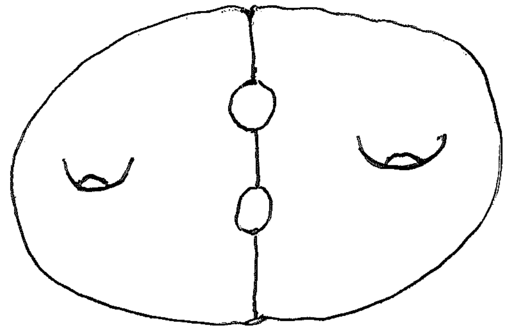
$$0 \longrightarrow \underset{\mathbb{Z}^{2g}}{\underset{\shortparallel}{H^1(\tilde{X})}} \longrightarrow H^1(X) \longrightarrow \underset{\bigoplus_i H^1(S_i)}{\underset{\shortparallel}{H^1(\partial X)}} \longrightarrow \underset{\mathbb{Z}}{\underset{\shortparallel}{H^2(\tilde{X})}} \longrightarrow 0$$

$$\bigoplus_i H^1(S_i) \longrightarrow \mathbb{Z}$$

and we can identify $H^1(\tilde{X})$ with $H^1_c(X)$ the cohomology with compact supports.

Now consider the double $X \cup \bar{X}$, where

$X \qquad \bar{X}$



$\bar{X}$ is the surface X with the conjugate complex structure. Then the MV sequence is

$$\begin{array}{ccccccc} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\ 0 & \longleftarrow & H_0(X\cup\bar{X}) & \longleftarrow & H_0(X) & \oplus & H_0(\bar{X}) \end{array}$$

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_2(X\cup\bar{X}) & \longrightarrow & H_1(\cup S_i) & \xrightarrow{(1,-1)} & H_1(X)\oplus H_1(\bar{X}) & \longrightarrow & H_1(X\cup\bar{X}) & \longrightarrow & H_0(\cup S_i) \\ & & \mathbb{Z} & & \mathbb{Z}^r & & \mathbb{Z}^{2g+r-1}\oplus\mathbb{Z}^{2g+r-1} & & & & \mathbb{Z}^r \end{array}$$

From this we see that the rank of $H_1(X\cup\bar{X})$ is $2g+2(r-1)$, i.e. $X\cup\bar{X}$ has $g+r-1$ handles, which is clear from the picture.

What can we say about harmonic forms on X? We always have the skew form on $H^1(X)$ given by wedge product **followed by** integrating. And as in the case $r=1$, this will be non-degenerate because the space of harmonic 1-forms splits into the direct sum $\Gamma(X,\Omega^1)\oplus\overline{\Gamma(X,\Omega^1)}$.

~~Next we have the subspace of harmonic forms given by the differentials of harmonic fns.~~

June 16, 1987

Let X be a connected Riemann surface with r boundary circles S_i and g handles. We have an exact sequence in cohomology

$$0 \longrightarrow H^1(X,\partial X) \longrightarrow H^1(X) \longrightarrow H^1(\partial X) \xrightarrow{\int_{\partial X}} \mathbb{C} \longrightarrow 0$$

(and similarly for any other coefficients like $\mathbb{R}, \mathbb{Z}$). Thus $H^1(X)$ has dimension $2g+r-1$.

We now propose to look at the Hodge theory for X. Everything we need seems to follow from

$$H^1(X,\mathcal{O}) = 0$$

where $\mathcal{O}$ is the sheaf of holomorphic functions. We shall assume this is a consequence of Stein manifold theory.

Let $\mathcal{H}$ be sheaf of harmonic functions. We have an exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \oplus \bar{\mathcal{O}} \longrightarrow \mathcal{H} \longrightarrow 0$$

whence a long exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \Gamma(X,\mathcal{O}) \oplus \Gamma(X,\bar{\mathcal{O}}) \longrightarrow \Gamma(X,\mathcal{H}) \longrightarrow$$

$$\longrightarrow H^1(X,\mathbb{C}) \longrightarrow 0 \longrightarrow H^1(X,\mathcal{H}) \longrightarrow \underset{\displaystyle 0}{\overset{}{H^2(X,\mathbb{C})}} \longrightarrow 0$$

This shows $H^1(X,\mathbb{C})$ measures the deviation of a harmonic function from being the sum of a holom. function and anti-holom. function. Also

$$H^1(X,\mathcal{H}) = 0.$$

Next we have the exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{H} \xrightarrow{d} \Omega^1 \oplus \bar{\Omega}^1 \longrightarrow 0$$

where $\Omega^1, \bar{\Omega}^1$ are respectively the sheaves of holomorphic and anti-holom. 1-forms. This yields.

$$0 \to \mathbb{C} \to \Gamma(X,\mathcal{H}) \xrightarrow{d} \Gamma(X,\Omega^1) \oplus \Gamma(X,\bar{\Omega}^1) \to H^1(X,\mathbb{C}) \to 0$$

$$\cancel{0 \to H^1(X,\Omega^1) \oplus H^1(X,\bar{\Omega}^1) \xrightarrow{\sim} H^2(X,\mathbb{C}) \to 0}$$

$$H^1(X,\Omega^1) = H^1(X,\bar{\Omega}^1) = 0$$

Thus we see that any elt of $H^1(X,\mathbb{C})$ is represented by a harmonic 1-form unique up to adding the differential of a harmonic function.

On the space of harmonic 1-forms $\Gamma(X,\Omega^1) \oplus \Gamma(X,\bar{\Omega}^1)$ we have the non-degenerate skew form $\int \omega_1 \omega_2$. Its restriction to $d\Gamma(X,\mathcal{H})$ is non-degenerate only for $r=1$. In effect

$$\int_X df \wedge dg = \int_{\partial X} f\,dg = -\int_{\partial X} g\,df = 0$$

if either f or g is locally constant on ∂X. This is consistent with there being no natural skew form on $H^1(X,\mathbb{C})$.

Let K denote the space of harmonic 1-forms vanishing on ∂X. Then

$$K \cap d\Gamma(X,\mathcal{H}) = \left\{ df \;\middle|\; \begin{matrix} f \text{ harmonic on } X \\ f \text{ loc. const on } \partial X \end{matrix} \right\} = H^0(\partial X,\mathbb{C})/\mathbb{C}$$

is the kernel of the skew form restricted to $d\Gamma(X,\mathcal{H})$.

Consider next the map

$$\alpha : K \longrightarrow H^1(X)$$

defined by taking $\omega \in K$ and looking at its cohomology class. This is the restriction to K of the similar map $\Gamma(\Omega^1) \oplus \Gamma(\bar{\Omega}^1) \longrightarrow H^1(X)$. Its kernel is $K \cap d\Gamma(\mathcal{H})$.

The image of $\alpha : K \longrightarrow H^1(X)$ is clearly contained in $H^1(X,\partial X) \subset H^1(X)$. NO To see it's onto $H^1(X,\partial X)$, take a class in this group as represent it by a harmonic 1-form ω. Then there is a function f on ∂X with $\omega|\partial X = df$. Extending f to a harmonic function, we then have that $\omega - df \in K$ and it represents the same cohomology class. Thus we have

<u>Prop</u>: We have

$$0 \longrightarrow \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ H^0(\partial X)/\mathbb{C}}}{K \cap d\Gamma(X,\mathcal{H})} \longrightarrow K \longrightarrow \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \mathbb{C}^{2g}}}{H^1(X,\partial X)} \longrightarrow 0$$

so $\dim(K) = 2g + r - 1$. $K \cap d\Gamma(X,\mathcal{H})$ is the kernel of the skew-form restricted to K.

The last statement is clear as if $df \in K \cap d\Gamma(X,\mathcal{H})$ and $\omega \in K$, then

$$\int_X df \cdot \omega = \int_{\partial X} f\omega = 0,$$

and because the skew form on $H^1(X,\partial X)$ is non-degenerate.

Apparently there is no natural splitting of the above sequence, i.e. no natural way to represent classes in $H^1(X,\partial X)$ by harmonic 1-forms vanishing on ∂X. There seems also to be no way to split off $H^1(X,\partial X)$ from $H^1(X)$. So the Hodge theory isn't very good.

See Dec 1987 p 401

Next we want to turn to the quantum field theory. First we look at the theory associated to harmonic functions, paying special attention to what happens when there are several boundary components.

~~Over a circle S we can consider pairs (f, ω) consisting of a function and a 1-form. We then have a canonical Fock space with operators $\phi(\omega) + i\pi(f)$~~

Over a circle S we can consider the spaces of functions f and 1-forms ω and the natural pairing $\int_S f\omega$ between them. There is a canonical Fock space with operators $\phi(\omega)$, $\pi(f)$ satisfying the CCR

$$\phi(\omega)^* = \phi(\bar{\omega}), \quad \pi(f)^* = \pi(\bar{f})$$

$$[\phi(\omega), \phi(\omega_1)] = [\pi(f), \pi(f_1)] = 0$$

$$[\phi(\omega), \pi(f)] = i\int_S f\omega$$

All this also holds for a union $\coprod S_i$ of circles.

Suppose that S, or more generally $\coprod_{i=1}^{n} S_i$, is the boundary of X. We would like to assign to the holomorphic structure on X a line in the Fock space. Given a harmonic $f \in \Gamma(X, \mathcal{H})$ we assign to f the operator

$$\phi(*df) + i\pi(f)$$

One has

$$① \quad [\phi(*df) + i\pi(f), \phi(*df_1) + i\pi(f_1)] = \int_{\partial X} f * df_1 - f_1 * df$$

$$= \int_X df \wedge (*df_1) - df_1 \wedge (*df) = 0$$

(In effect, recall that $*dx = dy$, $*dy = -dx$ so $(P\,dx + Q\,dy) \wedge {*}(R\,dx + S\,dy) = (PR + QS)\,dx\,dy$ is symmetric in the two 1-forms). Also one has

$$[\phi(*df) + i\pi(f), (\phi(*df) + i\pi(f))^*] = [\phi(*df) + i\pi(f), \phi(*d\bar{f}) - i\pi(\bar{f})]$$

$$= \int_{\partial X} f * d\bar{f} + \bar{f} * df = 2\int_X df * d\bar{f} = 2\int \left(|\partial_x f|^2 + |\partial_y f|^2\right) dx\,dy$$

(2)

$$\geqslant 0 \quad \text{with} = \iff f \text{ constant.}$$

Finally the solution of the Dirichlet problem tells us that any f on ∂X extends uniquely to a harmonic function on X.

In order that there exist a state ψ in Fock space be killed by the operators $\phi(*df) + i\pi(f)$, $f \in \Gamma(X, \mathcal{H})$ the commutation condition (1) and the positivity condition (2) are necessary. However the operator associated to the constant function $1/i$ namely $\pi(1)$, is hermitian with continuous spectrum so it has no normalizable eigenfunctions. This is analogous to having a system with translation degrees of freedom. $\pi(1)$ is like the total momentum. To proceed one has to remove this degree of freedom by fixing the total momentum and ignoring the position of the center of mass. In the present case one must work with a smaller set of operators, namely, one restricts to $\phi(\omega)$ where $\int_{\partial X} \omega = 0$ and requires $\pi(1)$ to be a certain constant, e.g. zero.

Thus we look at a different Fock space on which we have operators $\pi(f)$, $f \in C^\infty(\partial X)/\mathbb{C}$ and $\phi(\omega)$ where $\omega \in \Omega^1(\partial X)$ satisfies $\int_{\partial X} \omega = 0$. It is then clear that the obvious necessary conditions that there exist a state in the Fock space killed by the operators $\phi(*df) + i\pi(f)$, $f \in \Gamma(\mathcal{H})$ are satisfied.

Let's check the sufficiency. Let V be the real vector space of pairs (f, ω), $f \in C^\infty(\partial X, \mathbb{R})/\mathbb{R}$ and $\omega \in \Omega^1(\partial X, \mathbb{R})$, $\int \omega = 0$. Its complexification V_c consists of pairs (f, ω) which are complex-valued. I want to identify (f, ω) with the operator $\pi(f) + \phi(\omega)$ which is hermitian iff (f, ω) **is** real. Now let W be the space of pairs $(if, *df)$ where $f \in \Gamma(X, \mathcal{H})$. Then ①, ② above say that W is isotropic ~~and that~~ (i.e. $[\omega, \omega_1] = 0$) and positive (i.e. $[\omega, \omega^*] \geq 0$ with $=$ iff $\omega = 0$). Thus we have an **injection** $W \oplus \overline{W} \to V_c$

~~of the solution of the Dirichlet problem~~

To see it's surjective, we proceed as follows. Given $(f, \omega) \in V_c$ we extend f to a harmonic function. Then $\frac{1}{2i}(if, *df) \in W$ and $\frac{1}{2i}(-if, *df) \in \overline{W}$ and **their** difference gives $(f, 0)$ and sum $\times i$ gives $(0, *df)$. Thus we need to be able to solve the <u>Neumann problem</u>: Given $\omega \in \Omega^1(\partial X)$ with $\int_{\partial X} \omega = 0$, does there exist a harmonic function f such that $*df\big|_{\partial X} = \omega$?

To see the Neumann problem can be solved uniquely mod constants, let us consider the operator $T_X : C^\infty(\partial X) \longrightarrow \Omega^1(\partial X)$ which takes a function f on ∂X, extends it to a harmonic function on X, and then takes the normal derivative $*df \,|\, \partial X$. This operator is real, i.e. goes between real functions and 1-forms and it is symmetric relative to the natural pairing:

$$\int_{\partial X} f\, T_X g = \int_{\partial X} f * dg = \int_X df \wedge (*dg) = \int_X dg \wedge (*df)$$

$$= \int_{\partial X} g * df = \int_{\partial X} g\, T_X f$$

Now the kernel of T_X consists of the constant functions since if $f \in \Gamma(X, \mathcal{H})_{\text{real}}$ and $*df \,|\, \partial X = 0$, then

$$0 = \int_{\partial X} f * df = \int_X df \wedge * df \implies df = 0.$$

If we identify $C^\infty(\partial X)_{\text{real}}$ with $\Omega^1(\partial X)_{\text{real}}$ by choosing a metric, then T_X becomes a symmetric operator, more precisely, a real hermitian operator. It's a ψDO with symbol $\operatorname{sgn}(\xi)\xi$, so it should have an orthonormal sequence of eigenfunctions. Thus the kernel and cokernel of T_X are essentially identical, and the image of T_X is the orthogonal of the constant functions. This gives the solution of the Neumann problem.

Further work is needed to check that the polarization $W \oplus \bar{W} = V_c$ just defined is in the appropriate Shale class, but this should follow easily from the fact that T_X differs by a smooth kernel operator from the composition of d and the Hilbert transform defined relative to any parametrization of ∂X.

To summarize, we see that in the Fock space associated to the real symplectic space

$$③ \qquad C^\infty(\partial X)_{real}/\mathbb{R} \quad \oplus \quad \{\omega \in \Omega^1(\partial X)_{real} \mid \int_{\partial X} \omega = 0\}$$

there should be a unique line killed by the operators associated to pairs $(if, *df)|\partial X$ for $f \in \Gamma(X, \mathcal{H})$.

Now let's suppose $\partial X = \bigcup_{i=1}^{n} S_i$. In this case, instead of the total momentum operator $\pi(1)$ which we have fixed, we have momentum operators associated to each S_i. These are the operators $\pi(\chi_i)$, where χ_i is 1 on S_i and 0 on the others. The operator $\pi(\chi_i)$ commute and have continuous joint spectrum on the Fock space. We can fix the values of the $\pi(\chi_i)$ provided we ignore or forget about the conjugate position operators. Thus we can form the quotient (in the sense of Marsden-Weinstein) symplectic space

$$④ \qquad C^\infty(\partial X, \mathbb{R})/H^0(\partial X, \mathbb{R}) \quad \oplus \quad \{\omega \in \Omega^1(\partial X)_{real} \mid \int_{S_i} \omega = 0 \quad i = 1, \ldots, n\}.$$

The associated Fock space will be the tensor product of Fock spaces associated to each of the circles.

What I would like to understand is how the line in the Fock space associated to ③ gives rise to a line in the Fock space associated to the quotient ④. The latter line should be the unique line invariant under operators associated to pairs $(if, *df)|\partial X$ where f is harmonic and $\int_{S_i} *df = 0$ for $i = 1, \ldots, r$.

Let's try a finite-diml. analogy. Suppose V has basis $q_1, \ldots, q_n, p_1, \ldots, p_n$ and let us write

$$q = \underbrace{q_1, \ldots, q_m}_{q'}, \underbrace{q_{m+1}, \ldots, q_n}_{q''}$$

$$p = \underbrace{p_1, \ldots, p_m}_{p'}, \underbrace{p_{m+1}, \ldots, p_n}_{p''}.$$

Let U be spanned by $p'' = (p_{m+1}, \ldots, p_n)$. Then the annihilator U^0 is spanned by q', p', p''. We have the Marsden-Weinstein quotient setup

$$\begin{array}{c} U \subset U^0 \subset V \\ \downarrow \\ U^0/U \end{array}$$

We now take the Heisenberg representation $\mathcal{F}_V$ of V + restrict it to U^0. Since U is in the center of U^0, the representation will decompose according to the characters of U, and each component should be isomorphic to the Heisenberg representation of U^0/U.

There should be a map which takes a wave function $f(x', x'')$ in $\mathcal{F}_V$ to its component corresponding to ξ''

$$\int e^{-i\xi'' x''} f(x', x'')\, dx'' .$$

This will be in $\mathcal{F}_{\mathfrak{U}^0/\mathfrak{U}} = L^2$ functions of x' for a.e. ξ'' and elements of $\mathcal{F}_V$ are then L^2 fns. of ξ'', x'. If f is a Gaussian function associated to a polarization $V_c = W \oplus \overline{W}$, then clearly the partial transform above is also Gaussian.

To summarize, suppose we assign to any circle S the symplectic space of functions modulo constants $\oplus$ 1-forms with integral zero, and the associated Fock space $\mathcal{F}_S$. Then given an X with r boundary components S_i, and a collection of numbers ξ_i $i = 1, \ldots, r$ (such that $\sum \xi_i = 0$?) which we think of as momenta, there is a line in the tensor product $\bigotimes \mathcal{F}_{S_i}$ depending on the conformal structure of X and these momenta.

So far we have looked at the harmonic function conformal field theory. Next we want to look at holomorphic functions. Here we start with the real symplectic space $C^\infty(S, \mathbb{R})/\mathbb{R}$ equipped with the skew form $\int_S f\,dg$, and we let $\mathcal{F}_S$ be the associated Fock space representation.

Let X have boundary $\partial X = \bigcup_{i=1}^{r} S_i$. We want to produce a line in $\mathcal{F}_{\partial X} = \bigotimes_{i=1}^{r} \mathcal{F}_{S_i}$ associated to the conformal structure on X.

We start with the space $V_c = \Gamma(\Omega^1) \oplus \Gamma(\bar{\Omega}^1)$ of harmonic 1-forms on X. This has an underlying real subspace V and a skew form given by $i \int_X \omega_1 \omega_2$. The real symplectic space V is canonically polarized, so there is a canonical line in $\mathcal{F}_V$, namely the unique line killed by $\Gamma(\Omega^1)$. Let's denote by $|0\rangle$ a unit vector in $\mathcal{F}_V$ which generates this line. Now our goal is to obtain a line in $\mathcal{F}_{\partial X}$ from the line $\mathbb{C}|0\rangle \subset \mathcal{F}_V$.

As usual $C^\infty(\partial X)$ is identified with $\Gamma(\mathcal{H})$, the space of harmonic functions. We are interested in the real symplectic space $C^\infty(\partial X, \mathbb{R})/H^0(X, \mathbb{R})$ and we have

$$
\begin{tikzcd}
 & C^\infty(\partial X)/\mathbb{C} \arrow[d, equal] & & \\
0 \arrow[r] & d\Gamma(\mathcal{H}) \arrow[r] \arrow[d, "\substack{\text{kernel}\\ \text{has dim } r-1}"] & \overbrace{\Gamma(\Omega^1) \oplus \Gamma(\bar{\Omega}^1)}^{V_c} \arrow[r] & H^1(X) \arrow[r] & 0 \\
 & d\Gamma(\mathcal{H})/H^0(X, \mathbb{C}) & & H^1(X, \partial X) \arrow[u, phantom, "\bigcup \ \text{codim } r-1"] &
\end{tikzcd}
$$

This looks like a quotient situation à la Marsden-Weinstein.

Recall $K = \{\omega \in V \mid \omega|\partial X = 0\}$ maps onto $H^1(X, \partial X)$ with kernel $K \cap d\Gamma(\mathcal{H}) \simeq H^0(X, \mathbb{C})/\mathbb{C}$ which coincides with the kernel of the symplectic form on V_c restricted to $d\Gamma(\mathcal{H})$. This last statement follows by looking at the kernel of the pairing $\int_{S_i} f\,dg$ on each circle S_i. Taking the annihilator of $K \cap d\Gamma(\mathcal{H})$ will give a subspace of codim $r-1$

containing $d\Gamma(\mathcal{H})$. Now K clearly is contained in the annihilator of $d\Gamma(\mathcal{H})$, since if $df \in d\Gamma(\mathcal{H})$, $\omega \in K$, then

$$\int_X df\,\omega = \int_{\partial X} f\,\omega = 0 \qquad \text{as } \omega|\partial X = 0.$$

Thus $\left(K \cap d\Gamma(\mathcal{H})\right)^0 \supset K + d\Gamma(\mathcal{H})$ which maps onto $H^1(X,\partial X) \subset H^1(X)$. By dimension counting we have

$$\left(K \cap d\Gamma(\mathcal{H})\right)^0 = K + d\Gamma(\mathcal{H})$$

and so we have the following quotient setup

$$\begin{tikzcd}
0 \arrow[r] & d\Gamma(\mathcal{H}) \arrow[r] & \overbrace{\Gamma(\Omega^1)+\Gamma(\bar{\Omega}^1)}^{V_c} \arrow[r] & H^1(X) \arrow[r] & 0 \\
0 \arrow[r] & d\Gamma(\mathcal{H}) \arrow[u, equal] \arrow[r] \arrow[d] & K + d\Gamma(\mathcal{H}) \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d] & H^1(X,\partial X) \arrow[u, phantom, "\cup"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \underbrace{d\Gamma(\mathcal{H})/K \cap d\Gamma(\mathcal{H})}_{C^\infty(\partial X)/H^0(\partial X)} \arrow[r] & \dfrac{K + d\Gamma(\mathcal{H})}{K \cap d\Gamma(\mathcal{H})} \arrow[r] & H^1(X,\partial X) \arrow[r] & 0
\end{tikzcd}$$

This appears to produce a canonical line in the Heisenberg representation of $C^\infty(\partial X)/H^0(\partial X)$ in the following way. We start with the canonical line $\mathbb{C}|0\rangle$ in $\mathcal{F}_V$ and restrict this representation to $K + d\Gamma(\mathcal{H})$, where it decomposes according to the characters of $K \cap d\Gamma(\mathcal{H})$. We then project into the component corresponding to the trivial character of

$K \cap d\Gamma(\mathcal{H})$. This component can be identified with the Heisenberg representation of $A = \left(K + d\Gamma(\mathcal{H}) / K \cap d\Gamma(\mathcal{H})\right)_{real}$, call this $\mathcal{F}_A$. So we have a distinguished line in $\mathcal{F}_A$. But A splits naturally as the sum of the symplectic spaces $B = \left(d\Gamma(\mathcal{H}) / K \cap d\Gamma(\mathcal{H})\right)_{real} \simeq \left(C^\infty(\partial X)/H^0(\partial X)\right)_{real}$ and $C = \left(K / K \cap d\Gamma(\mathcal{H})\right)_{real} \simeq \left(H^1(X, \partial X)\right)_{real}$. Then the line in $\mathcal{F}_A \simeq \mathcal{F}_B \otimes \mathcal{F}_C$ can be viewed as an operator from $\mathcal{F}_C$ to $\mathcal{F}_B$ defined up to scalar factor. Corresponding to the self-dual lattice $H^1(X, \partial X; \mathbb{Z}) \subset H^1(X, \partial X; \mathbb{R})$ is a canonical "distributional" state in $\mathcal{F}_C$ which is mapped by the operator into a state in $\mathcal{F}_B$ as desired.

The problem is now to see that all this works and to characterize the line obtained in $\mathcal{F}_B$.

It seems there ought to be a polarization on $K + d\Gamma(\mathcal{H}) / K \cap d\Gamma(\mathcal{H})$ induced from the polarization of V.

Let's discuss this in finite dimensions. Let V be a real symplectic space with polarization $V_c = W \oplus \bar{W}$ where $[w, w^*] > 0$ for all $0 \neq w \in W$. Let $N \subset V$ be isotropic. Is there a polarization induced on N^0/N?

The obvious candidate is the subspace

$$(N_c + W \cap N_c^0)/N_c \subset (N^0/N)_c$$

To see if it works we first look at the case where N is 1-diml. Recall that W is a complex space with hermitian inner product and that an orthonormal basis for W is the same thing as a choice of annihilation operators $a_1, \ldots, a_g$. We can obviously choose this basis so that N is spanned by $a_1 + a_1^*$. Then $(N^0)_c$ is spanned by $a_1 + a_1^*, a_2, a_2^*, \ldots, a_g, a_g^*$ and $W \cap N_c^0$ is spanned by $a_2, \ldots, a_g$ so

$$N_c + W \cap N_c^0 / N_c \quad \text{is spanned by } a_2, \ldots, a_g.$$

As a check note that $N_c + W$ has the basis $a_1, a_1^*, a_2, \ldots, a_g$ so $(N_c + W) \cap N_c^0$ has the basis $a_1 + a_1^*, a_2, \ldots, a_g$ and so

$$(N_c + W) \cap N_c^0 / N_c = (N_c + W \cap N_c^0) / N_c$$

has the basis $a_2, \ldots, a_g$.

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Notation: Let V be a real symplectic vector space with symplectic form $\Omega(v,v')$ and let $\mathcal{F}_V$ be the associated Heisenberg representation. To each $v \in V$ belongs a (unbounded) hermitian operator $\psi \mapsto v\psi$ on $\mathcal{F}_V$ satisfying the canonical commutation relations

$$[v, v'] = i\,\Omega(v,v').$$

(In order to handle difficulties with unbounded operators, Weyl uses the unitary operators e^{iv} satisfying

$$e^{iv}e^{iv'} = e^{\underbrace{-\frac{1}{2}[iv,iv']}_{\frac{1}{2}i\Omega(v,v')}}\, e^{i(v+v')} \quad .)$$

Let $V_c = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V, and $v \to \bar{v}$ the conjugation on V_c fixing V. Then to each $v \in V_c$ belongs an operator $\psi \mapsto v\psi$ on $\mathcal{F}_V$ satisfying the CCR and such that the hermitian conjugate operator v^* is the operator $\bar{v}$.

A polarization of V is a complex subspace W such that $V = W \oplus \overline{W}$ and such that

$$[w, w'] = 0 \quad \text{for} \quad w, w' \in W$$
$$[w, w^*] > 0 \quad \text{for} \quad 0 \neq w \in W.$$

Then $\langle w' | w \rangle \overset{\text{def}}{=} [w, w'^*]$ defines a hermitian inner product on W. An orthonormal basis for W is a family of operators $a_1, \ldots, a_n$ satisfying

$$[a_i, a_j] = [a_i^*, a_j^*] = 0$$
$$[a_i, a_j^*] = \delta_{ij}$$

There is a unique unit vector $|0\rangle$ in $\mathcal{F}_V$ up to scalar factors which is killed by W. By letting the creation operators a_i^* act on $|0\rangle$ one obtains an isomorphism

$$S(\bar{W}) \simeq \mathcal{F}_V$$

where S denotes symmetric tensors in the Hilbert space sense.

Suppose now that N is an isotropic subspace of V, let $N^0 = \{v \in V \mid [v,x] = 0 \quad \forall x \in N\}$ be its annihilator, and let N^0/N be the "quotient" real symplectic space.

<u>Prop.</u> If W is a polarization of V, then $(N_c + W \cap N_c^0)/N_c$ = Image of $W \cap N_c^0$ in $(N^0/N)_c$ is a polarization of N^0/N.

Proof. We consider the image of N in W under the projection of $V_c = W \oplus \bar{W}$ onto W. This image is

$$I = \{ w \in W \mid w + w^* \in N \}.$$

It is a real subspace of W such that the hermitian inner product $\langle w' | w \rangle = [w, w'^*]$ takes real values when restricted to I, because

$$[w, w'^*] - \overline{[w, w'^*]} = [w, w'^*] - [w', w^*]$$
$$= [w + w^*, w' + w'^*] = 0$$

for $w, w' \in I$ since N is isotropic.

Thus if we take an orthonormal basis for I as a real vector space with inner product, we obtain an orthonormal sequence in W as a complex vector space with inner product. Thus we can choose an orthonormal basis $a_1, \dots, a_n$ of W such that $a_i + a_i^*$ for $i = 1, \dots, r$ is a basis for N. ~~Then we can identify $(N^\circ/N)_c$ with the subspace spanned by the a_i, a_i^* for $i > r$ and the image of $W \cap N^\circ_c$~~ Then N°_c has the basis $a_i + a_i^*$, $1 \le i \le r$ and a_j, a_j^* for $j > r$. Also $W \cap N^\circ_c$ has the basis a_j for $j > r$, and so the conclusion of the proposition is clear.

Perhaps the above would be clearer if one worked with the operator J on V_c which is i on W and $-i$ on $\overline{W}$. The polarization puts a structure of complex vector space with inner product on V. ~~that N is isotropic means that~~ Given N we split V into $N + JN$ and its orthogonal complement. The orthogonal complement can be identified with N°/N and it comes with the structure of complex space with inner product.

Let's now return to the Riemann surface X with g handles and r boundary circles. We have the real symplectic vector space given by real functions on ∂X modulo locally constant functions:

$$B = C^\infty(\partial X, \mathbb{R}) / H^0(\partial X, \mathbb{R})$$

We want to produce a line in $\mathcal{F}_B$ associated to holomorphic functions on X. We start with the real symplectic space V of real harmonic 1-forms, so $V_c = \Gamma\Omega^1 \oplus \Gamma\bar{\Omega}^1$. We have

$$\begin{array}{ccccccccc} 0 \to & \Gamma\mathcal{H}/\mathbb{C} & \xrightarrow{d} & V_c & \longrightarrow & H^1(X) & \longrightarrow & 0 \\ & \| & & & & & & \\ & C^\infty(\partial X)/\mathbb{C} & & & & & & \end{array}$$

and we want to divide out by N = locally constant functions.