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Let's start with the normalized acyclic Hochschild complex

$$\xrightarrow{b'} A\otimes\bar{A}^{\otimes 2}\otimes A \xrightarrow{b'} A\otimes\bar{A}\otimes A \xrightarrow{b'} A\otimes A \longrightarrow A \longrightarrow 0$$

When we apply $\mathrm{Hom}_{A\otimes A^{o}}(?,M)$ to this complex we obtain the complex $\bar{C}^{\bullet}(A,M)$ of normalized Hochschild cochains.

Better: Let's start with the A-bimodule Ω^1_A defined by the exact sequence

1) $$0 \longrightarrow \Omega^1_A \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0.$$

Because the standard complex $B(A)$ is acyclic, we have also an exact sequence

(*) $$A\otimes A^{\otimes 2}\otimes A \xrightarrow{b'} A\otimes A\otimes A \longrightarrow \Omega^1_A \longrightarrow 0$$

whence

$$\mathrm{Hom}_{A\otimes A^{o}}(\Omega^1_A, M) \cong Z^1(A,M) = \mathrm{Der}(A,M)$$

proving that Ω^1_A is a universal bimodule for derivation.

Let's determine the universal $d: A \to \Omega^1_A$.

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r] \arrow[rr, bend left, "b'"] & \Omega^1_A \subset & A\otimes A \\
A \arrow[u, "1\otimes\mathrm{id}\otimes 1"] \arrow[ur, "d"'] & &
\end{tikzcd}$$

Thus $\quad da = b'(1\otimes a\otimes 1) = a\otimes 1 - 1\otimes a.$

Notice that derivations are normalized 1-cochains since $D1 = D(1\cdot 1) = D1 + D1 \Rightarrow D1 = 0$. Thus we would have gotten the same result using the normalized acyclic Hochschild complex in (*).

Using the left A-module splitting $a \mapsto a\otimes 1$

of 1), we see that we have a left A-module isomorphism

$$A \otimes \bar{A} \xrightarrow{\sim} \Omega^1_A$$

$$a_0 \otimes a_1 \longmapsto a_0 \otimes a_1 - a_0 a_1 \otimes 1$$
$$= -a_0\, da_1$$

We change the sign and use the isom $a_0 \otimes a_1 \mapsto a_0\, da_1$.

Next define

$$\Omega^n_A = \Omega^1_A \otimes_A \cdots \otimes_A \Omega^1_A \qquad n\text{-times}$$

As a left A-module

$$\Omega^n_A \simeq A \otimes \bar{A}^{\otimes n}$$

$$a_0\, da_1 \cdots da_n \longleftarrow a_0 \otimes a_1 \otimes \cdots \otimes a_n .$$

The right A-module structure is given by

$$\begin{aligned}(-1)^n (a_0\, da_1 \cdots da_n)\, a_{n+1} = {} & a_0 a_1\, da_2 \cdots da_n\, da_{n+1} \\ & - a_0\, d(a_1 a_2)\, da_3 \cdots da_n\, da_{n+1} \\ & + a_0\, da_1\, d(a_2 a_3) \cdots \cdots da_{n+1} \\ & \cdots \\ & + (-1)^n a_0\, da_1 \cdots d(a_n a_{n+1})\end{aligned}$$

Thus a homomorphism $u : \Omega^n_A \to M$ of bimodules is the same thing as a multilinear map $\varphi : \bar{A}^{\otimes n} \to M$ (here $a_0 \varphi(a_1, \ldots, a_n) = u(a_0\, da_1 \cdots da_n)$) satisfying

$$0 = a_1 \varphi(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} \varphi(a_1, \ldots, a_n)\, a_{n+1}$$

Therefore

$$\boxed{\mathrm{Hom}_{A \otimes A^{\circ}}(\Omega^n_A, M) = \bar{Z}^n(A, M) \quad \text{normalized cocycles}}$$

Also we have

$$\Omega^n_A = \mathrm{Coker}\left\{ A\otimes\bar{A}^{\otimes n+1}\otimes A \xrightarrow{b'} A\otimes\bar{A}^{\otimes n}\otimes A \right\}$$

$$= \mathrm{Ker}\left\{ A\otimes\bar{A}^{\otimes n}\otimes A \xrightarrow{b'} A\otimes\bar{A}^{\otimes(n-1)}\otimes A \right\}$$

From another point of view we have a canonical derivation $d \in Z^1(A,\Omega^1_A)$ and its n-fold cup product is

$$\underbrace{d\cup\cdots\cup d}_{n} \in Z^n(A,\Omega^n_A)$$

$$\left(\underbrace{d\cup\cdots\cup d}_{n}\right)(a_1,\dots,a_n) = da_1\cdots da_n$$

This gives the universal normalized n-cocycle on A with values in Ω^n_A.

Discussion. In the above, we started with a derived category viewpoint, i.e., the derived category of A-bimodules with the tensor operation $\overset{L}{\otimes}_A$. We learned a lot of things:

action of $H^*(A,A)$ on objects

commutativity of $H^*(A,A)$

deformations of A obtained from derivations

The latter seems to be worthwhile pursuing since the relevant deformation of functions on $T^*(S^1)$ is of Poisson bracket type. What's needed now is to look at cyclic theory.

Consider the complex $\to \Omega^1_A \longrightarrow A\otimes A$ which resolves A, call this complex K', and take the tensor product

$$K'\otimes_A\cdots\otimes_A K' \qquad k \text{ times}$$

As on p 404 this is a resolution of A by projective $A\otimes A^{\circ}$-modules except at the top:

$$\Omega^n_A \longrightarrow \bigoplus_{i=1}^{n} \Omega^{i-1}_A \otimes_A (A\otimes A) \otimes_A \Omega^{n-i}_A \longrightarrow \cdots \longrightarrow P_0 \to A$$

Hence there is an isomorphism

$$0 \longrightarrow H_n(A,A) \longrightarrow \left(\Omega^1_A \otimes_A\right)^n \longrightarrow \bigoplus_{i=1}^{n} \Omega^{i-1}_A \otimes \Omega^{n-i}_A \otimes_A$$

Now as before the natural action of $\mathbb{Z}/n\mathbb{Z}$ on $\left(A \overset{L}{\otimes}_A\right)^n$ will be trivial modulo homotopy. Thus the image of $H_n(A,A)$ lands in $\left(\Omega^n_A \otimes_A\right)^{\sigma}$ and we have an exact sequence

$$0 \longrightarrow H_n(A,A) \longrightarrow \left(\Omega^1_A \otimes_A\right)^n_{\sigma} \xrightarrow{\alpha} \Omega^{n-1}_A$$

where the map α involves some kind of symmetrization, i.e. sum of n terms.

I don't yet understand what this means, except that a linear functional on $\left(\Omega^1_A \otimes_A\right)^n_{\sigma}$ should be essentially a cyclic n-cocycle.

Idea: Consider the DGA

$$\longrightarrow 0 \longrightarrow \overset{1}{A} \xrightarrow{u} \overset{0}{A}$$

where u is the identity. The cyclic complex functor applied to this DGA gives a bicomplex in which the rows are exact. The first column is the cyclic complex of A, and the columns of degree k represents $\left(A \overset{L}{\otimes}_A\right)^k_{\sigma}[k-1]$. Thus we get another double complex quasi-isomorphic to the cyclic complex, but where the columns are not the Hochschild complex although homotopic to it.

December 7, 1987

Consider the maps of DGA

$$\begin{array}{ccc} (\cdots \to I \to P) & \longrightarrow & (\to 0 \to A) \\ \downarrow & & \\ (\cdots \to P \to P) & & \end{array}$$

and apply the Connes complex functor

$$\begin{array}{ccc} \text{Connes}(I \to P) & \xrightarrow{\text{quis}} & \text{Connes}(A) \\ \downarrow & & \\ \text{Connes}(P \to P) & \overset{\text{quis}}{\simeq} & 0 \end{array} \qquad *$$

to obtain double complexes. We have

$$\text{Connes}(I \to P)_{(k)} = \begin{cases} \left(I \otimes_P^L\right)^k_\sigma [k-1] & k > 0 \\ \text{Connes}(P) & k = 0 \end{cases}$$

$$\text{Connes}(P \to P)_{(k)} = \begin{cases} \left(P \otimes_P^L\right)^k_\sigma [k-1] & k > 0 \\ \text{Connes}(P) & k = 0 \end{cases}$$

Let's take the cone on the vertical map in * ~~Since for $k \neq 0$ these are the same we should get~~ and denote it

$$R = \text{Cone}\left\{ \text{Connes}(I \to P) \longrightarrow \text{Connes}(P \to P) \right\}$$

Then R is a double complex with columns

$$R_{(k)} = \begin{cases} \mathrm{Cone}\left\{ (I \overset{L}{\otimes}_P)^k_\sigma \longrightarrow (P \overset{L}{\otimes}_P)^k_\sigma \right\} [k-1] & k > 0 \\ \mathrm{Cone}\left\{ \mathrm{Connes}(P) \rightarrow \mathrm{Connes}(P) \right\} & k = 0 \end{cases}$$

Moreover $\quad R \sim \Sigma\, \mathrm{Connes}(A) = \mathrm{Connes}(A)[1]$.

Actually we can do better for we can consider

$$\begin{tikzcd}
\mathrm{Connes}(I \rightarrow P) \arrow[r, phantom, "\sim"] \arrow[d] & \mathrm{Connes}(A) \arrow[d] \\
\mathrm{Connes}(P \rightarrow P) \arrow[r, phantom, "\sim"] \arrow[d] & \mathrm{Connes}(A \xrightarrow{1} A) \arrow[d] \\
R \arrow[r, "\sim"] & R'
\end{tikzcd}$$

and this shows that there's a consistency between the filtrations of R and R'.

December 8, 1987

The problem is to find the good foundations for cyclic theory. In studying extensions $P/I = A$, I have learned that the relative cyclic homology $HC(P, I)$ can be expressed in terms of the complexes $(I \overset{L}{\otimes}_P)^k_\sigma$. Thus $HC(A,A) = HC(A)$ can be expressed in terms of $(A \overset{L}{\otimes}_A)^k_\sigma$. This gives a different picture of the Connes bicomplex, which has the advantage of extending to the relative theory.

A natural program is the following. First there is the study of the functors

$$M \longmapsto (M \overset{L}{\otimes}_A)^k$$

on complexes of A-bimodules. Possibly, in analogy with the way $H^*(A,A)$ operates in the derived category of A-bimodules, there is some link between cyclic theory and these functors.

Secondly when one has an A-bimodule map $u : M \longrightarrow A$ satisfying $u(x)y = xu(y)$ for $x, y \in M$, there are maps (like B)

$$\Sigma\ (M \overset{L}{\otimes}_A)^{k+1}_\sigma \longrightarrow (M \overset{L}{\otimes}_A)^k_\sigma$$

which permit formation of a double complex. These have to be understood a lot better, especially from the derived category viewpoint.

Waldhausen's algebraic K-theory of spaces can be viewed as follows. If X is connected with basepoint, then ΩX can be viewed as a group up to homotopy. One can form a group ring by taking the free abelian group generated by ΩX, however the appropriate functor is $Q(Z_+) = \varinjlim_k \Omega^k \Sigma^k (Z \cup *)$ instead of free abelian group. Rationally these are the same.

So $Q(\Omega X_+)$ is a "ring up to homotopy", and there is an appropriate matrix group

$$GL_n\left(Q(\Omega X_+)\right) = \varinjlim_k \mathrm{Aut}_G\left(\bigvee^n S^k \wedge G_+\right)$$

where $G = \Omega X$. Then Waldhausen space is

$$A(X) = BGL(Q(\Omega X_+))^+$$

and its homotopy groups are Waldhausen's K-groups of X.

Rationally, one considers matrices over the simplicial ring $\mathbb{Q}[G]$, where G is a free simplicial group model for ΩX. One considers not invertible matrices, but rather, all matrices in $M_n(\mathbb{Q}[G])$ which have homotopy-inverses.

To simplify suppose X 1-connected whence G is connected. Then the Lie theory should be rather good. This means that the homotopy groups of $BGL(\;)^+$ can be reduced to the primitive part of the Lie algebra homology of $\mathfrak{gl}$, i.e. to the cyclic homology of $\mathbb{Q}[G] = \mathbb{Q}[\Omega X]$ (with a degree shift). Finally the cyclic homology of $\mathbb{Q}[\Omega X]$ is the equivariant homology of the free loop space:

$$HC(\mathbb{Q}[\Omega X]) = H\{ES^1 \times_{S^1} \Lambda(X)\}$$

(This was proved by Goodwillie and probably also Burghelea)

Let's review my viewpoint on the last result. One has a homotopy cartesian square

$$\begin{tikzcd} \Lambda(X) \arrow[r] \arrow[d] & X^I \sim X \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

so that by Eilenberg-Moore

$$\mathrm{Tor}^{H^*(X\times X)}(H^*(X), H^*(X)) \Longrightarrow H^*(\Lambda X)$$

Thus $H^*(\Lambda X)$ is linked to the Hochschild homology of $H^*(X)$.

Similarly the cyclic homology of $H^*(X)$ is linked to the equivariant cohomology $H^*_{S^1}(\Lambda X)$.

But this must be the wrong viewpoint, where one links the cyclic "coalgebra" homology of $H_*(X)$ to equivariant homology $H^{S^1}_*(\Lambda X) = H_*(ES^1 \times_{S^1} \Lambda X)$. A better starting point would be to consider the Hochschild homology of $H_*(\Omega X)$. This leads to looking at the cyclic homology of group rings.

Problem: What's the (reduced) cyclic homology of $k[G]$ where G is a free group?

Certainly what I've proved about the even cyclic homology groups and nilpotent extensions shows that the even groups except for $\overline{HC}_0$ are trivial. This is because any nilpotent extension $P/I^k \longrightarrow k[G]$ splits.

Look at the Hochschild homology over $A = k[G]$. An A-bimodule M is the same as a $G\times G$ module. In effect the action of $G \times G$ is $(g_1, g_2)(m) = g_1 m g_2^{-1}$.

Notice that

$$H^0(A, M) = \{m \mid am = ma, \forall a\}$$
$$= \{m \in M \mid gmg^{-1} = m\} = H^0(\Delta G, M)$$

and similarly

$$H_0(A, M) = H_0(\Delta G, M)$$

the point being that the subspace of M spanned by $[g, m] = gm - mg = g(mg)g^{-1} - mg$ is the same as the subspace spanned by $gmg^{-1} - m$. In general we have

$$H_*(A, M) = H_*(\Delta G, M)$$

In effect if P is a projective $(G \times G)$-module resolution of M, then it is also projective over ΔG.

If G is free, then $H_*(\Delta G, M) = 0$ for $* \geq 2$, just as if A were a free algebra.

In fact we can see easily that Ω_A^1 is a free A-bimodule, because a derivation of A with values in a bimodule M is an automorphism of the split extension

$$M \longrightarrow M \oplus A \longrightarrow A$$

and such autos correspond to liftings of $G \subset A$ into $(M \oplus A)^\times$. Such a lifting when G = free group generated by S is determined by the images of the elements of S in M, and these can be prescribed arbitrarily. Thus

$$\Omega_A^1 = A \otimes V \otimes A$$

where V is the $\mathbb{C}$ vector space with basis $\{ds \mid s \in S\}$.

Now that we know the Hochschild homology

$H_n(A) = 0$ for $n \geq 2$, we can use the Connes exact sequence to get that $\overline{HC}_n(A) = 0$ for $n \geq 1$, by showing that $B : \overline{HC}_0(A) \longrightarrow H_1(A)$ is an isom. This apparently requires a computation, since I can't see a direct argument that $HC_1(A) = 0$. We have an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_1(A) \arrow[r] & A \otimes V \arrow[r, "b"] & \bar{A} \arrow[r] & \underset{\overline{HC}_0(A)\,=}{\bar{A}/[\bar{A},\bar{A}]} \arrow[r] \arrow[llll, dashed, bend left, "B"] & 0 \end{tikzcd}$$

and B is induced by $d : A \longrightarrow \Omega^1_A = A \otimes V \otimes A \longrightarrow (A \otimes V \otimes A)\otimes_A = A \otimes V$

Recall that $\bar{A}/[A,A]$ is the free abelian group generated by the non-identity conjugacy classes of G. Let the generators of G be $\{x_i, i \in I\}$. A typical element of G is represented by a word in the elements x_i, x_i^{-1} for I, such that x_i and its inverse x_i^{-1} are not together. A conjugacy class is represented by a circular word of this sort. Normally the map from G to the set of its conjugacy classes preserves the lengths of words, except in the case where a word begins with an x_i (or x_i^{-1}) and ends with its inverse.

Next I want to identify the complex of length 1: $A \otimes V \xrightarrow{b} \bar{A}$ with the complex of chains on a directed graph. The vertices are the elements of G, and for each $g \in G$ and x_i there is an edge going from g to $x_i g x_i^{-1}$. So This is the graph associated to the conjugation action of G on itself.

The components of this graph are the conjugacy classes. Let's consider a conjugacy class γ and let $g \in \gamma$. Let's consider what happens to the length of g as we conjugate by x_i or x_i^{-1}. Let $g = a_1 \cdots\cdots a_n$ where the a_j are some x_k or x_k^{-1}, and $n = \text{length}(g)$ ~~so~~ that a generator and its inverse do not appear consecutively. Then

$$x_i g x_i^{-1} = x_i a_1 \cdots a_n x_i^{-1}$$

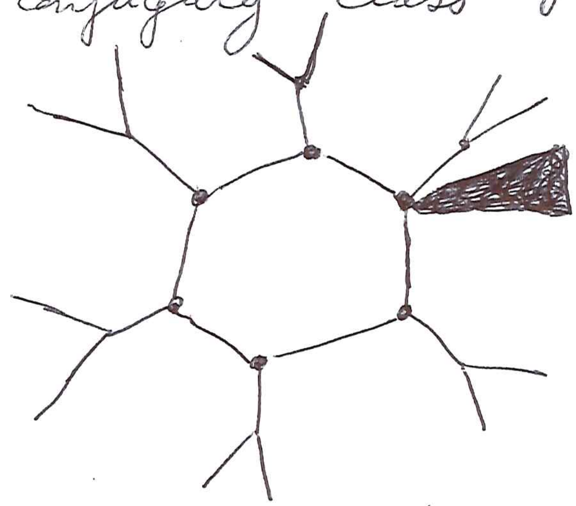
and there are four possibilities depending on whether $x_i a_1 = 1$ or not or $a_n x_i^{-1} = 1$ or not.

Let's first see what happens when $a_1 = a_n^{-1}$. Then if $x_i \neq a_1^{-1}$ we also have $x_i \neq a_n$, and the length of $x_i g x_i^{-1}$ is $n+2$. On the other hand if $x_i = a_1^{-1}$ then $x_i = a_n$ and the length of $x_i g x_i^{-1}$ is $n-2$. So we see that for a vertex $a_1 \cdots a_n$ with $a_1 = a_n^{-1}$, exactly one of the edges starting or ending at this vertex leads to a vertex of length $n-2$, and all the other edges lead to vertices of length $n+2$. So we have a contraction of our conjugacy class onto the subgraph whose vertices are the words of the form $g = a_1 \cdots a_n$ with $a_1 \neq a_n^{-1}$.

Now when we conjugate such a word by a generator x_i or its inverse, then we can't lower the length, and if we don't increase the length, then we cyclically permute the letters of g. This shows that the subgraph is an n-cycle * and that the conjugacy class γ looks as follows

should be an n/k-cycle, e.g. x^n will give:

n = 6



n = 1

~~*This assumes the letters of g are not all the same. See p. 428~~

So our graph has the homotopy type of a disjoint of circles, one for each non-identity conjugacy class.

Now if we write

$$A \otimes V = \bigoplus_i A\, dx_i = \bigoplus_i A\, x_i^{-1} dx_i$$

that is, we use the basis $x_i^{-1} dx_i$ for Ω_A^1 over $A \otimes A^{\circ}$, then $b : A \otimes V \longrightarrow \bar{A}$ sends

$$a\, x_i^{-1} dx_i \longmapsto [a x_i^{-1}, x_i] = a - x_i a x_i^{-1}$$

Then it follows that $A \otimes V \xrightarrow{b} \bar{A}$ can be identified with the chains on the graph. Hence we know the kernel and cokernel of $b : A \otimes V \to \bar{A}$ will have a basis indexed by the non-identity conjugacy classes. Finally, one has to check to $B : \bar{A} \longrightarrow A \otimes V$ assigns to each conjugacy class the cycle of lowest length words. This is more or less clear.

Examples. 1) $g = x y^{-1} x$. Then

$$d(xy^{-1}x) = dx\, y^{-1}x + x(-y^{-1}dy\, y^{-1})x + xy^{-1}dx$$
$$\approx (y^{-1}xx)(x^{-1}dx) - (y^{-1}xx)(y^{-1}dy) + (xy^{-1}x)(x^{-1}dx)$$

$$\begin{tikzcd}
 & & xxy^{-1} \\
xy^{-1}x \arrow[urr, bend left, "x"] & & \\
 & & y^{-1}xx \arrow[uu, "y"'] \arrow[ull, bend left, "x"]
\end{tikzcd}$$

2) $$d(xyxy) = dx\, yxy + x(dy)xy + xy(dx)y + xyx(dy)$$
$$\sim 2(yxyx)(x^{-1}dx) + 2(xyxy)(y^{-1}dy)$$

$$\begin{tikzcd}
xyxy \arrow[r, bend left, "y"] & yxyx \arrow[l, bend left, "x"]
\end{tikzcd}$$

This last example shows that $B = d$ will only be onto mod <u>torsion.</u>

So the (reduced) cyclic homology of the group ring of a free group should be trivial. NO see p. 444 To explore further

1) rational Waldhausen theory

2) Burghelea's thm. on cyclic homology of group rings. We have seen that for $A = \mathbb{C}[G]$, the Hochschild homology is the group homology of the conjugation action on the group ring, so

$$H_*(A,A) = H_*(G, \mathbb{C}[G])$$
$$= \bigoplus_{\gamma \text{ conjugacy class}} H_*(G_x, \mathbb{C}) \qquad \text{where } x \in \gamma.$$

<u>Correction:</u> Even though every every square zero extension $M \to Q \to A = \mathbb{C}[G]$, G free splits, this doesn't [redacted] show that $\overline{HC}_2(A) = 0$. All it says is that $\overline{HC}_2(A)$ is the equalizer of the two maps [redacted] on $\overline{HC}_0$ induced by $A \rightrightarrows A \sqcup A = A \oplus \Omega^1_A$. In other words $\overline{HC}_2(A) = \operatorname{Ker}\{\overline{HC}_0(A) \longrightarrow H_1(A)\}$. Thus one has to go through the calculation above.

Thus what is still missing in this whole business is a good approach to cyclic [redacted] homology which shows that free algebras and group rings of free groups have trivial cyclic homology.

December 9, 1987

Let A be the algebra of Schwartz functions $f(x,p)$ on $T^*(M)$, where $M = \mathbb{R}^n/\Gamma$ is a torus. Let P be the algebra of $f(h,x,p) = \sum_{n\geq 0} h^n f_n(x,p)$, [redacted] formal power series in h with coefficients in A, equipped with the twisted multiplication where

$$e^{-i\gamma x} p e^{i\gamma x} = p + h\gamma \qquad \gamma \in \Gamma^\vee.$$

The $P/I = A$ where $I = hP$.

On the algebra P we have a family of commuting derivations $\partial_{x_j}, \partial_{p_j}$ $j = 1, \dots, n$. To see these are derivations we express them as brackets

$$\begin{cases} \frac{1}{ih}[x,f] = \partial_p f \\ \frac{i}{h}[p,f] = \partial_x f \end{cases}$$

Using these I can construct a differential graded algebra

$$(*) \qquad P \xrightarrow{d} W^* \otimes P \xrightarrow{d} \Lambda^2 W^* \otimes P \longrightarrow \cdots \longrightarrow \Lambda^{2r} W^* \otimes P \to 0$$

which reduces where $h = 0$ to the DR complex of A. Here W is the $2r$-diml vector space with basis given by these derivations.

Let's recall that on the algebra $P/h^{r+1}P$ there is a trace given by taking the h^0 coeff. of

$$f \longmapsto \int [redacted] \left(\frac{dx\,dp}{2\pi h}\right)^n f(h,x,p)$$

which has a non-trivial effect on $K_0(A)$. We know this from the index theorem, as this trace gives the analytic index of an element of $K(T^*(M))$. Our problem has been to calculate this index via diffl geometry, i.e, using the DR complex of A. This is why I think that the DGA ⊛, which extends the DR complex of A to the deformation P, might be relevant.

According to my new theory the trace on the nilpotent extension $P/h^{r+1}P$ leads to a cyclic cohomology class of degree $2r$ on A, because there is a canonical homomorphism

$$HC_{2r}(A) \longrightarrow HC_0(P/h^{r+1}P).$$

Our ~~construction~~ of this homomorphism is a bit abstract, and seems quite far from DR complexes starting with A. It goes as follows. One forms the Connes complex of the DGA $I \to P$. This gives a double complex with an augmentation to the Connes complex of A which is a quis. One also forms the Connes complex of the DGA $P \to P$, which is a double complex such that each row is acyclic. The Connes complex for $I \to P$ embeds in that for $P \to P$, and we can form the quotient double complex

$$\begin{array}{|llll}
0 & & & P_\sigma^{\otimes 3}/I_\sigma^{\otimes 3} \\
0 & P\otimes_P/I & P_\sigma^{\otimes 2}/I_\sigma^{\otimes 2} & 0 \\
0 & P/I & 0 & 0 \\
\hline
\end{array}$$

The homology of the total complex is $HC_*(A)[1]$, so we get a spectral sequence for the double complex (shifted)

$$\begin{array}{|lll}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
P \otimes P/I & P_\sigma^{\otimes 2}/I_\sigma^{\otimes 2} & 0 \\
P/I & 0 & 0 \\
\hline
\end{array}$$

which abuts to $HC_*(A)$. Then one has an edge homomorphism

$$HC_{2k}(A) \longrightarrow E^1_{kk}$$

Clearly E^1_{kk} is a quotient of $P_\sigma^{\otimes(k+1)}/I_\sigma^{\otimes(k+1)}$ and a little more work should show it is $HC_0(P/I^{k+1})$.

This kind of construction is not very helpful, when A is a smooth commutative algebra, since one really has to bring in diff'l forms to describe the cyclic homology.

The spectral sequence above has edge homs.

$$H_n(P, A) \longrightarrow HC_n(A)$$

and there is an exact sequence of low degree terms

$$HC_0(A) = H_0(P, A)$$

$$0 \leftarrow HC_1(A) \leftarrow H_1(P, A) \leftarrow HC_0(P/I^2) \leftarrow$$

$$HC_2(A) \leftarrow H_2(P, A) \leftarrow E^1_{12}/\mathrm{Im}\, HC_0(P/I^3) \leftarrow HC_3(A)$$

$$E^1_{12} = H_1\left(\mathrm{Cone}\left((I \overset{L}{\otimes}_P)^2_\sigma \longrightarrow (P \overset{L}{\otimes}_P)^2_\sigma \right) \right)$$

A possible approach to link cyclic theory and differential forms is suggested by the fact that the standard bar resolution $B(A)$ of A as an $A\otimes A^{op}$ module involves the non-commutative diffl forms:

$$\begin{tikzcd}
{} \arrow[r] & A\otimes\bar{A}^{\otimes 2}\otimes A \arrow[rr] \arrow[dr, two heads] & & A\otimes\bar{A}\otimes A \arrow[r] \arrow[dr, two heads] & A\otimes A \arrow[r] & A \arrow[r] & 0 \\
& & \Omega^2_A \arrow[ur, hook] & & \Omega^1_A \arrow[u, hook] & &
\end{tikzcd}$$

In fact we saw that

$$\mathrm{Hom}_{A\otimes A^{op}}(\Omega^n_A, M) = Z^n(A,M)$$

I might hope that the double complex $\mathrm{Connes}(P \xrightarrow{1} P)$, whose kth column is a representative for $(P\overset{L}{\otimes}_P)^{\otimes k}_\sigma$, might be related nicely to the non-comm. DR complex of P. Then I might come in to provide a filtration.

Somewhere in all this the Bott theorem should enter?

Let's return to the DGA

$$(*) \qquad P \xrightarrow{d} W^*\otimes P \xrightarrow{d} \Lambda^2 W^*\otimes P \longrightarrow \cdots$$

and see if it helps to relate the trace on $P/h^{r+1}P$ to an element of $HC_{2k}(A)^*$. The quotient of $(*)$ by graded commutators is

$$P/[P,P] \longrightarrow W^*\otimes P/[P,P] \longrightarrow \cdots$$

Two ideas worth investigating.

1) What is $(A \overset{L}{\otimes}_P)^k_\sigma$ in relation to $(I \overset{L}{\otimes}_P)^k_\sigma$? More generally what is the cyclic cohomology of A over P, assuming this has a meaning? Given a homomorphism of rings $A \to B$, do we get a cyclic object $[k] \longrightarrow (B \otimes_A)^k$?

It would be nice to link the spectral sequence of the extension $P/I \to A$ to the Connes exact sequence for A, in particular to link the S operator to the I-adic filtration. You want to prove commutativity of

$$\begin{tikzcd} HC_{2k}(A) \arrow[r] \arrow[d, "S"'] & HC_0(P/I^{k+1}) \arrow[d] \\ HC_{2k-2}(A) \arrow[r] & HC_0(P/I^k) \end{tikzcd}$$

Also it would be nice to see $((I/I^2) \overset{L}{\otimes}_A)^k_\sigma$ appear, linked in a suitable way to the Hochschild homology.

2) Gauss-Manin: Goodwillie's theorem on derivations is only part of the picture. Think about the geometric situation where one defines the Gauss-Manin connection. One has a fibre bundle $X \to Y$ with compact manifolds as fibres, i.e. a proper smooth map in the algebraic geometry context. Then for each y one considers the DR cohomology of the fibre X_y. This gives a vector bundle over Y. This vector bundle has a natural connection because from the point of view of topology the bundle is locally trivial. This

connection can be defined purely algebraically and is the Gauss-Manin connection. The point is that as complex manifolds the fibres are not isomorphic; there are moduli around.

The idea is that it should be possible to treat the tangent groupoid deformation of Connes in a similar way. Goodwillie's theorem shows periodic cyclic theory is constant under such deformations, but a precise understanding of the mechanism of the Gauss-Manin connection might be very useful.

December 10, 1987

To understand the Gauss-Manin connection it might be possible to work in a C^∞ context with a fibre bundle $X \to Y$ with compact fibres, provided we don't use the local triviality of the fibration. More generally we might consider a foliated manifold X.

Suppose (X, F) is a foliated manifold. Here F is an integrable subbundle of T_X. Let Q be the normal bundle

$$0 \longrightarrow F \longrightarrow T_X \longrightarrow Q \longrightarrow 0$$

Then we have an exact sequence

$$0 \longrightarrow Q^* \longrightarrow T_X^* \longrightarrow F^* \longrightarrow 0$$

and integrability says that the ideal I generated by $\Gamma(Q^*)$ in $\Gamma(\Lambda^\cdot T_X^*) = \Omega_X^\cdot$ is closed under d. Then we can define a decreasing filtration

$$F_p\, \Omega_X^\cdot = I^p$$

$$F_p \Omega_X^\cdot / F_{p+1} \Omega_X^\cdot = \Gamma\left(\Lambda^p Q^* \otimes \Lambda^\cdot F^* [p] \right)$$

This filtration gives a spectral sequence

$$E_1^{pq} = H^{p+q}(gr_p\, \Omega_X^\cdot) = H^q(\Gamma(\Lambda^p Q^* \otimes \Lambda^\cdot F^*))$$

abutting to $H^*_{DR}(X)$.

~~We note that in the case where the foliation comes from a fibre bundle map $X \to Y$, more generally a submersion, F is the tangent bundle along the fibres and the normal bundle Q has a partial connection along the leaves which~~

Recall that the normal bundle Q to the foliation has a partial connection in the direction of the leaves which is flat. In general given a flat bundle there is a twisted DR complex with coefficients in this bundle which furnishes the cohomology of the local coefficient system associated to the flat bundle. The complex $gr_p \Omega_X^{\cdot} = \Gamma(\Lambda^p Q^* \otimes \Lambda^{\cdot} F^*)$ is the global sections of the family of these twisted DR complexes on the leaves associated to $\Lambda^p Q^*$.

Now let's suppose the foliation arises from a submersion $f: X \longrightarrow Y$ which is proper. Then $Q = f^* T_Y$ and F is the relative tangent bundle $T_{X/Y}$, so

$$E_1^{pq} = \Gamma(Y, \Lambda^p T_Y^* \otimes_{\mathcal{O}_Y} \underbrace{H^q(\Gamma \Lambda^{\cdot} F^*)}_{\text{bundle of cohomology of the fibres}})$$

and we are getting the Leray spectral sequence.

I think the important point is that the complex $\Gamma(X, \Lambda^{\cdot} F^*)$, which is the relative DR complex of X/Y, so that its cohomology groups are $\mathcal{O}_Y$ modules, inherits a flat connection on its cohomology. This connection is given by $d_1 : E_1^{0q} \longrightarrow E_1^{1q}$ and its flat because $d_1^2 = 0$.

Let's concentrate on this aspect. Let's start with a smooth projective variety X_0 over $\mathbb{C}$, and consider a deformation of it X over $\mathbb{C}[\varepsilon]$. Then we can form the relative de Rham complex $\Omega_X^{\cdot}$ which is a complex of sheaves over X_0 of modules over $\mathbb{C}[\varepsilon]$.

To reconstruct Grothendieck's version of the Gauss-Manin connection. Let X_0 be smooth projective over $\mathbb{C}$, and let X over $\mathbb{C}[\varepsilon]$ be a first order deformation. Then X and X_0 have the same underlying topological space, but different structural sheaves. One has a square zero extension

$$\begin{tikzcd} 0 \arrow[r] & \varepsilon\mathcal{O}_X \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_{X_0} \arrow[r] & 0 \\ & \mathcal{O}_{X_0} \arrow[u, "\wr"'] & & & \end{tikzcd}$$

of sheaves of algebras. The Kodaira-Spencer class associated to this deformation is an element of $H^1(X_0, T_{X_0})$, which is probably related to the exact sequence

$$0 \longrightarrow \varepsilon\mathcal{O}_X \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0} \longrightarrow \Omega^1_{X_0} \longrightarrow 0.$$

This is the basic long exact sequence for the cotangent complex; the fact it's injective at the left is because X_0 is smooth.

Besides the structural sheaves we can also consider the DR complexes, and these will fit into an exact sequence of $\mathbb{C}[\varepsilon]$-module sheaves

$$\begin{tikzcd} 0 \arrow[r] & \varepsilon\Omega^\bullet_X \arrow[r] & \Omega^\bullet_{X/\mathbb{C}[\varepsilon]} \arrow[r] & \Omega^\bullet_{X_0} \arrow[r] & 0 \\ & \Omega^\bullet_{X_0} \arrow[u, "\wr"'] & & & \end{tikzcd}$$

Our problem is to construct in the derived category of $\mathbb{C}$-sheaves on X_0 a canonical map

$$\Omega^\bullet_{X_0} \longrightarrow \Omega^\bullet_{X/\mathbb{C}[\varepsilon]}$$

such that

$$\Omega^\bullet_{X_0}\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon] \longrightarrow \Omega^\bullet_{X/\mathbb{C}[\varepsilon]}$$

is a quis. (I think this is what one means by the DR complex being a "crystal". More generally if X is a deformation of X_0 with parameter ring a local Artin algebra A, then we want ~~an~~ canonical isomorphism $\Omega^\bullet_{X_0}\otimes_{\mathbb{C}} A = \Omega^\bullet_{X/A}$ in the derived category of sheaves of A modules over X_0.)

To gain some insight let's suppose our first order deformation extends to a 1-parameter deformation. Let then X_∞ be over $\mathbb{C}[[\varepsilon]]$ and let X_1 be the deformation over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$. Then we can consider the ~~total~~ DR complex

$\Omega^\bullet_{X_\infty/\mathbb{C}}$.

In general one can consider an embedding of schemes $X_0 \xrightarrow{i} W$, and the cotangent complex exact triangle gives

$$0 \longrightarrow I/I^2 \longrightarrow i^*\Omega^1_W \longrightarrow \Omega^1_{X_0} \longrightarrow \overset{0}{\cancel{\Omega^1_{X_0/W}}} \to 0$$

assuming X smooth. This exact sequence gives an element of $\mathrm{Ext}^1_{O_{X_0}}(\Omega^1_X, I/I^2) = H^1(X_0, T_{X_0}\otimes I/I^2)$ which is probably the Kodaira-Spencer invariant.

Here seems to be the way to proceed. By the homotopy property of DR cohomology we get an equivalence between $\Omega^\bullet_{X_\infty}$ and $\Omega^\bullet_{X_0}$. Thus you have an explicit deformation ??

Let's consider a deformation X over $\Lambda = \mathbb{C}[[\varepsilon]]$ of $X_0 = X \times_\Lambda \mathbb{C}$. Then we can form the complex $\Omega^\bullet_X$ and we have an obvious restriction map $\Omega^\bullet_X \longrightarrow \Omega^\bullet_{X_0}$. The claim is that this is a quasi-isomorphism, and the proof should be some sort of deformation to the normal bundle. In practice this means that we filter $\Omega^\bullet_X$ by powers of ε, and exhibit a homotopy equivalence on the associated graded level.

Now we also have

$$0 \longrightarrow \underset{\underset{\mathcal{O}_X\, d\varepsilon}{\parallel}}{\Omega^1_{\Lambda/\mathbb{C}} \otimes_\Lambda \mathcal{O}_X} \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/\Lambda} \longrightarrow 0$$

so the codimension of the foliation is 1, and so we get a 2-step filtration on $\Omega^\bullet_X$:

$$0 \longrightarrow \underset{\underset{\Omega^\bullet_{X/\Lambda}[1]}{\parallel}}{I} \longrightarrow \Omega^\bullet_X \longrightarrow \Omega^\bullet_{X/\Lambda} \longrightarrow 0$$

In the derived category this means that there is a map $\nabla : \Omega^\bullet_{X/\Lambda} \longrightarrow \Omega^\bullet_{X/\Lambda}$ whose fibre is $\Omega^\bullet_X$ which is isomorphic to $\Omega^\bullet_{X_0}$. ∇ is in fact the Gauss-Manin connection.

December 12, 1987

Given any extension $I \to P \to A$ we've constructed a double complex whose k-th column is $\mathrm{Cone}\left\{ (I \overset{L}{\otimes}_P)^{k+1}_\sigma \longrightarrow (P \overset{L}{\otimes}_P)^{k+1}_\sigma \right\} [k]$ for $k \geq 0$ and whose total complex is quis to $\mathrm{Connes}(A)$. This gives a spectral sequence

$$
\begin{array}{|llll}
* & * & * & * \\
H_2(P,A) & * & HC_0(P/I^3) & 0 \\
H_1(P,A) & HC_0(P/I^2) & 0 & \\
H_0(P,A) & 0 & 0 & \\
\hline
\end{array}
\quad \Longrightarrow \quad HC_n(A)
$$

with a low degree exact sequence

$$
H_2(P,A) \to HC_2(A) \longrightarrow HC_0(P/I^2) \longrightarrow H_1(P,A) \longrightarrow HC_1(A) \longrightarrow 0
$$

It's more or less clear that when $P = A$ and $I = 0$ this five term exact sequence coincides with the Connes exact sequence; in fact the spectral sequence coincides with the Connes exact sequence.

It's becoming clear that a better understanding of $HC_1(A)$ is essential in order to develop the analogy between the cyclic and crystalline theories. The essential results that HC_1 vanishes on free algebras and group rings have to become transparent first.

The above exact sequence & 5 lemma imply

$$
H_1(P,A) \xrightarrow{\sim} H_1(P/I^2, A)
$$

One can derive this independently as follows. Let N be a $P/I = A$ module. Then

$$H_*(P,N) = H_*\left(N \overset{L}{\otimes}_P\right) = H_*\left(N \otimes_P \mathcal{B}(P) \otimes_P\right)$$

$$= H_*\left(N \otimes_A (A \otimes_P \mathcal{B}(P) \otimes_P A) \otimes_A\right)$$

$$= H_*\Big(N \overset{L}{\otimes}_A \underbrace{(A \otimes_P \mathcal{B}(P) \otimes_P A)}_{\text{because this is free over } A\otimes A^{op}} \overset{L}{\otimes}_A\Big) = H_*\left(N \overset{L}{\underset{A}{\otimes}} (A \overset{L}{\otimes}_P A) \overset{L}{\otimes}_A\right)$$

So there's a spectral sequence

$$E^2_{pq} = \operatorname{Tor}_p^{A\otimes A^{op}}\left(N, \operatorname{Tor}_q^{P}(A,A)\right)$$

$$\Longrightarrow H_*(P,N).$$

$$H_2(P,N) \longrightarrow E^2_{20} \xrightarrow{d^2} E^2_{01} \longrightarrow H_1(P,N) \longrightarrow E^2_{10} \longrightarrow 0$$

$$H_2(P,A) \longrightarrow H_2(A,A) \longrightarrow H_0(A, I/I^2) \longrightarrow H_1(P,A) \longrightarrow H_1(A,A) \longrightarrow 0$$

Again the five lemma yields $H_1(P,A) \xrightarrow{\sim} H_1(P/I^2, A)$.

We've seen that the important ingredient for the vanishing of cyclic homology for free rings is that the map

$$\overline{HC}_0(A) \longrightarrow H_1(A,A)$$

is an isomorphism. Now our study of Goodwillie's theorem gives some control on this map as follows. First we have the two canonical maps

$$\overline{HC}_0(A) \rightrightarrows \overline{HC}_0(A \oplus \Omega^1_A)$$

$$\overline{HC}_0(A) \oplus \overline{H}_0(A, \Omega^1_A)$$

induced by $1, 1+d : A \longrightarrow A \oplus \Omega^1_A$. The difference gives a map

$$\overline{HC}_0(A) \longrightarrow H_0(A, \Omega^1_A)$$

and on the other hand we have

$$0 \longrightarrow H_1(A,A) \longrightarrow H_0(A,\Omega^1_A) \xrightarrow{\circledast} A \longrightarrow H_0(A,A) \longrightarrow 0$$

Computation shows $\circledast$ kills the image of $HC_0(A)$ whence the desired map.

In the case of free algebras we can use Goodwillie's thm. to prove the higher cyclic homology is zero. We have the canonical derivation given by the grading of $T(V)$. This derivation acts trivially on the image of S. So the image of $\bar{HC}_2(A) \xrightarrow{S} \bar{HC}_0(A)$ has to be of degree 0 relative to the grading, hence this S map must be zero, showing (since $H_n(A,A) = 0$, $n \geq 2$) that $\bar{HC}_2(A) = \bar{HC}_4(A) = \cdots = 0$. Similarly

$$\underset{\overset{\|}{0}}{H_3(A,A)} \to \bar{HC}_3(A) \xrightarrow{S} \bar{HC}_1(A) \longrightarrow \underset{\overset{\|}{0}}{H_1(A,A)}$$

implies $\bar{HC}_1(A)$ has grading zero. But its a quotient of $H_1(A,A) \subset H_0(A,\Omega^1_A)$ which is positively graded, hence $0 = \bar{HC}_1(A) = \bar{HC}_3(A) = \cdots$.

December 14, 1987

I want to find a good proof that the reduced cyclic homology of a free group algebra is trivial in positive degrees. Let $A = \mathbb{C}[G]$, G free. For example $A = \mathbb{C}[z, z^{-1}]$. Let's write $A = P/I$, [redacted] where P is free. Then we can't lift the generators of G to invertibles in P, but we can lift 2×2 matrices of the form $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ to invertible matrices over P. Thus we can define a homomorphism

$$A \longrightarrow M_2(P)$$

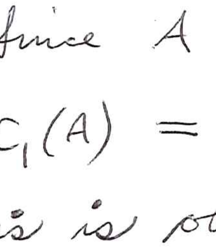
which might be useful.

Actually we know that $K_1(\mathbb{C}[z, z^{-1}]) = K_1(\mathbb{C}) \oplus \mathbb{Z}$ so we should check carefully that $HC_1(A) = 0$ in this case. Since A is a smooth algebra

$$HC_1(A) = \Omega^1_A / dA \qquad \Omega^1_A = \text{Kähler differentials}$$

[redacted] And this is obviously non-zero generated by $\frac{dz}{z}$. Hence there is a mistake in our earlier calculation of the cyclic homology of free group algebras.

The error occurs on p. 426 where we omitted the case of a word x_i^n, $n \neq 0$, where x_i is a generators. In this case the component of the graph contracts to a circle [redacted] and not to an $|n|$-cycle.



Now if we apply d to the conjugacy class of x_i^n we obtain for $n > 0$:

$$d(x_i^n) = \sum_{j=1}^{n} x_i^{j-1}\, dx_i\, x_i^{n-j}$$

$$= n\, x_i^n\, (x_i^{-1} dx_i)$$

and this formula also holds for $n<0$. So far we see that each generator of $HC_0(A) = \bar{A}/[A,A]$ $= \bar{H}_0(\text{graph})$ cancels each generator of $H_1(\text{graph})$. The error occurs when we identify the cycles on the graph with $A \otimes V$. This would be true if we were to use all elements of G as vertices including the identity.

Hence the correct way to proceed with the calculation is to define the graph to have all elements of G as vertices, and to have a directed edge from g to $x_i g x_i^{-1}$ for each generator x_i. Then the graph will have all the previous component together with the component with one vertex the identity

Conclusion:

<u>Theorem</u>: Let $A = \mathbb{C}[G]$ with G free. Then the reduced cyclic homology of A is

$\bar{H}C_0(A) = \bar{A}/[A,A] =$ vector space with basis the non-identity conjugacy classes of G

$$\bar{H}C_2(A) = \bar{H}C_4(A) = \cdots\cdots = 0$$

$\bar{H}C_1(A) = \bar{H}C_3(A) = \cdots\cdots =$ vector space with basis the generators of G

$= G_{ab} \otimes_{\mathbb{Z}} \mathbb{C}$.

December 15, 1987

If $A = \mathbb{C}[G]$, then there is a canonical map $G_{ab} \longrightarrow K_1(A)$, which when composed with the canonical maps to odd cyclic homology, gives canonical maps

$$G_{ab} \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow HC_{2k-1}(A).$$

Presumably this map is the isomorphism discussed yesterday when G is free.

Let's review our analysis of Goodwillie's thm. Let D be a derivation of A with values in the A-bimodule M. Then D can be viewed as a derivation of the extension

$$(*) \qquad M \longrightarrow A \oplus M \longrightarrow A$$

which is trivial on the ideal M and the ~~quotient~~ algebra A. Since D is nilpotent we obtain a 1-parameter group $e^{tD} = 1 + tD$ of automorphisms of the extension $(*)$. (Thus I have an action of $\mathbb{G}_a$ on $(*)$, but this has to be explained a bit.)

Consider the induced action on $HC_*(A \oplus M)$. First of all a derivation D of an algebra A extends to an endomorphism of the cyclic complex and so gives rise to an endomorphism $\rho(D)$ of the cyclic homology. In this way one gets an action of the Lie algebra of derivations on the cyclic homology.

In the particular case of a derivation D of $(*)$ acting trivially on M, A, we see that $\rho(D)$ is ~~nilpotent~~ on $HC_k(A \oplus M)$. In fact we can look at the action of the one parameter group e^{tD} of autos of $A \oplus M$ on the cyclic complex $(A\oplus M)_t^{\otimes(k+1)}$, and it's clear that

$$\rho(e^{tD}) = e^{t\rho(D)}$$

where $\rho(D)^{k+2} = 0$ on $(A\oplus M)_t^{\otimes(k+1)}$.

Another ingredient is the natural $\mathbb{Z}$ grading of the algebra $A \oplus M$, in which M has degree 1. This means we have an action of $\mathbb{G}_m$ on $A\oplus M$, hence an induced action on $HC_*(A\oplus M)$ which means there is a grading

$$HC_*(A\oplus M) = \bigoplus_{g\geq 0} HC_*(A\oplus M)_{(g)}$$

When we combine this with the action of $\mathbb{G}_a$ defined by $\rho(D)$, we get an action of $\mathbb{G}_m \ltimes \mathbb{G}_a$. All this means is that $\rho(D)$ is of degree 1 relative to the grading.

Then from the formula $\rho(e^{tD}) = e^{t\rho(D)}$ we conclude that a class $\xi \in HC(A\oplus M)$ is fixed under the 1-parameter autom. group $\rho(e^{tD})$ iff $\rho(D)\xi = 0$. In particular if $\xi \in HC(A\oplus M)_{(0)} = HC(A)$, then ξ is fixed under $\rho(e^{tD})$ iff

$$\rho(D) : HC(A) \longrightarrow HC(A\oplus M)_{(1)} = H(A,M)$$

kills ξ.

Thus what I proved is that given $D: A \to M$ it induces a map $\rho(D): HC_*(A) \longrightarrow H_*(A,M)$, and a class $\xi \in HC_*(A)$ is killed by $\rho(D)$ iff it is fixed under the maps $HC_*(A) \longrightarrow HC_*(A\oplus M)$ induced by the embeddings $\begin{array}{l} A \longrightarrow A\oplus M \\ a \longmapsto a + tDm \end{array}$.

If I take $M = \Omega^1_A$ and $D =$ the canonical derivation $d : A \to \Omega^1_A$, then $\rho(d) : HC_k(A) \to H_k(A, \Omega^1_A)$ is probably the same (up to embedding $H_{k+1}(A) \to H_k(A, \Omega^1_A)$) as the map $HC_k(A) \xrightarrow{B} H_{k+1}(A)$ in the Connes exact sequence. So we can conclude that $\xi \in HC_k(A)$ is killed by $B \implies \xi$ is equalized by the maps $A \rightrightarrows A \oplus M$, $a \mapsto a$, $a \mapsto a + Dm$, for any derivation $D : A \to M$.

It remains to deduce Goodwillie's thm. from this.

This time we start with a derivation D of A and we wish to understand its effect $\rho(D)$ on $HC_*(A)$. Let $\mathbb{C}[\varepsilon] = \mathbb{C} \oplus \mathbb{C}\varepsilon$ with $\varepsilon^2 = 0$, so that $A \otimes \mathbb{C}[\varepsilon] = A \oplus \varepsilon A =$ the semi-direct product of A and A considered as a bimodule. Then we can consider $e^{\varepsilon D}$ as an automorphism of $A \otimes \mathbb{C}[\varepsilon]$; and the effect of $e^{\varepsilon D}$ on $HC_*(A \otimes \mathbb{C}[\varepsilon])$ has been discussed above.

In fact we have the formula for $HC_*(A \oplus M)$ discovered by Goodwillie + Feigin + Tsygan, where

$$HC_*(A \oplus M) = HC_*(A) \oplus \bigoplus_{k \geq 1} \underbrace{H_*\left(\left(M \overset{L}{\otimes}_A\right)^k_t\right)[k-1]}_{HC_*(A \oplus M)_{(k)}}$$

When $M = A$ we know that the cyclic permutation σ acts trivially on $\left(A \overset{L}{\otimes}_A\right)^k$ (p. 405). t is the cyclic permutation with the sign $(-1)^{k-1}$, whence

$$H_*\left(\left(A \overset{L}{\otimes}_A\right)^k_t\right) = \begin{cases} H_*(A, A) & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Let's now take the Grothendieck viewpoint, in which a derivation D of an algebra A is to be viewed as an automorphism of the algebra $A \otimes \mathbb{C}[\varepsilon]$ over $\mathbb{C}[\varepsilon]$ which reduces to the identity modulo ε. The automorphism is $e^{\varepsilon D} = 1 + \varepsilon D$ and it acts not only on $HC_*(A \otimes \mathbb{C}[\varepsilon]$ over $\mathbb{C})$ but on $HC_*(A \otimes \mathbb{C}[\varepsilon]$ over $\mathbb{C}[\varepsilon])$. Because $\mathbb{C}[\varepsilon]$ is "flat" over $\mathbb{C}$ one has

$$HC_*(A \otimes \mathbb{C}[\varepsilon] \text{ over } \mathbb{C}[\varepsilon]) = HC_*(A) \otimes \mathbb{C}[\varepsilon]$$

and it's pretty clear that the effect of $e^{\varepsilon D}$ on the left is just $1 + \varepsilon\, \rho(D)$ on the right.

But we have an obvious map

$$\begin{array}{ccc} HC_*(A \otimes \mathbb{C}[\varepsilon]) & \longrightarrow & HC_*(A \otimes \mathbb{C}[\varepsilon] \text{ over } \mathbb{C}[\varepsilon]) \\ \| & & \| \\ HC_*(A) \oplus H_*(A, \varepsilon A) \oplus \cdots & & HC_*(A) \oplus \varepsilon\, HC_*(A) \end{array}$$

and because of the obvious $\mathbb{G}_m$-action on $\mathbb{C}[\varepsilon]$ the degree $k \geq 2$ terms go to zero. It seem reasonable to expect the map on degree 1 ~~terms~~

$$H_*(A, A) \longrightarrow HC_*(A)$$

to be the canonical map from Hochschild to cyclic homology. If so, then we have a commutative triangle

$$\begin{tikzcd} HC_*(A) \arrow[r, "\rho(D)"] \arrow[dr, "\rho(D)"'] & H_*(A,A) \arrow[d] \\ & HC_*(A) \end{tikzcd}$$

the are defined more generally for any $D: A \longrightarrow M$.

and Goodwillie's theorem follows.

If the above is all correct, we should have a fairly simple proof of Goodwillie's thm. as follows. First given $D: A \to M$ we define a map $\rho(D): HC_*(A) \longrightarrow H_*(A,M)$, say using the cyclic complex of $A \oplus M$. [redacted] Secondly if we take D to be, the canonical $d: A \longrightarrow \Omega^1_A$, then we want

$$\begin{tikzcd} HC_*(A) \arrow[r, "B"] \arrow[dr, "\rho(d)"'] & H_{*+1}(A,A) \arrow[d, phantom, "\wr"] \\ & H_*(A,\Omega^1_A) \end{tikzcd}$$

to commute. This shows that $\rho(d) = 0$ on $\mathrm{Ker}\, B = \mathrm{Im}\, S$. [redacted] It follows [redacted] from the universal character of d that $\rho(D): HC_*(A) \longrightarrow H_*(A,M)$ vanishes on $\mathrm{Ker}\, B$ for any derivation $D: A \to M$. [redacted] Finally if we take $M = A$, then we have to show

$$\circledast \qquad \begin{tikzcd} HC_*(A) \arrow[rr, "\rho(D)"] \arrow[dr, "\hat{\rho}(D)"'] & & H_*(A,A) \arrow[dl, "\text{canon. map } I"] \\ & HC_*(A) & \end{tikzcd}$$

commutes, where $\hat{\rho}(D)$ is the map on cyclic homology induced by the derivation D.

Notice that the image of $\hat{\rho}(D)$ is contained in $\mathrm{Im}(I) = \mathrm{Ker}(S)$, if the above triangle commutes, and this also implies $\hat{\rho}(D) = 0$ on $\mathrm{Im}(S)$. So it's enough to define $\rho(D)$ and check $\circledast$ commutes.

December 16, 1987

Yesterday we found a short proof of Goodwillie's theorem that derivations of A act trivially on the image of S in $HC_*(A)$. (This proof is probably implicit in Connes' ~~paper~~ Ch. I)

Let D be a derivation, and let $\rho(D)$ denote the induced action of D on $HC_*(A)$. The point is that $\rho(D)$ factors

$$\begin{tikzcd}
{} \arrow[r, "B"] & H_n(A) \arrow[r, "I"] \arrow[d] & HC_n(A) \arrow[r, "S"] \arrow[d, "\rho(D)"] \arrow[dl, dashed] \arrow[dl, phantom, "\text{commutes}"'] & HC_{n-2}(A) \arrow[r, "B"] \arrow[d, "\rho(D)"] & H_{n-1}(A) \arrow[r] \arrow[d] & {} \arrow[dl, dashed] \\
{} \arrow[r, "B"] & H_n(A) \arrow[r, "I"] & HC_n(A) \arrow[r, "S"] & HC_{n-2}(A) \arrow[r, "B"] & H_{n-1}(A) \arrow[r] & {}
\end{tikzcd}$$

where the dotted arrow can be more generally defined $\tilde{\rho}(D): HC_*(A) \longrightarrow H_*(A,M)$ for any derivation $D: A \to M$.

Defn. of $\tilde{\rho}(D)$:

$$HC_n(A) \xrightarrow{B} H_{n+1}(A,A) \overset{\text{isom for } n \geq 1}{\subset} H_n(A, \Omega^1_A) \longrightarrow H_n(A,M)$$

where the last map is induced by the unique A-bimod map $\Omega^1_A \xrightarrow{u} M$ such that $ud = D$.

Note that any derivation of A defines a map $H_{*+1}(A) \longrightarrow H_*(A)$ of degree -1 on Hochschild homology. Question: Can this map be combined with I to get a map $H_*(A) \longrightarrow HC_{*-1}(A)$ compatible in some sense with the above diagram?

December 18, 1987

To understand the Gauss-Manin connection.

The basic starting point is to consider a smooth proper map $X \xrightarrow{f} Y$, where Y is smooth. Then the fibrewise DR cohomology sheaves $R^i f_*(\Omega^\circ_{X/Y})$ are vector bundles on Y with an integrable connection, namely the Gauss-Manin connection which we want to construct. This means we want to make vector fields on Y act on $R^i f_*(\Omega^\circ_{X/Y})$ in a fashion consistent with their action on functions on Y.

So I have to discuss vector fields on Y first. Let $S_0 = Sp(k)$, $S = Sp(k[\varepsilon])$, where $\varepsilon^2 = 0$. A vector field on Y is a derivation of $\mathcal{O}_Y$, i.e. an automorphism of the extension

$$0 \longrightarrow \varepsilon\mathcal{O}_Y \longrightarrow k[\varepsilon]\otimes_k \mathcal{O}_Y \xrightarrow{\ \pi\ } \mathcal{O}_Y \longrightarrow 0$$

inducing the identity on $\varepsilon\mathcal{O}_Y$, $\mathcal{O}_Y$, or equivalently an autom. of $k[\varepsilon]\otimes_k \mathcal{O}_Y$ over $k[\varepsilon]$ inducing the identity modulo ε. Such an automorphism is determined by a homomorphism $\mathcal{O}_Y \to k[\varepsilon]\otimes_k \mathcal{O}_Y$ which is a section of π. Thus a vector field θ on Y can be identified with a retraction map for the square zero extension

$$Y = S_0 \times Y \overset{\theta}{\underset{}{\hookrightarrow}} S \times Y. \qquad (\text{products over } S_0 = Sp(k))$$

The zero vector field is given by the structural map $S \to S_0$. Thus when we have a vector field θ we have two maps $S \times Y \longrightarrow Y$, and hence we have a diagram

$$\begin{tikzcd}
Y \arrow[r, hook, "t", "\text{square zero}"'] \arrow[d, equal] & S \times Y \arrow[r, shift left, "\theta"] \arrow[r, shift right, "pr_2"'] \arrow[d] & Y \arrow[d, equal] \\
Y \arrow[r, hook] & Y \times Y \arrow[r, shift left] \arrow[r, shift right] & Y
\end{tikzcd}$$

which shows that the first infinitesimal neighborhood of the diagonal in $Y \times Y$ is sort of universal for vector fields.

Now returning to $X \xrightarrow{f} Y$ we can pull X back by θ, pr_2 and we get

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[r, hook] \arrow[d, "f"'] & \theta^*(X) \arrow[d, "f'"] & pr_2^*(X) = S \times X \arrow[dl, "f''"] \\
Y \arrow[r, hook] & S \times Y &
\end{tikzcd}$$

The Gauss-Manin connection we are after gives an isomorphism

$$R^i f'_* \left(\Omega^\bullet_{\theta^*(X)/S \times Y} \right) = R^i f''_* \left(\Omega^\bullet_{S \times X / S \times Y} \right)$$
$$\| $$
$$k[\varepsilon] \otimes_k R^i f_* \left(\Omega^\bullet_{X/Y} \right)$$

What Grothendieck does is to consider a more general situation:

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[r, hook] \arrow[d, "f"'] & X' \arrow[d, "f'"] & X'' \arrow[dl, "f''"] \\
S_0 \arrow[r, hook, "\text{square zero ext}"'] & S &
\end{tikzcd}$$

f, f', f'' smooth
squares cartesian

and he constructs a canonical isomorphism in the derived category $\Omega^\bullet_{X'/S} \simeq \Omega^\bullet_{X''/S}$

of sheaves of O_S-modules. Here one uses that X, X', X'' all have the same underlying topological space. The ingredients of his [redacted] construction are 1) locally X', X'' are isomorphic, [redacted] hence giving a torsor of isomorphisms of X', X'' under the [redacted] sheaf of derivations 2) Cartan's homotopy formula showing that derivations act trivially on $\Omega^{\circ}_{X'/S}$ in the derived category.

Now I want to find an analogue for cyclic theory, or rather I want to see if crystalline ideas are relevant to cyclic theory.

Let's concentrate on our main example which is the deformation of the commutative algebra of functions on the cotangent bundle. Let A = functions $f(x,p)$ on $T^*(S^1)$, P = deformation algebra of $f(h,x,p)$, and set $Q = P/h^2$, so that we have an extension

$$(*) \qquad 0 \longrightarrow \varepsilon A \longrightarrow Q \longrightarrow A \longrightarrow 0 \qquad \varepsilon = h \bmod h^2$$

(This is analogous to $\theta^*(X) = X'$ above). Thus we have two extensions namely $(*)$ and the split extension, I want to consider them as algebras over $k[\varepsilon]$. It doesn't seem possible to use Grothendieck's construction, since I can't see any way to localize and arrange that Q becomes isomorphic to $A \oplus \varepsilon A$.

December 23, 1987

The problem is to thoroughly understand the map

1) $$(I \overset{L}{\otimes}_P)^k_\sigma [1] \longrightarrow (I \overset{L}{\otimes}_P)^{k-1}_\sigma$$

which arises in studying extensions $A = P/I$. More generally we can consider a P-bimodule I equipped with a bimodule map $u : I \longrightarrow P$ satisfying

2) $$u(x)y = x\,u(y).$$

This allows us to consider the DGA

$$\longrightarrow 0 \longrightarrow I \xrightarrow{u} P$$

whose cyclic complex yields the maps 1).

Suppose $k=2$ to begin with. Let's look what can be done with a bimodule map $u : I \to P$ and later look at the consequences of 2). We have two maps in the derived category of P-bimodules

3) $$\begin{tikzcd} I \overset{L}{\otimes}_P I \arrow[r, "1\otimes u"] \arrow[d, "u\otimes 1"] & I \overset{L}{\otimes}_P P \arrow[d, "\sim"] \\ P \overset{L}{\otimes}_P I \arrow[d, "\sim"] & I \otimes_P P \arrow[d, equal] \\ P \otimes_P I \arrow[r, equal] & I \end{tikzcd}$$

and if we apply $\overset{L}{\otimes}_P$ we obtain two maps

4) $$I \overset{L}{\otimes}_P I \overset{L}{\otimes}_P \underset{u\otimes 1}{\overset{1\otimes u}{\rightrightarrows}} I \overset{L}{\otimes}_P$$

which are related by the cyclic permutation σ acting on the former complex.

(I can be more specific by using $\otimes_P \mathcal{B}(P) \otimes_P$ to calculate $\overset{L}{\otimes}_P$. Put $\mathcal{B} = \mathcal{B}(P)$. Then $u \otimes 1$ is

$$
\begin{tikzcd}
I \otimes_P \mathcal{B} \otimes_P I \otimes_P \mathcal{B} \otimes_P \arrow[r, leftrightarrow, "\sigma"] \arrow[d, "u\otimes 1 \otimes 1 \otimes 1 \otimes"] & I \otimes_P \mathcal{B} \otimes_P I \otimes_P \mathcal{B} \otimes_P \arrow[d, "1\otimes 1 \otimes u \otimes 1 \otimes"] \\
R \otimes_R \mathcal{B} \otimes_P I \otimes_P \mathcal{B} \otimes_P \arrow[r, leftrightarrow, "\sigma"] \arrow[d, "\varepsilon\otimes 1 \otimes 1 \otimes 1 \otimes"] & I \otimes_P \mathcal{B} \otimes_P P \otimes_P \mathcal{B} \otimes_P \arrow[d, "1\otimes 1 \otimes \varepsilon \otimes 1 \otimes"] \\
\cancel{P \otimes_P} I \otimes_P \mathcal{B} \otimes_P \arrow[r, equal] & I \otimes_P \mathcal{B} \otimes_P \cancel{P} \otimes_P
\end{tikzcd}
$$

where $\varepsilon : \mathcal{B}(P) \longrightarrow P$ is the augmentation.)

Actually it seems simpler to define the two maps in 3) by

$$
\begin{tikzcd}
I \overset{L}{\otimes}_P I \arrow[dr] & & \\
& I \otimes_P I \arrow[r, "1\otimes u"] \arrow[d, "u\otimes 1"] & I \otimes_P P \arrow[d, equal] \\
& P \otimes_P I \arrow[r, equal] & I
\end{tikzcd}
$$

In any case we have the two maps 4) related by σ.

Next suppose u satisfies 2): $u(x)y = xu(y)$. Then both maps $I \otimes_P I \rightrightarrows I$ coincide, hence the two maps $I \overset{L}{\otimes}_P I \rightrightarrows I$ in the derived category coincide, hence the two maps of complexes

4) $$I \overset{L}{\otimes}_P I \overset{L}{\otimes}_P \underset{1\otimes u}{\overset{u\otimes 1}{\rightrightarrows}} I \overset{L}{\otimes}_P$$

are the same in the derived category. If we compose with the inclusion $(I \overset{L}{\otimes}_P)^2_\sigma \hookrightarrow (I \overset{L}{\otimes}_P)^2$, then we have two reasons why the two maps become homotopic. One is because of the identity on u, the other is because for any bimodule ~~map~~ $u : I \to P$ the maps $u\otimes 1$, $1 \otimes u$ are intertwined by σ, hence the same on σ-invariants. The difference of these two reasons gives us a map

$$\left(I \overset{L}{\otimes}_P\right)^2_\sigma \longrightarrow I \overset{L}{\otimes}_P .$$

To be more specific, let $\tilde{I} \xrightarrow{\varepsilon} I$ be a P-bimodule resolution which is acyclic for $\overset{L}{\otimes}_P$, for example $\tilde{I} = I \otimes_P B(P)$. Actually this is suitable for $\overset{L}{\otimes}_P$ applied on the right, so it's probably better ~~to~~ take $\tilde{I}$ to be a flat P-bimodule resolution of I.

Then we have

$$\begin{tikzcd} \tilde{I}\otimes_P \tilde{I} \arrow[r, shift left, "u\varepsilon\otimes 1"] \arrow[r, shift right, "1\otimes u\varepsilon"'] \arrow[d, "\varepsilon\otimes\varepsilon"] & \tilde{I} \arrow[d, "\varepsilon"] \\ I\otimes_P I \arrow[r, "u\otimes 1 = 1\otimes u"] & I \end{tikzcd}$$

strictly $u\varepsilon\otimes 1$ is $\tilde{I}\otimes_P \tilde{I} \to P\otimes_P \tilde{I} \cong \tilde{I}$

assuming 2) holds

so assuming $\tilde{I}$ is a projective P-bimodule resolution of I, we know that $u\varepsilon \otimes 1$, $1 \otimes u\varepsilon$ are chain homotopic maps of complexes of P-bimodules. Let h be a chain homotopy. Then after applying $\otimes_P$, h gives a homotopy between the maps $\left(\tilde{I}\otimes_P\right)^2 \overset{u\varepsilon\otimes 1\otimes}{\underset{1\otimes u\varepsilon\otimes}{\rightrightarrows}} \tilde{I}\otimes_P$. But one is the σ-transform of the other, so restricting h to $\left(\tilde{I}\otimes_P\right)^2_\sigma$

we see $dh + hd = 0$, i.e. h defines a map $(\tilde{I}\otimes_P)^2_\sigma[1] \longrightarrow \tilde{I}\otimes_P$.

Next let's consider [redacted] higher k. First we look at the maps from $(\tilde{I}\otimes_P)^k$ to $(\tilde{I}\otimes_P)^{k-1}$ obtained by applying $u\varepsilon : \tilde{I} \longrightarrow P$ to one of the factors. There one such map for each cyclic embedding of $\{1,..,k-1\}$ in $\{1,..,k\}$. [redacted] Here cyclic means the cyclic order is preserved. The number of these embeddings is $k(k-1)$, since there are k possibilities for the point omitted, and $k-1$ possible cyclic embeddings with given omitted point. Clearly

$$\left\{\begin{matrix}\text{Cyclic embeddings} \\ \text{of } \{1,..,k-1\} \hookrightarrow \{1,..,k\}\end{matrix}\right\} \text{ is a } [redacted] \text{ torsor over } \mathbb{Z}/k \times \mathbb{Z}/k-1$$

which means that all the maps are obtained from $u\varepsilon \otimes (1\otimes)^{k-1} : (\tilde{I}\otimes_P)^k \longrightarrow (\tilde{I}\otimes_P)^{k-1}$ by acting with cyclic permutations on both sides. Thus there can be only one natural map we could write down (up to scalar ~~factor~~) from $(\tilde{I}\otimes_P)^k_\sigma$ to $(\tilde{I}\otimes_P)^{k-1}_\sigma$.

Now let's use the condition 2) which gives a homotopy between $u\varepsilon\otimes 1$ and $1\otimes u\varepsilon : \tilde{I}\otimes_P\tilde{I} \longrightarrow \tilde{I}$. Then we have successive homotopies

$$\tilde{I}\otimes_P \cdots \otimes_P \tilde{I} \quad \begin{matrix} \xrightarrow{u\varepsilon\otimes 1\otimes\cdots} \\ \simeq \\ \xrightarrow{1\otimes u\varepsilon\otimes 1\otimes\cdots} \\ \xrightarrow{1\otimes 1\otimes u\varepsilon\otimes 1\otimes\cdots} \\ \simeq \\ \longrightarrow \\ \cdots \\ \simeq \\ \xrightarrow{1\otimes 1\otimes\cdots\otimes u\varepsilon} \end{matrix} \quad \tilde{I}\otimes_P \cdots \otimes_P \tilde{I} \quad (k-1) \text{ times}$$

Now apply $\otimes_p$ whence we have k homotopic maps $\cdots \otimes 1 \otimes u\varepsilon \otimes 1 \otimes \cdots$

$$(\tilde{I} \otimes_p)^k \longrightarrow (\tilde{I} \otimes_p)^{k-1}$$

If we divide out by $\mathbb{Z}/k-1$ then these maps become conjugate under the $\mathbb{Z}/k$ action, and upon restricting to the invariant part $(\tilde{I} \otimes_p)^k_\sigma$ they become the same. But then we have $(k-1)$ self-homotopies of this single map from $(\tilde{I} \otimes_p)^k_\sigma$ to $(\tilde{I} \otimes_p)^{k-1}_\sigma$ indexed by $\mathbb{Z}/k - \{1\}$. ~~Probably one wants to take the sum in order to get the right B-like formula.~~

It seems likely that these self-homotopies, considered as maps $(\tilde{I} \otimes_p)^k_\sigma [1] \longrightarrow (\tilde{I} \otimes_p)^{k-1}_\sigma$, are the same.

December 24, 1987

It should be possible to construct a relative cyclic homology for A relative to P which is made up from the complexes

$$(A\overset{L}{\otimes}_P)^k$$

with their natural cyclic action. Here A is a P-bimodule equipped with a map of P-bimodules

$$A\otimes_P A \longrightarrow A$$

making A into an associative algebra. For example we can take $A = I$ or $A = P/I$ where I is an ideal in P.

Let's consider the case $A = P/I$ where P is free. Then

$$\circledast \quad A\overset{L}{\otimes}_P \overset{k}{\cdots} \overset{L}{\otimes}_P A = \overbrace{(P\leftarrow I)\otimes_P \cdots \otimes_P (P\leftarrow I)}^{k}$$

since P, I are projective left P-modules. The latter complex is a kind of Koszul-type complex, specifically in degree p it is

$$\bigoplus_{|S|=p} \overbrace{I\otimes_P \cdots \otimes_P I}^{p} = \bigoplus_{|S|=p} I^p = \Lambda^p(\mathbb{C}^k)\otimes I^p$$

If we take $k=2$ we have the complex

$$\begin{matrix} & & I & & \\ P & \leftleftarrows & \oplus & \leftleftarrows & I^2 \\ & & I & & \end{matrix}$$

whose homology is

$$A \qquad I/I^2$$

Consider the general case and define a decreasing filtration of ⊛ by

$$F_p : \begin{cases} \Lambda^q \mathbb{C}^k \otimes I^p & q \leq p \\ \Lambda^q \mathbb{C}^k \otimes I^q & q \geq p \end{cases}$$

Then

$$F_p / F_{p+1} = \begin{cases} \Lambda^q \mathbb{C}^k \otimes I^p / I^{p+1} & q \leq p \\ 0 & q > p \end{cases}$$

It's clear that if $I = P$ the complex ⊛ is contractible and the same is true if we replace $P \leftarrow I$ by $I^p \leftarrow I^p$. Thus it should be the case that if F_p / F_{p+1} were not truncated at degree p, there would be no homology. Thus

$$H_q (F_p / F_{p+1}) = \begin{cases} 0 & q \neq p \\ \underbrace{\operatorname{Ker}\{\Lambda^p \mathbb{C}^k \to \Lambda^{p-1} \mathbb{C}^k\}}_{Z_{p,k}} \otimes I^p / I^{p+1} & q = p \end{cases}$$

$$\therefore \quad F_p / F_{p+1} \sim Z_{pk}[p] \otimes I^p / I^{p+1}$$

We have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 / F_2 & \longrightarrow & F_0 / F_2 & \longrightarrow & F_0 / F_1 & \xrightarrow{\otimes A} & 0 \\ & & \wr & & & & \wr & & \\ & & Z_1[1] \otimes I/I^2 & & & & Z_0[0] & & \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_2 / F_3 & \longrightarrow & F_0 / F_3 & \longrightarrow & F_0 / F_2 & \longrightarrow & 0 \\ & & \wr & & & & \wr & & \\ & & Z_2[2] \otimes I^2/I^3 & & & & Z_0[0] \overset{\otimes A}{+} Z_1[1] \otimes I/I^2 & & \end{array}$$

Thus there is no interference between the $Z_p[p]$ for different p and we have a non-canonical isom. of $\otimes$ with $\oplus\, Z_p[p]$. In particular

$$H_q\big(A \overset{L}{\otimes}_P \overset{k}{\cdots} \overset{L}{\otimes}_P A\big) = \mathrm{Ker}\{\Lambda^q \mathbb{C}^k \to \Lambda^{q-1}\mathbb{C}^k\} \otimes I^q/I^{q+1}$$

Actually $Z_{q,k} = \mathrm{Ker}\{\Lambda^q \mathbb{C}^k \longrightarrow \Lambda^{q-1}\mathbb{C}^k\}$ should be isom. to $\Lambda^q(\mathbb{C}^{k-1})$. In effect the map is interior product with $v = (1, 1, \dots, 1) \in \mathbb{C}^k$, so if we pick a new bases for $\mathbb{C}^k$ containing v, it is clear that $\mathrm{Ker}\, i(v)$ is Λ of the space spanned by the other basis vectors.

Now let's go back and look at the exact sequences before as exact sequences of complexes of P-bimodules. Do the sequences split in the derived category? The first sequence corresponds to an element of

$$\mathrm{Ext}^2_{P \otimes P^{op}}\big(Z_0 \otimes A,\ Z_1 \otimes I/I^2\big)$$

which is possibly non-zero.

Let's consider $k=2$. Then

$F_0 = (P \leftarrow I) \otimes_P (P \leftarrow I)$ contains $(I \leftarrow I) \otimes_P (P \leftarrow I)$ which is contractible, so

$$F_0 \sim A \otimes_P (P \leftarrow I) = (A \xleftarrow{0} I/I^2)$$

which splits. Similarly

$$(P \leftarrow I) \otimes_P (P \leftarrow I) \otimes_P \cdots \otimes_P (P \leftarrow I)$$

$$\sim \quad A \otimes_P (P \leftarrow I) \otimes_P \cdots$$

$$= \big(A \xleftarrow{0} I/I^2\big) \otimes_P (P \leftarrow I) \otimes_P (P \leftarrow I) \otimes_P \cdots$$

$$= (A \xleftarrow{\circ} I/I^2) \otimes_A A \otimes_P (P \leftarrow I) \otimes_P \cdots$$

$$= (A \leftarrow I/I^2) \otimes_A (A \leftarrow I/I^2) \otimes_P (P \leftarrow I) \otimes_P$$

$$\simeq (A \xleftarrow{\circ} I/I^2) \otimes_A \cdots \otimes_A (A \leftarrow I/I^2)$$

Thus there is a canonical isomorphism of P-bimodule complexes

$$* \begin{cases} A \overset{L}{\otimes}_P \cdots \overset{L}{\otimes}_P A = (P \leftarrow I) \otimes_P \cdots \otimes_P (P \leftarrow I) \\ \qquad \simeq (A \xleftarrow{\circ} I/I^2) \otimes_A \cdots \otimes_A (A \xleftarrow{\circ} I/I^2) \\ \qquad = \bigoplus_{p=0}^{k-1} (\Lambda^p \mathbb{C}^{k-1}) \otimes I^p/I^{p+1}[p] \end{cases}$$

and so

$$H_*((A \overset{L}{\otimes}_P)^k) \simeq \bigoplus_{q=0}^{k-1} (\Lambda^q \mathbb{C}^{k-1}) \otimes H_{*-q}(P, I^q/I^{q+1})$$

only contributes in degrees 0, 1.

<u>December 25, 1987</u>

We are considering $A = P/I$ where P is free and trying to determine $H_*((A \overset{L}{\otimes}_P)^k)$ with its natural $\mathbb{Z}/k$ action. Yesterday we found the complex $A \overset{L}{\otimes}_P \overset{k}{\cdots} \overset{L}{\otimes}_P A$ is split, i.e. a direct sum of its homology groups (see * above). Set

$$K = (A \xleftarrow{\circ} I/I^2) \otimes_A \cdots \otimes_A (A \xleftarrow{\circ} I/I^2) \qquad k \text{ times}$$

for the module for $A \overset{L}{\otimes}_P \overset{k}{\cdots} \overset{L}{\otimes}_P A$ we found. Now we want the last $\overset{L}{\otimes}_P$ which we will do as

$$(A \overset{L}{\otimes}_P)^k = K \overset{L}{\otimes}_P = K \otimes_{P \otimes P^{op}} (P \otimes P \leftarrow \Omega^1_P)$$

$$= K \otimes_{A\otimes A^{op}} \Big(A\otimes A \longleftarrow \underbrace{A\otimes_P \Omega^1_P \otimes_P A}_{D}\Big)$$

But recall that one has exact sequences

$$0 \longrightarrow I/I^2 \longrightarrow D \longrightarrow \Omega^1_A \longrightarrow 0$$

$$* \qquad 0 \longrightarrow I/I^2 \longrightarrow D \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0$$

whence if $L_\cdot = (A\otimes A \longleftarrow D)$ we have a triangle of complexes of A-bimodules

$$I/I^2[1] \longrightarrow L_\cdot \longrightarrow A$$

and so also a triangle

$$\left(I/I^2 \overset{L}{\otimes}_A K \overset{L}{\otimes}_A\right)[1] \longrightarrow \left(A\overset{L}{\otimes}_P\right)^k \longrightarrow K\overset{L}{\otimes}_A$$

It almost looks as if $(A\overset{L}{\otimes}_P)^k$ is

$$(A \xleftarrow{0} I/I^2)\otimes_A \overset{k}{\cdots} \otimes_A (A\xleftarrow{0} I/I^2) \overset{L}{\otimes}_A$$

$$\simeq \left((A\xleftarrow{0} I/I^2)\overset{L}{\otimes}_A\right)^k$$

but more likely one is a twisted version of the other. The twisting is associated to the bimodule 2-extension $*$ which represents a canonical class in $H^2(A, I/I^2)$.

December 26, 1987

Remarks. Consider $H_*(P,M)$ where M is a bimodule over $A = P/I$. This is the homology of

$$M\overset{L}{\otimes}_P = M\overset{L}{\otimes}_{P\otimes P^o} P = M\otimes_P \mathcal{B}(P)\otimes_P$$
$$= M\otimes_{A\otimes A^o}\left(A\otimes_P \mathcal{B}(P)\otimes_P A\right)$$

and so there is a spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^{A\otimes A^o}\left(M, \mathrm{Tor}_q^P(A,A)\right) \Longrightarrow H_*(P,M)$$

When P is free $\quad \mathrm{Tor}_q^P(A,A) = \begin{cases} A & q=0 \\ I/I^2 & q=1 \\ 0 & q>1 \end{cases}$

so the spectral sequence gives a long exact sequence

$$\to \mathrm{Tor}_{n-1}^{A\otimes A^o}(M, I/I^2) \longrightarrow H_n(P,M) \longrightarrow H_n(A,M) \to$$
$$\to \mathrm{Tor}_{n-2}^{A\otimes A^o}(M, I/I^2) \longrightarrow \cdots$$

One can see this directly as follows

$$M\overset{L}{\otimes}_P = M\otimes_{P\otimes P^o}\left(P\otimes P \leftarrow \Omega^1_P\right)$$
$$= M\otimes_{A\otimes A^o}\left(\underbrace{A\otimes A \longleftarrow D}_{\text{has homology } A,\ I/I^2}\right) = M\overset{L}{\otimes}_{A\otimes A^o}\left(A\otimes A \leftarrow D\right)$$

so there is a triangle

$$\begin{array}{ccccc} M\overset{L}{\otimes}_{A\otimes A^o}\left(I/I^2[1]\right) & \longrightarrow & M\overset{L}{\otimes}_P & \longrightarrow & M\overset{L}{\otimes}_{A\otimes A^o}(A) \cdots \\ \| & & & & \| \\ (M\otimes_A I/I^2)\overset{L}{\otimes}_A[1] & & & & M\overset{L}{\otimes}_A \end{array}$$

where we have used that I/I^2 is left + right A-projective.

Thus we have

<u>Proposition:</u> The exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow D \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0$$

determines a class in $\mathrm{Ext}^2_{A\otimes A^\circ}(A, I/I^2) = H^2(A, I/I^2)$ i.e. a morphism $A \xrightarrow{\chi} I/I^2[2]$ in the derived category of A-bimodules. For any complex M of A-bimodules we have a triangle

$$M\overset{L}{\otimes}_P \longrightarrow M\overset{L}{\otimes}_A \xrightarrow{\chi_\#} \left(M\otimes_A I/I^2\right)\overset{L}{\otimes}_A [2]$$

I want to apply this to the complex

$$\overbrace{A\overset{L}{\otimes}_P \cdots \overset{L}{\otimes}_P A}^{k} = \overbrace{(A \xleftarrow{0} I/I^2)\otimes_A \cdots \otimes_A (A \xleftarrow{0} I/I^2)}^{k-1}$$

$$= \bigoplus_q \Lambda^q(\mathbb{C}^{k-1}) \otimes I^q/I^{q+1}[q].$$

This means looking at the proposition when $M = I^q/I^{q+1}$!

Actually the proposition really doesn't tell us anything about $H_*(M\overset{L}{\otimes}_P)$, rather the fact that this homology is zero outside degrees $0, 1$ tells us about $H_*(M\overset{L}{\otimes}_A)$. Put another way

$$M\overset{L}{\otimes}_P = M\otimes_A (A\otimes A \longleftarrow D)\otimes_A$$

$$= M \longleftarrow M\otimes_A D\otimes_A$$

is a pretty efficient description.

So our final formula is

$$\boxed{\begin{aligned} H_n\big((A\overset{L}{\otimes}_P)^k\big) &= \Lambda^n(\mathbb{C}^{k-1}) \otimes H_0(A, I^n/I^{n+1}) \\ &\quad \oplus \Lambda^{n-1}(\mathbb{C}^{k-1}) \otimes H_1(P, I^{n-1}/I^n) \end{aligned}}$$

Return to the double complex $\mathrm{Connes}(P \leftarrow I)$ and let's use the fact that on the Connes complex there is a S-operator lowering degree by 2. Presumably this extends to DGAlgebras. It would have to preserve the grading, hence apparently there is an S operator on the complex $(I\overset{L}{\otimes}_P)^k_\sigma$ for each k. What's involved here is the S-operator on the cyclic homology of the semi-direct product $P \oplus I[1]$ and by Goodwillie this has to be zero on the part of I-degree > 0. The S operator thus ought to be homotopic to zero on each of the columns of the double complex.

December 27, 1987

We have seen the importance of cyclic tensor products

$$* \qquad M_1 \overset{L}{\otimes}_A \cdots \overset{L}{\otimes}_A M_k \overset{L}{\otimes}_A$$

in the study of cyclic homology. It would be nice to be able to understand cyclic homology entirely in terms of such objects, which are intimately tied to the derived category.

Maybe we should think of a complex M of bimodules as an operator on say the derived category of left-modules:

$$** \qquad X \longmapsto M \overset{L}{\otimes}_A X$$

Then the operation $M_1, M_2 \longmapsto M_1 \overset{L}{\otimes}_A M_2$ corresponds to composition. And the cyclic tensor product corresponds to some sort of trace of the composition product. In particular $M \overset{L}{\otimes}_A$ corresponds to the trace of the operator $**$ in some sense.

Connes + Goodwillie have introduced cyclic objects, which [redacted] generalize [redacted] simplicial objects. A cyclic object is a contravariant functor on the category of cyclically-ordered finite sets, when this is suitably defined. The main example of a cyclic vector space is $\{1,2,\dots,k\} \longmapsto A^{\otimes k}$ where A is a unital algebra. [redacted] One has to be careful when defining the morphisms. There are two morphisms $\{1,2\} \rightrightarrows \{1\}$ in the category but only one maps of sets. An example of a cyclic set is $k \longmapsto G^k$, where G is a group, or even a monoid.

Graeme ~~[illegible]~~ pointed out that for $k \mapsto G^k$ with G a group, the underlying simplicial set is the nerve of the category defined by the conjugation action of G on itself. Thus the realization is $PG \times^G (G)_{conj}$ which $\sim$ ~~[illegible]~~ $L(BG)$, where L denotes free loop space. The cyclic realization then is maybe the homotopy orbit space $PS^1 \times^{S^1} (L(BG))$.

Goodwillie has shown that the complex $M \overset{L}{\otimes}_A$ is the homology complex of a cyclic vector space, namely

$$k \longmapsto \bigoplus_{i+j=k-1} A^{\otimes i} \otimes M \otimes A^{\otimes j}$$

More generally $M_1 \overset{L}{\otimes}_A \cdots \overset{L}{\otimes}_A M_n \overset{L}{\otimes}_A$ is the homology complex of the cyclic vector space

$$k \longmapsto \bigoplus_{i_0+\cdots+i_n+n=k} A^{\otimes i_0} \otimes M_1 \otimes A^{\otimes i_1} \otimes \cdots \otimes M_n \otimes A^{\otimes i_n}$$