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November 1, 1986

Connes S operator - can we explain it any better now? Here is a sample problem: Suppose the algebra A acts on H, and that F is an involution on H such that $[F,a]$ is in some Schatten ideals for all $a \in A$, allowing one to define cyclic cocycles. Geometrically one has a map

$$\mathcal{G} = U_n(A) \longrightarrow \mathcal{I}_{res}(H^{\oplus n}, F^{\oplus n}) \qquad g \mapsto gF^{\oplus n}g^{-1}$$

and one pulls back the character forms to get left-invariant forms on $\mathcal{G}$ which are primitive, hence correspond to cyclic cocycles on A. One thus gets cocycles φ_{k+2n} $n \geq 0$ and a natural question is whether $S^n\varphi_k = \varphi_{k+2n}$.

One might hope to prove this using the periodicity map

$$\mathrm{Grass}(V) \times P_1\mathbb{C} \longrightarrow \mathrm{Grass}(V \oplus V)$$

The character forms on the $\mathrm{Grass}(V\oplus V)$ can be pulled back and integrated over S^2 to get the character forms on $\mathrm{Grass}(V)$ with a degree shift of -2. The reason this looks promising is that Connes defn. of the S-operator involves doubling.

November 5, 1986

Problem: Given a unitary $g \in U_n$ produce a canonical deformation of $u_0 = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ such that u_t has all eigenvalues $\neq +1$ for $t \neq 0$.

Suppose g has all eigenvalues $\neq -1$ so that $g = \frac{1+X}{1-X}$ where $X = \frac{g-1}{g+1}$ is skew-adjoint. Set

$$u_z = \frac{1+L_z}{1-L_z}, \qquad L_z = \begin{pmatrix} X & -\bar{z} \\ z & -X \end{pmatrix} = X\varepsilon + \tfrac{1}{i}x\gamma^2 + iy\gamma^1$$

Since $L_z^2 = X^2 - x^2 - y^2 = X^2 - |z|^2 \leq |z|^2$, we see that L_z has all eigenvalues $\neq 0$ for $z \neq 0$, hence u_z has all eigenvalues $\neq 1$ for $z \neq 0$.

Let's check that u_z is defined for an arbitrary $g \in U_n$.

$$u_z = \frac{\left(1 + \left(\frac{g-1}{g+1}\right)\varepsilon + \frac{1}{i}x\gamma^1 + iy\gamma^2\right)^2}{1 - \left(\left(\frac{g-1}{g+1}\right)\varepsilon + \frac{1}{i}x\gamma^1 + iy\gamma^2\right)^2} \leftarrow 1 - \left(\frac{g-1}{g+1}\right)^2 + x^2 + y^2$$

$$\cancel{(g+1)^2 + (g-1)\varepsilon + \tfrac{1}{i}x\gamma^1 + iy\gamma^2}$$

$$= \frac{\left((g+1)\left(1 + \frac{1}{i}x\gamma^1 + iy\gamma^2\right) + (g-1)\varepsilon\right)^2}{(g+1)^2(1+|z|^2) - (g-1)^2}$$

This will be a smooth function on U_n provided the denominator doesn't vanish. This is clear as we have seen, since $\frac{\zeta+1}{\zeta-1} \in i\mathbb{R}$ for $|\zeta| = 1$.

Notice that as $z \longrightarrow \infty$, $u_z \longrightarrow -1$. Moreover if $g = -1$, then $u_z = -1$. Thus we have a map

$$S^2 \wedge \mathcal{U}_n \longrightarrow \mathcal{U}_{2n}$$

where S^2 is given the basepoint ∞ and the unitaries are given the basepoint -1.

Question: Is this a periodicity map?

The next step will be to look at the case where z goes to infinity along a real line in the plane, e.g. $y = 0$. In this case $L_x = X\varepsilon + \frac{1}{i} x\gamma^2$ anti-commutes with γ^1, so u_z is reversed by γ^1 and $u_x\gamma^1$ is a path of involutions. In fact as x ranges over $\mathbb{R}$ we get a loop ~~starting~~ and ending at $-\gamma^1$.

To understand this better we can conjugate ε, γ^2 to γ^1, γ^2 and so L_x becomes

$$L_\xi = X\gamma^1 + \xi\gamma^2 \qquad \frac{x}{i} = \xi \in i\mathbb{R}$$

Then we have the path of involutions

$$F_\xi = \frac{1+L_\xi}{1-L_\xi}\varepsilon = (1+L_\xi)\varepsilon\frac{1}{1+L_\xi} \qquad 1+L_\xi = \begin{pmatrix} 1 & X-i\xi \\ X+i\xi & 1 \end{pmatrix}$$

The involution F_ξ has the $+1$ eigenspace

$$\operatorname{Im}\begin{pmatrix} 1 \\ X+i\xi \end{pmatrix}$$

which is the graph of $X + i\xi = X + x$.

Thus if X ranges over $i\mathbb{R}$, which means that g ranges over $U(1)$ and if x also ranges over $\mathbb{R} \cup \infty$ it is clear that we we have a homeomorphism

$$S^1 \wedge U(1) \simeq \mathbb{C}P^1$$

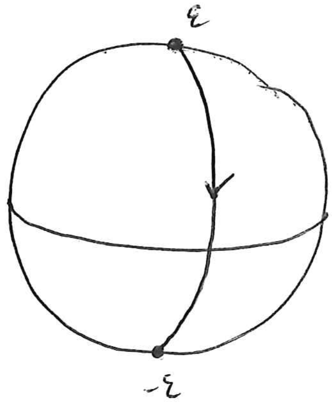
I would now like to relate the map just defined

$$S^1 \wedge U_n \longrightarrow Gr_n(\mathbb{C}^{2n})$$
$$\left(t, \frac{1+X}{1-X}\right) \longmapsto \operatorname{Im}\begin{pmatrix} 1 \\ X+t \end{pmatrix}$$

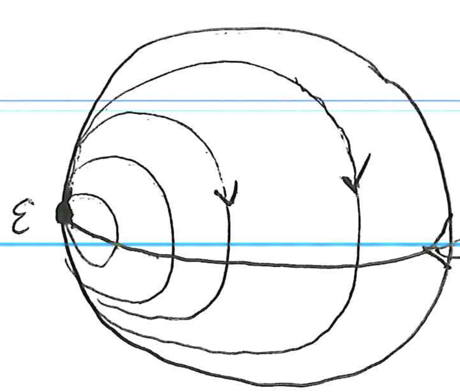
to the Bott map

$$\Sigma U_n \longrightarrow Gr_n(\mathbb{C}^{2n})$$
$$(t, g) \longmapsto \operatorname{Im}\begin{pmatrix} \sqrt{1-t} \\ \sqrt{t}\, g \end{pmatrix}$$

In order to see what's involved look at the case $n = 1$ where $U_n = U_1$ and $Gr_n(\mathbb{C}^{2n}) = \mathbb{P}^1(\mathbb{C}) = S^2$. In the latter case one has U_1 embedded as the equator of S^2 and the north & south poles correspond to the axes in $\mathbb{C}^2$.



As t varies one goes along the geodesic from ε to $-\varepsilon$. In the former case as t varies we get a circle on the Riemann sphere passing thru the north poles so the picture is



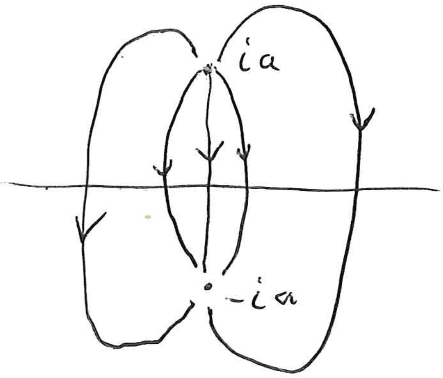
u, embedded as the equator.

It appears that we want to consider the flow on S^2 whose trajectories are the geodesics from ε to $-\varepsilon$. This is the flow given by

$$\begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \qquad \text{if} \quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We want to deform this flow into a parabolic flow, which should be possible because the unipotent elements in the Lie algebra are limits of semi-simple ones.

To find the formulas let's think of the equator as the real axis in $\mathbb{C}$ and let's find the flow going from ia to $-ia$ with these fix pts



$$\frac{z-ia}{z+ia} \quad \text{sends} \quad \begin{matrix} ia \mapsto 0 \\ -ia \mapsto \infty \end{matrix}$$

$$\frac{1}{2ia}\begin{pmatrix} ia & ia \\ -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -ia \\ 1 & ia \end{pmatrix}}_{\begin{pmatrix} e^t & -e^t ia \\ 1 & ia \end{pmatrix}} = \frac{1}{2ia}\begin{pmatrix} ia(e^t+1) & -a^2(-e^t+1) \\ -e^t+1 & ia(e^t+1) \end{pmatrix}$$

In order to have a limit $\overbrace{\text{we}}^{\text{as } a\to\infty}$ we must rescale t to t/a. The limit is then

$$\begin{pmatrix} 1 & -\frac{1}{2}it \\ 0 & 1 \end{pmatrix}$$

Let's now put this all together. We first need the "equator" that is the embedding of U_n in $Gr_n(\mathbb{C}^{2n})$. This is

$$g = \frac{1+X}{1-X} \longmapsto \operatorname{Im}\begin{pmatrix} 1 \\ X \end{pmatrix}$$

or

$$g \longmapsto \operatorname{Im}\begin{pmatrix} 1 \\ \frac{g-1}{g+1} \end{pmatrix} = \operatorname{Im}\begin{pmatrix} g+1 \\ g-1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \operatorname{Im}\begin{pmatrix} 1 \\ g \end{pmatrix}$$

This last formula shows that we have just rotated by $-45°$ the graph embedding of U_n

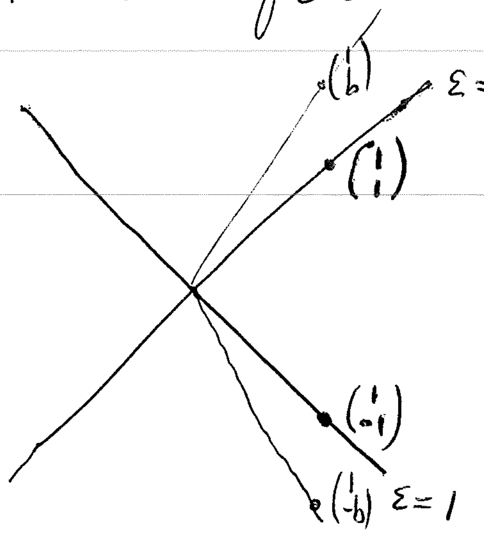
Now the flow at the Bott map end of the deformation we seek is

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{\begin{pmatrix} 1 & -1 \\ e^t & e^t \end{pmatrix}} = \begin{pmatrix} \frac{e^t+1}{2} & \frac{e^t-1}{2} \\ \frac{e^t-1}{2} & \frac{e^t+1}{2} \end{pmatrix}$$

and the flow at the other end is to be

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

Picture after rotation thru -45°



We want to obtain a flow with unique fixed line $\mathrm{Im}\begin{pmatrix}0\\1\end{pmatrix}$. So the thing to try is the flow moving from $\mathrm{Im}\begin{pmatrix}1\\-b\end{pmatrix}$ to $\mathrm{Im}\begin{pmatrix}1\\b\end{pmatrix}$ with these fixpoints as $b\to\infty$.

$$\frac{1}{2b}\begin{pmatrix}1 & 1\\ -b & b\end{pmatrix}\underbrace{\begin{pmatrix}1 & 0\\ 0 & e^t\end{pmatrix}\begin{pmatrix}b & -1\\ +b & 1\end{pmatrix}}_{\begin{pmatrix}b & -1\\ e^t b & e^t\end{pmatrix}} = \begin{pmatrix}\frac{e^t+1}{2} & \frac{e^t-1}{2b}\\ \frac{e^t-1}{2}b & \frac{e^t+1}{2}\end{pmatrix}$$

Now replace t by $\frac{2t}{b}$ and $b\to\infty$ and you get $\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$.

So we now put this as follows. We have for each b a map

$$U_n\times\mathbb{R}\longrightarrow Gr_n(\mathbb{C}^{2n})$$

$$(g,t)\longmapsto\begin{pmatrix}\frac{e^{2t/b}+1}{2} & \frac{e^{2t/b}-1}{2b}\\ \frac{e^{2t/b}-1}{2}b & \frac{e^{2t/b}+1}{2}\end{pmatrix}\mathrm{Im}\begin{pmatrix}g+1\\ g-1\end{pmatrix}$$

Now we should check that as $t \to \pm\infty$ the map has the constant limits $g \mapsto \mathrm{Im}\begin{pmatrix}1\\b\end{pmatrix}$, $\mathrm{Im}\begin{pmatrix}1\\-b\end{pmatrix}$. This seems OK so for each b we have a map

$$\psi_b : \Sigma^1 U_n \longrightarrow Gr_n(\mathbb{C}^{2n})$$

Then we should check that ψ_b is continuous in b and gives a deformation between the Bott map and the map

$$(*) \quad (g,t) \longmapsto \begin{pmatrix}1 & 0\\ t & 1\end{pmatrix} \mathrm{Im}\begin{pmatrix}g+1\\ g-1\end{pmatrix}$$

It's likely that one can check this for $n=1$ and then prove continuity by lifting back to $U_n/T_n \times T_n$.

It seems the above isn't useful. Certainly it doesn't have any nice invariance properties, "it" referering to the map $(*)$, whereas the usual Bott map is equivariant relative to the action of $U_n \times U_n$.

I originally started with the problem of deforming $\begin{pmatrix}g & 0\\ 0 & g^{-1}\end{pmatrix}$ canonically so that its eigenvalues were $\neq -1$. There's a simple solution ~~to~~ this question as follows. The point is that this unitary is reversed by g', hence

$$\begin{pmatrix}g & 0\\ 0 & g^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & g\\ g^{-1} & 0\end{pmatrix} \quad \text{is an involution.}$$

Then as this anti commutes with ε we get a family of involutions

$$\cos\theta\begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix} + \sin\theta\ \varepsilon = \begin{pmatrix} \sin\theta & (\cos\theta)g \\ (\cos\theta)g^{-1} & -\sin\theta \end{pmatrix}$$

which in turn correspond to unitaries

$$* \qquad \begin{pmatrix} (\cos\theta)g & \sin\theta \\ -\sin\theta & (\cos\theta)g^{-1} \end{pmatrix}$$

Notice that the eigenvalues of this are roots of

$$\lambda^2 - (\cos\theta)(\zeta + \zeta^{-1}) + 1 = 0$$

where ζ is an eigenvalue of g. So for $|\cos\theta| < 1$ the eigenvalues of $*$ are never ± 1.

November 6, 1986

Given a graded (E, D) we want to associate forms to any $\Gamma \subset E$. Let's first look at the ungraded case where Γ becomes a $g \in \mathrm{Aut}(E)$.

Let's consider $\mathrm{Aut}(E)$ as a manifold over M such that $\pi^*(E)$ has a tautological automorphism, as well as a connection induced by D. We have the Cayley embedding

$$\mathrm{Endsk}(E) \subset \mathrm{Aut}(E) \qquad X \to \frac{1+X}{1-X}$$

as the complement of the hypersurface where

$$\det(g-1) = 0.$$

Over this open set we have forms defined by

$$\frac{1}{-\lambda + X^2 + [D,X]_+ + D^2} \in \Omega\big(\mathrm{Endsk}(E),\ \pi^*\mathrm{End}\,E \otimes C_1\big)$$

What I propose to do is to show these forms extend smoothly to $\mathrm{Aut}(E)$.

To do this we choose an embedding $i : E \hookrightarrow \tilde{V}$ such that $D = i^* d\, i$. We then have maps

$$\mathrm{Endsk}(E) \subset \mathrm{Aut}(E) \xhookrightarrow{\varphi_\infty} M \times U(V) \xrightarrow{pr_2} U(V)$$

where the middle map extends a g on E by -1. In general let φ_t extend a g on E by the scalar $\frac{1+it}{1-it}$ so that $\varphi_t \to \varphi_\infty$ as $t \to \infty$.

Now on $U(V)$ we have the form ω_λ which restricts to

$$\frac{1}{-\lambda + X^2 + dX\sigma} \quad \text{on} \quad \mathrm{Endsk}(V)$$

Better diagram

$$\begin{tikzcd} \mathrm{Endsk}(E) \arrow[r, "\varphi_t"] \arrow[d, phantom, "\cap"] & \mathrm{Endsk}(V) \arrow[d, phantom, "\cap"] \\ \mathrm{Aut}(E) \arrow[r, "\varphi_t"] & U(V) \end{tikzcd}$$

Our basic lemma on behavior of the superconnection forms as part of the endo ~~the~~ goes to ∞ says that

$$\lim_{t\to\infty} \varphi_t^*\left(\frac{1}{-\lambda + X^2 + dX\sigma}\right)$$

$$= \begin{pmatrix} \frac{1}{-\lambda + X^2 + [DX]^\sigma + D^2} & 0 \\ 0 & 0 \end{pmatrix}$$

On the other hand φ_t is part of a smooth map $\mathrm{Aut}(E) \times S^1 \longrightarrow U(V)$, so

$$\lim_{t\to\infty} \varphi_t^*(\omega_\lambda) = \varphi_\infty^*(\omega_\lambda)$$

is a form on $\mathrm{Aut}(E)$ which restricts to over the open set $\mathrm{Endsk}(E)$.

An important point in the above is the fact that by passing to the bundle $\mathrm{Aut}(E)$ we effectively localize much better. This means that whereas a section of $\mathrm{Aut}(E)$ over M cannot be approximated even locally over M by a section of $\mathrm{Endsk}(E)$, nevertheless the bundle $\mathrm{Endsk}(E)$ is an open dense ~~subset~~ of $\mathrm{Aut}(E)$. ~~This is probably not a good argument for the non-commutative framework.~~

The next step is to handle the graded case. Here we have difficulties with the fact that the graph embedding

$$\mathrm{Hom}(E^0, E^1) \longrightarrow \mathrm{Gr}(E^0 \oplus E^1)$$

is dense only in the component of rank m subspaces, where $m = \operatorname{rank} E^0$. Similarly if we were to choose an embedding $i : E \hookrightarrow V$ then we can't ~~[illegible]~~ write -1 on $\mathrm{Ker}(i^*)$ as a limit of unitaries coming from graphs of maps $(E^0)^\perp \to (E^1)^\perp$ unless these two bundles have the same rank.

However suppose we consider the case of $\mathrm{Gr}_m(E^0 \oplus E^1)$ with $m = \operatorname{rank} E^0$ and let us choose an embedding $i : E \longrightarrow \tilde{V}$ (graded) with $D = i^* d i$. Then by adding a trivial bundle to V we can suppose $\operatorname{rank}(E^0)^\perp = \operatorname{rank}(E^1)^\perp$. If $\Gamma \subset E^0 \oplus E^1$ has rank m, then $i(\Gamma) \oplus (E^1)^\perp \subset V$ has rank $m + \operatorname{rank}(E^1)^\perp =$ ~~[illegible]~~ $\operatorname{rank}(E^0) + \operatorname{rank}(E^0)^\perp = \dim V^0$ and thus we have the diag.

$$\begin{array}{ccc} \mathrm{Hom}(E^0, E^1) & \xrightarrow{\varphi_t} & \mathrm{Hom}(V^0, V^1) \\ \cap & & \cap \\ \mathrm{Gr}_m(E^0 \oplus E^1) & \xrightarrow{\varphi_t} & \mathrm{Gr}_p(V) \end{array} \qquad p = \dim V^0$$

where φ_t will send Γ to the direct sum of Γ and the graph of tT where T is an isomorphism of $(E^0)^\perp$ and $(E^1)^\perp$ (T exists locally over M).

So now we can see as before that the forms

$$\begin{pmatrix} \dfrac{1}{-\lambda + X^2 + [D,X] + D^2} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{over } \mathrm{Hom}(E^0, E^1)$$

extends to the forms $\varphi_\infty^*(\omega_\lambda)$ on $Gr_m(E^0 \oplus E^1)$, where over $\mathrm{Hom}(V^0, V^1)$ we have

$$\omega_\lambda = \frac{1}{-\lambda + X^2 + dX}$$

This defines the form on $Gr_m(E^0 \oplus E^1)$ where $m = \mathrm{rank}(E^0)$. Now to handle the general case I would like to add trivial bundle. If $m < \dim E^0$ we use

$$Gr_m(E^0 \oplus E^1) \hookrightarrow Gr_{m+k}(E^0 \oplus (E^1 \oplus \tilde{\mathbb{C}}^k))$$

and if $m > \dim E^0$ we use

$$Gr_m(E^0 \oplus E^1) \hookrightarrow Gr_m((E^0 \oplus \tilde{\mathbb{C}}^k) \oplus E^1)$$

How do we see this works

November 7, 1986

On the unitary group we have the flow $g \longmapsto g_t$ corresponding under the Cayley transform to $X \longmapsto tX$. Thus

$$g_t = \frac{1+tX}{1-tX} \qquad \text{where} \qquad g = \frac{1+X}{1-X} \quad \text{or} \quad X = \frac{g-1}{g+1}$$

$$g_t = \frac{1 + t\,\frac{g-1}{g+1}}{1 - t\,\frac{g-1}{g+1}} = \frac{(g+1) + t(g-1)}{(g+1) - t(g-1)}$$

Let's see what happens to the matrix form $g^{-1}dg$ under this flow.

$$g = \frac{1+X}{1-X} = -1 + \frac{2}{1-X} \qquad\qquad dg = \frac{2}{1-X}\,dX\,\frac{1}{1-X}$$

$$g^{-1}dg = \frac{2}{1+X}\,dX\,\frac{1}{1-X}$$

$$X = \frac{g-1}{g+1} = 1 - \frac{2}{g+1} \qquad\qquad dX = \frac{2}{g+1}\,dg\,\frac{1}{g+1}$$

$$g_t^{-1}dg_t = \frac{2t}{1+tX}\,dX\,\frac{1}{1-tX}$$

$$= \frac{2t}{1+t\left(\frac{g-1}{g+1}\right)}\;\frac{2}{g+1}\,dg\,\frac{1}{g+1}\;\frac{1}{1-t\left(\frac{g-1}{g+1}\right)}$$

$$= \frac{4t}{(g+1)+t(g-1)}\,dg\,\frac{1}{(g+1)-t(g-1)}$$

Next set $\sqrt{\lambda} = \frac{1}{t}$ and the form $g_t^{-1} dg_t$ becomes

$$\theta_\lambda = \frac{4\sqrt{\lambda}}{\sqrt{\lambda}(g+1)+(g-1)} dg \frac{1}{\sqrt{\lambda}(g+1)-(g-1)}$$

Although θ_λ has been defined for $\lambda > 0$ we see that it admits an analytic continuation to $\mathbb{C} - \mathbb{R}_{<0}$. To see this we have to check that $\sqrt{\lambda}(g+1) \pm (g-1)$ never has the eigenvalue 0. Its eigenvalues are

$$* \qquad \sqrt{\lambda}\,(e^{i\theta}+1) \pm (e^{i\theta}-1)$$

If this is zero, then

$$\pm\sqrt{\lambda} = \frac{e^{i\theta}-1}{e^{i\theta}+1} = i \tan\left(\frac{\theta}{2}\right)$$

which implies $\lambda \leq 0$.

It seems to be interesting to be able to bound the eigenvalues $*$ away from zero by a function of λ, so that one would know the norm of $\frac{1}{\sqrt{\lambda}(g+1) \pm (g-1)}$ as a matrix functions on the unitary group.

So what I need to know is the minimum distance from the origin of

$$\sqrt{\lambda}\,(g+1) + (g-1) = (\sqrt{\lambda}+1)g + \sqrt{\lambda} - 1$$

as g ranges over the unit circle. This is the circle of radius $|\sqrt{\lambda}+1|$ about the point $\sqrt{\lambda}-1$. So we want the distance from the origin to the circle of radius $|\sqrt{\lambda}+1|$ with center $\sqrt{\lambda}-1$.

Translating we want the distance of $-(\sqrt{\lambda}-1)$ from the circle of radius $|\sqrt{\lambda}+1|$ with center 0. And this is clearly

$$\boxed{\Big|\,|\sqrt{\lambda}+1| - |\sqrt{\lambda}-1|\,\Big|}$$

For $\sqrt{\lambda}(g+1)-(g-1) = (\sqrt{\lambda}-1)g + (\sqrt{\lambda}+1)$

one gets the same minimum.

On the other hand when working with traces, say $\operatorname{tr} \Theta_\lambda^{odd}$, then one works the following conjugate of Θ_λ:

$$\frac{1}{\sqrt{\lambda}(g+1)-(g-1)}\ \overbrace{\frac{4\sqrt{\lambda}}{\sqrt{\lambda}(g+1)+(g-1)}\, dg\, \frac{1}{\sqrt{\lambda}(g+1)-(g-1)}}\ \big(\sqrt{\lambda}(g+1)-(g-1)\big)$$

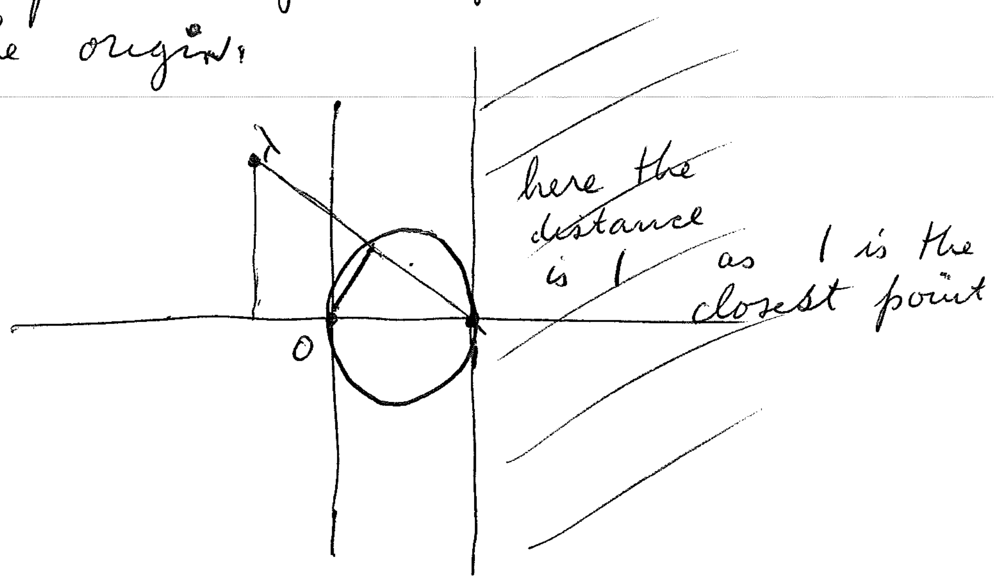
$$= \frac{4\sqrt{\lambda}}{\lambda(g+1)^2-(g-1)^2}\, dg$$

Here ~~the~~ we want the minimum of

$$|\lambda(e^{i\theta}+1)^2 - (e^{i\theta}-1)^2|$$

as $e^{i\theta}$ ranges over the unit circle. This is the minimum of $4\left|\lambda\left(\cos^2\frac{\theta}{2}\right)+\left(\sin^2\frac{\theta}{2}\right)\right|$

which is four times the distance of the segment joining 1 to λ from the origin.



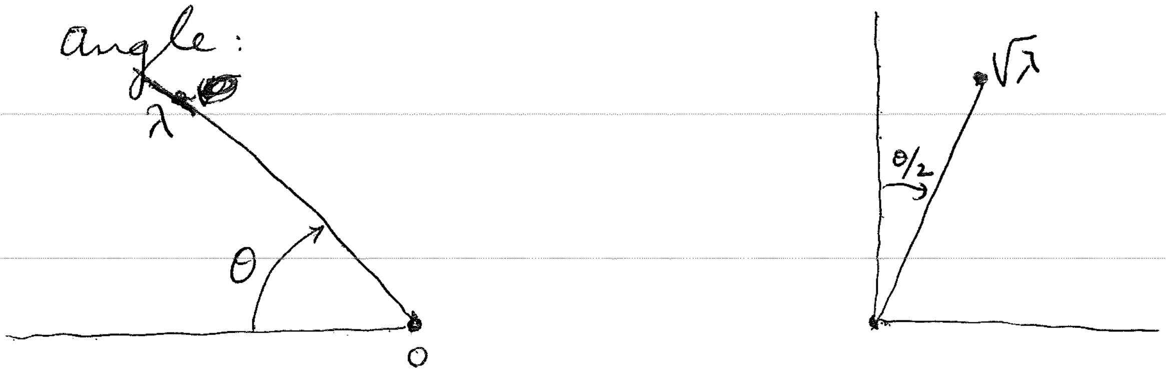
In the unit circle λ is the closest point on the segment, so the distance is $|\lambda|$.

If we are outside $|\lambda - \frac{1}{2}| < \frac{1}{2}$, $Re(\lambda) > 1$, then the closed point is where the segment meets the circle $|\lambda - \frac{1}{2}| = \frac{1}{2}$, and the distance is

$$d = \frac{|\mathrm{Im}\lambda|}{\sqrt{1-\lambda}}$$

The important point is that these matrix functions $\sqrt{\lambda}(g+1) \pm (g-1)$ and $\lambda(g+1)^2 - (g-1)^2$ are ~~...~~ invertible and that their inverses have bounds in λ which we can compute. I need to know that on a typical contour needed for the Laplace transform that the growth is a power of λ.

Next ~~...~~ suppose that λ heads to ∞ along a line. In the case of $\frac{|\mathrm{Im}\lambda|}{\sqrt{1-\lambda}}$ this is $\sim \frac{|\mathrm{Im}\lambda|}{|\lambda|}$ and is $\sin\Theta$ where Θ is the angle:



On the other hand if $\sqrt{\lambda} = x+iy$ $\quad x>0$

$$\Big| |\sqrt{\lambda}+1| - |\sqrt{\lambda}-1| \Big| = \sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2}$$

$$= r\left(\left(1+\frac{2x+1}{r^2}\right)^{1/2} - \left(1+\frac{-2x+1}{r^2}\right)^{1/2} + O\left(\frac{1}{r^2}\right)\right)$$

$$= r\left(\frac{1}{2}\,\frac{2x}{r^2}\cdot 2\right) + O\left(\frac{1}{r}\right)$$

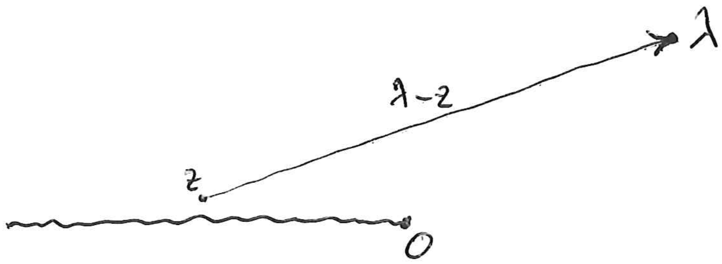
$$= \underbrace{\frac{2x}{r}}_{2\sin\frac{\theta}{2}} + O\left(\frac{1}{r}\right)$$

Thus if we want the inverses of $\sqrt{\lambda}(g+1) \pm (g-1)$ to remain bounded over the contour we want to keep θ away from zero.

November 9, 1986

Problem: $\log(\lambda - A)$ where A [redacted] has its spectrum in $\mathbb{R}_{\leq 0}$.

First: Notice that $\log(\lambda - z)$ is a holomorphic function of z in a nbd of $\mathbb{R}_{\leq 0}$. Here λ is a fixed point of $\mathbb{C} - \mathbb{R}_{\leq 0}$ and we choose $\arg(\lambda - z) \in (-\pi, \pi)$



So we can apply this holomorphic function to A to define $\log(\lambda - A)$. Here we use that A is a bounded operator, since obviously $\log(\lambda - x) \longrightarrow \infty$ as $x \longrightarrow -\infty$.

Secondly we have

$$\partial_\lambda \log(\lambda - A) = \frac{1}{\lambda - A}$$

which integrates to give

$$\log(\lambda - A) = \int_1^\lambda \frac{dz}{z - A} + \log(1 - A)$$

Now the resolvent $\frac{1}{\lambda - A}$ makes sense for certain unbounded operators, [redacted] for example

$$\frac{1}{\lambda - X^2} = \frac{1}{\lambda - \left(\frac{g-1}{g+1}\right)^2} = \frac{(g+1)^2}{\lambda(g+1)^2 - (g-1)^2}$$

[redacted] is defined for the whole unitary group.

Thus we might hope to find a renormalized logarithm [redacted] by suitably integrating the resolvent.

Next the resolvent is [redacted] equivalent to the operator e^{tA} via the Laplace transform

$$\int_0^\infty e^{-\lambda t} e^{tA}\, dt = \frac{1}{\lambda - A}$$

and so formally

$$\int_0^\infty e^{-\lambda t}\, e^{At}\, \frac{dt}{t} = -\log(\lambda - A)$$

Thus we have

$$\circledast \qquad \int_0^\infty e^{-\lambda t}\left(e^{At} - e^{Bt}\right)\frac{dt}{t} = -\log(\lambda - A) + \log(\lambda - B)$$

since $e^{At} - e^{Bt}$ vanishes at $t = 0$.

Notice that this assumes $e^{At} \to 1$ at $t \to 0$ which isn't true [redacted] when A has the eigenvalues $-\infty$.

The formula $\circledast$ suggests that when A, B are sufficiently close that $e^{At} - e^{Bt}$ is divisible by t, [redacted] then $\log(\lambda - A) - \log(\lambda - B)$ goes to zero as $\lambda \to \infty$. Then we [redacted] have

$$\log(\lambda - A) - \log(\lambda - B) = \int_\infty^\lambda \left(\frac{1}{z-A} - \frac{1}{z-B}\right) dz$$

[redacted] Let's now turn to our problem. We are concerned with the situation where $A = X^2 + [D, X] + D^2$ and $X = X' \oplus tX''$ relative to a decomposition $E = E' \oplus E''$. We suppose X'' invertible, whence we know

$$\frac{1}{\lambda - A} \longrightarrow \begin{pmatrix} \frac{1}{\lambda - A'} & 0 \\ 0 & 0 \end{pmatrix}$$

with $A' = X'^2 + [D', X'] + D'^2$.

We know $\log(\lambda - A)$ doesn't have a limit. However we would like to show that

$$\log(\lambda - A) - \log(\lambda - X^2)$$

converges to $\log(\lambda - A') - \log(\lambda - X'^2)$ extended by 0.

November 10, 1986 (Jeanie's birthday)

Let A, B be superalgebras and suppose that there is an involution ε in A^+ such that $\varepsilon(a)\varepsilon = \begin{cases} a & a \in A^+ \\ -a & a \in A^- \end{cases}$, for example $A = \mathrm{End}(V)$ where V is a super vector space. Then we have an isomorphism of superalgebras

$$(*) \qquad A \hat{\otimes} B \xrightarrow{\sim} A \otimes B$$

given by

$$\begin{array}{lcll} a \hat{\otimes} 1 & \longleftrightarrow & a \otimes 1 & a \in A \\ 1 \hat{\otimes} b & \longleftrightarrow & 1 \otimes b & b \in B^+ \\ 1 \hat{\otimes} b & \longleftrightarrow & \varepsilon \otimes b & b \in B^- \end{array}$$

Let's check this carefully. Let $\varphi(a) = a \otimes 1$ and let $\psi(b) = \begin{cases} 1 \otimes b & b \in B^+ \\ \varepsilon \otimes b & b \in B^- \end{cases}$. Then $\varphi : A \to A \otimes B$ and $\psi : B \longrightarrow A \otimes B$ are morphisms of superalgebras:

$$\psi((b+\beta)(b'+\beta')) = \psi(bb' + \beta\beta' + b\beta' + \beta b')$$
$$= 1 \otimes (bb' + \beta\beta') + \varepsilon \otimes (b\beta' + \beta b')$$

$$\psi(b+\beta)\psi(b'+\beta') = (1 \otimes b + \varepsilon \otimes \beta)(1 \otimes b' + \varepsilon \otimes \beta')$$
$$= 1 \otimes bb' + \varepsilon \otimes \beta b' + \varepsilon \otimes b\beta' + 1 \otimes \beta\beta'$$
$$= 1 \otimes (bb' + \beta\beta') + \varepsilon \otimes (\beta b' + b\beta').$$

Also the superbracket of $\varphi(a)$ and $\psi(b)$ is zero for any $a \in A$, $b \in B$:

$$\cancel{[\varphi(a), \psi(b)] = [a \otimes 1, 1 \otimes b] = 0 \quad \text{if } b \in B^+}$$

$$[\varphi(a+\alpha), \psi(b+\beta)]_s = [\varphi(a),\psi(b)] + [\varphi(a),\psi(\beta)] + [\varphi(\alpha),\psi(b)] + [\varphi(\alpha),\psi(\beta)]_+$$

$$= \cancel{[a\otimes 1, 1\otimes b]} + \cancel{[a\otimes 1, \varepsilon\otimes\beta]} + \cancel{[\alpha\otimes 1, 1\otimes b]} + [\alpha\otimes 1, \varepsilon\otimes\beta]_+$$

$$= (\alpha\otimes 1)(\varepsilon\otimes\beta) + (\varepsilon\otimes\beta)(\alpha\otimes 1)$$

$$= \alpha\varepsilon\otimes\beta + \varepsilon\alpha\otimes\beta = (\cancel{\alpha\varepsilon+\varepsilon\alpha})\otimes\beta = 0$$

Perhaps the easiest way to see the isom. ⊛ is to look at the supervector space $A\otimes B$ and to consider the algebra R of endomorphisms which commute with right multiplication by the elements $a\otimes 1$, $1\otimes b$, $a\in A$, $b\in B$. Then on one hand R is $A\otimes B$ acting by left mult. and on the other hand R is $A\hat{\otimes} B$ acting by

$$(a\hat{\otimes} b)(v\otimes w) = (-1)^{\deg b \deg v}\, av\otimes bw$$

e.g.

$$(1\hat{\otimes}\beta)(v\otimes w) = (-1)^{\deg v}\, v\otimes \beta w$$

$$= \varepsilon v\otimes\beta w$$

$$= (\varepsilon\otimes\beta)(v\otimes w)$$

Example: We can take $A = C_2$, whence we have

$$C_2\hat{\otimes} B = C_2\otimes B$$

which gives the periodicity of the Clifford algs. However C_1 doesn't have an ε and

$$C_1\hat{\otimes} C_1 = C_2$$

$$C_1\otimes C_1 = \mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}$$

are not isomorphic.

Next step will be to look at the algebra of matrix forms. This means the tensor product

$$\Omega(M)\ \hat{\otimes}\ \mathrm{End}(V)$$

where $V = V^+ \oplus V^-$. By the above this is isomorphic to ~~the~~ $\otimes$ product.

I think the way to view this is to consider everything as operators on the space of E-valued forms

$$\Omega(M,E) = \Gamma(M, \Lambda T^* \otimes E)$$

In my paper I defined left and right mult. by elements of $\Omega(M)$ on this space, so that

$$\omega\cdot(\eta s) = \omega\eta s \qquad \begin{matrix}\omega,\eta\in\Omega(M)\\ s\in\Gamma(E)\end{matrix}$$

(**) $$\eta s\cdot\omega = (-1)^{\deg s\,\deg\omega}(\eta\omega)s.$$

Then I identified $\Omega(M,\mathrm{End}\,E) = \Omega(M)\,\hat{\otimes}\,\underset{\Omega^0(M)}{\Omega^0(M,\mathrm{End}\,E)}$ with the operators commuting with right mult.

~~But the interesting point now is that the alg of actual right multiplication operators generated by the operators (**) is the same as the alg. of ops generated by~~

$\cancel{\eta s \longrightarrow \eta\omega s}$

Consider the algebra $A\hat{\otimes}B$ where now B has a involution ε of degree zero such that $\varepsilon b \varepsilon = (-1)^{\delta(b)} b$. Then we have isomorphism of superalgebras

$$A\hat{\otimes}B \simeq B\hat{\otimes}A \overset{\text{uses } \varepsilon}{\simeq} B\otimes A \simeq A\otimes B$$

$$a\hat{\otimes}1 \longrightarrow 1\hat{\otimes}a \longrightarrow 1\otimes a_+ + \varepsilon\otimes a_- \longrightarrow a_+\otimes 1 + a_-\otimes\varepsilon$$

$$1\hat{\otimes}b \longrightarrow b\hat{\otimes}1 \longrightarrow b\otimes 1 \longrightarrow 1\otimes b$$

Thus we claim there is an isomorphism

$$A\hat{\otimes}B = A\otimes B$$

with

$$a\hat{\otimes}1 = a_+\otimes 1 + a_-\otimes\varepsilon$$

$$1\hat{\otimes}b = 1\otimes b$$

Check:

$$(a\hat{\otimes}1)(a'\hat{\otimes}1) = (a_+\otimes 1 + a_-\otimes\varepsilon)(a'_+\otimes 1 + a'_-\otimes\varepsilon)$$

$$\| \qquad\qquad\qquad \|$$

$$aa'\hat{\otimes}1 \qquad a_+a'_+\otimes 1 + a_-a'_+\otimes\varepsilon + a_-a'_-\otimes\varepsilon^2 + a_+a'_-\otimes\varepsilon$$

$$\| \qquad\qquad\qquad \|$$

$$(aa')_+\otimes 1 + (aa')_-\otimes\varepsilon = (a_+a'_+ + a_-a'_-)\otimes 1 + (a_-a'_+ + a_+a'_-)\otimes\varepsilon$$

$$(1\hat{\otimes}b)(1\hat{\otimes}b') = (1\otimes b)(1\otimes b')$$

$$\| \qquad\qquad \|$$

$$1\hat{\otimes}bb' = (1\otimes bb')$$

~~$[a\hat{\otimes}1, 1\hat{\otimes}b]_s = [a_+\hat{\otimes}1 + a_-\hat{\otimes}1, 1\hat{\otimes}b]_s$~~

~~$(a_+\hat{\otimes}1)(1\hat{\otimes}b) - (1\hat{\otimes}b)(a_+\hat{\otimes}1)$~~

~~$[a\hat{\otimes}1,$~~

~~$[a_+\hat{\otimes}1, 1\hat{\otimes}b]_s + [a_+$~~

$$[a\hat{\otimes}1, 1\hat{\otimes}b]_s = [a_+\otimes 1 + a_-\otimes\varepsilon, 1\otimes b]_s$$

$$= \cancel{[a_+\otimes 1, 1\otimes b]_{ord}} + \cancel{[a_-\otimes\varepsilon, 1\otimes b_+]_{ord}} + [a_-\otimes\varepsilon, 1\otimes b_-]_+$$

$$= a_-\otimes b_-\varepsilon + a_-\otimes\varepsilon b_- = a_-\otimes(\cancel{b_-\varepsilon+\varepsilon b_-}) = 0$$

So we apply this to

$$\Omega(M)\hat{\otimes}\,\mathrm{End}(V) \simeq \Omega(M)\otimes\mathrm{End}(V)$$

$$\omega\hat{\otimes}1 = \omega_+\otimes 1 + \omega_-\otimes\varepsilon$$
$$1\hat{\otimes}u = 1\otimes u.$$

Thus in terms of matrix differential forms we have that a form ω is to be identified with

$$\begin{pmatrix}\omega & 0\\ 0 & \omega\end{pmatrix} \quad \text{if } \omega\in\Omega^+$$

$$\begin{pmatrix}\omega & 0\\ 0 & -\omega\end{pmatrix} \quad \text{if } \omega\in\Omega^-.$$

It seems desirable to set up the superconnection calculations in this notation if possible.

Let us therefore consider the superalgebra $\mathrm{End}(V)\hat{\otimes}\,\Omega(M)$ acting in the usual way on $V\otimes\Omega(M)$ which means

$$X\hat{\otimes}1 \quad \text{acts as} \quad X\otimes 1$$
$$1\hat{\otimes}\omega \quad \text{acts as} \quad 1\otimes\omega_+ + \varepsilon\otimes\omega_-$$

I want to identify $V\otimes\Omega(M)$ with column vectors of forms, and operators on $V\otimes\Omega(M)$ commuting with right multiplication by elts of $\Omega(M)$ are to be identified with left mult. by matrix forms. Then

$X \hat{\otimes} 1$ acts as the matrix X

$1 \hat{\otimes} \omega$ acts as the matrix $\omega_+ + \varepsilon\omega_-$ where as usual elts of Ω as interpreted as diagonal matrices.

(Question: Is there a better notation?) The algebra $\operatorname{End} V \otimes \Omega(M)$ of matrix forms is ~~generated~~ spanned by products $X\omega$ where $X\omega = \omega X$. The alg $\operatorname{End} V \hat{\otimes} \Omega(M)$ is spanned by the same products but where $X\omega = (-1)^{\delta(X)\delta(\omega)}\omega X$. We have an isomorphism of algebras

$$\Phi : \operatorname{End} V \hat{\otimes} \Omega(M) \xrightarrow{\sim} \operatorname{End} V \otimes \Omega(M)$$

such that
$$\Phi(X) = X$$
$$\Phi(\omega) = \omega_+ + \varepsilon\omega_-$$

What is the operator $[D, \ ]$ where $D = d + A$ is a connection? Let $D = dx^\mu \cdot D_\mu$. ~~Then we~~ ~~know that $[D, \ ]$ on End~~ We first have to recall how D acts on $\Omega(M, E) = \Omega(M) \otimes V$. We have

$$\begin{array}{ccccc} D(\omega s) & = & d\omega\, s & + & (-1)^{\delta(\omega)}\omega\, Ds \\ \| & & \| & & \| \\ (-1)^{\delta(\omega)\delta(s)} D(s\omega) & & (-1)^{(\delta(\omega)+1)\delta(s)} s\, d\omega & & (-1)^{\delta(\omega)}(-1)^{(1+\delta(s))\delta(\omega)} Ds \cdot \omega \end{array}$$

$$D(s\omega) = Ds \cdot \omega + (-1)^{\delta(s)} s \cdot d\omega \qquad ?$$

I guess it's clear that $[d, \ ]$ on the algebra $\operatorname{End} V \hat{\otimes} \Omega(M)$ acts as

$$\begin{array}{ccc} [d, X\omega] & = & (-1)^{\delta(X)} X \cdot d\omega \\ \Big\downarrow \Phi & & \Big\downarrow \Phi \\ X(\omega_+ + \varepsilon\omega_-) & = & (-1)^{\delta(X)} X(d\omega_- + \varepsilon\, d\omega_+) \end{array}$$

Therefore on $\operatorname{End} V \otimes \Omega(M)$ the operator $[d, \;]$ becomes simply $\varepsilon d = d_\varepsilon$:

$$X\omega \longmapsto \varepsilon d(X\omega) = \varepsilon X\, d\omega.$$

Thus if we want to pass from the algebra $\operatorname{End}(V) \hat{\otimes} \Omega(M)$ to matrix forms it appears that the operator d becomes εd.

It seems now that are two ways we could treat superconnections by working in the usual algebra

$$\Omega(M) \underset{\Omega^0(M)}{\otimes} \Omega^0(M, \operatorname{End} E)$$

of endomorphism-valued forms. The first is to adjoin to this algebra an element σ whose effect on $\Omega(M,E)$ is the involution which is $+1$ on even forms and -1 on odd forms. Then the operator on $\Omega(M,E)$ we associate to an $X \in \Omega^0(M, \operatorname{End} E)$ is $\sigma X = X\sigma$. Note σ commutes with $\Omega^0(M, \operatorname{End} E)$. Thus our superconnection is

$$D + X\sigma$$

and its curvature is

$$(D + X\sigma)^2 = D^2 + DX\sigma + \underbrace{X\sigma D}_{-XD\sigma} + X\sigma X\sigma$$

$$= D^2 + [D, X]\sigma + X^2$$

The second method is to let ε be the operator on $\Omega(M, E)$ giving the E grading and to form the operator

$$\varepsilon D + X$$

and call it the superconnection. The curvature is

$$(\varepsilon D + X)^2 = \varepsilon D \varepsilon D + \varepsilon DX + \underbrace{X\varepsilon}_{-\varepsilon X} D + X^2$$

$$= D^2 + \varepsilon [D, X] + X^2$$

~~Motivation for the second method. Suppose $E = \underline{V}$. Then $\Omega(M, \underline{V}) = \Omega(M) \otimes V$ is the space of vector valued forms.~~

November 11, 1986

I want to motivate $\varepsilon D + X$. We are working with various operators on the vector space $\Omega(M,E) = \Omega(M,E^0) \oplus \Omega(M,E^1)$ and hence it ought to be possible to ~~do things~~ write the operators in block form $\left(\begin{array}{c|c} \cdot & \cdot \\ \hline \cdot & \cdot \end{array}\right)$ in terms of operators we are used to from the ungraded case. For example, $D = \begin{pmatrix} D^0 & 0 \\ 0 & D^1 \end{pmatrix}$. Ultimately one wants to lift up to the principal bundle and work with matrix forms.

Let's look at $\Omega(M,\tilde{V})$. From the viewpoint of matrix forms we should write this as $V \otimes \Omega(M)$ and consider it as a right $\Omega(M)$-module. Those operators commuting with right $\Omega(M)$-multiplication form the algebra $\mathrm{End}(V) \otimes \Omega(M)$ of matrix forms.

In the superconnection business I considered various operators on $\Omega \otimes V$. Those satisfying $\cancel{T(\omega\alpha) = (-1)^{\delta(\omega)\delta(T)}\omega T(\alpha)}$ $\quad T(\alpha\omega) = T(\alpha)\omega$

form the algebra $\Omega \mathbin{\hat{\otimes}} \mathrm{End}(V)$ with generators

$$1 \mathbin{\hat{\otimes}} X = 1 \otimes X_+ + \sigma \otimes X_- \qquad \text{on } \Omega \otimes V$$

$$\omega \mathbin{\hat{\otimes}} 1 = \omega \otimes 1 \qquad \text{"}$$

As operators on $\Omega \otimes V$ we have

$$D = d + A = dx^\mu(\partial_\mu + A^a_\mu T_a)$$

$$= d \otimes 1 + A^a \otimes T_a$$

where $T_a \in \mathrm{End}^0(V)$, $A^a \in \Omega^1$. Also $X \in \Omega^0(M) \mathbin{\hat{\otimes}} \mathrm{End}^1 V$

say $X = f^b T_b$ is the operator

$$\text{"}\sigma X\text{"} = \sigma f^b \otimes T_b .$$

Now what I want to do is to produce an isomorphism

$$\Phi : \Omega \otimes V \xrightarrow{\sim} V \otimes \Omega$$

which will carry over the superconnection setup to matrix forms.

Clearly we need Φ to commute with right Ω-multiplication in order to get an isomorphism $\Omega \hat{\otimes} \mathrm{End}(V) \simeq \mathrm{End}(V) \otimes \Omega$. So

$$\Phi(\omega \otimes v) = \Phi\big((1 \otimes v)\,\omega\big)\,(-1)^{\delta(v)\delta(\omega)}$$

$$= \Phi(1 \otimes v)\cdot\omega\;(-1)^{\delta(v)\delta(\omega)}$$

It seems natural to have $\Phi(1 \otimes v) = v \otimes 1$, whence

$$\Phi(\omega \otimes v) = (-1)^{\delta(v)\delta(\omega)}\, v \otimes \omega$$

$$= (1 \otimes \omega_+ + \varepsilon \otimes \omega_-)(v \otimes 1)$$

Therefore, which is something I should have seen directly, the map Φ is the standard interchange for two super vector spaces. It has the property that

$$\Phi (1 \hat{\otimes} X)\, \Phi^{-1} = X \hat{\otimes} 1 = X \otimes 1$$

$$\Phi\ (\omega \hat{\otimes} 1)\ \Phi = 1 \hat{\otimes} \omega = 1 \otimes \omega_+ + \varepsilon \otimes \omega_-$$

Next

$$\Phi(d \hat{\otimes} 1)\,\Phi^{-1} = 1 \hat{\otimes} d = \varepsilon \otimes d$$

$$\Phi(A^a \hat{\otimes} T_a)\Phi^{-1} = \Phi(A^a \hat{\otimes} T_a)\Phi^{-1}$$

$$= T_a \hat{\otimes} A^a = (T_a \hat{\otimes} 1)(1 \hat{\otimes} A^a)$$

$$= (T^a \otimes 1)(\varepsilon \otimes A^a) = \varepsilon T^a \otimes A^a$$

Notice that $d + A = d + A^a T^a$ acts on $V \otimes \Omega(M)$ normally as $1 \otimes d + T^a \otimes A^a$, where normally means ~~...~~ without modification by the grading in V. Thus we see that

$$\Phi(D)\Phi^{-1} = \varepsilon D$$

and similarly, we find that

$$\Phi(\sigma X)\Phi^{-1} = \Phi(f^b \hat{\otimes} T_b)\Phi^{-1} = T_b \hat{\otimes} f^b = T_b \otimes f^b$$

$$= X$$

Thus under Φ the superconnection operator becomes $\varepsilon D + X$, so in block form it becomes

$$\begin{pmatrix} D & -T^* \\ T & -D \end{pmatrix}$$

Next we need the supertrace on the algebra

$$\begin{array}{ccccc} \Omega \hat{\otimes} \mathrm{End}(V) & \simeq & \mathrm{End}(V) \hat{\otimes} \Omega & \simeq & \mathrm{End}(V) \otimes \Omega \\ \omega \hat{\otimes} X & \longmapsto & X \hat{\otimes} \omega & = & X \otimes \omega_+ + X\varepsilon \otimes \omega_- \\ {\scriptstyle tr_s} \downarrow \quad \downarrow & & & & \\ \Omega \quad \omega\, tr(\varepsilon X) & & & & \end{array}$$

Thus we have tr_s : $X \otimes \omega_+ \longmapsto tr(\varepsilon X)\omega_+$

$X\varepsilon \otimes \omega_- \longmapsto tr(\varepsilon X)\omega_-$

Therefore we have on $\mathrm{End}(V)\otimes\Omega$, the supertrace

$$tr_s(X\otimes\omega_+) = tr(\varepsilon X)\,\omega_+$$
$$tr_s(X\otimes\omega_-) = tr(X)\,\omega_-$$

Check: Let's check this is a supertrace in $\mathrm{End}(V)\otimes\Omega$; we write $X\omega$ instead of $X\otimes\omega$. We want to see that $tr_s([X\alpha, Y\beta]_\bullet) = 0$ where the bracket is the supercommutator. A priori there are 16 possibilities two for each X, Y, α, β. But tr and $tr(\varepsilon\,?)$ vanish on odd endos, so we can suppose either both X and Y are even or both are odd. Suppose both X, Y even. 1) both α, β even

$$tr_s([X\alpha, Y\beta]) = tr_s(X\alpha Y\beta - Y\beta X\alpha) = \cancel{tr_s(XY-YX)}\,\alpha\beta$$

2) α, β odd

$$tr_s[X\alpha, Y\beta] = tr_s(X\alpha Y\beta + Y\beta X\alpha) = \cancel{tr_s(XY-YX)}\,\alpha\beta$$

3) α even, β odd or the reverse

$$
\begin{aligned}
tr_s[X\alpha, Y\beta] &= tr_s(X\alpha Y\beta - Y\beta X\alpha) = \\
&= tr_s((XY-YX)\alpha\beta) \\
&= tr(XY-YX)\,\alpha\beta = 0
\end{aligned}
$$

Next take X, Y both odd

$\alpha\beta$ even $\quad tr_s[X\alpha, Y\beta] = tr_s(X\alpha Y\beta + Y\beta X\alpha) = tr_s(XY+YX)\alpha\beta$

$\alpha\beta$ odd $\quad tr_s[\ \text{''}\ ] = tr_s(X\alpha Y\beta - Y\beta X\alpha) = tr_s(XY+YX)\alpha\beta$

α even β odd $\quad tr_s[\ \text{''}\ ] = tr_s(X\alpha Y\beta - Y\beta X\alpha) = tr_s((XY-YX)\alpha\underset{\text{odd}}{\beta})$

$$= tr(XY-YX)\,\alpha\beta$$

So it works.

As a final check let's prove

$$d\, tr_s(X\omega) = tr_s([\varepsilon d, X\omega])$$

Again there will be four cases but the supertrace vanishes on odd endos of V so we can suppose X even.

ω even
$$tr_s(X\omega) = tr(\varepsilon X)\omega$$
$$tr_s([\varepsilon d, X\omega]) = tr_s(\varepsilon d X\omega - X\omega\varepsilon d)$$
$$= tr_s(\varepsilon X \underbrace{d\omega - \omega d}_{d(\omega)}) = tr(\varepsilon X)\, d(\omega)$$

ω odd
$$tr_s(X\omega) = tr(X)\omega$$
$$tr_s([\varepsilon d, X\omega]) = tr_s(\varepsilon d X\omega + X\omega\varepsilon d)$$
$$= tr_s(\varepsilon X(d\omega + \omega d)) = tr_s(\varepsilon X\, d(\omega))$$
$$= tr(\varepsilon\varepsilon X)\, d(\omega) = tr(X)\, d(\omega)$$

So it's OKAY. Then it follows that

$$d\, tr_s(X\omega) = tr_s([\varepsilon D + L, X\omega])$$

and so one could carry out the super-connection paper in this notation.

But for my present purposes it's enough to observe that because tr_s vanishes on $End'(V)\otimes\Omega$ we have

$$tr_s\left(\frac{1}{\lambda - X^2 - \varepsilon[D,X] - D^2}\right) = tr\left(\varepsilon\frac{1}{\lambda - X^2 - \varepsilon[D,X] - D^2}\right)$$

Now we turn to the problem of relating the character forms on $Gr(V)$, which are defined using the Grassmannian connection on the subbundle, and the forms defined via the superconnection process.

Consider on $V = V^0 \oplus V^1$ the sesquilinear bilinear form

$$\begin{pmatrix} a' \\ b' \end{pmatrix} \underset{t}{\cdot} \begin{pmatrix} a \\ b \end{pmatrix} = a'^* a + t\, b'^* b$$

where $t \in \mathbb{C} - \mathbb{R}_{\leq 0}$. Then

$$\begin{pmatrix} a \\ b \end{pmatrix} \underset{t}{\cdot} \begin{pmatrix} a \\ b \end{pmatrix} = |a|^2 + t|b|^2 = 0 \implies a = b = 0.$$

which implies that the bilinear form is non-degenerate (sets up an isom. of V with its conjugate dual), and also that the orthogonal space to any subspace is actually complementary. Thus to any W we will have a complementary space $W^\perp$ associated and so there will be a connection on the subbundle over the Grassmannian.

Consider $Gr_m(V)$, where $m = \dim(V^0)$, and calculate $(\Gamma_T)^\perp$ i.e. $\begin{pmatrix} a \\ b \end{pmatrix} \ni$

$$\begin{pmatrix} x \\ Tx \end{pmatrix} \underset{t}{\cdot} \begin{pmatrix} a \\ b \end{pmatrix} = x^* a + t x^* T^* b = 0 \qquad \text{all } x$$

$$a = -tT^* b \qquad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -tT^* \\ 1 \end{pmatrix} b$$

So $\quad i = \begin{pmatrix} 1 \\ T \end{pmatrix} \qquad j = \begin{pmatrix} -tT^* \\ 1 \end{pmatrix}$

$$i^* = \frac{1}{1 + tTT^*}\begin{pmatrix} 1 & tT^* \end{pmatrix} \qquad j^* = \frac{1}{1 + tTT^*}\begin{pmatrix} -T & 1 \end{pmatrix}$$

and so the curvature can be found as follows

$$j^* di = \frac{1}{1+tTT^*}(-T \quad 1)\begin{pmatrix}0\\ dT\end{pmatrix}$$

$$i^* dj = \frac{1}{1+tT^*T}(1 \quad tT^*)\begin{pmatrix}-t\,dT^*\\ 0\end{pmatrix}$$

$$-i^*dj\, j^*di = \frac{t}{1+tT^*T}\, dT^* \frac{1}{1+tTT^*}\, dT$$

Actually I should do this calculation using

$$L = \begin{pmatrix}0 & -tT^*\\ T & 0\end{pmatrix} \qquad g = \frac{1+L}{1-L} \qquad F = g\varepsilon$$

In general the curvature of the direct sum of the subbundle and quotient bundles is

$$e\,de\,de + (1-e)d(1-e)d(1-e) = de\,de = \frac{1}{4}dF\,dF$$

$$= \frac{1}{4}\, dg\,\varepsilon\, dg\,\varepsilon = \frac{1}{4}dg\, dg^{-1}$$

and if

$$g = \frac{1+L}{1-L} = -1 + \frac{2}{1-L} \qquad dg = \frac{2}{1-L}\,dL\,\frac{1}{1-L}$$

$$g^{-1} = \frac{1-L}{1+L} = -1 + \frac{2}{1+L} \qquad dg^{-1} = \frac{2}{1+L}(-dL)\frac{1}{1+L}$$

and so

$$\frac{1}{4}dg\,dg^{-1} = -\frac{1}{1-L}\,dL\,\frac{1}{1-L^2}\,dL\,\frac{1}{1+L}$$

which is conjugate to

$$-\frac{1}{1-L^2}\,dL\,\frac{1}{1-L^2}\,dL$$

Anyway the upshot of this calculation is that the forms $tr_s\left(\frac{\sqrt{\lambda}}{\lambda - X^2}dX\right)^{2k}$ which I saw were well-defined on the whole Grassmannian for $\lambda \notin \mathbb{R}_{\leq 0}$ are in fact just character forms associated to a family of connections parametrized by $\lambda \in \mathbb{C} - \mathbb{R}_{\leq 0}$.

So now let us return to linking this family of character forms on the Grassmannian to those defined by superconnections. If D_λ is the connection on the subbundle associated to λ we have

$$tr\left(e^{uD_\lambda^2}\right) = \sum_{k\geq 0} \frac{u^k}{k!} \frac{(-1)^k}{2} tr_\varepsilon\left(\frac{\sqrt{\lambda}}{\lambda - X^2} dX\right)^{2k}$$

On the other hand we have

$$\int_0^\infty \left\{ tr_s\left(e^{u(X+\varepsilon d)^2}\right) - tr_s\left(e^{uX^2}\right) \right\} e^{-\lambda u} \frac{du}{u}$$

$$= \sum_{k>0} \frac{(-1)^k}{2k} tr_\varepsilon\left(\frac{1}{\lambda - X^2} dX\right)^{2k}$$

The problem is the $k!$ in the former versus the k in the latter makes it difficult to compare these forms nicely.

One method would be to transform the former

$$\int_0^\infty \left\{ tr\left(e^{uD_\lambda^2}\right)_{>0} \right\} e^{-\lambda u} \frac{du}{u} = \sum_{k>0} \frac{1}{k!} \frac{(k-1)!}{\lambda^k\, 2} tr_\varepsilon\left(\frac{\sqrt{\lambda}}{\lambda - X^2} dX\right)^{2k}$$

Thus we get

$$\boxed{\int_0^\infty tr\left(e^{uD_\lambda^2}\right)_{>0} e^{-\lambda u} \frac{du}{u} = \int_0^\infty tr_s\left(e^{u(X+\varepsilon d)^2}\right)_{>0} e^{-\lambda u} \frac{du}{u}}$$

Now at least for $\lambda > 0$ we know

$$\mathrm{tr}(e^{uD_\lambda^2}) = \varphi_{\sqrt{\lambda}}^* \, \mathrm{tr}(e^{uD_1^2})$$

so that this formula is equivalent to

$$\boxed{\int_0^\infty \mathrm{tr}(e^{uD_1^2})_{>0} \, e^{-\lambda u} \frac{du}{u} = \int_0^\infty \mathrm{tr}_s\left(e^{u(\sqrt{\lambda}X + \varepsilon d)^2}\right)_{>0} e^{-\lambda u} \frac{du}{u}}$$

This should hold for all $\lambda \in \mathbb{C} - \mathbb{R}_{\leq 0}$ by analytic continuation.

November 12, 1986

Set $A = (X + \varepsilon D)^2$, $A_0 = X^2$. The problem is to show that

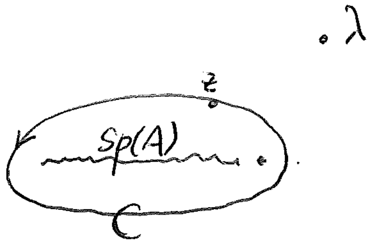
$$\log(\lambda - A) - \log(\lambda - A_0)$$

makes sense as a form not just on $\mathrm{Hom}(V^0, V^1)$ but on $\mathrm{Gr}_m(V)$.

Here $\log(\lambda - A)$ is defined by the holomorphic functional calculus

$$\log(\lambda - A) = \frac{1}{2\pi i} \oint \log(\lambda - z) \frac{1}{z - A}\, dz$$

where the contour circles $sp(A)$ which is a compact subset of $\mathbb{R}_{\leq 0}$, and where $\arg(\lambda - z) \in (-\pi, \pi)$. This defines $\log(\lambda - A)$ for $\lambda \notin \mathbb{R}_{\leq 0}$.



Suppose we move λ counterclockwise around $sp(A)$, then $\log(\lambda - z)$ jumps by $2\pi i$ in crossing the real axis to the left of $sp(A)$, so the operator $\log(\lambda - A)$ does the same. Since A_0, A have the same spectrum, we see that

$$\log(\lambda - A) - \log(\lambda - A_0)$$

is single-valued outside the convex hull of $sp(A_0)$.

So

$$\log(\lambda - A) - \log(\lambda - A_0) = \frac{1}{2\pi i} \oint \log(\lambda - z)\left(\frac{1}{z-A} - \frac{1}{z-A_0}\right) dz$$

$$= \frac{1}{2\pi i} \oint \log(\lambda - z) \frac{d}{dz}\left\{\log(z-A) - \log(z-A_0)\right\} dz$$

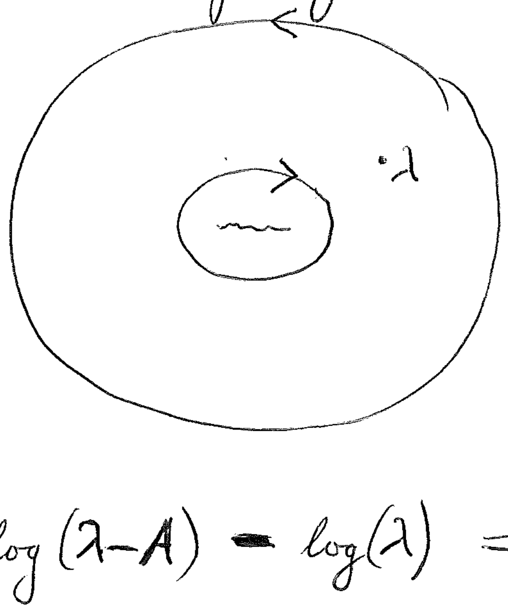
Now integrate by parts to get

$$= \frac{1}{2\pi i} \oint \underbrace{-\frac{d}{dz}\log(\lambda - z)}_{\frac{1}{\lambda - z}} \left\{\log(z-A) - \log(z-A_0)\right\} dz$$

Notice that $\log(\lambda - z)$ is singled valued as z runs around the contour. Thus it appears we have established a Cauchy formula

$$\log(\lambda - A) - \log(\lambda - A_0) = -\frac{1}{2\pi i} \oint \frac{1}{z-\lambda}\left\{\log(z-A) - \log(z-A_0)\right\} dz$$

Alternative proof: Usual Cauchy + fact that



$\log(\lambda - A) - \log(\lambda - A_0)$ goes to zero as $|\lambda| \to \infty$. This is proved rather simply as follows

$$\log(\lambda - A) - \log(\lambda) = \frac{1}{2\pi i} \oint \left\{\log(\lambda - z) - \log(\lambda)\right\} \frac{1}{z-A} dz$$

$$= \frac{1}{2\pi i} \oint_{|z|=r} \log\left(1 - \frac{z}{\lambda}\right) \frac{1}{z-A} dz$$

So $\boxed{\|\log(\lambda - A) - \log\lambda\| \leq C\left(-\log\left(1 - \frac{r}{|\lambda|}\right)\right) = O\left(\frac{1}{|\lambda|}\right)}$

So we conclude

$$\log(\lambda - A) - \log(\lambda) = \int_{\infty}^{\lambda} \left(\frac{1}{z-A} - \frac{1}{z} \right) dz$$

and the integral is abs. convergent since

$$\frac{1}{z-A} - \frac{1}{z} = \frac{A}{(z-A)z} = \frac{1}{z^2} \frac{A}{1-\frac{A}{z}} = O\left(\frac{1}{|z^2|}\right).$$

Thus I reach the formula

$$\boxed{\log(\lambda - A) - \log(\lambda - A_0) = \int_{\infty}^{\lambda} \left(\frac{1}{z-A} - \frac{1}{z-A_0} \right) dz}$$

which I want to use now to show that this operator is defined on the Grassmannian.

Let $A = (X + \varepsilon D)^2 = X^2 + \varepsilon [D, X] + D^2$.

$$\frac{1}{\lambda - A} = (g+1) \frac{1}{\lambda(g+1)^2 - (g-1)^2 - 2\varepsilon g^{-1}[D,g] + (g+1)D^2(g+1)} (g+1)$$

The first term is

$$\frac{1}{\lambda - A} - \frac{1}{\lambda - A_0} = (g+1) \frac{1}{\lambda(g+1)^2 - (g-1)^2} \left(2\varepsilon g^{-1}[D,g] + (g+1)D^2(g+1) \right)$$

$$\times \frac{1}{\lambda(g+1)^2 - (g-1)^2} (g+1) + \cdots$$

We must see this term decays sufficiently fast as $\lambda \to \infty$. Let's try to show

$$(g+1) \frac{1}{\lambda(g+1)^2 - (g-1)^2}$$

decays. $\dfrac{e^{i\theta}+1}{\lambda(e^{i\theta}+1)^2 - (e^{i\theta}-1)^2} = \dfrac{1}{2e^{i\theta}} \dfrac{\cos\frac{\theta}{2}}{\lambda(\cos\frac{\theta}{2})^2 + (\sin\frac{\theta}{2})^2}$

Thus we want

$$\circledast \qquad \max_{0 \le x \le 1} \frac{x}{|\lambda x^2 + 1 - x^2|}$$

and we might as well do our integrating over the positive real axis, so we can suppose $\lambda > 1$. Thus we want the minimum of

$$\frac{\lambda x^2 + 1 - x^2}{x} = (\lambda - 1)x + \frac{1}{x}$$

for $0 < x \le 1$. $\qquad \lambda - 1 - \frac{1}{x^2} = 0 \qquad x^2 = \frac{1}{\lambda - 1}$ min.

So for $\lambda > 1$

$$\circledast = \frac{\frac{1}{\sqrt{\lambda - 1}}}{2} = \frac{1}{2\sqrt{\lambda - 1}} = O\left(\frac{1}{\sqrt{\lambda}}\right)$$

It appears now that we have some problems already on the Grassmannian. I still have not succeeded in showing, or deciding whether, the matrix form

$$\circledast \qquad \log(\lambda - X^2 - \varepsilon dX) - \log(\lambda - X^2)$$

over $\mathrm{Hom}(V^0, V^1)$ extends to $Gr_m(V)$. I think I can prove $\circledast$ extends to the Grassmannian after applying tr_s as follows.

Set $A_t = X^2 + t\varepsilon dX$. Then

$$\log(\lambda - A_t) = \frac{1}{2\pi i} \int \log(\lambda - z) \frac{1}{z - A_t} dz$$

$$\partial_t \log(\lambda - A_t) = \frac{1}{2\pi i} \int \log(\lambda - z) \frac{1}{z - A_t} \dot{A}_t \frac{1}{z - A_t} dz$$

Now

$$\operatorname{tr}_s\left(\frac{1}{z-A_t}\mathring{A}_t\frac{1}{z-A_t}\right) = \operatorname{tr}_s\left(\frac{1}{(z-A_t)^2}\mathring{A}_t\right)$$

$$= -\partial_z \operatorname{tr}_s\left(\frac{1}{z-A_t}\mathring{A}_t\right)$$

So

$$\partial_t \operatorname{tr}_s\{\log(\lambda-A_t)\} = \frac{1}{2\pi i}\oint \log(\lambda-z)\,(-\partial_z)\operatorname{tr}_s\left(\frac{1}{z-A_t}\mathring{A}_t\right)dz$$

$$= \frac{+1}{2\pi i}\oint \frac{1}{z-\lambda}\operatorname{tr}_s\left(\frac{1}{z-A_t}\mathring{A}_t\right)\,dz$$

Now push the contour to ∞ and we get $\left(\operatorname{tr}_s = O\left(\frac{1}{|z|}\right)\right)$

$$\partial_t \operatorname{tr}_s\{\log(\lambda-A_t)\} = -\operatorname{tr}_s\left(\frac{1}{\lambda-A_t}\mathring{A}_t\right)$$

Thus

$$-\operatorname{tr}_s\{\log(\lambda-A)-\log(\lambda-A_0)\} = \int_0^1 \operatorname{tr}_s\left(\frac{1}{\lambda-A_t}\mathring{A}_t\right)dt$$

and this holds in general. Now in our case we can expand

$$\frac{1}{\lambda-A_t} = \sum_{k\geq 0}\left(\frac{1}{\lambda-A_0}tB\right)^k\frac{1}{\lambda-A_0}$$

and this is a finite series as B is a positive degree form. Thus doing the integration yields

$$\boxed{-\operatorname{tr}_s\{\log(\lambda-A)-\log(\lambda-A_0)\} = \sum_{k\geq 0}\frac{1}{k}\operatorname{tr}_s\left(\frac{1}{\lambda-A_0}B\right)^k}$$

and I know each of the terms in this series extends to the Grassmannian.

<u>November 13, 1986</u>

Problem: Try to understand whether $\log(\lambda - X^2 - dX\sigma) - \log(\lambda - X^2)$ is defined on the unitary group.

First we derive a formula for the first order perturbation of $\log(\lambda - A)$:

$$\delta \log(\lambda - A) = \frac{1}{2\pi i}\int \log(\lambda - z)\, \delta \frac{1}{z-A}\, dz$$

$$= -\frac{1}{2\pi i}\int \log(\lambda - z) \frac{1}{z-A}\, \delta A \frac{1}{z-A}\, dz$$

Thus we have

$$\log(\lambda - X^2 - dX\sigma) - \log(\lambda - X^2)$$

$$= -\frac{1}{2\pi i}\int \log(\lambda - z) \frac{1}{z-X^2}\, dX\sigma \frac{1}{z-X^2}\, dz \quad + \text{ higher order forms}$$

Now we propose to ~~evaluate~~ this by using a basis of eigenvector for X. We have by residues

$$-\frac{1}{2\pi i}\int \log(\lambda - z)\frac{1}{z-x}\frac{1}{z-y}\, dz = -\left(\frac{\log(\lambda - x)}{x-y} + \frac{\log(\lambda - y)}{y-x}\right)$$

$$= -\frac{1}{(x-y)}\log\left(\frac{\lambda - x}{\lambda - y}\right)$$

Now

$$\frac{1}{z - X^2}\, dX \frac{1}{z-X^2} = g(g+1)\frac{1}{z(g+1)^2 - (g-1)^2}\, g^{-1}dg \frac{1}{z(g+1)^2-(g-1)^2}(g+1)$$

Now $g^{-1}dg$ can be an arbitrary skew hermitian matrix. Lets look at an off diagonal entry where g has the value ρ for the row index and ρ' for the column index. Then we have to do the contour integral

$$-\frac{1}{2\pi i}\int \log(\lambda - z)\left\{\frac{\rho}{\rho+1}\;\frac{1}{z-\left(\frac{\rho-1}{\rho+1}\right)^2}\;\frac{1}{z-\left(\frac{\rho'-1}{\rho'+1}\right)^2}\;\frac{1}{\rho'+1}\right\}dz$$

$$= -\frac{\rho}{\rho+1}\;\frac{1}{\left(\frac{\rho-1}{\rho+1}\right)^2-\left(\frac{\rho'-1}{\rho'+1}\right)^2}\;\log\left(\frac{\lambda-\left(\frac{\rho-1}{\rho+1}\right)^2}{\lambda-\left(\frac{\rho'-1}{\rho'+1}\right)^2}\right)$$

Set $\rho' = 1$ and we get

$$-\frac{\rho}{\rho+1}\;\frac{1}{\left(\frac{\rho-1}{\rho+1}\right)^2}\;\log\left(\frac{\lambda-\left(\frac{\rho-1}{\rho+1}\right)^2}{\lambda}\right)$$

$$= -\frac{\rho(\rho+1)}{(\rho-1)^2}\;\log\left(\frac{\lambda-\left(\frac{\rho-1}{\rho+1}\right)^2}{\lambda}\right)$$

Let's put $\frac{\rho-1}{\rho+1} = \frac{i}{x}$ so that $\rho \longrightarrow -1$ is $x \longrightarrow 0$. Then $-1+\frac{i}{x} = -1+\frac{\rho-1}{\rho+1} = \frac{-2}{\rho+1}$,

$$\rho+1 = \frac{-2}{-1+\frac{i}{x}} = \frac{2x}{x-i}$$

so near $\rho = -1$ we have

$$x\log\left(\frac{\lambda-\frac{1}{x^2}}{\lambda}\right) = x\left[\log\left(\frac{1}{x^2}\right)+\log\left(\frac{\lambda x^2-1}{\lambda}\right)\right]$$

$$= -2x\log x \quad + \text{ smooth}$$

This shows that the function is not smooth.

So we conclude that

$$\log(\lambda - X^2 - dX\sigma) - \log(\lambda - X^2)$$

does not extend smoothly to a form on the unitary group.

This has some consequences because I was under the impression that I could go from the resolvent to the heat operator $e^{u(X^2+dX\sigma)}$ and then apply $\int_0^\infty ?\, e^{-\lambda u}\frac{du}{u}$. What this means therefore is that for these matrix valued forms there is already some sort of difficulty with the vanishing as $u \to 0$.

Return after finishing the paper.

Recall that you found the difficulty in trying to apply

$$\log(\lambda - A) - \log(\lambda - A_0) = \int_\infty^\lambda \left(\frac{1}{z-A} - \frac{1}{z-A_0}\right) dz$$

We have

$$\frac{1}{z-A} - \frac{1}{z-A_0} = \frac{1}{z-X^2}\, dX\sigma\, \frac{1}{z-X^2} +$$

$$= \underbrace{(g+1)\frac{1}{z(g+1)^2-(g-1)^2}}\; dg\,\sigma\, \underbrace{\frac{1}{z(g+1)^2-(g-1)^2}(g+1)}$$

The best estimate for this is $\frac{1}{\sqrt{z}}$ for each of these factors so the integral isn't convergent in an obvious way.

Already something interesting happens for the one forms on the unitary group U_2.

November 18, 1986

Friedan conversation:

A conformal field theory consists of a holom. vector bundle W over the modular space of "all" R.S. together with a hermitian metric h such that the connection on W is flat. In addition there is a holom. section ψ of $W \otimes E_c$ where $E_c =$ Hodge line bundle $(\lambda R^1_{\pi_*}(K))^{\otimes c/24}$. The partition function is something $h(\psi, \psi) =$ a section of $|E_c|$.

Chiral example: $\psi = \int e^{b \bar\partial c}$. Here everything is holomorphic. No h. Proof of Mumford theorem relating canonical line bundle on moduli space $\mathcal{M}$ to 13 th? power of Hodge line bundle.

Possible ideas: W with its flat connection should really be a D-module. To get positive inner product one maybe needs a piece of the Hodge filtration.

All the above is perturbative approach to the good theory, like finite dimensional approxs. to $\mathcal{L}(M)$. In particular get classical solutions (flat sections of W) from Calabi-Yau manifolds. Much too many.

Problem. Show Lattice approach to chiral fermions (neutrinos) doesn't work using spectral flow ideas. Start with cubic lattice introduce fermion field variables $\psi_x^* \; \psi_x$ for each lattice site and a Hamiltonian $\psi_x^* H(x-y) \psi_y$. Quantize to get tensor product of spinors at each site. Goal would be to obtain a different number of left + right fermions in the continuum limit Required by Weinberg-Salam.

Model. Fourier transform to momentum space = the dual torus. Over this one has a bundle of Hilbert spaces (fin. diml.) with endomorphism $H(p)$. Want to look at the low energy spectrum of $H(p)$ as p varies over the torus. You would like to see a non-trivial spectral flow, i.e. eigenvalues moving upward. This is impossible with finite-diml fibres, + was discovered by some Danes. Friedan gives diff form proof. Let P be projector on energies $\geq c$; he considers $\int_{\text{torus}} tr\, P[\nabla, P]^{odd}$ as the spectral flow and shows it is zero. Then he wants to remove assumptions about translation invariance so goes back to Hilbert space over lattice. Here he considers same projector (energies $\geq c$), ∇ is replaced by x = position, and he forms

$$Tr\,(P\underbrace{[x,P]^{odd}}_{\text{skew symmetrized over different coords.}})$$

which somehow measures the spectral flow.