January 4, 1986

Consider a $\mathbb{C}$-valued Gaussian process z_t. It is the same thing as a pair x_t, y_t of real valued Gaussian processes, jointly Gaussian: $z_t = x_t + iy_t$. We have equivalent conditions

1) The transformation $\begin{matrix} x_t \longmapsto -y_t \\ y_t \longmapsto +x_t \end{matrix}$ preserves the variance:

$$\langle x_t x_{t'} \rangle = \langle y_t y_{t'} \rangle$$

$$\langle x_t y_{t'} \rangle = -\langle y_t x_{t'} \rangle$$

2) $\langle z_t z_{t'} \rangle = 0.$

In effect $\uparrow = \langle x_t x_{t'} \rangle - \langle y_t y_{t'} \rangle + i\left(\langle x_t y_{t'} \rangle + \langle y_t x_{t'} \rangle \right).$

Let's form the real vector space spanned by the random variables x_t, y_t, ~~and~~ equip it with the inner product defined by the variance, ~~Condition 1) says that~~ and complete to obtain a real Hilbert space V. Condition 1) says V has a complex structure preserving the inner product (i.e. mult. by i is orthogonal with respect to the real inner product). We can then ~~define a he~~ extend the real inner product to a hermitian inner product with the same norm^2 by

$$\langle f | g \rangle = \langle fg \rangle - i \langle f Jg \rangle$$

$$\left(\langle f | Jg \rangle = \langle f Jg \rangle - i \langle f J^2 g \rangle \right.$$

$$\left. i \langle f | g \rangle = i \langle fg \rangle + \langle f Jg \rangle \right)$$

On the other hand let W be the complex Hilbert space obtained by taking complex linear combinations of the z_t with herm. inner product defined by $\langle \bar{z}_t z_{t'} \rangle$ and completing

We define a map from W to W by sending $w = \int f(t) z_t \, dt$ into

$$w + \bar{w} = \int \left(f(t) z_t + \overline{f(t)} \, \bar{z}_t \right) dt$$

This is a real linear combination of the x_t, y_t. Since

$$\langle (w + \bar{w})^2 \rangle = 2 \langle |w|^2 \rangle$$

if we use the map $w \longmapsto \frac{w + \bar{w}}{\sqrt{2}}$ we get an isometry of W with V. Formulas

$$z_t \longmapsto \frac{z_t + \bar{z}_t}{\sqrt{2}} = \sqrt{2} \, x_t$$

$$i z_t \longmapsto \frac{i z_t - i \bar{z}_t}{\sqrt{2}} = \frac{i 2 i y_t}{\sqrt{2}} = \sqrt{2}(-y_t)$$

shows that this map is compatible with the complex structures, if J on V is defined by $J x_t = -y_t$.

This is still a bit confused. However you do end up with the good notion of complex Gaussian process, namely are z_t such that $\langle z_t z_{t'} \rangle = 0$. What perhaps is missing is the idea that the path space for the process is a complex vector space of paths in $\mathbb{C}$ and the covariance is a hermitian inner product.

Question: Can these processes fill the gap between Dym McKean and DeBranges?

January 5, 1986

The problem is to construct some sort of theory for Gaussian integrals with complex exponent.

The exponent should be a quadratic form on a real vector space with positive definite real part. (Recall Siegel UHP consists of symmetric complex matrices with positive-definite imaginary part.) Examples:

① $$\int e^{-\frac{a}{2}x^2}\,dx \qquad Re(a) > 0.$$

$$\cancel{\int \mathcal{D}q\,\mathcal{D}q\; e^{\int\left(i p\dot{q} - \frac{p^2}{2} - V(q)\right)dt} = \int \mathcal{D}q\,\mathcal{D}p\; e^{\int (ip\dot{q} - H)\,dt}}$$

$$\cancel{= \int \mathcal{D}q\; e^{\int -}}$$

② $$\int \mathcal{D}q\,\mathcal{D}p \;\; e^{\int i p\,dq \,-\, H\,dt} \qquad H = \frac{p^2}{2} + V(q)$$

$$= \int \mathcal{D}q \;\; e^{\int -\frac{(dq)^2}{2dt} - V(q)\,dt} \qquad \text{Put } dt = a\,d\tau$$

$$= \int \mathcal{D}q \;\; e^{-\int\left[\frac{1}{a}\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + a\,V(q)\right]d\tau}$$

Here if $a = i$ we have real time + e^{-itH}

if $a = 1$ ——— imag. ——— $e^{-\tau H}$

Thus the problem includes constructing some sort of theory of path integrals for solving the equation

$$\partial_t u = a\,\partial_x^2 u$$

where $Re(a) > 0$.

Here's a possible approach. Let's consider

$$\int e^{-\frac{\sigma}{2}x^2}\,dx$$

with $Re(\sigma) > 0$. As is standard we keep track of this thing via its generating function

$$\int e^{-\frac{\sigma}{2}x^2 + iJx}\,dx = \frac{1}{\sqrt{2\pi\sigma}}\, e^{-\frac{J^2}{2\sigma}}$$

Now it seems desirable that σ should be allowed to specialize to purely imaginary values. So I think I want to restrict J to be real. I should think of having a real vector space and of constructing some kind of distributions on it which have Fourier transforms.

Next one notes that the Gaussian functions $e^{-\frac{\sigma}{2}x^2}$ $Re(\sigma) > 0$ are described nicely as follows. Among complex linear combinations of $q = x$, $p = \frac{1}{i}\partial_x$ are those α with $[\alpha, \alpha^*] > 0$, and such an α can be rescaled to be a destruction operator. $e^{-\frac{\sigma}{2}x^2}$ is just a ground state for a destruction operator:

$$e^{-\frac{\sigma}{2}x^2} \text{ is killed by } \alpha = ip + \sigma q$$

$$[\alpha, \alpha^*] = [ip + \sigma q, -ip + \bar{\sigma} q] = \bar{\sigma} + \sigma = 2\,Re(\sigma)$$

The generating function we are after is proportional to

$$(*) \quad \langle e^{-\frac{\tau}{2}x^2} \,|\, e^{iJx} \,|\, e^{-\frac{\sigma}{2}x^2} \rangle$$

in the limit as τ approaches 0.

So this suggests that I try to develop

a theory based upon evaluating ⊛. I should be thinking of Gaussian states on [redacted] a Weyl algebra.

Let's evaluate the analogue of ⊛ in the holomorphic representation:

$$\langle e^{\mu \frac{z^2}{2}} \,|\, e^{\gamma a^* - \bar{\gamma} a} \,|\, e^{\lambda \frac{z^2}{2}} \rangle / \text{norm}.$$

This is holomorphic in λ and $\bar{\mu}$, hence it suffices to do the evaluation for $\mu = \lambda$. Note

$$e^{\lambda \frac{z^2}{2}} \quad \text{is killed by} \quad a - \lambda a^*.$$

Let S be [redacted] a symplectic transformation such that

$$S|0\rangle = \text{const}\; e^{\lambda \frac{z^2}{2}}$$

Then

$$S\, a\, S^{-1} = \text{const} \quad a - \lambda a^*$$

$$S\, a^*\, S^{-1} = \text{const} \quad a^* - \bar{\lambda} a$$

$$S \begin{pmatrix} a \\ a^* \end{pmatrix} S^{-1} = \frac{1}{\sqrt{1-|\lambda|^2}} \begin{pmatrix} 1 & -\lambda \\ -\bar{\lambda} & 1 \end{pmatrix} \begin{pmatrix} a \\ a^* \end{pmatrix}$$

$$\begin{pmatrix} a \\ a^* \end{pmatrix} = \frac{1}{\sqrt{1-|\lambda|^2}} \begin{pmatrix} 1 & -\lambda \\ -\bar{\lambda} & 1 \end{pmatrix} S^{-1} \begin{pmatrix} a \\ a^* \end{pmatrix} S$$

$$\Rightarrow \quad S^{-1} \begin{pmatrix} a \\ a^* \end{pmatrix} S = \frac{1}{\sqrt{1-|\lambda|^2}} \begin{pmatrix} 1 & \lambda \\ \bar{\lambda} & 1 \end{pmatrix} \begin{pmatrix} a \\ a^* \end{pmatrix}$$

and

$$\frac{\langle e^{\lambda \frac{z^2}{2}} \,|\, e^{\gamma a^* - \bar{\gamma} a} \,|\, e^{\lambda \frac{z^2}{2}} \rangle}{\text{norm}} = \langle 0 | S^{-1} e^{\gamma a^* - \bar{\gamma} a} S | 0 \rangle$$

$$= \langle 0 | e^{\gamma \frac{1}{\sqrt{1-|\lambda|^2}} (\bar{\lambda} a + a^*) - \bar{\gamma} \frac{1}{\sqrt{1-|\lambda|^2}} (a + \lambda a^*)} | 0 \rangle$$

$$= \langle 0 | e^{\frac{1}{\sqrt{2}}[(\gamma-\bar\gamma\lambda)a^* - (\bar\gamma-\gamma\bar\lambda)a]} | 0 \rangle$$

$$= e^{-\frac{1}{2}\frac{(\gamma-\bar\gamma\lambda)(\bar\gamma-\gamma\bar\lambda)}{1-|\lambda|^2}}$$

$$= e^{-\frac{1}{2}\left\{|\gamma|^2\frac{1+|\lambda|^2}{1-|\lambda|^2} - \gamma^2\bar\lambda - \bar\gamma^2\lambda\right\}}$$

More generally

$$\boxed{\left\langle e^{\mu\frac{z^2}{2}} \middle| e^{\gamma a^* - \bar\gamma a} \middle| e^{\lambda\frac{z^2}{2}} \right\rangle \Big/_{norm} = e^{-\frac{1}{2}\frac{(\bar\gamma-\gamma\bar\mu)(\gamma-\bar\gamma\lambda)}{1-\bar\mu\lambda}}}$$

and if γ is real this becomes

$$e^{-\frac{1}{2}\gamma^2\frac{(1-\bar\mu)(1-\lambda)}{1-\bar\mu\lambda}}$$

$e^{\lambda\frac{z^2}{2}}$ killed by $a-\lambda a^* = \frac{1}{\sqrt{2\omega}}[(\omega q + ip) - \lambda(\omega q - ip)]$

" " $i(1+\lambda)p + (1-\lambda)\omega q$

" " $\frac{d}{dx} + \frac{(1-\lambda)}{1+\lambda}\omega x$

$$\therefore\ e^{\lambda\frac{z^2}{2}} \longleftrightarrow e^{-\frac{1-\lambda}{1+\lambda}\omega\frac{x^2}{2}}$$

Thus $\lambda = 1$ corresponds to the function 1

On the other hand $a^*+a = \sqrt{2\omega}\, q$, so that for γ purely imaginary $\gamma a^* - \gamma a$ is iyq, $y \in \mathbb{R}$.

So taking $\mu = 1$, γ purely imaginary in the formula outlined above get

$$e^{-\frac{\bar\gamma\gamma(1+\lambda)}{1-\lambda}} = e^{-|\gamma|^2\frac{1+\lambda}{1-\lambda}}$$

which checks as $\frac{1+\lambda}{1-\lambda}$ is any point in the UHP.

The formula above has exponent involving the mixed terms $\bar\lambda\gamma^2, \lambda\bar\gamma^2$. It's nicer to have cases where these don't occur.

Work with two sets a, a^*, b, b^* of ann. + creation operators and consider

$$e^{\lambda z w} = e^{\lambda a^* b^*}|0\rangle$$

This is killed by $a-\lambda b^*$ and $b-\lambda a^*$. I look for a metaplectic S such that

$$S|0\rangle = \text{const}\; e^{\lambda z w}$$

$$S^{-1}(a-\lambda b^*)S = \text{const}\; a$$

$$S^{-1}(b-\lambda a^*)S = \text{const}\; b$$

$$S\begin{pmatrix} a \\ a^* \\ b \\ b^* \end{pmatrix}S^{-1} = \text{const}\begin{pmatrix} a-\lambda b^* \\ a^*-\bar\lambda b \\ b-\lambda a^* \\ b^*-\bar\lambda a \end{pmatrix} = \text{const}\begin{pmatrix} 1 & & & -\lambda \\ & 1 & -\bar\lambda & \\ & -\lambda & 1 & \\ -\bar\lambda & & & 1 \end{pmatrix}\begin{pmatrix} a \\ a^* \\ b \\ b^* \end{pmatrix}$$

or
$$S^{-1}\begin{pmatrix} a \\ a^* \\ b \\ b^* \end{pmatrix}S = \frac{1}{\sqrt{1-|\lambda|^2}}\begin{pmatrix} 1 & & & \lambda \\ & 1 & \bar\lambda & \\ & +\lambda & 1 & \\ \bar\lambda & & & 1 \end{pmatrix}\begin{pmatrix} a \\ a^* \\ b \\ b^* \end{pmatrix}$$

$$\langle 0|S^{-1}e^{ha^*-\bar h a}S|0\rangle = \langle 0|e^{h\left(\frac{a^*+\bar\lambda b}{\sqrt{\ }}\right)-\bar h\left(\frac{a+\lambda b^*}{\sqrt{\ }}\right)}|0\rangle$$

$$= \langle 0|e^{\frac{ha^*-\bar h a}{\sqrt{\ }}+\frac{-\lambda\bar h b^*+\bar\lambda h b}{\sqrt{\ }}}|0\rangle$$

$$= e^{-\frac{1}{2}\left(\frac{|h|^2}{1-|\lambda|^2}+\frac{|\lambda\bar h|^2}{1-|\lambda|^2}\right)}$$

$$= e^{-\frac{1}{2}|h|^2\left(\frac{1+|\lambda|^2}{1-|\lambda|^2}\right)}$$

It follows that

$$\boxed{\left\langle e^{\mu \bar{z} w} \middle| e^{h a^* - \bar{h} a} \middle| e^{\lambda z w} \right\rangle \Big/ \mathrm{norm} = e^{-\frac{1}{2}|h|^2 \left(\frac{1+\bar{\mu}\lambda}{1-\bar{\mu}\lambda}\right)}}$$

January 6, 1986

The problem is to construct some sort of theory of Gaussian integrals in infinite dimensions with non real exponents. A particular problem would be to generalize the link between Brownian motion and the heat equation to the equation

$$\partial_t u = a\,\partial_x^2 u$$

where a is complex and $\mathrm{Re}(a) > 0$.

The approach will be to fix a pair of real vector space V, V^* with a non-degenerate ~~pairings~~ $V \times V^* \longrightarrow \mathbb{R}$. I want to ~~think of~~ the Gaussian integral as being a distribution of seme sort on V which will be determined by its generating function, the latter being a Gaussian function on V^*. The problem is to specify what is meant by distribution. Presumably we mean a linear functional suitably continuous on a linear space $\mathcal{L}$ of functions on V which does not depend on the Gaussian integral. $\mathcal{L}$ should contain the exponentials $e^{i(v,\xi)}$ for $\xi \in V^*$, and more generally suitable cylinder functions.

When the covariance Q is positive we know how to construct the distribution as a Gaussian cylinder measure on the Hilbert space obtained by completing V. Moreover we know the structure of $L^2(V)_Q$. To get started let's suppose from the outset that V is a Hilbert space with dual V^* and that we want to consider all inner products on V consistent with the topology. Then we have a family of Hilbert spaces $L^2(V)_Q$ depending on the inner product. We know there is a canonical isomorphism

$$\begin{array}{ccc} \hat{S}(V^*) & \xrightarrow{\sim} & L^2(V)_Q \\ 1 & & 1 \\ a_v & \longleftrightarrow & \partial_v \\ a_v^* & \longleftrightarrow & -\partial_v + \hat{v} \end{array} \qquad e^{\lambda} \longleftrightarrow e^{\lambda - \frac{1}{2}|\lambda|^2}$$

which is unitary provided $\hat{S}(V^*)$ is equipped with the inner product defined by Q.

Now it seems that $\hat{S}_n(V^*)$, which we can think of as a space of polynomial functions on V, is independent as a top. v.s. of Q. Maybe $\hat{S}(V) = \hat{\bigoplus} \hat{S}_n(V^*)$ depends on Q even for simple rescaling. However the subspace consisting of series $\sum p_n$ where p_n decays exponentially in n should be intrinsic.

Unfortunately by the time an elt of $\hat{S}_2(V^*)$ gets into $L^2(V)_Q$, it is dependent on Q, even in finite dimensions. One does have an intrinsic map

$$S(V^*) \hookrightarrow L^2(V)_Q$$

just because polynomials are cylinder functions.

Recall that the goal is to find a linear space $\mathcal{L}$ containing cylinder functions such that all the integrals for different Q can be interpreted as linear functionals on $\mathcal{L}$. I can take $\mathcal{L} = S(V^*)$ or $\mathcal{L} =$ all cylinder functions, but I'd rather find something bigger.

One idea would be to try to enlarge from $S(V^*)$ to $\oplus \hat{S}_n(V^*)$, but this runs into trouble in degree 2. This is the usual Hilbert-Schmidt versus trace class stuff. In effect $\hat{S}_2(V^*)$ can be identified with Hilbert-Schmidt quadratic forms on V, i.e. $(Bv,v)_Q$ where B is symmetric relative to Q and Hilbert Schmidt. The integral gives $\mathrm{tr}(B)$.

A better way to do this is to fix an inner product on V call it $(\ )$, and then represent Q as (Av,v) with $\varepsilon \le A \le \frac{1}{\varepsilon}$ some ε. Then integrating the quadratic fn (Bv,v) wrt the Q-Gaussian measure gives $\operatorname{tr}(AB)$. Now recall that the predual of the von Neumann algebra of bounded operators on Hilbert space is the Banach space of trace class operators in the trace norm. Thus we can extend the integral on $S_2(V^*)$ to the trace class version.

Next let's conjecture that a suitable trace class version of $S_n(V^*)$, denote it $S_n(V^*)^{(1)}$ can be found. These are the homogeneous polynomial functions (in infinitely many variables) which can be integrated with respect to any of the Gaussian measures.

Then we have to add up the different $S_n(V^*)^{(1)}$ in some way, say by taking series $\sum p_n$ such that p_n in trace norm decays exponentially. This means exponentials e^λ are in $\mathcal{L}$. Functions like e^B, where B is quadratic and is sufficiently small in trace norm, are not in $\mathcal{L}$ because the suff. small depends on the Gaussian integral.

This brings up the problem with this approach, namely, the kind of functions in $\mathcal{L}$ are "analytic"

One thing we haven't used is that V is real so that among the exponentials are the bounded ones. Also nowhere does positivity of the covariance occur.

January 8, 1986

Let V be a real vector space, say finite dim'l. If $Q(x) = |x|^2$ is an inner product on V, we can form $L^2(V, \mu_Q)$, where $\mu_Q = e^{-\frac{1}{2}Q(x)}\ \frac{\text{Lebesgue}}{\text{norm.}}$ is the associated Gaussian measure, and we can also form the Hilbert space symmetric tensor product $\hat{S}(V^*)$. One has a canonical isomorphism

$$\hat{S}^{Q}(V^*) \xrightarrow{\sim} L^2(V, \mu_Q)$$

$^{\wedge Q}$ indicates dependence on Q

such that

$$e^{\lambda} \longmapsto e^{-\frac{1}{2}Q(\lambda)} e^{\lambda} \qquad \text{for } \lambda \in V^*.$$

(Here L^2 and $\hat{S}^{Q}(V^*)$ are real Hilbert spaces.)

Now $\hat{S}^{Q}(V^*)$ is naturally a representation of the Weyl algebra of $V \oplus V^*$, that is, in addition to multiplication by $\lambda \in V^*$, we also have differentiation operators ∂_v. We therefore see that there is a natural representation of $W(V + V^*)$ on $L^2(V, \mu_Q)$. Since we also have the cyclic vector $1 \in L^2(V, \mu_Q)$ we get a state on the Weyl algebra. In fact the representation is irreducible so the state is irreducible (or extreme, as in Krein-Milman).

I want to emphasize this viewpoint, namely that a Gaussian prob. measure on V is a state on the Weyl algebra $W(V \oplus V^*)$. It is an irreducible "Gaussian" state.

Now I know what the irreducible Gaussian states on the Weyl algebra are. In fact they form a Siegel upper half plane.

<u>January 9, 1986</u>

Given a real vector space V (fin diml) and inner product $(\ ,\ )$ we consider the Hilbert space $L^2(V, e^{-\frac{1}{2}|x|^2}dx/\text{norm})$. The operator ∂_v has adjoint

$$\partial_v^* = e^{\frac{1}{2}|x|^2}(-\partial_v)e^{-\frac{1}{2}|x|^2} = -\partial_v + \hat{v}$$

where $\hat{v} \in V^*$ is the linear fnl $(v, ?)$. Hence we have the "translation operator" which is skew-adjoint

$$\frac{1}{2}(\partial_v - \partial_v^*) = \partial_v - \frac{1}{2}\hat{v}.$$

So we get a representation of the CCR with

$$q_v = \text{mult. by } \hat{v}$$
$$p_v = \frac{1}{i}\left(\partial_v - \frac{1}{2}\hat{v}\right)$$

and the Hilbert space is a module over the Weyl algebra. (What one has done is to use the isomorphism

$$L^2(V,dx) \xrightarrow{\sim} L^2(V, e^{-\frac{1}{2}|x|^2}dx/\text{norm})$$
$$e^{-\frac{1}{4}|x|^2} \longleftrightarrow 1$$

i.e. we extract the positive square root of the density $e^{-\frac{1}{2}|x|^2}$.)

Actually the formulas look nicer with $\frac{1}{2}$ replaced by ω, so that

$$\partial_v^* = e^{\omega|x|^2}(-\partial_v)e^{-\omega|x|^2} = -\partial_v + 2\omega\hat{v}.$$

Do for $V = \mathbb{R}$: Ground state $|0\rangle = e^{-\frac{1}{2}\omega|x|^2}$ is killed by $\partial_x + \omega x$. Put

$$a = \frac{1}{\sqrt{2\omega}}(\partial_x + \omega x) \qquad a^* = \frac{1}{\sqrt{2\omega}}(-\partial_x + \omega x)$$

$$p = \frac{1}{i}\partial_x = \frac{a - a^*}{2i}\sqrt{2\omega} \quad \Big| \quad q = x = \frac{a + a^*}{\sqrt{2\omega}}$$

Then

$$sp + tq = s\,\frac{a-a^*}{2i}\sqrt{2\omega} + t\,\frac{a+a^*}{\sqrt{2\omega}}$$

$$= \left(i\sqrt{\tfrac{\omega}{2}}\,s + \frac{t}{\sqrt{2\omega}}\right)a^* + \left(-i\sqrt{\tfrac{\omega}{2}}\,s + \frac{t}{\sqrt{2\omega}}\right)a$$

and so the generating function for this representation of the Weyl algebra is

$$\langle 0|\,e^{i(sp+tq)}\,|0\rangle = e^{-\frac{1}{2}\left\| i\sqrt{\frac{\omega}{2}}\,s + \frac{t}{\sqrt{2\omega}}\right\|^2}$$

$$= e^{-\frac{1}{4}\left(\omega s^2 + \frac{1}{\omega}t^2\right)}$$ (observe this is holom in ω)

Recapitulate: In a natural way L^2 of a Gaussian probability measure on V is a module over $W(V \oplus V^*)$.

My viewpoint is that to think of a Gaussian prob. measure as an irred Gaussian state on the Weyl algebra is better, because the object is more rigid. This viewpoint may not be reasonable.

On p. 125 we worked out the general formula

$$\left\langle \Phi_\lambda \middle|\, e^{\gamma a^* - \bar{\gamma} a} \,\middle| \Phi_\lambda \right\rangle = e^{-\frac{1}{2}\frac{|\gamma - \lambda\bar{\gamma}|^2}{1-|\lambda|^2}}$$

where $\Phi_\lambda = e^{\lambda\frac{z^2}{2}}/\text{norm}$. Here λ ranges over the disk $|\lambda| < 1$, and the 'variance' is the real quadratic form on the space of skew-adjoint ~~operators~~ linear combinations of a, a^* given by

$$\gamma a^* - \bar{\gamma} a \longmapsto \frac{|\gamma - \lambda\bar{\gamma}|^2}{1-|\lambda|^2}$$

I want now to understand this whole business better. First of all we have the real vector space, call it W, consisting of self adjoint operators $\gamma^t a^* + \bar{\gamma}^t a$ which are linear combinations of the operators a_i^*, a_i. On W we have the symplectic form given by commutator

$$\frac{1}{i}\left[\beta^t a^* + \bar{\beta}^t a, \gamma^t a^* + \bar{\gamma}^t a\right] = \frac{1}{i}(\beta^*\gamma - \gamma^*\beta)$$
$$= 2\, \mathrm{Im}\, \beta^*\gamma$$

Now a point of the Siegel UHP (or disk) can be identified with another choice of annihilation operators. Thus we seek a subspace M of W_c which is max. isotropic for the commutator and on which $[m, m^*] > 0$ for $m \neq 0$, $m \in M$. If M is spanned by $a_i + \lambda_{ik} a_k^*$ we see

$$0 = [a_i + \lambda_{ik} a_k^*, a_j + \lambda_{jk} a_k^*] = \lambda_{ji} - \lambda_{ij}$$

so λ_{ij} is symmetric, and the matrix

$$[a_i + \lambda_{ik} a_k^*, a_j^* + \bar{\lambda}_{jk} a_k] = \delta_{ij} - \lambda_{ik}\bar{\lambda}_{jk}$$

is positive-definite. $\therefore\ \lambda\lambda^* < 1$. Write $M = M_\lambda$

Associated to such an M is a positive quadratic form on W, namely $w \mapsto \langle \Phi_M | w^2 | \Phi_M \rangle$ where Φ_M is the state killed by M. To compute this use

$$W_c = M \oplus \bar{M}$$

to write $w \in W$ as

$$w = m + m^*.$$

Then $\langle \Phi_M | w^2 | \Phi_M \rangle = \langle \Phi_M | m m^* | \Phi_M \rangle$

$$= \langle \Phi_M | [m, m^*] | \Phi_M \rangle = [redacted] [m, m^*]$$

Thus you get a quadratic form on W by splitting w into its destruction and creation parts m, m^* and taking $[m, m^*]$.

Now I want to [redacted] understand exactly what quadratic forms on W are obtained in this way. In other words we are given a real symplectic vector space W and then we want to [redacted] investigate if [redacted] it comes from an M max isotropic in W_c $\ni$ $[m, m^*] > 0$ for $m \neq 0$, and if [redacted] so, how many M occur.

The idea is to look at the operator relating the quadratic and skew forms:

$$(w, w') = \frac{1}{i}[w, Tw']$$

$$\| \qquad\qquad\qquad \|$$

$$\langle w w' + w' w \rangle \qquad \frac{1}{i}[m + m^*, T(m' + m'^*)]$$

$$\|$$

$$[m + m', m^* + m'^*] - [m, m^*] - [m', m'^*]$$

$$\|$$

$$[m, m'^*] + [m', m^*] = [m, m'^*] - [m^*, m']$$

Thus

$$T(m' + m'^*) = i m'^* - i m'$$
$$= -i m' + i m'^*$$

so

$$T = \begin{cases} -i & \text{on } M \\ +i & \text{on } \overline{M} \end{cases}$$

and we see that $T^2 = -1$.

I conclude that the quadratic forms that occur ones whose associated operator T satisfies $T^2 = -1$. Also T preserves the symplectic forms.

$$[Tw, Tw'] = [-im + im^*, -im' + im'^*]$$

$$= [m, m'^*] + [m^*, m'] = [w, w'].$$

Conversely suppose T is a symplectic automorphism of W_c such that $T^2 = -1$. Then $W_c = M \oplus N$ where $T = -i$ on M and $T = +i$ on N. One has

$$[Tw, Tw'] = [w, w']$$

so M, N are isotropic. If we define $(\ ,\ )$ by

$$(w, w') = \frac{1}{i}[w, Tw']$$

$$= \frac{1}{i}[m+n, -im' + in']$$

$$= [m, n'] + [m', n]$$

we get a symmetric form. It is the hyperbolic form associated to the pairing $(m, n) \longmapsto [m, n]$ of $M \times N \longrightarrow \mathbb{C}$ which as non-degenerate as

$$[w, w'] = [m, n'] + [n, m'] = [m, n'] - [m', n]$$

is non-degenerate.

So I learn the following. Given W_c symplectic there is an equivalence between decompositions $W_c = M \oplus N$ with M, N isotropic, and between non degenerate quadratic forms on W_c represented by T symplectic with $T^2 = -1$.

January 10, 1986

Let W, Ω be a symplectic vector space (over $\mathbb{R}$ or $\mathbb{C}$). Then there is an equivalence between

1) symm. bilinear forms $(\ ,\ )$ on W
2) operators X on W such that

$$\circledast \qquad \Omega(Xw, w') + \Omega(w, Xw') = 0$$

The equivalence is given by

$$(w, w') = \Omega(Xw, w')$$

Also X gives the Hamiltonian flow assoc. to $H(w) = \frac{1}{2}(w,w)$. Clearly $(\ ,\ )$ is non-degenerate $\iff$ X is nonsingular.

For purposes of Q.M. we are interested in X such that $X^2 = -1$. Over $\mathbb{C}$, there is an equivalence between

1) X such that $X^2 = -1$ and X satisfies $\circledast$
2) Decompositions $W = W^+ \oplus W^-$ where W^+, W^- are isotropic relative to Ω.

The equivalence is given by associating to X the eigenspaces $W^\pm =$ where $X = \pm i$.

In the real case, X defines a complex structure on W and one obtains a hermitian form by

$$(w|w') = (w, w') + i\,\Omega(w, w')$$

<u>January 13, 1986</u>

Let V be a real vector space equipped with an inner product $(\ ,\ )$ and a skew-symmetric form $\Omega(\ ,\ )$. Using the skew-symmetric form we can construct the Weyl algebra $\mathrm{Weyl}(V,\Omega)$. For example, when Ω is zero, this algebra is commutative.

<u>Proposition</u>: There is a state φ on the Weyl algebra such that

$$\varphi(e^{iv}) = e^{-\frac{1}{2}(v,v)}$$

iff the skew-symmetric operator [redacted] K on V defined by

$$\Omega(v,v') = ([redacted]\, v, Kv')$$

satisfies

$$-K^2 \leq I.$$

To prove this we will prove:

<u>Lemma</u>: Let V, $(\ ,\ )$, Ω be as above. Then there exists an isometry $j: V \longrightarrow W$, where W is a [redacted] real vector space with [redacted] inner product and a complex structure J on W ($J^2 = -I$, and J is an isometry) such that $j^* J j = K$:

$$\Omega(v,v') = (jv, Jjv') \qquad \forall v, v' \in V.$$

Moreover if $W = jV + JjV$, then (W, J) is unique up to canonical isomorphism.

Proof: Observe [redacted] the inner product on the complex subspace $jV + JjV$ of W generated by jV is determined by $(\ ,\ )$ and $\Omega(\ ,\ )$:

(write $Jv = v$)

$$|v + Jv'|^2 = |v|^2 + (v, Jv') + (Jv', v) + |Jv'|^2$$
$$= |v|^2 + \Omega(v, v') + \Omega(v, v') + |v'|^2$$

Moreover we can use this formula to define an inner product on formal combinations $v + Jv'$ since

$$|v|^2 + 2\Omega(v, v') + |v'|^2 \geq |v|^2 - 2|v||v'| + |v'|^2 \geq 0.$$

The rest is clear. (Left out $|\Omega(v, v')| = |(v, Kv')| \leq |v| \cdot \|K\| |v'| \leq |v||v'|$)

For example, when $\Omega = 0$, we get simply the complex vector space $W = V \otimes \mathbb{C}$ with hermitian inner product associated to the Euclidean space V.

The existence part of the proposition is clear now because we take the irreducible representation of Weyl(W) with ground state $|0\rangle$ killed by $\frac{1}{2}$ of the complexification $W \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} W \oplus \overline{W}$ which is appropriate. This ~~state~~ has the Gaussian generating function

$$\langle 0 | e^{iw} | 0 \rangle = e^{-(?)|w|^2}$$

with variance determined by the inner product on W.

The only if part of the proposition should have a simple proof, which I will think about.

But the picture I want to think of is having a real vector space V with inner product and skew form. The skew form doesn't have to be non degenerate. I am reminded of having a line bundle over V with connection and constant curvature. Then I have translation operators, in fact two sets acting on sections of the line bundle. The two sets commute with each other but not with themselves. The quadratic form leads to a heat operator whose Weyl transform is a ~~state~~ which is Gaussian. (?)

Now let's turn to Gaussian measures. An inner product on V determines a Gaussian measure $d\mu$ on V^* whose variance is the inner product. The Hilbert space $L^2(V^*, d\mu)$ is the representation.

January 14, 1986

Let W be a complex Hilbert space with the hermitian inner product $(w|w')$; this is linear in w'. Then we can form the Fock space $\hat{S}(W)$ with operators a_w^*, a_w, where a_w^* is linear in w and a_w is anti-linear, satisfying

$$[a_w^*, a_{w'}^*] = [a_w, a_{w'}] = 0 \qquad [a_w, a_{w'}^*] = (w|w')$$

We have translation operators

$$T_w = e^{a_w^* - a_w} = e^{-\frac{1}{2}|w|^2} e^{a_w^*} e^{-a_w}$$

satisfying

$$T_w T_{w'} = e^{\frac{1}{2}[a_w^* - a_w, a_{w'}^* - a_{w'}]} T_{w+w'}$$

$$= e^{\frac{1}{2}\{-(w|w') + (w'|w)\}} T_{w+w'}$$

$$\boxed{T_w T_{w'} = e^{-i \operatorname{Im}(w|w')} T_{w+w'}}$$

also

$$\boxed{\langle 0 | T_w | 0 \rangle = e^{-\frac{1}{2}|w|^2}}$$

Next consider a real subspace V of W which generates W. ~~The Weyl algebra of W(V)~~

V inherits from W an inner product

$$(v, v') = \operatorname{Re}(v|v')$$

and a skew form with imaginary values $\Omega(v, v')$ given by

$$\Omega(v, v') = -2i \operatorname{Im}(v|v')$$

i.e.

$$\Omega(v, v') = [a_v^* - a_v, a_{v'}^* - a_{v'}]$$

One has

$$T_v T_{v'} = e^{\frac{1}{2}\Omega(v,v')} T_{v+v'}.$$

Think of Ω as curvature. Note that by the way we have set things up we have

$$|\Omega(v,v')| = 2|\mathrm{Im}\,(v|v')| \leq 2|v||v'|.$$

Let $\mathrm{Weyl}(V)$ be the C^* algebra generated by the operators T_v. Because V generates W it follows that the representation of $\mathrm{Weyl}(V)$ on Fock space $\hat{S}(W)$ is cyclic with generating vector $|0\rangle$ and generating function

$$\langle 0|T_v|0\rangle = e^{-\frac{1}{2}|v|^2}$$

On the other hand suppose given V real Hilb: with inner product $(\ ,\ )$ and imaginary-valued skew form $\Omega(\ ,\ )$ related by

$$\left|\tfrac{1}{2}\Omega(v,v')\right| \leq |v|\,|v'|.$$

Then we have seen how to reconstruct, uniquely up to canonical isomorphism a complex Hilbert space W generated by V such that

$$(v,v') = \mathrm{Re}\,(v|v')$$

$$\Omega(v,v') = -2i\,\mathrm{Im}\,(v|v').$$

Hence we know how to realize the function $v \mapsto e^{-\frac{1}{2}|v|^2}$ by means of a cyclic representation of $\mathrm{Weyl}(V,\Omega)$.

Special case where $\Omega = 0$.

Here $W = V \oplus iV = V \otimes \mathbb{C}$ and the hermitian inner product is the natural extension of the inner product given on V. In finite dimensions we also can form the Gaussian measure on V^* with variance $|v|^2$ on V, and this gives a cyclic representation of $\mathrm{Weyl}(V)$ with generating fn $e^{-\frac{1}{2}|v|^2}$.

<u>January 15, 1986</u>

KMS condition. This links a state φ and a one parameter group of automorphisms α_t of an algebra and says:

$$\varphi(\alpha_t(A)\,B) \quad \text{can be holomorphically}$$

continued to the strip $0 \leq \mathrm{Im}(t) \leq 1$ and

$$\varphi(\alpha_i(A)\,B) = \varphi(B\,A)$$

(Hence $\varphi(\alpha_{t+i}(A)\,B) = \varphi(B\,\alpha_t(A))$ which looks more like symmetry of the trace.)

For example consider a matrix algebra and $\varphi(A) = \mathrm{tr}(e^{-H}A)$. Write $\alpha_t(A) = e^{-itK}A\,e^{+itK}$. The KMS condition says

$$\mathrm{tr}(e^{-H}e^{K}A\,e^{-K}B) = \mathrm{tr}(e^{-H}BA)$$

for all A, B so

$$e^{-H}e^{K}A\,e^{-K} = A\,e^{-H}$$

or

$$e^{-H}e^{K}A = A\,e^{-H}e^{K}$$

for all A so K differs from H by a constant. Thus KMS implies $\alpha_t(A) = e^{-itH}A\,e^{itH}$.

Let us now consider a Gaussian state on a Weyl algebra and let us determine the corresponding ~~[illegible]~~ 1-parameter group α_t. We suppose that the Weyl algebra is associated to a real symplectic vector space (V, Ω). The Gaussian state determines an inner product on V. We now choose a complex structure on V as follows. The quadratic form given by $\frac{1}{2}$ the inner product can be viewed as a Hamiltonian, and we take the unique complex structure such that all the frequencies of the Hamiltonian flow are > 0. Using this complex structure we ~~[illegible]~~ ~~[illegible]~~ extend the symplectic form to a hermitian inner product.[1] The original inner product can then be written $(v|\rho v)$ where ρ is self-adjoint and ≥ 1.

These arguments prove the following:

<u>Prop</u>: Let φ be a Gaussian state on the Weyl alg. $Weyl(V)$ associated to a symplectic vector space V. ($Weyl(V)$ is generated by unitaries $W(v)$ ∋

$$W(v)\,W(v') = W(v+v')\; e^{\frac{1}{2}\Omega(v,v')}$$

where the symplectic form is $\frac{1}{i}\Omega(v,v')$.) Then there is a unique complex structure on V and hermitian inner product such that

$$\Omega(v,v') = -(v|v') + (v'|v) = -2i\,\mathrm{Im}(v|v')$$

(This comes from $W(v) = e^{a^*_v - a_v}$

$$\Omega(v,v') = [a^*_v - a_v,\; a^*_{v'} - a_{v'}] = -(v|v') + (v'|v)\;)$$

and such that

$$\varphi(W(v)) = e^{-\frac{1}{2}(v|\rho|v)}$$

where ρ is complex linear, hermitian and $\rho \geq 1$.

With this description of the Weyl algebra and state, let's now determine the α_t linked to this state by the KMS condition. We will assume that α_t comes from a 1-parameter group, also denoted α_t of symplectic autos of V.

$$\varphi\left(\alpha_t(W(v))\,W(v')\right) = \varphi\left(W(\alpha_t v)\,W(v')\right)$$

$$= e^{\frac{1}{2}\Omega(\alpha_t v, v')}\; e^{-\frac{1}{2}(\alpha_t v + v' | \rho | \alpha_t v + v')}$$

$$\varphi\left(W(v')\,\alpha_t(W(v))\right) = e^{\frac{1}{2}\Omega(v', \alpha_t v)}\; e^{-\frac{1}{2}(\alpha_t v + v' | \rho | \alpha_t v + v')}.$$

When the former is analytic continued to $t=i$ its supposed to be the same as the latter at $t=0$. Look at the exponent of the former $\times -2$:

$$(\alpha_t v|\rho|\alpha_t v) + (\alpha_t v|\rho|v') + (v'|\rho|\alpha_t v) + (v'|\rho|v')$$
$$+ (\alpha_t v|v') - (v'|\alpha_t v)$$

~~The KMS condition is $\varphi(\alpha_t(A))$~~

$$= (v|\rho|v) + (v|\rho\alpha_{-t}|v') + (v'|\rho\alpha_t|v) + (v'|\rho|v')$$
$$+ (v|\alpha_{-t}|v') - (v'|\alpha_t|v)$$

where I assume α_t preserves the state φ as well as Ω, hence α_t must be $\mathbb{C}$-linear and unitary. This analytically continues to

$$(v|\rho|v) + (v|\rho\alpha_{-i}|v') + (v'|\rho\alpha_i|v) + (v'|\rho|v')$$
$$+ (v|\alpha_{-i}|v') - (v'|\alpha_i|v).$$

The KMS condition implies this is to be

$$(v+v'|\rho|v+v') + (v'|v) - (v|v').$$

So looking at the $(o'|\,?\,|o)$ part we see that

$$\rho\alpha_i - \alpha_i = \rho + 1$$

or
$$\rho = \frac{\alpha_i + 1}{\alpha_i - 1}$$
, so that if α_t is given by e^{itH} on V, then

$$\boxed{\rho = \frac{e^H + 1}{e^H - 1}}$$

KMS link between generator H of the modular auto gp and the state ρ in the boson case

Now I recall the formula for the thermal state of the simple harmonic oscillator:

$$\circledast \quad \mathrm{tr}\left(e^{-\omega a^* a}\, e^{\gamma a^* - \bar\gamma a}\right)/\mathrm{norm} = e^{-\frac{1}{2}|\gamma^2|\left(\frac{e^\omega + 1}{e^\omega - 1}\right)}$$

This checks if we remember

$$\alpha_t\left(e^{\gamma a^* - \bar\gamma a}\right) = e^{-it(\omega a^* a)}\, e^{\gamma a^* - \bar\gamma a}\, e^{+it(\omega a^* a)}$$
$$= e^{(e^{-it\omega}\gamma)a^* - (e^{+it\omega}\bar\gamma)a}$$

so indeed $\alpha_i(\gamma) = e^{\omega}\gamma$.

We see in this example that the generator H of the [redacted] "modular automorphism group" α_t satisfies $H > 0$. (Note: $\rho > 1$, otherwise H doesn't exist.)

<u>Remark</u>: It would seem that one [redacted] might reverse the argument whereby we found H from ρ to actually prove $\circledast$. ?

Next let's discuss the fermion case. Given a complex Hilbert space V we can form the fermion Fock space $\hat{\Lambda} V$ with operators a_v (anti linear in v) and a_v^* (linear in v). The self adjoint operators $a_v + a_v^*$ satisfies

$$(a_v + a_v^*)^2 = |v|^2$$

and so generate a Clifford algebra. The vector $|0\rangle$ gives rise to a <u>state</u> on the Clifford algebra $C(V)$, which ought to be determined by the skew form

$$\langle 0 | [a_v + a_v^*, a_{v'} + a_{v'}^*]_- | 0 \rangle$$

$$= \langle 0 | a_v a_{v'}^* - a_{v'} a_v^* | 0 \rangle$$

$$= (v|v') - (v'|v) \quad = 2i \,\mathrm{Im}(v|v')$$

Thus we end up with the same kind of linear algebra as in the bosonic case. In effect the above representation of $C(V)$ can be restricted to a subspace W of V. W inherits the inner product, which determines the Clifford algebra $C(W)$, and the skew-form which (ought to) determine the state. Thus to a W, $(\,,\,)$, and $\Omega(\;)$ satisfying $\left|\frac{1}{2}\Omega(w,w')\right| \leq |w||w'|$ we should have associated a *cyclic* representation of $C(W)$.

Now we can put a complex structure on W consistent with the inner product such that the operator K representing Ω is $\mathbb{C}$-linear with eigenvalues in $i\mathbb{R}_{>0}$. Of course we have that the eigenvalues are in $i\,(0,1]$. (assuming Ω non-degenerate).

This gives the analogue of ρ before. I will work out for the fermion oscillator the state and use the formula to guess the general form of α_t.

We consider the state $tr(e^{-\omega a^*a}\ ?)/norm.$ The skew form evaluated on the obvious orthonormal basis of V is

$$tr(e^{-\omega a^*a}[a^*+a,\ ia^*-ia]_-)/norm$$

$$= i\langle -a^*a + aa^* + aa^* - a^*a\rangle$$

$$= 2i\langle 1-2a^*a\rangle = 2i\left\{1-2\frac{e^{-\omega}}{1+e^{-\omega}}\right\} = 2i\left\{\frac{e^{\omega}-1}{e^{\omega}+1}\right\}$$

Thus it appears that the state is to be determined in terms of the generator H of α_t by

$$\boxed{\rho = \frac{e^H-1}{e^H+1}}$$

KMS link between the generator ~~[illegible]~~ H of the modular auto group and the state ρ in the fermion case

Since $0<\rho\leq 1$ by assumption we see that $H\geq 0$ and that H exists and is unique when $\rho<1$.

Rmk: Curiously the boson + fermion formulas are essentially the same. In the former one normalizes the skew form to 1 and then the inner product is given by $\frac{e^H+1}{e^H-1}>1$. In the latter one normalizes the inner product to 1 and then the skew form is given by $\frac{e^H-1}{e^H+1}<1$. Clearly these are the same formulas in different normalizations.

<u>January 16, 1986</u>

I have been working on the problem of finding an adequate foundation for Gaussian integrals in infinite dimensions where the variance is complex valued, (say with positive real part although eventually I would like to have purely imaginary ~~exponent~~ variance.)

Here is a specific problem. The theory of Wiener measure gives a Gaussian probability measure on the Banach space X of continuous paths in $\mathbb{R}$, x_t $t\in[0,1]$ with $x_0 = 0$. One has the variance

$$\langle x_t x_{t'} \rangle = \min(t,t')$$

If one applies the homothety $x_t \longmapsto h x_t$ on X one obtains a family of Gaussian probability measures μ_h on X for $h>0$. A natural question is whether we can analytically continue this family for $\mathrm{Re}(h) > 0$ in some way. For example given a bounded continuous function f on X does

$$h \longmapsto \int f \mu_h$$

admit an analytic continuation to the RHP.

We see from this discussion (which applies to any Gaussian probability measure on a real vector space) that whatever an integral is it is a linear functional on some space, which is usually a commutative algebra. A Gaussian integral is associated to a real vector space V with a ~~va~~variance quadratic form, and one thinks of the

Gaussian integral as being defined over V^*. We have

$$\langle e^{i\sigma} \rangle = e^{-\frac{1}{2}\langle \sigma^2 \rangle}$$

so the space of test functions for the integral is to include the real exponentials $e^{i\sigma}$.

I think it is important to adopt the viewpoint of the theory of distributions. The integral is to be a suitable linear fnl on a space of test functions. We have to take this viewpoint even in finite dimensions, if we want to handle the case of purely imaginary variances.

<u>January 17, 1986</u>

I have decided to adopt a distribution theory viewpoint towards Gaussian integrals in infinite dimensions. This means that I need to find a space of test functions and then the integral will be constructed as a linear functional on the space of test functions.

Let me recall also a key question. Let μ be a Gaussian probability measure on a real Hilbert space W^*, i.e. the image of the Gaussian cylinder measure on V under a Hilbert Schmidt map $V^* \hookrightarrow W^*$ with dense image. Then we get a family μ_h of such measures for $h > 0$. Any bounded continuous f on W^* is integrable for all μ_h and the question is whether $\int f \, d\mu_h$

can be analytically continued to the RHP.
For example if $f = e^{i w}$, then

$$\int e^{iw}\, d\mu_h = e^{-\frac{1}{2} h^2 |w|_V^2}$$

This shows we only [redacted] expect a continuation for $|\arg(h)| < \frac{\pi}{4}$, and maybe things are quite subtle.

We have in any case a family of norms [redacted] $h^2|v|^2$ on V and we can construct a family of Hilbert spaces $\mathcal{H}_h$. $\mathcal{H}_h = L^2$ of V^* with resp. to the G. cylinder measure with variance $h^2|v|^2$. We know $\mathcal{H}_h$ [redacted] admits commuting unitary operators e^{iv}, $v \in V$, and a cyclic vector for these operators $|0\rangle_h$ such that

$$_h\langle 0|e^{iv}|0\rangle_h = e^{-\frac{1}{2}h^2|v|^2}.$$

So we have some sort of family of unitary representations. But something else is true, namely we have a way of relating elements of different Hilbert spaces, that is, we have certain sections of this family. For example [redacted] the family of cyclic vectors $|0\rangle_h$, [redacted] which all represent the function 1 on V^*. And when we have $V^* \hookrightarrow W^*$ Hilbert-Schmidt, then bounded continuous functions on W^* will give sections of this family.

This suggests that there might be some sort of connection on the family $\mathcal{H}_h$. It's not a unitary connection because the inner products of flat sections, such as $\langle 0|e^{iv}|0\rangle$, depend on h.

January 17, 1986 (cont.)

The idea: Consider the family of Gaussian cylinder measures μ_h on a real Hilbert space V^* ~~with variance~~ where μ_h has variance $h^2|v|^2$. To fix the ideas suppose V finite dimb. Then we have a family of Hilbert spaces $\mathcal{H}_h = L^2(V^*, \mu_h)$ each of which contains $S(V)$ as a dense subspace. This means that we have some sort of partially defined, better densely defined, operator

$$\begin{tikzcd} L^2(V^*, \mu_h) \arrow[rr, dashed] & & L^2(V^*, \mu_{h'}) \\ & S(V) \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

such that the triangle commutes. ~~On the other hand~~ What we can then do is to close up the operator (maybe) and ask for the subspace of $L^2(V^*, \mu_h)$ which is the intersection of the domains of all these operators. For example the exponential functions e^{iv} lie in this intersection.

But we also know that there is a standard isomorphism

$$(*) \qquad \begin{aligned} \hat{S}(V) &\xrightarrow{\sim} L^2(V^*, \mu_h) \\ |0\rangle &\longmapsto 1 \\ e^{ih(a_v^* + a_v)} &\longleftrightarrow \text{mult by } e^{iv} \end{aligned}$$

~~This means~~ Let $\rho_h : S(V) \longrightarrow \hat{S}(V)$ be the composition of the inclusion $S(V) \subset L^2(V^*, \mu_h)$ followed by $\circledast$.

Then I am trying to fill in

$$\begin{tikzcd} \hat{S}(V) \arrow[rr, dashed] & & \hat{S}(V) \\ & S(V) \arrow[ul, "\rho_h"] \arrow[ur, "\rho_{h'}"'] & \end{tikzcd}$$

Let's compute ρ_h on an exponential e^{iv}. Let's work out dimension 1, $V^* = \mathbb{R}$, $V = \mathbb{R}$(the function x).

$$\mu_h = e^{-\frac{1}{2}\frac{1}{h^2}x^2}\,dx/\text{norm}.$$

On $L^2(\mathbb{R}, \mu_h)$ we have

$$a = h\partial_x$$

$$a^* = e^{\frac{1}{2}\frac{x^2}{h^2}}(-h\partial_x)e^{-\frac{1}{2}\frac{x^2}{h^2}}$$

$$= +h\left(-\partial_x + \frac{x}{h^2}\right) = -h\partial_x + \frac{x}{h}.$$

So

$$e^{\gamma z} = e^{\gamma a^*} \longleftrightarrow e^{\gamma\left(\frac{x}{h} - h\partial_x\right)}1$$

$$\parallel$$

$$e^{-\frac{1}{2}\left[\gamma\frac{x}{h},\, \gamma h\partial_x\right]}e^{\gamma\frac{x}{h}} = e^{-\frac{\gamma^2}{2}}e^{\gamma\frac{x}{h}}$$

Thus

$$\rho_h\left(e^{\gamma\frac{x}{h}}\right) = e^{+\frac{\gamma^2}{2}}e^{\gamma z}$$

and so

$$\boxed{\rho_h\left(e^{\lambda x}\right) = e^{\frac{h^2\lambda^2}{2}}e^{h\lambda z}}$$

It would be nice if there were a flow on $\hat{S}(V)$ whose trajectories were $\rho_h(\alpha)$. Thus I want

$$h\frac{d}{dh}\rho_h(\alpha) = X\rho_h(\alpha)$$

where X is independent of h.

$$h\frac{d}{dh}\rho_h(e^{\lambda x}) = h\frac{d}{dh}\left(e^{\frac{h^2\lambda^2}{2}} e^{h\lambda z}\right)$$

$$= \left(h^2\lambda^2 + h\lambda z\right)\rho_h(e^{\lambda x})$$

$$= \left(\partial_z^2 + z\partial_z\right)\rho_h(e^{\lambda x}).$$

Now the operator $\partial_z^2 + z\partial_z$ (called the Cauchy operator is related to the harmonic oscillator. Notice that it **operates** on the space of polynomials in z and preserves the filtration by degree, and on the ~~homogeneous~~ quotient of degree n is multiplication by n.

Question: Can I determine within $\hat{S}(V)$ the space of α for which $e^{(\log h)X}$, $X = \partial_z^2 + z\partial_z$ is defined for all h?

January 18, 1986

On a real f.d. v.s. V^* with inner product we consider the family of Gaussian prob. measures $\mu_h = e^{-\frac{1}{2h^2}|x|^2}dx/\text{norm.}$ having variance $h^2|v|^2$. This gives a family of completions of the space $S(V)$ of poly. functions on V^*. We also know how to canonically identify $L^2(V,\mu_h)$ with $\hat{S}(V)$ the Fock space associated to real Hilbert space V. So we obtain maps

$$\begin{tikzcd} S(V) \arrow[r, hook] \arrow[dr, "\rho_h"'] & L^2(V^*,\mu_h) \arrow[d, "\wr"] \\ & \hat{S}(V) \end{tikzcd}$$

Yesterday we showed

$$\rho_h(e^{v}) = e^{\frac{h^2}{2}v^2}\, e^{hv} \qquad v \in V_c.$$

We know ρ_h is an isomorphism on polynomials. We compute the automorphism $\theta_h = \rho_h\rho_1^{-1}$

$$\rho_h\rho_1^{-1}(e^v) = \rho_h\left(e^{-\frac{1}{2}v^2}e^v\right)$$

$$= e^{+\frac{1}{2}(h^2-1)v^2}\, e^{hv}$$

$$\boxed{\theta_h(e^v) = e^{+\frac{1}{2}(h^2-1)v^2}\, e^{hv}}$$

Here v^2 refers to the $\mathbb{C}$-linear extension of $|v|^2$ on V to V_c, so that if we use an orthonormal basis, then we have $v^2 = \sum z_\mu^2$.

$$h\partial_h\, \theta_h(e^v) = \left(h^2v^2 + hv\right)\theta_h(e^v)$$

$$= \left(\partial_\mu^2 + z_\mu\partial_\mu\right)\theta_h(e^v).$$

Let's go back to dimension 1. Put $h = e^t$. If $\alpha \in \hat{S}(V)$ is such that $\Theta_h(\alpha)$ is defined for all $h > 0$, then $\Theta_h(\alpha)$ is a solution of

$$\circledast \qquad \partial_t f = \left(\partial_z^2 + z\partial_z\right) f$$

such that $f(t,z)$ is analytic in z, in fact $f_t(z)$ would be entire in z. Thus we want to investigate solutions of the above heat equation in entire functions.

This is a reasonably subtle question as for each complex number λ, there is a 2 diml space of entire functions which are eigenfunctions for $\partial_z^2 + z\partial_z$ with eigenvalue λ.

I want to discuss solutions of $\circledast$ formally. To solve $\circledast$ we take an entire function $f_0(z)$ and then look for a solution with the initial value $f_0(z)$. For example if $f_0(z) = e^{\lambda z}$, then

$$f_t(z) = e^{\frac{1}{2}(e^{2t}-1)\lambda^2} \cdot e^{(e^t\lambda)z}$$

Now if $f_0(z)$ is expressible in exponentials then we should obtain the solution $f_t(z)$ by putting together the exponential solutions.

Recall that any $f(z) \in \hat{S}(V)$ can be expressed in terms of exponentials as

$$f_0(z) = \int e^{-|w|^2} \frac{d^2 w}{\pi}\, e^{\bar{w} z}\, f_0(\bar{w})$$

from which we get the solution

$$(+) \qquad f_t(z) = \int e^{-|w|^2} \frac{d^2 w}{\pi} \, e^{\frac{1}{2}(e^{2t}-1)\bar{w}^2 + e^t \bar{w} z} f_0(w)$$

I believe that $e^{\alpha \frac{z^2}{2} + \beta z}$ belongs to $\hat{S}(V)$ iff $|\alpha| < 1$, since

$$\left| e^{\alpha \frac{z^2}{2}} \right|^2 = (1 - |\alpha|^2)^{-1/2}$$

Thus $f_t(z)$ (as a number) is the inner product of

$$w \longmapsto e^{\frac{1}{2}(e^{2t}-1) w^2 + e^t w \bar{z}} \quad \text{and } f_0(w)$$

from which it follows that for $|(e^{2t}-1)| < 1$, the values $f_t(\cancel{z})$ are defined.

It does not seem to be possible to use $(+)$ to select a class of $f_0 \in \hat{S}(V)$ such that $f_t(z)$ is defined for all t. In effect, even when f_0 is a polynomial the integral $(+)$ is not convergent once $\left|\frac{1}{2}(e^{2t}-1)\right| > 1$. The reason $(+)$ doesn't yield useful information is that it represents analytic fns. f_0 in an inefficient way (using all exponentials)

One defect with using exponentials is that one is led to believe that f_t is entire in t and periodic with period $2\pi i$. Yet we know that the equation

$$\partial_t f = (\partial_z^2 + z\partial_z) f$$

has solutions $f_t(z) = e^{st} f_0(z)$ if $(\partial_z^2 + z\partial_z) f_0 = s f_0$ is an eigenfunction.

Now let's review the standard method using contour integrals for constructing eigenfunctions. It is based on the fact that Fourier transform in z will convert $\partial_z^2 + z\partial_z$ to a first order operator. To carry this out in practice, we write

$$f(z) = \int e^{zw} g(w)\, dw$$

where g and the contour are to be chosen so that

$$\begin{aligned}
0 &= \left(\partial_z^2 + z\partial_z - s\right) f . \\
&= \int (w^2 + zw - s)\, e^{zw} \cdot g(w)\, dw \\
&= \int (w^2 + w\partial_w - s) e^{zw} \cdot g(w)\, dw \\
&= \int e^{zw} \underbrace{\left(-\partial_w(wg) + (w^2 - s) g\right)}_{(-w\partial_w + w^2 - s - 1) g}\, dw
\end{aligned}$$

assuming the endpoint contribution is zero by choice of the contour. Thus

$$\partial_w g = \left(w - \frac{(s+1)}{w}\right) g$$

$$\partial_w (\log g) = w - \frac{s+1}{w}$$

$$\log g = \frac{w^2}{2} - (s+1) \log w$$

$$g = e^{\frac{w^2}{2}} w^{-s-1}$$

So we get eigenfunctions for $\partial_z^2 + z\partial_z$ with eigenvalue s from contour integrals of the form

$$(++) \qquad f(z,s) = \int e^{\frac{w^2}{2} + zw} w^{-s} \frac{dw}{w}$$

when the contour is such that the endpoint contributions vanish. In practice this means we want g to vanish at the endpoints. The natural contours involve heading to ∞ in the directions where $e^{\frac{w^2}{2}}$ decays i.e. $\pm i\infty$.

(++) Proceed formally and combine the eigenfunctions with time dependence e^{st} to get

$$\int e^{\frac{w^2}{2}+zw}\frac{dw}{w}\underbrace{\int \underbrace{e^{st}\,w^{-s}}_{(e^{-t}w)^{-s}}\,k(s)\,ds}_{\hat{k}(e^{-t}w)}$$

$$= \int e^{\frac{1}{2}(e^{2t}-1)w^2+e^t wz}\frac{dw}{w}\,\hat{k}(w)\,e^{\frac{1}{2}w^2}$$

This agrees with what we have already found.

Actually this raises the question of resolving

$$\theta_{e^t}(e^{wz}) = e^{\frac{1}{2}(e^{2t}-1)w^2+(e^t w)z}$$

into exponentials e^{st}, which leads to

$$\int e^{\frac{1}{2}e^{2t}w^2+(e^t w)z}\,e^{-st}\,dt$$

$$= \int e^{\frac{1}{2}(hw)^2+(hw)z}\,h^{-s}\,\frac{dh}{h}$$

$$= w^s\int e^{\frac{1}{2}h^2+hz}\,h^{-s}\,\frac{dh}{h}$$ which is essentially the eigenfunction $f(z,s)$ of (++).

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It has occurred to me that instead of using the model $\hat{S}(V)$ for all the Hilbert spaces $L^2(V^*, d\mu_h)$ I can use rescaling transformations on the vector space V^*, at least in finite dimensions. Thus take $V^* = \mathbb{R}$ whence

$$d\mu_h = e^{-\frac{1}{2h^2}x^2} \frac{dx}{\sqrt{2\pi h^2}}$$

One then has

$$\cancel{\int f(hx)\, d\mu_h}$$

$$\int f(x)^2 \, d\mu_h = \int f(x)^2 e^{-\frac{1}{2h^2}x^2} \frac{dx}{\sqrt{2\pi h^2}}$$

$$= \int f(hx)^2 e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

and so we have an isomorphism

$$L^2(V^*, d\mu_h) \xrightarrow{\sim} L^2(V^*, d\mu_1)$$

$$f(x) \longmapsto f(hx).$$

So the map ρ_h studied above becomes

$$\begin{tikzcd} S(V) \subset L^2(V^*, d\mu_h) \arrow[r, "\sim"] & L^2(V^*, d\mu_1) \end{tikzcd}$$

with $S(V) \xrightarrow{\rho_h} L^2(V^*, d\mu_1)$

$$(\rho_h f)(x) = f(hx)$$

Hence the densely defined 1-parameter group Θ_h on $L^2(V^*, d\mu_1)$ $\ni$ $\Theta_h \rho_1 = \rho_h$ is

$$\boxed{(\Theta_h f)(x) = f(hx)}$$

Now the problem becomes to find a subspace of $L^2(\mathbb{R}, e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}})$ on which Θ_h is defined for all $h > 0$, and also admits the possibility of analytic continuation to $|\arg h| < \frac{\pi}{4}$. Notice the eigenfunctions of Θ_h given by the polynomials x^n.

Notice that we have reached homogeneous distributions and the Mellin transform. The eigenfunctions for the 1-parameter group Θ_h are characters

$$x \longmapsto |x|^s \, \mathrm{sgn}(x)^\varepsilon \qquad \varepsilon = 0, 1.$$

Spectral analysis of Θ_h involves taking the Mellin transform.

The actual Gaussian integral of $f(x)$

$$\int f(x) e^{-\frac{1}{2h^2}x^2} \frac{dx}{\sqrt{2\pi h^2}} = \int f(hx) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}$$

in good cases will be analytic in h, for $|\arg h| < \frac{\pi}{4}$ at least if f is bounded.