November 29, 1985

non-commutative residue. This is defined on the algebra of classical pseudo-differential operator symbols over M, denoted $CS(M)$. Classical means the homogeneous terms have integral degree. The residue sees the homogeneous term of degree $-\dim(M)$.

Example for S^1. The unit sphere in T^*_x consists of 2 points in dimension 1, so the space of homogeneous functions of degree m is 2-diml spanned by

$$|\xi|^m \qquad \operatorname{sgn}(\xi)|\xi|^m.$$

For m integral one has the basis

$$\xi^m \qquad \operatorname{sgn}(\xi)\,\xi^m.$$

Consequently a typical element of $CS(S^1)$ is of the form

$$\sum_{m\le N} a_m(x)\,\partial^m + \sum_{m\le N} b_m(x)\operatorname{sgn}(\partial)\,\partial^m$$

where a_m, b_m are smooth functions on S^1. Here $\operatorname{sgn}(\partial)$ is the Hilbert transform symbol.

Let's now consider the subalgebra consisting of symbols $\sum_{m\le N} a_m(x)\partial^m$. Call this subalgebra A, and let B be the constant coefficient symbols, i.e. the centralizer in A of ∂. Then B can be identified:

$$B \simeq \mathbb{C}[[\partial^{-1}]][\partial]$$

with the field of formal Laurent series in ∂^{-1}.

A also contains the subalgebra $C^\infty(S^1)$, which

I want to view as a kind of group algebra for $\mathbb{Z}$, the function e^{ix} being the generator. Since

$$e^{-i\xi x} P(\partial) e^{i\xi x} = P(\partial + i\xi)$$

we see that the exponential functions $\{e^{i\xi x} \mid \xi \in \mathbb{Z}\}$ normalize B.

In more detail one has

$$(\partial + i\xi)^{-1} = \partial^{-1}(1 + i\xi\partial^{-1})^{-1}$$

is a formal power series in ∂^{-1}. Thus there is a (unique continuous) automorphism of B obtained by substituting $(\partial + i\xi)^{-1}$ for ∂^{-1}. (General statement is that any formal power series $h(z) = a_1 z + a_2 z^2 + \dots$ with $a_1 \neq 0$ induces an automorphism of the Laurent series field $\mathbb{C}[[z]][z^{-1}]$ such that $z \mapsto h(z)$.)

The next point is that in some sence A is the crossed product of B by this action of $\mathbb{Z}$. Formally A has the basis $e^{i\xi x}\partial^m$, ξ and $m \in \mathbb{Z}$.

From this viewpoint we would expect to ~~define~~ obtain a trace on A starting from a trace on B which is $\mathbb{Z}$-invariant. It is not easy to construct linear functionals on the field of Laurent series which are invariant under substitutions; in fact the residue, which is defined on differentials $f(z)dz$, is the only thing that comes to mind.

Note that the differential dz is invariant under translations $z \mapsto z + \gamma$. Thus we see that on the field of formal Laurent series $\mathbb{C}[[z^{-1}]][z]$ obtained by completion $\mathbb{C}(z)$ at $z=\infty$, the linear functional

$$f(z) \longmapsto \operatorname{Res}\left(f(z)dz, \infty\right) = \begin{matrix}\text{coeff of } z^{-1} \text{ in} \\ \text{the L. series of } f\end{matrix}$$

is invariant under the substitutions $z \mapsto z + \text{const}$.

This means that we should get a trace on the algebra A by

$$\sum_{m \le N} a_m(x)\,\partial^m \longmapsto \int_{S^1} a_{-1}(x)\,dx .$$

In terms of the basis $e^{i\xi x}\partial^m$, everything goes to zero except $\xi = 0$, $m = -1$.

I would like to understand this algebra A a bit better, say by finding dense subalgebras, or similarly related algebras. I might try to lift symbols to operators.

First try to shrink B

December 6, 1985

Wodz... 's determinant for elliptic ψDO's.

Over a compact manifold X he considers elliptic ψDO's with classical type symbol, meaning that the terms are homogeneous fns. of degrees $m, m-1, m-2, \ldots$ where m is the order and m can be any complex number. For example, A^s where A is positive elliptic. Ell^m is the space of elliptic ψDO's of order m over X (acting on some v.b. E, or from E to F, in general.)

1. Given an elliptic A ~~with leading symbol $\sigma_m(A,\xi)$~~ suppose it admits complex powers A^s. For this to make sense one needs a curve γ in $\mathbb{C}$ from 0 to ∞ which doesn't meet the spectrum of A. Cutting $\mathbb{C}$ along γ one obtains a branch of z^s off the cut; this is a holom. fn. on the spectrum of A, so one can substitute A in this function to obtain A^s.

The same can be done on the complete symbol level. Working with the alg. of class. symbols $CS^{\cdot}$, the spectrum of A should be the union of the spectrum of the leading symbol $\sigma_m(A,\xi)$, the union being taken over $\xi \in T^*(X) - X$. For m real $\neq 0$ this should be a sector, so γ is an arc outside this sector.

But for $m=0$, the spectrum is some closed bdd. set not containing 0, and we just need a curve γ in order to define A^s.

Question: Assuming γ avoids the spectrum of A does it avoid the spectrum of the symbol of A? One apparently needs this to get that A^s is an elliptic ψDO.

2. A^s has order $s \cdot \mathrm{ord}(A)$. When this is an integer we can consider its non-commutative residue. Actually we can consider

$$\mathrm{res}(A^s)$$

in general, but it is zero unless $s \cdot \mathrm{ord}(A) \in \mathbb{Z}$. Here $A \in CS^{\cdot}$.

There is some result to the effect that

$$\mathrm{res}\left(\zeta_A(s)_{;s}\right) = \frac{\mathrm{res}(A^s)}{\mathrm{order}(A)}$$

Recall that Seeley showed $\zeta_A(s)$ to be meromorphic with poles at known places. Here $\zeta_A(s) = \mathrm{tr}(A^s)$ and A has to be an operator, although $\mathrm{res}(A^s)$ is defined for A a symbol.

Now for $s = 0$, $\mathrm{res}(A^s) = \mathrm{res}(I) = 0$, so one recovers the fact that $\zeta_A(s)$ is regular at $s = 0$.

<u>Question</u>: Why is $\mathrm{tr}(A^s)$ defined for order $A = 0$?
I think Wod.. defines (It isn't - we certainly can't divide by order(A) as above.)

$$Z(A) = \log\det(A) = -\zeta_A'(0)$$

for A having complex powers.

However $Z(A)$ is not best defined this way. A better method apparently is to define it as a kind of multiplicative invariant associated to the non-commutative residue.

3. Now suppose we work with the algebra CS^0 of complete classical ΨDO symbols of integral order. The invertible elts are the elliptic symbols Ell^0. We will be interested in the subgroup of Ell^0 consisting of A, order zero such that the leading symbol of A is homotopy through invertible leading symbols to I. This subgp is the identity component Ell^0_I of Ell^0. Given such an A we join it to the identity by a path A_t and then put

$$z(A) = \int_0^1 res(A_t^{-1} \dot{A}_t)\, dt$$

December 7, 1985

We are working with the algebra $CS^{\cdot}$ of classical symbols of ΨDO's, complete symbols and the terms are homogeneous of integral order. $Ell^{\cdot}$ is the group of invertible operators. Ell°_{0} is the identity component in $Ell^{\cdot}$. One has a group extension

$$1 \longrightarrow Ell^{\circ}_{0} \longrightarrow Ell^{\cdot} \longrightarrow \pi_0(Ell^{\circ}) \times \mathbb{Z} \longrightarrow 1.$$

Let E be the vector bundle over X on which these symbols operate, let $\pi : ST^*X \longrightarrow X$ be the cosphere bundle. The leading symbol of an element of Ell° is an invertible element of

$$\Gamma(ST^*X, \pi^* \operatorname{End} E) = \Gamma(ST^*X, \operatorname{Aut} \pi^* E).$$

We have

$$\pi_0 Ell^{\circ} = \pi_0 \Gamma(ST^*X, \operatorname{Aut} \pi^* E)$$

In the stable range we have

$$\pi_* \Gamma(ST^*X, \operatorname{Aut} \pi^* E) = K_1(ST^*X).$$

so in this case $\pi_0(Ell^{\circ})$ is abelian.

Wod... defines a "log det" function Z on Ell°_{0} in the following way. Given $A \in Ell^{\circ}_{0}$ we know there is a path A_t in Ell° joining I to A, and we put

$$Z(A) = \int_0^1 \operatorname{res}\left(A_t^{-1} \frac{d}{dt} A_t\right) dt$$

It is necessary to verify this is independent of the choice of path.

First consider the case where an arc γ going from 0 to ∞ exists not meeting the spectrum of A, so that complex powers A^s can be defined. Then we can take the path A^t $0 \leq t \leq 1$ and we find

$$Z(A) = \int_0^1 \operatorname{res}(\log A)\, dt = \operatorname{res}(\log A)$$

in analogy with the ~~well known~~ formula

$$\log \det A = \operatorname{tr} \log A$$

~~Now let's see why $Z(A)$ is well defined in Ell_0^0. Thus we want to take a loop A_t in Ell and see that~~

$$\cancel{\int \operatorname{res}(A_t^{-1} \dot{A}_t)\, dt}$$

~~is 0.~~

Next let's discuss why $Z(A)$ is well-defined on Ell_0^0. Wod... ~~claims that this~~ proves this by means of a general formula

$$\int \tau(A_t^{-1} \dot{A}_t)\, dt = \tau\left(\begin{array}{c}\text{projector assoc. to} \\ \text{the loop } A_t\end{array}\right)$$

and then by proving that $\operatorname{res}(\text{any projector in } CS^0) = 0$.

In general if $\mathcal{A}$ is say a Banach algebra one has a periodicity map

$$\beta : \pi_1(GL(\mathcal{A})) \longrightarrow K_0(\mathcal{A})$$

which views a loop in $GL(\mathcal{A})$ as a clutching function for a vector bundle on "$P_1(\mathcal{A})$" = "$S^2 \times \mathcal{A}$". It seems to be well-known that if τ is a trace on $\mathcal{A}$, then one has the formula

$$\int_0^1 \tau(A_t^{-1}\dot{A}_t)\,dt = \tau(\rho(A_t))$$

which is what is claimed. Connes uses this formula in discussing multiplicative characters.

Review Connes-Karoubi

$$\begin{tikzcd}
K_2^{top}(a) \arrow[r] \arrow[d, "\wr"] & K_1^{rel}(a) \arrow[r] \arrow[d, "\tau"] & K_1^{alg}(a) \arrow[r] & K_1^{top}(a) \\
K_0^{top}(a) \arrow[r, "\tau"] & \mathbb{C} & &
\end{tikzcd}$$

Thus one gets a map from the identity component of $GL(a)$ to $\mathbb{C}/\tau K_0(a)$. The other idea is to do this universally in a Fredholm operator situation where $K_{odd}^{top}(a) = 0$.

One sees from the way this is set up that
$$Z: Ell_0^\circ \longrightarrow \mathbb{C}$$
is a homomorphism. The derivative at the origin is the non commutative residue restricted to the algebra $CS^{\leq 0}$. (We see a little of the same kind of structure as for loop groups. Consider $\mathcal{L}GL_n$ as the units in the algebra of $n\times n$-matrices over the ring $\mathcal{L}\mathbb{C}$, or Laurent series. The Lie algebra is $\mathcal{L}(gl_n)$ and the exponential map is defined. If we restrict to the Laurent series $\mathbb{C}\{z\}[z^{-1}]$ which are meromorphic at $z=0$, then the exponential map will only be defined for the holomorphic (in $|z|\leq 1$) loops.)

So now we have an extension and homomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ell^0_0 & \longrightarrow & Ell & \longrightarrow & \pi_0(Ell^0) \times \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow Z & & & & \| & & \\ & & \mathbb{C} & & & & K_1(ST^*X) \quad \text{stably.} & & \end{array}$$

so we get a central extension by push forward

$$0 \longrightarrow \mathbb{C}_{add} \longrightarrow E \longrightarrow K_1(ST^*X) \times \mathbb{Z} \longrightarrow 0$$

Wod--- claims that this extension comes from an extension

$$0 \longrightarrow \mathbb{C} \longrightarrow E' \longrightarrow \mathbb{Z} \times \mathbb{Z} \longrightarrow 0$$

via the index homomorphism $\mathrm{Ind}: K_1(ST^*X) \longrightarrow \mathbb{Z}$. Specifically if $a, b \in Ell$, then

$$Z\big((a,b)\big) = \mathrm{Ind}(a)\cdot \mathrm{ord}(b) - \mathrm{Ind}(b)\cdot \mathrm{ord}(a)$$

where $(a,b) = aba^{-1}b^{-1}$.

This shows the impossibility of extending Z from Ell^0_0 to all of Ell. Apparently there is a natural extension to the subgroup mapping to zero in $\pi_0(Ell^0)$. In degree m this consists of operators whose leading symbols are homotopic to π^m where π is like $\Delta^{1/2}$. It contains operators A admitting complex powers, since

$$A^t\, \pi^{(1-t)m} \qquad 0 \le t \le 1$$

is a path between π^m and A.

Question: If A has degree $m > 0$, and admits complex powers, then what is Wod's formula for $Z(A)$ in terms of ζ_A? He apparently is able to show that if A_t joins A to B within the component of Ell containing π^m, then

$$\int_0^1 \operatorname{res}\left(A_t^{-1}\mathring{A}_t\right) dt = Z(B) - Z(A)$$

December 8, 1985

More Wodzicki:

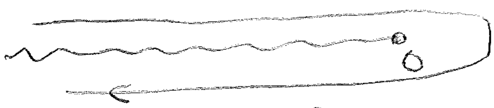
Here is perhaps one way to understand the analytic continuation of $\zeta_A(s)$. To keep things specific we consider the Riemann zeta

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

We want to obtain an expression for ζ as a contour integral using

$$a^{-s} = \frac{1}{2\pi i} \int_C \frac{1}{\lambda - a} \lambda^{-s} \, d\lambda$$

where C is the contour



and $Re(s) > 0$

Unfortunately $\sum_{n=1}^{\infty} \frac{1}{\lambda - n}$ doesn't make sense, but we can use

$$a^{-s} = \frac{1}{2\pi i} \int_C \frac{1}{(\lambda - a)^2} \frac{\lambda^{-s+1}}{-s+1} \, d\lambda$$

This gives

$$\sum_{n=1}^{N} \frac{1}{n^s} = \frac{1}{2\pi i} \int_C \sum_{n=1}^{N} \frac{1}{(\lambda - n)^2} \frac{\lambda^{-s+1}}{-s+1} \, d\lambda$$

and by dominated convergence one ought to be able to ~~take~~ the limit as $N \to \infty$ and get

$$\zeta(s) = \frac{1}{2\pi i} \int_C \sum_{1}^{\infty} \frac{1}{(\lambda - n)^2} \frac{\lambda^{-s+1}}{-s+1} \, d\lambda \qquad Re(s) > 1.$$

Also we should be able to get

$$\zeta(s) = \frac{1}{2\pi i} \int_C \sum_1^\infty \left(\frac{1}{s-n} + \frac{1}{n} \right) \lambda^{-s}\, d\lambda \qquad Re(s) > 1.$$

by integrating by parts.

Roughly then we are going to get at $\zeta(s) = tr(A^{-s})$ through the resolvent, more precisely $tr \frac{1}{(\lambda - A)^k}$, where k is such that this trace exists.

Now that we have ζ expressed as a contour integral we bring in the asymptotic expansion for $tr \frac{1}{(\lambda-A)^k}$ as $\lambda \to -\infty$:

$$\sum_{n=1}^\infty \frac{1}{(\lambda-n)^2} \simeq \frac{c_1}{\lambda} + \frac{c_2}{\lambda^2} + \frac{c_3}{\lambda^3} + \cdots$$

(Specifically:

$$\sum_{n=1}^\infty \frac{1}{(s+n)^2} = \int_0^\infty e^{-st} \sum_{n=1}^\infty e^{-nt}\, t^2 \frac{dt}{t}$$

$$= \int_0^\infty e^{-st} \underbrace{\frac{e^{-t}}{1-e^{-t}}}_{\left\{ \frac{1}{t} - \frac{1}{2} + b_1 t + b_3 t^3 + \cdots \right\}} t^2 \frac{dt}{t} \qquad b_1 = \frac{1}{12}$$

$$\sim \frac{1}{s} - \frac{1}{2s^2} + b_1 \frac{\Gamma(3)}{s^3} + b_3 \frac{\Gamma(5)}{s^5} + \cdots$$

so
$$\sum_{n=1}^\infty \frac{1}{(\lambda-n)^2} \sim -\frac{1}{\lambda} - \frac{1}{2\lambda^2} - b_1 \frac{\Gamma(3)}{\lambda^3} + \cdots \qquad \lambda \to -\infty \Big)$$

So we write

$$\zeta(s) = \frac{1}{2\pi i}\int_C \left(\sum_1^\infty \frac{1}{(\lambda-n)^2} - \frac{c_1}{\lambda} - \frac{c_2}{\lambda^2} \cdots - \frac{c_m}{\lambda^m} \right) \frac{\lambda^{-s+1}}{-s+1}\, d\lambda$$

$$+ \frac{1}{2\pi i}\int_C \left(\frac{c_1}{\lambda} + \frac{c_2}{\lambda^2} + \cdots + \frac{c_m}{\lambda^m} \right) \frac{\lambda^{-s+1}}{-s+1}\, d\lambda$$

Because the first integral is better at infinity we see it represents an analytic function for $Re(s) > -m+1$ divided by $\frac{1}{1-s}$. The second integral is easily done and found to be zero for $Re(s) > 1$. Thus one sees $\zeta(s)$ has a meromorphic continuation with only the simple pole at $s=1$, which is given by the first integral.

Now when s is integral the function λ^{-s+1} is analytic across the cut, so the above integral expression reduces to a integral over a circle around $\lambda = 0$.

Thus we find that $\quad \zeta(s) \sim \frac{-c_1}{1-s} = \frac{1}{1-s} \quad$ as $s \to 1$.

$$\zeta(0) = -c_2 = +\frac{1}{2}$$

$$\zeta(-1) = -\frac{1}{2}c_3 = \frac{1}{2} b_1 \Gamma(3) = \frac{1}{12}$$

$$\zeta(\underset{\geq 2}{\text{even}}) = 0.$$

December 10, 1985

Seeley stuff on $\zeta_A(s)$. The point I would like to understand is how to get from the symbol of A^{-s} = (symbol of $A)^{-s}$ to the analytic continuation of $\zeta_A(s)$. One one hand I have

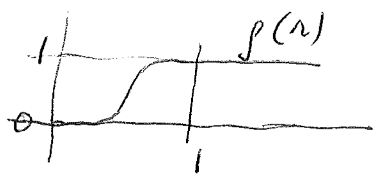
$$(\text{symbol } A)^{-s} = a_{-ms} + a_{-ms-1} + \cdots$$

a formal series of homogeneous functions on T^* and a process of cutoff + integration over T^*; this is how one generates the singularities, simple poles, of $\zeta_A(s)$, and the rest is supposed to be entire.

Consider the example of $\left(-\frac{d^2}{dx^2}\right)^{1/2}$ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

Then
$$\zeta_A(s) = 2\sum_{n=1}^{\infty} n^{-s}$$

symbol $A = |\xi|$ symbol $A^{-s} = |\xi|^{-s}$



$$(2\pi)^{-1}\int_{T^*(S^1)} \rho(|\xi|)\,|\xi|^{-s}$$

$$= 2\int_0^\infty \rho(x)\,x^{-s}\,dx$$

$$= 2\int_0^1 \rho(x)\,x^{-s}\,dx + 2\int_1^\infty x^{-s}\,dx$$

$$= \left(\begin{smallmatrix}\text{entire}\\ \text{fn}\end{smallmatrix}\right) + \frac{2}{s-1}$$

Thus I want an analytic proof that

$$\sum_{n=1}^{\infty} n^{-s} - \frac{1}{s-1} \quad \text{is entire.}$$

Now Wodzicki explained the method used. He constructs a parametrix $B(\lambda)$ for the resolvent $(A-\lambda)^{-1}$:

$$(A-\lambda)\,B(\lambda) \quad \text{and} \quad B(\lambda)(A-\lambda) \;=\; I + T(\lambda)$$

where $T(\lambda)$ is sufficient smooth to have a trace and [redacted] decays sufficiently in λ. λ has to be treated with a degree of homogeneity just as ξ. So now we argue that $I + T(\lambda)$ is invertible in a suff. nice class

$$\frac{1}{A-\lambda} = B(\lambda) - T(\lambda)\frac{1}{A-\lambda}$$

Then you use the contour integral

$$A^{-s} = \frac{1}{2\pi i}\int \frac{1}{\lambda - A}\,\lambda^{-s}\,d\lambda$$

to relate A^{-s} to the corresponding integral of $-B(\lambda)$. The error term involving a contour integral of $T(\lambda)$ will have contour integral analytic in a certain half plane $Re(s) > k$, where k can be made as negative as desired.

Somehow one can remove powers of λ from $B(\lambda)$ until the remainder has a trace. This is how one gets the analytic continuation.

Now it appears to me that an important ingredient is the fact that given a symbol one can construct a ψDO having that symbol, whose trace is given by integrating over T^*. I want to see this in the case of a circle.

In general given a manifold M one covers it by coordinate charts U and picks $\psi_U, \varphi_U \in C_0^\infty(U)$ such that

$$\psi_U = 1 \quad \text{in a nbd of } \operatorname{Supp} \varphi_U$$

$$\sum \varphi_U = 1.$$

If we identify U with an open subset of $\mathbb{R}^n$, then we can use $\psi_U P(x,D)\varphi_U$ defined using the local coordinates on U. This goes from $C^\infty(U)$ to $C_0^\infty(U)$. and has the symol which is

$$\psi_U \circ \text{symbol } P \circ \varphi_U = P(x,\xi)\circ\varphi_U(x)$$

on U. Then adding up we get the desired operator.

In terms of the circle what does this mean?

December 11, 1985

Here's is a good viewpoint for the construction $\sum \psi_U P(x,D)\varphi_U$. Look on $X\times X$ at the kernel. What has been done has been to take a kernel $K(x,y)$ defined in a nbd of Δ_X; this is $P(x,D)$. Then one multiplies by $\sum_U \psi_U(x)\varphi_U(y)$. Look at $\psi_U(x)\varphi_U(y)$. This is in $C^\infty(U\times U)$. If x is sufficiently close to y, then we have $\psi_U(x)\varphi_U(y) = \varphi_U(y)$. In effect this is true for $y \notin \operatorname{Supp}\varphi_U$, and if $y\in\operatorname{Supp}\varphi_U$, then by the assumption $\psi_U = 1$ near $\operatorname{Supp}\varphi_U$ one have $\psi_U(x) = 1$.

So we see that the function $\sum_U \psi_U(x)\varphi_U(y)$ is just a smooth function $\equiv 1$ near Δ_X with support in a nbd of Δ_X.

Consequently if we want to realize a constant coefficient ψDO we can take its Euclidean space kernel $K(x-y)$ multiply by a function $\rho(|x-y|)$, where $\rho(r)$: 

is a smooth function $\equiv 1$ near 0 and supported in $r < \varepsilon$. Then we make the resulting function $\rho(|x|)K(x)$ periodic to get a kernel on the torus.

Thus given $p(\xi)$ smooth with nice asymptotic expansion at ∞, we have

$$K(x) = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi x}\, p(\xi).$$

The operator $p(D)$ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ has kernel

$$\sum_m{}' K(x-y+2\pi m) = \frac{1}{2\pi}\sum_n{}' e^{in(x-y)}\, p(n)$$

and trace

$$2\pi \sum_m{}' K(2\pi m) = \sum_n{}' p(n) \qquad (\mathrm{vol}\, S^1 = 2\pi)$$

On the other hand we can form the kernel agreeing with $K(x-y)$ near the diagonal in $\mathbb{R}$, multiplied by $\rho(x-y)$ to get support in a nbd of the diagonal, and then descended to the circle. This gives the kernel

$$\sum_m{}' \rho(x-y+2\pi m)\, K(x-y+2\pi m).$$

By construction if ε is small ($< \pi$ say), then if $|x-y| < \varepsilon/2$ the above is just $K(x-y)$.

~~Consequently the operator~~

The trace of this operator on the circle is clearly

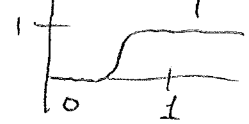
$$2\pi\, K(0) = 2\pi \int \frac{d\xi}{2\pi}\, p(\xi).$$

Next look at the difference of the kernels. If $|x-y| < \varepsilon/2$ the difference is

$$(*) \qquad \sum_{m\neq 0} K(x-y+2\pi m)$$

which is smooth (by fact $p(\xi)$ is smooth and has asymptotic expansion.

If we take $\quad p(\xi) = \varphi(|\xi|)\,|\xi|^{-s}$



then we want to see that the difference of the two operators, one with trace $\sum'_{n\neq 0} |n|^{-s}$, the other with trace $\int \varphi(|\xi|)\,|\xi|^{-s} \frac{d\xi}{2\pi}$, is a smooth kernel operator which is an entire function of s. From the formula we need to see that

$$K(x,s) = \int \frac{d\xi}{2\pi}\, e^{i\xi x}\, \varphi(|\xi|)\,|\xi|^{-s}$$

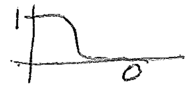
for $x \neq 0$ is entire in s and decays rapidly so that the series $(*)$ is also entire.

December 14, 1985

I recall that I was trying to understand Seeley's theorem for the zeta function. I was looking at constant coefficient operators on a torus, and I wanted to get a feeling for what can be proved using symbol calculations.

Given the function $p(\xi)$ on T^* which is smooth and has asymptotic expansion at ∞ we have the actual operator $p(D)$ which has the eigenvalues $p(\xi)$, ξ in the dual lattice. On the other hand we have the ψDO one constructs from this symbol by patching together local operators. I found a nice way to think of the latter, namely I take the kernel of the operator $p(D)$ on Euclidean space which is $K(x-y)$ where

$$K(x) = (2\pi)^{-n}\int d\xi\, e^{i\xi x} p(\xi)$$

then I multiply by $\rho(|x-y|)$, where ρ:  to get a kernel supported near the diagonal, and finally I descend this to the torus. This means I use the operator with kernel $L(x-y)$ where

$$L(x) = \sum_{\gamma\in\Gamma} \rho(|x+\gamma|)\, K(x+\gamma)$$

is periodic.

The point was the trace of L is

$$\operatorname{tr}(L) = \mathrm{vol}\cdot K(0) = \mathrm{vol}\ (2\pi)^{-n}\int d\xi\, p(\xi)$$

whereas $\operatorname{tr} p(D) = \sum_{\xi\in\hat{\Gamma}}' p(\xi)$. Hence it's not clear how much one can tell about $p(D)$ from the operator L.

One seems to know only that $p(D)-L$ has smooth kernel. When $p(\xi)$ is analytic in ξ. ~~then~~ $K(x)$ is also for $x \neq 0$ it seems, and one sees that $p(D)-L$ has smooth kernel analytic in ξ. So one can conclude that the singularities of $\operatorname{tr} p(D)$, $\operatorname{tr}(L)$ are the same. This is how one can calculate the singularities of the zeta function using the symbol. Wodzicki has a nice formula involving the noncommutative residue.

Now in the case of elliptic diff'l operators Seeley's method says something about the values of ζ at negative integers. This is maybe because the symbol of the resolvent involves terms

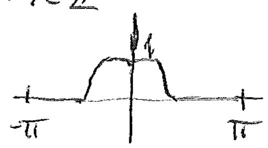
$$\frac{\xi^\alpha}{(\lambda - p(\xi))^k}$$

where p is a polynomial in ξ, and these terms are smooth in ξ. It's not clear.

December 16, 1985

I want to understand why Seeley's method say something about the value of ζ at negative integers for elliptic differential operators. First let's discuss the case of $A = -\frac{d^2}{dx^2}$ on S^1. Here the complete symbol of the resolvent is $\frac{1}{\lambda - \xi^2}$. The operator one constructs from this symbol has the kernel $\tilde{K}(x-y,\lambda)$ where

(1) $$\tilde{K}(x,\lambda) = \sum_{m\in\mathbb{Z}}{}' \varphi(x+2\pi m) \int \frac{d\xi}{2\pi} e^{i\xi x} \frac{1}{\lambda-\xi^2}$$

and φ: 

. (If $|x| < \pi+\varepsilon$ only $m=0$ term is $\neq 0$.) The operator $\frac{1}{\lambda+\partial^2}$ on S^1 has the kernel $K(x-y,\lambda)$ where

(2) $$K(x,\lambda) = \sum_n \frac{e^{inx}}{2\pi} \frac{1}{\lambda - n^2}.$$

I think the sort of thing we want to prove about $\zeta_A(s) = 2\zeta_R(2s)$ follows from the assertion that $K(x-y,\lambda) - \tilde{K}(x-y,\lambda)$ is smooth in x, y, λ and that it is $O\left(\frac{1}{|\lambda|^N}\right)$ for any N as $\lambda \to -\infty$. Assume this assertion holds, whence taking traces we have

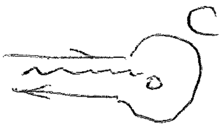
(3) $$\sum \frac{1}{\lambda - n^2} - \underbrace{\int d\xi \frac{1}{\lambda-\xi^2}}_{+\frac{\pi}{\sqrt{-\lambda}}} = O\left(\frac{1}{|\lambda|^N}\right) \quad \text{all } N$$

Actually we know (3) is true from the P.S. formula since $\frac{1}{\lambda-\xi^2}$ is smooth and integrable; also we can use the known formulas for $\sum \frac{1}{\lambda - n^2}$ in terms of the cotangent.

Next we see what we can prove about $\zeta_A(s) = 2\sum_{n=1}^{\infty} \frac{1}{n^{2s}} = 2\,\zeta_R(2s)$ at negative integers.

By contour integration

$$\zeta_A(s) = \frac{1}{2\pi i}\int_C \Big(\sum_n \frac{1}{\lambda - n^2}\Big)\lambda^{-s}\,d\lambda$$



This holds for $Re(s) > \frac{1}{2}$. (Do for partial sums $\sum'_{|n|\le N}$ and let $N \longrightarrow \infty$.)

Now we want to use the asymptotics (3); as $\frac{\pi}{\sqrt{-\lambda}}$ is not analytic on C, I seem to have to convert the integral to an integral over $0<t<\infty$ where $\lambda = -t$. In order to have the integral over the small circle go to zero we have to remove the $\frac{1}{\lambda}$ term. Thus we get

$$\zeta_A(s) = \frac{1}{2\pi i}\left((e^{i\pi})^{-s} + (e^{-i\pi})^{-s}(-1)\right)\int_0^{\infty}\Big(\sum_{n\neq 1}\frac{1}{-t-n^2}\Big)t^{-s}\,dt$$

$$= \frac{\sin(\pi s)}{\pi}\int_0^{\infty}\Big(\sum_{n\neq 1}\frac{1}{t+n^2}\Big)t^{-s}\,dt$$

This converges for $\frac{1}{2} \underset{t\to+\infty}{<} Re(s) \underset{t\to 0}{<} 1$. Analytically continuing thru $s = \frac{1}{2}$ gives

$$\zeta_A(s) = \frac{\sin(\pi s)}{\pi}\int_0^{\infty}\Big(\sum_{n\neq 1}{}'\frac{1}{t+n^2} - \frac{\pi}{\sqrt{t}}\Big)t^{-s}\,dt$$

This converges for $0 \underset{t\to\infty}{<} Re(s) \underset{t\to 0}{<} \frac{1}{2}$. Analytically continuing thru $S=0$ gives

$$\zeta_A(s) = \frac{\sin(\pi s)}{\pi} \int_0^\infty \left({\sum_n}' \frac{1}{t+n^2} - \frac{\pi}{\sqrt{t}} \right) t^{-s}\, dt$$

This integral converges for $Re(s) < 0$ and is holom. here. Since $\sin(\pi s)$ vanishes for $s \in \mathbb{Z}$, it follows that $\zeta_A(s)$ vanishes at $s = -1, -2, \ldots$, hence $\zeta_R(s)$ vanishes at $s = -2, -4, -6, -8, \ldots$.

The trouble with the above is that it seems to be fairly subtle, and it doesn't really explain why one can get the values of ζ at negative integers so readily.

Here's another approach. Let A be our positive elliptic operator with const coeff.; thus $A = p(D)$, where p is a polynomial. Seeley's method is to construct a parametrix $M(\lambda)$ for $(\lambda - A)^{-1}$ with estimates for the behavior of the difference as $\lambda \to \infty$. Then one will have control over

$$A^{-s} - \hat{M}(s) = \frac{1}{2\pi i} \int \left(\frac{1}{\lambda - A} - M(\lambda) \right) \lambda^{-s} d\lambda$$

What I have been missing is how, even knowing the analytic continuation of $\hat{M}(s)$ and its values at negative integers, does one know the value of the right side at negative integers is zero. One could argue by Cauchy if one knew $M(\lambda)$ were analytic on the negative real axis including 0

But maybe one could proceed as in the above example. One wants $M(\lambda)$ expressed in powers of λ. Somehow one tries to make the integral over $0<t<\infty$ and use the asymptotics as $\lambda \to \infty$ to get the poles of the Mellin transform. Then when we multiply by the $\frac{\sin(\pi s)}{\pi}$ we get values of zeta at negative integers.

Another (maybe better) idea. For $A = p(D)$ we can construct $\hat{M}(s)$ by taking the kernel of $p(D)^{-s}$ on [redacted] $\mathbb{R}^n$ and multiplying by φ to descend to the [redacted] torus. Let [redacted]

$$K(x,s) = (2\pi)^{-n} \int d\xi \, e^{i\xi x} \, p(\xi)^{-s}$$

so that the kernel of $\hat{M}(s)$ is

$$\sum_m \varphi(x + 2\pi m) \, K(x + 2\pi m, s)$$

or just $\varphi(x) K(x,s)$ if $|x| \leq \pi$. The kernel of $p(D)^{-s}$ on the torus is

$$\sum_m K(x + 2\pi m, s)$$

The idea here is that if s is a negative integer: $s = -k$, then $p(D)^{-s} = p(D)^k$ is a diff'l operator and so $K(x,-k)$ is a distribution supported at $x=0$. Therefore for s a negative integer the operators $\hat{M}(s)$ and A^{-s} are identical

December 17, 1985

We continue with the problem of understanding values of ζ at negative integers for elliptic differential operators.

Suppose to simplify that order of $A >$ dim so that $\operatorname{tr}\left(\frac{1}{\lambda - A}\right)$ exists, and that $A > 0$. Then somehow by contour integration we have

$$\operatorname{tr}(A^{-s}) = \frac{1}{2\pi i}\int \operatorname{tr}\left(\frac{1}{\lambda - A}\right)\lambda^{-s}\,d\lambda \qquad Re(s) > \frac{n}{m} \ ?$$

$$= \frac{\sin(\pi s)}{\pi}\int_0^\infty \operatorname{tr}\left(\frac{1}{t+A}\right)t^{-s}\,dt \qquad \frac{n}{m} < Re(s) < 1$$

where already I assume know that $\operatorname{tr}\left(\frac{1}{t+A}\right)$ grows like $t^{-n/m}$ as $t \to \infty$. This formula shows that the singularity structure of $\zeta_A(s)$ for $Re(s) < 1$ is determined by the asymptotic expansion of $\operatorname{tr}\left(\frac{1}{t+A}\right)$ as $t \to \infty$. Thus the residues at the non integral poles and the values at the integral poles result from the asymptotic expansion of the resolvent.

It might be even nicer to work with

$$A^{-s} = \frac{1}{2\pi i}\int \frac{1}{\lambda - A}\lambda^{-s}\,d\lambda$$

$$= \frac{\sin(\pi s)}{\pi}\int_0^\infty \frac{1}{t+A}t^{-s}\,dt \qquad Re(s) < 1?$$

Another try: suppose we can find an approximation $M(t)$ to $\frac{1}{t+A}$ such that

1) $tr\left\{\frac{1}{t+A} - M(t)\right\}$ exists and is $O(t^{-N})$ $t\to\infty$

2) $tr\left\{\int_0^\infty M(t)\, t^{-s}\, dt\right\}$ has a meromorphic continuation which is calculable in terms of the symbol of A

Then

$$tr(A^{-s}) = \frac{\sin(\pi s)}{\pi}\, tr\left\{\int_0^\infty \frac{1}{t+A}\, t^{-s}\, dt\right\}$$

$$= \frac{\sin(\pi s)}{\pi}\, tr\left\{\int_0^\infty \left(\frac{1}{t+A} - M(t)\right) t^{-s}\, dt + \int_0^\infty M(t)\, t^{-s}\, dt\right\}$$

$$= \underbrace{\frac{\sin(\pi s)}{\pi}}_{\substack{\text{vanishes}\\ \text{at integers}}} \underbrace{\int_0^\infty tr\left\{\frac{1}{t+A} - M(t)\right\} t^{-s}\, dt}_{\text{analytic for } s+N>-1} + \frac{\sin(\pi s)}{\pi}\, tr \int_0^\infty M(t)\, t^{-s}\, dt$$

and so we should determine the values of zeta at negative integers from the symbol.

Now let's carry this out where $A = \Delta$, say on a circle to begin with. $\frac{1}{t+A}$ has the symbol $\frac{1}{t+\xi^2}$. Suppose I were to realize this in the usual way

$$M(t,x) = \varphi(x) \int \frac{d\xi}{2\pi}\, e^{i\xi x}\, \frac{1}{t+\xi^2}\, \rho(\xi)$$

Then $$tr\, M(t) = vol \int \frac{d\xi}{2\pi}\, \frac{\rho(\xi)}{t+\xi^2}$$

$$= \frac{vol}{2\pi}\left\{\int \frac{d\xi}{\sqrt{t}}\, \frac{1}{1+\xi^2} + \int d\xi\, \frac{1}{t+\xi^2}\left(1-\rho(\xi)\right)\right\}$$

$$= \frac{vol}{2\pi}\left\{ \frac{\pi}{\sqrt{t}} + \sum_{k\geq 0} \frac{1}{t^{k+1}} \int d\xi\, (-\xi^2)^k (1-\rho(\xi)) \right\}$$

This shows that the residues of

$$\operatorname{tr} \int_0^\infty M(t)\, t^{-s}\, dt$$

at negative integers will depend on the smoothing at $\xi = 0$. So I must proceed more carefully in the construction of $M(t)$.

I think that perhaps the missing point is that the approximation $M(t)$ to $\frac{1}{t+A}$ is only needed for large t. Let's go over the steps carefully.

We start from the contour integration formula

$$A^{-s} = \frac{1}{2\pi i} \int_C \frac{1}{\lambda - A} \lambda^{-s}\, d\lambda$$

$$= \frac{1}{2\pi i} \int_{|\lambda|=\varepsilon} \frac{1}{\lambda - A} \lambda^{-s}\, d\lambda + \frac{\sin(\pi s)}{\pi} \int_\varepsilon^\infty \frac{1}{t+A} t^{-s}\, dt$$

To simplify suppose 0 is not an eigenvalue, and that $m = \operatorname{ord}(A) > n = \dim$. Then we should get

$$\operatorname{tr}(A^{-s}) = \frac{\sin(\pi s)}{\pi} \int_0^\infty \operatorname{tr}\left(\frac{1}{t+A}\right) t^{-s}\, dt \qquad \frac{n}{m} < \operatorname{Re} s < 1$$

I want to split this into

$$\frac{\sin(\pi s)}{\pi} \left\{ \underbrace{\int_0^1 \operatorname{tr}\left(\frac{1}{t+A}\right) t^{-s}\, dt}_{\substack{\text{merom. with simple} \\ \text{poles possibly at} \\ s = 1, 2, \ldots}} + \int_1^\infty \operatorname{tr}\left(\frac{1}{t+A}\right) t^{-s}\, dt \right\}$$

So we conclude that what is relevant to the values of $\zeta(s)$ at $s=0,-1,-2,\dots$ is the asymptotics of $\mathrm{tr}\left(\frac{1}{t+A}\right)$ as $t\to+\infty$. Let us look at this question in the special case $A=-\frac{d^2}{dx^2}+z$ on S^1.

The exact kernel for $\frac{1}{t+A}$ is

$$\sum \frac{e^{inx}}{2\pi}\,\frac{1}{t+n^2+z}$$

and

$$\mathrm{tr}\left(\frac{1}{t+A}\right)=\sum_{n\in\mathbb{Z}}\frac{1}{t+n^2+z}$$

The complete symbol is $\frac{1}{t+\xi^2+z}$ so the ψDO constructed from this symbol has kernel

$$\varphi(x)\int\frac{d\xi}{2\pi}\,e^{i\xi x}\,\frac{1}{t+\xi^2+z}.$$

Wait: In the Seeley setup one expands

$$\frac{1}{t+\xi^2+z}=\frac{1}{t+\xi^2}+\frac{-z}{(t+\xi^2)^2}+\frac{z^2}{(t+\xi^2)^3}+\cdots$$

so the symbol is this formal series. The ψDO constructed from this symbol is the sum of terms

$$\varphi(x)\,(-z)^k\int\frac{d\xi}{2\pi}\,e^{i\xi x}\,\frac{1}{(t+\xi^2)^{k+1}}$$

and this term has trace

$$(-z)^k\int\frac{d\xi}{2\pi}\,\frac{1}{(t+\xi^2)^{k+1}}=\frac{(-z)^k}{t^{k+\frac12}}\int\frac{d\xi}{2\pi}\,\frac{1}{(1+\xi^2)^{k+1}}$$

Compare with

$$\sum\frac{1}{t+n^2+z}=\frac{\pi}{\sqrt{t+z}}+\text{exp. decaying error as } t\to\infty$$

$$\therefore \quad \frac{\pi}{\sqrt{t+z}} = \frac{\pi}{\sqrt{t}}\left(1+\frac{z}{t}\right)^{-1/2}$$

$$= \frac{\pi}{\sqrt{t}}\left(1 - \frac{1}{2}\frac{z}{t} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\frac{z^2}{t^2} + \cdots\right)$$

Therefore the symbol calculus is supposed to yield the asymptotics of $\mathrm{tr}\left(\frac{1}{t+A}\right)$ as $t \longrightarrow \infty$.

December 18, 1985

On Gaussian measures again. The goal this time is to understand Malliavin's infinitely differentiable functions on Wiener space. I think the construction makes sense for any Gaussian measure.

Let V be a real vector space and suppose a Gaussian measure $d\mu$ is given on V. This Gaussian measure is specified by the variance, which is an inner product on $V^* = \mathrm{Hom}(V, \mathbb{R})$

$$|\lambda|^2 = \langle \lambda^2 \rangle$$

One has

$$\langle e^{i\lambda} \rangle = e^{-\frac{1}{2}|\lambda|^2}$$

for the F.T. of the Gaussian measure.

A basic result is the Hermite polynomial decomposition of $L^2(X, d\mu)$. If H is the Hilbert space obtained by completing V^* with respect to the inner product $|\lambda|^2$, then this decomposition is a canonical isomorphism

$$\hat{S}(H) \simeq L^2(X, d\mu)$$

where $S(H)$ is the symmetric tensor Hilbert space generated by H.

To derive this result let's first look at $V = \mathbb{R}$ with Gaussian measure $d\mu = e^{-\frac{1}{2}x^2}\frac{dx}{\sqrt{2\pi}}$ such that $\langle x^2 \rangle = 1$. Compute the adjoint of $\frac{d}{dx}$

$$\int \left(\frac{d}{dx} f\right) g \, e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}} = \int \left[f \left(-\frac{d}{dx} + x\right) g \right] e^{-\frac{1}{2}x^2} \frac{dx}{\sqrt{2\pi}}$$

whence if [redacted] we set $\partial = \frac{d}{dx}$

$$\partial^* = -\partial + x$$

Thus we obtain creation and annihilation operators

$$a = \partial \qquad\qquad [a, a^*] = 1.$$
$$a^* = -\partial + x$$

and we have a unique isomorphism

$$\underset{\substack{\parallel \\ \text{orth basis} \\ \frac{z^n}{\sqrt{n!}} \quad n \geq 0}}{\hat{S}(\mathbb{R})} \simeq L^2(\mathbb{R}, d\mu)$$

such that $1 \longmapsto 1$ which is compatible with a, a^*; here on the left $a = \frac{d}{dz}$, $a^* = z$.

Thus

$$z^n \longmapsto (x - \partial)^n 1$$

and

$$e^{\lambda z} \longmapsto \underbrace{e^{\lambda x - \lambda\partial}}_{e^{\lambda x} e^{-\lambda\partial} e^{-\frac{1}{2}[\lambda x, -\lambda\partial]}} 1$$

so

$$e^{\lambda z} \longmapsto e^{\lambda x - \frac{1}{2}\lambda^2}$$

December 19, 1985

Let V be a real vector space, let V^* be its dual, and let (v, λ) be the canonical pairing. On $S(V^*)$ we have the operators of multiplication by a $\lambda \in V^*$ and differentiation ∂_v in the direction $v \in V$. We suppose given an inner product on V^* which we write

$$\langle \lambda | \mu \rangle = \langle \lambda^*, \mu \rangle$$

in terms of a map $\lambda \mapsto \lambda^*$ from V^* to V. (This map is an isomorphism in the finite diml case.)

Assume the inner product on V^* comes from a Gaussian probability measure $d\mu$ on V. Then we obtain an inner product on $S(V^*)$ by means of the embedding

$$S(V^*) \hookrightarrow L^2(V, d\mu)$$

On the other hand the inner product on V^* extends to an inner product on $S(V^*)$ characterized by

$$\begin{cases} |\lambda^n|^2 = n!(|\lambda|^2)^n \\ S^n(V^*) \perp S^{n'}(V^*) \qquad n \neq n'. \end{cases}$$

<u>Assertion</u>: There is an isomorphism

$$S(V^*) \xrightarrow{\sim} S(V^*)$$

given by $\qquad e^\lambda \longmapsto e^{-\frac{1}{2}|\lambda|^2} e^\lambda$

which is unitary with respect to the symmetric tensor inner product on the left and the $L^2(V, d\mu)$ inner product on the right.

Unitarity: $\langle e^\lambda | e^\mu \rangle = e^{\langle \lambda | \mu \rangle}$ in the symmetric tensor inner product, whereas $\langle e^{-\frac{1}{2}|\lambda|^2} e^\lambda \, e^{-\frac{1}{2}|\mu|^2} e^\mu \rangle =$

$$= e^{-\frac{1}{2}(|\lambda|^2+|\mu|^2)} \underbrace{\langle e^{\lambda+\mu}\rangle}_{e^{\frac{1}{2}|\lambda+\mu|^2}} = e^{\langle\lambda|\mu\rangle}$$

in $L^2(V, d\mu)$.

Further formulas. Suppose we are in finite dimensional with $d\mu = e^{-\frac{1}{2}\langle x|x\rangle}\frac{dx}{N}$. Compute the adjoint of ∂_v for the inner product in $L^2(V, d\mu)$.

$$\int \partial_v f \cdot g \, e^{-\frac{1}{2}|x|^2} dx = \int f \cdot \left(e^{\frac{1}{2}|x|^2}(-\partial_v)e^{-\frac{1}{2}|x|^2} g\right) e^{-\frac{1}{2}|x|^2} dx$$

so

$$(\partial_v)^* = e^{\frac{1}{2}|x|^2}(-\partial_v)e^{-\frac{1}{2}|x|^2}$$

$$= -\partial_v + \partial_v\left(\tfrac{1}{2}|x|^2\right)$$

$$= -\partial_v + v^*$$

where $v^* \in V^*$ corresponds to v under the isom. $\lambda \mapsto \lambda^*$ from V^* to V.

On $L^2(V, d\mu)$ let us define operators

$$a_\lambda = \partial_{\lambda^*} \qquad \lambda \in V^*$$

$$a_\lambda^* = (\partial_{\lambda^*})^* = -\partial_{\lambda^*} + \lambda$$

Then

$$[a_\lambda, a_\mu] = 0$$

$$[a_\mu, a_\lambda^*] = [\partial_{\mu^*}, -\partial_{\lambda^*} + \lambda] = (\mu^*, \lambda) = \langle\mu|\lambda\rangle$$

$$[a_\lambda^*, a_\mu^*] = [-\partial_{\lambda^*} + \lambda, -\partial_{\mu^*} + \mu] = -\langle\lambda|\mu\rangle + \langle\mu|\lambda\rangle = 0$$

Thus we have creation (annihilation and) operators on $S(V^*) \subset L^2(V, d\mu)$, hence we know by the uniqueness of repns. of the CCR that there is an isometry

$$S(V^*) \longrightarrow S(V^*)$$

uniquely determined by

$$1 \longmapsto 1$$

$$\partial_{\lambda^*} \longleftrightarrow \partial_\lambda = a_\lambda$$

$$\lambda \longleftrightarrow -\partial_{\lambda^*} + \lambda = \text{[redacted]}\ a^*_\lambda$$

This [redacted] map sends

$$\lambda^n = (a^*_\lambda)^n 1 \quad \text{to} \quad (\lambda - \partial_{\lambda^*})^n 1$$

and hence it sends

$$e^\lambda \quad \text{to} \quad e^{\lambda - \partial_{\lambda^*}} 1 = e^{-\frac{1}{2}[\lambda, -\partial_{\lambda^*}]} e^\lambda e^{-\partial_{\lambda^*}} 1$$

$$= e^{-\frac{1}{2}|\lambda|^2} \cdot e^\lambda$$

December 20, 1985

If H is a real Hilbert space, then one can form its symmetric tensor Hilbert space $\hat{S}(H)$ and also the Hilbert space $L^2(H, d\mu)$, where $d\mu$ is the Gaussian cylinder measure on H. (I want to think of $\hat{S}(H)$ as a space of polynomial functions in H, so maybe I should write $\hat{S}(H^*)$ instead.) One has a canonical isomorphism

$$\hat{S}(H) \xrightarrow{\sim} L^2(H, d\mu)$$

such that

$$1 \longleftrightarrow 1$$

$$a_h \longleftrightarrow \partial_h$$

$$a_h^* \longleftrightarrow -\partial_h + h$$

here $h \in H$ is identified with the linear fnl $\langle h | ? \rangle$

$$e^h \longleftrightarrow e^{-\frac{1}{2}|h|^2} e^h$$

Let's now describe the Sobolev spaces à la Malliavin. Given an elt Φ of $L^2(H, d\mu)$ we can ask [redacted] that it be in the domain of all the differentiation operators ∂_h and that the resulting map $h \mapsto \partial_h \Phi$ be represented by an elt of $H^* \hat{\otimes} L^2(H, d\mu)$, called the gradient $\nabla\Phi$. Another way of putting this is to pick an orthonormal basis x^μ for H^* and ask that

$$\sum_\mu \|\partial_\mu \Phi\|^2 < \infty$$

$$\shortparallel$$

$$\langle \Phi | \sum a_\mu^* a_\mu \Phi \rangle$$

This defines the Sobolev 1-[redacted] norm, at least when added to $\|\Phi\|^2$.

A [redacted] basic fact about the gradient map [redacted] on $\hat{S}_k(H^*)$

which goes

$$\hat{S}_k(H^*) \xrightarrow{\nabla} H^* \otimes \hat{S}_{k-1}(H^*)$$

is that $\|\nabla \Phi_k\|^2 = k\|\Phi_k\|^2$ if $\Phi_k \in \hat{S}_k(H^*)$. Hence the Sobolev space of order 1 consists of $\Phi = \sum_{k\geq 0} \Phi_k$ such that $\sum k\|\Phi_k\|^2 < \infty$. Similarly the Sobolev space of order 2 will consist of Φ with $\sum k(k-1)\|\Phi_k\|^2 < \infty$, etc. The infinitely differentiable space consists of Φ such that $\|\Phi_k\|^2 k^n \to 0$ for any n.

For example exponential functions e^λ $\lambda \in H^*$ belong to the ∞-diff. space $\mathcal{H}^\infty$. But some Gaussian functions e^Q, $Q \in S^2(H^*)$ do not. NO For example

$$\| e^{\frac{\lambda}{2} a^{*2}} 1 \|^2 = \sum \frac{1}{k!}\left(\frac{\bar\lambda}{2}\right)^k \frac{1}{k!}\left(\frac{\lambda}{2}\right)^k \|a^{*2k}1\|^2$$

$$= \sum \frac{2k!}{k!\,k!}\left(\frac{|\lambda|^2}{4}\right)^k = \sum \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2k-1}{2}\right)}{k!}\left(-|\lambda|^2\right)^k$$

$$= \left(1-|\lambda|^2\right)^{-1/2}$$

I haven't done this correctly. We have

$$e^{\frac{\lambda}{2}z^2} = \sum_k \frac{\lambda^k}{k!\,2^k} z^{2k}$$

$$\| e^{\frac{\lambda}{2}z^2}\|^2 = \sum_{k=0}^{\infty} \frac{2k!}{k!\,k!}\frac{|\lambda|^{2k}}{4^k} = \sum_{k=0}^{\infty} \frac{\frac{1}{2}\frac{3}{2}\cdots\frac{2k-1}{2}}{k!}|\lambda|^{2k}$$

As

$$\frac{\frac{1}{2}\frac{3}{2}\cdots\frac{2k-1}{2}}{k!} = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2k}\right) \sim \exp\left(-\frac{1}{2}\log k\right) = \frac{c}{\sqrt{k}}$$

One sees that for $|\lambda| < 1$ the series $e^{\frac{\lambda}{2}z^2}$ does represent an inf. diff'ble element.

December 24, 1985

Program: Reconstruct the talk by Lewis (Dublin) on quantum stochastic processes, where he mentioned the quantization of a harmonic oscillator coupled to the EM field. There is a point in his notes I don't understand concerning a map between Weyl algebras $W(\mathcal{H}) \longrightarrow W(\mathcal{K})$ induced by a contraction operator from $\mathcal{H}$ to $\mathcal{K}$. (probably a completely positive map)

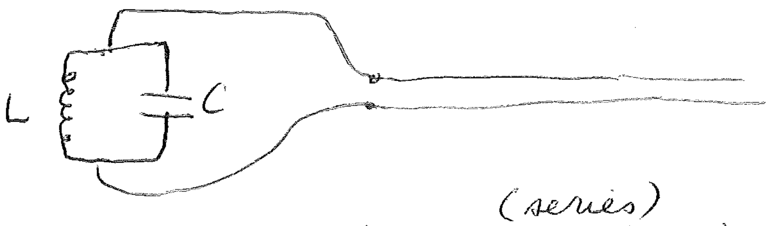
It seems to me that the way to handle things is to describe in my own terms the oscillator coupled to the EM field. This example occurred in my past work on trying to understand quantum mechanics. I recall feeling that new physics might result from a better approach to this example. The problem is to see how the oscillator radiates energy and decays. On the classical level things work well.

The classical picture can be described nicely from the viewpoint of hyperbolic DE's. The motions of the field plus oscillator form a real vector space with inner product given by the energy norm. Time translation gives a 1-parameter orthogonal group. One sees exponential decay of energy in any bounded region. The infinitesimal generator of time translation for the hyperbolic equation has both positive and negative energies. Quantization chops off the negative energies in a somewhat brutal way which introduces a discontinuity in the spectrum at 0.

(Question: Do solutions of Klein-Gordon have exponential decay?)

This discontinuity prevents the exponential decay of probability that the oscillator remain excited.

Let's now go over an example. I replace the EM field by a transmission line, and let the oscillator be a "tuned circuit"



If L, C are in parallel (series), the impedance of the transmission line should be large (small), so as to alter the resonant frequency $\frac{1}{\sqrt{LC}}$ only slightly.

I will assume unit impedance and speed on the transmission line, whence the equations of the voltage V and current I along the line are

$$-\frac{\partial V}{\partial t} = \frac{\partial I}{\partial x}$$

$$-\frac{\partial I}{\partial t} = \frac{\partial V}{\partial x}$$

whence

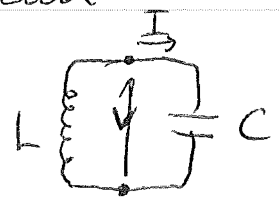
$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)(V + I) = 0 \qquad V + I = f(x - t) \text{ outgoing}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)(V - I) \qquad V - I = g(x + t) \text{ incoming}$$

The energy is

$$\int (V^2 + I^2)\,dx = \frac{1}{2}\int \left[(V+I)^2 + (V-I)^2\right] dx$$

Let's first see what happens with a "tuned circuit". The equations are



$$I = C\frac{dV}{dt}$$

$$V = -L\frac{dI}{dt}$$

This is a first order ODE with 2 unknowns + with constant coefficients. The space of solutions is 2 dimensional and a solution is specified by giving the values of V, I at a fixed time. The initial value problem may be solved by means of Laplace transform:

$$\frac{1}{C}\hat{I}(s) = \int_0^\infty e^{-st}V'(t)\,dt = -V_0 + s\hat{V}(s)$$

$$-\frac{1}{L}\hat{V}(s) = -I_0 + s\hat{I}(s)$$

$$\begin{pmatrix} s & -\frac{1}{C} \\ \frac{1}{L} & s \end{pmatrix}\begin{pmatrix} \hat{V} \\ \hat{I} \end{pmatrix} = \begin{pmatrix} V_0 \\ I_0 \end{pmatrix}$$

Natural modes occur when $\begin{vmatrix} s & -\frac{1}{C} \\ \frac{1}{L} & s \end{vmatrix} = s^2 + \frac{1}{LC} = 0$

i.e. at frequency $\frac{1}{\sqrt{LC}}$.

The energy of the circuit is

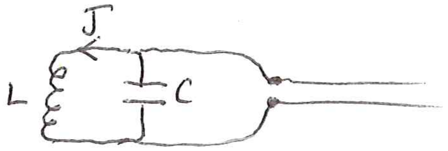
$$\frac{1}{2}LI^2 + \frac{1}{2}CV^2$$

and this is constant

$$\begin{aligned}\frac{d}{dt}\left(\tfrac{1}{2}LI^2 + \tfrac{1}{2}CV^2\right) &= LI\dot{I} + CV\dot{V} \\ &= LI\left(-\frac{V}{L}\right) + CV\left(\frac{I}{C}\right) \\ &= 0\end{aligned}$$

Thus in this example we have a 2 dimensional real Hilbert space with 1-parameter unitary (orthogonal) group.

Next let's consider the circuit coupled to the transmission line. Let J be the current flowing into the conductor. Then we have the equations



$x=0$

$$\star \quad \begin{cases} V(0) = L\dfrac{dJ}{dt} \\ C\dfrac{dV(0)}{dt} = -J - I(0) \end{cases} \qquad -\frac{\partial V}{\partial t} = \frac{\partial I}{\partial x} \qquad -\frac{\partial I}{\partial t} = \frac{\partial V}{\partial x}$$

The energy in the line is proportional to

$$\frac{1}{2}\int_0^\infty (V^2 + I^2)\,dx.$$

See how this changes in time

$$\partial_t \frac{1}{2}\int_0^\infty (V^2 + I^2)\,dx = \int (V\dot{V} + I\dot{I})\,dx$$

$$= -\int_0^\infty (V\,\partial_x I + I\,\partial_x V)\,dx = -[VI]_0^\infty$$

$$= V(0)\,I(0).$$

Let's try the energy

$$\boxed{E = \frac{1}{2}LJ^2 + \frac{1}{2}C\,V(0)^2 + \frac{1}{2}\int_0^\infty (V^2 + I^2)\,dx}$$

Then

$$\dot{E} = J\underbrace{L\dot{J}}_{V(0)} + V(0)\underbrace{C\dot{V}(0)}_{-J-I(0)} + V(0)\,I(0) = 0$$

so this is correct.

Question: Is there a natural symplectic structure on the space of solutions?

Origin of the question: Once one has the symplectic structure on the real Hilbert space of solutions the Weyl algebra is determined, and it appears that the quantization is completely fixed.

Let's leave this question aside and return to our Hilbert space given by completing solutions of the equations ☆ with respect to the energy E. I want next to understand the structure of this space, call it $\mathcal{K}$.

Sitting inside of $\mathcal{K}$ is a 1-dimensional subspace consisting of (J, V, I) where $V = I = 0$. In fact there is a 2 dimensional space which sits in the completion, but not in the space of initial data where $V(x), I(x)$ are supposed in $C_0^\infty(\mathbb{R}_{\geq 0})$. This is because $\mathcal{K}$ is clearly isomorphic as a Hilbert space to a direct sum of $\mathbb{R}$ with $\|J\|^2 = LJ^2$ and $L^2(\mathbb{R}_{\geq 0}, C\delta(0) + dx)$ and $L^2(\mathbb{R}_{\geq 0}, dx)$.

The program now is to describe carefully the solutions of our equation, and to investigate the nature of decay. I have been thinking that the Laplace transform is suitable for this purpose, since it is adapted to the initial value problem.

If we are given initial data $(J_0, V_0(x), I_0(x))$ we Laplace transform ☆ to get ODE's depending on the parameter s:

$$\begin{cases} \partial_x \hat{I} + s\hat{V} = V_0 \\ \partial_x \hat{V} + s\hat{I} = I_0 \end{cases} \qquad \begin{aligned} (\partial_x + s)(\hat{V} + \hat{I}) &= (V_0 + I_0) \\ (\partial_x - s)(\hat{V} - \hat{I}) &= -V_0 + I_0 \end{aligned}$$

$$\begin{cases} \frac{1}{L}\widehat{V(0)} = s\hat{J} - J_0 \\ -\frac{1}{C}\left(\hat{J} + \widehat{I(0)}\right) = s\hat{V} - V_0 \end{cases}$$

Notice that the first two equations determine $\hat{V}, \hat{I}$ up to $\widehat{V(0)}$, $\widehat{I(0)}$ and that the remaining two equations, when $\hat{J}$ is eliminated, yield only one relation on $\widehat{V(0)}$ and $\widehat{I(0)}$. Thus we are missing something, probably a boundary condition at $x = +\infty$.

Let's consider the case where we have an open transmission line, which means that we want to solve

$$\begin{aligned} (\partial_t + \partial_x)(V + I) &= 0 \\ (\partial_t - \partial_x)(V - I) &= 0 \end{aligned} \qquad \begin{aligned} &x \geq 0 \\ &t \in \mathbb{R} \end{aligned}$$

subject to requiring $I(0,t) = 0$ for all t.

Solutions of these equations are

$$\begin{aligned} (V + I)(x,t) &= f(x-t) \\ (V - I)(x,t) &= g(x+t) \end{aligned}$$

$$f(-t) = V(0,t) = g(t)$$

whence

$$V = \frac{f(x-t) + f(-x-t)}{2}$$

$$I = \frac{f(x-t) - f(-x-t)}{2}$$

is the solution. Then $\begin{aligned} V(x,0) &= (f(x) + f(-x))/2 \\ I(x,0) &= (f(x) - f(-x))/2 \end{aligned}$

and so f can be found from the initial values

$$f(x) = V(x,0) + I(x,0) \qquad x \geq 0$$

$$f(-x) = V(x,0) - I(x,0) \qquad x \geq 0$$

Now when it comes to the Laplace transform we have the transformed equation

$$(\partial_x + s)(\widehat{V+I})(x,s) = V(x,0) + I(x,0)$$

$$= f(x) \qquad x \geq 0.$$

This equation needs a boundary condition before it determines $\widehat{V+I}(x,s)$. Note that

$$\widehat{V+I}(x,s) = \int_0^\infty e^{-st} f(x-t)\, dt$$

$$= e^{-sx} \int_{-\infty}^{x} e^{st} f(t)\, dt$$

uses the values of $f(x)$ for $x < 0$

December 25, 1985

Yesterday I encountered difficulties with the Laplace transform approach to solving the IVP for a transmission line coupled to a tuned circuit. Let's drop this approach and instead concentrate on having a real Hilbert space with 1-parameter unitary group, and let's investigate the spectral decomposition.

The equations for the transmission line + circuit are

A

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)(V+I)=0 \qquad L\frac{dJ}{dt} = V(0)$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)(V-I)=0 \qquad C\frac{dV(0)}{dt} + J + I(0) = 0$$

To carry out the spectral decomposition we look for solutions with a given time dependence

$$V(x,t) = \hat{V}(x,\omega)\, e^{-i\omega t} \qquad \text{etc.}$$

so if we write $s=-i\omega$ we have

$$\left(\frac{\partial}{\partial x}+s\right)\widehat{V+I} = 0 \qquad \widehat{V+I} = Ae^{-sx} \text{ or } Ae^{i\omega x}$$

$$\left(\frac{\partial}{\partial x}-s\right)\widehat{V-I} = 0 \qquad \widehat{V-I} = Be^{sx} \text{ or } Be^{-i\omega x}$$

(I recall thinking of

$$\widehat{V+I} = A(k)e^{ikx} \qquad \text{as outgoing}$$

$$\widehat{V-I} = B(k)e^{-ikx} \qquad \text{as incoming}$$

because

$$f(x-t) = \int \frac{dk}{2\pi} e^{ik(x-t)}\hat{f}(k) \qquad \text{is an outgoing wave}$$

$$g(-x-t) = \int \frac{dk}{2\pi} e^{-ik(x+t)}\hat{g}(k) \qquad \text{is an incoming wave.)}$$

Can I say this better?

We have a Hilbert space with 1-parameter unitary group. There is some relation between the spectral decomposition of this unitary group and solutions of the wave equations with time dependence $e^{st} = e^{-i\omega t}$. So we calculate what these are. We find

$$\widehat{V+I} = Ae^{-sx} \qquad Ls\hat{J} = \widehat{V(0)} = \frac{A+B}{2}$$

$$\widehat{V-I} = Be^{sx} \qquad Cs\widehat{V(0)} + \hat{J} + \underbrace{\widehat{I(0)}}_{\frac{A-B}{2}} = 0$$

So $$\left(Cs + \frac{1}{Ls}\right)(A+B) + (A-B) = 0$$

$$\therefore \quad \underbrace{\frac{A+B}{A-B}}_{\frac{S+1}{S-1}} + \underbrace{\frac{1}{\frac{1}{Ls}+Cs}}_{Z(s)} = 0$$

$Z(s)$ impedance of 

where $S = \frac{A}{B}$ is the scattering. Thus the scattering is

$$\boxed{S = \frac{Z-1}{Z+1}}$$

in terms of the impedance of the circuit.

We have the incoming an outgoing representation. Given a solution (J, V, I) we know there are functions $f(x)$, $g(x)$ defined on $\mathbb{R}$ such that

$$(V+I)(x,t) = f(x-t)$$

$$(V-I)(x,t) = g(-x-t)$$

By associating to a solution (J, V, I) the functions f, g we get the outgoing and incoming representations. It seems nicer to write

$$\frac{V+I}{2}(x,t) = f(x-t) = \int \frac{dk}{2\pi} e^{ikx} e^{-ikt} \hat{f}(k)$$

$$\frac{V-I}{2}(x,t) = g(-x-t) = \int \frac{dk}{2\pi} e^{-ikx} e^{-ikt} \hat{g}(k)$$

for then the energy of the solution is

$$E = \lim_{t \to +\infty} \frac{1}{2} \int_0^\infty (V^2 + I^2)(x,t)\,dx = \int_{-\infty}^{\infty} f(x)^2 dx = \int \frac{dk}{2\pi} |\hat{f}(k)|^2$$

$$= \lim_{t \to -\infty} \int \left[\left(\frac{V+I}{2}\right)^2 + \left(\frac{V-I}{2}\right)^2\right] dx = \int_{-\infty}^{\infty} g(x)^2 dx = \int \frac{dk}{2\pi} |\hat{g}(k)|^2$$

Next we can consider the part of the Hilbert space consisting of solutions which at time $t=0$ are $\equiv 0$ for $x > 0$. (Actually I would like to speak of a filtration F_t increasing in t which for $t \geq 0$ consists of solutions with f satisfying $f(x) = 0$ for $x > t$. For $t \leq 0$ F_t consists of solutions with g satisfying $g(x) = 0$ for $x < -t$.)

The solutions with $f(x) = 0$ for $x > 0$ correspond to elements of the Hardy space $H_+ \subset L^2(\mathbb{R}, \frac{dk}{2\pi})$ consisting of $\hat{f}(k)$ which extend analytically (and bdd in some sense) to the UHP. Similarly the solutions with $g(x) = 0$ for $x > 0$ have $\hat{g}(k) \in H_+$ and hence $\hat{f}(k) = S(k)\hat{g}(k) \in SH_+$.

Therefore the subspace of solutions which at time 0 don't appear in the transmission line should be

$$H^+ \ominus SH^+ = H^+ \cap S(H^-).$$

As a check we should see that $SH^+ \subset H^+$.

Take the impedance $Z(s) = Ls$, whence

$$S = \frac{Z-1}{Z+1} = \frac{Ls-1}{Ls+1} = \frac{-Lik-1}{-Lik+1} = \frac{k - i\frac{1}{L}}{k + i\frac{1}{L}}$$

To simplify set $L = 1$, whence $S = \frac{k-i}{k+i}$. Then

$$SH^+ = \frac{k-i}{k+i} H^+ = (k-i) H^+$$

is the subspace of H^+ consisting of $\hat{f}(k)$ which vanish at $k = i$. Note that

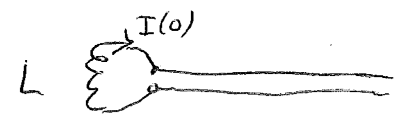
$$\frac{1}{k+i} = S \frac{1}{k-i} \in H^+ \cap SH^-$$

and the corresponding solution has

$$f(x) = \int \frac{dk}{2\pi} e^{ikx} \frac{1}{k+i} = \begin{cases} 0 & x > 0 \\ -ie^{+x} & x < 0 \end{cases}$$

$$g(x) = \int \frac{dk}{2\pi} e^{ikx} \frac{1}{k-i} = \begin{cases} 0 & x < 0 \\ ie^{-x} & x > 0 \end{cases}$$

What we have in this case is

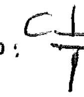


and the inductance is energized at $t = 0$ which means that current is flowing through the coils and there is no voltage + current in the transmission line. Now the situation is like a inductance losing energy to a resistance

$$-\underbrace{L}_{1}\frac{dI(0)}{dt} = \underbrace{R}_{1}I(0) \qquad I(0,t) = e^{-t} \quad \overset{\parallel}{V(0,t)}$$

whence $V(x,t) = I(x,t) = \begin{cases} e^{x-t} & x \le t \\ 0 & x > t \end{cases}$

for $t > 0$.

Other examples:  $Z(s) = \frac{1}{Cs}$. Then

$$S(k) = \frac{Z-1}{Z+1} = \frac{1+Cik}{1-Cik} = -\frac{k - i\frac{1}{C}}{k + i\frac{1}{C}}$$

is of the same sort.

 $Z = \frac{1}{\frac{1}{Ls} + Cs} = \frac{Ls}{CLs^2+1}$

$$S = \frac{Z-1}{Z+1} = -\frac{CLs^2 - Ls + 1}{CLs^2 + Ls + 1}$$

The ~~denominator~~ numerator has roots

$$s = \frac{+L \pm \sqrt{L^2 - 4CL}}{2CL}$$

satisfying $\mathrm{Re}(s) > 0$, which corresponds to $\omega = is$ being in the UHP. In this case H^+/SH^+ is 2 dimensional corresponding to the two zeros of $S(k)$ in the UHP.

Our next project will be to calculate the one parameter family of contractions on $\mathcal{H} = H^+ \ominus SH^+ = H^+ \cap SH^-$ induced by the one parameter unitary group $U_t : t \mapsto$ multiplication by e^{-ikt}. This [redacted] depends only on $S(k)$.

Example: $S(k) = \dfrac{k - ia}{k + ia}$, $a > 0$.

Then

$$H^+ \ominus SH^+ = H^+ \cap SH^- \ni \frac{1}{k+ia} = S\,\frac{1}{k-ia}$$

and

$$\left\langle \frac{1}{k+ia} \middle| e^{-ikt} \cdot \frac{1}{k+ia} \right\rangle = \int \frac{dk}{2\pi}\, \frac{e^{-ikt}}{k^2 + a^2}$$

$$= \frac{e^{-a|t|}}{2a}$$

$\mathcal{H}$ is generated by the unit vector $\dfrac{\sqrt{2a}}{k+ia} = \Phi$ and we have just seen that the contraction of U_t to $\mathcal{H}$

$$\langle \Phi | U_t | \Phi \rangle = e^{-a|t|}$$

This is certainly a 1-parameter semi-group for $t \geq 0$.

Actually it seems to be a general fact ~~that~~ connected with scattering that the contraction of U_t on $\mathcal{H}$ gives contraction semi groups for $t \geq 0$ and $t \leq 0$, one being the adjoint of the other. One has that $U_t = e^{-ikt}$ carries H^- into ~~itself~~ itself for $t \geq 0$. and U_t commutes with the scattering operator S. Thus U_t for $t \geq 0$ induces an operator semi group on SH^- / H^-; ~~other~~ ~~contracts~~ using the isom $SH^-/H^- = SH^- \cap H^+$ we see that T_t is a semi group for $t \geq 0$.

More precisely let $v \in \mathcal{H} = H^+ \cap SH^-$. As SH^- is stable under U_t $t \geq 0$ we [redacted] see from

$$U_t v = T_t v + w_t \qquad w_t \in \text{[redacted]}\ \mathcal{H}^\perp$$

that $w_t = U_t v - T_t v \in SH^-$, so $w_t \in SH^- \ominus \mathcal{H} = H^-$ (as $H^- \subset SH^-$ so $SH^- \ominus H^- = SH^- \cap H^+ = \mathcal{H}$)

Thus

$$U_{t'} U_t v = U_{t'} T_t v + \underbrace{U_{t'} w_t}_{\in H^-} \qquad t' \geq 0$$

and so projecting onto $\mathcal{H}$ one gets

$$T_{t'+t} v = T_{t'} T_t v .$$

Thus it follows that in these scattering examples we get contraction semi-groups. Let's consider this now in the case where

$$S(k) = \frac{(k-a)(k-b)}{(k-\bar{a})(k-\bar{b})}$$

where a, b belong to the UHP and are symmetrical with respect to the imaginary axis. Then

$$\underbrace{\frac{1}{k-\bar{a}}}_{H^+} = S(k) \underbrace{\frac{k-\bar{b}}{(k-a)(k-b)}}_{\in H^-}$$

so we see that $\mathcal{H}$ has the basis $\frac{1}{k-\bar{a}}, \frac{1}{k-\bar{b}}$.

In fact we have the isom

$$\mathcal{H} \longrightarrow H^+/SH^+ = H^+/(k-a)(k-b)H^+$$

which commutes with T_{-t} on $\mathcal{H}$ and e^{ikt} on the latter for $t \geq 0$. This enables us to find the eigenvector.

for T_{-t} namely

$$\frac{k-b}{(k-\bar{a})(k-\bar{b})} = S\frac{1}{k-a} \quad \text{has eigenvalue } e^{iat}$$

and $$\frac{k-a}{(k-\bar{a})(k-\bar{b})} = S\frac{1}{k-b} \quad \text{" \quad " } \quad e^{ibt}$$

On the other hand we have

$$H^+ \supset (k-a)H^+ \supset (k-a)(k-b)H^+$$

stable under e^{ikt} $t \geq 0$. The fn. $\frac{k-a}{(k-\bar{a})(k-\bar{b})}$ spans the orthognal complement for the second inclusion and $\frac{1}{k-\bar{a}}$ spans the orth complement for the first inclusion. The pair $\frac{1}{k-\bar{a}}, \frac{k-a}{(k-\bar{a})(k-\bar{b})}$ is an orthogonal basis for $\mathcal{H}$, the second vector being an eigenvector for T_{-t}. The function $\frac{1}{k-\bar{a}}$ has non-zero values at both a and b, hence we see that $\frac{1}{k-\bar{a}}$ is not an eigenvector for T_{-t}. Thus we conclude that the contraction semi-group T_{+t} on $\mathcal{H}$ for $t \leq 0$ is <u>not</u> normal.

<u>December 26, 1985</u>

What is the symplectic structure on the space of solutions of the wave equation for the transmission line:

$$\partial_t \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

$$\text{energy} = \frac{1}{2} \int (V^2 + I^2)\, dx$$

First consider the wave equation

$$\left[+\partial_t^2 + (-\partial_x^2 + V) \right] u = 0.$$

One views this as a harmonic oscillator. Configuration space is the vector space of functions $u(x)$ and on it we have the inner product $\int u^2 dx$ and the quadratic form $\int u(-\partial_x^2 + V) u \, dx$ represented by the self adjoint operator $-\partial_x^2 + V$. One has the Lagrangian

$$K.E. - P.E. = \int \left[\frac{1}{2} \dot{u}^2 - \frac{1}{2} u (-\partial_x^2 + V) u \right] dx$$

whose extrema are solution of the wave equation. Phase space is the space of Cauchy data $(u(x), \dot{u}(x))$; it carries the canonical 1-form $\alpha = \int \dot{u}(x)\, du(x)\, dx$ defining the symplectic structure, and the wave equation also results from the Hamiltonian flow on phase space associated to the function on phase space giving the total energy.

Now look at the transmission line. Here we have a first order equation, so phase space is the space M of $V(x), I(x)$; it is a real vector space with inner product given by the energy $(\times 2) = \int (V^2 + I^2)\, dx$. The generator of time evolution is a skew-adjoint operator. This skew-adjoint operator defines a skew symmetric form on phase space M, and it is natural to ask if the flow is then the Hamiltonian flow associated to the energy function.

Let's set this up abstractly. Suppose given a real vector space M with an inner product $(\,,\,)$ and a non-degenerate skew-symmetric form ω. We identify $T(M)_m$ with M in the usual way, whence ω can be viewed as a 2-form invariant under translation. Let $H(m) = \frac{1}{2}|m|^2$ and let's calculate the associated vector field X_H, which is defined by

$$* \qquad \iota(X_H)\,\omega = dH$$

If $v \in M$, let ∂_v be the corresponding vector field. Let A be the skew-adjoint operator on M which expresses ω using $(\,,\,)$:

$$\iota(\partial_w)\, i(\partial_v)\,\omega = (w, Av)$$

Then $*$ becomes at a point m

$$\begin{array}{ccccc} \iota(\partial_v)\, i(X_{H,m})\,\omega & = & (\partial_v H)_m & = & \frac{d}{dt}\left(\frac{1}{2}|m+tv|^2\right)\Big|_{t=0} \\ \| & & & & \| \\ (v, AX_{H,m}) & & & & (m,v) \end{array}$$

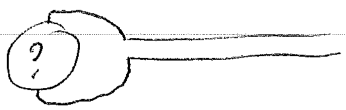
so we conclude $\quad AX_{H,m} = m \quad$ or

$$\boxed{X_{H,m} = A^{-1} m}$$

Thus the Hamiltonian flow is given by the inverse of the skew-adjoint operator representing ω in terms of the inner product.

It follows that the time evolution plus the inner product determine the symplectic structure.

Summarize: When we consider a circuit coupled to a transmission line



we get a real vector space M of solutions on which there is an inner product given by the energy and a time evolution preserving the inner product. The infinitesimal generator of the time evolution is skew-adjoint and injective (usually)), so there is a unique symplectic structure on M such that the time evolution is the Hamiltonian flow associated to the energy.

Let's use this in connection with the outgoing or incoming representation which tells us that M is isomorphic to the real $L^2(\mathbb{R})$ with time acting as translations, or via F.T. to the space of square integrable functions $f(k)$ on $\mathbb{R}$, complex-valued, such that $f(-k) = \overline{f(k)}$; here $U_t f(k) = e^{-ikt} f(k)$.

Recapitulate: The ultimate goal is to quantize the circuit + transmission line, and I want to keep track of the possibilities. What we have is a real Hilbert space M with 1-parameter orthogonal group having no fixed vectors. I have seen there is a symplectic structure on M, so I can form the Weyl algebra. There is also a! complex structure on M such that all frequencies are positive. On the other hand a complex Hilbert space has an underlying real symplectic structure. What is the relation between these viewpoints?

Given $(\, , \,)$ on M and the skew symmetric operator X generating the time evolution. This determines the symplectic form (Av, w), where A

is either X^{-1} (or $-X^{-1}$)

On the other hand the complex structure on M such that all eigenvalues are in $i\mathbb{R}_{>0}$ is

$$J = \frac{X}{|X|} \qquad |X| = (X^*X)^{1/2} = (-X^2)^{1/2}$$

since if multiplication by i on M is defined to be J, then we have $X = |X| i$ with $|X| > 0$.

Now given $(\,,\,)$ and J $(J^2 = -1,\ J = -J^*)$ the extension of $(\,,\,)$ to a hermitian inner product is

$$\begin{aligned} \langle v | w \rangle &= (v, w) + i(Jv, w) \\ &= (v, w) - i(v, Jw) \end{aligned}$$

(so that $\langle v | Jw \rangle = (v, Jw) + i(v, w) = i \langle v, w \rangle$)

Thus the imaginary part is

$$\mathrm{Im} \langle v | w \rangle = (Jv, w)$$

which is definitely different from the good symplectic structure $(X^{-1}v, w)$.

For quantization I have the wrong inner product on phase space. The inner product I want is connected with probability, i.e. the number of quanta. The energy is supposed to be put in the form

$$H = \sum h\omega_i \, a_i^* a_i$$

where a_i^*, a_i are creation + annih. ops. assoc. to an orthonormal basis of the 1-particle space. For a simple oscillator; let's review the calculation.

$$x = \mathrm{Re}(Ae^{-i\omega t})$$ describes the complex structure on the solutions.

$$\dot{x} = \omega \, \mathrm{Im}(Ae^{-i\omega t})$$

$$\omega x + i\dot{x} = \omega A e^{-i\omega t}$$

basic quantization rule

$$H = \tfrac{1}{2}(\dot{x}^2 + \omega^2 x^2) = \tfrac{1}{2}\omega^2 |A|^2 \;\overset{\downarrow}{=}\; \hbar\omega |a|^2$$

$$\omega A = \sqrt{2\hbar\omega}\, a$$

so $a = \frac{1}{\sqrt{2\hbar\omega}}(\omega x + i\dot{x})$ gives the good isomorphism between phase space and $\mathbb{C}$.

Let's now construct the quantization of the real vector space of real functions $f(x)$ with energy $H = \frac{1}{2}\int f^2 dx$ and with time evolution

$$e^{-t\partial_x} f(x) = f(x-t).$$

Using F.T. $\quad f(x-t) = \int \frac{dk}{2\pi} e^{ikx} e^{-ikt} \hat{f}(k)$

the energy is

$$H(f) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi} |\hat{f}(k)|^2 = \int_0^{\infty} \frac{dk}{2\pi} |\hat{f}(k)|^2.$$

Thus for each $k > 0$ we have an oscillator of frequency k.

Let $\mathcal{K}$ be this real vector space of $f(x)$. $f(x)$ real $\Leftrightarrow$ $\hat{f}(-k) = \overline{\hat{f}(k)}$, so $\mathcal{K}$ gets a complex structure by looking at the complex valued function $\hat{f}(k)$ for $k>0$.

To be more specific

$$f(x) = \int \frac{dk}{2\pi} e^{ikx} \hat{f}(k)$$

$$= \underbrace{\int_0^{\infty} \frac{dk}{2\pi} e^{ikx} \hat{f}(k)}_{h(x)} + \underbrace{\int_0^{\infty} \frac{dk}{2\pi} e^{-ikx} \hat{f}(-k)}_{\overline{h(x)}}$$

where $h(x) \in H^+$ has a natural extension to an analytic fn. in the UHP decaying as $\mathrm{Im}(z) \uparrow +\infty$.

Thus real function $f(x)$ are the real parts of the boundary values of a holomorphic functions in the UHP, so this gives a complex structure. Multiplying by i takes

$$h(x) = f(x) + i g(x)$$

to $\quad ih(x) = -g(x) + if(x)$, and hence transforms $f(x)$ to $-g(x)$ where $g(x)$ is the conjugate function.

Thus multiplying by i is essentially the Hilbert transform.

Continue with the quantization: I wish to associate to each real $f(x)$ an operator

$$\phi(f) = a^*(f) + a(f)$$

$$= \int_0^\infty dk \; c_k \left(\hat{f}(k) a_k^* + \overline{\hat{f}(k)}\, a_k \right)$$

which is self-adjoint such that

$$\langle 0 | \phi(f) H \phi(f) | 0 \rangle = \text{energy of } f.$$

But

$$\phi(f)|0\rangle = \int_0^\infty dk\, c_k \hat{f}(k) a_k^* |0\rangle$$

$$H = \int_0^\infty dk\, k\, a_k^* a_k$$

so

$$H\phi(f)|0\rangle = \int_0^\infty dk \; c_k^2 \hat{f}(k)\, k \; a_k^* |0\rangle$$

and so

$$\langle 0 | \phi(f) H \phi(f) | 0 \rangle = \int_0^\infty dk \; k c_k^2 |\hat{f}(k)|^2.$$

Thus up to some innocent constants we have $c_k = \frac{1}{\sqrt{k}}$

which means that the natural inner product on one particle states is

$$\langle 0|\,\phi(f)^2\,|0\rangle = \int_0^\infty dk\,\frac{1}{k}|\hat{f}(k)|^2$$

Therefore in the standard quantization the Hilbert space of 1 particles state is isomorphic to our space of real $f(x)$ with the above inner product. We have explained the time evolution and complex structure.

Next look at decay. Given $f(x)$, $g(x)$ lets find

$$\langle g|\,U_t f\rangle = \int_0^\infty dk\,\frac{1}{k}\,\overline{\hat{g}(k)}\,e^{-itk}\hat{f}(k)$$

$$(g, U_t f) = \int_{-\infty}^{\infty} dk\ \overline{\hat{g}(k)}\,e^{-itk}\hat{f}(k)$$

We run into the usual problem that $\langle g|U_t|f\rangle$ can't decay exponentially although $(g, U_t f)$ does.

December 27, 1985

It occurred to me that there ought to be some link between the transmission line setup and the loop group representation setup, since both have to do with real functions $f(x)$ with time action by translation. Rapid review of formulas:

$$L^2(\mathbb{R}/L\mathbb{Z}) \qquad |k\rangle = \frac{e^{ikx}}{\sqrt{L}} \qquad k \in \frac{2\pi}{L}\mathbb{Z}$$

$$\psi^*(x) = \sum_k \frac{1}{\sqrt{L}} e^{-ikx} \psi_k^* \qquad \psi(x) = \sum_\ell \frac{1}{\sqrt{L}} e^{i\ell x} \psi_\ell$$

$$\rho(x) = \psi^*(x)\psi(x) = \sum_q \frac{1}{L} e^{-iqx} \underbrace{\sum \psi_{\ell+q}^* \psi_\ell}_{\rho_q}$$

$$\rho(f) = \int f(x)\rho(x)\,dx = \sum_q \hat{f}(q)\rho_q \qquad \hat{f}(q) = \frac{1}{L}\int_0^L e^{-iqx} f(x)\,dx$$

$$f(x) = \sum e^{iqx} \hat{f}(q)$$

$$[\rho_{-q}, \rho_q] = \frac{qL}{2\pi} \qquad a_q^* = \sqrt{\frac{2\pi}{qL}}\,\rho_q \qquad q>0$$

$$[\rho(f), \rho(g)] = \frac{1}{2\pi i}\int_0^L f(x) g'(x)\,dx$$

$$H = \sum_{q>0} q\, a_q^* a_q = \sum_{q>0} \frac{2\pi}{L} \rho_q \rho_{-q}$$

$$= \pi \int \rho(x)^2 dx$$

Then

$$\dot{\rho}(f) = i[H, \rho(f)] = i\pi \int \left\{ \rho(x)\overbrace{[\rho(x), \rho(f)]}^{\frac{1}{2\pi i}\int \delta(x-y) f'(y)\,dy} + [\rho(x), \rho(f)]\rho(x) \right\} dx$$

$$= i\pi \frac{1}{2\pi i} \int 2\rho(x) f'(x)\,dx = \rho(f')$$

Let's now try to interpret things sensibly. We consider the real vector space, denote it W of real functions $f(x)$ on S^1 with time acting by translation and with the skew form

$$f, g \longmapsto \int f(x) g'(x)\, dx$$

Actually W should be functions modulo constants, both to make the skew form non-degenerate and to remove the zero mode for the time action. Since the circle action preserves the skew form, we can look for a Hamiltonian, i.e. a function H on W such that

$$dH = i_X \omega$$

The left-side applied to the tangent vector at the point f with direction g is

$$\frac{d}{d\varepsilon} H(f + \varepsilon g)\Big|_{\varepsilon=0}$$

At f the vector field X is f', so applied to the tangent vector g at the point f, the right side is

$$\omega(f', g) = \int f'(x) g'(x)\, dx$$

Thus

$$H(f) = \frac{1}{2}\int f'(x)^2\, dx$$

is the Hamiltonian function

So what this means is that we have the following link. From the transmission line we were led to real functions $f(x)$ with time action as translation and with energy given by $\frac{1}{2}\int f(x)^2 dx$. Thus if we use the map

$$f \longrightarrow f'$$

we go from the loop group case to the transmission line one. In particular we see that the symplectic form in the transmission line case is

$$f, g \longmapsto \int (D^{-1}f)(x)\, g(x)\, dx$$

(In both spaces we have to remove the 0 mode.)

What might be interesting about this observation is that ~~[illegible]~~ in the loop group setup one can work with the full group $\mathcal{L}U(1)$ instead of real functions without constant term in the Fourier series. There's more geometry.

For example, over ~~[illegible]~~ $\mathcal{L}U(1)$ is a line bundle with metric and connection which is equivariant for the circle action. Presumably ω is the curvature and H is the momentum associated to the circle action. This would perhaps give the constants on the other components of $\mathcal{L}U(1)$.

December 28, 1985

Standard model emerging from loop group representations: We consider the vector space of real functions $f(x)$ on the line, or complex functions $\hat{f}(k)$ satisfying $\overline{\hat{f}(-k)} = \hat{f}(k)$, the equivalence being

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \hat{f}(k)$$

On this vector space one has the symplectic form

$$\omega(f,g) = \int f g' \, dx = i \int_{-\infty}^{\infty} \frac{dk}{2\pi} k \hat{f}(-k) \hat{g}(k)$$

the energy

$$H(f) = \frac{1}{2} \int (f')^2 dx = \int_0^{\infty} \frac{dk}{2\pi} k^2 |\hat{f}(k)|^2.$$

These are linked by the fact that the Hamiltonian flow assoc. to H is translation:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} H(f+\varepsilon g) = \int f' g' \, dx = \omega(f', g)$$

There is a unique complex structure such that the frequencies of time translation are positive, and the symplectic form extends to a hermitian inner product:

$$\langle f | g \rangle = \int_0^{\infty} \frac{dk}{2\pi} k \overline{\hat{f}(k)} \hat{g}(k)$$

Then

$$\langle f|g\rangle - \langle g|f\rangle = \frac{1}{i} \omega(f,g)$$

$$H(f) = \langle f | \tfrac{1}{i}\partial_x | f \rangle$$

In order to quantize introduce a_k^*, a_k for $k>0$ normalized so that $[a_k, a_\ell^*] = 2\pi\,\delta(k-\ell)$, then set

$$\rho(f) = \int_0^\infty \frac{dk}{2\pi} \sqrt{k}\left(\hat{f}(k)\, a_k^* + \overline{\hat{f}(k)}\, a_k\right)$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, \rho(k) \qquad \begin{array}{ll} \rho(k) = \sqrt{k}\, a_k^* & k>0 \\ \rho(-k) = \sqrt{k}\, a_k & \text{"} . \end{array}$$

Then

$$[\rho(f), \rho(g)] = \int_0^\infty \frac{dk}{2\pi}\, k\left(\overline{\hat{f}(k)}\,\hat{g}(k) - \hat{f}(k)\,\overline{\hat{g}(k)}\right)$$

$$= \frac{1}{i}\, \omega(f,g)$$

Next I wanted to use this model to explore non-commutative stochastic processes. I have been thinking in terms of transmission lines where the standard model is real $f(x)$ with

$$\text{energy}(f) = \frac{1}{2}\int f^2\, dx = \int_0^\infty \frac{dk}{2\pi} \left|\hat{f}(k)\right|^2$$

In this model the hermitian inner product is

$$\langle f | g \rangle = \int_0^\infty \frac{dk}{2\pi} \frac{1}{k}\, \overline{\hat{f}(k)}\, \hat{g}(k)$$

For example if we take $\hat{f}(k) = \dfrac{i}{k+i\varepsilon}$, which spans $H^+ \cap SH^-$ for $S(k) = \dfrac{k-i\varepsilon}{k+i\varepsilon}$, then in the energy norm

$$\left(f, e^{-itH} f\right) = \frac{1}{2}\int \frac{dk}{2\pi}\, e^{-itk} \frac{1}{k^2+\varepsilon^2} = \frac{1}{2}\, \frac{e^{-\varepsilon|t|}}{2\varepsilon}$$

we have exponential decay in time, but for the hermitian inner product

$$\langle f | e^{-itH} | f \rangle = \int_0^\infty \frac{dk}{2\pi} \frac{1}{k} \frac{e^{-itk}}{k^2+\varepsilon^2}$$

we have a problem: f is not even in this Hilbert space.

The goal here is to construct the non-comm. stochastic process described by Lewis in his talk. The idea is to produce an embedding of Weyl algebras $j: W(\mathcal{H}) \subset W(\mathcal{K})$ where one has a time evolution on $\mathcal{K}$. Then we have a family of embeddings

$$j_t : W(\mathcal{H}) \longrightarrow W(\mathcal{K})$$

analogous to the evaluation at time t maps

$$\begin{aligned} \Omega &\longrightarrow X \\ x &\longmapsto x_t \end{aligned}$$

for a stochastic process. One also needs the analogue of the probability measure on Ω, which is a state on $W(\mathcal{K})$.

In this situation we need $\mathcal{H}$ and $\mathcal{K}$ to have compatible symplectic structures. Lewis mentioned starting with a complex Hilbert space $\mathcal{H}$ equipped with a 1-parameter group of contractions T_t, and then taking $\mathcal{K}$ to be the Sz.-Nagy one parameter unitary group of which T_t is the contraction.

But in the transmission line case one thinks of $\mathcal{H}$ as being the phase space of the oscillator, so one really wants it to be a symplectic subspace of $\mathcal{K}$.

Consider the example $S(k) = \dfrac{(k-a)(k-b)}{(k-\bar{a})(k-\bar{b})}$ where $a = \omega + i\varepsilon$, $b = -\omega + i\varepsilon$ hence $\bar{b} = -a$. In this case the (real) subspace [redacted] $H^+ \cap SH^-$ consists of

$$\frac{\alpha}{k-\bar a} - \frac{\bar\alpha}{k+a}$$

for different $\alpha \in \mathbb{C}$. Only for $\alpha = i\bar a$ does this vanish at $k=0$, which is a condition we need that it make sense in the symplectic form.

What seems to be true is that the transmission line example has to be approximated by a simpler example. For example start with $\mathcal{H} = \mathbb{C}$ with $T_t = e^{-\varepsilon|t|-i\omega t}$. In this case the Sz.-Nagy envelope is $\mathcal{K} = L^2(\mathbb{R}, \frac{dk}{2\pi})_{\mathbb{C}}$ where $\mathcal{H}$ is generated by $\varphi = \frac{1}{k-\bar a}$ suitably normalized

$$\langle \varphi | e^{-itH} | \varphi \rangle = \int \frac{dk}{2\pi} e^{-itk} \frac{1}{(k-a)(k-\bar a)}$$ for $t>0$ decays LHP

$$= \frac{-2\pi i}{2\pi} e^{-it\bar a} \frac{1}{\bar a - a} = \frac{1}{2\varepsilon} e^{-(\varepsilon+i\omega)t}$$

if $a = \omega + i\varepsilon$, $i\bar a = i(\omega - i\varepsilon) = \varepsilon + i\omega$.

To handle this [redacted] $\mathcal{K}$ quantum-mechanically I have to change the complex structure for $k<0$.

Here is the way to view non-commutative Gaussian processes, or Weyl processes. In the case of a Gaussian stochastic process [redacted] x_t one has an algebra of polynomials. [redacted] Specifically one has the degree 1 functions which are linear combinations

$$\int f(t) x_t \, dt$$

of the functions x_t. One takes various products to get polynomials. Expectation gives a trace which is determined by the variance

$$\langle x_t x_{t'} \rangle$$

by the standard contraction rules.

To generalize we want to replace the polynomials by the Weyl algebra! Just like the Gaussian setup is completely determined by linear combinations of the variables x_t, we will have linear combinations of variables q_t, p_t. The way this can be formulated is to suppose given symplectic spaces $\mathcal{H}, \mathcal{K}$ and a family of embeddings $j_t : \mathcal{H} \to \mathcal{K}$ which generate $\mathcal{K}$. Then we have induced algebra embeddings $j_t : W(\mathcal{H}) \longrightarrow W(\mathcal{K})$, which generalize the way the polynomials in the process x_t are obtained.

The next thing needed is a Gaussian state on the Weyl algebra $W(\mathcal{K})$. What this is is a state with Gaussian generating function. In the case of a complex Hilbert space $\mathcal{H}$ such a state on $W(\mathcal{H})$ has the generating fn.

$$\left\langle e^{a^*(h) - a(h)} \right\rangle = e^{-\frac{1}{2}\|Ah\|^2}$$

where $A \geq 1$; this is for states invariant under the circle action on $\mathcal{H}$. For example

$$\frac{\mathrm{tr}\left(e^{-\omega a^* a}\, e^{h a^* - \bar{h} a}\right)}{\mathrm{tr}\left(e^{-\omega a^* a}\right)} = e^{-|h|^2\left(\frac{1}{2} + \frac{e^{-\omega}}{1 - e^{-\omega}}\right)}$$

$$\frac{1}{2}\left\langle (h a^* - \bar{h} a)^2 \right\rangle = \frac{1}{2}|h|^2 \langle \underbrace{a^* a + a a^*}_{1 + 2a^* a} \rangle = |h|^2\left(\frac{1}{2} + \frac{1}{e^{\omega} - 1}\right)$$

December 29, 1985

Stationary Gaussian process: The space of real linear combinations $\int f(t) x_t \, dt$ is a real vector space V on which one has a one-parameter group of automorphisms given by time-translation. There is an invariant inner product given by the variance $\langle v^2 \rangle$. Also x_0 is a cyclic vector.

Stationary non-commutative Gaussian process. Here V is a real vector space with symplectic form and a one-parameter group of symplectic automorphisms. On V we have an invariant inner product given by $\langle v^2 \rangle$, where the square is taken in the Weyl algebra, and where $\langle \; \rangle$ denotes [redacted] an invariant Gaussian state on the Weyl algebra. This variance is not exactly an arbitrary invariant inner product on V, but is subject to some sort of inequality [redacted] (uncertainity principle?) relative to the symplectic form.

Problem: Let V be a real vector space (finite dimensional) equipped with a symplectic form ω and let $\langle \; \rangle$ be a state on $W(V)$. Let $Q(v) = \langle v^2 \rangle$. What conditions must Q satisfy? In other words, when ~~the~~ is a pos. def. quadratic form on V the variance of a state on $W(V)$?

Suppose $\dim(V) = 2$. Let q, p be an orthonormal basis for V with respect to Q:

$$Q(\lambda q + \mu p) = \lambda^2 + \mu^2$$

and assume q, p is oriented so that $[p,q] = \frac{h}{i}$ with $h > 0$. [redacted]

Then
$$0 \leq \langle (q+ip)^*(q+ip) \rangle = \langle q^2 + p^2 - i[p,q] \rangle$$
$$= 2 - h \quad \Longrightarrow \quad \boxed{h \leq 2}$$

In general I can "diagonalize" the skew form ω with respect to the inner product Q. This breaks V into a direct sum of 2 dim'l planes orthogonal for both ω and Q. In each plane we can choose a basis q_j, p_j orthonormal wrt Q and with $[p_j, q_j] = \frac{h_j}{i}$ where $h_j > 0$. Then clearly we have $h_j \leq 2$ for each j, if the quadratic form comes from a state on $W(V)$.

I think I can now describe the various non-commutative Gaussian processes. We have V with symplectic structure ω and time-evolution and with an invariant quadratic form Q. Using the time evolution we obtain a unique complex structure on V such that the frequencies are positive, and the symplectic form extends to give a canonical hermitian form. Because Q is time-invariant it commutes with the complex structure, so is related to the canonical inner product by a self-adjoint operator A. Because Q comes from a state on $W(V)$ we have $A \geqslant 1$.

So far I have just described what happens for $W(V)$, U_t, $\langle\ \rangle$ and have not brought in the subspace $\mathcal{H} \subset V$ representing q_0, p_0 at time $t=0$. I think however that scattering examples show lots of possibilities for $\mathcal{H}$ although V itself has a relatively standard structure if one uses the outgoing or incoming representation.

Discuss a simple example from a transmission line. Then I have identified V with the space of $\hat{f}(k) = \overline{\hat{f}(-k)}$ with $U_t = e^{-itk}$ and with

$$\langle f | g \rangle = \int_0^\infty \frac{dk}{2\pi} \frac{1}{k} \overline{\hat{f}(k)} \, \hat{g}(k)$$

This is the minimal Q I can use, namely

$$Q(f) = \langle 0 | (a^*(f) + a(f))^2 | 0 \rangle = \langle f | f \rangle$$

In general I can use any

$$Q(f) = \langle f | A | f \rangle$$

where A commutes with U_t and $A \geq 1$. For example we could use a thermal state

$$\langle f | f \rangle_\beta = \int_0^\infty \frac{dk}{2\pi} \frac{1}{k} \cdot \frac{e^{\beta k}+1}{e^{\beta k}-1} \cdot |\hat{f}(k)|^2$$

But we can't use the

$$\text{energy} = \int_0^\infty \frac{dk}{2\pi} |\hat{f}(k)|^2$$

as it doesn't satisfy the condition $A \geq 1$.