June 26, 1985

Let's review the mathematical picture of the quantum Hall effect. The simplest model is given by $M =$ 2-torus, say $\mathbb{R}^2/\Gamma$ where $\Gamma = \mathbb{Z}^2$. The magnetic field is given by a closed 2-form F on M, say $F = -iB\, dx^1 dx^2$. When the class of F is integral we can find a line bundle with connection over M having the curvature F. Such a line bundle with connection is unique up to a flat line bundle, and the group of flat line bundles over M is a 2-torus T.

June 27, 1985

I want to review the picture of the quantum Hall effect. The simplest model concerns motion of a charged particle in 2 dimensions in a constant magnetic field. On $\mathbb{R}^2$ we have a constant 2-form $F = \frac{1}{i} B\, dx^1 dx^2$. Up to isomorphism there is a unique line bundle with connection having F as curvature. The quantum mechanics is described by either the Dirac operator $\not{D}$ or the Laplacean D_μ^2 associated to this connection.

The line bundle may be trivialized using radial parallel transport from the origin. Then

$$D_\mu = \partial_\mu + A_\mu$$

$$A = \frac{B}{2i}(x^1 dx^2 - x^2 dx^1) \qquad\qquad dA = F.$$

acting on functions on $\mathbb{R}^2$. Also

$$\not{D} = \begin{pmatrix} 0 & D_1 - iD_2 \\ D_1 + iD_2 & 0 \end{pmatrix}$$

acting on pairs of functions, and

$$\begin{aligned} D_1 + iD_2 &= \partial_1 - \frac{B}{2i}x^2 + i\left(\partial_2 + \frac{B}{2i}x^1\right) \\ &= (\partial_1 + i\partial_2) + \frac{B}{2}(x^1 + ix^2) \\ &= 2\left(\partial_{\bar{z}} + \frac{B}{4}z\right) \end{aligned}$$

where $z = x^1 + ix^2$. Also

$$\gamma^1\gamma^2\varepsilon = i$$

$$\begin{aligned} \not{D}^2 &= D_\mu^2 + \tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} \\ &= D_\mu^2 + \gamma^1\gamma^2 \tfrac{1}{i} B = D_\mu^2 + \varepsilon B \end{aligned}$$

Let me now give the structure of these operators. First of all we have translation invariance. What this means is that the translations on the vector space $\mathbb{R}^2$ can be lifted to operators

$$e^{a_1 D_1 + a_2 D_2} = e^{a \cdot D}$$

on functions. We obtain an action of the central extension $\widetilde{\mathbb{R}}^2$ of $\mathbb{R}^2$ by S^1 defined by the skew-form F:

$$e^{a \cdot D} e^{b \cdot D} e^{-a \cdot D} e^{-b \cdot D} = e^{-iB(a_1 b_2 - a_2 b_1)}$$

The operators e^{aD} commute with the operators $\not{D}$ and D_μ^2. Actually this isn't correct - I need the operators on the right, denote them $\bar{D}_\mu$ which commute with the D_μ. Geometrically the line bundle with connection can be pulled back by a translation map, and then there is an isomorphism of the line bundle with its pull-back unique up to a scalar. Thus the group $\widetilde{\mathbb{R}}^2$ is simply the group of autos. of $(\mathbb{R}^2, L)$ covering translations in $\mathbb{R}^2$.

The two pieces of $\not{D}$, namely $D_1 \pm i D_2$, are essentially adjoints of each other, and have commutator a scalar. So they give rise to creation and annihilation operators as in the case of a simple harmonic oscillator. We note that the annihilation operator is proportional to $\partial_{\bar{z}} + \frac{B}{4} z$ which kills

$$e^{-\frac{B}{4}|z|^2} \{\text{holom. fns.}\}.$$

This space is ~~the~~ irreducible representation of the Heisenberg group $\widetilde{\mathbb{R}}^2$ on which the center acts as the identity

So the Hilbert space of L^2 sections of L splits into eigenspaces for $-D_\mu^2$ with the eigenvalues $0, 2B, 4B, \ldots$ and each eigenspace is an irreducible representation of $\widetilde{\mathbb{R}}^2$; I should say the standard irreducible representation of $\widetilde{\mathbb{R}^2}$.

Now I wish to consider a lattice $\Gamma \subset \mathbb{R}^2$. The important thing is whether $B \cdot \text{covolume}(\Gamma) \subset 2\pi\mathbb{Z}$ or not. If so, then the induced central extension $\tilde{\Gamma}$ of Γ by S^1, which we obtain from $\widetilde{\mathbb{R}^2}$ is trivial. We can therefore lift the translation action of Γ on $\mathbb{R}^2$ to an action of Γ on the line bundle L. Such a lifting is unique up to a character of Γ. One we have such a lifting we can descend and so obtain a line bundle with connection on the torus $M = \mathbb{R}^2/\Gamma$. There is a family of such line bundles with connection parametrized by the space of characters of Γ. The resulting Laplaceans are all like harmonic oscillators. The ground states have eigenvalue zero, and the zero eigenspace is the space of sections of the holomorphic line bundle. The dimension is given by the degree of the line bundle.

Next I wish to consider the other case, i.e. where $B \cdot \text{cov}(\Gamma) \not\subset 2\pi\mathbb{Z}$. The interesting case is where this is irrational, but maybe one can get some sensible ideas in the case where $\frac{1}{2\pi} B \cdot \text{cov}(\Gamma)$ is rational. Then $\tilde{\Gamma}$ comes from an extension of Γ by a μ_m-group.

June 29, 1985

Over $\mathbb{R}^2$ we consider the line bundle with connection (L, D) having constant curvature $F = \frac{B}{i} dx^1 dx^2$, for example take the trivial holomorphic line bundle with the metric $|1|_z^2 = e^{-\frac{B}{2}|z|^2}$. Let $\mathcal{H}_0$ be the Hilbert space of square integrable holomorphic sections of L. By scaling we can suppose $B = 2$, whence we are considering the Hilbert space of entire functions $f(z)$ on $\mathbb{C} = \mathbb{R}^2$ for which

$$\|f\|^2 = \int e^{-|z|^2} |f(z)|^2 \frac{d^2 z}{\pi} < \infty.$$

This is the holomorphic function representation of the CCR with

$$a^* = z \quad \text{and} \quad a = \frac{d}{dz}.$$

Hence we have translation operators

$$T_\gamma = e^{-\bar{\gamma} a^* + \gamma a} = e^{-\frac{|\gamma|^2}{2}} e^{-\bar{\gamma} a^*} e^{\gamma a}$$

$$(T_\gamma f)(z) = e^{-\frac{|\gamma|^2}{2}} e^{-\bar{\gamma} z} f(z+\gamma)$$

satisfying

$$T_{\gamma+\gamma'} = e^{-\frac{1}{2}[-\bar{\gamma} a^* + \gamma a, -\bar{\gamma}' a^* + \gamma' a]} T_\gamma T_{\gamma'}$$

$$= e^{\frac{1}{2}(\gamma \bar{\gamma}' - \bar{\gamma} \gamma')} T_\gamma T_{\gamma'}$$

Suppose we are given a lattice $\Gamma \subset \mathbb{C}$ whose covolume is in $2\pi\mathbb{Z}$. Then $\gamma \mapsto T_\gamma$ is a homomorphism and we can take the invariant elements under this action of Γ.

What we are getting is the space of holomorphic sections of the line bundle over the torus $\mathbb{C}/\Gamma$ obtained by descending L under the Γ-action.

Actually it seems that these Γ-invariant holomorphic sections of ${}^{O}L$ are not in the Hilbert space.

~~But in any case we can consider~~

June 30, 1985

Consider a torus $M = \mathbb{R}^2/\Gamma$ and a line bundle L over M with inner product + compatible connection having constant curvature > 0. We can then twist by flat line bundles on M, which are in 1-1 correspondence with characters of Γ. In this way we obtain a family of holomorphic line bundles over M of positive degree parameterized by the dual torus $\hat{\Gamma}$. Taking sections produces a vector bundle over $\hat{\Gamma}$ of rank $= \deg(L)$.

Notation: $\mathcal{L}$ total line bundle over $\hat{\Gamma} \times M$ and $E = \pi_*(\mathcal{L})$ is the v. bundle over $\hat{\Gamma}$.

The smooth sections of E form a finite projective module over $C^\infty(\hat{\Gamma})$, which by Fourier transform can be identified with the convolution algebra $\mathcal{S}(\Gamma)$ of rapidly decreasing functions on Γ. A natural problem is to describe this module directly so that it would make sense in the irrational case.

The original idea I had went as follows: Consider the Hilbert space of L^2 sections of L over $\mathbb{R}^2$ and consider the Dirac operator $D_1 + iD_2$ which one knows commutes with the translation operators.

Actually this raises the possibility of Atiyah's L^2-index theorem, but I was originally thinking in terms of the Laplacean $-D_\mu^2$. The first point is to somehow relate the Γ action on $L^2(\mathbb{R}^2, L)$ to the space $L^2(M, L)$ – to first do the descent on the level of smooth sections over $\mathbb{R}^2$, not holomorphic ones. So maybe I ought to calculate this descent in the case of $\mathbb{Z} \subset \mathbb{R} \longrightarrow S^1$.

Let us consider $\mathbb{Z}$ acting on $\mathcal{S}(\mathbb{R})$ by translation i.e. $f(x) \longmapsto f(x+1)$. There are two equivalent algebras in the sense of Morita equivalence

$$\underbrace{\mathcal{S}(\mathbb{R}) \times \mathbb{Z}}_{\substack{\text{crossed product} \\ \text{algebra}}} \qquad\qquad \mathcal{S}(S^1)$$

and the Morita equivalence is given by a suitable bimodule.

June 30, 1985

Short summary of the investigations over the past few days. I have been considering the family of line bundles over $M = \mathbb{R}^2/\Gamma$ equipped with a connection having a fixed constant curvature. The index of the corresponding family of $\bar\partial$ operators is rep. by a vector bundle over the torus $\hat\Gamma$. This vector bundle determines a finite proj. module over $C^\infty(\hat\Gamma)$ which by F.T. is isomorphic to the conv. alg. $\mathcal{S}(\Gamma)$.

My feeling is that it should be possible to describe this finite proj. module over $\mathcal{S}(\Gamma)$ directly, and that it makes sense in the irrational case, where $\mathcal{S}(\Gamma)$ is to be replaced by an irrational rotation algebra.

The first idea was that the module should be a smooth vector subspace of the Hilbert space $\operatorname{Ker}\bar\partial$ over $\mathbb{R}^2$. However calculation suggests this is wrong. If I take the degree of the line bundles over M to be 1, the index bundle is a trivial line bundle. The H^∞_{an} part of the holom. functions Hilbert space

$$H^0_{an} = \left\{ f \text{ anal on } \mathbb{C} \;\middle|\; \int e^{-|z|^2} |f|^2 \frac{d^2z}{\pi} < \infty \right\}$$

as a module over $\mathcal{S}(\Gamma)$, where $\Gamma = \mathbb{Z}^2$, I believe to be a non-free $\mathcal{S}(\Gamma)$-module.

Atiyah's L^2-index theorem suggests that the Hilbert space of square integrable sections over $\mathbb{R}^2$ in $\operatorname{Ker}\bar\partial$ is a good gadget. It occurs to me that Connes formulates this result without reference to a Γ.

The key idea in the L^2-index theorem is to assign a notion of dimension to $\operatorname{Ker} \bar\partial$ denoted $\dim_\Gamma$. One considers the kernel $e(x,y)$, $x,y \in \mathbb{R}^2$ of the orthogonal projector onto $\operatorname{Ker}\bar\partial$. Then

$$\dim_\Gamma(\operatorname{Ker}\bar\partial) = \int_M \operatorname{tr} e(x,x)$$

The theorem Atiyah proves is that

$$\operatorname{Ind}_\Gamma(D) = \operatorname{Ind}(D \text{ over } M)$$

Formally this impresses me as follows: It's like evaluating the rank of a finite projective module by specializing at one point.

But a first step is really to use the von Neumann algebra theory to understand $\dim_\Gamma$. Put $X = \mathbb{R}^2$ and consider $\mathcal{H} = L^2(X)$ with its natural Γ action. Use a Borel isomorphism of X with $M \times \Gamma$ we get

$$L^2(X) \simeq L^2(M) \hat{\otimes} \ell^2(\Gamma)$$ tensor product of Hilb. spaces

and hence

$$\left(\operatorname{End} L^2(X)\right)^\Gamma = \left(\operatorname{End} L^2(M)\right) \otimes \underbrace{\left(\operatorname{End} \ell^2(\Gamma)\right)^\Gamma}_{W(\Gamma)}$$ tensor prod of vN algs.

Inside this vN alg) one has trace class operators. On trace class operators one has a trace by combining the trace on $\operatorname{End} L^2(M)$ with the trace on $W(\Gamma)$ (eval. at $1 \in \Gamma$). It's this trace which is used in computing $\dim_\Gamma(\operatorname{Ker}\bar\partial)$.

At this point we could ask why the theorem is true. Analytically what is involved probably is the process of summing over Γ; this links kernels upstairs and downstairs. For example suppose I want to relate the heat kernels upstairs and downstairs. I start with $K_t(x,y)$ up on X and then the heat kernel downstairs is

$$\sum_\Gamma K_t(x, y+\gamma).$$

July 2, 1985:

The thread I have been pursuing is the idea that somehow that sections of family of spaces $\Gamma(M, L_\phi)$, $\phi \in \hat{\Gamma}$, $M = \mathbb{C}/\Gamma$ are equivalent to sections on $\mathbb{C}$. Here's a precise assertion:

Consider the lattice $\mathbb{Z} \subset \mathbb{R}$ and the two one-diml tori $\hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}, S^1)$ and $\mathbb{R}/\mathbb{Z}$. The 2-torus $\hat{\mathbb{Z}} \times \mathbb{R}/\mathbb{Z}$ will be coordinatized via $(x,y) \in \mathbb{R}^2/\mathbb{Z}^2$ so that y yields the character $n \to e^{2\pi i n y}$, and x yield $x \in \mathbb{R}/\mathbb{Z}$. Then we can consider over $\hat{\mathbb{Z}} \times \mathbb{R}/\mathbb{Z}$ the line bundle $\mathcal{L}$ whose sections are smooth functions F on $\mathbb{R}^2$ satisfying

$$F(x+1, y) = e^{2\pi i y} F(x,y)$$
$$F(x, y+1) = F(x,y)$$

(This line bundle has degree ± 1 ~~since~~ the clutching function $y \mapsto e^{2\pi i y}$ is of degree 1).

Claim is that we have an isomorphism

$$\mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \Gamma(\hat{\mathbb{Z}} \times \mathbb{R}/\mathbb{Z}, \mathcal{L})$$

given by $\quad f(x) \longmapsto F(x,y) = \sum_n e^{2\pi i n y} f(x-n)$

$$F(x,y) \longmapsto f(x) = \int_0^1 F(x,y)\, dy$$

It is clear that $f \mapsto F$ is well-defined, but to be precise one should relate operators on either side. Clearly

$$\partial_x f(x) \longmapsto \partial_x F(x,y)$$

Also

$$\partial_y F = \sum_n e^{2\pi i n y} \, 2\pi i n \; f(x-n)$$

$$(\partial_y - 2\pi i x) F = \sum_n e^{2\pi i n y} \, 2\pi i (n-x) f(x-n)$$

$$= (-2\pi i) \text{ transform of } x f(x).$$

Thus

$$x f(x) \longmapsto \left(\frac{i}{2\pi} \partial_y + x \right) F(x,y).$$

As a check note that

$$\left[\partial_x, \frac{i}{2\pi} \partial_y + x \right] = 1.$$

The above shows that if $f \in \mathcal{S}(\mathbb{R})$, then not only will $F(x,y)$ be continuous, but also when the operators ∂_x, $\frac{i}{2\pi}\partial_y + x$ are applied. $\therefore$ F is smooth.

Also the map $f \longmapsto F$ is unitary:

$$\int_0^1 dx \int_0^1 |F(x,y)|^2 dy = \int_0^1 \sum_n |f(x-n)|^2 dx = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Remark: $\Gamma(\hat{\mathbb{Z}} \times \mathbb{R}/\mathbb{Z}, \mathcal{L})$ reminds me of the fermion Fock space construction

$$\Lambda(V/W) \otimes \Lambda(W^*).$$

However there doesn't seem to be a canonical line in $\Gamma(\hat{\mathbb{Z}} \times \mathbb{R}/\mathbb{Z}, \mathcal{L})$, i.e. a line in $\mathcal{S}(\mathbb{R})$ which is attached to $\mathbb{Z}$.

Next I want to consider the 2-dimensional case. Here $M = \mathbb{C}/\Gamma$, say $\Gamma = \mathbb{Z}^2$ to begin with. Over M I have a line bundle L_0 of positive degree. This I lift back to $\mathbb{C}$ and take sections; in fact, it makes sense to take the Schwartz space of rapidly decreasing smooth sections. This is isomorphic to $\mathcal{S}(\mathbb{C})$ but the action of Γ is not just simple translation.

So what I am doing is to twist things a bit. If I were to consider the translation action of Γ on $\mathcal{S}(\mathbb{C})$ and were to perform the transform $f \longmapsto F$ as above, then I would get sections of the Poincaré line bundle over $\hat{\Gamma} \times \mathbb{C}/\Gamma$. Here I have a different action of Γ on $\mathcal{S}(\mathbb{C})$.

I seem to run into a contradiction, or paradox at least. On the one hand I think that if I take the family of degree 1 line bundles over M, then the line bundle over the parameter space $\hat{\Gamma}$ giving the holom. sections of the bundles in the family is trivial. (Thus the family $\mathcal{O}(P)$ as P ranges over M has the section given by the function 1.) On the other hand I think that this family of holomorphic sections is equivalent to a rapidly decreasing holomorphic section of L over $\mathbb{C}$. But the rapidly decreasing holom. sections of L over $\mathbb{C}$ is isomorphic to

$\mathcal{S}(\mathbb{R})$, and I can't see any way that $\mathcal{S}(\mathbb{R})$ is a trivial module over $\mathcal{S}(\Gamma)$.

I have to do the calculations carefully. I will take $\Gamma = \mathbb{Z}^2$ and take L to be the trivial bundle over $\mathbb{R}^2$ with the connection form

$$A = -2\pi i \, x\,dy \qquad\qquad F = dA = \frac{2\pi}{i} d^2 z$$

so that

$$D_x = \partial_x \;, \quad D_y = \partial_y + \frac{2\pi}{i} x$$

$$D_x + iD_y = \partial_x + i\partial_y + 2\pi x$$

Next I want ~~[illegible]~~ the translation action of $\widetilde{\mathbb{R}^2}$ which preserves this connection.

$$D_x = \partial_x \qquad\qquad \nabla_x = \partial_x + \frac{2\pi}{i} y$$

$$D_y = \partial_y + \frac{2\pi}{i} x \qquad\qquad \nabla_y = \partial_y$$

The translation operators corresponding to the generators of Γ are then

$$\left(e^{\nabla_y} F\right)(x,y) = F(x, y+1)$$

$$\left(e^{\nabla_x} F\right)(x,y) = \left(e^{\partial_x + \frac{2\pi}{i} y} F\right)(x,y)$$

$$= e^{-2\pi i y} F(x+1, y)$$

Thus sections of the line bundle over M obtained by descending this action of Γ on L over $\mathbb{R}^2$ are $F(x,y)$ [redacted] smooth on $\mathbb{R}^2$ such that

$$\begin{cases} F(x, y+1) = F(x,y) \\ F(x+1, y) = e^{2\pi i y} F(x,y) \end{cases}$$

as before.

Let's introduce a character of Γ:

$$(m,n) \longmapsto e^{2\pi i(am+bn)} = \chi_{ab}(m,n)$$

and the Eisenstein operation:

$$f \longmapsto \sum_{\gamma} \chi(\gamma)\, T_{-\gamma} f = F_\chi$$

which will take a rapidly decreasing f on R^2 to a function F_χ satisfying

$$T_\gamma(F_\chi) = \chi(\gamma)\, F_\chi$$

In this example $\Gamma = \mathbb{Z}\times\mathbb{Z}$ is generated by $(1,0)$ and $(0,1)$ and

$$\left(T_{(1,0)}F\right)(x,y) = e^{-2\pi i y} F(x+1, y)$$

$$\left(T_{(0,1)}F\right)(x,y) = F(x, y+1)$$

In general

$$\left(T_{(m,n)}F\right)(x,y) = e^{m(\partial_x - 2\pi i y)}\, e^{n\partial_y} F(x,y)$$

$$= e^{-2\pi i m y} F(x+m, y+n)$$

Thus we put for $f(x,y) \in \mathcal{S}(\mathbb{R}^2)$

$$F(x,y;a,b) = \sum_{(m,n)} e^{2\pi i(am+bn)}\, e^{+2\pi i m y} f(x-m, y-n)$$

Clearly
$$\begin{cases} F(x+1,y;a,b) = e^{2\pi i(a+y)} F(x,y;a,b) \\ F(x,y+1;a,b) = e^{2\pi i b} F(x,y;a,b) \end{cases}$$

which ought to mean that F is a section of the line bundle L_χ on M obtained by tensoring L/Γ by the flat line bundle corresponding to χ.

We can recover $f(x,y)$ from $F(x,y;a,b)$ by

$$f(x,y) = \int_0^1 da \int_0^1 db \; F(x,y;a,b)$$

Let's assume for the moment that this correspondence $f \mapsto F$ is an isomorphism, and let's try to see what happens to the holomorphic sections f.

First let's describe holom. sections for L/Γ.

July 3, 1985

Let's review yesterday's position. We begin with the constant 2-form on $\mathbb{R}^2$

$$F = \frac{2\pi}{i}\, dx\, dy$$

which is such that $\int \frac{i}{2\pi} F = 1$ for $M = \mathbb{R}^2/\mathbb{Z}^2$, and select a connection form having the curvature F, e.g.

$$A = \frac{2\pi}{i}\, x\, dy$$

This gives the connection D on the trivial bundle:

$$D_x = \partial_x \qquad\qquad \nabla_x = \partial_x + \frac{2\pi}{i} y$$

$$D_y = \partial_y + \frac{2\pi}{i} x \qquad\qquad \nabla_y = \partial_y$$

If ∇_x, ∇_y are defined as above, then they commute with D_x, D_y hence the operators

$$e^{a\nabla_x} e^{b\nabla_y} F(x,y) = e^{-2\pi i a y} F(x+a, y+b)$$

is a lifting of the translation $(x,y) \longmapsto (x+a, y+b)$ to sections of the trivial line bundle which preserves the connection D.

By the integrality choice of F one has that e^{∇_x} and e^{∇_y} commute, so we get an action of $\mathbb{Z}^2$ on the trivial line bundle equipped with the connection D covering the translation actions. The quotient by this action is a line bundle with connection over $M = \mathbb{R}^2/\mathbb{Z}^2$. Its sections are smooth $F(x,y)$ on $\mathbb{R}^2$ satisfying invariance under $\mathbb{Z}^2$:

$$\begin{cases} e^{-2\pi i y} F(x+1, y) = F(x,y) \\ F(x, y+1) = F(x,y) \end{cases}$$

Next we can twist by a character of $\mathbb{Z}^2$. Let the character be

$$(m,n) \longmapsto e^{2\pi i(am+bn)}$$

with $(a,b) \in \mathbb{R}^2$. The twisted action is given by the operators

$$e^{-2\pi i a} e^{\nabla_x}, \quad e^{-2\pi i b} e^{\nabla_y}$$

so that sections of the ~~quotient~~ line bundle $L_{a,b}$ on M are the same as smooth $F(x,y)$ satisfying

$$(*) \quad \begin{cases} F(x+1,y) = e^{2\pi i(y+a)} F(x,y) \\ F(x,y+1) = e^{2\pi i b} F(x,y) \end{cases}$$

By the way things have been constructed the connection D induces a connection on $L_{a,b}$ with D_x, D_y given by the same formulas on the F satisfying the above periodicity conditions $(*)$.

Next we produce sections $F(a,b,x,y)$ of $L_{a,b}$ depending smoothly on a,b by the Eisenstein process. Let $f(x,y) \in \mathcal{S}(\mathbb{R}^2)$ and put

$$F(a,b,x,y) = \sum_{m,n} \left(e^{-2\pi i a} e^{\nabla_x}\right)^{-m} \left(e^{-2\pi i b} e^{\nabla_y}\right)^{-n} f(x,y)$$

$$= \sum_{m,n} e^{2\pi i(am+bn)} e^{2\pi i m y} f(x-m, y-n).$$

This is a smooth function of a,b,x,y satisfying $(*)$, and periodicity in a,b. One can recover f from F by

$$f(x,y) = \int_0^1 da \int_0^1 db \; F(a,b,x,y).$$

Clearly $f \mapsto F$ commutes with D_x, D_y. This means we can obtain holomorphic sections of $L_{a,b}$ starting from an $f(x,y) \in \mathcal{S}(\mathbb{R}^2)$ satisfying

$$(D_x + iD_y) f = (\partial_x + i\partial_y + 2\pi x) f = 0$$

An obvious solution of this equation is $e^{-\pi x^2}$; the general solution is $e^{-\pi x^2} \cdot$ holom. For example

$$e^{-\pi x^2} e^{+\frac{\pi}{2}z^2} = e^{-\pi x^2 + \frac{\pi}{2}(x^2 - y^2 + 2ixy)}$$

$$= e^{-\frac{\pi}{2}(x^2+y^2) + \pi i x y}$$

which is rapidly decreasing.

Let's now try to produce holomorphic sections of $L_{a,b}$ for fixed a, b. Here we don't need to use an $f \in \mathcal{S}(\mathbb{R}^2)$ rather we can start with an $f \in \mathcal{S}(\mathbb{R})$. Set

$$F(x,y) = \sum_n e^{2\pi i (an + yn + yb)} f(x-n)$$

and note this defines a section of $L_{a,b}$. One has

$$f(x) = \int_0^1 F(x,y) e^{-2\pi i b y} \, dy$$

Now

$$\int_0^1 \left(\partial_y + \frac{2\pi}{i} x\right) F(x,y) \cdot e^{-2\pi i b y} \, dy$$

$$= \int_0^1 \left(\partial_y + \frac{2\pi}{i} x + 2\pi i b\right)\left(e^{-2\pi i b y} F\right) \cdot dy = 2\pi i (b - x) f(x)$$

Thus

$$D_x F \longleftrightarrow \partial_x f$$

$$D_y F \longleftrightarrow 2\pi i(b-x)f = \frac{2\pi}{i}(x-b)f$$

$$(D_x + iD_y)F \longleftrightarrow [\partial_x + 2\pi(x-b)]f$$

and so if we want F to be a holom. section of $L_{a,b}$ we want

$$[\partial_x + 2\pi(x-b)]f(x) = 0$$

$$f(x) = \text{const } e^{-\pi x^2 + 2\pi b x}$$

$$= \text{const } e^{-\pi(x-b)^2}$$

The F corresp. to $e^{-\pi(x-b)^2}$ is

$$F(a,b,x,y) = \sum_n e^{2\pi i(an + yn + yb)} e^{-\pi(x-n-b)^2}$$

This obviously satisfies

$$F(a+1,b,x,y) = F(a,b,x,y)$$

$$F(a,b+1,x,y) = e^{-2\pi i a} F(a,b,x,y)$$

We would like to modify F to

$$c(a,b)F(a,b,x,y)$$

so as to be periodic in a,b. Then c would have to satisfy

$$c(a+1,b) = c(a,b)$$

$$c(a,b+1) = e^{+2\pi i a} c(a,b).$$

So c would have to be a section of a non-trivial line bundle over the torus, which means it must vanish at some (a,b).

Notice that if we prove the formulas

$$f(x) \longmapsto F(x,y) = \sum_n e^{2\pi i(an + yn + yb)} f(x-n)$$

$$F(x,y) \longmapsto \int_0^1 e^{-2\pi i b y} F(x,y)\,dy$$

set up an isom. of $\mathcal{S}(\mathbb{R})$ with $C^\infty(M, L_{ab})$, then it follows that the space of holom. sections of L_{ab} is one-dimensional, and spanned by the image of $f(x) = e^{-\pi(x-b)^2}$.

So next I would like to summarize what I've learned. The preceding calculations were occasioned by a <u>paradox</u> which resulted because I thought that the determinant line bundle over the Jacobian of degree 1 line bundles was trivial. The reason is that ~~one~~ I thought in terms of the family $P \longmapsto \mathcal{O}(P)$, and if one views $\mathcal{O}(P)$ as sitting inside the fn. field, the function 1 of K gives a $\neq 0$ section of $\mathcal{O}(P)$ for each P, so the det. line for this family is trivial. However this is not the correct family — one can always take a family and tensor with a line bundle on the parameter space without changing what one sees at each parameter value.

To check the answer one use GRR. We're looking at $\mathcal{L} \otimes pr_2^* L_0$ over $\hat{\Gamma} \times M$, where $\mathcal{L}$ is the Poincare divisor, and L_0 is a fixed line bdle on M

of degree 1. Then

$$ch(\mathcal{L} \otimes pr_2^* L_0) = e^{c_1(\mathcal{L}) + 1 \otimes c_1(L_0)}$$

and we want $\quad (pr_1)_* \left[\frac{1}{2}(c_1(\mathcal{L}) + 1 \otimes c_1(L_0))^2 \right]$. Now

$$c_1(\mathcal{L}) = e_1 \otimes e_2 \pm e_2 \otimes e_1 \qquad e_i \text{ symplectic base of } H^1(M)$$

$$c_1(L_0) = \text{const } e_1 e_2$$

so $\quad c_1(\mathcal{L}) c_1(L_0) = 0$, and

$$(pr_1)_* \frac{1}{2}(\qquad)^2 = (pr_1)_* \frac{1}{2} c_1(\mathcal{L})^2$$

is independent of L_0.

Now that the paradox has been explained let's return to the original problem.

We start with a connection with constant curvature over $\mathbb{R}^2$ such that the curvature is integral with respect to the lattice Γ. Then we can lift translation by Γ on $\mathbb{R}^2$ to an action of Γ on the line bundle with connection. This is unique up to the different characters of Γ, and then by taking the quotient by the Γ action we obtain a family of line bdles L_χ with conn. on $M = \mathbb{C}/\Gamma$, param. by $\chi \in \hat{\Gamma}$. We are then interested in the v.b. V over $\hat{\Gamma}$ of holom. sections over the fibres (assume the degree > 0). The space of C^∞ sections of this v.b. V is a finite proj. $C^\infty(\hat{\Gamma})$-module. $C^\infty(\hat{\Gamma}) = \mathcal{S}(\Gamma)$. What we have verified is that

$$C^\infty(\hat{\Gamma}, V) \simeq \text{Space of rapidly decreasing holomorphic sections of } L \text{ over } \mathbb{C}.$$

The reason this is interesting is that it provides a generalization in the non-integral case. When Γ is not integral with respect to the curvature we obtain a smooth irrational rotation algebra. A central extension $\tilde{\Gamma}$ of Γ by S^1 acts on the line bdle L over $\mathbb{C}$, and the holomorphic rapidly decreasing sections of L should be a finite projective module over the quotient of $\mathcal{S}(\tilde{\Gamma})$ by the ideal forcing $S^1 \subset \tilde{\Gamma}$ to act via the identity character.

July 4, 1985

Problem. Consider the smooth irrational rotation algebra $\mathcal{A}_\theta$ consisting of $\sum\limits_{(m,n)\in\mathbb{Z}^2} a_{mn} U^m V^n$ with a_{mn} of rapid decrease, and where

$$UVU^{-1}V^{-1} = e^{2\pi i\theta} \qquad \theta \notin \mathbb{Q}.$$

This algebra has interesting projective f.g. modules and a trace. Supposedly the dimension of a finite projective module (defined as $\operatorname{tr}(id)$) is of the form $p+q\theta$ with $p,q\in\mathbb{Z}$. I would like to understand this result carefully.

Now an example of a finite projective $\mathcal{A}_\theta$-module is provided $\mathcal{S}(\mathbb{R})$ with

$$U = e^{\theta\partial} : f(x) \longmapsto f(x+\theta)$$

$$V = e^{2\pi i x} : f(x) \longmapsto e^{2\pi i x} f(x)$$

Let's recall how to define the trace of an endomorphism of a finite projective A-module P. One has (assume A acts on the right)

$$\begin{tikzcd} P\otimes_A \operatorname{Hom}_A(P,A) \arrow[r, "\sim"] \arrow[d] & \operatorname{End}_A(P) \\ A/[A,A] & \end{tikzcd} \qquad \begin{matrix} v\otimes\lambda \\ \downarrow \\ \lambda(v) \end{matrix}$$

Thus to compute the trace of an endom. u of P one writes u in the form $\sum v_i\lambda_i$ and then

$$\tau(u) = \sum_i \tau(\lambda_i(v_i)) \qquad \tau = \text{trace on } A.$$

The above example arises from a Morita

equivalence. Recall that two rings A, B are Morita equivalent when there module categories are equivalent.

$$\begin{array}{ccc} \mathrm{Mod}_A & \rightleftarrows & \mathrm{Mod}_B \\ X & \longmapsto & P \otimes_A X \\ Q \otimes_B Y & \longleftarrow & Y \end{array}$$

Here $P = {}_B P_A$, $Q = {}_A Q_B$ and one has

$$Q \otimes_B P \simeq A \qquad\qquad P \otimes_A Q \simeq B$$

$$Q \simeq \mathrm{Hom}_{A^\circ}(P, A) \qquad\qquad P \simeq \mathrm{Hom}_{B^\circ}(Q, B)$$

$$A = \mathrm{End}_B(P) \qquad\qquad B \simeq \mathrm{End}_A(Q)$$

Morita equivalences arise when one chooses a faithful projective A-module Q. Then $B = \mathrm{End}_A(Q)$ and one gets the equivalence of Mod_A and Mod_B.

Next let's discuss a non-algebraic kind of Morita equivalence having to do with the infinite cyclic covering $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. I want to proceed geometrically. A module over $\mathcal{S}(\mathbb{R}/\mathbb{Z})$ I'd like to think of as the space of sections of a family $\{\mathcal{H}_\xi\}$ of vector spaces. We want to compare modules over $\mathcal{S}(\mathbb{R}/\mathbb{Z})$ with modules over $\mathcal{S}(\mathbb{R}) \rtimes \mathbb{Z}$; we think of such a module as the space of sections (decaying at ∞) of a $\mathbb{Z}$-equivariant family of vector spaces over $\mathbb{R}$.

Given the family $\{\mathcal{H}_\xi\}$ over S^1, we lift it back to $\{\mathcal{H}_{\pi(x)}\}$. What are the sections? The obvious candidate is $\mathrm{Hom}_{\mathcal{S}(\mathbb{R}/\mathbb{Z})}(\mathcal{S}(\mathbb{R}), \mathcal{M})$, where $\mathcal{M}$ is the module of sections of $\{\mathcal{H}_\xi\}$. Why? Think of $\mathcal{S}(\mathbb{R})$ as being sections of the bundle over $\mathbb{R}/\mathbb{Z}$ whose fibre at ξ is $\bigoplus_{\pi^{-1}(\xi)} \mathbb{C}$ (or a Schwartz version). To map $\bigoplus_{\pi^{-1}(\xi)} \mathbb{C}$ into $\mathcal{H}_\xi$ is to select elements in $\mathcal{H}_{\pi(x)}$ for each $x \in \pi^{-1}(\xi)$.

Next we want to replace this Hom by a tensor product (so that our sections decay). For this one wants to identify the dual of $\mathcal{S}(\mathbb{R})$ over $\mathcal{S}(\mathbb{R}/\mathbb{Z})$ with $\mathcal{S}(\mathbb{R})$, the idea being that the fibres have distinguished bases. This duality is done by the trace map

$$\begin{aligned} \mathcal{S}(\mathbb{R}) &\longrightarrow \mathcal{S}(\mathbb{R}/\mathbb{Z}) \\ f(x) &\longmapsto \sum_n f(x-n) \end{aligned}$$

and the resulting pairing

$$\begin{aligned} \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}) &\longrightarrow \mathcal{S}(\mathbb{R}/\mathbb{Z}) \\ f \otimes g &\longmapsto \sum_n f(x-n)g(x-n). \end{aligned}$$

Now we have our map from $\mathcal{S}(\mathbb{R}/\mathbb{Z})$ modules to $\mathcal{S}(\mathbb{R}) \rtimes \mathbb{Z}$ modules, namely

$$\mathcal{M} \longmapsto \mathcal{S}(\mathbb{R}) \otimes_{\mathcal{S}(\mathbb{R}/\mathbb{Z})} \mathcal{M}$$

Suppose we want to go backwards, i.e. we start with sections of the equiv. family $\{\mathcal{H}_{\pi(x)}\}$, which decay, and we want sections of $\{\mathcal{H}_\xi\}$. We need to construct invariant sections, and the appropriate method seems to be to sum the $\mathbb{Z}$-translates. This is the same as applying

$$S(\mathbb{R}) \otimes_{S(\mathbb{R}) \rtimes \mathbb{Z}} (?) \;=\; \mathbb{C} \otimes_{S(\mathbb{Z})} (?)$$

A very important family of vector space $\{\mathcal{H}_x\}$ is given by functions

$$\mathcal{H}_x = \bigoplus_{\substack{\text{all arrows} \\ \text{ending at } x}} \mathbb{C}$$

The corresponding module should be $S(\mathbb{R}) \rtimes \mathbb{Z}$

Let's adopt the category viewpt. The idea here is that given a category there is a convolution algebra made of functions on its of arrows:

$$(f * g)(\gamma) = \sum_{\gamma = \alpha \cdot \beta} f(\alpha) g(\beta)$$

(This ignores convergence questions) We can use this viewpoint in the preceding examples. The algebra $S(\mathbb{R}) \rtimes \mathbb{Z}$ is the convolution algebra of the category defined by $\mathbb{Z}$ acting on $\mathbb{R}$. $S(\mathbb{R})$ is functions on the fibre category over $\mathbb{R}/\mathbb{Z}$ consisting of (ξ, x) with $\xi \in \mathbb{R}/\mathbb{Z}$, $x \in \mathbb{R}$ and $\pi(x) = \xi$.

Now we consider

$$S(\mathbb{R}) \otimes_{S(\mathbb{R}) \rtimes \mathbb{Z}} S(\mathbb{R})$$

This means we take pairs (ξ, x), (η, y) then force $x = y$ (tensoring over $S(\mathbb{R})$), then divide out by $\mathbb{Z}$ which is a kind of summing process and leads to $\mathcal{S}(\mathbb{R}/\mathbb{Z})$. The map is

$$\mathcal{S}(\mathbb{R}) \otimes_{\mathcal{S}(\mathbb{R})\rtimes\mathbb{Z}} \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}/\mathbb{Z})$$

$$f \otimes g \longmapsto \sum_n{}' f(x-n)g(x-n)$$

On the other hand

$$\mathcal{S}(\mathbb{R}) \otimes_{\mathcal{S}(\mathbb{R}/\mathbb{Z})} \mathcal{S}(\mathbb{R})$$

means we take pairs (ξ, x) and (η, y) and then force $\xi = \eta$. Clearly we are getting arrows in the category defined by $\mathbb{Z}$ acting on $\mathbb{R}$. Thus

$$\mathcal{S}(\mathbb{R}) \otimes_{\mathcal{S}(\mathbb{R}/\mathbb{Z})} \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{S}(\mathbb{R}) \rtimes \mathbb{Z}$$

July 6, 1985

$$\mathcal{A}_\theta = \{ \sum_{m,n} a_{mn} U^m V^n \mid a_{mn} \text{ rapid decrease} \}$$
$$UVU^{-1}V^{-1} = e^{2\pi i \theta}$$

One perhaps wants to think of $\mathcal{A}_\theta$ as an algebra of operators on $\mathcal{S}(\mathbb{R})$ with

$$\begin{cases} U \cdot f(x) = e^{\theta \partial_x} f(x) = f(x+\theta) \\ V \cdot f(x) = e^{2\pi i x} f(x) \end{cases}$$

(This enables one to interpret $\mathcal{A}_\theta$ as an alg. of operators on $L^2(\mathbb{R})$, hence to define the C^* algebra and von Neumann algebra as the uniform and weak closure of $\mathcal{A}_\theta$ in the bounded operators.)

Let's now describe the Morita equivalence of $A = \mathcal{A}_\theta$ and $B = \mathcal{A}_{\theta^{-1}}$ and use this to compute the dimension of the projective $\mathcal{A}_\theta$ module $\mathcal{S}(\mathbb{R})$. A Morita equivalence is given by ${}_B P_A$, ${}_A Q_B$ together with pairings.

$$Q \otimes_B P \xrightarrow{\sim} A \quad , \quad P \otimes_A Q \xrightarrow{\sim} B$$

Set $P = \mathcal{S}(\mathbb{R})$ with left $\mathcal{A}_{\theta^{-1}}$ and right $\mathcal{A}_\theta$ mult.

$$\begin{cases} U' \cdot f(x) = e^{\partial} f(x) = f(x+1) \\ V' \cdot f(x) = e^{2\pi i \theta^{-1} x} f(x) \end{cases} \qquad \begin{cases} f(x) U = e^{-\theta\partial} f(x) = f(x-\theta) \\ f(x) V = e^{2\pi i x} f(x) \end{cases}$$

Set $Q = \mathcal{S}(\mathbb{R})$ with

$$\begin{cases} U g(x) = e^{\theta\partial} f(x) = g(x+\theta) \\ V g(x) = e^{2\pi i x} g(x) \end{cases} \qquad \begin{cases} g(x) U' = e^{-\partial} g(x) = g(x-1) \\ g(x) V' = e^{2\pi i \theta^{-1} x} g(x). \end{cases}$$

Next define the pairings

$$(1)\qquad \begin{aligned} Q \otimes_B P &\longrightarrow A = \mathcal{A}_\theta \\ g \otimes f &\longmapsto \sum_{m,n} g(x+m)\, f(x+m+n\theta)\, U^n \end{aligned}$$

and

$$(2)\qquad \begin{aligned} P \otimes_A Q &\longrightarrow B = \mathcal{A}_{\theta^{-1}} \\ f \otimes g &\longmapsto \sum_{m,n} f(x+m\theta)\, g(x+m\theta+n)\, U'^n \end{aligned}$$

denote it $g \otimes f \mapsto \Phi(g,f)$

Let's check the properties of (1). If $h(x+1)=h(x)$, then

$$\Phi(g, fh) = \sum g(x+m)\, f(x+m+n\theta)\, h(x+n\theta)\, U^n$$

$$\Phi(g,f)h) = \sum g(x+m)\, f(x+m+n\theta)\, U^n\, h(x)$$

Similarly $\Phi(hg,f) = h\,\Phi(g,f)$. Next if $k(x+\theta)=k(x)$

$$\Phi(g,kf) = \sum g(x+m)\, \underset{\overset{\shortparallel}{k(x+m)}}{k(x+m+n\theta)}\, f(x+m+n\theta)\, U^n$$

$$= \Phi(gk,f)$$

and

$$\Phi(gU',f) = \sum g(x-1+m)\, f(x+m+n\theta)\, U^n$$

$$= \sum g(x+m)\, f(x+1+m+n\theta)\, U^n$$

$$= \Phi(g, U'f)$$

Now we can compute the trace of an endomorphism u of P as a right $\mathcal{A}_\theta$-module. First of all we have the trace on $\mathcal{A}_\theta$:

$$\operatorname{tr}\Big(\sum_m f_m(x)\, U^m\Big) = \int_0^1 f_0(x)\,dx$$

Now given an endomorphism u of P we try to represent it in the form $\sum |f_\alpha\rangle\langle g_\alpha|$ where $|f_\alpha\rangle \in P$ and $\langle g_\alpha|$ denote $\mathcal{A}_\theta$-linear maps $P \longrightarrow \mathcal{A}_\theta$. Because of the pairing (1): $Q \otimes P \longrightarrow \mathcal{A}_\theta$ any element of Q defines such an $\mathcal{A}_\theta$-linear map, so we will think of $\langle g_\alpha|$ as being given by $g_\alpha \in Q$ and this pairing.

What this means is that we want

$$K(x,y) = \sum_\alpha f_\alpha(x) g_\alpha(y) \qquad \in \mathcal{S}(\mathbb{R}\times\mathbb{R})$$

so that u is the endom. of P given by

$$\sum_{m,n} K(x+m\theta, x+m\theta+n)\, U'^{\,n} \in \mathcal{A}_{\theta^{-1}}$$

Once u has been represented in terms of $\sum |f_\alpha\rangle\langle g_\alpha|$ its trace is $tr\left(\sum \langle g_\alpha | f_\alpha\rangle\right)$, where $\langle g|f\rangle \in \mathcal{A}_\theta$ denotes the pairing (1). ~~Thus in terms of K we have~~

$$\cancel{tr\left(\sum K(x+m+n\theta, x+m)\, U\right)}$$

Now

$$tr\,\langle g|f\rangle = tr\left(\sum_{m,n} g(x+m) f(x+m+n\theta) U^n\right)$$
$$= \int_0^1 \left(\sum_m g(x+m) f(x+m)\right) dx \quad = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

and so we have

$$\boxed{tr\left\{\sum_{m,n} K(x+m\theta, x+m\theta+n)\, U'^{\,n}\right\} = \int_{-\infty}^{\infty} K(x,x)\,dx}$$

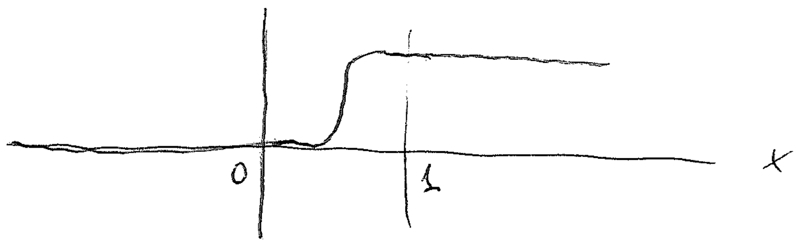
Now I have to find [redacted] a $K(x,y)$ that corresponds to the identity map of P. [redacted] You want $K(x,y) = 0$

on the lines $x+n=y$, $n \neq 0$. and then you want $\sum_m K(x+m\theta^{[redacted]}, x+m\theta^{[redacted]}) = 1$.

Construction of an $h(x) \in C_0^\infty(\mathbb{R})$ such that

$$\sum_{n\in\mathbb{Z}} h(x+n) = 1$$

Take a smooth "Heaviside" function $H(x)$



$H(x) = 0 \quad x \leq \frac{1}{2} - \varepsilon$

$H(x) = 1 \quad x \geq \frac{1}{2} + \varepsilon$

and put h [redacted] $(x) = H(x+1) - H(x)$; ~~this is a smooth~~ square wave ~~step~~ function; it's supported in $[-\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]$; $h(0)=1$.

So now we can take

$$K(x,y) = h(x-y)\big(H(x+\theta^{[redacted]}) - H(x)\big)$$

which has compact support. Let's compute the trace

$$\int_{-\infty}^{\infty} K(x,x)\,dx = \int_{-\infty}^{\infty} [H(x+\theta^{[redacted]}) - H(x)]\,dx$$

$$= \theta^{[redacted]}$$

For the last equality

$$\frac{d}{da} \int_{-\infty}^{\infty} [H(x+a) - H(x)]\,dx = \int_{-\infty}^{\infty} H'(x+a)\,dx$$

$$\int_{-\infty}^{\infty} H'(x)\,dx = H(\infty) - H(-\infty) = 1$$

Notice that we have tacitly assumed $\theta > 0$ in the above calculation. Where? Let's compute

$$\sum_{m \in \mathbb{Z}} \big(H(x + ma + a) - H(x + ma) \big)$$

carefully:

$$\sum_{m=-N}^{N-1} \big(H(x + (m+1)a) - H(x + ma) \big)$$

$$= H(x + Na) - H(x - Na)$$

$$\xrightarrow[N \to \infty]{\text{as}} \begin{cases} 1 & a > 0 \\ 0 & a = 0 \\ -1 & a < 0 \end{cases}$$

Towards a more geometric interpretation of the preceding. Let's ask what are the operators on $\mathcal{S}(\mathbb{R})$ which are endomorphism for the right $\mathcal{A}_\theta$-module structure. By the Schwartz kernel thm. such operators are represented by distributions $K(x,y)$ on $\mathbb{R}^2$:

$$f \longmapsto \int K(x,y) f(y)\, dy.$$

(smooth in x, tempered somehow). Then commuting with mult. by $e^{2\pi i x}$ means

$$\left(e^{2\pi i x} - e^{2\pi i y}\right) K(x,y) = 0$$

hence

$$K(x,y) = \sum_n{}' \, k_n(x)\, \delta(x + n - y)$$

where $k_n(x)$ is smooth. Commuting with $\rho u = e^{-\theta \partial}$ means

$$\int K(x-\theta, y) f(y)\, dy = \int K(x,y) f(y-\theta)\, dy = \int K(x, y+\theta) f(y)\, dy$$

or $$K(x, y+\theta) = K(x-\theta, y)$$
$$K(x+\theta, y+\theta) = K(x,y)$$
which means that $k_n(x+\theta) = k_n(x)$.

Conclude that any op. on $\mathcal{S}(\mathbb{R})$ commuting with right a_θ-mult. is described by a kernel in the form

$$(*) \qquad K(x,y) = \sum_n k_n(x)\,\delta(x+n-y)$$

where $k_n(x+\theta) = k_n(x)$. [redacted]f one shows [redacted] that $k_n \to 0$ rapidly, then such a K would [redacted] lie in $a_{\theta^{-1}}$.
For example
$$k_n(x) = \sum_m \tilde{K}(x+m\theta, x+m\theta+n)$$
with $\tilde{K} \in \mathcal{S}(\mathbb{R}^2)$.

What is the trace of $(*)$, the trace relative to a_θ? For operators coming from a $\tilde{K}(x,y)$, we know the trace is
$$\int_{-\infty}^{\infty} \tilde{K}(x,x)\,dx = \int_0^{|\theta|} \sum_m \tilde{K}(x+m\theta, x+m\theta)\,dx$$
$$= \int_0^{|\theta|} k_0(x)\,dx$$

This again gives $|\theta|$ for the $\operatorname{tr}(\mathrm{id}_P)$.

So at this point we see geometrically the operators on the projective module, but I don't yet see the trace geometrically.

<u>Problem</u>: Treat the case of a general (linear) transversal to the foliation.

Before immersing myself in computation again it would be nice to [redacted] discuss the situation in general terms. [redacted] I would like to link these foliation ideas originating with Connes to Schwinger's [redacted] idea that a space-like hypersurface gives a description of states of a QFT, and that [redacted] there is a transformation function [redacted] relating two descriptions belonging to different space-like hypersurfaces.

Let's begin with the [redacted] 2-torus and the Kronecker foliation. In Connes' game one looks at fields of Hilbert spaces over the torus with compatible isomorphisms when points are on the same leaf. By transversal to the foliation we shall mean an embedded circle which is a geodesic, so really it comes from a "rational" line in R^2. When we have a transversal we can restrict the field of Hilbert spaces to the transversal.

So what maybe is a good way to think is to regard a transversal as providing a presentation of the gadget of interest. Analogous to using an open covering in sheaf theory. Then there is the descent data for a given presentation, and also correspondences linking two different presentations.

[redacted]

Now what is interesting in this situation is the relative dimensions. Two transversals can be compared in size.

Next what can we say in the quantum situation? The first thing we might do is to attach to a transversal [redacted] the Fock space of functions on the circle. [redacted] Next given two transversals can we find a transformation between the Fock spaces? This is not obvious. One possible approach is to consider instead of these Fock spaces the Clifford algebras.

July 7, 1985

Consider the inf. cyclic covering $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. Then $\mathcal{S}(\mathbb{R})$ becomes an $\mathcal{S}(\mathbb{R}/\mathbb{Z})$-module. We can think of $\mathcal{S}(\mathbb{R})$ as a space of sections of an infinite-diml bundle over the circle, the fibre at $x+\mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ being the space of rapidly decreasing fns. on the coset $x+\mathbb{Z} \subset \mathbb{R}$. This is an infinite rank module so the identity endom. doesn't have a trace.

Actually what should the trace be? Endos. of $\mathcal{S}(\mathbb{R})$ as an $\mathcal{S}(\mathbb{R}/\mathbb{Z})$-module are described by kernels

$$K(x,y) = \sum_n k_n(x)\, \delta(x+n-y) \tag{1}$$

with $k_n(x)$ rapidly decreasing in n (I think). Fix a fibre $x_0+\mathbb{Z}$ over $\pi(x_0)$, and look at K on this fibre, i.e. restrict x,y to lie in this coset. One sees a matrix whose diagonals are the $k_n(x)$. The trace of this matrix is formally

$$\sum_n k_0(x_0+n)$$

Thus the trace of K relative to $\mathcal{S}(\mathbb{R}/\mathbb{Z})$ formally is the function on $\mathbb{R}/\mathbb{Z}$ given by

$$\mathrm{tr}_{\mathrm{rel}}(K) = \sum_n k_0(x+n) \tag{2}$$

If we combine this relative trace with

the trace on $\mathcal{S}(\mathbb{R}/\mathbb{Z})$ given by integration we get

(3) $$\operatorname{tr}(K) = \int_{\mathbb{R}/\mathbb{Z}} \sum_n k_0(x+n)\,dx = \int_{\mathbb{R}} k_0(x)\,dx.$$

So we learn that although K doesn't have a trace in the usual sense, because it lies in the commutant of $e^{2\pi i x}$, it has a different kind of trace.

Now we can go one stage further, namely, require that K commutes with $f(x)U = f(x-\theta)$. This means the kernel satisfies

$$k_n(x+\theta) = k_n(x) \qquad \text{for all } n.$$

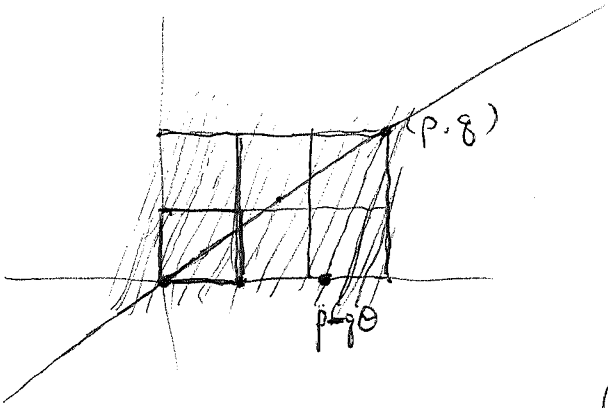
Now neither of the traces (2), (3) make sense but what does make sense is

(4) $$\int_{\mathbb{R}/\mathbb{Z}\theta} k_0(x)\,dx$$

What I saw by calculation yesterday is that (4) coincides with the actual trace of K as an endomorphism of $\mathcal{S}(\mathbb{R})$ as a right a_θ-module.

Morally one is using that the action is "finite" modulo $\mathbb{Z}\theta$ to define the trace.

Now it should be possible to compute the dimension of the projective module belonging to a transversal. Specifically let's take the circle in the torus obtained from the segment going from $(0,0)$ to (p,q) in the plane



So we again have a line $\mathbb{R}(p,q)$ mapping to $\mathbb{R}/\mathbb{Z}$, and we have a quotient of the line, namely by the subgroup $\mathbb{Z}(p,q)$. By the analogue of (4) we will map the segment $(0,0)$ to (p,q) onto the x axis, or if you want, parametrize the line by the x coordinate of the projection via the foliation, then take the length of the segment $(0,0)$ to (p,q) with this coordinate. So the answer is

$$|p-q\theta|$$

At this point I would like to try somehow to generalize the foliated torus to the case of an elliptic curve. θ has to become a point in the upper half plane.

Here are some ideas:

1) There is never a compact situation where one can pass to the quotient by the $\bar{\partial}$-operator.

2) Cauchy problem can't be solved for $\bar{\partial}$, but there's a version using subspaces of boundary values

3) I see how to think of two circles bounding an annular region. But how to handle two curves intersecting at a single point? It seems that the example of $S(\mathbb{R})$ as an $\mathcal{A}_{\theta^{-1}}-\mathcal{A}_\theta$ bimodule has to be generalized.

Recall that an element $f \in \mathcal{S}(\mathbb{R})$ is equivalent to a family smooth periodic functions for the different periodic boundary conditions. Thus for a given y

$$f(x) \longmapsto F(x,y) = \sum e^{2\pi i n y} f(x-n)$$

then $F(x+1,y) = e^{2\pi i y} F(x,y)$, and $F(x,y)$ depends only on $y \in \mathbb{R}/\mathbb{Z}$. The question is how this isomorphism behaves with respect to the Hardy space of $f(x)$ which are boundary values of holomorphic functions in the UHP (with some kind of boundedness condition)

The Hardy space H^2 consists of $f(x) \in L^2(\mathbb{R})$ such that $\hat{f}(\xi)$ vanishes for $\xi < 0$. Thus

$$f(x) = \int_0^\infty e^{2\pi i x \xi} \hat{f}(\xi)\, d\xi$$

and

$$\sum e^{2\pi i n y} f(x-n) = \int_0^\infty e^{2\pi i x \xi} \underbrace{\sum_n e^{2\pi i n y} e^{-2\pi i n \xi}}_{\sum_m \delta(y-\xi+m)} \hat{f}(\xi) d\xi$$

$$= \sum_m e^{2\pi i x(y+m)} \hat{f}(y+m)$$

So

$$F(x,y) = e^{2\pi i x y} \sum_m e^{2\pi i m x} \hat{f}(y+m)$$

Essentially one has formed the Fourier series in x whose coefficients are $\hat{f}(y+m)$, $m \in \mathbb{Z}$.

8 June 1985

Review of the loop group representation.

$S^1 = \mathbb{R}/L\mathbb{Z}$ $\qquad L^2(S^1)$ has orth. basis $|k\rangle$, $k \in \frac{2\pi}{L}\mathbb{Z}$

where
$$\langle x|k\rangle = \frac{1}{\sqrt{L}} e^{ikx}.$$

$$H = \frac{1}{i}\frac{d}{dx} \qquad H|k\rangle = |k\rangle$$

$$f(x) = \langle x|f\rangle = \sum_k \underbrace{\langle x|k\rangle}_{\frac{1}{\sqrt{L}}e^{ikx}}\langle k|f\rangle$$

$$\psi(x) = \frac{1}{\sqrt{L}}\sum_k e^{ikx}\psi_k$$

$$\psi(x)^* = \frac{1}{\sqrt{L}}\sum_k e^{-ikx}\psi_k^*$$

$$\{\psi(x), \psi(y)^*\} = \frac{1}{L}\sum_{k,l} e^{ikx-ily}\underbrace{\{\psi_k, \psi_l^*\}}_{\delta_{kl}}$$

$$= \frac{1}{L}\sum_k e^{ik(x-y)} = \sum_n \delta(x-y-nL)$$

Multiplication by f extends (formally) to

$$\rho(f) = \sum_{k,l}\psi_k^* \underbrace{\langle k|f|l\rangle}_{\frac{1}{L}\int_0^L e^{-i(k-l)x}f(x)\,dx \;=\; \hat{f}(k-l)}\psi_l$$

$$\rho(f) = \sum_q \hat{f}(q)\underbrace{\sum_l \psi_{l+q}^*\psi_l}_{\rho_q}$$

$$\hat{f}(q) = \frac{1}{L}\int_0^L e^{-iqx}f(x)\,dx$$
$$f(x) = \sum_q \hat{f}(q)e^{iqx}$$

Taking $f = \delta_x$ we get

$$\psi(x)^* \psi(x) = \sum_q{}' e^{-iqx} \rho_q$$

Commutation relations among the ρ_q:

Ground state $\quad \Phi_0 = |0, -\frac{2\pi}{L}, -2(\frac{2\pi}{L}), \dots \rangle$

is killed by $\rho_q = \sum_k{}' \psi_{k+q}^* \psi_k$ for $q < 0$. So these are the destruction operators.

$$[\rho_{-q}, \rho_q] = \lim \Big[\sum_{|k| \leq N} \psi_{k-q}^* \psi_k , \sum_{|l| \leq N} \psi_{l+q}^* \psi_l \Big]$$

$$= \lim \sum_{|k|, |l| \leq N} [\psi_{k-q}^* \psi_k , \psi_{l+q}^* \psi_l]$$

$$\shortparallel$$

$$\psi_{k-q}^* \delta_{k, l+q} \psi_l - \psi_{l+q}^* \delta_{l, k-q} \psi_k$$

$$\shortparallel$$

$$\delta_{k, l+q} \psi_l^* \psi_l - \delta_{l, k-q} \psi_k^* \psi_k$$

$$= \lim \sum_{-N \leq l \leq N-q} \psi_l^* \psi_l - \sum_{-N+q \leq k \leq N} \psi_k^* \psi_k$$

$$= \lim \underbrace{\sum_{-N \leq l < -N+q} \psi_l^* \psi_l}_{\text{number of } l \text{ in this range}} - \underbrace{\sum_{N-q < k \leq N} \psi_k^* \psi_k}_{0}$$

$$= \frac{qL}{2\pi}$$

$$\therefore \quad \boxed{[\rho_{-q}, \rho_q] = \frac{qL}{2\pi}}$$

So $\quad [\rho(f), \rho(g)] = \Big[\sum_{q'} \hat{f}(q') \rho_{q'} \sum_q \hat{g}(q) \rho_q \Big]$

$$= \frac{L}{2\pi} \sum_q \hat{f}(-q)\, q\, \hat{g}(q)$$

Now $\displaystyle \frac{1}{2\pi i}\int_0^L f(x) g'(x)\,dx = \frac{1}{2\pi i}\int_0^L \Big(\sum_{q'} \hat{f}(q') e^{iq'x} \sum_q iq\hat{g}(q) e^{iqx}\Big) dx$

$$= \frac{L}{2\pi} \sum \hat{f}(-q)\, q\, \hat{g}(q)$$

Thus

$$\boxed{[\rho(f), \rho(g)] = \frac{1}{2\pi i}\int_0^L f(x) g'(x)\,dx}$$

~~New idea. Recall that for each $\chi \in \hat{\mathbb{Z}}$ we can consider the (smooth) periodic functions wrt χ on $\mathbb{R}$: $f(x+n) = \chi(n) f(x)$ and that we can recover $\mathcal{S}(\mathbb{R})$ as the space of smooth ways of choosing such an f_χ for each χ.~~

~~χ defines a flat line bundle L_χ over $S^1 = \mathbb{R}/\mathbb{Z}$ whose sections are the χ-periodic smooth functions. Now the given Dirac op. $\frac{1}{i}\frac{d}{dx}$ induces one on~~

New idea: To each $\chi \in \hat{\mathbb{Z}}$ we have a flat line bundle L_χ over $S^1 = \mathbb{R}/\mathbb{Z}$ whose sections $\Gamma(S^1, L_\chi)$ are the χ periodic fns. on $\mathbb{R}$: $f(x+n) = \chi(n) f(x)$. We can recover $\mathcal{S}(\mathbb{R})$ as the smooth families $(f_\chi)_{\chi \in \hat{\mathbb{Z}}}$ where $f_\chi \in \Gamma(S^1, L_\chi)$.

Perform the Fock space construction on each of the spaces $L^2(S^1, L_\chi)$ with respect to the Dirac operator in each L_χ. The Dirac operator does what?

Sections of L_χ are $F(x,y)$ with

$$F(x+1,y) = e^{2\pi i y} F(x,y)$$

where $\chi(n) = e^{2\pi i n y}$. The Dirac operator is

$$\frac{1}{i} D_x = \frac{1}{i} \partial_x$$

and its eigenfunctions are $e^{2\pi i (y+n)x}$, $n \in \mathbb{Z}$

so that the eigenvalues are

$$y+n \qquad n \in \mathbb{Z}$$

Let's construct the Fock space for each y. It seems clear that we can fit them together to form a Hilbert space bundle over the circle $\hat{\mathbb{Z}}$.

Certainly the question is how does the Hilbert space of sections of this bundle compare to the Fock space that are ~~might~~ want to construct from $L^2(\mathbb{R})$?

Digression: Deligne's way to define the dilog pairing

$$(f,g) = e^{\oint \frac{\log f}{2\pi i} \frac{dg}{g}}$$

for invertible functions f,g over S^1. This is well-defined if f has degree zero, but not so obvious in general.

Deligne's idea is to lift back to the universal covering $\mathbb{R}$ and to introduce the connection

$$d - \frac{\log f}{2\pi i} \frac{dg}{g}$$

on the trivial line bundle bundle over $\mathbb{R}$. Using the gauge transformation given by g one can define an action of $\mathbb{Z}$ on the trivial bundle over $\mathbb{R}$. Then by descent one obtains a flat line bundle over S^1 whence an element of $\mathbb{C}^\times$ which is the monodromy for this flat line bundle.

Let's set this up in the universal situation. We have to construct a line bundle with connection over $\mathbb{C}^\times \times \mathbb{C}^\times$. We do this by lifting back to $\mathbb{C} \times \mathbb{C}^\times$ where we will use the connection

$$d - \frac{\log z}{2\pi i} \frac{dw}{w}$$

Let's put $z = e^{2\pi i x}$, $w = e^{2\pi i y}$ so that over $\mathbb{C} \times \mathbb{C} = \{(x,y)\}$ we have the connection

$$d - 2\pi i \, x \, dy$$

on the trivial bundle. We have the action of $\mathbb{Z} \times \mathbb{Z}$ generated by

$$e^{2\pi i y} e^{\partial_x} \quad , \quad e^{\partial_y}$$

and so this connection descends.

Another description: Consider the complex Heisenberg G group which is an extension of $\mathbb{C} \times \mathbb{C}$ by $\mathbb{C}^\times$. The Heisenberg algebra is made into a group by Campbell-Hausdorff and one divides by a $\mathbb{Z}$ in the center. Let Γ be the integral points of G. Then G/Γ is the principal $\mathbb{C}^\times$ bundle of a complex line bundle with connection over $\mathbb{C}^\times \times \mathbb{C}^\times$.

Interesting point: Why the symbol $(1-f, f)$ is trivial assume f, $1-f$ are invertible. This means that the line bundle with connection over $\mathbb{C}^\times \times \mathbb{C}^\times$ when restricted to $\mathbb{C} - \{0,1\} \subset \mathbb{C}^\times \times \mathbb{C}^\times$, $z \longmapsto (1-z, z)$ will give a flat line bundle with trivial monodromy.

Dilog function: Consider the multiple-valued 1-form

$$\frac{\log(1-z)}{2\pi i}\,\frac{dz}{z} \qquad \text{over} \quad \mathbb{C} - \{0,1\}$$

and [redacted] fix a base point and a value for $\log(1-z)$. Then integrate over curves to obtain a multiple-valued function.

A better thing to do would be to introduce the max. abelian covering over which one would obtain a single-valued holomorphic function.

Q: Is there any relation between this dilog pairing and the difficulty with constructing a 2 diml S^1-equivariant cohomology class over $\mathcal{L}BU(1)$?

July 9, 1985

L^2 index theorem - special case of $\bar\partial$ operator on sections of a constant curvature line bundle L over $\mathbb{C} = \mathbb{R}^2$. There is an action of $\tilde{\mathbb{R}}^2$ (Heisenberg group) on the pair $(\mathbb{R}^2, L)$ leaving the operator invariant. The kernel and cokernel of $\bar\partial$ within L^2 are modules over the von Neumann Weyl algebra R generated by $\tilde{\mathbb{R}}^2$. This is of type I, in fact the L^2 sections of L give the regular representation of R. The kernel and cokernel will have finite length. It seems that in the example at hand the length is one for the kernel when the curvature is >0.

Thus assuming the curvature is >0, the $\operatorname{Ker}\bar\partial$ is an irreducible R-module, and this is the index of $\bar\partial$ in this example.

Now if you have a $\Gamma \subset \mathbb{R}^2$ one can view $\operatorname{Ker}(\bar\partial)$ as a $R(\tilde\Gamma)$ - module $\left(R(\tilde\Gamma) \subset R(\tilde{\mathbb{R}}^2) = R\right)$ and take the dimension of this module using the trace on $R(\tilde\Gamma)$. It seems that the dimension is always the volume of $\mathbb{R}^2/\Gamma$ times some constant depending on the curvature. In fact $\frac{i}{2\pi}$ times $\int F$ over $\mathbb{R}^2/\Gamma$.

What is the first Chern class?

Let's try to understand the family of Fock spaces which one obtains by considering all connections modulo gauge transformations over the circle. What this means is that I fix the trivial line bundle over S^1 and look at connections on it which preserve the metric. For each of these I get a Dirac operator on the L^2-sections and I can ask about the Fock space.

There are two ways to present the family. The first is to keep the Hilbert space $L^2(S^1)$ fixed but allow the Dirac operator to vary:

$$\frac{1}{i}\frac{d}{dx} + a \qquad\qquad a \in \mathbb{R}$$

but then one has to use gauge transformations:

$$e^{-2\pi i x}\left(\frac{1}{i}\frac{d}{dx} + a\right) e^{2\pi i x} = \frac{1}{i}\frac{d}{dx} + a + 2\pi.$$

The second is to vary the boundary conditions i.e. consider $f(x)$ on $\mathbb{R}$ satisfying

$$f(x+1) = e^{2\pi i y} f(x)$$

and keep the operator $\frac{1}{i}\frac{d}{dx}$ fixed. In the first presentation the eigenfunctions are $e^{2\pi i n x}$ with corresp. eigenvalues $a + 2\pi$. In the second the eigen-functions and eigenvalues are

$$e^{2\pi i (n+y)x} \qquad 2\pi(n+y)$$

Now I want to bring in the uniqueness of the Fock space as a repn. of the CAR. Starting from the Hilbert space $\mathcal{H}$ of L^2 sections of our flat line bundle we form the Clifford algebra with the generators $\psi(f)$, $\psi(f)^*$. Then Fock space is characterized as a module over the Clifford algebra generated by a ground state, or vector annihilated by the $\psi(f)$ for f of pos. energy, and $\psi(f^-)^*$ for f^- of neg. energy. Here $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is the decomposition defined by the Dirac operator.

Now one knows also that the Fock space can be obtained as an irred. repn. of the canonical "boson" commutation relations, associated to the loop group $\mathcal{L}\,U(1)$ and the dilog skew form. It seems that this second description is independent of the flat line bundle as this group acts on the sections of the line bundle. The formulas stay the same

$$\rho_q = \sum_k \psi_{k+q}^* \psi_k$$

but now k runs over a coset of $\frac{2\pi}{L}\mathbb{Z}$.

So we have the following picture. For each character $e^{2\pi i y x/L}$ of $\mathbb{Z}L$ we have the flat line bundle over $S^1 = \mathbb{R}/\mathbb{Z}L$ with the orthonormal basis of sections e^{ikx}, $k \in \frac{2\pi}{L}y + \frac{2\pi}{L}\mathbb{Z}$. We form the Fock space based on the $k \leq 0$. Then as y goes from 0 to 1, we have to use the shift operator to form a bundle of Fock spaces.

10 July 1985

I want to review a bunch of examples which I would eventually like to fit together in a coherent theory.

1) Connes treatment of the torus with Kronecker foliation. Each circle transverse to the foliation can be veiwed as a presentation. One gets the algebra of functions on the circle acted on by a shift which comes from the foliation. Two different presentations are linked by a Morita equivalence.

So ultimately one has an intrinsic category of modules attached to the foliation.

2) Fock space associated to $L^2(S^1)$ with $H = \frac{1}{i}\frac{d}{dx}$. The extension of H to the Fock space is of positive energy, hence e^{-tH} makes sense on the Fock space level, even though e^{-tH} doesn't make sense on $L^2(S^1)$.

I want to think of the Fock space together with this operator e^{-tH}, more generally $e^{i\tau H}$ with $\tau \in UHP$ as being a presentation of the torus $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ using the real circle $\mathbb{R}/\mathbb{Z}$. Here the complex structure is supposed to generalize a foliation.

3) Tame symbol theory (Bloch-Deligne). If X is a R.S. we have

$$\begin{tikzcd}
& K_2X \arrow[r] \arrow[d, dashed] & K_2(X-S) \arrow[r] \arrow[d, "\alpha"] & \bigoplus_{x\in S}\mathbb{C}^* \arrow[r] \arrow[d, equal] & K_1X \\
0 \arrow[r] & H^1(X,\mathbb{C}^*) \arrow[r] & H^1(X-S,\mathbb{C}^*) \arrow[r] & \bigoplus_{x\in S}\mathbb{C}^* \arrow[r, "+"] & \mathbb{C}^*
\end{tikzcd}$$

Passing to the limit we obtain

$$\begin{tikzcd}
 & K_2X \arrow[r] \arrow[d, dashed] & K_2F \arrow[r] \arrow[d, "\alpha"] & \bigoplus_{x\in X} \mathbb{C}^* \arrow[d, equal] & & \\
0 \arrow[r] & H^1(X,\mathbb{C}^*) \arrow[r] & \varinjlim_S H^1(X-S,\mathbb{C}^*) \arrow[r] & \bigoplus_{x\in X} \mathbb{C}^* \arrow[r] & \mathbb{C}^* \arrow[r] & 0
\end{tikzcd}$$

α is defined by the symbol

$$(f,g) = e^{\int \frac{\log f}{2\pi i} \frac{dg}{g}}$$

over the different curves in X.

~~Now I believe that if I take the field F and the various spaces of~~

Now the tame symbol $K_2F \to \mathbb{C}^*$ assoc. to a point x of X I know how to interpret as a central extension of $F^\times$ associated to a Fock space associated to F and the family of $\mathcal{O}_x$ lattices. But what we learn from the above exact sequence is that there is an even bigger central extension of $F^\times$ than what you would get by taking the tensor product of all these representations.

Would you might like to do would be to start with the cocycle on $F^\times$ with values in the space of flat line bundles on complements of finite sets. This space of flat line bundles is a rather big torus which has lots of characters each of which gives an irreducible repn.

However I think this is a bit uncountable and one ought to be able to find something a bit more analytic.

July 15, 1985

Araki - Woods, Repns of the CCR describing a non-rel. Infinite Free Bose gas. J. Math Phys 4 (1963) p.637.

Let V be a complex vector space with inner product. One is interested in finding operators $a^*(f)$, $a(f)$ depending linearly and conjugate linearly on $f \in V$ such that

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0 \quad , \quad a(f)^* = a^*(f).$$

$$[a(f), a^*(g)] = \langle f | g \rangle$$

(The operators $a(f)$, $a^*(f)$ are unbounded, but these commutation relations can be rigorously treated in the Weyl form using the unitary operators $e^{a^*(f) - a(f)}$.)

Fock space is a representation of these CCR having a cyclic vector Φ_F annihated by all the $a(f)$. The generating functional for this representation is

$$\langle \Phi_F | e^{a^*(f) - a(f)} | \Phi_F \rangle = e^{-\frac{1}{2}\|f\|^2}$$

Note that

$$e^X e^Y = e^{X+Y} e^{\frac{1}{2}[X,Y]}$$

when $[X,Y]$ commutes with X, Y. Hence

$$e^{a^*(f) - a(f)} = e^{-\frac{1}{2}[a^*(f), -a(f)]} e^{+a^*(f)} e^{-a(f)}$$

$$= e^{-\frac{1}{2}\|f\|^2} e^{a^*(f)} e^{-a(f)}$$

Here is the first construction of Araki and Woods. One starts with V as $L^2(R/LZ)$ and looks at Fock spaces. There is a Hamiltonian around say $H|k\rangle = \frac{k^2}{2m}|k\rangle$ where $\langle x|k\rangle = \frac{1}{\sqrt{L}} e^{ikx}$, and one looks at the ground state in the n-particle space which is

$$\frac{1}{\sqrt{n!}} \left(a_0^*\right)^n \Phi_F$$

where $a_0^* = a^*(|0\rangle)$. One computes the generating functional:

$$\frac{1}{n!} \left\langle (a_0^*)^n \Phi_F \middle| e^{a^*(f) - a(f)} \middle| (a_0^*)^n \Phi_F \right\rangle$$

as follows. We want $n!$ times the coefficient of $(\bar{v}u)^n$ in

$$\left\langle e^{v a_0^*} \Phi_F \middle| e^{a^*(f) - a(f)} \middle| e^{u a_0^*} \Phi_F \right\rangle$$

$$= \left\langle \Phi_F \middle| e^{\bar{v} a_0} \underbrace{e^{a^*(f) - a(f)}}_{e^{-\frac{1}{2}\|f\|^2} e^{a^*(f)} e^{-a(f)}} e^{u a_0^*} \middle| \Phi_F \right\rangle$$

$$= e^{-\frac{1}{2}\|f\|^2} \left\langle \Phi_F \middle| e^{\bar{v} a_0} e^{a^*(f)} e^{u a_0^*} e^{-\langle f | u | 0 \rangle} \middle| \Phi_F \right\rangle$$

$$= e^{-\frac{1}{2}\|f\|^2 - u\langle f | 0\rangle + \bar{v}(\langle 0 | f\rangle + u\langle 0 | 0\rangle)}$$

$$= e^{-\frac{1}{2}\|f\|^2 - u\langle f | 0\rangle + \bar{v}\langle 0 | f\rangle + \bar{v}u}$$

Thus we get

$$n!\, e^{-\frac{1}{2}\|f\|^2} \sum_{k=0}^{n} \frac{\left(-\langle f | 0\rangle\right)^k \left(\langle 0 | f\rangle\right)^k}{k!\; k!} \frac{1}{(n-k)!}$$

The idea is now to let n, L go to ∞ in such a way that $\frac{n}{L} \longrightarrow \rho$. Note that

$$\langle 0|f\rangle = \frac{1}{\sqrt{L}}\int_{-\frac{L}{2}}^{L/2} f(x)\,dx \sim \frac{1}{\sqrt{L}}\int_{-\infty}^{\infty} f(x)\,dx \sim \frac{1}{\sqrt{L}}\hat{f}(0)$$

so we get

$$e^{-\frac{1}{2}\|f\|^2}\sum_{k=0}^{n}\frac{\left(-\overline{\hat{f}(0)}\right)^k\left(\hat{f}(0)\right)^k}{(k!)^2}\underbrace{\frac{1}{L^k}\frac{n!}{(n-k)!}}_{\frac{n(n-1)\cdots(n-k+1)}{L^k}\to\rho^k}$$

$$\longrightarrow e^{-\frac{1}{2}\|f\|^2}\underbrace{\sum_{k=0}^{\infty}\frac{(-1)^k\left(|\hat{f}(0)|^2\right)^k\rho^k}{(k!)^2}}_{\text{0th order Bessel fn. } J_0(\rho|\hat{f}(0)|^2)}$$

An alternative approach is to use instead of the ground state of particle number precisely n a mixture. We can introduce a chemical potential and afterwards adjust it so that the average number of particles is n.

In doing this calculation we can split V into the line spanned by $|0\rangle$ and its orthogonal complement. We thereby reduce to the case of a simple h. oscillator. The number of particles operator is a^*a, where now a stands for a_0. We use the density operator

$$\frac{e^{-\beta a^*a}}{tr(e^{-\beta a^*a})} \qquad tr(e^{-\beta a^*a}) = \sum_0^\infty e^{-\beta n} = \frac{1}{1-e^{-\beta}}$$

to weight states appropriately, where β is adjusted to give

$$n = \langle a^*a\rangle = -\frac{d}{d\beta}\log tr(e^{-\beta a^*a}) = \frac{e^{-\beta}}{1-e^{-\beta}} = \frac{1}{e^{\beta}-1} \sim \frac{1}{\beta}$$

We have to compute

$$\mathrm{tr}\left(e^{-\beta a^* a}\, e^{f a^* - \bar{f} a}\right) \Big/ \mathrm{tr}\left(e^{-\beta a^* a}\right)$$

Steps:

$$:e^{-t a^* a}:\; e^{\lambda z} = \sum \frac{(-t)^n}{n!} (a^*)^n a^n e^{\lambda z} = e^{-t\lambda z} e^{\lambda z} = e^{(1-t)\lambda z}$$

$$e^{-\beta a^* a} e^{\lambda z} = e^{(e^{-\beta}\lambda) z} \implies \boxed{\begin{array}{l} e^{-\beta a^* a} = \,:e^{-t a^* a}: \\ \iff e^{-\beta} = 1 - t \end{array}}$$

$$\mathrm{tr}\left(e^{-\beta a^* a} e^{f a^* - \bar{f} a}\right) = e^{\frac{1}{2}|f|^2} \mathrm{tr}\left(e^{f a^*} :e^{-t a^* a}: e^{-\bar{f} a}\right)$$

$$= e^{\frac{1}{2}|f|^2} \mathrm{tr}\left(:e^{\underbrace{-t a^* a + f a^* - \bar{f} a}_{-t\left(a^* + \frac{\bar{f}}{t}\right)\left(a - \frac{f}{t}\right) - \frac{1}{t}|f|^2}}:\right)$$

$$= e^{-|f|^2\left[\frac{1}{t} - \frac{1}{2}\right]} \mathrm{tr}\left(:e^{-t(a^* + t^{-1}\bar{f})(a + t^{-1} f)}:\right)$$

$$\boxed{\mathrm{tr}\left(e^{-\beta a^* a} e^{f a^* - \bar{f} a}\right) = \frac{1}{1 - e^{-\beta}}\, e^{-\frac{1}{2}|f|^2 \frac{\cosh(\beta/2)}{\sinh(\beta/2)}}}$$

$$\therefore \frac{\mathrm{tr}\left(e^{-\beta a^* a} e^{f a^* - \bar{f} a}\right)}{\mathrm{tr}\left(e^{-\beta a^* a}\right)} = e^{-\frac{1}{2}|f|^2 \coth(\beta/2)} \sim e^{-\frac{1}{n}|f|^2} \quad \text{as } n \to \infty$$

Conclusion is that this ensemble, or mixture, approach leads to a simpler formula which is closer to the later discussion in Araki-Woods.

5

July 16, 1985

Let's start with the complex Hilbert space $L^2(\mathbb{R}/L\mathbb{Z})$ form its Fock space $\mathcal{H}_F$ with vacuum Φ_F and generating functional

$$\langle \Phi_F |\, e^{a^*(f)-a(f)}\, | \Phi_F \rangle = e^{-\frac{1}{2}\|f\|^2}$$

We let $\langle x | k \rangle = \frac{1}{\sqrt{L}} e^{ikx}$, $k \in \frac{2\pi}{L}\mathbb{Z}$, as usual and write a_0, a_0^* for $a(|0\rangle)$ and $a^*(|0\rangle)$ resp. We are interested in the state on the Weyl alg generated by $e^{a^*(f)-a(f)}$ as $f \in L^2(\mathbb{R}/L\mathbb{Z})$ given by the generating function

$$\operatorname{tr}\left(e^{-\beta a_0^* a_0}\, e^{a^*(f)-a(f)}\right) \Big/ \operatorname{tr}\left(e^{-\beta a_0^* a_0}\right).$$

This is a [redacted] simpler version of what Araki and Woods look at. The value of this generating function is

$$e^{-\frac{1}{2}\|f\|^2}\; e^{-\frac{1}{2}|\hat{f}(0)|^2 \coth(\beta)}$$

where $\hat{f}(0) = \langle 0 | f \rangle = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} f(x)\,dx$, $\|f\|^2 = \int_{-L/2}^{L/2} |f(x)|^2 dx$

Now following A-W we let $n = \langle a_0^* a_0 \rangle = \frac{1}{1-e^{-\beta}} \sim \frac{1}{\beta}$ tend to ∞, and let L tend to ∞ so that $\frac{n}{L} \to \rho$. I want f to be in $C_0^\infty(\mathbb{R})$. Clearly

$$\frac{1}{2}\coth(\beta)\,|\hat{f}(0)|^2 \sim \underbrace{\frac{1}{\beta}}_{n} \cdot \frac{1}{L} \left| \int_{-L/2}^{L/2} f(x)\, dx \right|^2 \longrightarrow \rho \left| \int_{-\infty}^{\infty} f(x)\,dx \right|^2$$

so the limiting generating function is

$$e^{-\frac{1}{2}\|f\|^2}\; e^{-\rho \left|\int_{-\infty}^{\infty} f(x)\,dx\right|^2}$$

Let's look at repns. of the CCR $[a, a^*] = 1$. There is the Fock space repn. with its Φ_F such that $a\Phi_F = 0$, whence

$$\langle \Phi_F | e^{\lambda a^* - \bar{\lambda} a} | \Phi_F \rangle = e^{-\frac{1}{2}|\lambda|^2}.$$

Another possibility has the generating function

$$\frac{\operatorname{tr}\left(e^{-\beta a^* a} e^{\lambda a^* - \bar{\lambda} a}\right)}{\operatorname{tr}\left(e^{-\beta a^* a}\right)} = e^{-|\lambda|^2\left(\frac{1}{2} + \frac{1}{e^\beta - 1}\right)}$$

Here $\frac{1}{2} + \frac{1}{e^\beta - 1}$ is any no $> \frac{1}{2}$.

Another way to get this same generating function is to use $[a^* + z, a + \bar{z}] = 1$; here $a^* + z$, $a + \bar{z}$ act on $\mathcal{H}_F \otimes L^2\left(\mathbb{C}, \frac{e^{-\frac{1}{\rho}|z|^2} d^2 z}{\pi \rho}\right)$ and the cyclic vector is $\Phi_F \otimes 1$. The generating function is

$$\langle \Phi_F \otimes 1 | e^{\lambda a^* - \bar{\lambda} a} \otimes e^{\lambda z - \bar{\lambda} \bar{z}} | \Phi_F \otimes 1 \rangle$$

$$= e^{-\frac{1}{2}|\lambda|^2} \int e^{-\frac{1}{\rho}|z|^2 + \lambda z - \bar{\lambda}\bar{z}} \frac{d^2 z}{\pi \rho} = e^{-\frac{1}{2}|\lambda|^2 - \rho|\lambda|^2}$$

A further way to obtain this generating fn. is to consider on $\mathcal{H}_F \otimes \mathcal{H}_F$ the cyclic vector $\Phi_F \otimes \Phi_F$ and the operators

$$\tilde{a}^* = s(a^* \otimes 1) + t(1 \otimes a)$$

$$\tilde{a} = \bar{s}(a \otimes 1) + \bar{t}(1 \otimes a^*)$$

Then $[\tilde{a}, \tilde{a}^*] = |s|^2 - |t|^2 = 1$, and the generating function is:

$$\lambda \tilde{a}^* - \bar{\lambda} \tilde{a} = (\lambda s a^* - \bar{\lambda} \bar{s} a) \otimes 1 + 1 \otimes (\lambda t a - \bar{\lambda} \bar{t} a^*)$$

$$\langle \Phi_F \otimes \Phi_F | e^{\lambda \tilde{a}^* - \bar{\lambda} \tilde{a}} | \Phi_F \otimes \Phi_F \rangle = \langle \Phi_F | e^{\lambda s a^* - \bar{\lambda} \bar{s} a} | \Phi_F \rangle \langle \Phi_F | e^{\lambda t a - \bar{\lambda} \bar{t} a^*} | \Phi_F \rangle$$

$$= e^{-\frac{1}{2}|\lambda s|^2} e^{-\frac{1}{2}|-\bar{\lambda} \bar{t}|^2} = e^{-\frac{1}{2}(|s|^2 + |t|^2)|\lambda|^2}$$

Now Araki + Woods take this algebra for each momentum k. Thus they deal with the generating function

$$e^{-\int |f_k|^2 \left(\frac{1}{2} + \rho(k)\right)\, dk}$$

which they handle by taking $s_k = \sqrt{1+\rho(k)}$, $t_k = \sqrt{\rho(k)}$. So one has made a symplectic transformation which is evidently not unitarily implementable

July 21, 1985

Consider the finite covering $\mathbb{R}/n\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$. Then $C^\infty(\mathbb{R}/n\mathbb{Z})$ is a free module of rank n over $C^\infty(\mathbb{R}/\mathbb{Z})$; a basis is e^{ikx}, $k = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{2\pi(n-1)}{n}$. Hence the ring of endomorphisms of this module is isom. to $M_n(C^\infty(\mathbb{R}/\mathbb{Z}))$. We also know that

$$C^\infty(\mathbb{R}/n\mathbb{Z}) \rtimes \mathbb{Z}/n\mathbb{Z}$$

is isom. to the ring of endos. Also these are isomorphisms of $*$-algebras. Thus the group of unitary elements in the above cross-product is isom. to the smooth loop group $\mathcal{L}U(n)$ of $U(n)$.

Now we can form the Fock space of $L^2(\mathbb{R}/n\mathbb{Z})$. This gives a unitary representation of a central extension of $\mathcal{L}U(n)$. This representation is irreducible because it is already irreducible over the central extension of the subgroup $U(1, C^\infty(\mathbb{R}/n\mathbb{Z})) \simeq \mathcal{L}U(1)$

The preceding is Graeme's way to construct the fundamental repn. of $\mathcal{L}U(n)^\sim$ from that of $\mathcal{L}U(1)^\sim$.

Next we would like to do the same thing for the infinite covering $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. Here we consider $\mathcal{S}(\mathbb{R})$ as a module over $C^\infty(\mathbb{R}/\mathbb{Z})$, and instead of all endomorphisms we look at the algebra

$$\mathcal{S}(\mathbb{R}) \rtimes \mathbb{Z}$$

which gives the smooth or rapidly-decreasing endos. The corresponding unitary group consisting of unitary operators on $L^2(\mathbb{R})$ of the form $1+K$ with K in $\mathcal{S}(\mathbb{R})\rtimes\mathbb{Z}$ should be a version of the free loop group on the infinite unitary group U. Call this group $U(\mathcal{S}(\mathbb{R})\rtimes\mathbb{Z})$.

The problem is now whether $U(\mathcal{S}(\mathbb{R})\rtimes\mathbb{Z})^{\sim}$ acts on the Λ-Fock space belonging to $L^2(\mathbb{R})$. Actually there is probably nothing special about $\mathbb{Z}$ here, and perhaps it is more natural to look at $\mathcal{S}(\mathbb{R})\rtimes\mathbb{R}$ which should be the algebra of operators on $L^2(\mathbb{R})$ which have kernels $K(x,y)$ lying in $\mathcal{S}(\mathbb{R}^2)$. Note the same as $\mathcal{S}(\mathbb{R})$ & $\mathcal{S}(\mathbb{R})\rtimes\mathbb{R}$.

The other possibility is to try to form some sort of limit of the Λ-Fock space of $L^2(\mathbb{R}/n\mathbb{Z})$ as $n\longrightarrow\infty$. Here I guess n goes to ∞ in the sense of divisibility, and one would like an exterior algebra of some sort where the rank is continuous, something very type II. One is trying to take $L^2(\mathbb{R})$ with the Hardy half-space spanned by the e^{ikx} with $k\le 0$, and then construct a Fock space which multiplication by e^{itx} would extend. This sort of thing might be natural in connection with the inverse spectral problem