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$$\int_0^1 dt \ \mathrm{tr}\left(e^{D^2 + t[D,\theta] + (t^2-t)\theta^2}\theta\right)$$ 757

We attempt to relate Chern character forms associated to a super connection $tr_s\left(e^{(D+L)^2}\right)$, $L = i\begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$ with what one gets from the Grassmannian graph construction.

Let us start off with an odd degree endom.

$$L = i\begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix} \quad \text{of} \quad E = E^0 \oplus E^1$$

and connection $D = D^0 + D^1$ on E. The idea of the Grassmannian graph construction is to consider the Grassman bundle $G(E)$ of subspaces of $E^0 \oplus E^1$ of dimension $= \dim E^0 = \dim E^1$. Then we define a section s of $G(E)$ over M by letting $s(m) = $ graph of $u(m) \subset E_m$.

$$\begin{tikzcd} G(E) \arrow[d, "\pi"] \\ M \arrow[u, bend left, "s"] \end{tikzcd}$$

Then we have $s^*(S) = \text{graph}(u) \simeq E^0$, where S is the subbundle. Now S is canonically a direct summand of $\pi^* E$, so it inherits a connection from the connection on $\pi^* E$ obtained from the given connection D on E. In fact pulling back by s, we get the connection on E^0 obtained by the isomorphism

$$E^0 \simeq \text{graph}(u)$$

and the connection on the latter obtained from its being a direct summand of E, using the metric.

Recall that the projector onto graph u (p. 728) and onto its orthogonal complement $= \{-u^*(y) + y \mid y \in E'\}$ are

$$e = \begin{pmatrix} 1 \\ u \end{pmatrix} (1+u^*u)^{-1} \begin{pmatrix} 1 & u^* \end{pmatrix}$$

$$1-e = \begin{pmatrix} -u^* \\ 1 \end{pmatrix} (1+uu^*)^{-1} \begin{pmatrix} -u & 1 \end{pmatrix}$$

If D is the connection, then the induced connection on $\operatorname{Im} e = \operatorname{graph}(u)$ is $e.D.e$ and the curvature is

$$\begin{aligned} e.D.e.D.e &= e.D^2.e + e[D,e].D.e \\ &= \quad '' \quad + [D,e](1-e).D.e \\ &= \quad '' \quad + [D,e](1-e)[D,e] \\ &= e.D^2.e + e[D,e]^2 \end{aligned}$$

Let's calculate the second term

$$[D,e] = \begin{pmatrix} 0 \\ [Du] \end{pmatrix} (1+u^*u)^{-1} \begin{pmatrix} 1 & u^* \end{pmatrix} + \begin{pmatrix} 1 \\ u \end{pmatrix} [D,(1+u^*u)^{-1}] \begin{pmatrix} 1 & u^* \end{pmatrix}$$
$$+ \begin{pmatrix} 1 \\ u \end{pmatrix} (1+u^*u)^{-1} \begin{pmatrix} 0 & [Du^*] \end{pmatrix}$$

$$(1-e)[D,e] = \begin{pmatrix} -u^* \\ 1 \end{pmatrix} (1+uu^*)^{-1} [Du] (1+u^*u)^{-1} \begin{pmatrix} 1 & u^* \end{pmatrix}$$

$$[D,e](1-e)[D,e] = \begin{pmatrix} 1 \\ u \end{pmatrix} (1+u^*u)^{-1} [Du^*] (1+uu^*)^{-1} [Du] (1+u^*u)^{-1} \begin{pmatrix} 1 & u^* \end{pmatrix}$$

Now let us use the isom. $E^0 \simeq \operatorname{graph} u$ given by $\begin{pmatrix} 1 \\ u \end{pmatrix}$. Then

$$[D,e](1-e)[D,u] \longleftrightarrow (1+u^*u)^{-1} [Du^*] (1+uu^*)^{-1} [Du]$$

$$e\, D^2\, e \longleftrightarrow (1+u^*u)^{-1} \begin{pmatrix} 1 & u^* \end{pmatrix} D^2 \begin{pmatrix} 1 \\ u \end{pmatrix} = (1+u^*u)^{-1} \left(D_0^2 + u^* D_1^2 u \right)$$

It seems useful to consider the case where E^0, E^1 are both trivial and $D = d$. In this case the curvature of the subbundles E^0, E^1 identified with graph u and $[\text{graph}(-\overset{*}{u})]^t$ with the induced connection are respectively

$$(1+u^*u)^{-1} du^* (1+uu^*)^{-1} du$$

$$(1+uu^*)^{-1} du \,(1+u^*u)^{-1} du^*$$

In terms of $L = i\begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$ we have

$$1 - L^2 = \begin{pmatrix} 1+u^*u & 0 \\ 0 & 1+uu^* \end{pmatrix} \qquad dL = i\begin{pmatrix} 0 & du^* \\ du & 0 \end{pmatrix}$$

and so the curvature matrix on $E^0 \oplus E^1$ for the above connection is

$$\begin{pmatrix} (1+u^*u)^{-1} du^* (1+uu^*)^{-1} du & 0 \\ 0 & \cdots\cdots\cdots \end{pmatrix} = -\frac{1}{1-L^2}\, dL \,\frac{1}{1-L^2}\, dL$$

Hence the n-th Chern character form for $E^0 - E^1$ computed with this connection is

$$\frac{(-1)^n}{n!} \frac{1}{2} \operatorname{tr}_s \left(\frac{1}{1-L^2} dL\right)^{2n}.$$

The $\frac{1}{2}$ comes from the fact that the character forms for E^0 and E^1 are of opposite sign; $\operatorname{Tr}(XY)^n = -\operatorname{Tr}(YX)^n$ if X, Y are matrices of 1-forms.

On the other hand

$$\int_0^\infty e^{\beta(L^2 + dL)} e^{-\beta} d\beta = \frac{1}{1 - L^2 - dL}$$

$$= \frac{1}{1-L^2} + \frac{1}{1-L^2} dL \frac{1}{1-L^2} + \frac{1}{1-L^2} dL \frac{1}{1-L^2} dL \frac{1}{1-L^2} + \cdots\cdots$$

which gives the wrong number of factors of $(1-L^2)^{-1}$, as well as the wrong constants (factorials have disappeared).

Conclusion: We still have no progress on the link between the heat kernel and parametrix approach. $\blacksquare$

Let's now go over Connes approach which seems to be nicely related to the Grassmannian forms. We start with an operator

$$\mathcal{D} : \mathcal{H}^+ \longrightarrow \mathcal{H}^-$$

whose index is to be computed, and ultimately its cyclic cocycles. We can replace it by an invertible operator reduced by an idempotent in the following way. We add trivial pieces the simplest being

$$P : \begin{tikzcd} \mathcal{H}^0 \arrow[r, "\mathcal{D}"] \arrow[dr, "+id"] & \mathcal{H}^1 \\ \oplus & \oplus \\ \mathcal{H}^1 \arrow[r, "\mathcal{D}^*"'] \arrow[ur, "-id"'] & \mathcal{H}^0 \end{tikzcd} \qquad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ on both sides.}$$

Thus $$P = \begin{pmatrix} \mathcal{D} & -1 \\ 1 & \mathcal{D}^* \end{pmatrix} \qquad \text{so} \qquad ePe = [redacted]\ \mathcal{D}$$

Then $$P^* = \begin{pmatrix} \mathcal{D}^* & 1 \\ -1 & \mathcal{D} \end{pmatrix} \qquad \text{so}$$

$$P^*P = \begin{pmatrix} \mathcal{D}^* & 1 \\ -1 & \mathcal{D} \end{pmatrix}\begin{pmatrix} \mathcal{D} & -1 \\ 1 & \mathcal{D}^* \end{pmatrix} = \begin{pmatrix} 1+\mathcal{D}^*\mathcal{D} & 0 \\ 0 & 1+\mathcal{D}\mathcal{D}^* \end{pmatrix}$$

is ≥ 1 and so is invertible.

At this point we can use the formula

$$\text{Index } ePe = \operatorname{Tr}\left(P^{-1}[P,e]\right)^{2n+1}$$

for n large enough so the trace makes sense.

To calculate $\quad P^{-1}[P,e] = (P^*P)^{-1}P^*[P,e]$.

$$[P,ef] = \left[\begin{pmatrix}\mathcal{D} & -1\\ 1 & \mathcal{D}^*\end{pmatrix}, \begin{pmatrix} f & 0\\ 0 & 0\end{pmatrix}\right] = \begin{pmatrix}\mathcal{D}f & 0\\ f & 0\end{pmatrix} - \begin{pmatrix} f\mathcal{D} & -f\\ 0 & 0\end{pmatrix} = \begin{pmatrix}[\mathcal{D},f] & -f\\ f & 0\end{pmatrix}$$

$$P^*[P,e] = \begin{pmatrix}\mathcal{D}^* & 1\\ -1 & \mathcal{D}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & \mathcal{D}^*\\ \mathcal{D} & -1\end{pmatrix}$$

$$P^*[P,ef] = \begin{pmatrix}\mathcal{D}^* & 1\\ -1 & \mathcal{D}\end{pmatrix}\begin{pmatrix}[\mathcal{D},f] & -f\\ f & 0\end{pmatrix} = \begin{pmatrix}\mathcal{D}^*[\mathcal{D},f]+f & -\mathcal{D}^*f\\ f\mathcal{D} & f\end{pmatrix}$$

Thus even for $f=1$ it seems that $\operatorname{Tr}\left(P^{-1}[P,e]\right)^{odd}$ is fairly complicated.

Except the following happens. Note that

$$P^*[P,e] = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & \mathcal{D}^*\\ \mathcal{D} & 0\end{pmatrix}$$

commutes with $\quad P^*P = \begin{pmatrix}1+\mathcal{D}^*\mathcal{D} & \\ & 1+\mathcal{D}\mathcal{D}^*\end{pmatrix}$.

This means we have lousy notation. Let's instead try

$$P = \begin{pmatrix}1 & \mathcal{D}^*\\ \mathcal{D} & -1\end{pmatrix} : \begin{matrix}\mathcal{H}^+\\ \oplus\\ \mathcal{H}^-\end{matrix} \longrightarrow \begin{matrix}\mathcal{H}^+\\ \oplus\\ \mathcal{H}^-\end{matrix}$$

with $\quad e = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ on the source of P and $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$ on the target of e. Then

$$[P,ef] = \begin{pmatrix}1 & \mathcal{D}^*\\ \mathcal{D} & -1\end{pmatrix}\begin{pmatrix}f & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & f\end{pmatrix}\begin{pmatrix}1 & \mathcal{D}^*\\ \mathcal{D} & -1\end{pmatrix}$$

$$= \begin{pmatrix}f & 0\\ \mathcal{D}f & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ f\mathcal{D} & -f\end{pmatrix} = \begin{pmatrix}f & 0\\ [\mathcal{D},f] & f\end{pmatrix}$$

so $\quad [P,e] = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Also $\quad P^2 = \begin{pmatrix}1 & \mathcal{D}^*\\ \mathcal{D} & -1\end{pmatrix}\begin{pmatrix}1 & \mathcal{D}^*\\ \mathcal{D} & -1\end{pmatrix} = \begin{pmatrix}1+\mathcal{D}^*\mathcal{D} & 0\\ 0 & \mathcal{D}\mathcal{D}^*+1\end{pmatrix}$

Thus $$\mathrm{Tr}\left(\left(P^{-1}[P,e]\right)^{2n+1}\right) = \mathrm{Tr}\,(P^2)^{-n}P$$

$$= \mathrm{Tr}\left(\begin{pmatrix} 1+\mathcal{D}^*\mathcal{D} & 0 \\ 0 & 1+\mathcal{D}\mathcal{D}^* \end{pmatrix}^{-n}\begin{pmatrix} 1 & \mathcal{D}^* \\ \mathcal{D} & -1 \end{pmatrix}\right)$$

$$\boxed{\text{Index} = \mathrm{Tr}\,(1+\mathcal{D}^*\mathcal{D})^{-n} - \mathrm{Tr}\,(1+\mathcal{D}\mathcal{D}^*)^{-n}}$$

April 29, 1984

Summary of ideas occurring before & after the Eilenberg trip:

1) Here is some basic algebra: One has the derivation of odd degree $[\not{D}, ?]$ on the algebra of asymptotic operators. It square $[\not{D}, [\not{D}, ?]]$ is $-[H, ?]$ where H belongs to the algebra. In the classical limit one gets $[D, ?]$ on $\Omega(M, \operatorname{End} E) \otimes$ functions on T^* with square given by the inner derivation belonging to $-|p|^2 + D^2$.

The important point is that $\not{D}$ does not become D in the classical limit, since $\not{D}$ is not an asymptotic operator, and since $\not{D}^2$ does not become D^2. So the relevant algebra is the operators $[\not{D}, ?]$ and H such that $[\not{D}, \;]^2 = -[H, \;]$.

A simpler example of this algebra is $[D, \;]$ on $\Omega(M, \operatorname{End} E)$ with $[D, ?]^2 = [K, ?]$, K being the curvature.

2) My original idea that there should be some sort of limiting heat kernel attached to a $\not{D}$ which gives all the index information attached to $\not{D}$ gets modified. The good limiting gadget will be essentially (if we remove the p's which introduce the 2π factors) the operator $[D, \;]$ on, and the element D^2 of, $\Omega(M, \operatorname{End} E)$. This gadget is enough to describe the equivariant form representing the character of the index bundle at $\not{D}$, which is an element of $S(\Omega^0)^* \otimes \Lambda(\Omega^1)^*$ where $\Omega^q = \Omega^q(M, \operatorname{End} E)$. Formula:

$$\int_M \hat{A}(M)\, \operatorname{tr}\left(e^{D^2 + [D, \delta D] + X}\right) \qquad \begin{array}{l} X \in \Omega^0 \\ \delta D \in \Omega^1 \end{array}$$

3) One of my goals is to understand properly what Connes is doing with his cyclic cocycles. I believe it is related to the local index theorem for families of Dirac operators by transgression. The families index gives a sequence of cohomology classes on $B\mathcal{G}$, or on $\mathcal{G}$, where $\mathcal{G} = \mathrm{Map}(M, U)$ which are related by periodicity. By using periodicity any class on $\mathcal{G}$ can be pushed into the range where it can be represented by left-invariant differential forms.

4) I reviewed what I did in Nov + Dec on transgressing the equivariant forms on $(\mathcal{A}, \mathcal{G})$, and found (p. 746) that given $D \in \mathcal{A}$ one gets the Lie cochain on $\tilde{\mathfrak{g}} = \Omega^0(M, \mathrm{End}\,E)$

$$(*) \qquad \int_0^1 dt\, tr\left(e^{D^2 + t[D,\theta] + (t^2-t)\theta^2}\theta\right)$$

with values in $\Omega(M)$. This is a cocycle modulo some filtration. This cochain has the virtue that if one restricts it to $\Omega^0(M) \subset \Omega^0(M, \mathrm{End}\,E)$, it becomes

$$tr(e^{D^2}) \int_0^1 dt\, tr\left(e^{t d\theta + (t^2-t)\theta^2}\theta\right)$$

which is nice because we want the process of extension from $\Omega^0(M)$ to $\Omega^0(M, \mathrm{End}\,E)$ (using D) followed by the restriction, to multiply by $tr(e^{D^2})$.

Connes also has a way to use D on E to extend cyclic cocycles on $\Omega^0(M)$ to $\Omega^0(M, \mathrm{End}\,E)$. He embeds E in a trivial bundle so that D becomes the Grassmannian connection. Question: Is there a way to define a trace

$$\underbrace{\{\Omega(M, \mathrm{End}\,E), [D, ?], D^2\}} \longrightarrow \{\Omega(M), d\} \quad ?$$

Maybe this should be the non-comm. diffl forms for $\Omega^0(M, \mathrm{End}\,E)$

5) The fundamental weakness in my differential form approach is <u>the lack of understand</u> of periodicity.

$ad(\phi) = [\phi, \ ]$, with $ad(\phi)^2 = -ad(H)$ is a <u>quantized version</u> of $ad(D)$, with $ad(D^*)^2 = ad(F)$. Find trace concept in this algebra.

April 30, 1984

Problem: I suppose given a superalgebra R with a odd degree derivation D such that D^2 is the inner derivation $[K, \,]$. Given a representation $\mathfrak{g} \to R$ I want to construct cocycles in $C^*(\mathfrak{g})$. The representation is an element θ of $C^1(\mathfrak{g}, R)$ with $\delta\theta + \theta^2 = 0$. So from the element $\theta, D\theta, K$ and a suitable trace map $R \to k$ I can construct elements of $C^*(\mathfrak{g})$. I would like to know when I get cocycles.

A first thing I could do is to adjoin D to R. A basic property of the trace is that it vanishes on commutators and the image of D. I think this should be the only property I want to use. Thus I cam thinking in terms of the "universal" trace

$$R \longrightarrow R/[R,R] + DR.$$

In applications the trace will not be everywhere defined, and especially not on D.

So let's adjoin D to R with $D^2 = K$. Then in $C^*(\mathfrak{g}, R)$ we have two elements D, θ of odd degree to play around with. The first thing I could do is to take the tensor algebra

$$T(D,\theta) \twoheadrightarrow T(D,\theta)/[\ ,\ ]$$

and look at the effect of δ on this, where δ is defined by

$$\delta(D) = 0$$
$$\delta\theta = -\theta^2$$

Perhaps things simplify if we ~~[illegible]~~ assume $D^2 = 0$.

The universal algebra:

Connes constructs cyclic cocycles starting with the following. One has a superalgebra R equipped with an odd degree derivation D satisfying $D^2=0$. Given then a representation $\theta : \mathfrak{g} \to R$ of a Lie algebra, he shows that

$$\operatorname{tr}\left(\theta (D\theta)^j\right) \in C^{j+1}\left(\mathfrak{g},\ R/[R,R]+DR\right)$$

is a Lie cocycle.

Let's go over this in a simple case. We have the Lie coboundary δ on $C^*(\mathfrak{g}, R)$ defined by $\delta\theta + \theta^2 = 0$. Then

$$\delta\left[\theta(D\theta)\right] = -\theta^2 D\theta - \theta \underbrace{\delta D\theta}_{+D(+\theta^2)}$$

mod commutators

$$= -\theta^2 D\theta - \theta(D\theta)\theta - \theta^2 D\theta \sim -\theta^2 D\theta$$

$$[D,\theta^3] = (D\theta)\theta^2 - \theta(D\theta)\theta + \theta^2 D\theta \sim 3\theta^2 D\theta$$

so $\quad \delta \operatorname{tr} \theta D\theta = -\frac{1}{3} \operatorname{tr}[D,\theta^3]$

These computations are done in an algebra generated by elements θ, $D\theta$ where $D^2\theta$ is defined to be zero, and δ is defined by

$$\delta\theta = -\theta^2, \qquad \delta D + D\delta = 0$$

I would like to do calculations without assuming $D^2=0$. Thus I want to consider the universal algebra with odd degree derivation D generated by elements θ, K satisfying $D^2\theta = [K,\theta]$, $DK=0$. Thus the algebra will be built up out of the monomials in $\theta, D\theta, K$.

I feel that one should extend Connes framework where cyclic cocycles are attached to differential graded algebras with trace to include an algebra R with D, K. Here D is a degree 1 derivation of R and $K \in R^0$ is such that $D^2 = [K, \ ]$. The concept of a trace is similar; a linear fnl on R vanishing on $[R,R]$ and $D(R)$.

If this extension is done, then one can always add to K a generic central element, whence from any procedure for defining cyclic cocycles one can generate a sequence of other cocycles. I think this should explain the S-operator.

<u>Ex</u>: Suppose we consider the algebra $R = \Omega(M, \mathrm{End}\,E)$ with D given by $[D, \ ]$ and $K = D^2$ where D is a connection. We can suppose a trace given by a closed current in M.

Now I consider the homomorphism $\Omega^0(M) \longrightarrow R$ given by $f \longrightarrow f \cdot id_E$. In fact consider more generally the map $i : \Omega^{\cdot}(M) \longrightarrow \Omega^{\cdot}(M, \mathrm{End}\,E)$, $\omega \longrightarrow \omega \cdot id_E$. Then we have

$$[D, \omega] = d\omega$$

so that the embedding ι is compatible with D on both algebras. But it is not compatible with K.

Let's try to relate this to the S-operator. Suppose that E is defined via an idempotent matrix e and that D is the Grassmannian connection. Inside $M_n\{\Omega^{\cdot}(M)\}$, we consider the differential algebra $\tilde{R}$ generated by fe, $f \in \Omega^0(M)$. Now f, df and any $\omega \in \Omega^{\cdot}(M)$ commute with e, de, and so we get a map of differential algebras

$$\Omega^\cdot(M) \otimes \hat{\Omega}(ke) \longrightarrow M_k\{\Omega^\cdot(M)\}$$

with image $\tilde{R}$. Now we can also consider the subalgebra R' of $M_k\{\Omega^\cdot(M)\}$ obtained by adjoining de to $eM_k\{\Omega^\cdot(M)\}e = \Omega(M, \operatorname{End} E)$. ~~This should~~ ~~be the same as $\tilde{R}$~~ This contains $\tilde{R}$. ?

Twisted polynomial rings. Let V be a real vector space equipped with a skew form $\omega \in \Lambda^2 V^*$. Then one forms a twisted polynomial ring, or Weyl algebra, in the same way one can form a Clifford algebra for a degenerate quadratic form. This much is old hat, however we now have a new approach to this algebra based upon the line bundle with connection over V having the curvature form ω. In particular there is a different concept of kernel for the heat operator. Before, the only way I could discuss the heat kernel of the harmonic oscillator, say, is by means of a 'polarization', i.e. writing ω as a hyperbolic symplectic form. But now it seems that we might get a uniform way of writing the integral operators belonging to this Weyl algebra, where the non-commutativity is built into the parallel transport.

Let's fix some notation: Let L be the line bundle with the connection D satisfying $D^2 = \omega$. Then for each $v \in V$ we get a first order operator $D_v = i(X_v)D$ where X_v is the constant vector field on V associated to $v \in V$.

Because L has constant curvature we get a group of automorphisms of V, L which induce translations on V. (General picture: We get a central extension by S^1 of the group of autos. of V preserving ω). This group will be a central extension of V by S^1, and the inf generators of this group give another family of first order operators on $\Gamma(V,L)$, commuting with the family $\{D_v\}$.

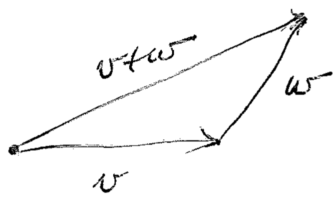
So the first problem will be to describe the two sets of operators. I have the feeling that one set is the commutant of the other, and that those integral operators commuting with the translation action form a twisted convolution algebra.

Let D be the connection on L and $D_v = i(X_v)D$ be covariant differentiation in the direction v, where X_v is the constant vector field given by the vector v. Then it is clear that e^{D_v} is [redacted] the parallel translation operator which results from parallel translating along the line segment from x to $x+v$. To see this just arrange [redacted] a trivialization of the line bundle along the lines parallel to v.

Then we have the Weyl commutation relations

$$e^{D_v} e^{D_w} = e^{D_{v+w}} e^{\frac{1}{2}[D_v, D_w]}$$

since by assumption $[D_v, D_w]$ is a scalar. One can picture this relation using the triangle:



Here $\frac{1}{2}[D_v, D_w]$ is the curvature integrated over the triangle, it gives the difference in phase between the parallel transport [redacted] along $\longrightarrow\nearrow$ and $\nearrow$.

Now given this family of translation operators one can form the operators

$$\int dv\, f(v)\, e^{D_v}$$

$u = v+w$

$v = u-w$

where $f \in C_0^\infty$. Then

$$\int dv\, f(v) e^{D_v} \int dw\, g(w) e^{D_w} = \int dw\, dv\, f(v) g(w)\, e^{D_{v+w}} \times e^{\frac{1}{2}[D_v, D_w]}$$

$$= \int du \left\{ \underbrace{\int dw\, f(u-w)\, g(w)\, e^{\frac{1}{2}[D_{u-w}, D_w]}}_{\text{twisted convolution } f*g} \right\} e^{D_u}$$

so we get a twisted convolution algebra.

Finally we need the kernel of this operator

$$\langle x \Big| \int dv\, f(v)\, e^{D_v} \Big| 0 \rangle = \int dv\, f(v) \langle x | e^{D_v} | 0 \rangle$$

Now e^{D_v} is just a parallel translation operator, so when applied to the δ fn. at 0 it gives a δ fn. at v. So it's pretty clear that this integral has to be $f(x) \cdot T_{x,0}$, where $T_{x,0}$ is $\parallel$ transport from 0 to x. In general

$$\langle x | \int dv\, f(v)\, e^{D_v} | x' \rangle = f(x-x')\, T_{x,x'}$$

So we conclude that writing an operator in the form $\int dv\, f(v)\, e^{D_v}$ is equivalent to giving its Schwartz kernel.

So now the problem arises as to how to compute the kernel for the heat kernel $e^{tD_\mu^2}$, i.e. to represent this operator as

$$e^{tD_\mu^2} = \int dv\, f_t(v)\, e^{D_v}$$

I know I can handle this by the Mehler formula as Getzler does, but I would really like a direct proof.

Notice that when the curvature is zero we are computing the ordinary heat kernel

$$e^{t\partial_\mu^2} = \int d^n v\, f_t(v)\, e^{v^\mu \partial_\mu} \qquad e^{-t\xi_\mu^2} = \int d^n v\, f_t(v)\, e^{iv^\mu \xi_\mu}$$

from which we get

$$f_t(v) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{v^2}{4t}}$$

as usual using the Fourier transform.

Now an interesting question is whether there is a Fourier or Weyl transform in this situation. In the zero curvature case we can identify our algebra with functions of ξ, then to do the Fourier transform we multiply by $e^{-ix\xi}$ and integrate $\int \frac{d^n\xi}{(2\pi)^n}$ which is a kind of a trace. So it would be interesting to see whether the same formalism worked.

Let's take the symplectic case where the trace should be the trace on the unique irreducible representation of the Weyl algebra. I seem to remember deriving a formula

$$\mathrm{Tr}\left(\int dv\, f(v)\, e^{Dv}\right) = \text{const.}\, f(0)$$

May 3, 1989

We are given a real vector space V with skew form ω and then we consider the "Weyl algebra" consisting of operators

$$\int dv\, f(v)\, e^{D_v}$$

on sections of the line bundle over V with connection having the constant curvature ω. Here $f \in \mathcal{S}(V)$, and using the formula

$$e^{D_v} e^{D_w} = e^{D_{v+w}} e^{\frac{1}{2}[D_v, D_w]}$$

$$[D_v, D_w] = \omega(v, w)$$

one sees that one gets a twisted convolution algebra structure on $\mathcal{S}(V)$.

Now the basic problem I want to solve is to show the heat operator $e^{tD_\mu^2}$ exists in this Weyl algebra and to exhibit its kernel. Now the idea I had to get the kernel is to use the following trace on the Weyl algebra

$$tr : \int dv\, f(v) e^{D_v} \longrightarrow f(0)$$

This is a trace because from the convolution formula

$$(f * g)(v) = \int dw\, f(v-w) g(w) e^{\frac{1}{2}[D_v, D_w]}$$

one gets

$$(f*g)(0) = \int dw\, f(-w) g(w)$$

which is symmetric in f and g.

When the form ω is symplectic, i.e. nondegenerate there is a unique trace up to a multiplicative constant, namely the trace as an operator on the unique irreducible representation. Let's compute what this is. Write $V = \mathbb{R}^{2n}$ with coordinates $q_1, \dots, q_n, p_1, \dots, p_n$ and suppose the corresponding operators satisfy the canonical commutation relations with an h:

$$[p_i, q_j] = \frac{h}{i}\delta_{ij}$$

We write $D_v = v \cdot D$ acting on the irred. repn. as $i(u \cdot q + v \cdot p)$. Then we want to compute the trace of

$$\int du\, dv\ f(u,v)\, e^{i(uq+vp)}$$

$$= \int du\, dv\ f(u,v) \underbrace{e^{-\frac{1}{2}uv[iq, ip]}}_{e^{-\frac{i}{2}uv}}\ e^{iuq}\, e^{ivp}$$

Suppose $f(u,v)\, e^{-\frac{i}{2}uv} = f(u)\, g(v)$, whence we want the trace of the operator

$$* \qquad \int du\ f(u) e^{iuq} \int dv\ g(v) e^{ivp}$$

acting on $L^2(\mathbb{R}^n)$, where $q_i =$ mult by. x_i, $p_j = \frac{h}{i}\partial_j$.

We compute the kernel

$$\langle x | \int dv\, g(v) e^{ivp} | x' \rangle = \int \frac{d^n p}{(2\pi)^n h^n}\, e^{\frac{i}{h}p(x-x')} \int dv\, g(v)\, e^{ivp}$$

$$= \frac{1}{h^n} \int g(v)\ \delta\left(\frac{x-x'}{h} + v\right) dv$$

$$= \frac{1}{h^n}\, g\left(-\frac{x-x'}{h}\right).$$

So the kernel of $*$ is

$$\hat{f}(-x)\, \frac{1}{h^n}\, g\left(-\frac{x-x'}{h}\right)$$

and so the trace is

$$\int dx\, \hat{f}(-x)\, \frac{1}{h^n}\, g(0) = \frac{(2\pi)^n}{h^n} f(0)g(0).$$

Thus

$$\cancel{tr \int dv\, f(v)\, e^{D_v} = \frac{(2\pi)^n}{h^n} f(0)}$$

$$\boxed{\text{trace} \int du\,dv\, f(u,v)\, e^{-i(uq+vp)} = \frac{(2\pi)^n}{h^n} f(0,0)}$$

New idea: Treat the Weyl algebra by following the analogy with the Clifford algebra. Recall that one makes $C(V,Q)$ act on ΛV by defining Clifford multiplication by v to be

$$c(v) = e(v) + i(\lambda_v)$$

where $\lambda : V \longrightarrow V^*$ is such that

$$\left(c(v)^2 = \right) \quad \lambda_v(v) = Q(v)$$

Similarly given a skew form ω on V we define a differential operator D_v on functions on V by

$$D_v = \partial_v + \lambda_v$$

where $\lambda : V \longrightarrow V^*$ is such that we get the correct commutation relations

$$\left([D_u, D_v] = \right) \quad \partial_u \lambda_v - \partial_v \lambda_u = \omega(u,v)$$

(Thus what we are doing is to trivialize the line bundle in such a way that the connection form is linear.)

If we use the sychronous framing at O, then $\lambda_v = \frac{1}{2}\omega(\, , v)$. So

$$e^{D_v} = e^{\partial_v + \lambda_v} = e^{\partial_v} e^{\lambda_v} e^{\cancel{\frac{1}{2}[\partial_v, \lambda_v]}}$$

$$e^{D_v}\delta(x) = e^{\partial_v}\delta(x) = \delta(x+v)$$

$$\int dv\, f(v) e^{D_v}\delta(x) = \int dv\, f(v)\,\delta(x+v)$$

$$= f(-x)$$

Thus the identification of the Weyl algebra with functions on V defined in analogy with $C(V) \simeq \Lambda V$ is essentially $\int dv\, f(v) e^{D_v} \longmapsto f(v)$.

Big problem: Construct $e^{tD_\mu^2}$ in this algebra. I want to do it directly without passing through the harmonic oscillator.

Because the kernel of $e^{tD_\mu^2}$ involves sinh + cosh it is clear that one wants to do the time evolution somewhere else. The time evolution is exponential e^{tF}, and the sinh comes from algebra, possibly a Gaussian integral.

May 4, 1984

I want to derive the kernel for $e^{tD_\mu^2}$ using the path integral & evaluating the path integral over the critical path. This means I need the action, Hamiltonian, etc.

Suppose we review the physics

$$H = \frac{1}{2}(p_\mu - A_\mu)^2 \qquad \text{here } A_\mu \text{ real fn. of } x.$$

$$\dot{q}_\mu = \frac{\partial H}{\partial p_\mu} = p_\mu - A_\mu$$

$$\dot{p}_\mu = -\frac{\partial H}{\partial q_\mu} = +(p_\nu - A_\nu)\,\partial_\mu A_\nu = (\partial_\mu A_\nu)\,\dot{q}_\nu$$

$$\dot{p}_\mu = (\dot{q}_\mu + A_\mu)^{\cdot} = \ddot{q}_\mu + (\partial_\nu A_\mu)\,\dot{q}_\nu$$

$$\ddot{q}_\mu = (\partial_\mu A_\nu - \partial_\nu A_\mu)\,\dot{q}_\nu = F_{\mu\nu}\,\dot{q}_\nu$$

Here $F_{\mu\nu}$ is a real skew-symmetric matrix. Thus if
F is constant

$$\dot{q} = e^{tF} c$$

where c is the intial velocity, and

$$q = c_0 + \frac{e^{tF} - I}{F}\, c$$

where c_0 is the initial position. Take $c_0 = 0$.

Notice that time evolution is expenential for the velocity $\dot{q}$. The velocity vector rotates according to the skew-symmetric matrix F.

$$L = p\dot{q} - H = (\dot{q} + A)\dot{q} - \tfrac{1}{2}\dot{q}^2 = \tfrac{1}{2}\dot{q}^2 + \dot{q}A$$

In the synchronous framing at 0 one has

$$A_\mu = \tfrac{1}{2} x^\nu F_{\nu\mu} \qquad \Longrightarrow \qquad A = -\tfrac{1}{2} F q$$

so $$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\dot{q} F q$$

$$Fq = F\,\frac{e^{tF}-1}{F}\,c = e^{tF}c - c = \dot{q} - c$$

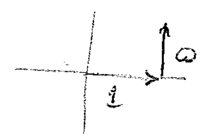
$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\dot{q}(\dot{q}-c) = \frac{1}{2}\dot{q}c$$

$$S = \int_0^t L\,dt = \frac{1}{2} q c$$

q final position at time t.

$$\boxed{S = \frac{1}{2}\, q \,\frac{F}{e^{tF}-1}\, q}$$

suppose $$F = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \quad \text{in } \mathbb{R}^2$$



$$e^{tF}-1 = \begin{pmatrix} (\cos\omega t)-1 & -\sin\omega t \\ \sin\omega t & (\cos\omega t)-1 \end{pmatrix}$$

$$\det(e^{tF}-1) = \cos^2 - 2\cos + 1 + \sin^2 = 2(1-\cos\omega t)$$

$$F(e^{tF}-1)^{-1} = \frac{1}{2(1-\cos)}\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} \cos-1 & \sin \\ -\sin & \cos-1 \end{pmatrix}$$

$$= \frac{1}{2(1-\cos)}\begin{pmatrix} \omega\sin & -\omega(\cos-1) \\ \omega(\cos-1) & \omega\sin \end{pmatrix}$$

$$\frac{1}{2} q F(e^{tF}-1) q = \frac{1}{4(1-\cos)}(\omega\sin)(x^2+y^2)$$

$$= \frac{1}{4}\,\frac{\omega\; 2\sin\frac{\omega t}{2}\cos\frac{\omega t}{2}}{2\sin^2\frac{\omega t}{2}}\,(x^2+y^2)$$

$$\boxed{S = \frac{1}{4}\left(\frac{\omega\cos\frac{\omega t}{2}}{\sin\frac{\omega t}{2}}\right)(x^2+y^2)}$$

A more intelligent computation is to go back to

$$S = \frac{1}{2}\, q\, \frac{F}{e^{tF}-1}\, q$$

and observe that the matrix is not symmetric. Let's symmetrize it

$$\frac{1}{2}\left(\frac{F}{e^{tF}-1} + \frac{+F}{-e^{-tF}+1}\right) = \frac{F}{2}\left(\frac{e^{-\frac{t}{2}F}}{e^{\frac{t}{2}F}-e^{-\frac{t}{2}F}} + \frac{e^{\frac{t}{2}F}}{e^{\frac{t}{2}F}-e^{-t/2F}}\right)$$

$$= \frac{F\cosh\left(\frac{t}{2}F\right)}{2\sinh\left(\frac{t}{2}F\right)}$$

so that

$$\boxed{S = \frac{1}{4}\, q\, \frac{F\cosh\left(\frac{t}{2}F\right)}{\sinh\left(\frac{t}{2}F\right)}\, q}$$

for a general F.

At this point one really ought to go back and understand the determinant factor in front of the Gaussian expression in the heat kernel.

The general story goes as follows for Gaussian integrals. One is given an affine space and a quadratic function Q over it, say with positive-definite leading terms. Then one wants to integrate e^{-Q} over this affine space with respect to some Haar measure. The answer is that

$$\int e^{-Q} = e^{-Q(x_c)} \left(\det \tfrac{1}{2}\partial_i\partial_j Q\right)^{-1/2} (2\pi)^{n/2}$$

where $\frac{1}{2}\partial_i\partial_j Q$ represents the quadratic form on the associated linear space, and the det has to be computed relative to the Haar measure. Here x_c is the critical point.

Now I want to apply this to the path integral giving the heat kernel $e^{tD_\mu^2}$. First do the physics notation: $H = \frac{1}{2}(p-A)^2$ where $A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu$ is real. Then the path integrals are of two kinds

$$\int \mathcal{D}q\,\mathcal{D}p \; e^{i\int (p\dot{q} - H)dt}$$

or
$$\int \mathcal{D}q \; e^{i\int L(q,\dot{q})\,dt}$$

so let's see what these are. We have

$$p = \dot{q} + A$$

$$L = (\dot{q}+A)\dot{q} - \frac{\dot{q}^2}{2} = \frac{1}{2}\dot{q}^2 + \dot{q}A = \frac{1}{2}\left(\dot{q}^2 - \dot{q}Fq\right)$$

The critical points for $\int L\,dt$ are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\dot{q} - \frac{1}{2}Fq\right) = \frac{\partial L}{\partial q} = +\frac{1}{2}F\dot{q}$$

or
$$\ddot{q} = F\dot{q}$$

and we pin one of these down by requiring q to start at 0 and end at x at time t.

Take $x = 0$, then the critical path is $q(\tau) \equiv 0$ and we want the determinant of the operator

$$-\partial_\tau^2 + F\partial_\tau = (-\partial_\tau + F)\partial_\tau$$

on paths with $q(0) = q(t) = 0$. This has clearly to do with the operator e^{tF}. As $\dot{q}$ seems to be a better variable to work with, one should perhaps identify the path $q(\tau)$ starting and ending at zero with the functions $\dot{q}(\tau)$ which are periodic and have $\int_0^t \dot{q}(\tau)d\tau = 0$. (?)

May 5, 1984

The problem is to construct the operator $e^{tD_\mu^2}$ in the Weyl algebra generated by D_μ satisfying the commutation relations

$$[D_\mu, D_\nu] = F_{\mu\nu}$$

where $F_{\mu\nu}$ is a constant skew-symmetric matrix. If I use a synchronous frame at the origin, then

$$D_\mu = \partial_\mu + A_\mu \qquad A_\mu = -\tfrac{1}{2} F_{\mu\nu} x^\nu$$

so that we are dealing with the operators

$$\tfrac{1}{i} D_\mu = \tfrac{1}{i}\partial_\mu + \tfrac{1}{i} A_\mu = p_\mu - \tfrac{1}{2i} F_{\mu\nu} q_\nu$$

and the Hamiltonian $H = -D_\mu^2$ is a quadratic function of the q, p's. Thus the Hamiltonian belongs to the Lie alg. of the symplectic group, and constructing $e^{itD_\mu^2}$ is the same as giving the 1-parameter family of unitary operators corresponding to this Hamiltonian.

In the above A, F are purely imaginary.

Next observe what we have [redacted] above. We have a real vector space V with a metric, i.e. positive definite form, and a skew-form F. Suppose this time that F is real-valued. For example we can take the harmonic oscillator. Here V has basis p, q the metric is p^2+q^2 and the skew form is $\{p,q\} =$ [redacted] ω. Using the positive definite form as a Hamiltonian one gets classical equations of motion on V. The point of the harmonic oscillator is that because of the curvature ω there is a complicated quantization. There is no simple description of the quantum mechanical time evolution. (What I mean by simple is a description of the Hilbert space in terms of a complete commuting family of operators on which the time evolution is very simple.)

This seems to be a good framework to think of the motion in a constant magnetic field as well as the harmonic oscillator. One has a Hamiltonian which is a positive quadratic function on a linear space of operators whose commutators is given by a skew-form.

The only question I have is whether it is possible to the representation of $e^{tD_\mu^2}$ in terms of the $e^{x^\mu D_\mu}$ in some direct way. I'd love to use the mutual commutant idea in some way.

Derivation of Schroedinger kernel for harmonic oscillator using the explicit metaplectic formula. Suppose one is given a symplectic transformation

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q' \\ p' \end{pmatrix}$$

Then this is unitarily implemented by the operator with kernel

$$\frac{1}{\sqrt{2\pi}}\,\frac{1}{\sqrt{\beta}}\,e^{i\left(\alpha\frac{q^2}{2}+\beta q q'+\gamma\frac{q'^2}{2}\right)}.$$

where $S(q,q') = \alpha\frac{q^2}{2}+\beta q q'+\gamma\frac{q'^2}{2}$ is related to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by the formulas

$$p = \frac{\partial S}{\partial q} = \alpha q + \beta q'$$

$$-p' = \frac{\partial S}{\partial q'} = \beta q + \gamma q' \quad \Longrightarrow \quad q = -\frac{\gamma}{\beta}q' - \frac{1}{\beta}p'$$

$$p = \alpha\left(-\frac{\gamma}{\beta}q' - \frac{1}{\beta}p'\right) + \beta q' = \left(\beta - \frac{\alpha\gamma}{\beta}\right)q' - \frac{\alpha}{\beta}p'$$

Actually I want to go the other way, namely from $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to S: Thus I start with

$$q = aq' + bp' \qquad\qquad p' = \frac{1}{b}q - \frac{a}{b}q'$$

$$p = cq' + dp' \qquad\qquad p = cq' + d\left(\frac{1}{b}q - \frac{a}{b}q'\right)$$

$$= \left(c - \frac{da}{b}\right)q' + \frac{d}{b}q$$

$$p = \frac{d}{b}q - \frac{1}{b}q'$$

$$-p' = -\frac{1}{b}q + \frac{a}{b}q'$$

$$\alpha = \frac{d}{b}, \quad \beta = -\frac{1}{b}, \quad \gamma = \frac{a}{b}$$

Now consider $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$:

$$\dot{q} = \frac{\partial H}{\partial p} = p$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\omega^2 q$$

$$\begin{pmatrix} q \\ p \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

so the symplectic transformation is

$$\exp t\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} = \begin{pmatrix} \cos\omega t & \frac{\sin\omega t}{\omega} \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix}$$

and so the action is

$$S(q, q') = \frac{\omega\cos\omega t}{\sin\omega t}\frac{q^2}{2} - \frac{\omega}{\sin\omega t}qq' + \frac{\omega\cos\omega t}{\cancel{\sin\omega t}}\frac{q'^2}{2}$$

etc.

May 6, 1984

Let's try to understand a bit about weights on a von Neumann algebra, the KMS condition, etc.

Let $\mathcal{A}$ be an involutive algebra, which means, at least, that it is endowed with an anti-linear involution $x \mapsto x^*$ such that $(xy)^* = y^*x^*$.

Let φ be a linear functional on $\mathcal{A}$ such that

$$\varphi(x^*x) \geq 0.$$

Then we get a scalar product on $\mathcal{A}$:

$$\langle x, y \rangle = \varphi(x^*y)$$

satisfying

$$\langle xy, z \rangle = \langle y, x^*z \rangle$$

Example: Let $\mathcal{A}$ be the algebra of bounded operators on Hilbert space and

$$\varphi(x) = tr(e^{-H}x)$$

where e^{-H} is a trace class operator ≥ 0.

First, recall GNS construction. Suppose $\mathcal{A}$ acts on a Hilbert space $\mathcal{H}$ compatibly with $*$. Let $v \in \mathcal{H}$, and put

$$\varphi(A) = \langle v, Av \rangle \qquad A \in \mathcal{A}.$$

Then

$$\varphi(A^*A) = \langle v, A^*Av \rangle = \|Av\|^2$$

so that what we get by completing $\mathcal{A}$ is the cyclic subspace of $\mathcal{A}$ <u>spanned</u> by v.

Let's return to the example where

$$\varphi(x) = \operatorname{tr}(\rho x)$$

where $\mathcal{A}$ = bdd operators and ρ is a trace class operator ≥ 0. Then

$$\|x\|^2 = \varphi(x^* x) = \operatorname{tr}(\rho x^* x)$$

$$= \operatorname{tr}\left(\rho^{\frac{1}{2}} x^* x \rho^{1/2}\right) = \operatorname{tr}\ (x\rho^{1/2})^*(x\rho^{1/2})$$

which means that the Hilbert space obtained from the GNS construction applied to φ is just the closed left ideal in the space of Hilbert-Schmidt operators generated by $\rho^{1/2}$.

Let's return to the general theory, where one starts with $\mathcal{A}$ and the weight φ which might only be densely defined (e.g. tr on bdd operators). Then one constructs a Hilbert space $\mathcal{H}$ by completing wrt

$$\|x\|^2 = \varphi(x^* x)$$

and a star representation of $\mathcal{A}$ on $\mathcal{H}$ given by left multiplication. Now the theory of Tomita-Takesaki says that the map $x \longrightarrow x^*$ in $\mathcal{A}$ has a closure S as an operator on $\mathcal{H}$ which is an anti-linear involution. S has a polar decomposition

$$S = J\Delta^{1/2} \qquad \begin{aligned} \Delta &= (\text{adj. } S)\circ S \\ J &= \text{isometric involution} \end{aligned}$$

What does this mean in the example? $\mathcal{A}$ = bdd operators, assume $\operatorname{Ker}\rho = 0$, then $\mathcal{A}\rho^{1/2} \subset$ Hilbert Schmidt is dense, so $\mathcal{H}$ becomes the Hilbert-Schmidt operators with $\mathcal{A}$ acting via left multiplication. The extra piece of information is the ~~[illegible]~~ embedding $\mathcal{A} \longrightarrow \mathcal{H}$, $x \longmapsto x\rho^{1/2}$. The map $x \longmapsto x^*$ on $\mathcal{A}$ becomes on $\mathcal{H}$:

$$S: \quad y \longmapsto y\rho^{-1/2} \longmapsto \rho^{-1/2}y^* \longmapsto \rho^{-1/2}y^*\rho^{1/2}$$

Then

$$\|Sy\|^2 = \mathrm{tr}\,(\rho^{-1/2}y^*\rho^{1/2})^*(\rho^{-1/2}y^*\rho^{1/2}) = \mathrm{tr}(\rho^{1/2}y\rho^{-1}y^*\rho^{1/2})$$

$$= \mathrm{tr}\,(y^*\rho y\rho^{-1}) = \langle y, \rho y\rho^{-1}\rangle = \langle y, \Delta y\rangle$$

so $\Delta y = \rho y\rho^{-1}$. When transported back to $\mathcal{A}$ we get

$$x \longmapsto x\rho^{1/2} \longmapsto \rho(x\rho^{1/2})\rho^{-1} = \rho x\rho^{-1}\rho^{1/2} \longrightarrow \rho x\rho^{-1}.$$

and so the modular automorphism group is

$$\Delta^{it}x\Delta^{-it} = \rho^{it}x\rho^{-it} = e^{-itH}xe^{+itH}$$

May 7, 1984

Problem: Compute $Tr(e^{-H} e^{v.D})$ where $H = -\frac{1}{2}D_\mu^2$ and $v.D = v^\mu D_\mu$ and the D_μ are generators of a Weyl algebra: $[D_\mu, D_\nu] = F_{\mu\nu}$ constant.

I want to see what can be accomplished using the KMS condition. This is formulated as follows. Consider for x, y fixed bdd operators the function

$$\Phi(t) = Tr(e^{-H} x e^{itH} y e^{-itH})$$
$$= Tr(x e^{itH} y e^{-itH-H})$$

and notice that as $H > 0$, that it extends holomorphically to the strip $0 < \operatorname{Im} t < 1$. Then

$$\Phi(t+i) = Tr(e^{-H}(e^{itH} y e^{-itH})x)$$

and this is the KMS condition:

$$Tr(e^{-H} x \alpha_t(y)) \xrightarrow[\text{continues from } t \text{ to } t+i \text{ and}]{\text{analytically}} = Tr(e^{-H}\alpha_t(y)x)$$

So now consider

$$\Phi(t, v, w) = Tr(e^{-H} e^{v.D} \quad e^{itH} e^{w.D} e^{-itH})$$

I need $e^{itH}(w.D)e^{-itH}$.

$$\frac{d}{dt} e^{itH}(w^\nu D_\nu) e^{-itH} = e^{itH} \underbrace{[itH, w^\nu D_\nu]}\, e^{-itH}$$

$$-\frac{i}{2} \underbrace{[D_\mu^2, D_\nu]}_{2F_{\mu\nu} D_\mu} w^\nu = (-i F_{\mu\nu} w^\nu) D_\mu$$

$$\therefore \quad e^{itH}(w^\nu D_\nu) e^{-itH} = (e^{-itF} w) \cdot D$$

Next I will need

$$[u^\mu D_\mu, v^\nu D_\nu] = u^\mu v^\nu F_{\mu\nu} = u \cdot F v.$$

$$\Phi(t, v, w) = Tr\left(e^{-H} e^{vD} e^{e^{itF} w \cdot D}\right)$$

$$= \underbrace{Tr\left(e^{-H} e^{(v + e^{-itF} w) D}\right)}_{\Phi(t)} \underbrace{e^{\frac{1}{2} v \cdot F e^{-itF} w}}_{h(t)}$$

Apply KMS to see that the above analytically continues to

$$\Phi(t+i, v, w) = Tr\left(e^{-H} e^{e^{-itF} w \cdot D} e^{v \cdot D}\right)$$

$$= \underbrace{Tr\left(e^{-H} e^{(e^{-itF} w + v) D}\right)}_{\Phi(t)} \underbrace{e^{-\frac{1}{2} v \cdot F e^{-itF} w}}_{h(t)^{-1}}$$

Then I have that

$$\Phi(t+i)\, h(t+i) = \Phi(t)\, h(t)^{-1}$$

so

$$\frac{\Phi(t)}{\Phi(t+i)} = h(t)\, h(t+i) = e^{\frac{1}{2} v \cdot F e^{-itF} (1 + e^F) w}$$

If this is iterated formally one gets

$$\Phi(t) = h(t)\, h(t+i)^2\, h(t+2i)^2 \cdots$$

so one needs

$$1 + 2e^F + 2e^{2F} + \cdots = 1 + \frac{2e^F}{1 - e^F} = \frac{1 + e^F}{1 - e^F}$$

and we are able to conclude that

$$\Psi(t) = \left(\text{fn. of } e^{2\pi t}\right) e^{\frac{1}{2} v\cdot F\, e^{-itF} \frac{1+e^F}{1-e^{F}} w}$$

which partially explains the Gaussian dependence.

What about the convergence of the geometric series? The matrix F is purely imaginary and skew-symmetric, so a typical example is $F = i\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$. This is hyperbolic: the eigenvalues are $\pm\omega$ so e^F will have an eigenvalue > 1, so the geometric series doesn't converge.

I am still missing what I think should exist, namely, a formula for $e^{-\beta H}$ starting from, or using, the time evolution

$$e^{itH} e^{v\cdot D} e^{-itH} = e^{(e^{-itF} v)\cdot D}$$

Somehow once I know that the effect of e^{-H} on $e^{v\cdot D}$ is given by $v \to e^{-F} v$, then the rest should be a consequence of the "graph" of this transf.

May 8, 1984

Back to the problem of the existence of a heat operator. I really want to carry through the idea that if L_t $t \geq 0$ is a path starting at I with tangent vector $-H$, then

$$e^{-tH} = \lim_{n\to\infty} \left(L_{\frac{t}{n}}\right)^n.$$

To make this precise one requires $L_t(x,y) \longrightarrow \delta(x-y)$ with first + 2nd moments $\sim t$.

Example: Suppose L_t is a convolution operator with kernel $L_t(x,y) = L_t(x-y)$ over $\mathbb{R}^n$ or a torus.

$$L_t(x-y) = \int \frac{d\xi}{2\pi} e^{i\xi(x-y)} f_t(\xi) \qquad n=1$$

$$\left(L_{\frac{t}{n}}\right)^n (x-y) = \int \frac{d\xi}{2\pi} e^{i\xi(x-y)} f_{\frac{t}{n}}(\xi)^n$$

Let's be more specific and think in terms of a random walk. Let $\varphi \in C_0^\infty(\mathbb{R})$ and put

$$L_t(x-y) = \varphi\left(\frac{x-y}{\sqrt{t}}\right)\frac{1}{\sqrt{t}}$$

I want to assume that $\int dx\, \varphi(x) = 1. = \hat{\varphi}(0)$.

$$L_t(x-y) = \int \frac{d\xi}{2\pi} e^{i\xi\frac{x-y}{\sqrt{t}}} \hat{\varphi}(\xi)\frac{1}{\sqrt{t}}$$

$$= \int \frac{d\xi}{2\pi} e^{i\xi(x-y)} \hat{\varphi}(\sqrt{t}\,\xi)$$

$$\left(L_{\frac{t}{n}}\right)^{*n}(x-y) = \int \frac{d\xi}{2\pi} e^{i\xi(x-y)} \hat{\varphi}\left(\sqrt{\tfrac{t}{n}}\,\xi\right)^n$$

So what is happening to $\hat{\varphi}\left(\sqrt{\frac{t}{n}}\,\xi\right)^n$ as $n \to \infty$. First look pointwise. For any value of ξ we have that $\frac{1}{\sqrt{n}}\xi \longrightarrow 0$ and since $\hat{\varphi}$ is smooth at 0 we have

$$\hat{\varphi}(\xi) = 1 + a\xi + \tfrac{1}{2}b\xi^2 + O(\xi^3)$$

$$\hat{\varphi}\left(\frac{1}{\sqrt{n}}\xi\right) = 1 + \frac{a\xi}{\sqrt{n}} + \frac{1}{2}\frac{b}{n}\xi^2 + O\left(\frac{1}{n^{3/2}}\right).$$

Assume $a = 0$. Then

$$\hat{\varphi}\left(\frac{\sqrt{t}}{\sqrt{n}}\xi\right)^n = \left(1 + \frac{1}{2}\frac{bt}{n}\xi^2 + O\left(\frac{1}{n^{3/2}}\right)\right)^n$$

$$= e^{\frac{1}{2}bt\xi^2 + O\left(\frac{1}{n^{1/2}}\right)}$$

Familiar example: $\varphi(x) = \frac{1}{2}\left(\delta(x-a) + \delta(x+a)\right)$. This is not in C_0^∞, but we know it gives the random walk on the line. Then

$$\hat{\varphi}(\xi) = \int dx\, e^{-i\xi x}\varphi(x) = \frac{1}{2}\left(e^{-ia\xi} + e^{ia\xi}\right) = \cos(a\xi)$$

$$\sim 1 - \frac{a^2\xi^2}{2} + \cdots$$

and

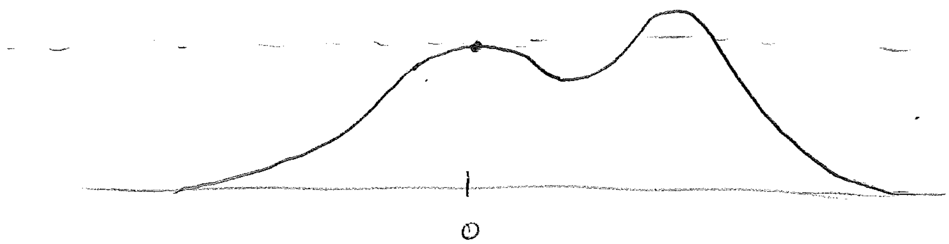
$$\hat{\varphi}\left(\sqrt{\frac{t}{n}}\xi\right)^n = \cos\left(\sqrt{\frac{t}{n}}a\xi\right)^n = \left(1 - \frac{a^2t}{2n}\xi^2 + \cdot\right)^n$$

$$= e^{-t\frac{a^2}{2}\xi^2 + O\left(\frac{1}{n}\right)}$$

In this case we see that the convergence is far from uniform, since the cosine takes the value 1 infinitely often.

Now suppose one started with a $\varphi \in C_0^\infty(\mathbb{R})$, or maybe, more generally, $\varphi \in \mathcal{S}(\mathbb{R})$. Then one can ask, assuming $\varphi(0) = 1$, $\varphi'(0) = 0$, whether $\varphi\left(\frac{\xi}{\sqrt{n}}\right)^n \longrightarrow e^{\frac{1}{2}\varphi''(0)\xi^2}$ as $n \longrightarrow \infty$ in $\mathcal{S}$. Clearly we need also $\varphi''(0) < 0$. We can obviously construct $\hat{\varphi}$

with graphs like



for which the convergence isn't uniform. So it looks necessary to have additional information on φ.

May 9, 1984

What I learned from Melrose yesterday:

$\left(\frac{|x|^2}{t}\right)^k e^{-|x|^2/t}$ is bounded $(t>0)$ because

$y^k e^{-y}$ is bounded for $y \geq 0$ (max occurs at $y=k$).

Hence $\boxed{|x|^\ell e^{-|x|^2/t} \leq \text{Const} \cdot t^{\ell/2}}$

Hadamard's way of constructing a fundamental solution to the wave equation becomes even simpler for the heat equation. (For the heat equ. the bicharacteristics are tangent to the real cotangent space at zero but head into the complex domain)

We want a solution of

$$(\partial_t - \Delta) E = \delta \qquad \delta = \delta(t)\,\delta(x)$$

where E is supported in $t \geq 0$. Use normal coords. at $x=0$ and let Δ_0 be the flat Laplacean. Then $\Delta - \Delta_0$ will have a Taylor exp. as a sum of terms $x^\alpha \partial^\beta$ whose coefficients involve the curvature and its derivatives. Now set up the Neuman series for

$$(\partial_t - \Delta_0) E = \delta + (\Delta - \Delta_0) E$$

$$E = E_0 + (\partial_t - \Delta_0)^{-1} (\Delta - \Delta_0) E$$

$$= E_0 + E_1 + E_2 + \cdots$$

where $E_n = (\partial_t - \Delta_0)^{-1} (\Delta - \Delta_0) E_{n-1}$.

Now the basic idea is that the E_n are each of the form of a series of terms $t^k x^\alpha E_0$

so it seems one is grinding out the asymptotic expansion for the heat kernel. Notice what one needs to make this work is a suitable filtration (or weighting) of the terms $t^k x^\alpha E_0$ together with an understanding of

$$(\partial_t - \Delta_0)^{-1}\, t^k x^\alpha E_0 \,.$$

Example: Consider the physics way of putting Planck's constant in on quantizing

$$H = \frac{1}{2}(p_\mu - A_\mu)^2$$

One gets the operator

(1) $$H = \frac{1}{2}\left(\frac{\hbar}{i}\partial_\mu - A_\mu\right)^2$$

which is quite different from

(2) $$-\frac{1}{2}\hbar^2 D_\mu^2 \qquad D_\mu = \partial_\mu - iA_\mu$$

which I encounter in the square of the Dirac operator.

One can check that (1) is correct and not (2) as follows. We know that when F is constant

$$\langle x | e^{-\frac{i}{\hbar}Ht} | x' \rangle = \int \mathcal{D}q(t)\, e^{\frac{i}{\hbar}\int(\frac{1}{2}\dot{q}^2 + A\dot{q})\,dt}$$

$$= e^{\frac{i}{\hbar}S(t,x;x')} \cdot \text{det factor}$$

On the other hand

$$\langle x | e^{-\frac{i}{\hbar}(-\frac{1}{2}\hbar^2 D_\mu^2)t} | x' \rangle = \langle x | e^{\frac{i}{2}D_\mu^2(\hbar t)} | x' \rangle = e^{i S(\hbar t, x, x')} \cdot \text{det factor}$$

so it's clear.

What is fascinating about this example is that changing $\partial_\mu - iA_\mu$ to $\partial_\mu - \frac{i}{h}A_\mu$ destroys the integrality conditions that must hold for a connection on a line bundle. In other words physically Planck's constant is not a free parameter when there is topology.

Today I had the idea of trying to construct the heat kernel at least formally by using that we know what it is for $h=0$, and then correcting to get the higher powers of h. The point is that to use the expression $e^{-\frac{(x-y)^2}{4t}}$ **is awkward** from the viewpoint of composition. So we really should be starting with the $h=0$ algebra and writing e^{-tp^2}

Problem: Construct the heat operator e^{-tH} where $H = p^2 + V(q)$ and where the construction is to take place in the algebra of operators depending on a Planck's constant.

Idea: We know what to do when $h=0$, so why not first proceed formally, treating h as an adjoined nilpotent element.

At $h=0$ we have the algebra of smooth functions on the cotangent bundle rapidly decreasing at ∞. The next question is to treat the first order deformation of this algebra.

Possibility: Let us work on a torus, e.g. the circle. Then we can form the following algebra. Start with the smooth functions on the torus M and form a crossed product with the Schwartz space of functions on the translation group of the torus under convolution. So what I get is $\mathcal{S}(\mathbb{R}^n)$ which I can think of a functions $f(p)$ like e^{-tp^2} mixed with smooth fns. of the q's which are periodic. The algebra structure then comes from the standard commutation relations

$$[p,q] = h/i$$

and hence I should add in functions of h.

So what I see from this example is that I should be able to start with the functions on the cotangent bundle of rapid decrease, and construct a deformation of it to all orders in Planck's constant.

Now here is the basic deformation. Suppose we divide out by h^2. Then we will get

$$0 \longrightarrow hA/h^2A \longrightarrow A/h^2A \longrightarrow A/hA \longrightarrow 0$$

where A/hA is the commutative algebra of functions of p, q. Let $H = p^2 + V$ in A/h^2A. We are assuming $e^{-t\bar{H}}$ exists in $\bar{A} = A/hA$. We need to have a path $\varphi(t)$ in A/h^2A which lifts $e^{-t\bar{H}}$ and has $\varphi(0) = I$, $\varphi'(0) = -H$. Then we look for a modification of φ:

$$\varphi(t) + \bar{\eta}(t)h \qquad \bar{\eta}(t) \in \bar{A}$$

which satisfies $\bar{\eta}(0) = \bar{\eta}'(0) = 0$ and

$$(\partial_t + H)(\varphi(t) + \bar{\eta}(t)h) = 0.$$

Now as $\overline{\varphi(t)} = e^{-t\bar{H}}$ it follows that

$$(\partial_t + H)\varphi(t) = \bar{f}(t)h \qquad \bar{f}(t) \in \bar{A}$$

and so we have to solve

$$\bar{f} + (\partial_t + \bar{H})\bar{\eta} = 0$$

The solution is obviously

$$\bar{\eta} = -\int_0^t dt' \, e^{-(t-t')\bar{H}} \bar{f}(t')$$

Also
$$\bar{\eta}(0) = \int_0^0 \cdots = 0$$
$$\bar{\eta}'(0) = -\bar{f}(0) = 0$$

Let's do this in detail for $H = p^2 + V(q)$, to see what we are getting. What is the algebra A/h^2A?

We identify A/hA with functions in p's q's.

May 11, 1984

I want to go over what Melrose told me about Hadamard's method for constructing the heat kernel, say for the Laplacean H on a curved manifold. The heat kernel $\langle x|e^{-tH}|y\rangle$ is ~~equivalent to a~~ just a forward fundamental solution $E(t,x,y)$ for $\partial_t + H$:

$$(\partial_t + H)E(t,x,y) = \delta(t)\,\delta(x-y)$$
$$E(t,x,y) = 0 \qquad \text{for} \quad t<0.$$

By the Volterra stuff it suffices to construct an E satisfying

$$(\partial_t + H)E(t,x,y) = \delta(t)\,\delta(x-y) + \text{smooth}$$
$$E(t,x,y) = 0 \qquad \text{for} \quad t<0$$

i.e. a so-called forward parametrix for $\partial_t + H$. One does this for each y, treating y as a parameter, so one might as well suppose $y=0$. Then one lets H_0 be the negative (const. coeff) Laplacean on the tangent space at 0, where we use the exponential map to identify the manifold and the tangent space.

Now the construction proceeds as follows. The equation to be solved is

$$(\partial_t + H)u = f \qquad\qquad f = \delta(t)\delta_0 \qquad \delta_0(x)=\delta(x)$$
$$u = 0 \qquad t<0$$

and we will treat H as a deformation of H_0. This means we write the equation

$$(\partial_t + H_0)u = f + (H_0 - H)u.$$

Let E_0 be the known forward fundamental solution for $\partial_t + H_0$. Then as $u=0$ for $t<0$ we must have

$$u = E_0 f + E_0 (H_0 - H) u \qquad W = \ldots$$

which has the formal solution

$$u = E_0 f + E_0 W E_0 f + E_0 W E_0 W E_0 f + \cdots$$

The next step will be to analyze the terms of this equation to find the singularity at $(t,x) = (0,0)$. To simplify the discussion, let's suppose

$$H_0 = -\frac{1}{4}\partial_\mu^2 \quad \text{so} \quad E_0(t,x,y) = \frac{e^{-\frac{(x-y)^2}{t}}}{(\pi t)^{n/2}}$$

$$H = H_0 - W \quad \text{where} \quad W = W(x).$$

Then

$$(E_0 f)(t,x) = E_0(t,x,0) = \frac{e^{-\frac{x^2}{t}}}{(\pi t)^{n/2}}$$

Digression on the Volterra business: We suppose found a forward parametrix for $\partial_t + H$: This is an operator valued function $E(t)$ such that

$$E(t) = 0 \qquad \text{for} \quad t < 0$$

$$(\partial_t + H) E(t) = \delta(t) - K(t)$$

where $K(t)$ has some properties to be determined. What we want to do is to solve the equation

$$(\partial_t + H)\tilde{E}(t) = \delta(t)$$

$$\tilde{E}(t) = 0 \qquad t < 0.$$

What I have to do is to think of a function like $K(t)$ $t \geq 0$ as being a convolution operator on $f(t)$, $t \geq 0$: $\quad (Kf)(t) = \int_0^t dt'\, K(t-t') f(t')$.

Then if $P = \partial_t + H$ on such $f(t)$, we have

$$P\tilde{E} = 1 - K$$

as convolution operators, hence if $(1-K)^{-1}$ exists

$$P\tilde{E}(1-K)^{-1} = (1-K)(1-K)^{-1} = 1.$$

Moreover the essence of Volterra equations is that the Neumann series converges

$$(1-K)^{-1} = 1 + K + K^2 + K^3 + \cdots .$$

Now I would like to understand why this works.

First go over the classical argument. Suppose $|K(t)| \leq M$, on the interval $0 \leq t \leq a$. Then

$$(K * K)(t,t') = \int_{t'}^{t} dt_1\, K(t,t_1)\, K(t_1,t')$$

$$|K^2(t,t')| \leq M^2 \int_{t'}^{t} dt_1 = M^2(t-t')$$

$$|K^3(t,t')| \leq \int_{t'}^{t} dt_1\, |K(t,t_1)|\,|K^2(t_1,t')|$$

$$\leq \int_{t'}^{t} dt_1\, M \cdot M^2(t-t') = M^3 \frac{(t-t')^2}{2!}$$

So

$$\Big|\Big(\sum_{n \geq 1} K^n\Big)(t,t')\Big| \leq M + M^2(t-t') + M^3\frac{(t-t')^2}{2!} + \cdots$$

$$= M e^{M(t-t')}$$

Now we are interested in the case of a convolution kernel $K(t-t')$ and so we see that the Neumann series converges provided

$$|K(t)|$$

is bounded on any interval $0 \leq t \leq a$.

But what happens if I treat the convolution algebra via the Laplace transform? Then given a function of t such as $K(t)$ for $t \geq 0$ we define

$$\hat{K}(\lambda) = \int_0^\infty e^{-\lambda t} K(t)\, dt .$$

Assuming this converges for $Re(\lambda)$ large we have the formula

$$\widehat{Kf}(\lambda) = \hat{K}(\lambda)\hat{f}(\lambda) .$$

Now why should $1 + \hat{K} + \hat{K}^2 + \cdots\cdots$ converge? The idea is that all we need is $|\hat{K}(\lambda)| < 1$ for $Re(\lambda)$ large enough.

Let's go back to

$$(\partial_t + H)\, E(t) = \delta(t) - K(t)$$

and choose a smooth fn $\rho(t)$ with $\rho(t) = 1$ $t \leq 0 + \varepsilon$ and $\rho(t) = 0$ for $t \geq 1$. Then

$$\begin{aligned}(\partial_t + H)(\rho E) &= (\partial_t \rho) E + \rho(\partial_t + H) E \\ &= (\partial_t \rho) E + \rho\,(\delta(t) - K(t)) \\ &= \delta(t) + (\partial_t \rho) E - \rho K\end{aligned}$$

and the new K vanishes for $t \geq 1$. Thus there is for this K no problem with the Laplace transform, and what we see is important is the behavior of what $K(t)$ as $t \downarrow 0$.

Basically what I need is that $K(t)\, dt$ is absolutely continuous with respect to dt. More accurately you need to have $\hat{K}(\lambda)$ topologically nilpotent, actually tending to zero as $\lambda \longrightarrow \infty$. The important thing is to avoid a jump at $t=0$. In fact if we can integrate by parts

$$\int_0^\infty e^{-\lambda t} K(t)\, dt = \frac{K(0)}{\lambda} + \int_0^\infty \frac{e^{-\lambda t}}{\lambda} K'(t)\, dt$$

then one should win.

Let's try to summarize this digression. I am trying to construct e^{-tH} for $t \geq 0$ or equivalently the forward fundamental solution for $\partial_t + H$. In other words we are trying to invert the operator $\partial_t + H$ acting on $f(t)$, $t \geq 0$. Now I propose to find a forward parametrix

$$(\partial_t + H) E(t) = \delta(t) - K(t)$$

where $1 - K$ can be inverted because of Volterra's arguments. It is sufficient for $K(t)$ to be bdd for $0 \leq t \leq \varepsilon$.

Now if we pass via the Laplace transform, then we are trying to show $\lambda + H$ is invertible on the space of $\hat{f}(\lambda)$:

$$\int_0^\infty e^{-\lambda t} \partial_t f\, dt = \left[e^{-\lambda t} f\right]_0^\infty + \lambda \int_0^\infty e^{-\lambda t} f(t)\, dt$$
$$= \lambda \hat{f}(\lambda) \qquad \text{if } f(0) = 0.$$

So I am trying to produce $(\lambda+H)^{-1}$ knowing that there is an $\hat{E}(\lambda)$ with

$$(\lambda+H)\hat{E}(\lambda) = 1-\hat{K}(\lambda)$$

Something strange is happening. It seems that what this Volterra argument amounts to is an argument that if $E(t)$ is a any curve in operators starting at the identity with the correct behavior at $t=0$ in the sense that

$$(\partial_t+H)E(t) = \delta(t)-K(t)$$

where $K(t)$ as $t\downarrow 0$ is sufficiently nice as to permit the Volterra theory to work, then applying this iterative process leads from $E(t)$ to e^{-tH}.

This is exactly the same sort of thing I wanted to do discretely, namely, given $\varphi(t)$ starting with $\varphi(0)=I$ and tangent vector $-H$, then

$$\lim \varphi\left(\tfrac{t}{n}\right)^n = e^{-tH}.$$

Let's examine the situation in the finite dimensional matrix case and assume $\varphi(t)$ is a smooth function of t with formal power series at $t=0$:

$$\varphi(t) = I - Ht + \tfrac{a}{2}t^2 + \cdots\cdots$$

Then

$$(\partial_t+H)\varphi = \begin{array}{l} -H + at + \cdots\cdots \\ +H - H^2t + \cdots\cdots \end{array} = (a-H^2)t + \cdots$$

and

$$\begin{aligned}(\partial_t+H)\,\theta(t)\varphi(t) &= \delta(t)\varphi(t) + \theta(t)(\partial_t+H)\varphi(t) \\ &= \delta(t) + \theta(t)\left[(a-H^2)t+\cdots\right].\end{aligned}$$

Hence the $K(t)$ in this case will be $\theta(t)t\cdot$ smooth fn. This seems to be more restrictive than requiring

$K(t)$ to be bounded at $t \to 0$. In fact take $E(t) = \theta(t)$, then

$$(\partial_t + H)\, E(t) = \delta(t) + \theta(t) H$$

so $K(t) = -\theta(t) H$ and iteration produces the exponential series for e^{-tH}.

This example shows that the "Volterra method" is fundamentally more powerful in the sense that we can start with much cruder paths $E(t)$.

So for example take $E(t) = \theta(t)\, e^{-H_0 t}$. Then

$$(\partial_t + H)\, \theta(t) e^{-H_0 t} = \underbrace{\delta(t) e^{-H_0 t}}_{\delta(t)} + \underbrace{\theta(t)(-H_0 + H)\, e^{-H_0 t}}_{-K(t)}$$

so if $-H_0 + H$ is bounded and $e^{-H_0 t}$ is a contraction operator, then

$$\| K(t) \| \leq \| -H_0 + H \| \qquad \text{for all } t$$

and so the Volterra method should work.

If we take the Laplace transform

$$\hat{K}(\lambda) = \int_0^\infty e^{-\lambda t} (-H_0 + H)\, e^{-H_0 t}\, dt = (-H_0 + H) \frac{1}{\lambda + H_0}$$

then the Volterra iteration is the geometric expansion

$$\frac{1}{\lambda + H} = \frac{1}{\lambda + H_0} + \frac{1}{\lambda + H_0} (-H_0 + H) \frac{1}{\lambda + H_0} + \cdots\cdots .$$

Hence I want to know that in some way as $\lambda \to \infty$

$$\| \hat{K}(\lambda) \| \to 0 .$$