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The problem is [redacted] to compute the character of the Thom class $i_! 1$ in K-theory of a vector bundle N over M equipped with metric, connection, and Spin^c structure. (More generally one might want the character of the virtual bundle on N corresponding to a $\mathrm{Cliff}(N)$-module.)

Let's first consider the case of a $U(m)$-bundle E/M and take the Spin^c structure represented by the $\mathrm{Cliff}(E)$ module ΛE. In this case I want to do the Clifford algebra calculations using the creation and annihilation operators a_i^*, a_i instead of the Dirac operators γ^μ.

Let me first compute the character of the Bott element in this approach. [redacted] In this $M = pt$ and $E = \mathbb{C}^m$ and $\mathrm{Cliff}(E) = C_{2m}$ has the generators a_i^*, a_i with usual (anti-) commutation relations. Take $m=1$ and recall that

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = a^* + a$$

$$\gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i a^* - i a$$

so that over $\mathbb{C} \ni z = x + iy$ we have the Clifford mult.

$$x\gamma^1 + y\gamma^2 = \begin{pmatrix} 0 & \bar{z} \\ z & 0 \end{pmatrix} = z a^* + \bar{z} a$$

on the spinors. In general for any m we have that the Clifford multiplication on $S = \Lambda \mathbb{C}^m$ at $z \in \mathbb{C}^m$ will be

$$z a^* + \bar{z} a = z^\mu a_\mu^* + \bar{z}^\mu a_\mu$$

With the flat connection d on $\pi^*(S)$ over $\mathbb{C}^m$ we get the superconnection $d + i(za^* + \bar{z}a)$ which has the curvature

$$\left(d + i(za^* + \bar{z}a)\right)^2 = i(dz\, a^* + d\bar{z}\, a) - |z|^2 .$$

$[idz a^*, id\bar{z} a]$

The Chern character is then

$$tr_s\left(e^{-|z|^2 + idz\, a^* + id\bar{z}\, a}\right) = e^{-|z|^2}\, tr_s\left(e^{idz a^*} e^{id\bar{z} a} e^{-\frac{1}{2}[\ ,\ ]}\right)$$

Now $-[idz\, a^*, id\bar{z}\, a] = -dz\,[a^*, a]\,d\bar{z} = -dz\, d\bar{z}$, so this becomes

$$e^{-|z|^2 - \frac{1}{2} dz\, d\bar{z}}\; tr_s\left(e^{idz\, a^*}\, e^{id\bar{z}\, a}\right).$$

[redacted]

Next I need to know what tr_s does to monomials in the a_i^*, a. If $m = 1$ we have the basis $1, a^*, a, aa^*$

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} \quad \begin{pmatrix}0&0\\1&0\end{pmatrix} \quad \begin{pmatrix}0&1\\0&0\end{pmatrix} \qquad \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

and hence tr_s kills $1, a^*, a$ and $tr_s(aa^*) = 1$. It follows by tensor product considerations that

$$\begin{cases} tr_s\left(a_{i_1} \cdots a_{i_p}\, a^*_{j_1} \cdots a^*_{j_q}\right) = 0 & \text{if } p+q < 2m \\ tr_s\left(a_1 \cdots a_m\, a_m^* \cdots a_1^*\right) = tr_s\left(a_1 a_1^*\, a_2 a_2^* \cdots a_m a_m^*\right) = 1 \end{cases}$$

So the above supertrace becomes

$$e^{-|z|^2 - \frac{1}{2} dz\, d\bar{z}}\; tr_s\left(\prod (1 + i dz^\mu a_\mu^*) \prod (1 + i d\bar{z}^\mu a_\mu)\right)$$

$$= \underline{\qquad\qquad}\; i^{2m}\, tr_s\left(dz^1 a_1^* \cdots dz^m a_m^*\, d\bar{z}^m a_m \cdots d\bar{z}^1 a_1\right)$$

$$= e^{-|z|^2 - \frac{1}{2} dz\, d\bar{z}} (-1)^m\, dz^1 \cdots dz^m\, d\bar{z}^m \cdots d\bar{z}^1\; tr_s\left(a_1 \cdots a_m\, a_m^* \cdots a_1^*\right).$$

Now $dz\,d\bar z = -2i\,dx\,dy = -2i\,d^2 z$, so that we have finally

$$tr_s\left(e^{-|z|^2 + i\,dz\,a^* + i\,d\bar z\,a}\right) = e^{-|z|^2}(-1)^m(-2i)^m\,d^{2m}z$$

and applying $\int\left(\frac{i}{2\pi}\right)^m$ then gives $(-1)^m$ as it should.

Next I want to consider the general case. This means E is a complex m-diml bdle with inner product and connection. Let $i: M \to E$, $\pi: E \to M$ be the obvious maps, and consider the bundle $\pi^*(\Lambda E)$ over E equipped with Clifford multiplication, i.e. the odd degree endom which at $\xi \in E$ is the operator $e_\xi + \iota_{\xi^*}$ on $\pi^*(\Lambda E)_\xi = (\Lambda E)_{\pi\xi}$. The connection we have on E induces one on ΛE, which in turn gives us a connection on $\pi^*(\Lambda E)$. We want to calculate the character of $\pi^*(\Lambda E)$ equipped with this connection + odd degree endom.

(Point out that $\pi^*(\Lambda E)$ with the odd endo given by Clifford multiplication represents the Thom class $i_! 1 \in K(E)$. So we are computing $ch(i_! 1)$ as a diffl. form on E.)

Let's do the calculation locally where we can choose a local trivialization $E \simeq M \times \mathbb{C}^m$. Then the connection on E can be written $D = d + \theta$ where θ is an skew-hermitian matrix of 1-forms on M. Let's write

$$\theta = dx^\mu\,\theta_{\mu,ij} \qquad x^\mu \text{ coords on } M.$$

where $\theta_{\mu,ij}$ is skew hermitian in i,j. Then I have to [redacted] extend D on E to ΛE. The extension is a derivation and is going to be

$$d_M + \theta a^* a = d_M + dx^\mu \theta_{\mu ij} a_i^* a_j .$$

Here I use that $\alpha_{ij} a_i^* a_j$ is the degree 0 derivation of ΛE extending the endom. of E given by the matrix α_{ij}.

Next we lift this connection back to E, whence the d_M becomes d_E. Our superconnection is therefore

$$\tilde{D} = d_E + \theta a^* a + i(z a^* + \bar{z} a)$$

and its curvature is

calculation with indices $\rightarrow$

$$\underbrace{\left(d_E + \theta a^* a\right)^2}_{(d\theta + \theta^2)\, a^* a} + i \underbrace{\left[d_E + \theta a^* a, z a^* + \bar{z} a\right]}_{(dz + \theta z) a^* + (d\bar{z} - \theta \bar{z}) a} - |z|^2$$

Here $dz + \theta z$, $d\bar{z} - \theta\bar{z}$ can be explained with indices if [redacted] we want but we [redacted] might as well think in terms of a line bundle at this stage.

So our [redacted] character is

$$e^{-|z|^2} \operatorname{tr}_s \left(e^{i(dz + \theta z) a^* + i(d\bar{z} - \theta\bar{z}) a + (d\theta + \theta^2) a^* a} \right)$$

which reminds me of a forced harmonic oscillator.

Lets begin by evaluating this in the [redacted] case $m = 1$, where we are dealing with 2×2 matrices. Except that we have to remember the anti-commutation of the 1-forms with a, a^*.

Let's start with the ordinary (boson) harmonic oscillator with a forcing term:

$$H = \omega a^* a + a^* J + \bar{J} a$$

We know that

$$\operatorname{tr}\left(e^{-\omega a^* a}\right) = \sum_{n \geq 0} e^{-n\omega} = \frac{1}{1-e^{-\omega}}$$

Now we complete the square

$$H = (a^* + \bar{J}\omega^{-1})\,\omega\,(a + \omega^{-1}J) - \bar{J}\omega^{-1}J$$

[redacted] and we get

$$\operatorname{tr}\left(e^{-H}\right) = \operatorname{tr}\left(e^{-(a^* + \bar{J}\omega^{-1})\omega(a+\omega^{-1}J)}\right) e^{\bar{J}\omega^{-1}J}$$

But one has the same commutation relation $[a+\omega^{-1}J, a^* + \bar{J}\omega^{-1}]$ so that this becomes

$$\operatorname{tr}\left(e^{-\omega a^* a - a^* J - \bar{J} a}\right) = \frac{1}{1-e^{-\omega}}\, e^{\bar{J}\omega^{-1}J}$$

Now I want to do the same thing in the fermion situation to evaluate

$$\operatorname{tr}_s\left(e^{\omega a^* a + i\eta a^* + i\bar{\eta} a}\right)$$

where we are working in a tensor product

$$\Omega \,\hat{\otimes}\, C_2$$

where is a commutative superalgebra, η $\bar{\eta}$ are of odd degree, ω is of even degree in Ω. In the application $\Omega = \Omega(E)$. Now I am going to argue that I can assume ω is invertible. [redacted]

We can replace Ω by a free commutative superalg and then ω is [redacted] a non-zero divisor, then we can localize to render it invertible without loss of information. (In the example I could have taken M to be $BU(n)$, and the forms on M to be the basic part of the Weil algebra.)

Complete the square

$$\omega a^* a - i\,\text{[redacted]}\,a^*\eta + i\bar\eta a$$
$$= (a^* + i\bar\eta\omega^{-1})\,\omega\,(a - i\omega^{-1}\eta) - \bar\eta\omega^{-1}\eta$$

Now $[a^* + i\bar\eta\omega^{-1}, a - i\omega^{-1}\eta] = [a^*, a] = 1$, so we should be able to conclude that

$$tr_s\left(e^{\omega a^* a + i\eta a^* + i\bar\eta a}\right) = e^{-\bar\eta\omega^{-1}\eta}\, tr_s\left(e^{(a^* + i\bar\eta\omega^{-1})\omega(a - i\omega^{-1}\eta)}\right)$$
$$= e^{-\bar\eta\omega^{-1}\eta}\, tr_s\left(e^{\omega a^* a}\right) = e^{-\bar\eta\omega^{-1}\eta}\left(1\,\text{[redacted]}\,e^{\omega}\right)$$
$$= (1 - \bar\eta\omega^{-1}\eta)(1 - e^{\omega})$$
$$= (\omega - \bar\eta\eta)\left(\frac{1-e^{\omega}}{\omega}\right) = (\omega + \eta\bar\eta)\left(\frac{1-e^{\omega}}{\omega}\right)$$

Recall yesterday that I wanted the formula

$$ch\,(i_! 1) = \underbrace{i_* 1}_{\text{homogeneous form}} \cdot \pi^*(\text{Todd } i)$$

Thus we seem to obtain this formula for a line bundle.

March 1, 1984

On the Berezin determinant: One place this arises is when one wants to define integration on a supermanifold.

Lets start with an open set of the supermanifold $\mathbb{R}^{p,q}$. This means we have an open set U of $\mathbb{R}^p$ equipped with the superalgebra

$$A(U) = C^\infty(U) \otimes \Lambda[y_1, \ldots, y_q].$$

If we restrict to the compactly supported elements, then we have a natural linear functional on this space $C_0^\infty(U) \otimes \Lambda[y]$ namely

$$(+) \quad \int_U : \quad \sum f_I(x)\, y^I \longmapsto \int d^n x\, f_{1,\ldots,n}(x).$$

Let's define a density to be a linear functional on $C_0^\infty(U) \otimes \Lambda[y]$ which is given by a distribution which is smooth, i.e.

$$\sum_I f_I(x)\, y^I \longmapsto \int d^n x \sum_I g_I(x) f_I(x)$$

where the $g_I \in C^\infty(U)$. The set of densities is clearly a free module over $A(U)$ with the generator $(+)$. This uses the fact that $\Lambda[y]$ is a Gorenstein ring, i.e. the dual is isomorphic as a module to the ring.

Now given a super diffeomorphism φ between open subsets U, V of $\mathbb{R}^{p,q}$ we can transport $\int_V$ via φ to get a density on U which we can then write as multiplication by an element J_φ of $A(U)$ followed by $\int_U$. Then J_φ is ~~the~~ Jacobian associated to

φ.

Associated to $A(U)$ is a super-module of differentials which is free with basis dx_i, dy_j. Presumably the Jacobian J_φ can be computed somehow from the effect of φ on differentials. The point is that the isomorphism identifies the U and V situations, but that over U and V we have different bases. Thus we have an invertible matrix, or better, an isomorphism of a free super module over the superalgebra $A(U)$.

So the Berezin determinant should be defined on the even degree autos. of a free supermodule over a commutative superalgebra. It should be an exponentiated version of the supertrace. In fact one can even make sense out of it for non commutative superalgebras A, the idea being that

$$tr_s : \mathrm{End}_A(E) \longrightarrow A/[A,A]$$

is a map of Lie superalgebras and hence should correspond to a morphism of algebraic supergroups. I suppose here that A is finite-diml over the ground field.

We really need to compute examples of the superdeterminant especially of the form $\langle 0|S|0 \rangle$ and also for free modules over Clifford algebras.

March 3, 1984

Review the correspondence approach to index problems in the algebraic geometry context or RR for complex manifolds.

Let E be a holom. vector bundle over a compact holom. manifold M. By Serre duality

$$H^\cdot(M,E)^* = H^\cdot(M,E') \qquad E' = E^* \otimes \omega$$

and so using Kunneth we have

$$\begin{aligned}
\mathrm{Hom}\left(H^\cdot(M,E), H^\cdot(M,E)\right) &= H^\cdot(M,E)\otimes H^\cdot(M,E') \\
(\star) \qquad &= H^\cdot(M\times M, E \boxtimes E') \\
&= H^\cdot(M\times M, \underline{\mathrm{Hom}}(p_2^*E, p_1^*E)\otimes p_2^*\omega)
\end{aligned}$$

Thus endos of $H^\cdot(M,E)$ are represented by cohomology classes of a holom. bundle on $M\times M$.

Now the trace of an endo on $H^\bullet(M,E)$ is given, relative to the above isomorphism, by restricting the class on $M\times M$ to the diagonal and then taking the "integral over M" or "trace" map:

$$H^\cdot(M, \mathrm{End}\, E\otimes\omega) \xrightarrow{tr} \mathbb{C} .$$

Next in order to compute the index we need to take the identity map on $H^\cdot(M,E)$ realize it as a cohomology class on $M\times M$ and then go through this trace process.

We claim there is a canonical class

$$U \in H^n(M\times M, \underline{\mathrm{Hom}}(pr_2^*E, pr_1^*E)\otimes pr_2^*\omega))$$

which, when interpreted as a correspondence, gives the identity map on $H^*(M,E)$. This assertion is clear from the chain of isomorphisms (A). What is less clear is that the map

$$H^n(M\times M, \underline{\mathrm{Hom}}(pr_2^*E, pr_1^*E)\otimes pr_2^*\omega) \longrightarrow H^n(M, \mathrm{End}\,E\otimes\omega) \xrightarrow{tr} \mathbb{C}$$

applied to this canonical class gives the trace of the identity on $H^*(M,E)$, i.e. the index. But this must come out of the duality + Kenneth formalism.

So if I want the index thm. what I must do is to compute the [redacted] image of U under this restriction to Δ map. [redacted] Suppose $E = \mathcal{O}$. Then I must produce an element [redacted]

$$U \in H^n(M\times M, pr_2^*\omega)$$

and I must compute its image under

$$H^n(M\times M, pr_2^*\omega) \longrightarrow H^n(M,\omega) \xrightarrow{\int} \mathbb{C}.$$

I don't need to do the last integral since already in $H^n(M,\omega) = H^n(M,\Omega^n)$ must appear the class $Td(M)_{(n)}$.

Grothendieck's method of constructing U is to use the maps

$$\mathrm{Ext}^n_{M\times M}(\mathcal{O}_M, pr_2^*\omega) \longrightarrow H^n_{M\times M}(pr_2^*\omega)$$

$$\mathrm{Ext}^n_{M\times M}(\mathcal{O}_M, pr_2^*\omega) \xrightarrow{\sim} H^0(M\times M, \underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_M, pr_2^*\omega))$$
$$\simeq H^0(M\times M; \mathcal{O}_M) = H^0(M,\mathcal{O}_M).$$

The bottom arrow comes from the local to global Ext spectral sequence; it is an edge homomorphism which is an isomorphism as the local extes are concentrated in dim n by Koszul complexes.

So our problem comes to understanding the image of U under the following composition

$$H^0(M\times M, \underline{\mathrm{Ext}}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_M, pr_2^*\omega)) \xleftarrow{\sim} \mathrm{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_M, pr_2^*\omega)$$

$$\to H^n(M\times M, pr_2^*\omega) \xrightarrow{\Delta^*} H^n(M,\omega).$$

Now I can't use Grothendieck's localization process but I want to find analogues of the arguments. So the first step will be to think of U as a cohomology class on $M\times M$ hence as a zero mode of the appropriate Dirac operator. This is the harmonic representative and the idea will be somehow that the heat kernels will somehow give a Witten type deformation of the zero mode U to something concentrated along the diagonal.

So the first thing to write out carefully is the exact relation between the operators with smooth kernel on the Dolbeault complex on M and some Dolbeault complex on $M\times M$.

March 4, 1984

Here is the program: Given $\not{D} = h\gamma^\mu D_\mu + \varepsilon L$ I know that $\langle x|e^{-H}|y\rangle$ in the limit $\frac{x-y}{h} \to \sigma$, $h \to 0$ approaches a kernel on the tangent bundle. I want to describe this limiting kernel precisely.

My first guess was that it is the Thom class of the tangent bundle in some sense. Here I want to consider the simplest operator, namely the Dirac operator on a Spin^c manifold. The limiting kernel is an n form on the tangent bundle which has dimension $2n$. ~~the Thom class~~ If this is restricted to the zero section, one gets the n-form which is the index density, i.e. $[\hat{A}(M)]_n$.

I can ask how to obtain n-forms [redacted] on the tangent bundle with Gaussian shape. If the n form is closed its class is unique by the Thom isomorphism.

I have been trying to understand the index theorem for Dirac operators from the viewpoint provided by what Bott did for the De Rham complex. The idea is that we want to deform the idea to a smooth kernel operator then restrict this to the diagonal and integrate ~~to~~ get the index.

In the case of DR and $\bar\partial$, the smooth kernel operator is given by a smooth representative for the [redacted] cohomology class of the diagonal. For DR we take the Thom class of the diagonal embedding $\Delta: M \longrightarrow M \times M$. In the $\bar\partial$ case we take a class

$$U \in H^n(M \times M, pr_2^* \omega) \qquad \omega = \Omega^n \quad \text{canonical bdle.}$$

which can be constructed in [redacted] $\mathrm{Ext}^n_{\mathcal{O}_{M\times M}}(\mathcal{O}_M, pr_2^*\omega)$

[redacted] via the local to global spectral sequence.

What's important I think is the correspondence idea, namely, that we construct kernels on the product $M \times M$. Heat operators e^{-tH} give such kernels and they have a Gaussian shape peaking along the diagonal. For small t one would like to see these kernels [redacted] constructed by a Witten procedure of tensoring with a [redacted] Gaussian in the normal direction.

Repeat: Start with a Dirac operator $\not{D} = h\gamma^\mu D_\mu + \epsilon L$ form the heat kernel $\langle x | e^{-H} | y \rangle$ and take the limit as $h \longrightarrow 0$. Then I should obtain a kernel on the tangent bundle and the problem is to explain exactly what this kernel is. This will be the theorem to prove as it simultaneously will yield the index theorem and the Lefschetz fixpoint theorem.

It [redacted] might be enough to prove the theorem for the Dirac operator on spinors, at least if $L \neq 0$. In effect given the Dirac operator on $S \otimes E$ belonging to a connection D on E we know that E, D are given by the Grassmannian connection on a direct summand of a trivial bundle. So how [redacted] do we obtain the [redacted] heat kernel corresponding to $e \not{D} e$ where e is an orthogonal idempotent?

What sort of answer do I expect to obtain? I sort of believe that the Dirac operator on S contributes the basic [redacted] kernel, and that tensor with E, D simply multiplies by the character of E, D. Let's assume this is the case, and test it out for the De Rham operator. The operator $d+d^*$ is the Dirac operator with coefficients in the spinors. The character of the spinors times $\hat{A}$ should give the Pfaffian or Euler class.

So what seems to be the case, since we know already the answer to expect for $d+d^*$, is that the limiting kernel for the Dirac operator will be the heat kernel for the harmonic oscillator with frequency given by the curvature.

~~February~~ March 5, 1984:

Concept of Dirac operator. Let M be a Riemannian manifold, $|\xi|^2$ the associated norm on T^*. By a Dirac operator on M we mean a first order operator Q on a vector bundle E such that $\sigma(Q^2, \xi) = |\xi|^2 \mathrm{Id}_E$. Then [redacted] the symbol of Q defines a module structure on E over the Clifford algebra of T^*.

If D is a connection on E, then we get a first order operator $\not{D}$

$$\Gamma(E) \xrightarrow{D} \Gamma(T^* \otimes E) \xrightarrow[\text{mult.}]{\text{Cliff}} \Gamma(E)$$

which has the symbol given by Clifford mult, i.e. the same symbol as Q. I claim that there is a unique connection on E such that Q^2 and $\not{D}^2$ agree modulo zeroth order operators.

Uniqueness: Introduce local coordinates x^μ, let $\gamma^\mu = \sigma(Q, dx^\mu)$ be the corresponding operators on E. Then $\not{D} = \gamma^\mu D_\mu$ where $D = dx^\mu D_\mu$. Let's consider adding a 0-th order operator B on E to $\not{D}$, and [redacted] let's see how the square changes:

$$(\not{D} + B)^2 = \not{D}^2 + \not{D}B + B\not{D} + B^2$$

This will agree with $\not{D}^2$ modulo zeroth order provided

$$\begin{aligned} \not{D}B + B\not{D} &= \gamma^\mu D_\mu B + B\gamma^\mu D_\mu \\ &= \gamma^\mu [D_\mu, B] + (\gamma^\mu B + B\gamma^\mu) D_\mu \end{aligned}$$

is zeroth order. Now $[D_\mu, B]$ is zeroth order as $D_\mu = \partial_\mu + A_\mu$, hence we must have $\gamma^\mu B + B\gamma^\mu = 0$.

So we reach the following problem. Given a Clifford module classify or describe all endomorphisms B anticommuting with Clifford multiplication. I suppose I am in even dimensions so that at each point the Clifford algebra is simple. Fix a point of M and let S be the spinors, whence $E = S \otimes F$ where F is a vector space. In even dimensions there is in the Clifford algebra the element

$$\varepsilon = i^{-n/2} \gamma^1 \cdots \gamma^n$$

which anti commutes with all the γ^μ. (At this point I can assume the metric is $g^{\mu\nu} = \delta^{\mu\nu}$) Then εB will commute with the γ^μ:

$$\gamma^\mu \varepsilon B = -\varepsilon \gamma^\mu B = \varepsilon B \gamma^\mu$$

and so we know that $\varepsilon B = id_S \otimes T$ where T is an endomorphism of F.

I wanted to show that I couldn't have $B = \gamma^\mu A_\mu$, but since the A_μ are completely arbitrary endos. of E, it's clear this is nonsense.

Suppose F is one-dimensional, say $E \simeq S$, then we see the only possibility for B is ε. This can be written $\gamma^\mu A_\mu$ in many ways.

The **point** I am missing is that I want the connection D **to** be compatible with Clifford multiplication. So in the flat case this means that $D_\mu = \partial_\mu + A_\mu$ where A_μ are endos of E as a Clifford module.

Conversation with Graeme:

characters of representations of loop groups, characters of the discrete series repns. + Atiyah Bott fixpoint formula. Lie superalgebra formed from vector fields and $-\frac{1}{2}$ densities on the circle. Schwarzian derivative and action of diffeomorphism on Hill's operators, Miura transformation.

The Kac character formula for repns. of loop groups is a generalization of the Weyl charater formula. The latter follows by applying the AB fixpt formula to the line bundle over G/B giving the repn. by the Borel-Weil thm. In the character formula for the loop group one extends the translation circle action to the inside of the circle. Something similar happens with the discrete series repns. of $SL_2(\mathbb{R})$. These are realized by holomorphic functions in the disk, really, sections of homogeneous line bundles. Some elements of $SL_2(\mathbb{C})$ act, like those which shrink the disk. One can compute the fixpt. contribution and see that the AB formula works.

Fundamental representation of $\mathcal{G} = (S^1)^{S^1} = Map(S^1, U(1))$ can be constructed in 2 ways. Heisenberg structure on the Lie algebra, fermion Fock space associated to the action of $\mathcal{G}$ on $L^2(S^1)$. One knows these are isomorphic canonically hence for the action of diffeomorphisms. Graeme says that the vertex operators, which form the tool to map Fock space into the Heisenberg type representation, can be defined intrinsically by specifying what one means by a sequence of elements of $\mathcal{G}$ converging to a "blip."

March 6, 1984

I have to learn how to do Riemannian geometry calculations. I want to be able to handle families of Dirac operators. This means that the metric will be varying, and I want tools to handle a varying metric. So if I use the orthonormal frame bundle, this itself will be changing with the parameters.

In order to get a feeling for what is needed I should think in terms of the index thm. for families. So let us suppose that we have a fibre bundle $f: X \to Y$ with compact Riemannian fibres. In fact suppose the total space is equipped with a Dirac operator, in particular a metric. To be more specific suppose the base is a Spin^c-manifold with spinors S_Y. At each point of X the Clifford algebra of T_X is the tensor product of the Clifford algebras of the horizontal + vertical tangent spaces, so we can write $E = S_Y \otimes E'$ where E' is a module over the vertical Clifford algebra. At this point we have ignored the difference of metrics on the horizontal tangent bundle and the metric on the base. Let's assume that the metric on X is such that $T_{X,\text{horiz}} \xrightarrow{\sim} f^*T_Y$ is an isometry so that we can get to the main point.

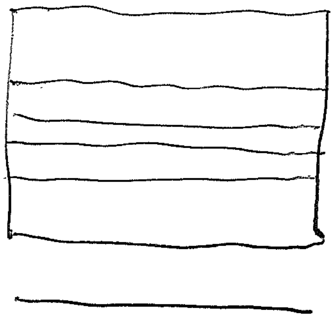
I want to think of the Dirac operator on X as being the Dirac operator on Y with coefficients in a Hilbert bundle with superconnection given by the Dirac operators along the fibres. For this to be so, I have to see the Hilbert bundle of sections of E' over the fibres as being equipped naturally with a connection.

Repeat: Consider a fibre bundle $f: X \to Y$ with Riemannian structures on the fibres. Because the existence of a spin structure is a topological obstruction, a spin structure on one fibre should extend to one on nearby fibres at least, so we can form the family of Dirac operators on the fibres. Now I want to construct the Chern character of the index of this family. So I need a connection on the Hilbert bundle over Y which consists of L^2 spinor fields over the fibres. It seems that I need a connection in the fibre bundle X/Y at least. This means that along curves in Y I get trivializations of the fibre bundle. The problem now is to extend the connection to the spinors.

Suppose I have a way to extend the connection to the spinors. Then I have a connection also on the endo ring of the spinors which is the Clifford algebra. Hence also probably a connection on the longitudinal tangent bundle which is compatible with the metric since it sits inside the Clifford algebra.

So the problem seems to be roughly the following. A connection in the fibre bundle X/Y allows one to trivialize X over a curve in Y, so in particular one has a way to identify tangent vectors to the fibers at the ends of the curve. But this identification is not compatible with the metric. There should be a way to get a metric isomorphism of the tangent bundles.

Assume for example that we have a metric given in X and that this defines the connection in X/Y. The curve in Y lifts to curves in X which cross the fibres orthogonally. Let's assume Y is a curve and let us rescale the metric in the transverse direction so that it agrees with the metric on Y. Then $X = \mathbb{R} \times M$



and the lines $\mathbb{R} \times m$ all geodesics. The metric on $\mathbb{R} \times M$ is such that the obvious longitudinal and transverse directions are $\perp$.

$$ds^2 = dt^2 + g_{t\mu\nu}\, dx^\mu dx^\nu$$

Now consider parallel transport along a geodesic $\mathbb{R} \times m$. Then it preserves the tangent vector to the geodesic, hence must carry the vertical tangent space at one end to the vertical tangent space at the other. We know it is an isometry so we get an isometric isomorphism for the two metrics at the ends.

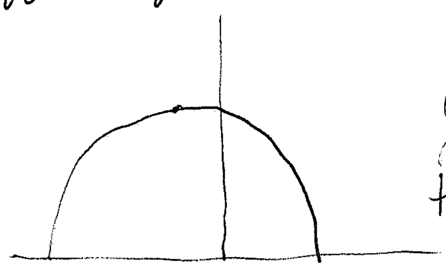
We can get a simple model for this. We have a path in the space of positive-definite quadratic forms on a real vector space, and we have some way of lifting this into the general linear group. Thus we have $q_t(v)$ and we seem to get an isom g_t of V such that $q_t(g_t(v)) = q_0(v)$. Thus we have

$$\begin{tikzcd} & GL_n(\mathbb{R}) \arrow[d] \\ \mathbb{R} \arrow[ur, "g"] \arrow[r, "q"'] & GL_n(\mathbb{R})/O_n. \end{tikzcd}$$

There is an obvious way to do this using a connection in the principal bundle. There should be a unique $GL_n(\mathbb{R})$-invariant connection; it's enough to give the complementary space at the identity, you need an O_n-invariant complement to $\mathrm{Lie}(O_n) \subset \mathfrak{gl}_n(\mathbb{R})$, it seems clear.

At this point we see the following. If we start with a connection in X/Y and with metrics along the fibres, then we have a way to lift the connection to the tangent bundle along the fibres. This will preserve the metric and so will give a connection in the horizontal <u>direction</u> in the frame bundle.

Consider the UHP with usual metric $\frac{dx^2+dy^2}{y^2}$. and let f be the projection onto the real axis. Then any horizontal line crosses the fibres orthogonally but is not a geodesic. This is what can happen when the metric on the horizontal vectors differs from the metric on Y.



On the other hand if we consider the projection onto the y-axis, then the metric is the same and curves $\perp$ to the fibres are geodesics.

Let's now review the program. I have decided that I want to learn how to do Riemannian geom. calculations. In other words I really want to be able write down Dirac operators without getting lost with indices. There is some hope that things can be done working in the frame bundle. However, I want a machine that will work well for families, so it seemed wise to start with the index problem for a family of Dirac operators. This raises the problem of a connection in the Hilbert bundle of spinor fields along the fibres. To solve this one looks more generally at the bundle of orthonormal frames in the fibres of X/Y. Given a connection in X/Y it seems that one gets a connection in this fibre frame bundle

~~84 ff added Mar 71 He used up paper.~~
~~Suppose~~ $\cancel{a^* \xi a}$

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Go back to the computation of $ch(i_! 1)$ where $i : M \to E$ is the zero-section of a complex bundle. We know this leads to computing

$$tr_s\left(e^{i(dz\, a^* + d\bar{z}\, a) + \Omega a^* a}\right)$$

which is a differential form on $E = M \times \mathbb{C}^n$. Here Ω is the curvature [redacted] of E which is a skew-hermitian matrix of [redacted] 2-forms on M, the z^μ are the coords on $\mathbb{C}^n$, and a^*, a denote the standard creation & annihation operators on $\Lambda \mathbb{C}^n$.

Notice that what we have done is to take the Clifford algebra $\mathrm{End}(\Lambda\mathbb{C}^n)$ generated by the a^*, a tensor it with the commutative superalgebra R generated by the elements dz^μ, $d\bar{z}^\mu$ and $\Omega_{\mu\nu}$. Then we are constructing a point in the group of points with values in R.

Let's see if we can express this more clearly. We have a super vector space $\Lambda\mathbb{C}^n = V$ which is our basic Hilbert space of states. Then there is a Lie superalgebra [redacted] $\mathrm{End}(V)$ whose points over a comm. superalgebra R are the degree 0 autos of $V \otimes R$ over R. A typical way to construct such points is to take an element of $\mathrm{End}(V) \hat{\otimes} R = \mathrm{End}_R(V \otimes R)$ of degree zero and to exponentiate it. (We ignore the convergence question.

To be very specific, let us take $n = 1$ whence $\mathrm{End}\, V$ is the Clifford algebra C_2 with basis $1, a^*, a, a^*a$. [redacted] (Interesting point: As a Lie algebra $\mathrm{End}(V)$ contains sl_2 as a factor, so it is [redacted] not solvable. But as a Lie superalgebra ~~it~~ has the [redacted] filtration

$$\langle 1 \rangle \subset \langle 1, a^*, a \rangle \subset \langle 1, a^*, a, a^*a \rangle$$

and so it is solvable as a Lie superalgebra.)

Next I want to discuss

$$e^{\omega a^* a + a^* J + \tilde{J} a}$$

algebraically. In the application we have in mind $\omega, J, \tilde{J}$ are forms on a manifold, so we are ~~exponentiating~~ in the algebra $\mathrm{End}(V) \hat{\otimes} R$ where R is a superalgebra containing $\omega_{ij} \in R^0$ and $J_i, \tilde{J}_i \in R^1$, and ω is nilpotent. So there is no problem in the exponentials being defined.

Let's look carefully at what we are doing. We start with a Lie superalgebra $\mathrm{End}(V)$ and then are calculating something in the group of points with values in R. To get a feeling for the setup I should first look at ordinary Lie theory.

Let $\mathfrak{g}$ be a Lie algebra. Then $U(\mathfrak{g})$ is a Hopf algebra which is cocommutative and which when dualized $U(\mathfrak{g})^* = S(\mathfrak{g}^*)^\wedge$ is the formal functions on the associated formal group. In fact if $\mathfrak{g} = \mathrm{End}(V)$, then the obvious algebraic group to look at is the group $GL(V)$ and then the ring of algebraic functions A on $GL(V)$ is an affine group scheme. In this case

$$U(\mathfrak{g})^* = A^\wedge$$

where $\wedge$ denotes completion at the identity.

The process I am looking at is to take an Artin ring R, then take an element of $X \in \mathfrak{g} \otimes R$ with nilpotent entries, and exponentiate to get an element of the group of points $G(R)$. This means that to

X in $\mathfrak{g}\otimes R$, we get a map $\mathfrak{g}^* \longrightarrow R$ which extends to $S(\mathfrak{g}^*) \longrightarrow R$, and therefore gives $A \longrightarrow R$ which is an element of $G(R)$. Thus we identify maps $A \longrightarrow R$ carrying the augmentation ideal to the nilradical of R with elements of $\mathfrak{g} \otimes \mathrm{Nil}(R)$. The universal situation, then, is when $R = \hat{A}$.

Let's try another viewpoint namely that R is to be a complete algebra, say a super algebra of bounded operators on a super Hilbert space closed under uniform convergence. (If you want R to be commutative, then it can't be a $*$ algebra since $JJ^* + J^*J = 0$ would force $J = 0$.)

So let's take the case where $V = \Lambda\mathbb{C}$ is 2-dim'l spin space with the operators $1, a^*, a, a^*a$. If R is complete commutative superalgebra, then we can exponentiate in

$$\mathrm{End}_R(V \otimes R) = R \hat{\otimes} \mathrm{End}(V)$$

and so form $\quad e^{\omega a^* a + a^* J + \tilde{J} a} \quad$ for $\omega \in R^0$; $J, \tilde{J} \in R^1$.

Now you want a formula for this in terms of the natural basis for $\mathrm{End}(V)$.

Let's complete the square assuming that ω is invertible. Better, suppose $J = \omega K$, $\tilde{J} = \tilde{K}\omega$.

$$\begin{aligned} & \omega a^* a + a^* \omega K + \tilde{K} \omega a \\ = \ & \omega (a^* + \tilde{K})(a + K) - \omega \tilde{K} K \end{aligned}$$

Now $\quad \dfrac{d}{dt}\, e^{t\tilde{K}a} a^* e^{-t\tilde{K}a} = e^{t\tilde{K}a} [\tilde{K}a, a^*] e^{-t\tilde{K}a} = e^{t\tilde{K}a} \tilde{K} e^{-t\tilde{K}a}$

$$= \tilde{K} \qquad \Rightarrow \qquad e^{t\tilde{K}a} a^* e^{-t\tilde{K}a} = a^* + t\tilde{K}$$

$$e^{tKa^*} a\, e^{-tKa^*} = a + tK.$$

$$\therefore \quad e^{t(Ka^* + \tilde{K}a)} \begin{Bmatrix} a \\ a^* \end{Bmatrix} e^{-t(Ka^* + \tilde{K}a)} = \begin{cases} a + tK \\ a^* + t\tilde{K} \end{cases}$$

so we conclude that

$$e^{\omega(a^*a + a^*\tilde{K} + Ka)} = e^{Ka^* + \tilde{K}a}\, e^{\omega a^* a}\, e^{-(Ka^* + \tilde{K}a)}\, e^{-\omega \tilde{K}K}$$

So

$$tr_s\left(e^{\omega a^* a + a^* \tilde{J} + Ja}\right) = tr_s\left(e^{\omega a^* a}\right) e^{-\frac{\tilde{J}J}{\omega}}$$

But now I really look carefully at the last term

$$e^{-\frac{\tilde{J}J}{\omega}} = 1 - \frac{\tilde{J}J}{\omega} = \frac{1}{\omega}(\omega - \tilde{J}J)$$

What exactly does this mean in several dimensions? Let's go back to our differential form example

$$tr_s\left(e^{\Omega a^* a + i dz\, a^* + i d\bar{z}\, a}\right)$$

Here $\quad J = i d\bar{z}, \quad \tilde{J} = -i dz$

Let's consider the following problem first. Let ω_{jk} be an invertible matrix and form the exterior algebra with generators J_j, $\tilde{J}_j$. Consider if

$$e^{+\tilde{J}_j (\omega^{-1})_{jk} J_k} = \frac{\det(\omega_{jk} + \tilde{J}_j J_k)}{\det(\omega)}$$

is true.

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$$tr_s\left(e^{\omega a^*a + a^*J + \tilde{J}a}\right) = tr_s\left(e^{\omega a^* a}\right)e^{-\tilde{J}\omega^{-1}J}$$

$$= tr_s(\Lambda e^{\omega})\, e^{-\tilde{J}\omega^{-1}J} = \det(1-e^{\omega})e^{-\tilde{J}\omega^{-1}J}$$

$$= \det\left(\frac{1-e^{\omega}}{\omega}\right)\ (\det\omega)e^{-\tilde{J}\omega^{-1}J}$$

So the problem is to find a formula for

$$\det\omega\ e^{-\tilde{J}\omega^{-1}J}$$

which makes it obvious that ω doesn't have to be invertible.

Recall the fermion integration formulas

$$\cancel{\int \mathcal{D}\psi\mathcal{D}\psi^*\ e^{-A\psi^*\psi} +}$$

$$\int \mathcal{D}\psi\mathcal{D}\psi^*\ e^{-A\psi^*\psi} = c\,\det(A)$$

$$\frac{\int \mathcal{D}\psi\mathcal{D}\psi^*\ e^{-A\psi^*\psi}(-\psi_i^*\delta A_{ij}\psi_j)}{\int\mathcal{D}\psi\mathcal{D}\psi^*\ e^{-A\psi^*\psi}} = \delta\log\det A = Tr(A^{-1}\delta A)$$

so that $\quad (A^{-1})_{ij} = \int \mathcal{D}\psi\mathcal{D}\psi^*\ e^{-\psi^*A\psi}\ \psi_i\psi_j^* \Big/ \mathcal{D}\psi\mathcal{D}\psi^*\, e^{-\psi^*A\psi}$

The trouble is that these formulas are inherently imprecise. However the idea I had is that we do know something like

$$(\det\omega)e^{-\tilde{J}\omega^{-1}J} = \int\mathcal{D}\psi\mathcal{D}\psi^*\ e^{\psi^*\omega\psi + \psi^*J + \tilde{J}\psi}$$

because the fermion integral on the right is evaluated by completing the square and using translation invariance

So we get the formula

$$tr_s\left(e^{a^*\omega a + a^*J + \bar{J}a}\right) = \det\left(\frac{1-e^{\omega}}{\omega}\right)\int D\psi D\psi^*\, e^{\psi^*\omega\psi + \psi^* J + \bar{J}\psi}$$

which suggests some sort of classical limit process.

If we quantized differently so that $[a^*, a] = h$, then the Todd factor would be

$$\det\left(1 - e^{h\omega}\right).$$

Today's lecture. The missing point is how to describe

$$\det(\omega)\, e^{-\bar{J}\omega^{-1}J}$$

when ω is not invertible. Here $\omega = \omega_{jk}$ is a matrix and $\bar{J}_j, J_k$ are odd elements of a super algebra which is commutative.

So far the simplest version I have is

$$\det(\omega)\, e^{-\bar{J}\omega^{-1}J} = \int D\psi^* D\psi\, e^{\psi^*\omega\psi + \psi^* J + \bar{J}\psi}$$

where $\int D\psi^* D\psi$ picks out the coefficient of $\prod_j \psi_j^*\psi_j$. In other words you take the exterior algebra generated by ψ_j^*, ψ_j tensor R with it and then pick out the appropriate coefficient.

One possibility is to think of J as a row vector $|J\rangle = (J_k)$ and $\bar{J}$ as a column vector $(\bar{J}_k) = \langle\bar{J}|$ and then to form

$$\det\left(\omega + |J\rangle\langle\bar{J}|\right).$$

In other words one has a matrix $\omega_{jk} + J_j\bar{J}_k$ in R^{even} and one takes the determinant.

$$\det\left(\omega + |J\rangle\langle\bar J|\right) = \det(\omega)\,\det\left(1 + \underbrace{\omega^{-1}|J\rangle\langle\bar J|}_{A}\right)$$

$$A = \omega^{-1}_{j\ell}\, J_\ell \bar J_k$$

$$\log\det(1+A) = \mathrm{tr}\,\log(1+A)$$

$$= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}\,\mathrm{tr}(A^n)$$

$$\mathrm{tr}(A^2) = \omega^{-1}_{j\ell} J_\ell \bar J_m\, \omega^{-1}_{mp} J_p \bar J_j$$

$$= -\bar J_j\, \omega^{-1}_{j\ell} J_\ell \cdot \bar J_m\, \omega^{-1}_{mp} J_p$$

$$\mathrm{tr}\,A^2 = \mathrm{tr}\;\omega^{-1}|J\rangle\langle\bar J|\,\omega^{-1}|J\rangle\langle\bar J|$$

$$= -\langle\bar J|\omega^{-1}|J\rangle^2$$

In general

$$\mathrm{tr}\,A^n = \mathrm{tr}\;\underbrace{\omega^{-1}|J\rangle\langle\bar J|\,\omega^{-1}|J\rangle\langle\bar J|\cdots\langle\bar J|}_{n \text{ factors}}$$

$$= -\langle\bar J|\omega^{-1}|J\rangle^n$$

So therefore it seems we get

$$\det\left(\omega + |J\rangle\langle\bar J|\right) = \det(\omega)\, e^{-\langle\bar J|\omega^{-1}|J\rangle + \frac{1}{2}\langle\bar J|\omega^{-1}|J\rangle^2 - \cdots}$$

$$= \det(\omega)\left(1 + \langle\bar J|\omega^{-1}|J\rangle\right)^{-1}$$

Check: $$\det\left(I + |J\rangle\langle\bar J|\right) = \begin{vmatrix} 1 + J_1\bar J_1 & J_1\bar J_2 \\ J_2\bar J_1 & 1 + J_2\bar J_2 \end{vmatrix}$$

$$= 1 + \left(J_1\bar J_1 + J_2\bar J_2\right) + \underbrace{\left(J_1\bar J_1 J_2\bar J_2 - J_2\bar J_1 J_1\bar J_2\right)}_{2 J_1\bar J_1 J_2\bar J_2}$$

$$\frac{1}{1 - (J_1\bar J_1 + J_2\bar J_2)} = 1 + \left(J_1\bar J_1 + J_2\bar J_2\right) + \left(J_1\bar J_1 + J_2\bar J_2\right)^2 + \cdots$$

OKAY

March 9, 1984

Let us go over the classical mechanics quantum mechanics, & classical limit formalism.

Classical mechanics deals with the cotangent bundle or more generally a symplectic manifold. The functions on T^* form a Lie algebra under Poisson bracket. In particular for $M = \mathbb{R}^n$, the functions on T^* of degree ≤ 2 form a Lie algebra under Poisson bracket. Those of degree ≤ 1 form the Heisenberg alg. spanned by $p, q,$ [redacted] 1 where z is the constant function 1. Let's set the notation so that Hamilton's eqns. are

$$\dot{q} = \{H, q\} = \{\tfrac{p^2}{2} + V, q\} = p$$
$$\dot{p} = \{H, p\} = \{\tfrac{p^2}{2} + V, p\} = -\frac{\partial V}{\partial q} \qquad \Longrightarrow \quad \{p, q\} = \text{[redacted]}\, 1$$

The quadratic functions under Poisson bracket form the Lie algebra of the symplectic group. Hence the functions of degree ≤ 2 form the semi-direct ~~of~~ the symplectic Lie algebra and the Heisenberg Lie algebra.

In quantum mechanics we have operators, and it seems desirable to think of them as operators on the intrinsic Hilbert space of ½ densities [redacted] on the manifold. To be specific we take [redacted] $M = \mathbb{R}^n$ and then have the operators

$$q = x \qquad p = \frac{h}{i}\partial_x \qquad \Longrightarrow \quad [p, q] = \frac{h}{i}$$

on $L^2(\mathbb{R}^n)$. Given a Hamiltonian H which is now a self-adjoint operator, the equations of motion are

$$\dot{q} = \frac{i}{h}[H, q] = \frac{i}{h}[\tfrac{p^2}{2} + V(q), q] = p \qquad \text{[redacted]}$$

$$\dot{p} = \frac{i}{h}[H, p] = \frac{i}{h}[\tfrac{p^2}{2} + V(q), p] = -V'(q)$$

Notice that we get a different quantum mechanics for each value of h, so perhaps we should think of H as depending on h. [redacted] Thus over the $h \neq 0$ line, I have a family of Hilbert spaces all equipped with q, p operators. The Hamiltonian is an operator on this bundle and so depends on h in general.

The next point will be to bring in the symplectic group which is acting on the real q, p vector space. On each Hilbert space there is a lifting [redacted] as follows. [redacted] Each symplectic transformation of $V = \mathbb{R}q + \mathbb{R}p$ is implemented by a unitary operator on the representation belonging to a given h, call this Hilbert space $\mathcal{H}_h$. This gives a projective repn. of $Sp(V)$ on $\mathcal{H}_h$ [redacted] which lifts canonically to the metaplectic group $\widetilde{Sp}(V)$. (Any central extension of the Lie algebra is trivial canonically, so the Lie algebra lifts, etc.). Therefore to each homog. quadratic function of the q, p belongs a Hamiltonian operator in each of the spaces $\mathcal{H}_h$.

Now the question arises as to whether we have for each h a representation of the Lie algebra of [redacted] degree ≤ 2 polys in q, p under Poisson bracket.

Let's start again, this time from the viewpoint of representation theory. We have this Lie algebra over $\mathbb{R}$ which is the semi-direct product of the symplectic algebra $sp(V)$ acting on the Heisenberg algebra of V. We want its irreducible representations and the characters.

By Mackey theory one begins with the reps. of the Heisenberg algebra. This has the basis q, p, z where $[p,q] = z$, and z is central. In an irreducible representation z [redacted] must be a scalar h, [redacted] and if $h = 0$ things are trivial, so we suppose $h \neq 0$. From the unitary [redacted] viewpoint we want p, q, z to be realized by skew-hermitian operators. So the scalar value of z [redacted] is purely imaginary.

At this point I have to resolve a conflict in notation. I want to write the Lie algebras over $\mathbb{R}$ but to associate to skew-adjoint operators to elements of the Lie algebra. But physicists use self-adjoint operators.

Let's reserve $\{\ ,\ \}$ to denote the Poisson bracket so that we are dealing with a Lie algebra with basis q, p, z such that $\{p,q\} = z \quad \{p,z\} = \{q,z\} = 0$. We now look for a homomorphism of this Lie alg. into the Lie alg. of skew-hermitian matrices. [redacted] Let us agree that skew-hermitian matrices [redacted] will always be written iA, where A is hermitian. Then the representation will be

$$q \longmapsto i\hat{q}\ , \quad p \longmapsto i\hat{p}\ , \quad z \longmapsto i\hat{z}$$

where $\hat{q}, \hat{p}, \hat{z}$ are hermitian operators satisfying

$$[i\hat{p}, i\hat{q}] = i\hat{z} \qquad \text{or} \qquad [\hat{p},\hat{q}] = \frac{\hat{z}}{i} \qquad \text{etc.}$$

In an irreducible representation $\hat{z}$ will be a real number h.

The Stone-von Neumann theorem tells us that for $h \neq 0$ there is only one irreducible repn., namely $L^2(\mathbb{R})$ with

$$\hat{q} = \text{mult. by } x \qquad \hat{p} = \frac{h}{i}\partial_x$$

Now by Mackey theory a representation of the semi-direct product of the symplectic group by the Heisenberg group is [redacted] obtained from an orbit of the symplectic group an irred. repn. of the normal subgroup. In this case if $\hat{z} = h \neq 0$, then the orbit is trivial, and ~~so~~ up to a projective repn. problem, we get a unique way to extend the repn. of the Heisenberg [redacted] group to the semi-direct product.

So let us take $\frac{p^2}{2}$ in our Lie algebra $\mathfrak{g}$ = functions of q, p of degree ≤ 2 with $\{\,,\,\}$. Then we have

$$\left\{\frac{p^2}{2}, q\right\} = p \qquad \left\{\frac{p^2}{2}, p\right\} = \left\{\frac{p^2}{2}, z\right\} = 0.$$

So we seek an operator $\widehat{\frac{p^2}{2}}$ such that

$$\left[\widehat{\frac{p^2}{2}}, \hat{q}\right] = \frac{1}{i}\hat{p} \qquad \left[\widehat{\frac{p^2}{2}}, \hat{p}\right] = \cancel{\left[\widehat{\frac{p^2}{2}}, h\right]} = 0$$

Now $$\left[\tfrac{1}{2}(\hat{p})^2, \hat{q}\right] = \hat{p}\,[\hat{p}, \hat{q}] = \hat{p}\,\frac{h}{i}$$ so it is clear that

$$\widehat{\frac{p^2}{2}} = \frac{1}{h}\frac{(\hat{p})^2}{2} + \text{scalar}.$$

Actually if we want $\widehat{\frac{p^2}{2}}$, $\widehat{qp}$, $\widehat{\frac{q^2}{2}}$ to be the image of the symplectic Lie algebra, then we must take them to be

$$\widehat{\frac{p^2}{2}} = \frac{1}{h}\frac{(\hat{p})^2}{2} \qquad \widehat{qp} = \frac{1}{h}\,\frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2} \qquad \widehat{\frac{q^2}{2}} = \frac{1}{h}\frac{(\hat{q})^2}{2}$$

as one can easily check.

Now I know how the symplectic group, or better metaplectic group acts on $\mathcal{H}_h$. The rule seems to be that the flow corresponding to an element X of $\mathfrak{g}$ is conjugation by $e^{\frac{i}{h}t\hat{X}}$

It would be better to say that we have

described an irreducible repn. of $\mathfrak{g}$ for each h. The next project is to compute the character of this representation. The character is supposed to be some sort of central distribution on the group. In fact Graeme tells me that as a representation of the metaplectic group we have one of the discrete series, so in principle a lot could be learned from this example.

Let's use the exponential map to identify the Lie algebra and the group around the identity. Among the elements of $\mathfrak{g}$ are those quadratic functions of q, p which have positive definite leading term. This gives an open subset where one might hope to get the character via a boundary value of something analytic.

What interests me is the following. Take a positive definite H_0 say $H_0 = \frac{p^2}{2} + \frac{q^2}{2}$. Then in $\mathfrak{g}$ I can consider $H_0 + Jq + Kp$ where $J, K \in \mathbb{R}$. This is a forced oscillator and so should be essentially equivalent to H_0.

Start again. $\mathfrak{g}$ = Lie alg of quadratic fns of q, p under $\{\ \}$, where $\{p,q\} = 1$. For each $h \neq 0$ we get a representation of $\mathfrak{g}$ as operators on $L^2(\mathbb{R})$ by

$$X \longmapsto i\widehat{X} \qquad , \quad 1 \longrightarrow ih$$

where

$$\hat{q} = \text{mult by } x$$

$$\hat{p} = \frac{h}{i}\partial_x$$

$$\widehat{\frac{p^2}{2}} = \frac{1}{2h}(\hat{p})^2 \qquad \widehat{qp} = \frac{1}{2h}(\hat{q}\hat{p} + \hat{p}\hat{q}) \qquad \widehat{\frac{q^2}{2}} = \frac{1}{2h}(\hat{q})^2$$

Then I can ask for the character

$$e^X \longmapsto tr\left(e^{i\hat{X}}\right) \qquad X \in \mathfrak{g}.$$

Let's take the case where we know that $\hat{X}$ has discrete spectrum. $X = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$. Then

$$\hat{X} = \frac{1}{2h}(\hat{p})^2 + \frac{\omega^2}{2h}(\hat{q})^2 \qquad \text{where } [\hat{p},\hat{q}] = \frac{h}{i}$$

and I know the eigenvalues of $\frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2$ are $(n+\frac{1}{2})h\omega$. Hence the eigenvalues of $\hat{X}$ are $(n+\frac{1}{2})\omega$, $n \geq 0$, which leads to the character

$$tr\left(e^{i\hat{X}}\right) = \sum_{n\geq 0} e^{i(n+\frac{1}{2})\omega} = \frac{e^{i\frac{\omega}{2}}}{1-e^{i\omega}}$$

Notice that this is independent of h

Summarize: I originally thought that I could treat the classical limit by introducing the Heisenberg algebra semi-direct product with symplectic Lie algebra.

The Connes approach to the classical limit is to consider the algebra generated by $\hat{q}$ and $\hat{p}$ and $\hat{h}$ satisfying the relations $[\hat{p},\hat{q}] = \frac{\hat{h}}{i}$, $[\hat{p},\hat{h}] = [\hat{q},\hat{h}] = 0$. Thus we get the universal enveloping algebra of the Heisenberg algebra. This algebra has irreducible representations for each $h \in \mathbb{R}-0$ given by $\hat{q} \longmapsto x$, $\hat{p} \longmapsto \frac{h}{i}\partial_x$, $\hat{h} \longrightarrow h$.

In the Connes theory we take an H in this algebra for example $H = \frac{\hat{p}^2}{2} + \frac{1}{2}\omega^2\frac{\hat{q}^2}{2}$, then form the partition function $tr(e^{-H})$ which is a functor of h with a limit given by the classical partition fn.

But H physically generates time evolution namely the time evolution in the representation belonging to Planck's constant h is given by the unitary operator $e^{\frac{i}{h}tH}$. Thus, take the Hamiltonian classically to be the generator of the time evolution, then the generator for the unitary group of time evolution is $\frac{1}{h}H$ so you have to multiply by h before taking the quantum partition function. (Energies depend on Planck's constant but the frequencies are more or less determined classically.)

The next project will be to understand the fermion variant of this. Start with the Heisenberg superalgebra which will be spanned by $c^*, c, \hat{h}$ subject to

$$c^2 = (c^*)^2 = 0 \quad , \quad c^*c + cc^* = \hat{h} \quad , \quad [\hat{h}, c] = [\hat{h}, c^*] = 0 .$$

We get an irreducible representation for each $h \neq 0$ by letting $\hat{h} \longmapsto h$, $c^* \longmapsto a^*$, $c \longmapsto ha$ where a^*, a are the usual creation and annihilation operators.

So we now have the Heisenberg alg. and its family of irred. repns. We next bring in the orthogonal Lie algebra which in the present case is generated by $\frac{1}{2}(c^*c - cc^*) = c^*c - \frac{h}{2}$. Take a specific one parameter family with frequency ω. This means the operator

$$e^{-\frac{i}{h}t\omega(c^*c - \frac{h}{2})} = e^{-\frac{i}{h}tH} \quad , \quad H = \omega\left(c^*c - \frac{h}{2}\right).$$

Note that $c^*c - \frac{h}{2}$ has the eigenvalues $\pm\frac{h}{2}$, so the energies are $\pm\frac{h\omega}{2}$. The quantum partition function is

$$\mathrm{tr}_s\left(e^{-\omega\left(c^*c-\frac{h}{2}\right)}\right) = e^{\frac{h\omega}{2}} - e^{-\frac{h\omega}{2}} \sim h\omega$$

The classical partition function is

$$\int d\psi\, d\psi^*\, e^{-\omega\psi^*\psi} = \int d\psi\, d\psi^* \left(1-\omega\psi^*\psi\right) = \omega$$

Possible attack: $\mathrm{tr}_s\left(e^{-\frac{i}{h}tH}\right)$ is directly related to the character of the element $e^{-\frac{i}{h}tH}$ of the spin group in the spin representation. So what I am after is a feeling for these partition functions as characters on the spinor or metaplectic groups.

March 10, 1984

Let's review the Thom class formula, check that it is closed, and see what happens as $t \to \infty$. Take the case of a line bundle, say trivial: $L = M \times \mathbb{C}$, with connection $D = d + \theta$. Then on $\pi^*(\Lambda L)$ we have

$$D = d + \theta a^* a \qquad\qquad D^2 = \omega a^* a \;, \quad \omega = d\theta$$

$$L = i\,(z a^* + \bar{z} a) \qquad\qquad [D, L] = i\,(dz + \theta z)\,a^* + i\,(d\bar{z} - \theta\bar{z})\,a$$

$$L^2 = -|z|^2 \qquad\qquad = a^* \underbrace{(-i(dz + \theta z))}_{J} + \underbrace{(i(d\bar{z} - \theta\bar{z}))}_{\bar{J}} a$$

and [redacted]

$$tr_s\left(e^{(D+L)^2}\right) = \frac{1 - e^{\omega}}{\omega}\, e^{-|z|^2}\left(\omega - \bar{J}J\right).$$

Now [redacted]

$$-\bar{J}J = (-1)\,i\,(d\bar{z} - \theta\bar{z})(-i)(dz + \theta z) = (dz + \theta z)(d\bar{z} - \theta\bar{z})$$

Thus

$$\boxed{i_* 1 = e^{-|z|^2}\left(\omega + \theta(z\,d\bar{z} + \bar{z}\,dz) + dz\,d\bar{z}\right)}$$

Let's check this is closed.

$$e^{-|z|^2}\left(\omega + \theta\, d|z|^2\right) = e^{-|z|^2}\left(d\theta - d|z|^2\theta\right)$$

$$= e^{-|z|^2}\left(d - d|z|^2\right)\theta$$

But $\quad e^{|z|^2} d\, e^{-|z|^2} = d - d|z|^2 \quad$ so the above is

$$= d\left(e^{-|z|^2}\theta\right).$$

Hence

$$\boxed{i_* 1 = d\left(e^{-|z|^2}\theta\right) + e^{-|z|^2}\, dz\, d\bar{z}}$$

which is obviously closed and has $\int_{\mathbb{C}} \frac{i}{2\pi}(i_* 1) = 1$

Comparison with Bott's construction of the Thom class. He uses the fact that when the Euler class is pulled back to the bundle minus the zero section it becomes exact as the bundle has a section. Thus on $L-M$ we have the connection form (globally defined)

$$\tilde{\theta} = \frac{dz}{z} + \theta = \text{M.C. form} + \theta$$

and $\pi^*(\omega) = d\tilde{\theta}$. Now the Thom class is

$$i_*1 = \pi^*(\omega) - d(f\tilde{\theta})$$

where f is a smooth function on L which is zero near the zero section and 1 far out. ~~Now my formula for the Euler class is~~ This is

$$\pi^*(\omega) - df\cdot\left(\frac{dz}{z}+\theta\right) - f\pi^*(\omega) = (1-f)\pi^*(\omega) - df\left(\frac{dz}{z}+\theta\right)$$

Now my formula is

$$e^{-|z|^2}\pi^*(\omega) - \left(e^{-|z|^2}d|z|^2\right)\theta + e^{-|z|^2}dz\,d\bar{z}$$

so a close match would be

$$1-f = e^{-|z|^2} \qquad df = e^{-|z|^2}d|z|^2.$$

However the two formulas then differ by

$$\underbrace{\left(e^{-|z|^2}d|z|^2\right)\frac{dz}{z}}_{\text{Bott's}} \qquad \underbrace{\left(e^{-|z|^2}dz\,d\bar{z}\right)}_{\text{mine}}$$

Actually these two are equal since

$$-d|z|^2\frac{dz}{z} = -(\bar{z}dz + zd\bar{z})\frac{dz}{z} = dz\,d\bar{z}$$

Hence we have learned that my way of producing the Thom class $i_* 1$ coincides with Bott's in the case of complex line bundles. Now we want to see if this holds in general. Some goals of this line of investigation might be: to understand Bott's formulas, possibly to connect his approach to the fermion integration formula, maybe construct $i_* 1 \in H^p_c(E, \Omega^p)$ when E is a holomorphic vector bundle.

The idea of Bott's construction is that $\pi^* E$ where $\pi : E \longrightarrow M$ is the projection has a canonical section which is non-zero off the zero section. This gives a reason for the Euler class c_n of $\pi^* E$ over $E - M$ to vanish, and analyzing this will show $\pi^*(c_n) = db$ giving the desired form b over $E-M$ of degree $2n-1$.

M compact + oriented. According to the Schwartz kernel theorem any linear map $T : \Omega(M) \longrightarrow \Omega(M)$ which is continuous from the $(-\infty)$ topology to the $(+\infty)$ topology is given by a kernel $\hat{T}$ which is a smooth differential form on the product. The formula is

$$T\omega = \int_{y \in M} \hat{T}(x,y)\,\omega(y) \qquad \text{or} \qquad T = (pr_1)_* (\hat{T} \cdot) \, pr_2^*$$

On the endomorphisms T we have the operations $T \mapsto d \circ T$, $T \mapsto T \circ d$. It is clear that

$$d(T\omega) = (-1)^n (pr_1)_* \left(d\hat{T} \cdot pr_2^* \omega \right) + (-1)^{n + \deg \hat{T}} (pr_1)_* \left(\hat{T} \cdot pr_2^* d\omega \right)$$

Note that $\deg(T) = -n + \deg(\hat{T})$, hence this says

$$[d,T] = \left(d\circ T - (-1)^{\deg T}\, T\circ d\right) = (pr_1)_* \circ (-1)^n d\hat{T} \circ pr_2^*$$

It is more or less clear that $T \mapsto d\circ T$ and $T \longmapsto T\circ d$ are given up to sign with $d'T$ and $d''T$ respectively. In the case of a Riemannian manifold we also have d^* on the forms and so we can consider $T \longmapsto d^*\circ T,\ T\circ d^*$. On the other hand we also have $d^* = (d^*)' + (d^*)''$ on the product. So it is again that $T \mapsto d^*\circ T,\ T\circ d^*$ coincide up to sign with the effect of $(d^*)'$ and $(d^*)''$. But it also must be the case that $[T, d^*]=0$ is <u>not</u> equivalent to $d^*(\hat{T}) = 0$, since $T = e^{-t\Delta}$ commutes with both d and d^*. Hence $d(\hat{T}) = 0$, but if also $d^*(\hat{T}) = 0$, then $\hat{T}$ would be harmonic. But there is only a finite-diml. space of harmonic forms, so one couldn't have a family approaching a distribution, or current, with support along the diagonal.

The general idea goes as follows. From Connes + Getzler's ideas, it should be the case that the Schwartz kernels belonging to the heat operators $e^{-t\Delta}$, $\Delta = (d+d^*)^2$, when scaled suitably have definite limits as kernels on the tangent bundle.

March 11, 1984

Signature operator: I believe that the heat operator $e^{-t\Delta}$ where $\Delta = (d+d^*)^2$ on a Riem. manifold M has a limiting kernel which is a form on T_M representing the Thom class. In fact [redacted] I want to write down this limiting kernel using my Gaussian formulas for the Thom class. If this is so, then the index theorem for the signature operator (Hirzebruch signature thm.) has to result from the explicit [redacted] Thom class form on T_M. This is because the signature operator is $d+d^*$ on forms, but with a different grading. So the Laplaceans for the signature and DR operators are the same.

Similarly if I have a Kahler manifold M, then I can take the $\bar\partial$-complex for Ω^p, i.e. the Dolbeault complex

$$\Omega^{p,0} \xrightarrow{\bar\partial} \Omega^{p,1} \longrightarrow \cdots\cdots$$

take the Dirac sum over p and get the forms with operator $\bar\partial + \bar\partial^*$. Now it is fundamental for a Kähler manifolds that the $\bar\partial$-Laplacean $(\bar\partial+\bar\partial^*)^2$ is $\frac{1}{2}(d+d^*)^2$. So the Hirzebruch RR theorem should also fall out somehow from the explicit Gaussian-Thom form on T_M.

Let's begin with a Riemann manifold, let's form the Clifford algebra $C(T^*)$, and use the left multiplication map $C(T^*) \longrightarrow \Lambda(T^*)$, $\alpha \longmapsto \alpha\cdot 1$ to identify the two. Suppose given a connection on T^* preserving the metric, it extends to $C(T^*)$ and $\Lambda(T^*)$ compatible with the above isomorphism.

Let's agree to [redacted] write a connection on a bundle E as an operator $D : \Gamma(E) \longrightarrow \Gamma(T^* \otimes E)$ with T^* on the left. If E is a $C(T^*)$-module, then we can compose D with Clifford multiplication $T^* \otimes E \to E$ to get a first order differential operator $\not{D}$ on E.

Let ω^i be an orthonormal frame for T^* [redacted] on some open set. Then $D = \omega^i D_i$ where $D_i = i(e_i) D$ and e_i is the dual frame in T. D_i is a first order operator on E and we have $\not{D} = \gamma^i D_i$ where $\gamma^i =$ Clifford multiplication by ω^i on E.

Given a connection D on $E = T^*$, it is [redacted] a first order operator $D : \Gamma(T^*) \longrightarrow \Gamma(T^* \otimes T^*)$, which can be composed with exterior product $T^* \otimes T^* \xrightarrow{m} \Lambda^2 T^*$ to get an operator from $\Gamma(T^*)$ to $\Gamma(\Lambda^2 T^*)$ whose symbol is clearly the same as d. Then $mD - d$ is a vector bundle homomorphism $T^* \longrightarrow \Lambda^2 T^*$ which should be identifiable with the torsion of the connection. Thus when the torsion is zero the operator d on $\Gamma(\Lambda T^*)$ coincides with the connection followed by exterior multiplication:

$$\Gamma(\Lambda^p T^*) \xrightarrow{D} \Gamma(T^* \otimes \Lambda^p T^*) \xrightarrow{m} \Gamma(\Lambda^{p+1} T^*)$$

Cartan notation: ω^i orthonormal frame. Then the connection is

$$D\omega^i = \theta^i_\mu \otimes \omega^\mu \qquad \text{in } T^* \otimes T^*$$

so that relative to this frame $D = d + \theta$ and the curvature is $d\theta + \theta^2$. ~~[illegible]~~

~~[illegible]~~

<u>Note:</u> Given a connection D on T^* one can extend it to ΛT^* as a derivation, getting maps

$$\Lambda T^* \longrightarrow T^* \otimes \Lambda T^*$$

On the other hand one can also extend D on $\Lambda^{\cdot}T^*$ as a derivation over the forms

$$\Lambda^{\cdot}T^* \xrightarrow{D} T^* \otimes \Lambda^{\cdot}T^* \xrightarrow{D} \Lambda^2 T^* \otimes \Lambda^{\cdot}T^* \xrightarrow{D} \cdots .$$

All this could be done for $D: E \to T^* \otimes E$; we can then get an extension to $\Lambda^i T^* \otimes \Lambda^{\cdot} E \longrightarrow \Lambda^{i+1} T^* \otimes \Lambda^{\cdot} E$.

Let's go back to $D\omega^i = \theta^i_\mu \otimes \omega^\mu \in T^* \otimes T^*$. As D preserves the metric and ω^i is an orth. frame, it follows $\theta^i_\mu = -\theta^\mu_i$. On the other hand torsion zero means, or at least should imply,

$$d\omega^i = \theta^i_\mu \wedge \omega^\mu$$

Let's write $\theta^i_\mu = c^i_{\mu\nu}\,\omega^\nu$, and suppose we have two connections $D, \tilde{D}$ on T^* preserving the metric and with 0 torsion. Then the difference gives a $c^i_{\mu\nu}$ which is symmetric under μ,ν interchange and skew-symm. under ν, i interchange. But then

$$c^i_{\mu\nu} = -c_\mu{}^\nu{}_i = -c_i{}^\nu{}_\mu = +c^\mu_{i\nu} = c^\mu_{\nu i} = -c^i_{\nu\mu} = -c^i_{\mu\nu}$$

so c must be zero and we get the uniqueness of the LC connection.

Summarize: Start with a connection D on T^*, the torsion is the difference of

$$T^* \xrightarrow{D} T^* \otimes T^* \xrightarrow{m} \Lambda^2 T^*$$

and d. When the torsion is zero we have the formula

$$d = e(\omega^\mu) D_\mu \qquad D_\mu \text{ extended to } \Lambda T^*$$

since both agree on Λ^0 and Λ^1.

So now I want to calculate the Thom class on the tangent bundle of a Riemann surface. We think of the [redacted] tangent bundle as being a complex line bundle, in fact even a holomorphic line bundle.

Review our previous calculations. L is a complex line bundle with connection and metric, we choose a local trivialization $L = M \times \mathbb{C} / M$ compatible with the metric whence the connection becomes $D = d + \theta$ where $\bar{\theta} = -\theta$. Let $z : L \longrightarrow \mathbb{C}$ be the second projection. Then we [redacted] found the Thom class to be (p. 589)

$$i_* 1 = d\left(e^{-|z|^2}\theta\right) + e^{-|z|^2} dz\, d\bar{z} .$$

Now this can be written as follows. Recall that the connection form on [redacted] $L - M$ is

$$\tilde{\theta} = \theta + \frac{dz}{z}$$

Then

$$d\left(e^{-|z|^2}\tilde{\theta}\right) = d\left(e^{-|z|^2}\theta\right) + d\left(e^{-|z|^2}\frac{dz}{z}\right)$$

$$= d\left(e^{-|z|^2}\theta\right) + \underbrace{e^{-|z|^2}(-z\,d\bar{z} - \bar{z}\,dz)\frac{dz}{z}}_{e^{-|z|^2} dz\, d\bar{z}} = i_* 1$$

This gives an intrinsic [redacted] way of describing the Thom form namely take connection form on $L - M$, multiply by $e^{-|\xi|^2}$, where $|\xi|^2$ is the norm, and then take d.

Now we can apply this to the tangent bundle of our Riemann surface and so obtain the Thom form. The question now is what do I want to do with it.

(Notice that $d\left(e^{-|z|^2}\tilde{\theta}\right)$ gives a Thom form even if the connection does not preserve the metric, since two connections differ by a form η on the base and $d(e^{-|z|^2}\eta)$ represents zero in the n cohomology of L with proper supports over M.)

March 12, 1984

Let's start with a Riemannian manifold M and form the Clifford algebra $C(T^*)$. Choose a connection on T^* preserving the metric; it then extends to a connection on $C(T^*)$ and we get a Dirac operator on $C(T^*)$ by

$$\not{D}: \quad C(T^*) \xrightarrow{D} T^* \otimes C(T^*) \xrightarrow{m} C(T^*).$$

If we use the Levi-Civita connection on T^*, then under the canonical identification $C(T^*) \simeq \Lambda(T^*)$, this Dirac operator is $d+d^*$. Since any connection differs from the Levi-Civita connection by the torsion, it follows that the Dirac operator in general should be $d+d^*$ + torsion term.

Derivation property for $\not{D}$: Let α and $\omega \in \Gamma C(T^*)$

$$\not{D}(\alpha\omega) = (\not{D}\alpha)\,\omega + (\gamma^\mu\alpha)\cdot D_\mu\omega$$

(Proof. $\not{D}(\alpha\omega) = \gamma^\mu D_\mu(\alpha\omega) = \gamma^\mu(D_\mu\alpha\cdot\omega + \alpha\cdot D_\mu\omega)$.)

The important conclusion is that if ω is a section of $C(T^*)$ which is flat, then it defines an endomorphism of $(\not{D}, C(T^*))$. We have the following two examples

1) For an oriented manifold we can take ω to be the orientation element, i.e. the element corresponding to the orientation section of $\Lambda^n(T^*)$.

2) For an almost complex manifold, we take a connection preserving the almost complex structure (and the metric of course). Associated to these is a two form ω of type $(1,1)$ which will be flat. Exponentiating ω in the Clifford algebra will give a homomorphism $S^1 \longrightarrow \Gamma(C(T^*))$ which is flat. Thus by right multiplication we will get a circle action on $C(T^*)$ commuting with $\not{D}$.

This is the sort of thing that takes place with a Kähler manifold. Except we now have a much simpler way, and slightly different in fact, to think about things. The usual way of thinking is to regard the forms as a double complex.

Linear algebra: Let T^* be a real vector space with inner product and $C(T^*)$ its Clifford algebra. An orientation on T^* determines an embedding of $\mathbb{Z}/2$ in $C(T^*)^\times$, and a complex structure on T^* compatible with the metric determines a circle $S^1 \subset C(T^*)^\times$. (What does one need to get an $S^3 \subset C(T^*)^\times$? Is this related to the new index theorem of Alvarez-Gaumé that I heard about from Freedman?)

Let's work out the circle case. Then we might as well identify $C(T^*)$ with the Clifford algebra $\mathrm{End}(\Lambda \mathbb{C}^m)$ with the generators a_i^*, a_i, and T^* with the self-adjoint linear combinations of these generators. The circle action is naturally associated to the complex structure + hermitian inner product, hence it must commute with the action of $U(m)$ on $\mathrm{End}(\Lambda \mathbb{C}^m)$. Since the various exterior powers are distinct irreducible representations of $U(m)$, the commutant of $U(m)$ consists of the scalar operators in each $\Lambda^g \mathbb{C}^m$. So now it's clear that the circle action just gives the natural grading on the exterior algebra.

This is confirmed by the fact that the generator of the circle action is supposed to be a degree 2 element of the Clifford algebra, namely the element of $\Lambda^2 T^*$ which is associated to the imaginary part of the hermitian inner product. The only thing one can write down in an intrinsic way is the number operator $N = \sum a_i^* a_i$.

Next let's compare the orientation element with the circle actions. The orientation element is essentially

$$\gamma^1 \cdots\cdots \gamma^n$$

which is the element of $C(T^*)$ mapping to the generator $\omega^1 \cdots \omega^n$ of $\Lambda^n T^*$ determined by the orientation. Now

$$(\gamma^1 \cdots\cdots \gamma^n)^2 = (-1)^{\frac{n(n-1)}{2}} = (-1)^m$$

Actually I should really first ask about the commutant of $SO(n)$ in the Clifford algebra. For n even we know the Clifford algebra is the algebra of Endos of the spinors S, and that this splits into two irreducible representations S^+, S^- of $Spin(n)$. Thus the commutant must be $\mathbb{C} \oplus \mathbb{C}$ and it is generated by the involution ε giving the grading $S = S^+ \oplus S^-$. This gives our $\mathbb{Z}/2\mathbb{Z}$ inside of the Clifford algebra. Changing the orientation amounts to changing ε to $-\varepsilon$. One needs ~~to~~ make some sort of convention about ε: One has

$$\varepsilon = i^{\pm m} \gamma^1 \cdots \gamma^n$$

and one has to pick the sign somehow.

Let's see the connection with the number operator N. Recall that

$$\gamma^1 = a^* + a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^2 = ia^* - ia = \begin{pmatrix} & -i \\ i & \end{pmatrix}$$

$$\gamma^1 \gamma^2 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = e^{\frac{\pi}{2} i (1 - 2a^*a)}$$

$$= e^{-\pi i (a^*a - \frac{1}{2})}$$

So in general using $C_n = C_2 \hat{\otimes} \cdots \hat{\otimes} C_2$ m times, we have

$$\gamma^1 \cdots \gamma^n = e^{-\pi i \sum_j (a_j^* a_j - \frac{1}{2})} = i^m (-1)^N$$