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February 5, 1984

Problem: Given a vector bundle E over M, set $\mathcal{A} = \Omega^0(M)$, $\mathcal{B} = \Omega^0(M, \mathrm{End}\, E)$. Given a cyclic cocycle on $\mathcal{A}$ [redacted] one wants to extend it to $\mathcal{B}$, and establish the Morita invariance of cyclic cohomology.

Connes approach is to construct a differential graded algebra containing $\mathcal{B}$, and then construct the cocycle as a trace on this algebra. One starts with the algebra of End E-valued forms

$$\mathcal{B} \otimes_{\mathcal{A}} \Omega(M) = \Omega(M, \mathrm{End}\, E).$$

A connection D on E gives rise to a degree one derivation $[D, \,]$ of $\Omega(M, \mathrm{End}\, E)$ whose square is $[D^2, \,]$. Connes constructs a larger algebra obtained by adjoining an element to $\Omega(M, \mathrm{End}\, E)$ on which one has a degree one derivation d satisfying $d^2 = 0$.

My first idea is that this larger diffl. algebra can contain $\Omega(M, \mathrm{End}\, E)$ as neither a sub or quotient, but it could [redacted] as a compression contain $\Omega(M, \mathrm{End}\, E)$. In the same way that we can obtain non-trivial curvature from a flat connection by compression.

Compression means reducing with respect to an idempotent. [redacted] Write E as the image of a projector e on the trivial bundle $M \times V$. Thus $e \in \Omega^0(M) \otimes \mathrm{End}\, V$, $e^2 = e$. Then

$$\Omega^0(M, \mathrm{End}\, E) = e(\Omega^0(M) \otimes \mathrm{End}\, V)e$$

Now I want to do something similar with forms. We have

$$\Omega(M, \mathrm{End}\, E) = e\,(\Omega(M) \otimes \mathrm{End}\, V)\,e$$

and these algebras operate on

$$\Omega(M,E) = e\big(\Omega(M)\otimes V\big)$$

(I want to check that if $D = e \cdot d \cdot e$ is the Grassmannian connection, then $[D, \cdot\,]$ on $\Omega(M, \operatorname{End} E)$ is the same as $\varphi \longmapsto e(d\varphi)e$ on $e(\Omega(M)\otimes \operatorname{End} V)e$.

Let $\varphi \in e(\Omega(M)\otimes \operatorname{End} V)e$. Then as operators on $\Omega(M)\otimes V$

$$ede\varphi = ed\varphi = ed\varphi e$$
$$\varphi ede = \varphi de = e\varphi de$$
$$\Longrightarrow \quad [ede, \varphi] = e[d,\varphi]e = e(d\varphi)e.\,)$$

Thus if we reduce $\Omega(M)\otimes \operatorname{End} V$, the diff'l algebra of matrix forms, with respect to the idempotent e, we get the algebra $\Omega(M, \operatorname{End} E)$ with the derivation $[D, \,]$ where $D = ede$ is the Grassmannian connection.

Now if we want a differential algebra containing $\Omega^0(M, \operatorname{End} E)$, the obvious candidate appears to be to adjoin to $\Omega(M, \operatorname{End} E)$ the element de. I would have to do this if I wanted an diff'l subalgebra of $\Omega(M)\otimes \operatorname{End} V$ closed under d containing $\Omega^0(M, \operatorname{End} E)$.

But check now that $\Omega(M, \operatorname{End} E)\langle de\rangle$ is closed under d. Enough to show that $d\varphi$ is in this algebra when $\varphi \in \Omega(M, \operatorname{End} E)$. But then $\varphi = e\varphi = \varphi e$ so

$$d(\varphi) = \underbrace{d(e\varphi e)}_{} = (de)\varphi e + e(d\varphi)e + (-1)^{\deg \varphi} e\varphi(de)$$

and so it is clear. $\qquad (de)\varphi + (-1)^{\deg\varphi}\varphi(de) + e(d\varphi)e$

Something is strange, because I would have liked the bracket of de and φ in this formula.

<u>Second idea</u>: Let P be the principal bundle of E, and let's suppose a metric is given on E so that the structural group G is compact. Then we have

$$\Omega(M, \operatorname{End} E) = \text{basic elements of } \Omega(P) \otimes \operatorname{End} V$$

and so we have an embedding of the algebra $\Omega(M, \operatorname{End} E)$ inside a differential algebra.

There are two possibilities. One is to take the algebra

$$\left(\Omega(P) \otimes \operatorname{End} V\right)^G$$

of invariant forms. The other is to take the algebra generated by $\Omega(M, \operatorname{End} E)$ and the connection form θ. Clearly the second algebra is contained in the first. Since two connection forms differ by an element of $\Omega^1(M, \operatorname{End} E)$ it follows that the second algebra is independent of the connection.

Question: Is $\left(\Omega(P) \otimes \operatorname{End} V\right)^G$ generated by $\Omega(M, \operatorname{End} E)$ and the connection form Θ?

Let R be the subalg. gen. by $\Omega(M, \operatorname{End} E)$ and Θ. Because $\left(\Omega(P) \otimes \operatorname{End} V\right)^G$ is the smooth sections of a bundle of algebras, it is clear by partitions of 1 etc, that one can suppose E is trivial. I can suppose Θ is the connection obtained from the trivialization as pointed out above. Then

$$\left(\Omega(P) \otimes \operatorname{End} V\right)^G = \left(\Omega(M) \otimes \Omega(G) \otimes \operatorname{End} V\right)^G$$

$$= \Omega(M) \otimes \left(\Omega(G) \otimes \operatorname{End} V\right)^G$$

and so it's enough to consider $M = pt$, whence θ is the MC form. Write $\theta = \theta^a X_a$ where X_a is a basis for $\mathfrak{g} \subset \operatorname{End} V$. Now we have an isomorphism

$$\left(\Omega(G) \otimes \operatorname{End} V\right)^G \xrightarrow{\sim} \Lambda \mathfrak{g}^* \otimes \operatorname{End} V$$

by restricting to the identity of G. And we have $\Omega(M, \operatorname{End} E) = \operatorname{End} V$ sitting inside. But now if you multiply $\theta = \theta^a X_a$ by elements of $\operatorname{End} V$ you can get all form $\theta^a Y$ with $Y \in \operatorname{End} V$; this is because $e_i X e_j$ picks out the (i,j)-th entry of the matrix X. In particular you can get $\theta^a I$ and you are finished.

It is necessary to get a better hold on the algebra $(\Omega(P) \otimes \operatorname{End} V)^G$. In particular I want to see if a connection actually gives a projection back onto $\Omega(M, \operatorname{End} E)$.

February 6, 1984

Connes construction to kill curvature: [redacted]

Let $\Omega^* = \{\Omega^n, n \geq 0\}$ be a [redacted] graded algebra, let D be a degree one derivation such that $D^2 = [K, \ ]$ where K is an element of Ω^2. For example

$$\Omega = \Omega(M, \operatorname{End} E) \qquad D = [\nabla, \ ]$$

where ∇ is a connection on E; then $K = \nabla^2 =$ curvature

Then Connes adjoins an element X of degree one to Ω satisfying the following relations

$$X^2 = K \qquad \Omega X \Omega = 0.$$

Notice that even if Ω is unital, then this new algebra is non-unital. So we are dealing again with one of his non-unital tricks.

Call the new algebra $\Omega(X)$. Any element of $\Omega(X)$ is of the form

$$\omega_{11} + \omega_{12} X + X\omega_{21} + X\omega_{22} X$$

$$= \begin{pmatrix} 1 & X \end{pmatrix} \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} \text{[redacted]}\,1 \\ X \end{pmatrix}$$

and this representation is unique. This representation by 2×2 matrices over Ω is suggested by

$$\begin{pmatrix} \Omega & \Omega X \\ X\Omega & X\Omega X \end{pmatrix} \cdot \begin{pmatrix} \Omega & \Omega X \\ X\Omega & X\Omega X \end{pmatrix} \subset \begin{pmatrix} \Omega & \Omega X \\ X\Omega & X\Omega X \end{pmatrix}$$

and by $X\Omega \cdot X\Omega = 0$, $\Omega X \cdot \Omega X = 0$.

What it maybe means is that we are taking the direct sum of a module with itself, but where the second copy has a funny multiplication. In the example $\Omega(M, \operatorname{End} E)$ we know this algebra can be identified with endomorphisms of $\Omega(M,E)$ as a (right) $\Omega(M)$-module. Then we are taking two copies of this module $\Omega(M,E)$ perhaps. If so you would want to explain what sort of gadget X is.

Let's go back to the example where E is the image of a projector $e \in \Omega^0(M) \otimes \operatorname{End} V$. Then I work in the algebra of matrix forms $\Omega(M) \otimes \operatorname{End} V$ which operates on $\Omega(M) \otimes V$. Look at

$$de \in \Omega^1(M) \otimes \operatorname{End} V$$

and think of things in terms of operators on $\Omega(M) \otimes V$ relative to the decomposition

$$\Omega(M,V) = e\,\Omega(M,V) \oplus (1-e)\,\Omega(M,V).$$

Then because $e\,de\,e = 0$ the operator de is off diagonal.

Now $\Omega = \Omega(M, \operatorname{End} E) = e\big(\Omega(M) \otimes \operatorname{End} V\big)e$.

Let $\Omega\langle de \rangle$ be the algebra of endos of $\Omega(M,V)$ generated by Ω and de. This isn't correct because $(de)^2$ is not in Ω^2

The point seems to be that $\Omega\langle de\rangle$ does not contain de

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February 7, 1984

Let $e \in \Omega^0(M) \otimes \operatorname{End} V$ be a projector, and E its image in the trivial bundle $M \times V$ over M. Then $\Omega(M,E) = e(\Omega(M) \otimes V)$ and $\Omega(M, \operatorname{End} E) = e\,(\Omega(M) \otimes \operatorname{End} V)\,e$. We have the Grassmannian connection $D = e.d.e$ on E. It has the curvature

$$e.d.e.d.e =$$

$$= e(\cancel{e}d + [d,e])\,de$$

$$= e\,[d,e](ed + [d,e])$$

$$= e\,[d,e][d,e]$$

because of $e\,[d,e]\,e = e(1-e)[d,e] = 0$. Put $de = [d,e]$.

Let us now go over Connes construction. Put $\Omega = \Omega(M, \operatorname{End} E)$. Now Connes constructs an algebra, which he says is obtained by adjoining to Ω an odd element X, satisfying the relations

$$X^2 = K \quad , \quad \Omega X \Omega = 0 .$$

The elements of this algebra are uniquely written in the form

$$\omega_{11} + \omega_{12} X + X \omega_{21} + X \omega_{22} X .$$

Now it is important to notice that X does not belong to this algebra. If e denotes the unit in Ω, then the algebra contains the elements eX and Xe.

Thus it would be better to describe the algebra as being generated by Ω and elements eX, Xe with the relations

$$eX \cdot Xe = K$$

$$e(eX) = eX \qquad (eX)e = 0$$
$$e(Xe) = 0 \qquad (Xe)e = Xe.$$

We now represent this algebra as operators on $\Omega(M) \otimes V$ by interpreting $\Omega = \Omega(M, \mathrm{End}\, E)$ $= e(\Omega(M) \otimes \mathrm{End}\, V)e$ in the obvious way, and let

$$(Xe) \longmapsto de \cdot e$$
$$(eX) \longmapsto e \cdot de.$$

Then the above relations are satisfied. (One shouldn't think of X as represented by de, as X is not in the algebra. Also $(de)^2 \neq K$, whereas $X^2 = K$. One thing one could is to replace the relation $X^2 = K$ by $eX^2e = K$. If I define in Connes algebra X to be $X = (eX) + (Xe)$, then I have the relations

$$X \cdot e = (eX)e + (Xe)e = Xe \quad , \quad e \cdot X = (eX)$$

and hence $\quad (1-e)X = X \cdot e \quad , \quad eX = X(1-e).)$

Now I have to define a differential on Connes algebra. The problem here is that his formulas diverge from the standard model I have.

Recapitulate: Let $\Omega = \Omega(M, \mathrm{End}\, E)$; this is a graded algebra with unit e and it is equipped with the degree one derivation $D\varphi = e\, d\varphi\, e$ which satisfies

$$D^2\varphi = e\, d(e(d\varphi)e)e = e\, de \underbrace{(d\varphi)}_{d(e\varphi) = de\,\varphi + e\,d\varphi} e + (-1)^{\deg d\varphi} e \underbrace{d\varphi\, de}_{d(\varphi e) = d\varphi\, e + (-1)^{\deg\varphi}\varphi\, de} e$$

$$= e\, de^2 \varphi - (-1)^{\deg\varphi}(-1)^{\deg\varphi} e\varphi\, de^2 e = [\underbrace{e\, de^2 e}_{K}, \varphi]$$

Now we construct a new algebra by adjoining an element X of degree 1 to Ω satisfying

$$\begin{cases} X = Xe + eX & \Rightarrow \quad eXe = 0 \\ eX^2e = K \end{cases}$$

Then $\Omega\langle X\rangle$ consists of elements $\omega_{11} + \omega_{12}X + X\omega_{21} + X\omega_{22}X$ and we have a natural homomorphism

$$\begin{aligned} \Omega\langle X\rangle &\longrightarrow \Omega(M)\otimes \operatorname{End} V \\ X &\longmapsto de \end{aligned}$$

Next I want to define a ~~differential~~ degree one derivation

$$\Omega \xrightarrow{\ d\ } \Omega\langle X\rangle.$$

We already have one which is given by D on Ω

Start again: Let me work in the image of $\Omega\langle X\rangle$ in $\Omega(M)\otimes \operatorname{End} V$. The image is closed under d which is determined by the formulas

$$\begin{aligned} d\varphi = d(e\varphi e) &= \overset{de}{\cancel{X}}\varphi + ed\varphi e + (-1)^{\deg\varphi}\varphi\overset{de}{\cancel{X}} \\ &= [\overset{de}{\cancel{X}}e, \varphi] + (-1)^{\deg\varphi}[\varphi, \overset{de}{\cancel{X}}e] + D\varphi \\ &= [de\cdot e - e\cdot de, \varphi] + D\varphi \end{aligned}$$

$$d(de) = 0.$$

February 8, 1984

Problem: Let $\mathcal{A} = \Omega^0(M)$, $\mathcal{E} = \Omega^0(M,E)$ where E is a vector bundle over M, $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}) = \Omega^0(M, \mathrm{End}\,E)$. Given a cyclic cocycle on $\mathcal{A}$ and a connection on E, Connes has a way to extend this cocycle to $\mathcal{B}$. This is how he establishes Morita invariance. I want to properly understand this process.

Suppose the cyclic cocycle on $\mathcal{A}$ is given by an elliptic operator P over M. One would like the cocycle on $\mathcal{B}$ to be given by some version of tensoring P with the vector bundle E.

The problem I have is with the definition of the cocycle. If P is invertible, then the cocycle is defined easily using the operator of degree 1

$$F = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix}$$

satisfying $F^2 = I$.

(Review formulas.

$$\begin{tikzcd} e\mathcal{H}^0 \arrow[r, "ePe"] \arrow[d] & e\mathcal{H}^1 \arrow[d] \arrow[dl, dotted] \\ e\mathcal{H}^0 \arrow[r, "ePe"] & e\mathcal{H}^1 \end{tikzcd}$$

Identity on this complex is homotopic to

$$K = e - eFeFe = eF[F,e]e = \underbrace{eF(1-e)}_{eeF-eFe}[F,e]e$$
$$= e[e,F][F,e]e = -e[F,e]^2$$

so
$$\text{Index} = \mathrm{tr}_{\mathcal{E}}(K^n)$$
$$= (-1)^n\,\mathrm{tr}_{\mathcal{E}}\left(e[F,e]^{2n}\right)$$

for n large enough that the trace makes sense.

More generally put

$$\varphi(a_0, \dots, a_{2n}) = (-1)^n \operatorname{tr}_\varepsilon\left(a_0 [F, a_1] \cdots [F, a_{2n}]\right)$$

$$= (-1)^n \operatorname{tr}_\varepsilon\left(F^2 a_0 [F, a_1] \cdots [F, a_{2n}]\right)$$

$$= (-1)^{n+1} \operatorname{tr}_\varepsilon\left(F a_0 [F, a_1] \cdots [F, a_{2n}] F\right)$$

$$= (-1)^{n+1} \operatorname{tr}_\varepsilon\left(F a_0 F [F, a_1] \cdots [F, a_{2n}]\right)$$

$$= (-1)^n \frac{1}{2} \operatorname{tr}_\varepsilon\left(F [F, a_0] \cdots [F, a_{2n}]\right)$$

$$= (-1)^n \frac{1}{2}\left\{ \operatorname{tr}\left(P [P^{-1}, a_0] \cdots [P^{-1}, a_{2n}]\right) + \operatorname{tr}\left(P^{-1} [P, a_0] \cdots [P, a_{2n}]\right)\right.$$

$$= (-1)^n \frac{1}{2}\left\{ (-1)^{n+1} \operatorname{tr}\left([P, a_0] P^{-1} \cdots [P, a_{2n}] P^{-1}\right) \right.$$
$$\left. + (-1)^n \operatorname{tr}\left(P^{-1}[P, a_0] \cdots [P, a_{2n}]\right) \right\}$$

$$= \text{[redacted]} \operatorname{tr}\left(P^{-1}[P, a_0] \cdots P^{-1}[P, a_{2n}]\right)$$

So the formulas I want are

$$\boxed{\begin{aligned} \operatorname{Index}(ePe) &= (-1)^n \operatorname{tr}_\varepsilon\left(e [F, e]^{2n}\right) = \varphi(e, \dots, e) \\ \varphi(a_0, \dots, a_{2n}) &= (-1)^n \operatorname{tr}_\varepsilon\left(a_0 [F, a_1] \cdots [F, a_{2n}]\right) \\ &= (-1)^n \frac{1}{2} \operatorname{tr}_\varepsilon\left(F [F, a_0][F, a_1] \cdots [F, a_{2n}]\right) \\ &= \text{[redacted]} \operatorname{tr}\left(P^{-1}[P, a_0] \cdots P^{-1}[P, a_{2n}]\right) \end{aligned}}$$

Notice the absence of any sort of normalization constants (i.e. factorials) in these formulas for the index. This surprises me when I think of [redacted] $\operatorname{tr}_\varepsilon(e[F,e]^{2n})$ as being $\operatorname{tr}_\varepsilon(K^n)$ <u>where</u> K is the curvature.

Review: [redacted] $\mathcal{A} = \Omega^0(M)$, $\mathcal{E} = \Omega^0(M, E)$, $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ $= \Omega^0(M, \mathrm{End}\, E)$. Then we know that $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent. In particular the categories of finite projective [redacted] modules over $\mathcal{A}$ and $\mathcal{B}$ are equivalent. [redacted] So there has to be an operator over $\mathcal{B}$ corresponding to the operator P over $\mathcal{A}$ in so far as the index pairing is concerned.

Geometrically what is happening is the following. A finite projective $\mathcal{B}$-module ~~corresponds to~~ is the same as a vector bundle V over M with a right action of the alg. ~~bundle~~ $\mathrm{End}(E)$. It corresponds to a vector bundle F over M, in fact, we have

$$V = F \otimes E \qquad\qquad F = V \otimes_{\mathrm{End}\, E} E$$

Now the idea is that [redacted] an operator over $\mathcal{A}$ or M can be tensored with a vector bundle F to get a new operator, although this requires some choice of connection, or something similar, on F.

Morita invariance is easily understood from the Lie algebra viewpoint. Cyclic cocycles are primitive cocycles on the Lie algebra $gl(A)$ which are invariant under $gl(k)$. Now given a finite projective A-module E one writes it in the form eA^k with e an idempotent in $M_k(A)$. (I seem to be assuming A has a unit.) Then $\mathrm{End}_A(E)$ is isomorphic to the subring $eM_k(A)e$ of $M_k(E)$. So we get a Lie algebra map

$$gl(\mathrm{End}_A E) \hookrightarrow gl(A)$$

and so a map on cyclic homology. Clearly we pull-back cyclic cocycles on A to get ones on $\mathrm{End}_A(E)$. What we are saying is that a cyclic n cocycle on A can be extended to $M_k(A)$ by the trace; then restricted to $eM_k(A)e$.

Let's first see how this works in the case that I have been interested in mainly. Let $\mathcal{A} = \Omega^0(M)$, let E be a vector bundle over M, $\mathcal{E} = \Omega^0(M,E)$ the corresponding finite projective $\mathcal{A}$-module. ~~We could~~ suppose $E = \text{Im}\, e$ where e is an idempotent in $\Omega^0(M) \otimes \text{End}\, V$. Take the cyclic cocycle on $\mathcal{A}$ given by a n-cycle γ on M:

$$\varphi(a^0, \ldots, a^n) = \int_\gamma a^0 da^1 \ldots da^n$$

Then we want the corresponding cocycle on

$$\mathcal{B} = \text{End}_{\mathcal{A}}(\mathcal{E}) = \Omega^0(M, \text{End}\, E).$$

φ extends to the matrix fns. $\Omega^0(M) \otimes \text{End}\, V$ by

$$\varphi(X^0, \ldots, X^n) = \int_\gamma tr(X^0 dX^1 \ldots dX^n)$$

so all that we do is to restrict this φ to the subring of matrix functions X satisfying $eX = Xe = X$.

February 9, 1984

Connes' approach begins with the idea of ~~tensoring~~ an elliptic operator with a vector bundle. Let's examine this process. $\mathcal{A} = \Omega^0(M)$, let P be an elliptic 0th order pseudo-diffl operator between vector bundles E^+, E^-, let $\mathcal{H}^{\pm} = L^2(M, E^{\pm})$. Then $\mathcal{H}^+, \mathcal{H}^-$ are $\mathcal{A}$-modules and

$$\mathcal{H}^+ \xrightarrow{P} \mathcal{H}^-$$

satisfies $[P, a]$ compact $\forall a \in \mathcal{A}$.

Now given a vector bundle E we let $\mathcal{E} = \Omega^0(M, E)$ and want to extend P to the tensor product: $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}^+ \dashrightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}^-$. If we express E as the image of a projector $e \in M_k(\mathcal{A})$, then $\mathcal{E} = e\mathcal{A}^k$ and $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} = e\mathcal{H}^k$. So suppose $k = 1$ to simplify.

Thus we suppose we are given

$$\mathcal{H}^+ \xrightarrow{P} \mathcal{H}^-$$

where $\mathcal{A}$ acts on $\mathcal{H}^{\pm}$ and $[P, a]$ is compact $\forall a \in \mathcal{A}$. From P I construct cocycles on $\mathcal{A}$. Say that P is invertible, whence I can put $F = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix}$ and then the cocycles are

$$\varphi(a^0, \ldots, a^{2n}) = (-1)^n \operatorname{tr}_\varepsilon \left(a^0 [F, a^1] \cdots [F, a^{2n}] \right)$$

$$= (-1)^n \frac{1}{2} \operatorname{tr}_\varepsilon \left(F [F, a^0] \cdots [F, a^{2n}] \right)$$

$$= \operatorname{tr} \left(P^{-1}[P, a^0] \cdots P^{-1}[P, a^{2n}] \right)$$

Now one can restrict these cocycles to the subalgebra $e\mathcal{A}e$ of $\mathcal{A}$. But notice what has happened to our original operator $P : \mathcal{H}^+ \to \mathcal{H}^-$. It hasn't changed, and it remains invertible. What has changed is the fact that $e\mathcal{A}e$ no longer acts unitally on $\mathcal{H}$. If we try to restrict to the subspace $e\mathcal{H}^+, e\mathcal{H}^-$ on which $e\mathcal{A}e$ acts unitally then P has to be replaced by ePe, which is not necessarily invertible.

What we learn is that we can reverse this process to handle a non-invertible operator. The idea will be to add to our $\mathcal{A}$-modules spaces on which $\mathcal{A}$ acts trivially.

Let us start then with $F = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix}$ on $\mathcal{H}^+ \oplus \mathcal{H}^-$ and reduce it with respect to e.

$$P = \begin{pmatrix} ePe & eP(1-e) \\ (1-e)Pe & (1-e)P(1-e) \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} eP^{-1}e & eP^{-1}(1-e) \\ (1-e)P^{-1}e & (1-e)P^{-1}(1-e) \end{pmatrix}$$

Conversely suppose I am given just what one sees on $e\mathcal{H}^+, e\mathcal{H}^-$; call these space $\mathcal{H}_1^{\pm}$. One is given the operator

$$eFe = \begin{pmatrix} 0 & eP^{-1}e \\ ePe & 0 \end{pmatrix} = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} = F_1 \quad \text{on } \mathcal{H}_1$$

such that $e - F_1^2$ is compact. I want to enlarge $\mathcal{H}_1$ so that F_1 becomes the reduction $\underbrace{}_{\text{or compression}}$ of an invertible F: Two obvious possibilities

$$\begin{pmatrix} p & 1 \\ 1-qp & -q \end{pmatrix}\begin{pmatrix} q & 1 \\ 1-pq & -p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} p & 1-pq \\ 1 & -q \end{pmatrix}\begin{pmatrix} q & 1-qp \\ 1 & -p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Here

$$P = \begin{pmatrix} p & 1 \\ 1-qp & -q \end{pmatrix} : \begin{matrix} \mathcal{H}_1^+ \\ \oplus \\ \mathcal{H}_1^- \end{matrix} \longrightarrow \begin{matrix} \mathcal{H}_1^- \\ \oplus \\ \mathcal{H}_1^+ \end{matrix}$$

and the identity e of $\mathcal{A}$ is acting as 0 on the lower factors and zero on the upper ones.

Let's now start work on trying to relate cyclic cocycles and heat kernels. The goal will be to work with

$$\not{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \qquad \text{and the heat kernel } e^{-t\not{D}^2}$$

rather than the $\begin{pmatrix} 0 & D^{-1} \\ D & 0 \end{pmatrix}$ that Connes uses.

It is my hope that I can define directly a cyclic cocycle attached to $\not{D}$. One might hope to use the identity

$$\int_0^\infty e^{-tD^*D} D^* \, dt = (D^*D)^{-1} D^* = D^{-1}$$

The basic operation consists of tensoring an operator with a vector bundle E. This operation looks very nice for Dirac operators. Let's try to get this clear.

Let us consider a Dirac operator $\not{D}$ over the manifold M. Then $\not{D}$ acts on a vector bundle S which is a module over the Clifford algebra $C(T)$, and the symbol of $\not{D}$ is Clifford multiplication. To keep things simple, suppose I consider just <u>the</u> Dirac operator on a

Riemannian Spin^c manifold, so that S is the bundle of spinors. In fact let the manifold be **flat**, e.g. $\mathbb{R}^{2n}$, so that I can write

$$\not{D} = -i\gamma^\mu \partial_\mu$$

Now I want to tensor with the bundle E. Let us suppose E is the image of a projector $e \in \Omega^0(M) \otimes \operatorname{End} V$. Then the result of tensoring $\not{D}$ with E (in the Connes way) is the operator on $S \otimes E = e(S \otimes V)$ given by

$$e\not{D}e = -i\gamma^\mu e \cdot \partial_\mu \cdot e .$$

In other words I have the Dirac operator with coefficients in E using the Grassmannian connection $e \cdot d \cdot e$ on E.

Next suppose we had solved our problem of attaching a cocycle to [redacted] $\not{D}$ and $e\not{D}e$. In fact it seems that I would like to [redacted] regard $e\not{D}e$ on $S \otimes E$ as an operator over the ring $\Omega^0(M, \operatorname{End} E)$.

February 10, 1984

Problem: Characterize Dirac operators.

Let M be a Riemannian manifold, let $\not{D}$ be a first order differential operator on a bundle E whose square has for symbol multiplication by the metric viewed as a quadratic function on T^*. The symbol of $\not{D}$ [redacted] which is a map $T^* \otimes E \to E$ then extends to an action of the Clifford algebra bundle $C(T^*)$ on E. On the other hand I know already that there is a unique connection ∇ on E such that $\not{D}^2$ differs from the trace Laplacean of ∇ by a zeroth order operator. So we have two structures on E, namely a Clifford multiplication $T^* \otimes E \to E$ and a connection $\nabla : J_1(E) \longrightarrow T^* \otimes E$. There are various compatibility questions one can [redacted] pose:

1) If T^* is equipped with the Levi-Civita connection, is the map $T^* \otimes E \longrightarrow E$ compatible with the connections?

2) If $J_1(E) \xrightarrow{\nabla} T^* \otimes E \longrightarrow E$ the same as the original operator $\not{D}$?

3) The curvature of ∇ is ~~an element of $Hom(\Lambda^2 T, End E)$,~~ [redacted] a section of $\Lambda^2 T^* \otimes End\, E$, Clifford mult. gives a map $\Lambda^2 T^* \longrightarrow End\, E$, so we get a product elt. in $End\, E \otimes End\, E \longrightarrow End\, E$. Is this endo. of E the difference of $\not{D}^2$ and [redacted] $\nabla \cdot \nabla$?

To study these questions one should first understand what a Clifford module looks like. Let's suppose we are in the even case, which means that

M is even-dimensional and E is $\mathbb{Z}_2$-graded and $\not{D}$ is of odd degree. Then E becomes a super module over $C(T^*)$. Let's suppose M is a Spin^c manifold in which case we have the vector bundle of spinors S which is irreducible over $C(T^*)$. Then E can be written

$$E = S \otimes F$$

where F is a supervector bundle, canonically isomorphic to $\mathrm{Hom}_{C(T^*)}(S, E)$.

Now S comes with the **canonical Levi-Civita** connection, so one sees that on an Clifford module bundle E there is a distinguished family of Dirac-type operators, namely ones obtained from a connection on F. (Possibly one wants also superconnections)

Let's now do the calculation in the Euclidean case, whence S is the trivial bundle with fibre the spinors. Assume F is trivial with fibre V. Then an operator $\not{D}$ on $S \otimes F$ with symbol given by Clifford multiplication is of the form

$$\not{D} = \gamma^\mu \partial_\mu + L$$

where L is a function on Euclidean space M to odd endos. of $S \otimes V$. Then

$$\begin{aligned}\not{D}^2 &= \partial_\mu^2 + \gamma^\mu \cdot \partial_\mu L + L \cdot \gamma^\mu \cdot \partial_\mu + L^2 \\ &= \partial_\mu^2 + (\gamma^\mu L + L \gamma^\mu)\partial_\mu + \gamma^\mu \partial_\mu(L) + L^2\end{aligned}$$

If $\nabla_\mu = \partial_\mu + A_\mu$ is a connection on $S \otimes F$, so that $A_\mu : M \longrightarrow \mathrm{End}(S \otimes V)$, then the trace Laplacean

is
$$\nabla_\mu^2 = \partial_\mu^2 + \partial_\mu A_\mu + A_\mu \partial_\mu + A_\mu^2$$
$$= \partial_\mu^2 + 2A_\mu \partial_\mu + \partial_\mu(A_\mu) + A_\mu^2$$

Thus if $\not{D}^2$ and ∇_μ^2 are to agree up to zeroth order we must have
$$A_\mu = \tfrac{1}{2}(\gamma^\mu L + L\gamma^\mu)$$

Let's write down our two compatibity conditions. The first is that Clifford mult $T^*\otimes E \longrightarrow E$ should be compatible with the connection ∇. This says that γ^μ on E should commute with ∇ as γ^μ corresponds to the flat section dx^μ of T^*. Thus we have
$$\nabla_\nu \gamma^\mu = \gamma^\mu \nabla_\nu \implies [A_\nu, \gamma^\mu] = 0$$

But A_μ is an endom. of $S\otimes F$ and to say it commutes with the γ^μ means that it is of the form $1\otimes \bar{A}_\mu$ where $\bar{A}_\mu$ is an endom. of F. It seems that as A_μ is of even degree, so must be $\bar{A}_\mu$.

So we conclude from compatibility condition 1) that our connection on $E = S\otimes F$ comes from a connection on F preserving the grading. Let's now put
$$\not{D} = \gamma^\mu \partial_\mu + L = \gamma^\mu(\partial_\mu + A_\mu) + \tilde{L}$$
so that $\tilde{L}$ is now the deviation for compatibility condition 2), namely whether $\not{D} = \gamma^\mu \nabla_\mu$. Then we calculate

$$\not{D}^2 = \left(\gamma^\mu \nabla_\mu + \tilde{L}\right)^2$$

$$= \nabla_\mu^2 + \tfrac{1}{2}\gamma^\mu\gamma^\nu[\nabla_\mu, \nabla_\nu] + \underbrace{\gamma^\mu \nabla_\mu \tilde{L} + \tilde{L}\gamma^\mu \nabla_\mu}_{(\gamma^\mu \tilde{L} + \tilde{L}\gamma^\mu)\nabla_\mu + \gamma^\mu \nabla_\mu(\tilde{L})} + \tilde{L}^2$$

Since $\not{D}^2$ and ∇_μ^2 agree up to a zeroth order operator, we have

$$\gamma^\mu \tilde{L} + \tilde{L}\gamma^\mu = 0$$

so we have $\tilde{L} = 1 \otimes L$ where L is an odd degree endom. of F.

<u>Summary</u>. A Clifford module bundle E is of the form $S \otimes F$ with F a super vector bundle. A Dirac operator on E is given (or associated to) a connection D_μ on F preserving the grading, and a degree 1 endom. L of F by the formula

$$\gamma^\mu D_\mu + \varepsilon_S L \qquad \text{on } S \otimes E$$

where ε_S is the grading on S. We have characterized these operators as first order operators $\not{D}$ on E of odd degree with certain properties. Namely $\not{D}^2$ has symbol $|\xi|^2$ and so determines a connection ∇ in E as well as a Clifford module structure. These structures are compatible in the sense that the Clifford mult. $T^* \otimes E \to E$ is compatible with connections. Finally $\not{D}$, $\gamma.\nabla$ differ by an odd endomorphism of E as a Clifford module.

My ultimate project is to attach to such a Dirac operator on $S \otimes F = E$ a sequence of cyclic cocycles defined on the algebra

$$\mathrm{End}_{C(T^*)}(E) = \mathrm{End}(F)$$

(Think of this algebra as acting on the [redacted] left; an element of $KK(A,B)$ is represented by left A-, right B-modules.)

The next project will be to consider the operation of tensoring a Dirac operator $\not{D}$ on $S \otimes F = E$ with a vector bundle E' equipped with a connection ∇ (or maybe superconnection). We define $\tilde{\not{D}}$ on $E \otimes E'$ by the formula

$$\tilde{\not{D}}(u \otimes v) = \not{D}u \otimes v + \gamma^\mu u \otimes \nabla_\mu v.$$

Let's check it is well-defined-

$$\tilde{\not{D}}(fu \otimes v) = \underbrace{\not{D}(fu)}_{[f \not{D}u + \partial_\mu f \, \gamma^\mu u]} \otimes v + \gamma^\mu fu \otimes \nabla_\mu v$$

$$= f\not{D}u \otimes v + \gamma^\mu u \otimes (f\nabla_\mu v + \partial_\mu f \, v)$$

$$= \not{D}u \otimes (fv) + \gamma^\mu u \otimes \nabla_\mu (fv) = \tilde{\not{D}}(u \otimes fv)$$

Suppose that E' is the image of an idempotent e operating on a trivial bundle $\mathbb{1}^k$ and that $\nabla = e \cdot d \cdot e$ is the Grassmannian connection.

Much simpler would be to take a $\not{D}$ on $E = S \otimes F$ and an idempotent degree zero operator e on E as a $C(T^*)$-module. Then if $\not{D} = \gamma^\mu D_\mu + \varepsilon L$ we have $e\not{D}e = \gamma^\mu e D_\mu e + \varepsilon \cdot eLe$, so $e\not{D}e$ is the Dirac operator associated to the connection eDe on eF

and the odd endo. eLe on eF.

Here is the key idea I had yesterday. It starts with Connes' [redacted] perception of Morita invariance which goes as follows. A cocycle (cyclic) on [redacted] A extends in a trivial way using the trace to $M_k(A)$. Then it can be restricted to the (non-unital) subalgebra

$$eM_k(A)e = \mathrm{End}_A(eA^k)$$

This agrees with the idea that we have Lie algebra maps

$$[redacted]\ \mathfrak{gl}\,\mathrm{End}_A(eA^k) \subset \mathfrak{gl}\,M_k(A) \xrightarrow{\sim} \mathfrak{gl}\,A.$$

He actually sees the Morita invariance on the cochain level as follows.

$$\begin{tikzcd}
eM_k(A)e \arrow[d, hook] & & & & \\
M_k(A) \arrow[d, "\rho"] & & & & \\
M_k(\Omega^0 \arrow[r] & \Omega^1 \arrow[r] & \cdots \arrow[r] & \Omega^n \arrow[r] \arrow[d, "tr"] & \cdots\cdots) \\
& & & \Omega^n/[\Omega,\Omega]^n &
\end{tikzcd}$$

The lesson I learn from this is when an idempotent is given to define E, a fin. proj. A-module, then there is an obvious way to pass from cocycles on A to cocycles on $\mathrm{End}_A(eA^k) = \mathrm{End}_A(E)$. Namely given $\varphi(a_0,\ldots,a_n)$ on A you get $tr\,\varphi(b_0,\ldots,b_n)$

on $M_k(A)$ and then you just restrict to $eM_k(A)e$.

This fact I will regard as basic, namely, given e such that $E = eA^k$, then a cocycle on A induces one on $eM_k(A)e \cong \mathrm{End}_A(E)$. Now the process, according to Connes, depends only on the Grass. connection associated to e. In other words, given now a finite projective module E with a connection, then a cocycle on A induces one on $\mathrm{End}_A(E)$.

[(<u>Problem</u>: Given a connection on a vector bundle E preserving an inner product, is it the Grassmannian connection for some embedding of E as a direct summand of a trivial bundle? This should be analogous to the isometric embedding problem for Riemannian manifolds.)]

February 11, 1984

The main problem: Construct cyclic cocycles belonging to a Dirac operator. There should be a canonical construction. A Dirac operator is of the form $\not{D} = \gamma \cdot D + \varepsilon \cdot L$ on $S \otimes E$, where E is a super vector bundle. The cyclic cocycles belonging to $\not{D}$ should be cocycles on $\Gamma(M, \operatorname{End} E)$. Notice that is a super algebra, so there arises the problem of doing cyclic theory for superalgebras. I assume that Burghelea, Dwyer, etc. have done this.

Ex. The cyclic homology for **End**(V) considered as a superalgebra, where $V = V^0 \oplus V^1$. ~~The Hochschild homology depends only on the algebra [illegible] superalgebra, [illegible] End(V) [illegible]~~ Consider the Hochschild homology first. This is obtained from the standard resolution

$$\longrightarrow A \otimes A \otimes A \xrightarrow{b'} A \otimes A \xrightarrow{b'} A$$

of A as an A-bimodule. We have a functor on A-bimodules, namely $X \longmapsto X/[A,X]$ of which we are taking derived functors. Here $[A,X]$ is graded commutators and it arises as follows:

$$(A \otimes_A X) \otimes_A = X/[A,X]$$

In any case the Hochschild homology arises by applying this functor to the above standard resolution. Now for $A = \operatorname{End}(V)$ we know any left A-module is of the form $V \otimes W$, where W is a super vector space. Hence an A-bimodule is of the form $V \otimes Z \otimes V$ where

Z is a supervector space. Thus the category of A-bimodules is semi-simple and so the Hochschild homology is just $A/[A,A]$ in degree zero. It follows from the Connes sequence that the cyclic homology of $\mathrm{End}\, V$ is the same as that of $\mathbb{C}$, the relevant map being the super-trace.

Notice that even when the manifold is a point there are non-trivial Dirac operators, namely odd degree endos. L of a super vector space. Hence a family of Dirac operators over a point parametrized by Y should be a super vector bundle $E = E^0 \oplus E^1$ over Y together with an odd degree endomorphism L of E. I want the character of the index of this family, so I choose a connection D in E preserving the grading and construct the form

$$\mathrm{tr}_s\, e^{(D+L)^2} .$$

The idea I had is to take the case where $M = pt$, $V = V^0 \oplus V^1$, and put $\mathfrak{a} = (\mathrm{End}\, V)^1$, $\mathcal{G} = (\mathrm{Aut}\, V)^0$ and go thru the construction of invariant differential forms on $\mathcal{G}$. I am not yet happy with the transgression process I have used before.

Take a vector bundle which is the image of an idempotent e and compute the restriction of the basic cocycles $\mathrm{tr}(\theta (d\theta)^n)$ to $e M_k(A) e$. We have

$$d\theta = d(e\theta e) = (de)e \cdot \theta - \theta \cdot e de + \underbrace{[D,\theta]}_{e d\theta e}$$

$n=1$ $\quad \operatorname{tr} \text{[redacted]}\, \theta = \boxed{\operatorname{tr} \theta}$

$n=2$ $\quad \operatorname{tr} \theta d\theta = \operatorname{tr} \theta[\text{[redacted]}(de)e\theta - \theta e(de) + [D\theta]] = \boxed{\operatorname{tr} \theta[D,\theta]}$

$n=3$ $\quad \operatorname{tr} \text{[redacted]}\, \theta(d\theta)^2 = \operatorname{tr}\left(-\theta^2 e(de) + \theta[D,\theta]\right)\left(de\, e\theta - \theta\, ede + [D,\theta]\right)$

$$= \operatorname{tr}\left(-\theta^2 K\theta - \theta[D,\theta]\theta\, ede + \theta[D,\theta]^2\right)$$

$$= \boxed{\operatorname{tr}\left(\theta[D,\theta]^2 - \theta^3 K\right)}$$

$n=4$ $\quad \operatorname{tr} \theta(d\theta)^3 = \operatorname{tr}\left(-\theta^2 K\theta - \theta[D\theta]\theta\, ede + \theta[D,\theta]^2\right)\left((de)e\theta - \theta e(de) + [D,\theta]\right)$

$$= \operatorname{tr}\big(-\theta[D,\theta]\theta K\theta + \theta^2 K\theta^2\, ede \text{ [redacted] } \theta[D,\theta]^2\theta\, ede$$
$$- \theta^2 K\theta[D,\theta] + \theta[D,\theta]^3\big)$$

$$= \boxed{\operatorname{tr}\left(\theta[D,\theta]^3 - (\theta^2[D,\theta]\theta + \theta[D,\theta]\theta^2)K\right)}$$

Now the other approach to constructing cocycles on $\Omega^{\circ}(M, \operatorname{End}_{\text{[redacted]}}(E))$ given a connection D on E is to use the path of connections $\delta + D + t\theta$, where these are interpreted as operators in

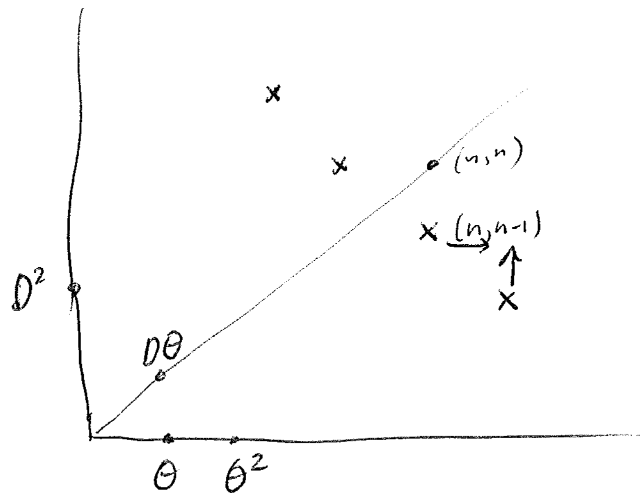
$$C(\breve{g}, \Omega(M, \operatorname{End} E)).$$

One has $\quad (\delta + D + t\theta)^2 = D^2 + t[D,\theta] + (t^2 - t)\theta^2$

and the formula

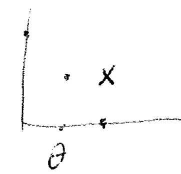
$$\operatorname{tr}\left(D^2 + [D,\theta]\right)^n - \operatorname{tr}(D^2)^n = (\delta + d)\int_0^1 dt\, n \operatorname{tr} \theta\left(D^2 + t[D\theta] + (t^2 - t)\theta^2\right)^{n-1}$$

Picture of the range



The component of degree $(n, n-1)$ will be carried by δ into the image of d.

$$\int_0^1 dt\; 2\,\mathrm{tr}\; \theta\left(D^2 + tD\theta + (t^2-t)\theta^2\right)$$



has 2,1 component

$$\int_0^1 dt\,2t\;\; \mathrm{tr}\,(\theta[D,\theta]) = \mathrm{tr}(\theta[D,\theta])$$

Also $\int_0^1 dt\;\; 3\;\mathrm{tr}\;\overset{1,0}{\theta}\left(\overset{0,2}{D^2} + t\overset{1,1}{D\theta} + (t^2-t)\overset{2,0}{\theta^2}\right)^2$ has 3,2 component

$$\underbrace{\int_0^1 dt\; 3t^2}_{3\cdot\frac{1}{3}}\; \mathrm{tr}\;\theta(D\theta)^2 \;+\; \underbrace{\int_0^1 dt\; 3(t^2-t)}_{3\left(\frac{1}{3}-\frac{1}{2}\right)\; =\; 3\left(\frac{-1}{6}\right) = -\frac{1}{2}}\;\; \mathrm{tr}\left(\theta D^2\theta^2 + \theta^3 D^2\right)$$

$$= \mathrm{tr}\left(\theta[D,\theta]^2 - \theta^3\cdot D^2\right)$$

So there is a chance that the two procedures will in fact lead to exactly the same cocycles.

The first situation to understand is something I think we have checked namely the $(t^2-t)\theta^2$ business and the Connes S-operator.

February 12, 1984

The problem: We have made precise what are Dirac operators on a Riemannian manifold M and shown that they correspond to super vector bundles E equipped with super-connection. The problem remains to construct a sequence of cyclic cocycles representing the Chern character in homology of the Dirac operator. The cocycles should be defined on $\Gamma(\mathrm{End}\,E)$ or $\Gamma(\mathrm{End}\,E)^{\circ}$ and a first question is which of these two should it be?

Let's start from the fact that we do know how to get cocycles from invertible operators. After all, Connes develops the theory of cyclic cocycles starting from <u>invertible</u> operators and ~~idempotents~~.

Question: What is the K-theory of a superalgebra? For example, consider a superalgebra like the Clifford algebras C_n. For n even, the Grothendieck group of super C_n modules is $\mathbb{Z}\oplus\mathbb{Z}$. On the other hand the cyclic homology of C_n I computed yesterday and found to be $\mathbb{C}$ in even degrees and 0 in odd degrees. For example $HC_0(C_n) = C_n/[C_n,C_n]$ is 1-dimensional generated by a degree zero element, and the HC_1 is 0.

For n odd the Grothendieck group of super C_n modules is $\mathbb{Z}$ and $HC_0(C_n)$ is one dimensional generated by an odd degree element. Let's compute the map

$$K_0(C_n\text{-}\overset{\text{super}}{\text{modules}}) \longrightarrow HC_0(C_n)$$

for $n=1$. In general recall the s-trace map on endos. of a finite projective P is defined by

$$\begin{array}{ccc} x\otimes\lambda & P\otimes_A \operatorname{Hom}_A(P,A) \xrightarrow{\sim} \operatorname{End}_A(P,P) \\ \downarrow & \downarrow \\ (-1)^{(\deg x)(\deg\lambda)}\lambda(x) & A/[A,A] \end{array}$$

In particular if P is free with the basis x_i and if λ_i is the dual basis of $\operatorname{Hom}_A(P,A)$:

$$\lambda_i(x_j) = \delta_{ij}$$

then the identity map of P is $\sum_i x_i \otimes \lambda_i$ and so its super trace is

$$tr_s(id_P) = \sum_i (-1)^{\deg x_i}$$

For C_n n-odd, the generator of the Grothendieck group is a free right module with 1-generator which can be either even or or odd degree. This is enough to show that the super trace of the identity is zero.

So
$$K_0(\text{super } C_n\text{-modules}) \longrightarrow HC_0(C_n) \quad \text{is zero}$$
$$\mathbb{Z} \qquad\qquad \mathbb{C} \quad \text{for } n \text{ odd}$$

and
$$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{m_1 - m_2} \mathbb{C} \quad \text{for } n \text{ even.}$$

This seems to show that K_0 is wrong for superalgebras.

February 13, 1984

Here is something I missed. Let $\not{D}$ be a Dirac operator over the manifold M. Then I believe that $\not{D}$ should determine a sequence of cocycles on the ring $\mathcal{A} = \Omega^0(M)$. Moreover these cocycles should be given by differential forms in the expected way. I don't have to get involved with transgression mechanisms in order to check my conjectures.

Let's be more precise. Start with a Riemannian spin^c manifold M and the Dirac operator $\not{D}_0$ on S. Then we have an even diff'l form $\hat{A}(M)$ on M such that

$$\text{Index}\,(\not{D}_0 \otimes E) = \int_M ch(E)\hat{A}(M).$$

This is the global index thm, but the heat equation method will give a local index thm. which goes as follows. Given a connection D on E one gets [redacted] a definite Dirac operator $\not{D}$ on $S \otimes E$ and

$$tr_S \langle x | e^{-t\not{D}^2} | x \rangle \xrightarrow[t \to 0^+]{\text{as}} [redacted] \Big[\Big(tr_E(e^{D^2}) \cdot \hat{A}(M)\Big)\Big]_{2n}(x)$$

Let me try to get the logic straight.

1) topological problem: Consider all Dirac operators over M, [redacted] and consider the index map to $\mathbb{Z}$. This index map is completely equivalent to the $\hat{A}$-genus of M as a [redacted] cohomology class.

2) local index problem. Consider all Dirac operators on M and the map which assigns to an operator $\not{D}$ its index density which is the n-form

$$\left[tr_{s} (e^{D^2}) \cdot \hat{A}(M) \right]_n \qquad n = \dim M.$$

This map is determined by the form $\hat{A}(M)$. One might hope that this local index density map determines $\hat{A}(M)$.

3) I am after 'the' sequence of cyclic cocycles on $\mathcal{A} = \Omega^0(M)$ determined by the Dirac operator. The first cocycle should be of dimension n and the rest should be obtained from this one by the S operator.

Unfortunately this doesn't work for cyclic cocycles defined by an F such that $F^2 = I$. In his Ch. I, §8 Connes shows that if

$$\tau_{2n}(\theta) = tr_{\varepsilon}\left(F[F,\theta]^{2n}\right)$$

then it is **not** true that $S\tau_{2n} = \tau_{2n+2}$, but rather they differ by a coboundary.

February 14, 1984

Let $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ be a graded vector space and $F = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}$ an odd degree involution on it. Then Connes forms the differential graded algebra

$$(*) \qquad (\operatorname{End}\mathcal{H})^0 \xrightarrow{[F,} (\operatorname{End}\mathcal{H})^1 \xrightarrow{[F,} (\operatorname{End}\mathcal{H})^0 \longrightarrow \cdots\cdots$$

and uses it to [redacted] define cyclic cocycles on $(\operatorname{End}\mathcal{H})^0$. Notice that the super commutator quotient is

$$\mathbb{C}\oplus\mathbb{C} \qquad 0 \qquad \mathbb{C} \qquad 0$$

(assuming $\mathcal{H}$ finite dimensional). This is because

$$\operatorname{End}\mathcal{H} / [\operatorname{End}\mathcal{H}, \operatorname{End}\mathcal{H}] \xrightarrow[\sim]{tr_s} \mathbb{C}$$

Now take $\mathcal{H}$ to be $\mathbb{C}\oplus\mathbb{C}$ and put

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

whence

$$[F,e] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$e[F,e] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$e[F,e][F,e] = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$[F,e]^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So we find that the algebra $(*)$ is exactly the algebra of non-commutative differential forms for the algebra $\mathbb{C} \oplus \mathbb{C}e$ with $d1 = 0$.

This is perhaps a good fact for explaining the S-operator.

Now I want to seriously look at what takes place over a point. First take an ordinary vector bundle, i.e. a vector space V. The gauge transf. group is the unitary group, which has interesting left-invariant forms.

How can we construct these forms? There is my method wherein one works with the DG algebra

$$C^*(\mathfrak{g}) \otimes \operatorname{End} V$$

and the MC form $\theta \in C^1(\mathfrak{g}) \otimes \operatorname{End} V$ which represents a flat connection on $G \times V$ over G. Then one uses the path of connections $\delta + t\theta$ to get

$$\delta \int_0^1 dt \, tr\left(e^{(t^2-t)\theta^2}\theta\right) = 0$$

whence one gets a collection of forms

$$\int_0^1 dt \, \frac{(t^2-t)^n}{n!} \, tr\left(\theta^{2n+1}\right) \in C^{2n+1}(\mathfrak{g})$$

There is also Connes method which consists in using the non-commutative forms $\hat{\Omega}(\mathbb{C}e)$ over the non-unital ring $C = \mathbb{C}e$. One works with the double differential algebra

$$C^*\left(\mathfrak{g}, \hat{\Omega}^*(\mathbb{C}e)\right) \otimes \operatorname{End} V$$

and the MC form $\theta = \theta e \in C^1(\mathfrak{g}, \hat{\Omega}^0(\mathbb{C}e)) \otimes \operatorname{End} V$. One gets the cocycles

$$\operatorname{tr}\ \theta(d\theta)^n \quad \in \quad C^{n+1}(\mathfrak{g}, \hat{\Omega}^n/[\,,\,]^n + d\hat{\Omega}^{n-1})$$

In the case of $\hat{\Omega}(ke)$ one knows that $\hat{\Omega}^n/[\hat{\Omega},\hat{\Omega}]^n$ is zero for n odd and generated by $e(de)^n$ for n even.

Kumar: Math Ann 1982 refers to earlier work by two Indians: Amer J. Math (1961) who prove that any connection is a Grassmannian connection. (Ramanan ?) Kumar [redacted] shows that the Weil algebra $W(\mathfrak{g})$, $\mathfrak{g} = \mathrm{Lie}_n(U(k))$ becomes isomorphic to the forms on the Stieffel manifold of k-frames in $\mathbb{C}^N$ which are invariant under $U(N)$, as one takes the $N \to \infty$ limit.

Important problem: Given $e \in \operatorname{Proj} M_k(A)$ one knows how to take cyclic cocycles on A and [redacted] 'restrict' them to $\operatorname{End}_A E = eM_k(A)e$ where $E = eA^k$. This is obvious from the Lie algebra interpretation of cyclic cocycles.

Now Connes shows this restriction process makes sense for a finite projective module E equipped with a connection, the above corresponding to the Grassmannian connection associated to e.

My problem is to find a differential geometric version of the restriction process.

February 15, 1984

Let F be an odd degree endo of a super Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ such that $F^2 - I = K$ is in a Schatten class so that for large n K^n is of trace class. Let us consider the graded algebra with degree one derivation

1) $$(\operatorname{End}\mathcal{H})^0 \xrightarrow{[F} (\operatorname{End}\mathcal{H})^1 \xrightarrow{[F} (\operatorname{End}\mathcal{H})^2 \longrightarrow \cdots$$

as analogous to

2) $$\Omega^0(M, \operatorname{End}E) \xrightarrow{[D} \Omega^1(M, \operatorname{End}E) \xrightarrow{[D} \Omega^2(M, \operatorname{End}E) \to \cdot$$

The latter algebra has a supertrace with values in $\Omega(M)$, and we have the basic identity

3) $$d\, tr_s \phi = tr_s [D, \phi]$$

which enables one to prove

4) $$d\, tr_s (K^n) = 0$$

5) DR class of $tr_s(K^n)$ is ind. of D

where $K = D^2 \in \Omega^2(M, \operatorname{End}E)$ satisfies
$$[D,[D,\phi]] = [K, \phi]$$

The algebra 1) has a super trace with values in

$$\mathbb{C} \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow \cdots$$

so that 3) is trivially satisfied. It should, by analogy, be true that the analogue of 5) is true namely that $tr_s(K^n)$ is independent of the choice of F. Now we know this is the case because the index of F is $(-1)^n tr_s(K^n)$, and the index is constant under perturbations.

But so far one has no relation between the classes $(-1)^n tr_s(K^n)$ for different n, and this relation doesn't seem to be the sort of thing that comes out of the differential form ideas.

Algebraic proof that $(-1)^n tr_s(K^n)$ is independent of n: If A is an endo of degree 0 commuting with F, then

$$tr_s(F^2A) = tr_s(FAF) = -tr_s(F^2A) \implies tr_s(F^2A) = 0$$

Thus $$tr_s(KA) = tr_s((F^2-I)A) = -tr_s A \qquad \text{etc.}$$

Now one of my goals will be to write down completely formulas for the cyclic cocycles determined by such an F even when $F^2 \neq I$, in which case the formulas are to be modified by the presence of the curvature term $K = F^2 - I$.

Narasimhan + Ramanan – Existence of Universal connections, Amer J. Math (1961) 83

Let $V = V^0 \oplus V^1$ be a super vector space which is finite dimensional. Consider the superalg End V. I believe the cyclic cohomology of this superalgebra is the same as that of $\mathbb{C}$. In degree zero one has the super trace and the rest should be generated by periodicity.

Problem: Construct these basic ~~[illegible]~~ cyclic cocycles.

February 16, 1984

The ultimate goal of my present work will be to achieve some understanding of Dirac operators. In particular I want to prove [redacted] a local index theorem for families of Dirac operators, [redacted] find the connection of cyclic cohomology and [redacted] higher anomalies.

A rough idea is that one is interested in local descriptions of K-classes, and that a good local description gives rise to local invariants such as cyclic cocycles with support in the diagonal, or differential forms.

In some sense a Dirac operator is an unbounded version of a Fredholm operator F. Instead of $F^2 - I$ compact, one has to impose a positivity condition, namely $\not{D}^2 \geq 0$. This is Connes' idea for an unbounded Kasparov type theory.

Now, we have discovered a generalization of Dirac operator. The original idea was that one is over a Riemann. spin^c-manifold and has the basic Dirac operator $\not{D}_0$ on the spinors S. Then given a vector bundle E with connection, one can form $\not{D} = \not{D}_0$ [redacted] $+ \gamma \cdot D$ on $S \otimes E$. However, more generally, given a super vector bundle E with superconnection $D + L$, then one obtains $\not{D} = \not{D}_0 + \gamma \cdot D + \varepsilon L$ on $S \otimes E$ and the most general Dirac operator is of this form.

February 17, 1984

Let's consider the Kasparov type K-theory of graded C_k-modules. This should be equivalent to [redacted] what's in the Atiyah-Singer paper on the index theory of skew-adjoint Fredholm operators.

First start with finite-dimensional graded C_k-modules. The Grothendieck group is $\mathbb{Z}\oplus\mathbb{Z}$ if k is even and $\mathbb{Z}$ if k is odd. The supertrace for [redacted] endos. of f.d. graded C_k-modules takes values in

$$C_k/[C_k, C_k] = \mathbb{C}\cdot\gamma^1\cdots\gamma^k .$$

(To see this, note [redacted] that $[C_k, C_k]$ is spanned by

$$[\gamma^j, \gamma^{i_1}\cdots\gamma^{i_p}] = \begin{cases} 0 & j\notin\{i_1,\dots,i_p\} \\ \pm 2\gamma^{i_1}\cdots\widehat{\gamma^{i_a}}\cdots\gamma^{i_p} & \text{otherwise}\end{cases}$$

so that [redacted] $[C_k, C_k]$ is spanned by [redacted] all $\gamma^{i_1}\cdots\gamma^{i_p}$ with $p<k$.)

Normally one is used to the idea that for semi-simple rings the trace, or character, determines the isomorphism class of the [redacted] module or representation. This is not true for graded C_k-modules, moreover one knows from topology the good K-theory of graded C_k-modules is the cofibre theory for the forgetful map

$$C_{k+1}\text{ mods} \longrightarrow C_k \text{ mods}.$$

This is the Atiyah-Bott-Shapiro viewpoint, namely, that the K-theory of a vector bundle E (with compact supports) is the relative K-theory of $C(E)$-modules modulo $C(E\oplus 1)$-modules.

(Question: $C(E)$ and $P(E)$?)

Now the Kasparov-Atiyah-Singer idea is to go to infinite dimensional graded C_k-modules, but to include a Fredholm operator F. Thus one considers a graded Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ which is a C_k-module together with an operator F of degree one such that $F^2 - I$ is compact and also F is a degree one homomorphism of C_k-modules. This means F anti-commutes with the γ^i.

If $k=0$, then we have a pair of Hilbert spaces $\mathcal{H}^0, \mathcal{H}^1$ and a Fredholm operator $P: \mathcal{H}^0 \longrightarrow \mathcal{H}^1$ and a parametrix $Q: \mathcal{H}^1 \longrightarrow \mathcal{H}^0$. One can fix the C_k-module $\mathcal{H}$ without affecting homotopy questions using Kuiper's thm. So the key point is the homotopy relation which is imposed on the Fredholm operators F in order to get K-classes. If F is an involution, then it can be deformed to a fixed such F; better, the space of involutions F is contractible by Kuiper's thm.

Thus the homotopy relation on the F's builds in the requirement that we should be working modulo C_{k+1}-modules.

Notice that we must pass to infinite dimensions in order that Kuiper's thm. holds. For example, if $k=0$, then the set of involutions F of odd degree on $V^0 \oplus V^1$ is the space of isomorphisms of V^0 with V^1, and this isn't contractible.

Let's summarize the above. I am looking at the K-theory of graded C_k-modules. If I take the Grothendieck group of f.g. graded C_k-modules, then I obtain the wrong answer. One way to correct this is to form a relative theory of C_k-modules modulo C_{k+1}-modules, as was done by ABS although this has some difficulties, namely [redacted] a class in the relative theory doesn't necessarily come from a C_k-module, so one [redacted] doesn't have representatives for the K-classes. The AS+Kasparov way is to allow Hilbert C_k-modules, but to put in the finiteness by means of an [redacted] odd degree C_k-endomorphism F such that F^2-I is compact. Then the homotopy equivalence relation gives the K-classes, and by Kuiper's thm. the invertible F's [redacted] represent zero.

In my super connection paper I discuss the situation for families of C_1-modules, i.e. a super vector bundle $E^0 \oplus E^1$ with an action of C_1. Then this [redacted] super C_1-bundle is the same as [redacted] the ordinary bundle E^0. The analogue of Kasparov's F is a self adjoint operator A on E^0. F is invertible where A is invertible. I construct [redacted] odd forms on the manifold supported near the set where F isn't invertible.

What I would like to [redacted] make precise is the feeling that I really don't care about what happens on the set where F is invertible. In fact I can completely forget that E^0 is given on the whole manifold. My idea is that there should be some category of objects which are roughly C_k-bundles with an F, except that we don't care about ones where F is invertible. Maybe these are the Hilbert C^*-modules of Kasparov.

In any case what seems clear at the moment is that the sort of objects I want are $t \to \infty$ limits of things such as the following: An open set U in M, a super C_k-bundle E over U, an odd self-adjoint endom. tL on E such that the subset where tL is not invertible is compact.

There are only two cases to consider because of the periodicity of the algebras C_k. For $k=0$ one has then over U two vector bundles E^0, E^1 and a self-adjoint $L = \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}$, where Z = singular set of u, is compact. So what we have over U is a map $u: E^0 \to E^1$ which is an isomorphism off Z.

In this case we can extend to the whole manifold as follows. If we add to u the identity map of a bundle F such that $E^1 \oplus F$ is trivial over U, then we may suppose E^1 extends to all of M. But on $U-Z$, E^0 is isomorphic to E^1 and so E^0, u extend over M. Thus we have realized excision

$$K(M, M-Z) \xrightarrow{\sim} K(U, U-Z)$$

explicitly on the level of difference elements: $[E^0 \xrightarrow{u} E^1]$.

For $k=1$ a C_1-bundle with L can be identified with a single v.b. E and a self adjoint endom. L. So we are given this picture over U where the singular set Z of L is compact. I would like to know whether this setup extends to M. There is an obstruction in this case which we can see in the following way.

We can add a bundle F to E such that $L = +1$ on F, and so suppose that E is trivial, hence extends to M.

Now take a [redacted] trivial bundle over M and a non-trivial unitary automorphism of it ?

The problem is the following: Given an open set U in M and a [redacted] bundle E [redacted] over U together with $L =$ a self-adjoint endo. of E whose singular set is compact. Then by adding to E a bundle with invertible self-adjoint endo I would like to see (E, L) the ~~restriction~~ to U of a $(\tilde{E}, \tilde{L})$ over M such that $\tilde{L}$ is invertible off U. We can assume E is trivial whence $\tilde{E}$ can be a trivial bundle. Let Z be the singular set of L so that Z is compact inside U, hence also inside M. Then we have a map from $U - Z$ into invertible self-adjoint matrices which we want to extend to a map [redacted] defined on $M - Z$. There are some obvious problems connected with the fact that U is open, so replace U by a compact nbd. of Z. [redacted] Then we have a familiar extension type problem where the space we are mapping into is invertible self-adjoint matrices. This has the homotopy type of the disjoint union of Grassmannians, because the eigenvalues can be pushed to ± 1.

Now I am allowed to add to (E, L) any invertible bundle over U with self-adjoint endom. This is how I can assume E is trivial. But now let us assign to (E, L) the element of $K_0($[redacted]$U - Z)$ represented by the difference of the positive and negative subbundles of E wrt L over $\underline{U - Z}$.

Repeat: Suppose we are given over $U \subset M$ a pair (E,L) consisting of a **vector** bundle and self-adjoint endomorphism L whose singular set Z is compact. Then I get an element of $K^0(U-Z)$ given by the difference of the positive and negative subbundles for L.

I allow myself to change (E,L) by adding an (E',L') over U where L' is invertible. This changes the element of $K^0(U-Z)$ by the image of an element of ~~[illegible]~~ $K^0(U)$.

Suppose (E,L) is the restriction to U of an $(\tilde{E},\tilde{L})$ over M where $\tilde{L}$ has the same singular set as L. Then clearly the element of $K^0(U-Z)$ lies in the image of $K^0(M-Z)$.

Thus my problem leads to an obstruction in the cokernel of

$$K^0(M-Z) \oplus K^0(U) \longrightarrow K^0(U-Z) \longrightarrow K^1(M)$$

which by the Mayer-Vietoris sequence ~~[illegible]~~ identifies the obstruction with an element of $K^1(M)$. Now actually the original data I thought would give me an element of

$$K^1(M,M-Z) \xrightarrow{\sim} K^1(U,U-Z)$$

which is consistent as one has a map $K^1(M,M-Z) \to K^1(M)$.

Mayer-Vietoris diagram

$$\begin{tikzcd}
 & K^0(M-Z) \arrow[r] \arrow[d] & K^1(M,M-Z) \arrow[r] \arrow[d, "\wr"] & K^1(M) \arrow[d] \\
K^0(U) \arrow[r] & K^0(U-Z) \arrow[r] & K^1(U,U-Z) \arrow[r] & K^1(U)
\end{tikzcd}$$

Now I started with an (E, L) over U with singular set Z. [redacted] The element of $K^1(U)$ corresponding to (E, L) is trivial, because L can be deformed to $+$ id.

So one seems to have the following geometry. Take an open covering $M = U \cup (M - Z)$ and a vector bundle over the intersection $U \cap (M - Z) = U - Z$. Such a thing determines an element of $K^1(M)$ because the suspension of [redacted] BU [redacted] embeds in U (the inf. unitary gp), by the Bott map. However certainly the suspension of BU is not h.eq to U so we don't get all elements of $K^1(M)$.

Now the standard way to realize an elt of $K^1(M)$, or really $K^{-1}(M)$, is via a map from M to the unitary group $U(N)$. [redacted] The set of unitary matrices without the eigenvalue 1 is contractible. So a map $f : M \longrightarrow U(N)$ with Z the subset of $x \in M \ni \det\{1 - f(x)\}$ [redacted] $= 0$ [redacted] represents an element of $K^{-1}(M, M - Z)$. Now [redacted] you would also like to find a [redacted] nbd U of Z such that over U none of the matrices $f(x)$ has the eigenvalue -1. If so then one can take $L =$ logarithm of U, and then we are in the situation already described.

So we see the problem at least. The finite diml situation doesn't really allow [redacted] one to ignore the eigenvalues at -1. Roughly I would like to forget eigenvalues $\neq 1$ which is what the Kuiper thm. permits.

Summarize what I've learned today:

I began with the problem of K-theory of the Clifford algebras C_k and C_k-bundles. In order to get sensible answers it is necessary to introduce an odd degree endom. F or L and to allow homotopies of this. This achieves the effect of working modulo C_{k+1}-modules.

The Kuiper thm. says that in infinite dimensions the invertible F carry no homotopy information.

I would like to be able to do the same with some kind of $t \to \infty$ limit.

Finally I analyzed elements of $K^1(M, M-Z)$ which come from (E,L) defined in a nbd U of Z (Z supposed compact) such that L is invertible off Z. These give elements of $K^1(M,M-Z) \simeq K^1(U,U-Z)$ which go to zero in $K^1(U)$, and hence come from a bundle over $U-Z$ via the Bott map.

Somehow I feel that the differential forms or limiting currents as $t \to \infty$ carry the essential information