Jan. 10, 1984 — Feb. 4, 1984

396–454

It seems that I have a new proof that the Chern character cohomology class is independent of the choice of connection.

Let $E^+ = E^-$ have two connections $D^\pm$ put

$$D = \begin{pmatrix} D^+ & \\ & D^- \end{pmatrix} = d + \begin{pmatrix} A^+ & \\ & A^- \end{pmatrix} \qquad \text{on} \qquad E^+ \oplus E^- = M \times (V^+ \oplus V^-)$$

$$L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad [D, L] = \begin{pmatrix} 0 & A^+ - A^- \\ A^- - A^+ & 0 \end{pmatrix} = B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $B = A^+ - A^-$

$$D^2 = \begin{pmatrix} (D^+)^2 & 0 \\ 0 & (D^-)^2 \end{pmatrix} = \begin{pmatrix} F^+ & \\ & F^- \end{pmatrix}.$$

We know that if $\Omega = D^2 + t[D,L] + t^2 L^2$, then

$$\frac{d}{dt} \operatorname{tr}_s(e^{\Omega}) = d \operatorname{tr}_s(e^{\Omega} L)$$

hence

$$\operatorname{tr}_s(e^{D^2}) = d\left\{ -\int_0^\infty dt \operatorname{tr}_s(e^{\Omega} L) \right\}$$

provided we integrate along a path along which $\operatorname{tr}_s(e^{\Omega})$ goes to zero. More precisely

$$e^{\Omega} = e^{D^2 + t[D,L] + t^2 L^2} \qquad L^2 = I$$

so we take the path in the complex plane to be the positive imaginary axis: 0 to $i\infty$.

Let's recall that e^{Ω} is computed in the algebra $\Omega(M) \hat\otimes \operatorname{End}(V)$, which acts as operators on $\Omega(M) \otimes V$. The natural way to write such operators is as matrix forms. To incorporate the signs we should write $\varepsilon_\Omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for L. Then

$$\mathrm{tr}_s\left(e^{\Omega}L\right) = \mathrm{tr}_V\left(\underbrace{e^{t^2+D^2+tB\varepsilon_\Omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}}\varepsilon_\Omega\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}}\right)$$

$$= e^{t^2}\,\mathrm{tr}_V\left(e^{F+tB\varepsilon_\Omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}}\varepsilon_\Omega\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)$$

Now we expand the exponential using the standard perturbation series

$$e^{F+G} = \sum_n \int_{1\geq t_1\geq\cdots\geq t_n\geq 0} dt_1\cdots dt_n\; e^{(1-t_1)F}\underset{tB\varepsilon_\Omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}}{G}\, e^{(t_1-t_2)F}G\cdots G e^{t_nF}$$

Bring all the $\varepsilon_\Omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ to the right and use that

$$\varepsilon_\Omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}\varepsilon_\Omega\begin{pmatrix}0&1\\-1&0\end{pmatrix} = -\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

together with the fact that ε_Ω anti-commutes with B. Only an odd number of B's will have a non-zero trace. Then

$$\mathrm{tr}_V\left(e^{F+tB\varepsilon_\Omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}}\varepsilon_\Omega\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right) = \sum_{n\ \text{odd}}\int_{1\geq t_1\geq\cdots\geq t_n\geq 0} dt_1\cdots dt_n\,\mathrm{tr}\Big\{e^{(1-t_1)\begin{pmatrix}F^+&\\&F^-\end{pmatrix}}tB$$

$$\times\, e^{(t_1-t_2)\begin{pmatrix}F^-&\\&F^+\end{pmatrix}}tB\cdots tBe^{t_n\begin{pmatrix}F^-&\\&F^+\end{pmatrix}}\Big\}$$

It seems that this is the same as

$$\mathrm{tr}_V\left(e^{F+tB\begin{pmatrix}0&1\\1&0\end{pmatrix}}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \mathrm{tr}_V\left(e^{\begin{pmatrix}F^+&tB\\tB&F^-\end{pmatrix}}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$$

Thus we reach the following formula for the difference of character forms computed with respect to two connections:

$$\boxed{\operatorname{tr}\left(e^{F^+}\right) - \operatorname{tr}\left(e^{F^-}\right) = d\left\{ -\int_0^{i\infty} dt\, e^{t^2} \operatorname{tr}\left(e^{\begin{pmatrix} F^+ & tB \\ tB & F^- \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right\}}$$

This should be contrasted with the old formula

$$\operatorname{tr}\left(e^{F^+}\right) - \operatorname{tr}\left(e^{F^-}\right) = d\left\{ \int_0^1 dt \operatorname{tr}\left(e^{(1-t)F^- + tF^+ + (t^2-t)B^2} B \right) \right\}$$

I think these formulas are different, i.e. the forms in braces are different in general. For example to first order in B we have

$$\operatorname{tr} e^{\begin{pmatrix} F^+ & tB \\ tB & F^- \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \int_0^1 ds \operatorname{tr}\left[e^{(1-s)\begin{pmatrix} F^+ & 0 \\ 0 & F^- \end{pmatrix}} tB\, e^{s\begin{pmatrix} F^- & 0 \\ 0 & F^+ \end{pmatrix}} \right]$$

$$= \int_0^1 ds \operatorname{tr}\left[e^{(1-s)F^+} tB\, e^{sF^-} + e^{(1-s)F^-} tB\, e^{sF^+} \right]$$

$$= t \int_0^1 ds \operatorname{tr}\left[\left(e^{sF^-} e^{(1-s)F^+} + e^{(1-s)F^+} e^{sF^-} \right) B \right]$$

Now
$$-\int_0^{i\infty} dt\, e^{t^2} t = -\frac{1}{2}\int_0^{-\infty} du\, e^{u} = \frac{1}{2}\int_0^{\infty} e^{-u} du = \frac{1}{2}.$$

So to first order in B

$$-\int_0^{\infty} dt\, e^{t^2} \operatorname{tr}\left[e^{\begin{pmatrix} F^+ & tB \\ tB & F^- \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \int_0^1 ds \operatorname{tr}\left[\frac{\{e^{sF^-}, e^{(1-s)F^+}\}}{2} B \right]$$

whereas the Chern-Simon form gives

$$\int_0^1 dt \operatorname{tr}\left(e^{(1-t)F^- + tF^+} B \right)$$

to first order in B. (Note that at a given pt. B

can be small, even zero, while $F^+ - F^-$ is large.)

Example: Consider the case where $F^+ = F^- = 0$, for example when we have a flat connection on the trivial bundle. Then

$$\operatorname{tr} e^{\begin{pmatrix} 0 & tB \\ tB & 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sum_{m \text{ odd}} \frac{t^m}{m!} \, 2 \operatorname{tr}(B^m)$$

$$-\int_0^{i\infty} dt \; e^{t^2} \, t^{2n+1} \, 2 = -\int_0^{-\infty} du \; e^u u^n = \int_0^{\infty} du \, e^{-u} (-u)^n$$

$$= (-1)^n n!$$

so

$$-\int_0^{i\infty} dt \; e^{t^2} \operatorname{tr} e^{\begin{pmatrix} 0 & tB \\ tB & 0 \end{pmatrix}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(2n+1)!} \operatorname{tr}(B^{2n+1}).$$

The Chern-Simons form is

$$\int_0^1 dt \operatorname{tr}\left(e^{(t^2-t)B^2} B\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{tr}\left(B^{2n+1}\right) \underbrace{\int_0^1 dt \, (t^2-t)^n}_{\substack{(-1)^n t^n (1-t)^n \\ \frac{n!\,n!}{(2n+1)!}}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(2n+1)!} \operatorname{tr}\left(B^{2n+1}\right).$$

<u>Example</u>: Consider over $\mathbb{R}^n$ n even the virtual bundle with compact support given by the spinors S. Thus we have the trivial graded bundle over $\mathbb{R}^n$ with fibre $S = S_+ \oplus S_-$ and the odd degree endomorphism

$$L = \gamma^\mu x^\mu \qquad \text{at} \qquad x = (x^\mu) \in \mathbb{R}^n.$$

Take $D = d$. Then $D^2 = 0$, $L^2 = x^2$

$$[D, L] = \text{[redacted]}\; dx^\nu \, [\partial_\nu, \gamma^\mu x^\mu] = dx^\mu \gamma^\mu$$

so the character form is

$$tr_s\left(e^{u(D^2 + t[D,L] + t^2L^2)}\right) = e^{ut^2x^2}\, tr_s\left(e^{ut\, dx^\mu \gamma^\mu}\right)$$

$$= e^{ut^2x^2} (ut)^n \, tr_s\left(dx^1\gamma^1 \, \underbrace{dx^2\underbrace{\gamma^2 \cdots dx^n\gamma^n}}\right)$$

$$= e^{ut^2x^2} (ut)^n \, dx^1 \cdots dx^n \, \underbrace{tr_s(\gamma^n \cdots \gamma^1)}_{(2i)^{n/2}(-1)^{\frac{n(n-1)}{2}}}$$

Change $t \longmapsto \frac{it}{\sqrt{u}}$, the above becomes

$$= e^{-t^2x^2} \, t^n \left(\frac{-u}{\text{[redacted]}}\right)^{n/2} (2i)^{n/2} \; dx^1 \cdots dx^n \, (-1)^{\frac{n(n-1)}{2}}$$

$$\frac{i}{2\pi}$$

$$= e^{-t^2x^2} \, t^n \, \frac{dx^1 \cdots dx^n}{\pi^{n/2}} \, (-1)^{\frac{n(n-1)}{2}}$$

For $n = 2$ we have

$$L = \gamma^1 x + \gamma^2 y = \begin{pmatrix} 0 & x - iy \\ x + iy & 0 \end{pmatrix}$$

so we have the K-element corresponding to the complex

$$\mathbb{1} \xrightarrow{z} \mathbb{1}.$$

This looks like it should have degree -1 on $\mathbb{C}$ as an element of K. This should explain the factor $(-1)^{\frac{n(n-1)}{2}}$.

One of the things I should go over carefully is the limit process where t goes to infinity. The problem is that we can only show that a non-homogeneous expression goes to zero. So let me fix a number u and consider the DR class of the form

$$tr_s\left(e^{u(D^2 + t[D,L] + t^2L^2)}\right)$$

This is an even form, and I know its DR class is independent of t, so on letting $t \to 0$ I see it represents the character of E^+ − the character of E^-. On the other hand if $L^2 > 0$, then by letting $t \longrightarrow \infty$ in such a way that $ut^2 \to -\infty$, we can conclude that the form goes to zero.

This last step requires some estimates. Let's work in a normed algebra so that we have inequalities like

$$\|A+B\| \leq \|A\| + \|B\|$$
$$\|AB\| \leq \|A\| \cdot \|B\|.$$

Then use

$$e^{A+B} = e^A + \int_0^1 dt\, e^{(1-t)A} B e^{tA} + \int_0^1 dt_1 \int_0^{t_1} dt_2\, e^{(1-t_1)A} B \times e^{(t_1-t_2)A} B e^{t_2 A}$$
$$+ \cdots$$

and assume known that

$$\|e^{tA}\| \leq e^{ct} \qquad \text{for } t \geq 0 \qquad c \in \mathbb{R}.$$

and we get

$$*\left\{\begin{aligned} \|e^{A+B}\| &\leq \|e^{A}\| + \int_0^1 dt\, \|e^{(1-t)A}\|\, \|B\|\, \|e^{tA}\| + \cdots\cdots \\ &\leq e^{c} + \int_0^1 dt\; e^{c}\|B\| + \int_0^1 dt_1 \int_0^{t_1} dt_2\; e^{c}\|B\|^2 + \cdots\cdots \\ &= e^{c}\left(1 + \|B\| + \frac{\|B\|^2}{2} + \cdots\right) = e^{c} e^{\|B\|}. \end{aligned}\right.$$

Now we apply this to the operator

$$e^{u(D^2 + t[D,L] + t^2L^2)}$$

where $ut^2 \longrightarrow -\infty$. Put $A = -L^2$. If $L^2 > 0$ is hermitian, then by using eigenvectors we can see that

$$\|e^{-tL^2}\| \leq e^{-\lambda t} \qquad \text{for } t \geq 0, \text{ with } \lambda > 0.$$

So the above estimate will give us

$$\left\| e^{uD^2 + ut[D,L] + (-ut^2)(-L^2)} \right\| \leq e^{-\lambda(-ut^2)}\, e^{\|uD^2 + ut[D,L]\|}$$

$$\leq e^{-\lambda(-ut^2) + \|uD^2\| + |t|\,\|u[D,L]\|}$$

and as $t \to \infty$ the quadratic term swamps the rest.

More precise & rigorous version of $*$. Use Duhamel formula

$$e^{t(A+B)} = e^{tA} + \int_0^t ds\; e^{(t-s)A} B\, e^{s(A+B)}$$

whence we get the inequality

$$\|e^{t(A+B)}\| \leq \|e^{tA}\| + \int_0^t ds \|e^{(t-s)A}\| \|B\| \|e^{s(A+B)}\|$$

$$\leq e^{ct} + \int_0^t ds\, e^{ct-cs} \|B\| \|e^{s(A+B)}\|.$$

So if we put $f(t) = e^{-ct}\|e^{t(A+B)}\|$ we have

$$f(t) \leq 1 + \|B\| \int_0^t ds\, f(s)$$

Now iterate this inequality:

$$f(t) \leq 1 + \|B\| \int_0^t dt_1 \left(1 + \|B\| \int_0^{t_1} dt_2\, f(t_2)\right)$$

$$\leq 1 + \|B\|t + \|B\|^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, f(t_2)$$

$$\leq 1 + \|B\|t + \frac{\|B\|^2}{2} t^2 + \|B\|^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3\, f(t_3).$$

Thus as in Picard's theorem one finds that if $|f(t)| \leq M$ on ~~the~~ an interval, then in fact

$$|f(t)| \leq e^{\|B\|t}.$$

So again one gets

$$\|e^{t(A+B)}\| \leq e^{+ct} e^{\|B\|t}$$

~~The only point left over is whether in finite dimensions one has~~

~~Remark:~~

January 11, 1984

On p. 367 the signs aren't quite correct. Let D be a connection on $E = E^0 \oplus E^1$, a $\mathbb{Z}_2$-graded bundle. We extend D to

$$\Omega(M,E) = \Omega^{\bullet}(M) \otimes_{\Omega^0(M)} \Omega^0(M,E)$$

so that

$$D(\omega\alpha) = d\omega \cdot \alpha + (-1)^{\deg\omega}\, \omega \cdot D\alpha .$$

However we can also regard $\Omega(M,E)$ as a right $\Omega(M)$-module by defining

$$\alpha \cdot \omega = (-1)^{\deg\alpha \deg\omega}\, \omega\alpha$$

Here $\deg\alpha$ is the total $\mathbb{Z}_2$ degree on $\Omega(M,E)$.

Then

$$D(\alpha\omega) = (-1)^{\deg\alpha\deg\omega}\, D(\omega\alpha)$$

$$= (-1)^{\deg\alpha\deg\omega}\left(d\omega\cdot\alpha + (-1)^{\deg\omega}\,\omega\, D\alpha\right)$$

$$= (-1)^{\deg\alpha\deg\omega}\left[(-1)^{\deg\alpha(\deg\omega+1)}\alpha\cdot d\omega + (-1)^{\cancel{\deg\omega}}(-1)^{(\deg\alpha\cancel{+1})\deg\omega}\, D\alpha\cdot\omega\right]$$

$$= (-1)^{\deg\alpha}\,\alpha\cdot d\omega + D\alpha\cdot\omega$$

so that

$$\boxed{D(\alpha\omega) = D\alpha\cdot\omega + (-1)^{\deg\alpha}\,\alpha\cdot d\omega}$$

Conclude: The difference of two connections is a right $\Omega(M)$-homomorphism of $\Omega(M,E)$ of degree 1. I think I can conclude that we have an algebra isomorphism

$$\boxed{\Omega(M)\,\hat{\otimes}_{\Omega^0(M)}\,\Omega^0(M,\operatorname{End}E) \xrightarrow{\sim} \operatorname{End}_{\Omega(M)^{op}}(\Omega(M,E))}$$

Check that the map is well-defined. First given

$X \in \Omega^0(M, \mathrm{End}E)$ I want to check it is a right $\Omega(M)$-module map

$$X(\alpha\omega) = (-1)^{\deg\alpha\deg\omega} X(\omega\alpha)$$

$$= (-1)^{\deg\alpha\deg\omega}(-1)^{\deg\omega\deg X}\,\omega X\alpha$$

$$= X\alpha\cdot\omega$$

Given $\omega \in \Omega(M)$ I want to check that left mult by ω is a right $\Omega(M)$-module map:

$$\omega(\alpha\eta) = (-1)^{\deg\alpha\deg\eta}\,\omega\eta\alpha$$

$$= (-1)^{\deg\alpha\deg\eta + (-1)\deg\omega\deg\eta}\,\eta\omega\alpha$$

$$= (\omega\alpha)\eta$$

Finally I want to compare the super-trace on $\mathrm{End}_{\Omega(M)^{op}}(\Omega(M,E))$ with the way I have defined it on $\Omega(M)\hat{\otimes}_{\Omega^0(M)}\Omega^0(M,\mathrm{End}E)$. So take an endo of the form $e\otimes\omega\lambda$, $e\in E, \omega\in\Omega, \lambda\in E^*$. Its supertrace as an endo is $(-1)^{\deg e(\deg\omega+\deg\lambda)}\,\omega\lambda(e)$. It comes from $(-1)^{\deg\omega\cdot\deg e}\,\omega\otimes e\cdot\lambda \in \Omega(M)\hat{\otimes}\Omega^0(M,\mathrm{End}E)$, and I calculate the super-trace here as

$$(-1)^{\deg\omega\cdot\deg e}\,\omega\,(-1)^{\deg e\deg\lambda}\,\lambda(e).$$

Thus things check.

January 13, 1984

If A is an element of a Banach algebra, then

$$e^{tA} = \frac{1}{2\pi i} \oint_C e^{t\lambda} (\lambda - A)^{-1} d\lambda$$

where the contour C encloses $Sp(A)$. So if $Re(\lambda) < a$ for all $\lambda \in Sp(A)$ we can choose C so that

$$|e^{t\lambda}| = e^{t\, Re(\lambda)} \leq e^{ta} \qquad \begin{matrix} t \geq 0 \\ \lambda \in C. \end{matrix}$$

Then

$$\|e^{tA}\| \leq \frac{1}{2\pi} \int_C e^{ta} \|(\lambda - A)^{-1}\| |d\lambda|$$

$$\leq \text{Const } e^{ta}, \qquad t \geq 0.$$

Examples such as $A = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ with α large and a small > 0, show that the Constant cannot be replaced by one.

Suppose now that we have

$$\|e^{tA}\| \leq M e^{ta} \qquad t \geq 0.$$

Then consider the Dyson expansion

$$e^{t(A+B)} = e^{tA} + \int_0^t dt_1\, e^{(t-t_1)A} B e^{t_1 A} + \cdots$$

$$+ \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, e^{(t-t_1)A} B \cdots B e^{t_n A} + \cdots$$

Then

$$\|e^{t(A+B)}\| \leq M e^{ta} + \cdots + \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, M e^{(t-t_1)a} \|B\| \cdots \|B\| e^{t_n a} + \cdots$$

$$\leq M e^{ta} \left(1 + \underline{tM\|B\| + } \frac{t^2}{2!} M^2 \|B\|^2 + \cdots \right) = M e^{ta} e^{tM\|B\|}$$

Next: I want to consider the case where A is an endomorphism of a vector bundle, hence is a section of the algebra $\Omega^0(M, \operatorname{End} E)$. This space has a C^k topology and can be completed to be a Banach space. I want to see that it actual has a C^k norm which makes it into a Banach algebra. For then, if I know that the spectrum of A at each point is contained in $\operatorname{Re}(\lambda) < a$, I can conclude that

$$\|e^{tA}\| \leq \text{Const}\, e^{ta} \qquad t \geq 0.$$

It is clear that I can define some sort of k-norm which will satisfy

$$(*) \quad |xy| \leq C|x||y| \qquad x, y \in \mathcal{A} = \Omega^k(M, \operatorname{End} E)$$

and the point will be to modify the norm so as to obtain a normed algebra. Obvious choice is the norm of left multiplication

$$\|x\| = \sup_y \left\{ \frac{|xy|}{|y|} \right\}$$

which is just the pull-back of the norm on the algebra $\mathcal{L}(\mathcal{A}, \mathcal{A})$ of bounded operators on $\mathcal{A}$. Hence it will satisfy

$$\|xy\| \leq \|x\| \, \|y\|.$$

On the other hand from $(*)$ above we have

$$\|x\| < C|x|$$

and using the $1 \in \mathcal{A}$, we hav

$$|x| = |x1| \leq \|x\| \cdot |1|$$

so that we conclude the norms $|x|$, $\|x\|$ are equivalent.

January 15, 1984:

Odd case. Let C_1 be the first Clifford algebra. It is super: $C_1 = \mathbb{C} \oplus \mathbb{C}\alpha$, where $\alpha^2 = 1$. Hence it is not supercommutative, although it is commutative.

For the odd case we want to consider super vector bundles $E = E^0 \oplus E^1$ which are C_1-modules. This means that multiplication by α is an isomorphism of E^0 with E^1. The first problem is whether E should be a left C_1-module or a right C_1-module. At this point it makes no difference as in either case we have an isomorphism of E^0 with E^1.

It makes a difference when we try to describe endomorphisms. Suppose $V = V^0 \oplus V^1$ is a graded C_1-module. Multiplication by α is an isomorphism of V^0 with V^1. Let T be an odd degree endo. of V. If V is a right C_1-module we have $T\alpha = \alpha T$, but if V is a left C_1-module, then $T\alpha = -\alpha T$.

In the interests of operators I should first work out the theory for right C_1-modules. We can write
$$V = V^0 \otimes C_1$$
and then
$$V^0 \otimes C_1 \otimes (V^0)^* \xrightarrow{\sim} \mathrm{End}_{C_1^{op}}(V^0 \otimes C_1).$$
This is really
$$V \otimes_{C_1} \underbrace{\mathrm{Hom}_{C_1^{op}}(V, C_1)}_{V^*} \xrightarrow{\sim} \mathrm{End}_{C_1^{op}}(V)$$
so to get the super-trace we use the map
$$\begin{array}{ccccc} V \otimes V^* & \longrightarrow & V^* \otimes V & \longrightarrow & C_1 \\ v \otimes \lambda & \longmapsto & (-1)^{\deg v \deg \lambda} \lambda \otimes v & \longrightarrow & (-1)^{\deg v \deg \lambda} \lambda(v) \end{array}$$

and then we have to divide out by $[C_1, C_1]$ to get something well-defined.

Notice: the category of graded C_1-modules is equivalent to the category of ungraded C_0-modules, i.e. vector spaces. However the graded C_1-modules have a richer Hom of some sort. What does this mean?

It seems that I should concentrate on right super C_1-modules. The super algebra C_1 is non-commutative, so you have to keep right and left straight. Then a super C_1-module is of the form

$$V = V^0 + V^1 = V^0 \otimes C_1$$

and if we identify V^0 and V^1 via α, then endos with commute with α. So $\mathrm{End}_{C_1^{op}}(V)$ consists of operators

$$\underset{\text{even}}{\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}} \qquad\qquad \underset{\text{odd}}{\begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}}$$

with $u, v \in \mathrm{End}(V^0)$. Examples of super commutators are

$$\left[\begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix}, \begin{pmatrix} u_2 & 0 \\ 0 & u_2 \end{pmatrix}\right] = \begin{pmatrix} [u_1, u_2] & 0 \\ 0 & [u_1, u_2] \end{pmatrix}$$

$$\left[\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & [u, v] \\ [u, v] & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 0 & v_1 \\ v_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & v_2 \\ v_2 & 0 \end{pmatrix}\right]_+ = \begin{pmatrix} v_1 v_2 + v_2 v_1 & 0 \\ 0 & v_1 v_2 + v_2 v_1 \end{pmatrix}$$

so the only possible super trace is $\begin{pmatrix} u & v \\ v & u \end{pmatrix} \longmapsto \mathrm{tr}\, v$

Now the idea is that when we come to a super C_1-vector bundle $E = E^0 \oplus E^1$, the super algebra $\Omega(M)$ being commutative, it does not matter whether we consider

$$\Omega(M,E) = \Omega(M) \otimes_{\Omega^0(M)} \Omega^0(M,E)$$

as a right or left super $\Omega(M)$ module. But we still want C_1 to stay on the right.

<u>January 16, 1984</u>

Some points to discuss in the odd case.

Consider a graded C_1-bundle $E = E^0 \oplus E^1$ over M. I think of C_1 as acting on the right. Then when I compute endomorphisms I consider endos. commuting with α. So then when I consider connections I won't violate the sign rule. More precisely if I start with a connection on E commuting with α, so that it is the same on E^0 and E^1 under the isomorphism given by α, then we extended to $\Omega(M,E)$ in the obvious way we have D commuting with α. Writing α on the right, this becomes

$$D(\eta\alpha) = (D\eta)\alpha \qquad \eta \in \Omega(M,E)$$

and there is no violation of the sign rule.

Then if we identify E^0, E^1, and endomorphism of E as C_1-module is of the form

$$\begin{pmatrix} u & v \\ v & u \end{pmatrix}$$

with $u, v \in \operatorname{End}(E^0)$. In particular we get

$$L = i\begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}$$

where $v = v^*$, the same sort of formula encountered in the even case.

Thus what we have essentially is a single vector bundle E^0 with a connection D^0 and self-adjoint endomorphism u. Such data determine an odd K-class on M with supports in the singular set of u. In effect we associate to u the bdle automorphism $e^{[redacted]iu}$. Over the open set where u is invertible we have a definite deformation of this automorphism to the identity, namely we [redacted] push the eigenvalues of u to $\pm 2\pi$ without crossing zero. So this defines an odd K-class on M modulo the open set where u is invertible.

Next I want to work through the super-connection business. The point will be that the calculations in the odd case are exactly the same except they take place in a smaller algebra, namely

[redacted]

$$\Omega(M) \underset{\Omega^0(M)}{\otimes} \Omega^0(M, \mathrm{End}_\alpha(E)) = \Omega(M, \mathrm{End}_\alpha E)$$

and this time the super trace [redacted]

$$tr_s : \Omega(M, \mathrm{End}_\alpha E) \longrightarrow \Omega(M)$$

is of odd degree.

<u>January 17, 1984</u>

The odd case: Given a vector bundle F with metric I get a super vector bundle

$$E = F \otimes C_1 \qquad C_1 = \mathbb{C} \oplus \mathbb{C}\gamma$$

with right C_1 action. We've already understood the algebra

$$\Omega(M, \operatorname{End} E) = \Omega(M) \underset{\Omega^0(M)}{\hat{\otimes}} \Omega^0(M, \operatorname{End} E)$$

as an algebra of operators on $\Omega(M,E)$. Now I want to consider the subalgebra

$$\mathcal{A}_\gamma = \Omega(M, \operatorname{End}_\gamma E) = \Omega(M) \underset{\Omega^0(M)}{\hat{\otimes}} \Omega^0(M, \operatorname{End}_\gamma E)$$

of those operators which commute with right mult. with γ. It's clear that

$$\operatorname{End}_\gamma E = \operatorname{End} F \otimes C_1$$

so that

$$\mathcal{A}_\gamma = \Omega(M, \operatorname{End}_\gamma E) = \Omega(M, \operatorname{End} F) \hat{\otimes} C_1$$

In other words we can represent the operators in $\mathcal{A}_\gamma$, or elements in $\mathcal{A}_\gamma$ in the form

$$\alpha + \beta j$$

where $\alpha, \beta \in \Omega(M, F)$ and $j = 1 \otimes \gamma$ in . Thus j commutes with even α and anti-commutes with odd α.

So now consider super connections on E commuting with γ. The curvature + difference of two superconnections is in $\mathcal{A}_\gamma$.

The supertrace on $\mathcal{A}$ vanishes on $\mathcal{A}_\gamma$; the appropriate super trace on $\mathcal{A}_\gamma$ is the map

$$\mathcal{A}_\gamma = \Omega(M, \mathrm{End}_\gamma E) = \Omega(M, \mathrm{End} F)\,\hat{\otimes}\, C_1 \longrightarrow \Omega(M) \otimes C_1/[C_1,C_1]$$
$$\alpha + \beta j \longmapsto \mathrm{tr}(\beta)\,[redacted]\,\gamma$$

So when we take the curvature $\tilde{D}^2$ of a superconnection and [redacted] form $\mathrm{tr}_s\, e^{\tilde{D}^2}$ using this trace, we get an odd form on M. $\quad L = i\begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}$

Take now $\tilde{D} = D + L = d + A + Bj$ on $\Omega(M,F) \otimes C_1$. D is connection on F. Then curvature is

$$\tilde{D}^2 = \underbrace{\left(D^2 + B^2\right)}_{\alpha} + \underbrace{[D,B]}_{\beta}\, j$$

where α is even, hence commutes with j, and β is odd, hence anti-commutes with j. Now in

$$e^{\alpha+\beta j} = e^{\alpha} + \int_0^1 dt\, e^{(1-t)\alpha}\beta j\, e^{t\alpha} + \int_0^1 dt_1 \int_0^{t_1} dt_2\, e^{(1-t_1)\alpha}\beta j\, e^{(t_1-t_2)\alpha}\beta j\, e^{t_2\alpha} + \cdots$$

we move all the j's to the right. The signs are

$$\begin{aligned} &\beta j \\ &(\beta j)^2 = -\beta^2 \\ &(\beta j)^3 = -\beta^3 j \\ &(\beta j)^4 = -\beta^3 j \beta j = \beta^4 \end{aligned}$$

It's clear then that

$$\mathrm{tr}_s\left(e^{\alpha+\beta j}\right) = \frac{1}{i}\left(\text{odd part of } \mathrm{tr}\, e^{\alpha + i\beta}\right)\gamma$$

$$tr_s(e^{\alpha+\beta j}) = \frac{1}{2i}\left(tr\, e^{\alpha+i\beta} - tr\, e^{\alpha-i\beta}\right)\bar{\gamma}$$

Let's check this out

$$tr_s\left(e^{(D+tB_j)^2}\right) = tr_s\left(e^{t^2B^2 + t[D,B]j + D^2}\right)$$

$$= \frac{1}{2i}\, tr\left(e^{t^2B^2 + t[D,iB] + D^2} - e^{t^2B^2 - t[D,iB] + D^2}\right)$$

So take $L = i\begin{pmatrix} 0 & u \\ u & 0 \end{pmatrix}$, or $B = iu$ and this becomes

$$\frac{1}{2i}\, tr\left(e^{-t^2u^2 - t[D,u] + D^2} - e^{-t^2u^2 + t[D,u] + D^2}\right)$$

which up to a normalization constant is just

odd degree part of $tr\left(e^{-t^2u^2 + t[D,u] + D^2}\right)$

Here D is a connection and u is a self-adjoint operator.

January 18, 1984

Conversation with Connes: Topics

trick to get from $F^2 \equiv I \pmod{\mathcal{K}}$ to $F^2 = I$.

Kasparov cup product, positivity condition, convexity (with Skandalis)

bordism (with [redacted] Baum) represents $R(G)$ not $K_\circ(BG)$.

description of BG, G a semi-simple Lie gp., G/K is the final [redacted] object for proper G-actions, barycenters.

pseudo diffl operators + blow up of $V \times V$ along Δ

physics: space-time doesn't exist, there is a groupoid of charts, why [redacted] does structure group reduce to $SO(3,1)$, C^*-alg. cosheaf over space time

coaction of a Lie group on a C^* algebra and the non-commutative picture of principal bundles, how to see the C^*-alg in physics are real.

[redacted] YM fncl. on $\mathcal{a}$ like $d(x,0)$ on G/K.

Gelfand-Fuks + cyclic cocycles, interpretation of Godbillon-Vey.

Actually masses should be points for the groupoid, i.e. independent of the observer.

Let me see how much I can reconstruct.

Consider his trick for replacing a Fredholm operator P by an invertible one, in order to obtain a cocycle. This involves the ring $A \oplus A\varepsilon$ with $\varepsilon^2 = 0$. Somehow he now does this ~~[illegible]~~ for $A = \mathbb{C}$ and then takes cup product. The idea is to consider the operator of multiplication by a real number λ on $\mathbb{C}$. Then upon cup product one has something like $P \otimes 1 + 1 \otimes \lambda$ whose square is

$$(P \otimes 1 + 1 \otimes \lambda)^2 = P^2 + \lambda^2 \geqslant \lambda^2$$

hence it is invertible. Now one can compute the cyclic cocycle using the invertible operator. Since the class is independent of λ, ~~[illegible]~~ and is a (polynomial?) function in λ, one gets the same class by letting $\lambda = 0$, or ∞.

Kasparov cup product: When constructing the Kasparov product one must make choices. It is important that the choices lies in a contractible set, e.g. a convex set, so that the result is unique up to homotopy.

An example of a Kasparov product is furnished by tensoring an operator P with a vector bundle. If the v.b. is described by an idempotent e, then ~~[illegible]~~ I have seen that the interaction of e and P is more or less the same as a connection D. Therefore the Kasparov product is something like the sum $D \otimes 1 + 1 \otimes P$.

Connes and Skandalis characterize when an operator like $D\otimes 1 + 1\otimes P$ represents the Kasparov product. The conditions are that the operator have a derivation (or connection) property, and also some sort of positivity property. Note that a choice of connection + positive operator are convex choices.

Their result describes the Dirac operator with coefficients in a complex of v.b.'s as the Kasparov product of the Dirac K-homology class with the K-cohomology class represented by the complex of vector bundles. So given a total Dirac operator in a fiber bundle $X \longrightarrow M$, one checks it has the derivation property and positivity property, so is the Kasparov cup product of the elements:

$$\underbrace{KK(\mathbb{C}(X), C(M))}_{\substack{\psi \\ \text{family of Diracs} \\ \text{on the fibres}}} \times \underbrace{KK(C(M), \mathbb{C})}_{\substack{\psi \\ \text{Dirac on } M \\ \text{K-homology class}}} \longrightarrow \underbrace{KK(\mathbb{C}(X), \mathbb{C})}_{\substack{\psi \\ \text{Dirac on } X}}.$$

Pseudo diffl operators + $V\times V$ blown up along ΔV.

~~We form a topological groupoid. The objects are triples (t,x,y) with $t\in[0,1]$ and $(x,y)\in V\times V$ and pairs $(0,v)$ with v a tangent vector in V.~~

We form a topological groupoid. The objects are pairs (t,x) with $t\in[0,1]$, $x\in V$. For $t>0$ there is a unique morphism from (t,x) to (t,y), and for $t=0$ there is one morphism from $(0,x)$ to $(0,x)$ given by each tangent vector at x. This is a smooth groupoid and has an associated convolution algebra which can be completed to a C^*-algebra.

We have two subgroupoids given by $t=0$, $t=1$.

The groupoid of $(0, x)$, $x \in V$ has for convolution algebra the smooth families over V of [redacted] convolution algebras in the tangent spaces. By Fourier transform this algebra is isomorphic to functions on T_V^* vanishing at ∞.

Let us now consider the restriction map from the C^*-algebra associated to the full groupoid to the $t=0$ subgroupoid. The kernel consists of families with parameter t, vanishing at $t=0$, of ordinary smooth kernels on $V \times V$. The K-theory is the same as that of $[0,1]$ mod $\{0\}$, and this is trivial.

I should have mentioned that the algebra for $t=1$ or for any $t>0$ is just the algebra of kernels on $V \times V$ under convolution, hence its K-theory is that of a point.

Thus we get

$$K^*(\text{fns. on } T^*) \xleftarrow{\sim} K^*(V \times V, \Delta V_{\text{alg}}) \longrightarrow K^*(pt)$$

which is just the index map defined analytically. In effect defining a ψDO from a symbol involves choosing a section of the restriction of the algebra of the $(V \times V, \Delta V)$ groupoid to the T_V groupoid.

This construction is needed when there is a [redacted] gp. acting

Connes would like to lift all the C^*-alg. stuff back to (smooth ?) geometry. Central to this aim is a conjecture to the effect that the K-cohomology of the C^*-algebra is the same as the K-homology of the classifying space, at least up to some orientation business. At the moment he doesn't have a perfect description of the classifying space K-homology, only his work with Baum.

The first point is that K-homology classes of BG are compactly supported, hence are realized by compact things mapping in. A second point is that one doesn't want just free actions. For example, when G is compact a map $M \to BG$ is a principal G-bundle over M, so one is getting the K-theory of BG which is a big limit (direct for homology, inverse for coh.). For the G-index thm. one wants arbitrary G-actions on manifolds & equivariant bundles.

A fact is that if one asks for a way to assign an element of an abelian gp to a compact G-manifold & bundle, compatible with Gysin maps, then one gets $R(G)$. Here compatible with Gysin maps is easier to work with, but equivariant to, bordism.

For G a Lie gp. one considers proper G-actions with compact quotient. Up to homotopy there is a unique map to G/K. This explains the importance of the Dirac operator on G/K as a means to go from $R(K)$ to operators (?), and lets to the conjecture

$$R(K) \xrightarrow{\sim} K(C^*_{red}(G))$$

proved by Pennington + Wasserman.

Connes says that on a 1-conn. Riemannian manifold, if the curvature is ≤ 0, then the weighted sum

$$f(x) = \sum t_i \, d(x, x_i)$$

has a unique critical point, so defines barycentric coords.

All this uses is convexity of the function $x \mapsto d(x,y)$ along geodesics.

Physics: The standard C^*-alg approach to quantum field theory says one should have a cosheaf of C^* algebras over space-time, such that two open sets which are space-like separated have commuting algebras. Finally there should be some global Lorentz invariance.

Connes objects to the latter as being inconsistent with general relativity. His viewpoint is that there is no space-time in the same way that there is no quotient to a foliation in general. What one has is observers; each observer has a chart, hence his C^*-algebra. Two observers that can relate ~~to~~ each other have overlapping charts. Thus there is a smooth groupoid. (Notice the connection with Kaluza-Klein.)

Why is the local ~~structural~~ group $SO(3,1)$? There perhaps is a ergodic type obstruction to reducing the group further, and this ergodic stuff should be essential for the existence of thermodynamics.

I asked him how one sees that the C^*-alg on space-time is actually comes from a real three dimensional time-like section. He said that there is a coaction of $SO(3)$ on the C^*-algebra.

Connes proof of the index theorem.

M manifold, consider Clifford algebra of T_M to define a K-homology class [redacted] in this algebra. operator $D = d + \delta$ on forms on M, tensors with complex $\mathbb{C} \xrightarrow{\lambda} \mathbb{C}$ where $\mathbb{C}$ acts as zero on the second. This gives $D \otimes 1 + 1 \otimes \lambda$ which is invertible; form cocycle let $\lambda \to 0$ (or ∞). Evaluating cocycle in the limit involves asymptotic diffl operators; local on M; Getzler proof is simpler with same result. Works equivariantly, important that one has only to compute cocycle in this canonical $d+\delta$ situation. Morita equivalence of $\mathcal{A}$ with endos. of a vector bundle E involves character of E; this introduces $\hat{A}$ somehow.

Jan 19, 1984 Hodge $*$ operator is Clifford multiplication with the orientation.

Let's go over Connes' proof of the index theorem. One idea which seems to be very good is the following:

The basic operator will be $d+\delta$ on forms on M. But you can think of this operator in two ways. The normal way is as an [redacted] operator on an $\Omega^0(M)$-module. The interesting way is as an operator on [redacted] a module over the Clifford algebra. In the first case we know the index problem is relatively trivial, because the $\hat{A}$ denominator is missing. However this is exactly what is killed [redacted] under the forgetful map.

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dihedral cohomology

$$\int_Z f_0\, df_1 \cdots df_n = \text{area inside } (f_0, \dots, f_n)(Z) \subset R^{n+1}$$

Novikov conjecture: (Morgan. last remaining step in classification of manifolds up to finite stuff.)

It has to do with signature of inverse images. Connes says it as follows: Given a map $f: M \longrightarrow S^n$ the signature of the inverse image of a gen. point is known not ~~to~~ be homotopy invariant of (M,f). This is because the pontryagin classes, better the L-classes, are not homotopy invariant except the top one. But given $M \xrightarrow{f} K(\pi,1)$ and a cohomology class in $K(\pi,1)$, say represented by a submanifold, transversally oriented, then the signature of the inverse image is supposed by the Novikov conjecture to be a homotopy invariant.

Morgan: For example the Novikov Farrell fibring theorem says that an M homotopy equivalent to $N \times S^1$ is of the form $M = N' \times S^1$, and $N_1 \times S^1 \sim N \times S^1 \Longrightarrow N_1 \sim N'$. Hence $M \longrightarrow \text{Signature}(N')$ is a homotopy invariant. This result proves Novikov for $\pi = \mathbb{Z}$, and similarly for π abelian.

group cocycles give cyclic cocycles on the group algebra
You could describe these for Heisenberg groups.

Odd conventions. Given $n = 2m-1$ and an ungraded C_n-module W, the involutions being α^μ we get a graded $C_{n+1} = C_{2m}$-module

$$V = W \oplus W$$

with the following.

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \alpha^\mu \\ \alpha^\mu & 0 \end{pmatrix} = \alpha^\mu \gamma \qquad 1 \le \mu < 2m$$

$$\gamma^{2m} = -i\varepsilon\gamma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Thus for example if $n=1$ and $\alpha^1 = 1$ on $\mathbb{R}$ then we get for $\gamma^1, \gamma^2, \varepsilon$ the three Pauli matrices. Also we have

$$tr_V(\varepsilon\gamma^1 \cdots \gamma^{n+1}) = tr_V(\alpha^1\gamma \cdots \alpha^n\gamma\underbrace{(-i\varepsilon\gamma)\varepsilon}_{-\varepsilon\gamma})$$

$$= 2i\, tr_W(\alpha^1 \cdots \alpha^n)$$

so that if we take the preferred ungraded C_n-module to have dim 2^{m-1} with $\alpha^1 \cdots \alpha^n = i^{m-1}$, then V becomes the spinors: Good choice is

$$tr_W(\alpha^1 \cdots \alpha^n) = (2i)^{m-1}.$$

Notice that
$$tr_W(\alpha^{i_1} \cdots \alpha^{i_p}) = \begin{cases} 0 & \begin{matrix} i_1 < \cdots < i_p \\ 1 \le p < n \end{matrix} \\ 2^{m-1} & p = 0 \\ (2i)^{m-1} & p = n \end{cases}$$

In effect if p is odd and $< n$, then $\alpha^{i_1} \cdots \alpha^{i_p}$ anti-commutes with a missing α^j; thus it is odd relative to the α^j grading and has zero trace. If p is even and >0, then $tr(\alpha^{i_1}\cdots\alpha^{i_p}) = -\,tr(\alpha^{i_2}\cdots\alpha^{i_p}\alpha^{i_1})$ and also this, hence we get 0.

Finally if I take the corresponding character I get

$$tr_W\left(e^{-t^2x^2 + t\,dx^\mu \alpha_\mu}\right)$$

$$= e^{-t^2x^2}\, tr_W \prod_\mu \left(1 + t\,dx^\mu \alpha_\mu\right)$$

$$= e^{-t^2x^2}\left[2^{m-1} + t^n\, tr_W\left(dx^1\alpha_1 \cdots dx^n\alpha_n\right)\right]$$

$$dx^1\cdots dx^n \underbrace{tr_W(\alpha_n\cdots\alpha_1)}_{(2i)^{m-1}(-1)^{\frac{(2m-1)(2m-2)}{2}}} \quad = (-2i)^{m-1}$$

Thus the character form is

$$\text{odd part of } tr_W\left(e^{-t^2x^2 + t\,dx^\mu\alpha^\mu}\right) = (-2i)^{m-1}\, e^{-t^2x^2}\, t^n\, dx^1\cdots dx^n$$

January 25, 1984

Problem: Consider a Clifford algebra bundle $C(T)$ where T is a vector bundle over M. Then super $C(T)$-modules give rise to elements of $K(T)$. The problem will be to construct their Chern characters.

Start with the simplest case where T is trivial. Then I have a super vector bundle E with C_n-action. Suppose n is even. Then I know that super C_n-modules are the same as super C_0-modules. On the other hand I know how to associate to a super C_n-module an element of $K(\mathbb{R}^n)$. This should somehow be Bott periodicity.

Therefore Bott periodicity seems to be a kind of Morita equivalence.

January 27, 1984

I want to consider the local index formula for the family of all Dirac operators [redacted] with coefficients in the vector bundle E. Let $\mathcal{H} = L^2(M, S\otimes E)$, $\mathcal{G}$ = gauge transf. gp, $\mathcal{A}$ = connections, $A \longmapsto \not{D}_A$ the equivariant map from $\mathcal{A}$ to $(\mathrm{End}\,\mathcal{H})'$. Then by passing to the quotient we [redacted] get a Fredholm operator on the induced Hilbert bundle $\mathcal{A}\times^{\mathcal{G}}\mathcal{H}$ over $\mathcal{A}/\mathcal{G}$. [redacted] To apply the character formula [redacted] I need a connection on $\mathcal{A}\times^{\mathcal{G}}\mathcal{H}$, i.e. a connection on $\mathcal{A}\times\mathcal{H}$ which descends.

Look at this [redacted] problem first. Suppose we have a representation $\rho: G \longrightarrow \mathrm{Aut}(V)$ of G and a principal G-bundle P/M. Then we want a connection on the bundle $P\times^G V$ over M. Start with the invariant connection d on $P\times V$ and let $\theta \in \Omega^1(P)\otimes \mathfrak{g}$ be a connection form in P. Then $d+\rho(\theta)$ is a connection which descends.

Start again: Let $\mathcal{H}$, $\mathcal{G}$, $\mathcal{A} \xrightarrow{\not{D}} (\mathrm{End}\,\mathcal{H})'$ be as above. Then we get a Fredholm map F on the super Hilbert bundle $\mathcal{A}\times^{\mathcal{G}}\mathcal{H}$ over $\mathcal{A}/\mathcal{G}$. This defines an element of $K^0(\mathcal{A}/\mathcal{G})$. We are trying to transgress this K-element to something perhaps in $K^{-1}(\mathcal{G})$ which would be roughly the difference of the two maps of $\mathcal{G}$ to the unitary group.

Look at ~~finite~~ dimensions first. Given a repn [redacted] of G on V and a principal G bundle P/M, we get the bundle $P\times^G V$

representing an elt of $K^0(M)$. This transgresses to an elt $K^{-1}(G)$ which is the map $G \longrightarrow \mathcal{U}$ given by the representation.

In terms of characteristic classes we get the following. If θ is a connection for P/M, then $d + \rho(\theta)$ on $\Omega(P) \otimes V$ descends to give a connection on $P \times^G V$. Drop ρ from the notation, so that $D = d + \theta$. Then the Chern character is the DR class on M represented by the form

$$tr\left(e^{d\theta + \theta^2}\right) \quad \text{on } P.$$

To compute the transgression we use the family of connections $d + t\theta$ which gives

$$tr\left(e^{d\theta+\theta^2}\right) - n = d\int_0^1 dt \, tr\left(e^{t d\theta + t^2\theta^2}\theta\right).$$

When we restrict to a fibre, the form θ becomes the MC form, and $d\theta = -\theta^2$ so that we get the form

$$\int_0^1 dt \, tr\left(e^{(t^2-t)\theta^2}\theta\right)$$

representing the trangression.

Now I want to do something similar in the infinite-dimensional setup. Again let θ be a connection form in the principal $\mathcal{G}$-bundle $\mathcal{A}$ over $\mathcal{A}/\mathcal{G}$. Then on $\mathcal{A} \times \mathcal{H}$ I have the connection $d + \theta$ which descends to gives a connection on $\mathcal{A} \times^{\mathcal{G}} \mathcal{H}$ over $\mathcal{A}/\mathcal{G}$. I would like to use the forms

$$tr_s\left(e^{d\theta + \theta^2}\right)$$

but the trace doesn't make sense. $d\theta + \theta^2 = \Omega$ is a two-form with values in $(\mathrm{End}\,\mathcal{H})^0$. It is essentially an operator on $L^2(S \otimes E)$ arising from a vector bundle endo. of E, so it's a multiplication operator and doesn't have a trace. Somehow only the difference of the operators on $\mathcal{H}^0$ and $\mathcal{H}^1$ has a trace. (?)

The next step is then to use the Chern character form with the ~~odd~~ degree endo $\not{\partial}$:

$$tr_s\left(e^{-t^2\not{\partial}^2 + it[D, \not{\partial}] + D^2}\right),$$

where $D = d + \theta$. This is a form on $\mathcal{A}$ which descends to $\mathcal{A}/\mathcal{G}$. I want to go thru the transgression process, which means that I have to write this form as d of something, and then restrict the something to $\mathcal{G}$. So how does this go?

The only technique we have is to change the superconnection

$$\tilde{D} = d + \theta + it\not{\partial}$$

January 28, 1984

Bott's theorem on Chern numbers. Let M^{2n} be a $2n$-diml manifold on which the circle S^1 acts, and let E be an equivariant vector bundle. Let p be an invariant polynomial of degree n. Bott's theorem expresses the Chern number $p(E)$ of E as sums of contributions over the fixpoints. I want to see if I can see this result geometrically. More precisely I would like to deform the differential form representing $p(E)$ into a neighborhood of the fixpoint set.

So assume the action is free. Then E descends to $\bar{E}$ over the orbit space M/S^1 which has dimension $2n-1$, hence $p(\bar{E}) = 0$ and $p(E) = 0$ for diml reasons. On the differential form level, let D be a connection on E which is invariant, and let θ be a connection form for the S^1 action. Then $D + \theta\varphi$, where φ is the Higgs field, descends. So $p((D+\theta\varphi)^2)$ is a $2n$ form which is basic. But a $2n$ form killed by $i(X)$ has to be zero. Then

$$p((\underset{\overset{\|}{0}}{D+\theta\varphi})^2) - p((D)^2) = d\int_0^1 dt\ p'((D+t\theta\varphi)^2, \theta\varphi)$$

allows one to express $p(D^2)$ as a coboundary.

January 29, 1984.

Consider a circle action on M and an equivariant bundle E. The localization theorem says that in some sense equivariant cohomology classes on M when localized are described by their restriction to the fixpoint set. I would like to see this done geometrically, for example I would like to have differential forms representing equivariant classes, such as the equivariant characteristic classes of E, given by forms concentrated near the fixpoints.

Now one has to incorporate the localization process into the picture. I can't expect to represent a class by a form near the fixpts unless it somehow vanishes when the fixpoint set is removed.

Witten seems to do the localization process in an analytical way which goes as follows. He considers the forms on M with the differential operator $d + t\iota_X$ where t is a parameter. Notice that this preserves only the $\mathbb{Z}_2$-grading on forms. He has some argument to get to the case of invariant forms in which case, because

$$(d+t\iota_X)^2 = t(d\iota_X + \iota_X d) = t\mathcal{L}_X$$

one gets $(d+t\iota_X)^2 = 0$ on $\Omega(M)^S$.

In order to keep the connection with equivariant forms I should use $d - t\iota_X$, so let's change the notation.

So now we have

$$\Omega^{ev}(M)^S \rightleftarrows \Omega^{odd}(M)^S$$

and we can form the Laplacean and do Hodge theory to realize the cohomology H_t with respect to $d - t\iota_X$ via harmonic forms.

Atiyah and Bott have pointed out that H_t for $t \neq 0$ is essentially the localized equivariant cohomology. The equivariant cohomology $H_S(M)$ is a graded module over $H_S(pt) = k[u]$ where $\deg u = 2$. When localized $H_S(M)[u^{-1}]$ is a module over $k[u,u^{-1}]$ which one can think of as a vector bundle over the u-plane minus the origin. So I can specialize at a value t of u and get a super vector space.

They show this is the same as Witten's H_t as follows. Introduce the complex of equivariant forms

$$k[u] \otimes \Omega(M)^S \qquad \text{with diffl} \quad d - u\iota_X$$

Localization is exact so $H_S(M)[u^{-1}]$ is the homology of

$$k[u,u^{-1}] \otimes \Omega(M)^S \qquad \text{with diffl} \quad d - u\iota_X .$$

These homology groups are free over $k[u,u^{-1}]$, so specialization is also exact, i.e.

$$H\Big[k_t \otimes_{k[u,u^{-1}]} k[u,u^{-1}] \otimes \Omega(M)^S\Big] = k_t \otimes_{k[u,u^{-1}]} H_S(M)[u^{-1}]$$

$$\Vert$$

$$H\big(\Omega(M)^S \text{ with } d - t\iota_X\big) = H_t .$$

So what we see is that Witten's cohomology H_t can be identified with the localized equivariant cohomology.

I gather that Witten shows as $t \to \infty$ that harmonic forms for $d + t\iota_X$ actually concentrate at the fixpoints. In this way he is able to show that the cohomology coincides with that of the fixpoint set.

But I can ask a related question: What happens to characteristic forms for an equivariant vector bundle as $t \to \infty$. The equivariant curvature of E relative to an invariant connection D is $D^2 + u\varphi$, where $\varphi = -[\iota_X, D]$ is the Higgs field. So we can look at the form

$$\operatorname{tr} e^{D^2 + t\varphi}$$

as $t \to \infty$.

Let's be specific and take an equivariant line bundle. We have $d\varphi = \iota_X D^2$, and we are looking at the form

$$e^{t\varphi + D^2}$$

In practice we write $iD^2 = \omega$ and $i\varphi = H$, so that ω and H are real satisfying $\iota_X \omega = dH$. This means that H is the Hamiltonian generating X when ω is symplectic. So we have the form

$$e^{-itH - i\omega}$$

which doesn't concentrate at the stationary points as $t \to \infty$

On the other hand when M has dim $2n$

$$\left(\frac{i}{2\pi}\right)^n \int e^{-itH - i\omega} = \int_M e^{-itH} \frac{\omega^n}{(2\pi)^n}$$

is the quantity which has the exact stationary phase approximation, so something is happening.

Atiyah mentioned the following. Frankel's basic observation is that one has inequalities:

Morse: $\quad H(M) \leq H(M^{crit})$

equivariant: $\quad H(M^S) \leq H(M)$

so in the case of an S-action on a Kahler manifold (with at least one fixpt so that the Hamiltonian H exists), the critical points for H are the same as the fixpoints for the circle action; hence the Morse theory is perfect.

(Why doesn't this work for circle actions on symplectic manifolds?)

We have seen that Witten's $\mathbb{Z}_2$-complex

$$\Omega^{ev}(M)^S \rightleftarrows \Omega^{odd}(M)^S$$

with differential $d - t\iota_X$ represents essentially the localized equivariant cohomology. In particular any class for this complex should be represented by a form supported near the fixpoint set. In fact he must have a way to identify harmonic forms on the fixpt set with zero modes for $d - t\iota_X$ for large t.

The simplest case would be when the fixpoint set is empty. Then he must see that the Laplacean of this operator is strictly positive and so there is no cohomology. Can I see the same thing algebraically?

If there are no fixpoints, then X is never zero, so we can find an invariant 1-form $\theta \in \Omega^1(M)^S$ such that $\iota_X \theta = 1$. But then

$$[d - u\iota_X, \theta] = d\theta - u$$

so that multiplying by $d\theta$ and by u on [redacted] $k[u] \otimes \Omega(M)^S$ are homotopic. On the other hand $d\theta$ is nilpotent since $d\theta \in \Omega(M)^S$ which is finite dimensional. Thus u is nilpotent [redacted] on $H_S(M)$, etc.

The same argument works if u is a parameter t and we consider $d - t\iota_X$, θ on $\Omega(M)^S$.

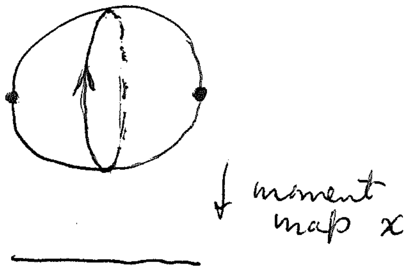
January 30, 1984 Conversation with Michael

Frankel observation holds for a circle action on a symplectic manifold. Also the spectral sequence for equivariant cohomology is degenerate, i.e. additively

$$H_S(pt) \otimes H(M) = H_S(M)$$

and hence there is no $H_S(pt)$ torsion in $H_S(M)$.

Yang-Mills = norm squared of the moment map. It leads to equivariantly perfect Morse function. Example



Then x^2 has maximum at N, S poles and minimum on equator.

$$\begin{array}{ccc} \text{min} & & \text{max} \\ 1 & + & \dfrac{2t^2}{1-t^2} \end{array} = \frac{1+t^2}{1-t^2} = \text{P.S. } H_S(S^2)$$

[redacted]

Morse index

This is quite different from the Morse function x

$$\begin{array}{ccc} \text{min} & & \text{max} \\ \dfrac{1}{1-t^2} & + & \dfrac{t^2}{1-t^2} \end{array} = \frac{1+t^2}{1-t^2}$$

In the Kähler case the Morse cells are complex and the circle action extends to a $\mathbb{C}^*$-action preserving the cells. In Kirwan's work the cells are affine bundles over the [redacted] critical submanifolds, affine relative to nilpotent groups. This explains why the counting formulas over finite fields works. (There are two logical problems relating Morse results with counting formulas. The Morse decompositions give [redacted] long exact sequences, hence are not a priori additive. The finite field methods are additive but

, a priori, for the stratum attached to a critical set there might be a complicated relation between the numbers of rational points. Both difficulties don't occur in Kirwan's setup so that is why results agree.)

Review of Donaldson's proof that stable bundles have unique YM-connections. Picture from Mumford-Kirwan etc theory: YM = square of moment map which is defined by K action (K = max. comp. of G); moduli space is X^s/G or YM minima/K. The unique K orbit which minimizes YM in a G-orbit is also the part of the G-orbit in the line bundle (or vector space if you think of G as acting linearly) closest to the zero section.

Donaldson therefore has to minimize the distance to the origin on the space of metrics on a fixed holomorphic bundle. He deforms the Kähler metric until it becomes a δ-function concentrated on a hyperplane section. He can control this change, and then using that a stable bundle on the surface restricts to a stable bundle on the hyperplane section, he sees the limit is bounded below.

Novikov conjecture. Let π be a f.p. group. Given a manifold M with fundamental group π and a coh. class on $B\pi$, one takes the submanifold dual to this coh. class and computes the index. Is this operation $\{M\} \longrightarrow \mathbb{Z}$ on the manifolds with fundamental group π a homotopy invariant? An equivalent description is to push forward the K-homology class on M given by the signature operator into $B\pi$ and ask whether this is a homotopy invariant. Pairing with a representation of π gives a homotopy invariant, since it gives the signature of M

with coefficients in the representation. More generally this conclusion holds for "representation" generalized to "element of $K(C_r(\pi))$", where $C_r(\pi)$ is the reduced C^*-algebra of π. So we want to know the relation between

$$K_*(B\pi) \quad \text{and} \quad K^*(C_r(\pi)).$$

I probably have K homology + cohomology for $C_r(\pi)$ confused. Connes' conjecture is that

$$\mu : K_*(B\pi) \longrightarrow K^*(C_r(\pi)) \qquad \text{is an isom.}$$

and a representation of π should pair with a finite projective module to give an integer. Thus "representations" are in K-homology $KK(C_r(\pi), \mathbb{C})$, and so we see that if μ is injective, then the Novikov conjecture follows (more or less).

John Roe: Let $\mathcal{A}$ be an algebra contained in $\mathcal{B}$ such that $D \in \mathcal{B}$ is invertible mod $\mathcal{A}$, $\mathcal{A}$ being assumed to be an ideal in $\mathcal{A}$. Then from the exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{A}/\mathcal{B} \longrightarrow 0$$

is a connecting homomorphism

$$K_1(\mathcal{A}/\mathcal{B}) \xrightarrow{\delta} K_0(\mathcal{A}).$$

Roe describes this connecting homomorphism as follows. Given D and a $P \in \mathcal{B}$ with $DP \equiv PD \equiv I$ mod $\mathcal{A}$ one obtains the image of D under δ

as the difference of two 2x2 matrices over $\mathcal{A}$ which are projectors

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} PD-1 & ? \\ ? & DP \end{pmatrix}.$$

January 31, 1984.

Scott Petrarch: He explained to me what a Duistermatt - Heckman type proof of the index theorem would amount to. Namely an actual evaluation at any time of the traces of the Dirac heat kernel. This is instead of an asymptotic evaluation.

On spinors over a Riemann. spin manifold with Killing vector field X, we should have

$$[X, \not{D}] = \mathcal{L}_X$$

where X is identified with an element of the Clifford algebra.

Atiyah summarizes talk by Berry: Phase + Degeneracy. Deform a Hamiltonian ~~with~~ many parameters, following the eigenvector one gets a line bundle, a connection, hence going around a loop results in a phase shift. Generic case encounters degeneracy in codim 3.

Classical example of a bead running around a closed curve. When rotated adiabatically there is a phase shift $\pm\left(1 - \frac{L^2}{4A}\right)$ which is always ≤ 0 and $= 0$ for the circle by the isoperimetric inequalities.

Localization thm. in equivariant cohomology implies [redacted] that in the localized cohomology we have

$$i_* \frac{i^*(\alpha)}{e(\nu)} = \alpha$$

The problem I would like to understand is how to do this explicitly in Witten's cohomology. For example, can I define i_*.

Cyclic cohomology = Witten cohomology of the loop space.

February 1, 1984

Conversation with Connes: Kasparov KK, [redacted] Seminar on Hörmander's Several Complex Variables in order to get the Grauert result, Morita equivalence formulas in Ch. II., calculation of cocycle for $\mathbb{C}/\Gamma$ and the $\bar{\partial}$ operator, bivariant cyclic homology, rotation alg. on S^1 and index thm. for Mathieu. Example of Hilbert Schmidt [redacted] operators, cyclic cohomology of $C_c^\infty(V)$ = proper homology, [redacted] circle bundle

Glenys: Loday wants to relate cyclic homology to some kind invented by Deligne + Bylinski

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Problem with cyclic theory when the ring has no unit. Consider the algebra of Hilbert Schmidt operators. One has cyclic cocycle $tr(x^0x^1x^2) = \tau(x^0, x^1, x^2)$ which is the Hochschild coboundary $b\varphi$ where $\varphi(x^0, x^1) = tr(x^0x^1)$. But then the cocycle τ should be in the image of S. For some reason the candidate should be $tr(x)$ which isn't defined for Hilbert-Schmidt operators.

Morita equivalence: See Chap. II. [redacted] Note that the element he adjoins to make a complex, i.e. to kill the curvature, [redacted] is something like the $d1$ or ede. This builds in the S-operator. He said something about passing from A to $\underset{A}{\mathrm{End}}(E) = B$ and then from B back to A.

If an abelian group G acts on A, then $A \rtimes G$ has a natural $\hat{G}$ action. Moreover $(A \rtimes G) \rtimes \hat{G}$ is Morita equiv. to A and this equiv. is compatible with the G-actions.

Given a circle bundle P/M, the crossed product of the functions on P with S admits a $\mathbb{Z}$-action (which is inner: as the action on the base is trivial?) Somehow the effect on cyclic cohomology has to do with multiplication by the Euler class of the circle bundle.

Other topics: computing the cocycles for the Hilbert transform on $\mathbb{R}$, and $\partial_{\bar{z}}$.

<u>Idea</u>: Can his new theory be used to construct characteristic classes for flat bundles in "real" $\otimes \mathbb{R}$ K-theory? Atiyah + Segal's question for non-unitary representations.

February 2, 1984.

Lunch with Katz, Gabber, Deligne.

Differential Galois theory. Start with a linear DE over $\mathbb{C}$. This is a holomorphic vector bundle E with flat connection. We are primarily interested in the case where there is a non regular singular point at ∞.

There is an intelligent way to take the Zariski closure of the monodromy based on Tannakian ideas. The monodromy is trivial for $\mathbb{C}$, but one can look at flat sections of $E^{\otimes N}$ which are algebraic. Put another way, if one started with an algebraic variety over a field of char. zero, and a flat bundle, then by embedding the ground field in $\mathbb{C}$ and taking Zariski closure of the monodromy, one gets various algebraic groups depending on the embedding. A good algebraic definition based on alg. flat sections of $E^{\otimes N}$ solves this difficulty.

One uses the Chevally theorem, that an algebraic subgroup G of GL_N is the stabilizer of a line in some repr.

Example: Take the fn. e^z which belongs to the operator $\frac{d}{dz} - 1$ on $E = \mathbb{1}$. Tensoring N-times leads to e^{Nz}. None of these are algebraic for $N \neq 0$, hence the differential Galois group is $GL_1 = \mathbb{G}_m$.

Example: Airy: $\frac{d^2}{dz^2} - z$. Here the differential Galois group is all of $SL_2(\mathbb{C})$.

Effect of wild ramification on H^*: Take an affine non-singular curve and an irreducible repr. of its fundamental group over $\mathbb{Q}_\ell$, so that one has an ℓ-adic étale sheaf F. Then H^0_c and H^2_c are zero, hence the Euler char. χ is $-\dim H^1_c$. Naively one expects $\chi(F) = \chi(\mathbb{Q}_\ell)\cdot \operatorname{rank} F$, but there is

If one wants something like the Γ function factor in the usual Fourier transform between x^s and x^{1-s}, then one has to put some kind of rational structure into these spaces.

Deligne: $\pi_1(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $\pi_1(\mathbb{P}^1-\{0,1,\infty\},x)_{\ell\text{-completion}}$.
$\pi_1(\mathbb{P}^1-\{0,1,\infty\},x)_\ell$ is a free pro-ℓ-group on 2-generators, isomorphic to free Lie on $H^1 \sim \mathbb{Z}_\ell(1)^{\oplus 3}/\text{diag}$. The isomorphism is non-canonical and upon writing out exact sequences using the descending central series, one finds classes in

$$H^1(\bar{\mathbb{Q}}/\mathbb{Q}, T_\ell(n)) \sim K_{1-2n}(\mathbb{Q})$$

One can view $\pi_1(\mathbb{P}^1-0,1,\infty)_\ell$ as representing an elt. in a non-abelian coh. gp $H^1(\bar{\mathbb{Q}}/\mathbb{Q}, G)$ where G is a group of auto of the free Lie algebra. The idea is to reduce G as small as possible, in which case it should have generators corresponding to the infinite generators of $K_*(\mathbb{Q})$.

Over $\mathbb{C}$ one can do the mixed Hodge analogues. On $X=\mathbb{P}^1-0,1,\infty$ there is an H^1 and an H_1 both 2 diml, and the pairing gives a 1-form on X with values in the H_1. Then construct a differential equation on X whose monodromy is the gp. assoc. to the free Lie algebra gen. by H_1. The point is that $FL\ H_1$ or $(\pi_1)_\ell$ has a mixed Hodge structure, the integrality coming from ~~topology~~ - Chen + Sullivan + Morgan; really I should say rational lattice. On the other hand upon integrating the DE ~~one gets~~ a transcendental function (essentially the polylogarithm). Then one gets another rational structure from the natural diff. form basis times $2\pi i$ factors. Ramakrishna computes the ~~ratio~~ and gets ζ(integer) mod $\mathbb{Q}^*$.

an error term due to wild ramification. The formula involves the multiplicity of the Swan repn., and this is zero iff the ramification is tame.

$\exists$ Analogy between wild ramif. in char. p and irregular singular points over $\mathbb{C}$. For example at an irregular singular point one can define an integer (called the Swan ?) which measures the irregularity. Two approaches 1) look for largest lattice stable under $z^n z\frac{d}{dz}$ ($n \in \mathbb{Q}$ by adjoinig $z^{1/N}$), get a Newton polygon and compare breaks. 2) take $\mathcal{D}$-module viewpoint and look at the characteristic variety (which sits in the bundle $T^*(\log z)$ dual to the operator $z\frac{d}{dz}$). Then the Swan is the multiplicity of the vertical component ~~over~~ $z=0$.

Fourier transform for $\mathcal{D}$-modules: In the case of $\mathbb{C}$ a $\mathcal{D}$-module is simply a f.g. module over the Weyl alg $W(x, \frac{d}{dx})$. It's F.T. is obtained by $\frac{d}{dx} \mapsto -x$, $x \mapsto \frac{d}{dx}$. A general definition is possible using the appropriate derived category language.

Over a finite field a sheaf gives a function on the set of rational points, and then the function belonging to the F.T. is the F.T. of the function. Example: Take the sheaf F on the line extension by zero from the sheaf over $\mathbb{G}_m$ associated to a complex character of μ_m and the principal μ_m-bundle

$$0 \longrightarrow \mu_m \longrightarrow \mathbb{G}_m \xrightarrow{m} \mathbb{G}_m \longrightarrow 0.$$

Maybe this should be the Lang isogeny.

Then the F.T. of the sheaf F is essentially itself twisted by a line bundle over the finite field. The corresponding formula for functions involves the Gaussian sum which occurs because of this line bundle.

February 3, 1984

Connes: before lunch; if Γ torsion-free, conjecture $H^*(\Gamma) \simeq H^*_\lambda(\mathbb{C}[\Gamma])$, important to have statements using the group ring as an algebra without the augmentation given by the trivial repn. because $C^*_r(\Gamma)$ doesn't map to $\mathbb{C}$, example of free group.

If G acts on A, then $A \rtimes G$ admits a dual action of G. Dual action can be described in three equivalent ways: 1) using positive functionals 2) using representations: A dual action of G on B allows one to associate to each representation of G a B-bimodule, so $R(G) \longrightarrow KK(B,B)$ is an alg. map. When G is abelian $\hat{G}$ acts on B. 3) using Hopf algebra type maps: $B \longrightarrow B \otimes C(G)$.

For a circle bundle $P \xrightarrow{S} M$, let $A = C(P)$. Then $C(P) \rtimes G \approx C(M)$. We can now define an element of $KK(M,M)$ as follows

$$K\,C(M) \xrightarrow[\substack{\text{of the}\\ \text{center}}]{\text{inclusion}} KC(P) \rtimes G \xrightarrow[\substack{\text{the generator}\\ \text{of } \hat{G} = \mathbb{Z}}]{\text{action of}} KC(P) \rtimes G \xrightarrow{\text{tr}} KC(M)$$

On K-theory and also on cohomology this is multiplication by (1, euler class).

Problem of the conjecture $K_*(B\Gamma) = K_*(C_r(\Gamma))$ when Γ not torsion-free. Here one replaces $K_*(B\Gamma)$ by a geometric group consisting of classes of proper Γ-actions on (M, E), instead of free actions. Gives correct answer for G finite.

Novikov conjecture is now proved for elements of $H^*(B\Gamma)$ which are represented by the following "geometric cycles for Γ": $(V, E^\pm, \sigma, \omega)$, where V is a Γ-manifold

$E^\pm$ are Γ-bundles over V, σ = degree 1 map on $E = E^+ \oplus E^-$, which is an isomorphism outside a compact subset, and such that $\sigma - \sigma^g \longrightarrow 0$ at ∞ for each $g \in \Gamma$, (in sense of orientation, not degree) [redacted]
ω is an odd form on V Γ-invariant.

Here is how such a geometric cycle for Γ determines an element of $H^*(B\Gamma)$. Form over $P_\Gamma \times^\Gamma V$ the K-element represented by $P_\Gamma \times^\Gamma E^\pm$ and the map σ. It will have support proper over $B\Gamma$, hence so will its character. Then as ω is an odd form Γ-inv. it gives an odd form on $P_\Gamma \times^\Gamma V$, so it makes sense to integrate over the fibre:

$$\pi_*\left(ch(E^\pm, \sigma)\ \omega \right) \in H^*(B\Gamma).$$

group cocycles and cyclic cocycles. G Lie gp, van Est:

$$H^n_{cont}(G, \mathbb{R}) = H^n\left(\Omega(G/K)^G \right)$$

Let $c(g^1, \ldots, g^n)$ be a differentiable group cocycle. Then it defines a cyclic cocycle on $C_c^\infty(G)$ [redacted] by

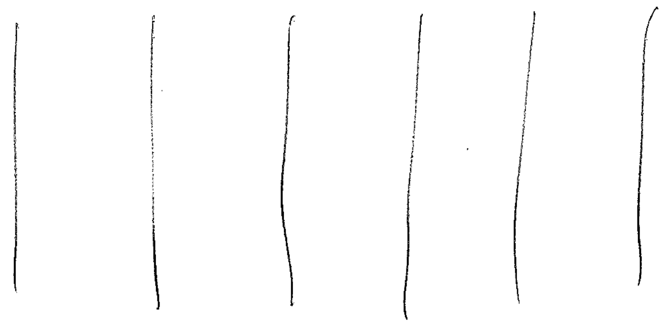
$$\tau(f^0, f^1, \ldots, f^n) = \int_{G^n} f^0(g^0) \cdots f^n(g^n)\, c(g^1, g^2, \ldots, g^n)\, dg^1 \cdots dg^n$$

Haar measures

Somewhere there is a pairing $\langle\ , \tau \rangle$ probably with representations of G. For example if one started with $1 \in H^0(\Omega(G/K)^G)$, then $\tau(f^0) = f^0(e)$ is somehow related to [redacted] the Plancherel formula.

Connes says my Chern character formula should be useful in the transversal elliptic operators for the action of a compact group. The symbol is defined on a singular subset of T^*. Explicit formulas are lacking when the fibres <u>jump</u>, even for S!

Back to G Lie gp. If $G = SL_2(\mathbb{C})$, then the dual of G is the space of principal series:



union of lines except for some stuff with the Weyl gp. Then one has the volume class in $H^3((G/K)^G)$. One has from $K_*(C_r(G)) = K_*(\hat{G})$, and the fact $\hat{G}$ is a union of lines, various elements in $H_1(\hat{G})$. The pairing somehow involves S.

Connes: during lunch + at tea

He explained to Deligne + Gromov how Kasparov improved Douglas etc. to include the even case. [redacted] The rough idea: Start with a BDF extension, say a map $A \xrightarrow{u}$ Calkin algebra. Then under certain conditions (A a nucleare C^*-algebra) one can lift this map to a ~~positive~~ map $A \xrightarrow{\tilde{u}} \mathcal{B}(\mathcal{H})$. Now use Stinespring to construct a larger Hilbert space $\mathcal{H}'$ such that $\tilde{u}$ is the compression of a representation of A in $\mathcal{H}'$. I think this means that [redacted] $\tilde{u} = eve$ where $v: A \longrightarrow \mathcal{B}(\mathcal{H}')$ is a homomorphism of algebras and e is the projection of $\mathcal{H}'$ onto $\mathcal{H}$. Then e commutes with A modulo compacts, so we now have an odd Kasparov element.

Note that the effect of the above construction is to produce the additive inverse of the original BDF extension.

(A C^* algebra A is nucleaire if its tensor product with another C^*-algebra has a unique norm, e.g. $C_r(\Gamma)$ where Γ is amenable. Γ Amenable means that given any finite subset S and $\varepsilon > 0$ there is a larger finite set X such that the [redacted] size of $(X - X^s) \cup (X^s - X)$ divided by the size of X is $< \varepsilon$, for each $s \in S$. Solvable groups are amenable, but free groups are not. The reduced and maximal [redacted] group rings are equal $\Longleftrightarrow$ group is amenable). Connes claim $C_r(\Gamma)$ is the correct thing since you [redacted] have some control over the coefficients of an elements.

I asked whether you lose ~~re~~presentations not contained in the regular repn. He replied given one such $\mathcal{H}_\pi$ you can consider $\mathcal{H}_\pi \otimes \ell^2(\Gamma)$ as a bimodule over $C_r(\Gamma)$.

[redacted]

Further example: Condition T says trivial repn. is isolated in the regular repn. Introduce

$$R(\Gamma) = \text{von Neumann alg} = \text{commutant of right action of } \Gamma \text{ on } \ell^2(\Gamma).$$

This is of type II_1 (unless Γ has an abelian subgrp. of finite index). Claim that $R(\Gamma)$ satisfies $T \Longleftrightarrow$ [redacted] Γ satisfies T. Here one considers $\ell^2(\Gamma)$ as a bimodule over $R(\Gamma)$ as the analogue of the trivial representation. By looking at coefficients one can describe what it means for bimodules to be close. Application: Take $\Gamma = Sl_3(\mathbb{Z})$ which satisfies T, then $R(\Gamma)$ is of type II_1 but is not isom. to $R(\Gamma) \otimes M_2$.

Wave packet transform, strong Gårding inequality, positive lifting, blowup of $V \times V$ along ΔV, Maslov formula + Lagrangian submanifolds as $h \to 0$.

Wave packet transform. To each element of T^* one constructs a wave packet. Then one use these wave packets to decompose the identity operator into rank one operators

$$(*) \qquad id = \int_{\xi \in T^*} d^{2n}\xi \; |\xi\rangle\langle\xi| \qquad d^{2n}\xi = \text{symplectic volume}$$

Enough to do approximately with a partition of 1, then using $\sqrt{}$ you can make it exact.

Strong Gårding inequality proved by Friedrichs. Given A with positive symbol $a(\xi)$, one introduces

$$B = \int_{\xi \in T^*} d^{2n}\xi \; |\xi\rangle \, a(\xi) \langle\xi|$$

so that $B \geq 0$ and $A-B$ is lower order. In fact we get an estimate

$$(f, Bf) = \int d^{2n}\xi \quad a(\xi) |\langle \xi | f \rangle|^2$$

from which one can compare (f, Af) [redacted] and derive the inequality.

The basic decomposition $(*)$ provides a quantization process. In fact it gives a positive lifting

$$\text{functions on } T^* \longrightarrow \text{operators on } L^2$$

This is a concrete realization of the Stinespring + Kasparov business (Glenys says to look at Kasparov's version in the Roumanian Journal of [redacted] Operator Theory.)

Asymptotic differential operators: We start with the groupoid obtained from $V \times V \times [0,1]$ consisting of x, y, ε by adjoining $T_V \times \{0\}$ as the limits of $\frac{x-y}{\varepsilon}$ as $\varepsilon \to 0$. The point is now to [redacted] study asymptotic behavior of an operator as $\varepsilon \to 0$. Somehow what happens is the Maslov theory in which what occurs in the classical limit is [redacted] a Lagrangian submanifold of T^*, phase problems along this submanifold, and some recursive calculations of coefficients in the asymptotic expansion.

Geometric proof of the index thm: Connes proposes to replace all C^* algebras by geometrically defined groups. There is a functor μ from the geometrical to the analytic (or C^*) K-groups, so when one has a geometric isomorphism one gets also an analytic isom.

Discussion of the Novikov conjecture. Think of **defining for a manifold** with fundamental group π an equivariant signature. This lies in the Witt group $W(\mathbb{C}\pi)$, and is obviously a homotopy invariant of the operator. On the other hand the equivariant signature operator of the manifold pushes forward under the **classifying** map $f: M \longrightarrow B\pi$ to give an element of $K_0(BG)$ which rationally is $H_*(BG)$.

Somehow the problem comes down to mapping the Witt ring to K-theory, i.e. taking a quadratic form and splitting it into positive + negative bundles. This has to be done via Cauchy's theorem so can't be done over $\mathbb{C}\pi$ but rather something analytical like $C_r(\pi)$. Thus one gets a map

$$W(\mathbb{C}\pi) \longrightarrow K_0(C_r(\pi))$$

such that the two signatures have the same image:

$$\begin{tikzcd}[row sep=small]
W(\mathbb{C}\pi) \arrow[dr] & \\
& K_0(C_r(\pi)) = KK(\mathbb{C}, C_r(\pi)) \\
K_0(B\pi) \arrow[ur, "\mu"] &
\end{tikzcd}$$

Thus Novikov follows from injectivity of μ.
In **turn** we **can** try to construct enough elements of $KK(C_r(\pi), \mathbb{C})$ (generalized representations of π) to detect non-zero elements of $K_0(B\pi)$. Thus it is really a question of generating $H^{ev}(B\pi)$ by "repns. of π".

Connes claims that the Novikov conjecture is true for the following cohomology classes of π. Take a family of representations of π parametrized by a space X; [redacted] the character of the family is a cohomology class on $X \times B\pi$, and capping with a homology class on X gives a cohomology class of π. (This seems to be a less general form of his "geometric cycles for π" where π is allowed to act on X and one takes a π-invariant chain on X, see p. 444.)

Let M be a compact oriented manifold of dim $\equiv 0 \mod 4$. Given a vector bundle E over M I can twist the signature operator on forms and get an index. This is given by

$$\text{index} = \int_M ch(E)\, \mathcal{L}(M)$$

where $\mathcal{L}(M)$ is the L-genus. These indices are supposedly not usually homotopy-invariant in the sense that given a heq $f: M' \longrightarrow M$, then the index of M', f^*E is not the same as that for M, E.

Actually I first should have taken a cohomology class α on M and looked at the number

$$(*) \qquad \int_{M'} f^*\alpha \cdot \mathcal{L}(M') = \int_M \alpha \cdot f_* \mathcal{L}(M')$$

for $f: M' \longrightarrow M$ a heq. Rationally α is realized by an embedded submanifold with trivial normal bundle, or more accurately a framed map $N \longrightarrow M$, and then the number is the signature of $f^{-1}N$. This is in fact the way Thom defines rational Pontryagin classes.

The Novikov conjecture asserts the homotopy invariance of the above number $(*)$ when α comes from $H^*(B\pi)$, π = fundamental group of M. Equivalently that

$\gamma_*(\mathcal{L}(M) \cap [M]) \in H_*(B\pi)$ (in homology), $\gamma : M \to B\pi$ the canonical map is a [redacted] homotopy-invariant.

Now if E is a flat unitary bundle on M, then I think the cohomology $H(M,E)$ in the middle dimension is a finite-dimensional vector space with hermitian inner product, so there is a signature. This signature is given by the index of the signature operator, but perhaps coincides with $\operatorname{sign}(M) \cdot \operatorname{rank} E$ as $\operatorname{ch}(E) = \operatorname{rank} E$ for flat bundles.

Question: What can we say when we have a family of flat unitary bundles?

So we suppose given a family of representations of π parametrized by Y. Then we get a bundle $\tilde{E}$ over $Y \times M$ with partial flat connection in the M-direction. Then I can do two things:

1) twist the signature operator on M with $\tilde{E}$ to get a family of operators parametrized by Y. The index is an element of $K(Y)$ which we get from the index thm. for families:

$$\operatorname{ch}(\text{index}) = \int_M \operatorname{ch}(\tilde{E})\,\mathcal{L}(M) \quad \in H^*(Y)$$

If I look at $\operatorname{ch}(\tilde{E}) \in H^*(Y \times M) = H^*(Y) \otimes H^*(M)$, then there are $\alpha \in H^*(M)$ which are obtained from $\operatorname{ch}(\tilde{E})$ and cycles on Y. So if $\operatorname{ch}(\text{index})$ is a homotopy-invariant of M, then we see $(*)$ for these α is homotopy-inv. This follows from the second way of getting at $\operatorname{ch}(\text{index})$.

2) We look at the K-element on Y obtained from the family of cochains on M with coefficients in E_y with its hermitian structure. This is the topological signature for the family. (Lusztig's work)

February 4, 1984

I want to go over Connes method of extending a cyclic cocycle on A to one on $\mathrm{End}_A(E)$, where E is a finite projective module over A. I have the idea that this can be understood by analogy with defining invariant forms on the gauge group associated to an arbitrary, not necessarily trivial vector bundle.

Review the construction: Let E be a vector bundle over M, and $\mathcal{G}$ be the group of automorphisms, say unitary relative to some inner product. Let $\tilde{\mathfrak{g}} = \mathrm{Lie}(\mathcal{G})$. In order to produce Lie algebra forms, better, invariant differential forms on $\mathcal{G}$, we consider a flat connection on the trivial $\mathcal{G}$-bundle over Y. This is the same as giving $\theta \in \Omega^1(Y, \Omega^0(M, \mathrm{End}\, E))$ skew-adjoint such that $d'\theta + \theta^2 = 0$. Then [redacted] $d' + \theta$ can be viewed as a flat partial connection on the bundle $pr_2^*(E)$ over $Y \times M$ [redacted] in the Y-direction. Now choose a connection on E, call it D, and let D'' be the obvious vertical connection on $pr_2^*(E)$ which is the pull-back of D. Then we get two connections

$$d' + D'' \quad , \quad d' + \theta + D''$$

on $pr_2^*(E)$ over $Y \times M$ which are flat in the Y-direction.

A better notation seems as follows. [redacted] Write $\tilde{d} = \delta + d$ over $Y \times M$, where $\delta = d_Y$, $d = d_M$. Then the pull-back of D on E to $pr_2^*(E)$ is $\tilde{D} = \delta + D$ and [redacted] we consider the 1-parameter family of connections

$$\tilde{D} + t\theta$$

The curvature is

$$\tilde{D}^2 + t\tilde{D}\theta + t^2\theta^2 = D^2 + tD\theta + (t^2 - t)\theta^2 .$$

Hence the basic formula is

$$\operatorname{tr}\left(e^{D^2+D\theta}\right) - \operatorname{tr}\left(e^{D^2}\right) = d\int_0^1 dt \ \operatorname{tr}\left(e^{D^2+tD\theta+(t^2-t)\theta^2}\theta\right)$$

Viewpoint: I am reviewing the way to construct invariant forms on G. I recall having too ways to do this, one ~~being based on [redacted] flat connections~~ ~~the other~~ being the above which leads to left-invariant forms, the other being the one I used in the letter to Singer which produced right invariant forms.

[redacted] Now ultimately the idea is that we produce two connections [redacted] on [redacted] $pr_2^*(E)$ over $G\times M$ which are partially flat in the G-direction. Then the Bott theorem gives vanishing for characteristic classes, and so we get [redacted] difference elements. This means that the ch_n class gives a form on $G\times M$ of dimension $2n-1$ whose boundary has filtration $\geq n$. Hence integrating over a p-cycle in M with $p<n$ will give a closed form over G.

So start with a p-cycle γ in M. Then pick an n and we get a [redacted] closed invariant form on G of degree $2n-1-p$. It should be a cyclic $2n-1-p-1$ cocycle. (~~It~~ depends on $2n-1-p$ elements of $\tilde{\mathfrak{g}}$ and so is a $2n-1-p-1$ cochain on the ring $\Omega^0(M,\operatorname{End}E)$.) The obvious choice seems to be ~~to~~ take $n=p+1$, for than $n>p$ and $2n-2-p=p$, so that the p-cycle γ gives rise to a cyclic p-cocycle

$$\frac{1}{p!}\int_0^1 dt \int_{\gamma_p} \operatorname{tr}\left(\left(D^2+tD\theta+(t^2-t)\theta^2\right)^p\theta\right).$$

Now $\quad \theta^2 \in C^2(\tilde{\mathfrak{g}}, \Omega^0(M, \operatorname{End} E))$

$$D\theta \in C^1(\tilde{\mathfrak{g}}, \Omega^1(M, \operatorname{End} E))$$

$$D^2 \in C^0(\tilde{\mathfrak{g}}, \Omega^2(M, \operatorname{End} E))$$

and in

[redacted] $\displaystyle\int_0^1 dt \int_{\gamma_p} \operatorname{tr}\left(D^2 + tD\theta + (t^2-t)\theta^2\right)^p \theta \frac{1}{p!}$

we want to collect terms of degree p in M so as to get a non-trivial integral over γ_p. These will be of the following sort

$p \quad D\theta$'s $\qquad \displaystyle\int_0^1 dt \operatorname{tr} \frac{(D\theta)^p\theta}{p!} t^p = \frac{1}{(p+1)!} \operatorname{tr} \theta (D\theta)^p$

$p-2 \quad D\theta$'s ; this will have symmetrized products of

D^2 , $(D\theta)^{p-2}$, θ^2

$p-4 \quad D\theta$'s ; this will involve

$(D^2)^2 \quad (D\theta)^{p-4} \quad (\theta^2)^2 \qquad$ etc.

Notice that there are p θ-terms as there should be.

Now I can't seem to see a simple formula for this thing so I should look at Connes approach