August 1, 1984

Problem: Bismut claims that

$$\mathrm{tr}\ T\left\{ e^{\int_0^1 [-\dot{x}^\mu A_\mu(x) + \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}(x)]dt} \right\}$$

is a closed equivariant form on the loop space.

Let's consider the case of a line bundle, where this becomes the exponential of the form

$$\int_0^1 [-\dot{x}^\mu A_\mu(x) + \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}(x)]\,dt$$

having components of degrees 0, 2. From the structure of this it looks as if it came from a form on $\Omega M \times S^1$ having components of degrees 1 and 3.

Here's a possible construction. We have the evaluation map $\Omega M \times S^1 \longrightarrow M$, $(x,t) \mapsto x_t$, and if we pull the bundle back it acquires a canonical automorphism, namely, the auto of E_{x_t} given by parallel transport along the loop x starting at t. From a line bundle, however, this gives only a 1-form on $\Omega M \times S^1$.

What is it? The automorphism is of E_{x_t} is the number $\exp \int_t^{t+1} [-\dot{x}^\mu A_\mu(x)]\,dt$ which is an S^1-valued function on $\Omega M \times S^1$ coming from ΩM. The corresponding 1 form is its logarithmic derivative, which will come from ΩM, hence it will integrate over the circle to zero.

New approach: Starting from a 1 form $A = dx^\mu A_\mu$ on M, one manages to construct forms

$$\int_0^1 -\dot{x}^\mu A_\mu\, dt \qquad \int_0^1 \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}\, dt$$

of degrees 0 and 2 on ΩM. Recall what

Bott said about this. In general, one can call a differential form on ΩM a natural transformation which assigns to any map

$$Y \times S^1 \xrightarrow{f} M$$

a differential form on Y. Given a form ω on M we have two simple ways to produce a form on Y, namely

$$(pr_1)_* f^* \omega \qquad , \quad (pr_1)_* (dt \cdot f^* \omega)$$

besides restriction to a point of S^1. More generally one can form the map

$$Y \times (S^1)^k \longrightarrow (Y \times S^1)^k \xrightarrow{f^k} M^k$$

and so given forms $\omega_1, \dots, \omega_k$ one can pull back the product $\prod_j pr_j^* \omega_j$ and integrate over $(S^1)^k$. These have something to do with Chen's iterated integrals.

Next let's see what can be done with a one-form $A = dx^\mu A_\mu$ on M. Pulling back via the evaluation

$$\Omega M \times S^1 \xrightarrow{ev} M$$

we obtain the 1-form

$$\delta x^\mu A_\mu + dt \, \dot{x}^\mu A_\mu$$

where $\delta x^\mu = \psi^\mu$. Then applying pr_{1*} we get

$$pr_{1*}(ev^* A) = \int_0^1 dt \, \dot{x}^\mu A_\mu \qquad \text{(the desired 0 form)}$$

Then

$$d \, pr_{1*}(ev^* A) = - pr_{1*} \, ev^* \, dA$$

$$= -pr_{1*}\left\{ \tfrac{1}{2}\, \delta x^\mu \delta x^\nu F_{\mu\nu} + \tfrac{1}{2}\, dt\, \dot{x}^\mu \delta x^\nu F_{\mu\nu} + \tfrac{1}{2}\, \delta x^\mu\, dt\, \dot{x}^\nu F_{\mu\nu} \right\}$$

$$= -\int_0^1 dt\ \dot{x}^\mu \delta x^\nu F_{\mu\nu}$$

On the other hand

$$pr_{1*}\, dt\ ev^* F = \int_0^1 dt\, \tfrac{1}{2}\delta x^\mu \delta x^\nu F_{\mu\nu}$$

and if X is the time translation vector field:

$$\iota_X\, \delta x_t^\mu = \dot{x}_t^\mu$$

then

$$\iota_X\, pr_{1*}\, dt\ ev^* F = \int_0^1 dt\ \dot{x}^\mu \delta x^\nu F_{\mu\nu}.$$

Hence we see that

$$d \int_0^1 dt\ \dot{x}^\mu A_\mu + \iota_X \int_0^1 dt\ \tfrac{1}{2}\delta x^\mu \delta x^\nu F_{\mu\nu} = 0$$

Finally as F is closed

$$d\ pr_{1*}\, dt\ ev^* F = -pr_{1*}\, d(dt\ ev^* F) = 0$$

It seems that the operator $d + \iota_X$ on forms on ΩM is essentially the super-symmetry operation. In order to see this I have to think of the forms as an algebra with certain generators, namely, the functions x_t^μ for different μ, t together with their differentials $\delta x_t^\mu = \psi_t^\mu$. Then $d + i_X$ is an odd degree derivation such that

$$(d + i_X)\, x_t^\mu = \delta x_t^\mu = \psi_t^\mu$$

and $$(d+i_X)\,\psi^\mu_t = \dot{x}^\mu_t$$

Now let us consider the supersymmetry operator which is defined by

$$\delta x^\mu = \varepsilon\psi^\mu \qquad \delta\psi^\mu = -\varepsilon\dot{x}^\mu$$

(Check: $$\delta\left(\dot{x}^2+\psi\dot{\psi}\right) = 2\dot{x}\,\varepsilon\dot{\psi} + (-\varepsilon\dot{x})\dot{\psi} + \psi(-\varepsilon\ddot{x})$$
$$= \varepsilon\left[\dot{x}\dot{\psi} + \ddot{x}\psi\right] = \varepsilon\,\partial_t(\dot{x}\psi)$$
so the action $\int(\dot{x}^2+\psi\dot{\psi})\,dt$ is supersymmetric.)

Then
$$\varepsilon(d - i_X)\begin{cases} x^\mu_t \\ \psi^\mu_t \end{cases} = \begin{cases} \varepsilon\psi^\mu_t \\ -\varepsilon\dot{x}^\mu_t \end{cases}$$

so that the supersymmetry operator is
$$\delta = \varepsilon(d-i_X).$$

Check:
$$d\int_0^1 dt\ \dot{x}^2 = \int_0^1 dt\ 2\dot{x}(\delta x)^{\cdot}$$
$$i_X\int_0^1 dt\ \delta x(\delta x)^{\cdot} = \int_0^1 dt\left(\dot{x}(\delta x)^{\cdot} - \delta x\,\ddot{x}\right)$$

so that
$$(d-i_X)\int_0^1 dt\left(\dot{x}^2 + \delta x\,\delta\dot{x}\right) = 0.$$

Note that if we describe the algebra of forms on the loop space as being generated by the superfield $X = x+\theta\psi$, then the operator $d-i_X$ which sends $x \mapsto \psi$, $\psi \mapsto -\dot{x}$ is given by

$$\underset{x+\theta\psi}{\underset{\parallel}{X}} \longmapsto \underset{\psi+\theta\dot{x}}{\underset{\parallel}{(\partial_\theta+\theta\partial_t)X}}$$

(Note the sign is alright as an odd derivation δ would satisfy

$$\delta(x+\theta\psi) = \delta x - \theta\,\delta\psi\,.)$$

Thus we have

$$d - \iota_X \longleftrightarrow \partial_\theta + \theta\partial_t$$

relative to these pictures.

Recall that the superfield action is

$$-\int dt \int d\theta \; \underbrace{(\partial_\theta - \theta\partial_t)X}_{(\psi - \theta\dot{x})} \cdot \underbrace{\partial_t X}_{(\dot{x} + \theta\dot{\psi})} \;=\; +\int dt\,(\dot{x}^2 + \psi\dot{\psi}).$$

We can now see why this action has to be killed by $d - i_X \longleftrightarrow \partial_\theta + \theta\partial_t$ in some heuristic sense – We know $\partial_\theta + \theta\partial_t$ commutes with $\partial_\theta - \theta\partial_t$, ∂_t so the action is obviously unchanged under the infinitesimal supersymmetry $\delta X = \varepsilon(\partial_\theta + \theta\partial_t)X$.

So the main problem is to understand Bismut's theorem, that the differential form on ΩM

$$\mathrm{tr}\,T\{\, e^{\int_0 dt\,(-\dot{x}_\mu A_\mu(x) + \frac{1}{2}\delta x^\mu \delta x^\nu F_{\mu\nu}(x))} \}$$

is killed by $d - i_X$. (i.e. that it is supersymmetric) Suppose we introduce $\bar{\eta}, \eta$ fields, so that this form can be obtained as the coefficient of $\bar{e}^\mu$ in

$$\int \mathcal{D}\bar\eta \mathcal{D}\eta \; e^{-\int_0^1 dt\, \bar\eta (\partial_t + \mu + \dot{x}^\mu A_\mu - \frac{1}{2}\delta x \delta x F)\eta}$$

$$= \det\left(1 - e^{-\mu} T\left\{ e^{-\int_0^1 dt (\dot{x}^\mu A_\mu - \frac{1}{2}\delta x \delta x F)} \right\}\right)$$

So it would seem to be enough to prove

$$e^{-\int_0^1 dt\, \bar\eta(\partial_t + \mu)\eta} \quad e^{\int_0^1 dt\, \bar\eta(-\dot{x}^\mu A_\mu + \frac{1}{2}\delta x^\mu \delta x^\nu F_{\mu\nu})\eta}$$

is killed by $d - \iota_X$. So it seems this reduces us to the abelian case, i.e. to the form

$$e^{\int_0^1 dt\, (-\dot{x}^\mu \bar\eta A_\mu \eta + \frac{1}{2}\delta x^\mu \delta x^\nu \bar\eta F_{\mu\nu}\eta)}$$

where we have checked the result, at least for rank 1.

Let's explore the topology behind constructing equivariant (for S^1) classes on $\tilde\Omega M$ which restrict over the fixpt submanifold M to the character of E. One can suppose E is the canonical bundle over M = Grassmannian. Then $\tilde\Omega M$ [redacted] $\tilde\Omega BU(k)$ has the homotopy type of the fibre bundle over $BU(k)$ associated to the conjugation action of $U(k)$ on itself. So one sees that cohomologically

$$H^*(\tilde\Omega BU(k)) = S[c_1, c_2, \dots] \otimes \Lambda[e_1, e_3, \dots]$$

We want the equivariant cohomology of $\tilde\Omega BU(k)$ for the natural S^1 action. This is given by the standard spectral sequence

c_1	0		0	
e_1	0	$e_1 u_1$	0	
1	0	u	0	u^2

Notice first that e_1 comes from an equivariant class. [redacted] To see this, recall that we are looking at the bundle over ΩM obtained by taking the fibre at the $t=0$ point of a loop. Wait: Is this an equivariant bundle? A point in the bundle is a loop x in M together with an element $\xi \in E_{x_0}$. Given $a \in S^1$ we can translate the loop: $x_t^a = x_{t+a}$ and then map $\xi \in E_{x_0}$ to the point of $E_{x_0^a} = E_{x_a}$ corresp. by parallel transport along x_t between [redacted] $t=0$ and $t=a$. This gives an $\mathbb{R}$-action on E it seems, but not an S^1-action since full parallel transport along the [redacted] loop will not give the identity.

To show e_1 comes from an equivariant class on ΩM I will exhibit an invariant map from ΩM to S^1, namely, the determinant of the monodromy around a loop.

Lets review the problem. We have this v.b. E with (unitary) connection over M. We know how to define classes c_j, e_j in $H^*(\Omega M)$ associated to E. To a loop x in M we can associated the fibre E_{x_0} together with the autom. of this fibre given by the monodromy along the loop. Thus over ΩM we get a vector bundle (namely the pull back of E via the map $\Omega M \to M$, $x \to x_0$) together with an automorphism of this bundle. Now [redacted] to a bundle with autom. we can assign classes c_j, e_j.

The problem is whether these classes can be realized by equivariant cohomology classes for the

S^1 action on ΩM. The e_1 class can be concretely described as the map $\Omega M \to S^1$ giving the determinant of the bundle automorphisms. The next class of interest is the first Chern class c_1.

Let's then consider just the case of a line bundle E over M. Then we wish to construct an equivariant class in $H^2_{S^1}(\Omega M)$ which becomes c_1 of the bundle $ev_0^*(E)$ if we forget the S^1-action, and which restricts to the element of $H^2_{S^1}(M)$ given by c_1 of E with the trivial S^1-action.

$$\begin{tikzcd} H^2_{S^1}(\Omega M) \arrow[r] \arrow[d] & H^2_{S^1}(M) = H^2_{S^1} \oplus H^2(M) \arrow[d] \\ H^2(\Omega M) \arrow[r] & H^2(M) \end{tikzcd}$$

Now if we want c_1 to exist in integral cohomology, then it means there is an S^1-equivariant line bundle on ΩM. No this is wrong because the action of S^1 isn't free on ΩM.

Here is a potential cohomology argument that the cohomology of $\Omega BU(k)$ is realized by equivariant classes. One uses that the E_2 term of the relevant spectral sequence is torsion-free over $\mathbb{Z}$, so that if the spectral sequence degenerates rationally it degenerates. So if one can produce differential forms one wins. This should be enough to handle line bundles. First obstruction should be for $SU(2)$ with the differentials

?

August 2, 1984

New viewpoint: The reason you have difficulties with the S^1-action is perhaps because you haven't worked out the formalism of gravity.

To begin, let's consider the setup without the S^1 action. We have a vector bundle E/M and want to construct in a natural way cohomology classes on ΩM. One way to proceed is to consider ΩE as a bundle over ΩM for the ring $\Omega\mathbb{C} = C^\infty(S^1)$ and to use what we know about characteristic classes for families of vector bundles on a fixed manifold. In general given a family of bundles on N parametrized by Y, we identify this with a v.b. on $Y \times N$, then take Künneth components of the latter wrt homology classes in N to get characteristic cohomology classes in Y for the family. In our case N is the circle, so we obtain two sorts of classes, even degree ones by restriction to a point of N, and odd degree ones by integrating over S^1.

Now given E/M we have over ΩM a canonical family of vector bundles on S^1, whose corresponding bundle over $\Omega M \times S^1$ can be identified with the pull-back of E under the evaluation map. Hence from E we can construct a family of even and a family of odd Chern type classes on ΩM.

These are the only (essentially) possibilities, as clearly the universal case is where $M = BU(k)$, in which case we know the cohomology of $\Omega BU(k)$. Also $\Omega BU(k)$ is the classifying space of $\Omega U(k)$, so that there is no difference between the characteristic classes for the following two problems:

1) characteristic classes on ΩM attached to E/M

2) characteristic classes for families of bundles over S^1.

(This must hold in general, since $Map(N, BU(k))$ classifies families of bundles on N. Also a natural class on $Map(N,M)$ attached to a E/M is determined by the universal case $M = BU(k)$. Thus both types of char. classes can be identified with coh. classes of $Map(N, BU(k))$.)

Now we wish to bring in the circle action. We want S^1-equivariant cohomology classes on ΩM attached to a v.b. E/M, which are natural. Hence we only have to worry about the universal case $M = BU(k)$, and so we are interested in the equivariant cohomology of $\Omega BU(k)$, i.e., the cohomology of

$$Z = PS^1 \times^{S^1} \Omega BU(k)$$

A map from Y into this space Z determines a circle bundle P/Y together with an equivariant map $P \longrightarrow \Omega BU(k)$, equivalently $P \times S^1 \longrightarrow BU(k)$. Such a map which is S^1-equivariant is the same as a map $P \longrightarrow BU(k)$. Thus it seems that what we get is that maps $Y \to Z$ are the same up to homotopy as giving a circle bundle P over Y and a vector bundle over P.

More generally:

<u>Prop</u>: Equivariant cohomology classes of ΩM can be identified with characteristic classes for pairs $(P/Y, P \to M)$ where P/Y is an S^1-bundle and $P \to M$ is a map.

In effect given P/Y and $P \to M$, a point of P determines a map $S^1 \to P$, and hence a loop in M. Thus we get an equivariant map $P \longrightarrow \Omega M$. Choosing a classifying map $P \to PS^1$ over $Y \to BS^1$, we get an equivariant map $P \to PS^1 \times \Omega M$ hence $Y \longrightarrow PS^1 \times^{S^1} \Omega M$. The converse process is also clear.

At this point we see the topological problem, namely we are trying to determine characteristic classes for pairs consisting of a circle bundle and vector bundle over it. Let's look at this in the case of line bundles:

We have

$$\begin{array}{c} P \; L \\ \downarrow{\scriptstyle \pi} \\ Y \end{array}$$

where P is a circle bundle over Y and L is a line bundle on P.

The problem is whether L comes from a line bundle over Y. Look at the Gysin sequence

$$H^2(Y) \longrightarrow \underset{\substack{\cup \\ c_1(L)}}{H^2(P)} \xrightarrow{\delta = \pi_*} H^1(Y) \xrightarrow{e} H^3(Y)$$

in integral cohomology. The obstruction is $\pi_* c_1(L)$ in $H^1(Y)$. If this is non-zero it can be detected by a map $S^1 \longrightarrow Y$. But any circle bundle over a circle is trivial, so where are we?

In the case where $P = Y \times S^1$, then we have two characteristic classes we can attach to Y, L. We take $c_1(L)$ and either integrate over the fibre to get an element $\pi_* c_1(L) \in H^1(Y)$, or we can pull c_1 back by a <u>section s to get</u> $s^* c_1(L) \in H^2(Y)$.

Review: [redacted] I am given an S^1-bundle P/Y and a line bundle L over P. I want to construct a closed 2-form on Y. (I would really like to construct a line bundle on Y which is somehow the average of the line bundles $R_t^*(L)$, $t \in S^1$ where $R_t : p \mapsto pt$. [redacted] But I don't see how to do this, and maybe it is impossible except [redacted] on the differential form level.)

[redacted] Let x^μ be coordinates on P and let $x^\mu_t(p) = x^\mu_{pt}$. We define on P a function

$$f : p \longmapsto \int_0^1 dt \ \dot{x}^\mu(pt) A_\mu(x_{pt})$$

which is S^1-invariant.

Better, let us assume L is trivial and let $dx^\mu A_\mu$ be the connection form. Then the above function is the push down of the 1-form to Y.

Next we define a 2-form on P

$$\omega = [redacted] \int_0^1 dt \ \tfrac{1}{2} dx^\mu_{pt} \, dx^\nu_{pt} \, F_{\mu\nu}(x_{pt})$$

which I think is just the average of the curvature form $F = dA$.

Repeat: We have P/Y, L over P, and are given a connection in L. The curvature is a 2-form F on P which can be averaged to get an S^1-invariant closed 2-form ω. If we trivialize the line bundle L locally over Y, this is possible as a line bundle over a circle is trivial, then the connection [redacted] gives a connection form A which can be integrated over the fibres of $\pi : P \longrightarrow Y$ to get a function

$$f = \pi_*(A) \quad \text{on } Y$$

locally defined. A different trivialization changes A by

$d \log g$ where g is an S^1-valued function on P whence the change in f is $\pi_*(d\log g)$ which is a locally constant function on Y with values in $2\pi i\mathbb{Z}$. Thus e^{f} is an intrinsic function on Y, which I now recognize as the monodromy in the fibres relative to the given connection.

Therefore intrinsically associated to the connection in L are this monodromy function e^f and the averaged curvature ω which is an invariant closed 2-form.

<u>Proposition:</u> $\quad -df = i_X \omega$

<u>Proof:</u> Let $R : P \times S^1 \longrightarrow P$ be $R(p,t) = pt$. Then

$$\omega = \int_0^1 dt \; j_t^* R^* F \qquad j_t : \begin{array}{l} P \longrightarrow P \times S^1 \\ p \longmapsto (p,t) \end{array}$$

$$= \int_0^1 dt \; R_t^* F$$

so

$$i_X \omega = \int_0^1 dt \; i_X R_t^* F$$

$$= \int_0^1 dt \; j_t^* \, i\!\left(\tfrac{\partial}{\partial t}\right) R^* F$$

$$= (pr_1)_*(R^* \underbrace{F}_{dA}) = -d\,(pr_1)_*(R^* A) = -df$$

In coordinates.

$$\omega = \int_0^1 dt \, \tfrac{1}{2}\, dx_t^\mu \, dx_t^\nu \, F_{\mu\nu}(x_t)$$

$$i_X \omega = \int_0^1 dt \;\; \dot{x}_t^\mu \, dx_t^\nu \, F_{\mu\nu}(x_t)$$

$$f = \int_0^1 dt \;\; \dot{x}_t^\mu \, A_\mu(x_t)$$

$$\cancel{A = dx^\mu A_\mu(x)}$$

$$\cancel{R^*A = dx_t^\mu A_\mu(x_t) + dt\,\dot{x}_t^\mu A_\mu(x_t)}$$

$$df = \int_0^1 dt \left[d\dot{x}_t^\mu A_\mu(x_t) + \dot{x}_t^\mu\, dx_t^\nu\, \partial_\nu A_\mu(x_t) \right]$$

$$= \int_0^1 dt \left[dx_t^\mu (-\dot{x}_t^\nu)\partial_\nu A_\mu(x_t) + \text{———} \right]$$

$$= -\int_0^1 dt\ \dot{x}_t^\mu\, dx_t^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)(x_t)$$

$$= -i_X \omega .$$

Now we have to recall how we reconstruct from such a pair f, ω a form on Y using the connection form θ in P/Y. Namely we modify ω by an exact form so that it becomes killed by i_X.

$$\iota_X[\omega - d(f\theta)] = \iota_X \omega - \iota_X(df\,\theta) - \cancel{\iota_X(f\,d\theta)}$$

$$= -df + df \underbrace{(\iota_X \theta)}_{1} = 0$$

Thus $\omega - d(f\theta) = (\omega - \theta i_X \omega) - f(d\theta)$ is a closed 2-form which descends to Y. The only problem is that f is defined up to $2\pi i\mathbb{Z}$.

Consequently we see that the only way we can obtain something even partially defined is to exponentiate in a setting where $d\theta = 1$. ?

Remarks for further research: An interesting problem is to find the precise nature of the kind of class just constructed. Is it an element of $H^2(Y, \mathcal{a})$ for some twisted coefficients $\mathcal{a}$?

1) If one restricts ~~attention~~ to those situations $(P/Y, L)$ where the monodromy element in $H^1(Y, \mathbb{Z})$ is trivialized, i.e. one gives a logarithm for the monodromy map $Y \to S^1$, then one can modify the connection so the mono-dromy becomes trivial. In this case by taking flat sections over the circle fibres, we see that the line bundle L descends.

So we can compute the cohomology of the universal covering of $PS^1 \times^{S^1} \Omega BU(1)$. Analogy with orientations and spin structures.

So now let's go on to Bismut's construction. We start with E/M and let P be the principal bundle; the group is $U(k)$. In P we have the connection form $A \in \Omega^1(P) \otimes \mathfrak{g}$ and the curvature $F = dA + A^2 \in \Omega^2(P) \otimes \mathfrak{g}$.

Next consider ΩP as a principal $\Omega U(k)$ bundle over ΩM. Let's first recall what we know about characteristic classes for principal P/Y bundles over gauge groups such as $\Omega U(k)$. One identifies P with a v.b. over the product $Y \times S$, takes the character and integrates over cycles in S. In the present case the bundle over $\Omega M \times S$ is the pull-back of E/M under the evaluation map.

Let's also construct the characteristic forms on ΩM according to the general ideas. We choose a connection in $\Omega P/\Omega M$, this is the same as a **partial** ΩM connection in the bundle over the product, one extends to a full connection to get character forms on the product $\Omega M \times S^1$. For the present example we already have the **needed** full connection, and we see we are taking the character of E, pulling back via evaluation and integrating over cycles.

The interesting thing here is that nowhere appears the idea of parallel transport along the loops, i.e. adding up the curvature of E along a path. This **is a** new feature of the Bismut setup **—** clearly reminiscent of quantization & the $\bar{\eta}\,\eta$ term.

Put another way Chern characters are usually found by exponentiating a **single** curvature, not by some time-ordered exponential.

August 3, 1984

Some differential geometry. Let Z be a submanifold of a Riemannian manifold X, let ν be the normal bundle to Z. What structure does ν have? By definition ind. of metric

$$0 \longrightarrow T_Z \longrightarrow T_X|Z \longrightarrow \nu \longrightarrow 0$$

and because of the metric $\nu = (T_Z)^{\perp} \subset T_X|Z$. ν inherits a inner product and a connection. Parallel transport of a normal vector v along a curve in Z is ~~the same as parallel~~ infinitesimally given by parallel transport in X followed by projecting onto ν.

Given a vector bundle E with connection and inner product over a Riemannian manifold M, we can define a metric on E. What is the curvature of E? Problem: Are straight line segments in the fibres of E geodesics?

(Example: Take E to be a line bundle over a circle, and consider two points ξ_0, ξ_1 in the fibre of E over the basepoint x_0 of the circle, such that $\parallel$ transport around the circle carries ξ_0 to ξ_1. Any horizontal curve in E has the same length as its projection to the base. Thus the horizontal curve γ over the circle going from ξ_0, ξ_1 is a geodesic in E. Now scaling we get a curve $\lambda\gamma$ which is a geodesic in E joining $\lambda\xi_0$ to $\lambda\xi_1$ and these curves all have the same length. For large λ the straight line in E_{x_0} from $\lambda\xi_0$ to $\lambda\xi_1$ has large length. This shows straight lines the fibres of E are not necessarily geodesics of the shortest length. However it is the case in this example and possibly in general that the straight lines in the fibres of E are geodesics.)

<u>Question</u>: Consider the differential forms on ΩM with the operator $d - u\iota_X$ where u is a non-zero scalar. Is the cohomology of $(\Omega(\Omega M))^{S^1}$ with the differential $d-ui_X$ isomorphic to $H_{DR}(M)$ with its natural $\mathbb{Z}_2$-grading?

Assuming this is so, it means that this Witten complex $\left((\Omega^{\cdot}(\Omega M))^{S^1}, d-ui_X\right)$ is isomorphic to the localized cyclic homology of $C^\infty(M)$. This is consistent with various analogies:

(i) way ΩM enters into the index thm.

(ii) analogy of the basic exact sequence + spectral sequence of cyclic theory with equivariant cohomology for a circle action.

What Bismut's result does is to show that the natural restriction to the fixpoint set map

$$H(\Omega^{\cdot}(\Omega M)^{S^1}, d-ui_X) \longrightarrow H_{DR}(M)$$

is onto $H^{ev}(M)$, because he constructs liftings for the classes $ch(E)$.

Natural question: Can you generalize somehow from $\mathcal{A} = C^\infty(M)$ to a general algebra $\mathcal{A}$?

Comments on "Index theorem and equivariant cohomology on the loop space" by J.-M. Bismut.

There is some confusion about the term 'equivariant cohomology' as used in this paper, which might be clarified after Definition 2.2 or before Theorem 3.9 . Usually by the equivariant cohomology $H^*_S(M)$, $* \in \mathbb{N}$ for a circle action one means the cohomology of the fibre space over the classifying space of the circle with fibre M . According to the Atiyah-Bott paper (reference [4]), the idea of introducing the $\mathbb{Z}_2$-graded complex of invariant forms with the differential $d + i_X$ is due to Witten, who argued that for a compact manifold the cohomology of this complex is the cohomology of the fixpoint set. The link between Witten's cohomology and equivariant cohomology is discussed carefully in [4] for the case of finite dimensional manifolds. However, for the loop space $\mathcal{L}M$, I believe it is not known how the Witten cohomology is related to the equivariant cohomology of the loop space. One expects that the restriction to the fixpoint map

$$H^*_{\text{Witten}}(\mathcal{L}M) \longrightarrow H^*(M) \qquad , \qquad * \in \mathbb{Z}_2$$

is an isomorphism. An immediate consequence of Theorem 3.9 is that this map surjects onto the even cohomology of M , since the even cohomology is spanned by Chern character classes of vector bundles. One can probably show in a similar way that the odd cohomology is in the image of the above map, by using the fact that the odd cohomology is spanned by the Chern character classes of invertible matrix-valued functions on M .

Let's return to the index thm. for families and try to write down an account before leaving here.

We start with a family of Dirac operators. This means that we have a family of compact Riemannian manifolds, i.e. a differentiable fibre bundle X/Y with compact fibres, and an inner product on the relative tangent bundle $T_{X/Y}$. We can then form the Clifford algebra bundle $C(T_{X/Y})$ and we suppose given a super-bundle ξ over X which is a module over $C(T_{X/Y})$. For example we could ask that at each point x ξ is an irreducible $C(T_{X/Y})_x$ module, in which case the restriction of ξ to each fibre X_y is a module of spinors. The obstruction to the existence of ξ is a topological one.

Given ξ, we also need a partial connection on ξ in the longitudinal directions, compatible with the Levi-Civita connections in the fibres. Then we can write down the Dirac operator on sections of ξ; it linear over functions on Y.

We have an infinite-diml vector bundle over Y, which one might denote $\mathcal{H} = \pi_*(\xi)$, whose fibre at y, is the sections of ξ over X_y. (In order to see $\mathcal{H}$ is a bundle one needs local trivializations of (X, ξ) over Y.)

In order to set up the theorem we need a connection in $\mathcal{H}$, i.e. we must trivialize $\mathcal{H}$ over curves in Y.

Start again: We begin with a differentiable fibre fibre X/Y and a familiys of Dirac operators on the fibres. This means giving Riemannian metrics on the fibres, and a bundle ξ over X which is a Clifford module over the longitudinal Clifford algebra, and finally a longitudinal connection ξ compatible with the LC connection.

But now we need a connection on $\mathcal{H} = \pi_*(\xi)$. This means I have to trivialize the setup over curves. So let me suppose the base Y is a curve. What is confusing is that I cant trivialize the bundle of Riemannian manifolds, since Riemannian manifolds are distinguished by their curvature.

Consider a connection on X/Y and metrics along the fibres. Then from a metric on Y we can define a metric on X, by defining horizontal + vertical subbundles of T_X to be orthogonal, and the metric on $T_{X,\text{horiz}}$ to be such that $T_{X,\text{horiz}} \longrightarrow \pi^* T_Y$ is an isometry. To handle this I take an orthonormal coframe in X of the form

$$\eta^1, \dots, \eta^p \ \omega^1, \dots, \omega^n$$

where the η^j are the pull-backs of an orth. coframe in Y. These are now sections of T_X^*, and the ω^k restrict on each fibre to an orthonormal coframe.

Now we have to write the LC connection on X, which means I have to give $D : \Gamma(T_X^*) \longrightarrow \Gamma(T_X^* \otimes T_X^*)$ characterized by the requirements that it lift d and preserve the metric. To simplify let's assume that the base Y is flat, say $\eta^j = dy^j$, so that $d\eta^j = 0$.

At this point it becomes clear that we might as well fix the η^j and instead introduce the frame bundle of the tangent bundle along the fibres.

August 4, 1984

Consider a differentiable fibre bundle X/Y, [redacted] $\pi : X \to Y$ and a family of Dirac operators on the fibres. This means one has metrics given on the fibres, a vector bundle ξ on X which is a module over the Clifford algebra of $T_{X/Y}$ and a partial connection along the fibres on this vector bundle compatible with the LC connection on [redacted] the fibres. Let $\mathcal{H}$ be the infinite-dimensional bundle $\pi_*(\xi)$.

The problem is to get detailed descriptions so that I can see what is happening. For example, how are we to describe concretely the case where ξ restricts to the spinors [redacted] on each fibre? In practice one doesn't describe spinors concretely very often. Rather, like in physics, one describes the Clifford algebra which operates on the spinors by generators and "commutation" relations.

So $\mathcal{H}$ also will not be described concretely but rather it will understood by a certain algebra of operators. In fact we are interested in the algebra of forms on Y with values in End $\mathcal{H}$.

Let's recall that the algebra of diff'l operators on the spinors over a Riemannian Spin$_c$ manifold M is described by operators f, γ^μ, D_μ corresponding to an orthonormal frame ω^μ. [redacted] [redacted] We need the Levi-Civita connection:

$$D_\mu(\omega^\alpha) = \Gamma^\alpha_{\mu\nu}\,\omega^\nu$$

This is uniquely characterized by

(i) preserves metric $\iff$ $\Gamma^\alpha_{\mu\nu} = -\Gamma^\nu_{\mu\alpha}$

(ii) torsion $= 0$ $\iff$ [redacted] $\Gamma(T^*) \xrightarrow{D} \Gamma(T^* \otimes T^*) \longrightarrow \Gamma(\Lambda^2 T^*)$ is d

$\iff$ $\quad d\omega^\alpha = \Gamma^\alpha_{\mu\nu}\,\omega^\mu\omega^\nu$

Thus if the numbers $\hat{\Gamma}^{\alpha}_{\mu\nu}$ are given by

$$d\omega^{\alpha} = \tfrac{1}{2}\hat{\Gamma}^{\alpha}_{\mu\nu}\,\omega^{\mu}\omega^{\nu} \qquad \hat{\Gamma}^{\alpha}_{\mu\nu} = -\hat{\Gamma}^{\alpha}_{\nu\mu}$$

then we have the following conditions determining $\Gamma^{\alpha}_{\mu\nu}$:

$$\text{(i)} \qquad \Gamma^{\alpha}_{\mu\nu} = -\Gamma^{\nu}_{\mu\alpha}$$

$$\text{(ii)} \qquad \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu} = \hat{\Gamma}^{\alpha}_{\mu\nu}$$

In fact the formula for Γ is

$$\boxed{\Gamma_{ab}{}^{c} = \tfrac{1}{2}\left(\hat{\Gamma}_{ab}{}^{c} - \hat{\Gamma}_{ac}{}^{b} - \hat{\Gamma}_{bc}{}^{a}\right)}$$

(Check: The right side is clear skew-sym in b,c. If we skew-symmetrize the above over a,b the last two terms drop out and we get $\hat{\Gamma}_{ab}^{c}$ as desired.) (Finally the uniqueness of Γ is proved because assuming $f(a,b,c)$ is skew in b,c and symm. in a,b then it changes sign under the product of these two transposition which is a 3-cycle

$$f(a,b,c) = -f(a,c,b) = -f(c,a,b)$$

which is of order $3 \Rightarrow f = -f \Rightarrow f = 0$.)

Now I propose to approach the problem of a differentiable fibre bundle, or rather, a family of Dirac operators, by starting with a Dirac operator on the total space X, and then rescaling in the transverse direction.

Review of Godbillon-Vey: Let X be a manifold with transversally oriented foliation of codim 1. Let's trivialize the normal bundle $Q = \nu$ of the foliation and pick a 1-form ω on X spanning Q^* (= annihilator of longitudinal vectors) at each point. This is possible as we assume ν oriented, and such a ω corresponds also to a choice of scalar product on ν.

Let's review Bott's thm. in general. In general the foliation gives the **exact** sequence

$$0 \longrightarrow S \longrightarrow T \longrightarrow Q \longrightarrow 0$$

where $\Gamma(S)$ is closed under bracket, or equivalently

$$d\{\Gamma(Q^*)\} \subset \Gamma(T^* \wedge Q^*).$$

It follows that the ideal I of $\Omega^*(X)$ generated by $\Gamma(Q^*) \subset \Omega^1(X)$ is stable under d. Get complex:

$$I^p/I^{p+1}: \text{[redacted]} \quad \underline{\Lambda^p Q^*} \xrightarrow{\bar{d}} \underline{S^* \otimes \Lambda^p Q^*} \xrightarrow{\bar{d}} \underline{\Lambda^2 S^* \otimes \Lambda^p Q^*} \longrightarrow \cdots$$

with $\bar{d}^2 = 0$. In particular [redacted] the bundle [redacted] Q^* has a canonical flat partial connection along the leaves.

In the oriented codim 1 case we take the frame ω for Q^*, then we can write

$$d\omega = \theta\omega$$

for some $\theta \in \Omega^1(X)$. The class of θ [redacted] in $\Gamma(S^*)$ is the partial connection form, so choosing θ amounts to lifting the partial connection on Q^* to a full connection. The curvature of the connection is $d\theta$ which lies in the ideal generated by Q^* (since we know the partial connection is flat, $d\theta$ goes to zero in $\Gamma(\Lambda^2 S^*)$, so lies in $\Gamma(T^* Q^*)$.) Directly:

$$(d\theta)\omega \text{ [redacted] } - \theta \underbrace{d\omega}_{\theta\omega} = d(d\omega) = 0$$

$$\therefore \ (d\theta)\,\omega = 0 \implies d\theta = \theta_1 \omega \quad \text{for some } \theta_1.$$

Botts theorem now says that $(d\theta)^2 = 0$, and it follows because $d\theta$ is in the ideal generated by Q^* and this has zero square.

Now to construct the Godbillon-Vey, or other secondary classes, one needs two reasons for the primary class to vanish. In the case of $(d\theta)^2$, Bott's theorem gives one reason, another comes from the triviality of Q^*. So if we use the linear path $t\theta$, we get the formula

$$0 = (d\theta)^2 = d\int_0^1 dt\ 2(t\,d\theta\cdot\theta) = d(\theta\, d\theta)$$

and the secondary class is then:

$$\theta\, d\theta .$$

Generalization: Consider a foliation with a trivialized normal bundle, say via forms ω^μ which are a frame for Q^*, then we have

$$d\omega^\mu = \theta^\mu_\nu \omega^\nu$$

and the matrix form $\theta = (\theta^\mu_\nu)$ constitutes an extension of the given flat partial connection to a full connection. Thus we get a map

$$W(\mathfrak{gl}_n) \longrightarrow \Omega^*(X)$$

which according to Bott's thm. will kill $S_k(\mathfrak{g}^*)$ for $k > n$ ($n =$ codim foliation). So if we put

$$W_n = S(\mathfrak{g}^*)/S_{>n}(\mathfrak{g}^*) \otimes \Lambda\mathfrak{g}^* \qquad \mathfrak{g} = \mathfrak{gl}_n$$

then we get a map of complexes

$$W_n \longrightarrow \Omega^*(X).$$

Now in fact we know that $W(\mathfrak{g})$ contains a minimal subalgebra generated by character forms $c_{2k} = tr\left(\frac{\Omega^k}{k!}\right)$ and the transgression or Chern-Simons forms

$$h_{2k-1} = \int_0^1 dt \; tr\left(\frac{(t\Omega + (t^2-t)\theta^2)^{k-1}}{(k-1)!}\theta\right)$$

which satisfy $dh_{2k-1} = c_{2k}$, so we get a smaller model for W_n, namely,

$$S(c_2, c_4, \dots, c_{2n}) \otimes \Lambda[h_1, h_3, \dots, h_{2n-1}] \Big/ \text{ideal generated by monomials in the } c_{2j} \text{ of degree } > 2n.$$

For example with $n=1$ we get the complex with basis

$$1,\ h \xrightarrow{d} c,\ hc,\ \cancel{c^2},\ \cancel{hc^2},\ \cancel{\cdots}$$

and so the cohomology is like that of S^3.

<u>August 5, 1984</u>

Consider a differentiable fibre bundle X/Y where Y is 1-diml, suppose we are given a connection on it, and [redacted] metrics on the fibres, and a metric on Y. There is then a metric on X determined.

Since Y has dimension 1, locally over Y we have a trivialization of X, so let's suppose

$$X = Y \times M \qquad\qquad Y = \mathbb{R}$$

whence we are in the situation of being given a 1-parameter family of metrics on M and studying the metric on $\mathbb{R} \times M$ given by

$$ds^2 = dt^2 + g_{\mu\nu}(t,x)\, dx^\mu dx^\nu .$$

Let's now consider a horizontal curve in X/Y, [redacted] and consider the tangent bundle to X along this curve, [redacted] and parallel translation along it relative to the LC connection on X. Because the horizontal curves are geodesics in X, the horizontal tangent space [redacted] is preserved under parallel transport, hence its orthogonal complement, the vertical tangent space, is also preserved. Thus along the curve we get a trivialization of the vertical tangent bundle, and this trivialization is compatible with the metric.

On the other hand the trivialization $X = Y \times M$ of the differentiable fibre bundle also trivializes the vertical tangent bundle along horizontal curves, but this trivialization is not compatible with the metrics.

[redacted]

This becomes interesting when one considers the infinite-dimensional bundle $\mathcal{H}$ over Y obtained by taking the differential forms on the fibres. On one hand we use the trivialization $X = Y \times M$ to see that $\mathcal{H}$ is trivial, and this trivialization is compatible with d. It won't be compatible with $d+d^*$, since d^* depends on the metric.

On the other hand, we can use the other connection to trivialize $\mathcal{H}$. This preserves the metric on $T_{X/Y}$, hence will preserve Clifford multiplication by $T^*_{X/Y}$ on $\Lambda T^*_{X/Y}$. Thus the symbol of $d+d^*$ on $\pi_*(\Lambda T^*_{X/Y})$ will be preserved. It would be very interesting to know the actual change in the operator $d+d^*$. This uses the connection on $T^*_{X/Y}$ in the vertical direction.

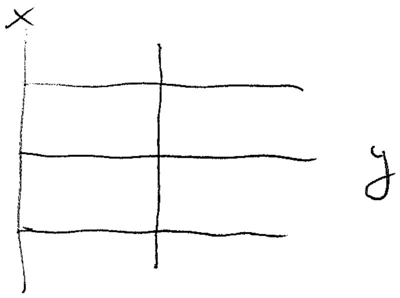
We learn from the above discussion that the key situation to understand is the following. We are given a one-parameter family of metrics g_t on a fixed manifold M. Then we can form a metric on $\mathbb{R}\times M$, namely,

$$ds^2 = dt^2 + g(t,x)_{\mu\nu}\, dx^\mu dx^\nu$$

and from this we get for each m a metric on the trivial bundle $\mathbb{R}\times T_m(M)\,/\,\mathbb{R}$ as well as a connection preserving this metric. The problem is to determine this connection. It is obtained from the LC connection belonging to the metric ~~on~~ $\mathbb{R}\times M$, but hopefully depends only on the family of inner products on $T_m(M)$.

Example: Lets consider the UHP with the Poincaré metric $ds^2 = \frac{dx^2+dy^2}{y^2} = dt^2 + e^{-2t}dx^2$. Think of the UHP as being projected on the y-axis. We have the orthonormal frame

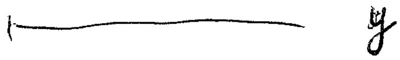
$$\omega^1 = dt \qquad \omega^2 = e^{-t}dx$$

$X =$ 

$$d\omega^1 = 0 \Rightarrow \hat{\Gamma}_{ab}^{\ \ 1} = 0$$

$$d\omega^2 = -e^{-t}\,dt\,dx = -\omega^1\omega^2$$

$$\hat{\Gamma}_{12}^{\ \ 2} = -1 \qquad \hat{\Gamma}_{21}^{\ \ 2} = 1$$

$Y =$ 

$$\Gamma_{ab}^{\ \ c} = \tfrac{1}{2}\left(\hat{\Gamma}_{ab}^{\ \ c} - \hat{\Gamma}_{ac}^{\ \ b} - \hat{\Gamma}_{bc}^{\ \ a}\right)$$

So one has $\Gamma_{ab}^{\ \ c} = 0$ unless two of a,b,c are 2 and one is a 1.

Skew in b,c so the only non-zero ones are

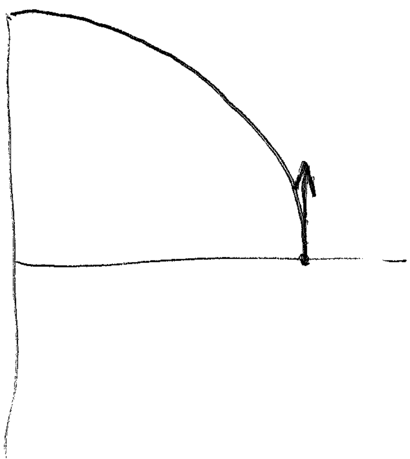
$$\Gamma_{22}^{\ \ 1} = \tfrac{1}{2}\left(\cancel{\hat{\Gamma}_{22}^{\ \ 1}} - \hat{\Gamma}_{21}^{\ \ 2} - \hat{\Gamma}_{21}^{\ \ 2}\right) = -1$$

$$\Gamma_{21}^{\ \ 2} = \tfrac{1}{2}\left(\hat{\Gamma}_{21}^{\ \ 2} - \cancel{\hat{\Gamma}_{22}^{\ \ 1}} - \hat{\Gamma}_{12}^{\ \ 2}\right) = 1$$

which gives

$$\boxed{\begin{aligned} D_2\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} &= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\otimes\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} \\ D_1\begin{pmatrix}\omega^1\\ \omega^2\end{pmatrix} &= 0\end{aligned}}$$

Geometrically I know that if I transport a unit vector pointing vertically in the vertical direction it rotates to the left, because that is the way the geodesic goes. This means that



$$D_2(e_2) = -e_1$$

$$D_2(e_1) = e_2$$

For the families theorem I have a space $\mathcal{H}$ in which I have operators L, $D_\mu L$, $F_{\mu\nu}$ and I have to calculate

$$\lim_{h\to 0} tr_s \left(e^{h^2L^2 + h\, dy^\mu D_\mu L + \frac{1}{2} dy^\mu dy^\nu F_{\mu\nu}} \right)$$

where the exponential is compute in the algebra

$$End(\mathcal{H}) \otimes \Lambda[dy^\mu]$$

and the super trace is taken of the $End(\mathcal{H})$ factor.

The important thing then is to describe the space $\mathcal{H}$ and these operators, and in some sense the algebra $End(\mathcal{H})$ with its supertrace is more important than $\mathcal{H}$.

L will be a Dirac operator over M. To keep thing simple suppose it is the Dirac operator belonging to a $Spin_c$ structure, whence $\mathcal{H}$ will be the sections over M of the spinor bundle.

The next step will be to describe what the operator $D_\mu L$ on $\mathcal{H}$ looks like. I believe that an infinitesimal change in metric on M will give such an operator. Such an infinitesimal change is described by a self-adjoint operator on the tangent bundle.

We start with a description of the Dirac operator. M is a Riemannian manifold, it has a principal frame bundle with group $SO(n)$, and one supposes given a double covering which is a principal $Spin(n)$ bundle P. Then the spinor bundle on M is the associated vector bundle associated to the spin representation. The LC connection on T_M induces one in the spinor frame bundle P

so one can define the horizontal vector fields X_μ complementary to the vertical ones. The Dirac operator is $\gamma^\mu X_\mu$ acting on invariant spinor-valued functions on P.

Now suppose we have a one-parameter family of metrics on M. Over $\mathbb{R}\times M$ we will get a principal $SO(n)$-bundle of orthonormal frames in T_M for the corresponding metric, and we can also get a double covering $\tilde{P}$ which is a principal $\mathrm{Spin}(n)$ bundle. $\tilde{P}$ will have its usual collection of vertical vector fields and the X_μ. But what we need is a full connection in $\tilde{P}$ over $\mathbb{R}\times M$, i.e. I need a vector field invariant under $\mathrm{Spin}(n)$ which lifts d/dt on $\mathbb{R}\times M$. Once I have this $\tilde{P}$ is of the form $\mathbb{R}\times P$ where $P = \tilde{P}_{t=0}$.

But I have discussed the only reasonable way an orthonormal frame at $t=0$ can be extended to be an orthonormal frame for all t. Let's review the underlying linear algebra.

Given an inner product on $V=\mathbb{R}^n$ it is of the form $\langle X|Y\rangle = X^* b Y$ with b pos. definite symmetric. An orthonormal frame can be identified with a g in $GL_n\mathbb{R}$ such that $g^*g = b$. The simplest choice is $g = b^{1/2}$, the positive square root of b. (More generally this "simplest" way to go from X^*b_0Y to X^*bY appears as follows. First write

$$X^*bY = X^*b_0(b_0^{-1}b)Y$$

so that $b_0^{-1}b$ is the pos. def. matrix with respect to b_0 representing the inner product X^*bY. Then if

$$g = (b_0^{-1}b)^{1/2}$$

we have
$$(gX)^*b_0(gY) = X^*[(b_0^{-1}b)^{1/2}]^*b_0(b_0^{-1}b)^{1/2}Y$$

$$= X^* \underbrace{(b\, b_0^{-1})^{1/2}\, b_0}_{b_0 \underbrace{b_0^{-1}(b b_0^{-1})^{1/2} b_0}_{(b_0^{-1} b)^{1/2}}} (b_0^{-1} b)^{1/2} Y \quad \text{[redacted]}$$

$$= X^* b_0\, b_0^{-1} b\, Y = X^* b\, Y.$$

Note that this means the [redacted] simplest way to go from X^*Y to $X^* b_0 Y$, which is $b_0^{1/2}$, followed by the simplest way from X^*b_0Y to X^*bY, which is $(b_0^{-1}b)^{1/2}$ is the product

$$b_0^{1/2}(b_0^{-1}b)^{1/2} \neq b^{1/2} \quad \text{in general.)}$$

The principal is that a family of "inner products" b_t is to be lifted to a family of orthonormal frames g_t by the rules

$$g_{t+dt} = g_t(1 + g_t^{-1} dg_t)$$

$$1 + g_t^{-1} dg_t = (b_t^{-1} b_{t+dt})^{1/2}$$

$$= (1 + b_t^{-1} db_t)^{1/2} = 1 + \tfrac{1}{2} b_t^{-1} db_t$$

i.e.

$$g^{-1}\dot{g} = \tfrac{1}{2} \underbrace{b^{-1}\dot{b}}_{}$$

symmetric wrt X^*bY

When translated in terms of frames this means the following. Given a 1-parameter family of orthonormal frames ω_t^μ for a ~~family~~ of metrics, then this family is [redacted] a lifting of the metrics provided [redacted] we have

$$\partial_t \omega^\mu = b^\mu_\nu\, \omega^\nu$$

with b^μ_ν symmetric.

Now we have, at least in principal, a way to trivialize $\tilde{P}$ in the t direction. We have a vector field K on $\tilde{P}$ invariant under $Spin(n)$ which covers $\frac{d}{dt}$ on $\mathbb{R} \times M$. Now I [redacted] want to know how to calculate

$$[K, X_\mu]$$

Problem: Given a 1-parameter family of metrics g_t on M, compute $\partial_t \not{D}$, where $\not{D}$ is the Dirac operator on spinors.

The main difficulty is how to identify the spinor bundles at different times, so we can talk about $\partial_t \not{D}$. We form the principal $SO(n)$-bundle P over $\mathbb{R}\times M$ whose fibre at t,m is the set of orthonormal frames in $T_m(M)$ for the metric g_t. A lifting of $P|_{t=0}$ to a $Spin(n)$-bundle will extend uniquely to give a lifting of P to a principal $Spin(n)$-bundle $\tilde{P}$. Then the assoc. vector bundle S to the spin representation $\tilde{P}\times^{Spin(n)} S_n$ is the vector bundle over $\mathbb{R}\times M$ which at time t is the spinor bundle over M for the metric g_t. Our problem is now to define a partial connection on S in the $\mathbb{R}$-direction, i.e. to define a differential operator D_t on $\Gamma(S)$ with symbol multiplication by dt. As we have the LC connection in the M-direction already, we are to extend the latter to a full connection. So our problem becomes one of extending the given LC connection on P in the M-direction to a full connection.

Let's now get specific, and compute with a local trivialization on M. We suppose given an orth. coframe ω_0^μ on M at time $t=0$. The connection we want will tell us how to extend this coframe to a framing of $\mathbb{R}\times T(M)$. The condition is, I believe, that ω_t^μ is to be orthonormal for g_t and that

$$\partial_t \omega^\mu = b^\mu_\nu \omega^\nu \qquad \text{where } b^\mu_\nu \text{ is } \underline{\text{symmetric}}.$$

This condition defines a partial connection in the transverse direction:

$$D_t(\omega^\mu) = 0.$$

Now I would like to check that this definition of D_t

coincides with other reasonable ways of defining such a connection.

The candidate I had in mind comes from [redacted] the LC connection on $\mathbb{R}\times M$ belonging to the metric

$$dt^2 + g_{t,\mu\nu}(x)\,dx^\mu dx^\nu .$$

What would be an orthonormal coframe for such a metric? Clearly dt, ω^μ, where ω^μ is a 1-form on $\mathbb{R}\times M$ killed by d/dt, i.e. ω^μ is a 1-parameter family of forms on M.

So suppose we start with the metric on $\mathbb{R}\times M$ defined by [redacted] requiring the coframe dt, ω^μ to be orthonormal. What does it mean for $D_t(\omega^\mu)=0$?

Let's use [redacted] $t = x^0$. We want $D_0(\omega^\alpha) = \Gamma_{0\nu}{}^\alpha\,\omega^\nu$.

Now $$d\omega^\alpha = dt\cdot\underbrace{\partial_t\omega^\alpha}_{b^\alpha_\beta\omega^\beta} + \tfrac{1}{2}\hat{\Gamma}^\alpha_{\beta\gamma}\,\omega^\beta\omega^\gamma$$

so we see that $\hat{\Gamma}_{0\beta}{}^\alpha = b^\alpha_\beta$. But

$$\Gamma_{0\beta}{}^\alpha = \tfrac{1}{2}\left(\hat{\Gamma}_{0\beta}{}^\alpha - \hat{\Gamma}_{0\alpha}{}^\beta - \cancel{\hat{\Gamma}_{\beta\alpha}{}^0}\right) = \tfrac{1}{2}\left(b^\alpha_\beta - b^\beta_\alpha\right)$$

So we do see that $D_t(\omega^\mu)=0 \iff b^\alpha_\beta$ is symmetric.

Consider now a fibre bundle X/Y and a metric on X which is constructed from a metric on Y, a connection in X/Y, and a metric in the fibres.
Question: Does parallel transport in the horizontal direction preserve the decomposition of T_X into horizontal and vertical subbundles?

To handle this let ω^a denote the pullback of an orthonormal coframe on Y, and let this be extended by ω^μ to become an orthonormal coframe ω^a, ω^μ in X. The horizontal subbundle of T_X is the kernel of the ω^μ.

To see whether the vertical subbundle $T_{X/Y} \subset T_X$ is preserved under parallel transport in the horizontal direction we compute

$$D_a\, \omega^c = \Gamma_{ab}{}^c\, \omega^b + \Gamma_{a\mu}{}^c\, \omega^\mu$$

and see whether

$$\Gamma_{a\mu}{}^c = \tfrac{1}{2}\left(\hat{\Gamma}_{a\mu}{}^c - \hat{\Gamma}_{ac}{}^\mu - \hat{\Gamma}_{\mu c}{}^a \right)$$

vanishes. Now

$$d\omega^a = \tfrac{1}{2}\, \hat{\Gamma}_{bc}{}^a\, \omega^b \omega^c$$

since the forms ω^a come from Y. Thus

$$\hat{\Gamma}_{\mu *}{}^a = 0$$

and so

$$\Gamma_{a\mu}{}^c = -\tfrac{1}{2}\, \hat{\Gamma}_{ac}{}^\mu .$$

Now

$$d\omega^\mu = \tfrac{1}{2}\, \hat{\Gamma}_{ab}^\mu\, \omega^a\omega^b + \hat{\Gamma}_{a\nu}^\mu\, \omega^a \omega^\nu + \tfrac{1}{2}\, \hat{\Gamma}_{\lambda\nu}^\mu\, \omega^\lambda \omega^\nu$$

and $\hat{\Gamma}_{ab}^\mu$ is related to the vertical part of the bracket of two horizontal vector fields:

$$\Gamma_{ab}^\mu = i(X_a)\, i(X_b)\, d\omega^\mu = -\omega^\mu([X_a, X_b])$$

Thus we see that the curvature in X/Y (which is the 2-form on Y with values in vector fields along the fibre: $\tfrac{1}{2}\, \omega^a \omega^b \otimes [X_a, X_b]$) prevents the horizontal parallel transport of horizontal vectors from being horizontal.

Let's consider a Riemannian X equipped with a foliation, and work locally so that we can view X as a fibre bundle over Y the quotient space. Choose an orthonormal frame in $Q^* \subset T^*_X$, call it ω^a and extend to an orthonormal coframe ω^a, ω^μ in X. Then we have

$$d\omega^a = \frac{1}{2} \hat{\Gamma}^a_{bc} \omega^b \omega^b + \hat{\Gamma}^a_{b\mu} \omega^b \omega^\mu$$

(there is no $\frac{1}{2}\hat{\Gamma}^a_{\mu\nu} \omega^\mu \omega^\nu$ term as we have a foliation)

$$d\omega^\mu = \frac{1}{2} \hat{\Gamma}^\mu_{ab} \omega^a \omega^b + \hat{\Gamma}^\mu_{a\nu} \omega^a \omega^\nu + \frac{1}{2} \hat{\Gamma}^\mu_{\lambda\nu} \omega^\lambda \omega^\nu$$

Now what we want to do is to rescale the metric in the transverse direction so that the Dirac operator becomes

$$\not{D} = h \gamma^a D_a + \gamma^\mu D_\mu$$

What is the new metric? The symbol is

$$\sigma(\not{D}, \xi) = h \gamma^a \xi_a + \gamma^\mu \xi_\mu \qquad \xi_a = i(X_a)\xi \text{ etc.}$$

So $\quad \|\xi\|^2 = h^2 \xi_a^2 + \xi_\mu^2 \quad$ and we see that an orthonormal frame for the new metric is

$$h X_a, \; X_\mu$$

hence a orthonormal coframe for the new metric is

$$\frac{1}{h}\omega^a, \quad \omega^\mu$$

What is the new LC connection?

$$d\left(\frac{\omega^a}{h}\right) = \frac{1}{2} h \hat{\Gamma}^a_{bc} \frac{\omega^b}{h} \frac{\omega^c}{h} + \hat{\Gamma}^a_{b\mu} \frac{\omega^b}{h} \omega^\nu$$

$$d\omega^\mu = \frac{1}{2} h^2 \hat{\Gamma}^\mu_{ab} \frac{\omega^a}{h} \frac{\omega^b}{h} + h \hat{\Gamma}^\mu_{a\nu} \frac{\omega^a}{h} \omega^\nu + \frac{1}{2} \hat{\Gamma}^\mu_{\lambda\nu} \omega^\lambda \omega^\nu$$

Notice that if we want there to be a **limit** as $h \to 0$, we have already used the foliation condition: $\hat{\Gamma}^a_{\lambda\mu} = 0$. What am interested in is when there is an induced connection on $T_{X/Y}$ in the limit. First consider the vertical direction

$$D_\lambda(\omega^\mu) = \Gamma^\mu_{\lambda a}\frac{\omega^a}{h} + \Gamma^\mu_{\lambda\nu}\,\omega^\nu$$

where $\Gamma^\mu_{\lambda\nu}$ is independent of h but

$$\Gamma^\mu_{\lambda a} = \frac{1}{2}\Big(h\hat{\Gamma}^\mu_{\lambda a} - \underbrace{\frac{1}{h}\hat{\Gamma}^a_{\lambda\mu}}_{0} - h\hat{\Gamma}^\lambda_{a\mu}\Big)$$

this uses the foliation condition otherwise it is $\frac{1}{h}$.

is $O(h)$. So as $h \to 0$ we get

$$D_\lambda(\omega^\mu) = \Gamma^\mu_{\lambda\nu}\,\omega^\nu$$

which is just the LC connection in the leaf.

Now for the horizontal direction

$$\left(\frac{\omega^a}{h}\otimes D_a\right)(\omega^\mu) = \frac{\omega^a}{h}\otimes\Gamma^\mu_{ab}\frac{\omega^b}{h} + \frac{\omega^a}{h}\otimes\Gamma^\mu_{a\nu}\,\omega^\nu$$

where

$$\Gamma^\mu_{ab} = \frac{1}{2}\left(h^2\hat{\Gamma}^\mu_{ab} - \hat{\Gamma}^b_{a\mu} - \hat{\Gamma}^a_{b\mu}\right)$$

$$\Gamma^\mu_{a\nu} = \frac{1}{2}\Big(h\hat{\Gamma}^\mu_{a\nu} - h\hat{\Gamma}^\nu_{a\mu} - \underbrace{\frac{1}{h}\hat{\Gamma}^a_{\nu\mu}}_{0}\Big)$$

Note: We could always take the connection in $T_{X/Y}$ which is the Grassmannian connection. This means working with the ω^μ modulo the ω^a and gives:

$$D_\lambda(\omega^\mu) \equiv \Gamma^\mu_{\lambda\nu}\,\omega^\nu \qquad (\mathrm{mod}\ \omega^a)$$

$$\left(\frac{\omega^a}{h}\otimes D_a\right)(\omega^\mu) \equiv \frac{\omega^a}{h}\otimes\Gamma^\mu_{a\nu}\,\omega^\nu \qquad (\mathrm{mod}\ \omega^a)$$

One sees the foliation condition is necessary for this connection to make sense in the limit, in which case $\Gamma^{\mu}_{a\nu} = O(h)$ becomes zero and the ω^{μ} are flat.

The other thing to do might be to take the limiting formulas

$$\begin{cases} d\left(\dfrac{\omega^a}{h}\right) = \hat{\Gamma}^{a}_{b\mu}\,\dfrac{\omega^b}{h}\,\omega^{\mu} \\[2ex] d\left(\omega^{\mu}\right) = \dfrac{1}{2}\,\hat{\Gamma}^{\mu}_{\lambda\nu}\,\omega^{\mu}\omega^{\nu} \end{cases}$$

whence the limiting connection is

$$\frac{\omega^a}{h}\otimes D_a\left(\frac{\omega^c}{h}\right) = \frac{\omega^a}{h}\otimes\Gamma^{c}_{a\mu}\,\omega^{\mu}$$

$$\omega^{\mu}\otimes D_{\mu}\left(\frac{\omega^c}{h}\right) = \omega^{\mu}\otimes\Gamma^{c}_{\mu b}\,\frac{\omega^b}{h}$$

where
$$\Gamma^{c}_{a\mu} = \frac{1}{2}\left(\hat{\Gamma}^{c}_{a\mu} - \hat{\Gamma}^{a}_{\mu c}\right)$$
$$\Gamma^{c}_{\mu b} = \frac{1}{2}\left(\hat{\Gamma}^{c}_{\mu b} - \hat{\Gamma}^{b}_{\mu c}\right)$$

Notice that $\Gamma^{c}_{\mu a} = \frac{1}{2}\left(\hat{\Gamma}^{c}_{\mu a} - \hat{\Gamma}^{a}_{\mu c}\right) = \frac{1}{2}\left(-\hat{\Gamma}^{c}_{a\mu} + \hat{\Gamma}^{a}_{c\mu}\right)$ $= -\Gamma^{c}_{a\mu}$.

So we seem to get

$$\boxed{D\left(\frac{\omega^c}{h}\right) = \frac{\omega^a}{h}\otimes\Gamma^{c}_{a\mu}\,\omega^{\mu} - \omega^{\mu}\otimes\Gamma^{c}_{a\mu}\,\frac{\omega^a}{h}}$$

Also

$$\frac{\omega^a}{h}\otimes D_a\left(\omega^{\mu}\right) = \frac{\omega^a}{h}\otimes\Gamma^{\mu}_{ab}\,\frac{\omega^b}{h} \qquad \Gamma^{\mu}_{ab} = -\frac{1}{2}\left(\hat{\Gamma}^{b}_{a\mu} + \hat{\Gamma}^{a}_{b\mu}\right)$$

$$D_{\lambda}\left(\omega^{\mu}\right) = \Gamma^{\mu}_{\lambda\nu}\,\omega^{\nu}$$

$$\boxed{D\left(\omega^{\mu}\right) = \frac{\omega^a}{h}\otimes\Gamma^{\mu}_{ab}\otimes\frac{\omega^b}{h} + \omega^{\lambda}\otimes\Gamma^{\mu}_{\lambda\nu}\,\omega^{\nu}}$$

I don't know if this has any meaning in general, however if we [redacted] know the ω^a came from Y we have $\hat{\Gamma}^a_{b\mu} = 0$ so the connection becomes

$$D\left(\frac{\omega^a}{h}\right) = 0 \qquad D(\omega^\mu) = \omega^\lambda \Gamma^\mu_{\lambda\nu}\, \omega^\nu$$

Moral: The procedure of rescaling the metric in the ~~longitudinal~~ transversal direction and trying to take the limit of the LC connection as $h \to 0$ is for the birds

(see p. 205)

August 7, 1984

What I learned yesterday: We start with X/Y with metrics on the fibres and a connection. Then we can choose an orthonormal vertical coframe ω^μ and a horizontal coframe ω^a which comes from Y. (The important thing here is the bundle $T^*_{X/Y}$.) Then we have

$$d\omega^\mu = \tfrac{1}{2}\hat{\Gamma}^\mu_{ab}\,\omega^a\omega^b + \hat{\Gamma}^\mu_{a\nu}\,\omega^a\omega^\nu + \tfrac{1}{2}\hat{\Gamma}^\mu_{\lambda\nu}\,\omega^\lambda\omega^\nu$$

(I should really think of $\omega^a = dy^a$, since I really don't care about the metric on Y.)

Each of the three terms will be of importance for the families index theorem; they correspond to the three pieces D^2, $[D,L]$, L^2 of the curvature. However we must apply the LC process to get the corresponding operators.

In more detail, let's look at the middle term $\hat{\Gamma}^\mu_{a\nu}\,dy^a\omega^\nu$. This gives the connection in the bundle $T^*_{X/Y}$ in the horizontal direction. (Perhaps I should work with the orthonormal frame X_μ in $T_{X/Y}$).

Let's start again and try to see what comes out of the connection in X/Y. Let X_μ be a vertical frame, i.e. a frame in $T_{X/Y}$, and let Y_a be the horizontal lift of $\partial/\partial y^a$. Then we have a partial connection in $T_{X/Y}$ in the horizontal direction given by $D_a X_\mu = [Y_a, X_\mu]$. The curvature of this partial connection is

$$[D_a, D_b]X_b = [[Y_a, Y_b], X_\mu]$$

hence the curvature can be identified with the bracket $[Y_a, Y_b]$ which is a vertical vector field.

We see therefore that the connection + curvature for the horizontal direction on $T_{X/Y}$ is represented, or described by brackets. Hence it is captured by the formula for $d\omega^\mu$, or really the first two terms of the formula.

But now comes the LC process. We are [redacted] given metrics on the fibres, hence an inner product on $T_{X/Y}$. The transverse connection $D_a X_\mu = [Y_a, X_\mu]$ doesn't preserve the inner product.

Let's review the program. We have X/Y with metrics along the fibres. Thus we can form the principal $SO(n)$ bundle, $n = \dim(X/Y)$, consisting of orthonormal frames in [redacted] $T_{X/Y}$. Call this P and assume given a lifting to a $Spin(n)$-bundle $\tilde{P}$, so that we can form the bundle $\tilde{P} \times^{SO(n)} S_n = S$ of spinors on the fibres of X/Y. $\mathcal{H} = \pi_*(S)$ and a section of $\mathcal{H}$ over Y is a section of S over X.

I now want to define a connection in [redacted] $\mathcal{H}$, i.e. to assign to each vector field v on Y an operator D_v on $\Gamma(Y, \mathcal{H}) = \Gamma(X, S)$ with the derivation property relative to functions on Y. [redacted] Suppose given a connection in X/Y, whence the vector field v lifts to a horizontal vector field on X. Hence it is enough to give a [redacted] connection on S in the horizontal, or transverse, direction. This in turn would come from a transverse connection on $T_{X/Y}$ preserving the metric.

What we have from the connection in X/Y is [redacted] a transverse connection in $T_{X/Y}$ which does not necessarily preserve the metric. Specifically if X_a is a horizontal vector field (lifting $\partial/\partial y^a$) and X_μ is a vertical vector field, then $[X_a, X_\mu]$ is again vertical. (I see this because the 1-parameter gp. generated by X_a covers [redacted] a 1-parameter grp in Y generated by $\partial/\partial y^a$, so the image of X_μ under the Y_a flow remains vertical.)

Now use the idea of projecting from gl_n to the Lie algebra of $SO(n)$ = skew-symmetric matrices. This map is

$$\alpha_\mu{}^\nu \longmapsto \frac{1}{2}\left(\alpha_\mu{}^\nu - \alpha_\nu{}^\mu\right).$$

Then using this we can obtain from our transverse connection in $T_{X/Y}$ a transverse connection preserving the metric.

In details, let's first define the transverse connection. Let H be a horizontal v.f., let $V \in \Gamma(T_{X/Y})$ be a vertical vector field, let $\pi : T_X \to T_{X/Y}$ be the projection onto the vertical tangent vectors. Consider

$$\hat{D}_H(V) = \pi[H, V] \in \Gamma(T_{X/Y})$$

Then

$$\hat{D}_{fH}(V) = \pi[fH, V] = \pi\big(f[H,V] + \underbrace{[f,V]}_{-Vf}H\big)$$

$$= f\,\hat{D}_H(V)$$

$$\hat{D}_H(fV) = \pi\big([H,f]V + f[H,V]\big)$$

$$= (Hf)V + f\,\hat{D}_H(V)$$

so we see that $\hat{D}$ is a transverse connection in $T_{X/Y}$.

In component notation, let X_a = horizontal lift of $\partial/\partial y^a$, and let X_μ be a vertical frame. Let ω^a, ω^μ be the dual coframe and put

$$d\omega^\mu = \frac{1}{2}\hat{\Gamma}^\mu_{ab}\,dy^a dy^b + \hat{\Gamma}^\mu_{a\nu}\,dy^a\omega^\nu + \hat{\Gamma}^\mu_{\lambda\nu}\,\omega^\lambda\omega^\nu$$

Then

$$\hat{D}_{X_a}(X_\mu) = \pi[X_a, X_\mu]$$

$$\omega^\mu[X_a, X_\nu] = -i(X_\nu)\,i(X_a)\,d\omega^\mu = -\hat{\Gamma}^\mu_{a\nu}$$

so

$$\hat{D}_{X_a}(X_\nu) = -\hat{\Gamma}^\mu_{a\nu}\,X_\mu$$

Now suppose X_μ is orthonormal, then this connection won't preserve the metric because $-\Gamma_{a\nu}^{\ \ \mu}$ isn't necessarily skew in μ, ν. So we project, getting a new connection

$$\bar{D}_{X_a}(X_\nu) = -\tfrac{1}{2}\left(\hat{\Gamma}_{a\nu}^{\ \ \mu} - \hat{\Gamma}_{a\mu}^{\ \ \nu}\right) X_\mu$$

which preserves the metric. On the dual bundle $T^*_{X/Y}$ we get

$$\boxed{D_a(\omega^\mu) = \underbrace{\tfrac{1}{2}\left(\hat{\Gamma}_{a\nu}^{\ \ \mu} - \hat{\Gamma}_{a\mu}^{\ \ \nu}\right)}_{\Gamma_{a\mu}^{\ \ \nu}} \omega^\nu}$$

(Notice this is not the same as $T^*_{X/Y} \subset T^*_X \xrightarrow{D_a} T^*_X$ which would be

$$D_a(\omega^\mu) = \Gamma_{a\nu}^{\ \ \mu}\,\omega^\nu + \Gamma_{ab}^{\ \ \mu}\,\omega^b$$

$$\Gamma_{ab}^{\ \ \mu} = \tfrac{1}{2}\Big(\underset{\substack{\uparrow \\ \text{curvature of } X/Y.}}{\hat{\Gamma}_{ab}^{\ \ \mu}} - \underbrace{\hat{\Gamma}_{a\mu}^{\ \ b} - \hat{\Gamma}_{b\mu}^{\ \ a}}_{0 \text{ if } \omega^a = dy^a}\Big)$$

So the above D_a is the LC D_a in T^*_X followed by the projection back to $T^*_{X/Y}$.)

<u>Summary</u>: I have just constructed the transverse connection on $T_{X/Y}$. It is described by the following formulas. Let X_μ be an orthonormal frame in $T_{X/Y}$, and ω^μ the dual coframe in $T^*_{X/Y}$ which we embed in T^*_X using the connection. Then we have

$$\begin{cases} D_a(\omega^\mu) = \Gamma^\mu_{a\nu}\,\omega^\nu & \Gamma^\mu_{a\nu} = \frac{1}{2}\left(\hat\Gamma^\mu_{a\nu} - \hat\Gamma^\mu_{a\nu}\right) \\ \text{where} \quad d\omega^\mu = \frac{1}{2}\hat\Gamma^\mu_{ab}\,dy^a dy^b + \hat\Gamma^\mu_{a\nu}\,dy^a\,\omega^\nu + \frac{1}{2}\hat\Gamma^\mu_{\lambda\nu}\,\omega^\lambda\omega^\nu \end{cases}$$

Let's recall that I am trying to assemble enough operators on the sections of the spinor bundle S to describe the connection D in $\mathcal{H}$ and the Dirac operator L. We now have the operators D_a on $\Gamma(T_{X/Y})$; these extend to the Clifford algebra $C(T_{X/Y})$ and the spinors.

What is the curvature of the connection $D = dy^a D_a$ and how is it related to the curvature of X/Y which is $\pm\frac{1}{2}dy^a dy^b\,\Gamma^\mu_{ab}X_\mu$? The first guess is that this vector field should be projected somehow.

I have produced a connection in the vector bundle $T_{X/Y}$ and the associated bundle of spinors S. From the viewpoint of the local index formula, what is important is the curvature of this connection, because the curvature is substituted into the A-hat series and integrated over the fibre to get the family index. Thus we must now become interested in a 2nd order effect like the curvature. Let's start by working out what happens for the usual curvature of a Riemannian manifold.

Let's carefully go thru the limiting process we did yesterday. We start with the orth. coframe on X given by dy^a, ω^μ. Then we change the metric so that a new orthonormal coframe is $\frac{dy^a}{h}$, ω^μ. Now compute the LC connection for the new metric

$$d\omega^\mu = \tfrac{1}{2}h^2\hat{\Gamma}_{ab}{}^{\mu}\frac{dy^a}{h}\frac{dy^b}{h} + h\hat{\Gamma}_{a\nu}{}^{\mu}\frac{dy^a}{h}\omega^\nu + \tfrac{1}{2}\hat{\Gamma}^{\mu}_{\lambda\nu}\,\omega^\lambda\omega^\nu$$

$$D_a(\omega^\mu) = \Gamma_{ab}{}^{\mu}\frac{dy^b}{h} + \Gamma_{a\nu}{}^{\mu}\,\omega^\nu$$

where
$$\Gamma_{ab}{}^{\mu} = \tfrac{1}{2}\left(h^2\hat{\Gamma}_{ab}{}^{\mu} - \cancel{\hat{\Gamma}_{a\mu}{}^{b}} - \cancel{\hat{\Gamma}_{b\mu}{}^{a}}\right)$$

$$\Gamma_{a\nu}{}^{\mu} = \tfrac{1}{2}\left(h\hat{\Gamma}_{a\nu}{}^{\mu} - h\hat{\Gamma}_{a\mu}{}^{\nu} - \cancel{\hat{\Gamma}_{\nu\mu}{}^{a}}\right).$$

Now D_a itself has no meaning as $h\to 0$, but we have

$$D'(\omega^\mu) = \frac{dy^a}{h}\otimes D_a(\omega^\mu) = \frac{dy^a}{h}\otimes\left(\tfrac{1}{2}h^2\hat{\Gamma}^{\mu}_{ab}\frac{dy^b}{h} + \frac{h}{2}\left(\hat{\Gamma}^{\mu}_{a\nu} - \hat{\Gamma}_{a\mu}^{\nu}\right)\omega^\nu\right)$$

$$\boxed{D'(\omega^\mu) = dy^a\otimes\left[\tfrac{1}{2}\hat{\Gamma}^{\mu}_{ab}\,dy^b + \tfrac{1}{2}\left(\hat{\Gamma}^{\mu}_{a\nu} - \hat{\Gamma}_{a\mu}{}^{\nu}\right)\omega^\nu\right]}$$

and this is independent of h.

Next
$$D_\mu(\omega^\alpha) = \Gamma^{\alpha}_{\mu,a}\frac{dy^a}{h} + \Gamma^{\alpha}_{\mu\nu}\,\omega^\nu$$

$$\Gamma^{\alpha}_{\mu a} = \tfrac{1}{2}\left(h\hat{\Gamma}^{\alpha}_{\mu a} - \cancel{\hat{\Gamma}^{a}_{\mu\alpha}} - h\hat{\Gamma}^{\mu}_{a\alpha}\right)$$

$$\Gamma^{\alpha}_{\mu\nu} = \tfrac{1}{2}\left(\hat{\Gamma}^{\alpha}_{\mu\nu} - \hat{\Gamma}_{\mu\alpha}{}^{\nu} - \hat{\Gamma}^{\mu}_{\nu\alpha}\right)$$

usual LC connection in the fibre

So

$$D''(\omega^\alpha) = \omega^\mu \otimes D_\mu(\omega^\alpha) = \omega^\mu \otimes \left[\frac{h}{2}\left(\hat\Gamma^\alpha_{\mu a} - \hat\Gamma^\mu_{a\alpha}\right)\frac{dy^a}{h} + \Gamma^\alpha_{\mu\nu}\,\omega^\nu\right]$$

$$\boxed{D''(\omega^\alpha) = \omega^\mu \otimes \left[\frac{1}{2}\underbrace{\left(\hat\Gamma^\alpha_{\mu a} - \hat\Gamma^\mu_{a\alpha}\right)}_{-\left(\hat\Gamma_{a\mu}{}^{\alpha} + \hat\Gamma_{a\alpha}{}^{\mu}\right)} dy^a + \Gamma^\alpha_{\mu\nu}\,\omega^\nu\right]}$$

is also independent of h.

Next

$$D_a\left(\frac{\omega^c}{h}\right) = \Gamma^c_{ab}\,\frac{\omega^b}{h} + \Gamma^c_{a\mu}\,\omega^\mu$$

$$\Gamma^c_{ab} = \frac{1}{2}\left(\cancel{\hat\Gamma^c_{ab}} - \cancel{\hat\Gamma^b_{ac}} - \cancel{\hat\Gamma^a_{bc}}\right) = 0$$

$$\Gamma^c_{a\mu} = \frac{1}{2}\left(\cancel{\hat\Gamma^c_{a\mu}} - h^2\hat\Gamma^\mu_{ac} - \cancel{\hat\Gamma^a_{\mu c}}\right) = -\frac{1}{2}h^2\hat\Gamma^\mu_{ac}$$

$$\frac{\omega^a}{h}\otimes D_a\left(\frac{\omega^c}{h}\right) = \frac{\omega^a}{h}\otimes\left[-\frac{1}{2}h^2\hat\Gamma^\mu_{ac}\,\omega^\mu\right]$$

so that

$$\boxed{D'(\omega^c) = O(h^2)}$$

Finally

$$D_\mu\left(\frac{\omega^c}{h}\right) = \Gamma^c_{\mu a}\,\frac{\omega^a}{h} + \Gamma^c_{\mu\nu}\,\omega^\nu$$

$$\Gamma^c_{\mu a} = \frac{1}{2}\left(\cancel{\hat\Gamma^c_{\mu a}} - \cancel{\hat\Gamma^a_{\mu c}} - h^2\hat\Gamma^\mu_{ac}\right) = -\frac{1}{2}h^2\hat\Gamma^\mu_{ac}$$

$$\Gamma^c_{\mu\nu} = \frac{1}{2}\left(\cancel{\hat\Gamma^c_{\mu\nu}} - h\hat\Gamma^\nu_{\mu c} - h\hat\Gamma^\mu_{\nu c}\right)$$

$$\omega^\mu\otimes D_\mu\left(\frac{\omega^c}{h}\right) = \omega^\mu\otimes\left[-\frac{1}{2}h^2\hat\Gamma^\mu_{ac}\,\frac{\omega^a}{h} + \left(-\frac{1}{2}\right)h\left(\hat\Gamma^\nu_{\mu c} + \hat\Gamma^\mu_{\nu c}\right)\omega^\nu\right]$$

$$\boxed{D''(\omega^c) = O(h^2)}$$

As usual with these calculations I don't understand what is going on. I really should be doing the calculations in the algebra of differential operators on the spinors.

This time let's start with the limiting process used to prove the index theorem. The algebra of differential operators on the spinors is generated by functions, γ^μ, D_μ subject to the following commutation relations

$$
(1) \qquad
\begin{aligned}
[D_\mu, f] &= X_\mu f \\
[\gamma^\mu, \gamma^\nu] &= \delta^{\mu\nu} \\
[D_\mu, \gamma^\alpha] &= \Gamma^\alpha_{\mu\nu}\gamma^\nu \\
[D_\mu, D_\nu] + \hat{\Gamma}^\alpha_{\mu\nu} D_\alpha &= \tfrac{1}{4}\gamma^a\gamma^b R_{\mu\nu a}{}^b
\end{aligned}
$$

The algebra of asymptotic diff'l operators on the spinors is generated by h, functions, $h\gamma^\mu$, hD_μ. In the $h \to 0$ limit, set $h\gamma^\mu = \psi^\mu$, $hD_\mu = ip_\mu$. (This means in the associated graded algebra.) Then the limiting algebra is generated by functions, ψ^μ, p_μ with the relations

$$
\begin{aligned}
[p_\mu, f] &= 0 \\
[\psi^\mu, \psi^\nu] &= 0 \\
[p_\mu, \psi^\mu] &= 0 \\
-[p_\mu, p_\nu] &= \tfrac{1}{4}\psi^a\psi^b R_{\mu\nu a}{}^b
\end{aligned}
$$

Now let me see if I can do the same thing in the families situation. The algebra of diff'l operators on the spinors on X is generated by functions, γ^a, γ^μ, D_a, D_μ satisfying the full set of relations (1). The algebra of asymptotic operators is generated by h, functions on X, $h\gamma^a$, γ^μ, hD_a, D_μ. I want to take the $h \to 0$ limit, i.e. mod out by h. We get an algebra generated by functions on X, ψ^a which I can identify with dy^a, γ^μ, ip_a, D_μ. Now I have to determine the commutation relations in the limit.

Let's consider the four relations of type $[D_*, \gamma^*]$.

$$[hD_a, \gamma^\mu] = \Gamma^\mu_{ab}(h\gamma^b) + h\Gamma^\mu_{a\nu}\gamma^\nu$$

$$\Gamma^\mu_{ab} = \frac{1}{2}\left(\hat\Gamma^\mu_{ab} - \cancel{\hat\Gamma^{b}} - \cancel{\hat\Gamma^{a}}\right)$$

$$\boxed{[ip_a, \gamma^\mu] = \frac{1}{2}\hat\Gamma^\mu_{ab}\psi^b}$$

$$[D_\mu, h\gamma^a] = \Gamma^a_{\mu b}(h\gamma^b) + h\Gamma^a_{\mu\nu}\gamma^\nu$$

$$\Gamma^a_{\mu b} = \frac{1}{2}\left(\cancel{\hat\Gamma^{a}} - \cancel{\hat\Gamma^{b}} - \hat\Gamma^\mu_{ba}\right) = \frac{1}{2}\hat\Gamma^\mu_{ab}$$

$$\boxed{[D_\mu, \psi^a] = \frac{1}{2}\hat\Gamma^\mu_{ab}\psi^b}$$

$$[hD_a, h\gamma^b] = h\Gamma^b_{ac}h\gamma^c + h^2\Gamma^b_{a\mu}\gamma^\mu$$

$$\boxed{[ip_a, \psi^b] = 0}$$

$$[D_\mu, \gamma^\nu] = \frac{1}{h}\Gamma^\nu_{\mu a}(h\gamma^a) + \Gamma^\nu_{\mu\lambda}\gamma^\lambda$$

$$\Gamma^\nu_{\mu a} = \frac{1}{2}\left(\hat\Gamma^\nu_{\mu a} - \hat\Gamma^\mu_{a\nu}\right) = -\frac{1}{2}\left(\hat\Gamma^\nu_{a\mu} + \hat\Gamma^\mu_{a\nu}\right)$$

So we run into trouble with this relation, unless for some reason we manage to arrange that $\hat\Gamma^\nu_{a\mu}$ be skew in μ, ν.

We have seen that when the base is a curve

$$d\omega^\mu = \underbrace{\hat{\Gamma}^{\mu}_{0\nu}\, dt\, \omega^\nu}_{} + \cdots$$

$$dt\, \partial_t \omega^\mu = b^\mu_\nu \omega^\nu$$

the good condition, saying that ω^μ is horizontal, is that $b^\mu_\nu = \hat{\Gamma}_{0\nu}{}^{\mu}$ be symmetric. In the above we want $\hat{\Gamma}^{\nu}_{a\mu}$ to be skew-symmetric, and this probably means that the metric doesn't vary.

Comparing the formulas on page 199, 200 for $D(\omega^\mu)$, one sees that there is no way to eliminate the deviation of D from preserving the horizontal-vertical splitting by rescaling the metric.

August 8, 1984

We are given a fibre bundle X/Y with a connection and metrics on the fibres, i.e. a scalar product on the bundle $T_{X/Y}$. Associated to $T_{X/Y}$ with its inner product is the bundle S of spinors on the fibres. We have defined a connection on $T_{X/Y}$ extending the **Levi-Civita** connection along the fibres, which preserves the metric, hence induces a connection on S. The curvature of this connection can be substituted in the $\hat{A}$-series to yield a form on X, which can be integrated over the fibre to yield a form on Y, which is the RHS of the local families index formula.

Consider now the infinite dimensional bundle $\mathcal{H} = \pi_* S$ such that $\Gamma(Y, \mathcal{H}) = \Gamma(X, S)$. On $\mathcal{H}$ we have a connection D defined as follows. Given a vector field on Y it lifts to a horizontal vector field on X, which then acts on sections of S via the connection in S; this gives the action of vector fields on sections of $\mathcal{H}$ defining the connection. We also have the endomorphism L of $\mathcal{H}$ given by the Dirac operators in each of the fibres of X/Y. Then $D+hL$ constitutes a super-connection on $\mathcal{H}$ over Y, whose Chern character form is a differential form on Y. The DR class of this form is independent of h.

The families index formula should say that as $h \to 0$, the Chern character form of $D+hL$ approaches the RHS form on Y described above.

Where does one compute the Chern character form? One has $D^2 + h[D,L] + h^2L^2 \in \Gamma(Y, \Lambda T_Y^* \hat{\otimes} \operatorname{End}(\mathcal{H}))$; this is a differential form on Y with values in $\operatorname{End}(\mathcal{H})$, so at a given pt $y \in Y$ it lies in the algebra

$$\Lambda T^*_{Y,y} \ \hat{\otimes}\ \operatorname{End}(\mathcal{H}_y)$$

which is generated by dy^a and endos. of $\mathcal{H}_y$. In fact

$$D^2 + h[D,L] + h^2L^2 = \tfrac{1}{2}dy^a dy^b[D_a,D_b] + h\,dy^a[D_a,L] + h^2L^2$$

so the sort of operators in $\mathrm{End}(\mathcal{H}_y) = \{$operators on the spinors on the fibre $X_y\}$ we must be concerned with are ~~things like~~ h^2L^2, $[D_a, hL]$, $[D_a,D_b]$ which are differential operators. The hope will be that they are asymptotic differential operators, i.e. they land in the algebra generated by functions, h, $h\gamma^\mu$, hD_μ. This is what we must check. If this is the case then we must determine the limits of these operators as $h\to 0$, i.e. their image in the associated graded algebra discovered by Getzler.

One problem is the following. $[D_a,D_b]$ is the operator on spinors associated to the same operator on $T_{X/Y}$. In fact $[D_a,D_b]$ on $T_{X/Y}$ is the a,b component of the curvature, so it is a vector bundle endomorphism of $T_{X/Y}$. If we write it $R_{ab\lambda}{}^{\mu}$, then $[D_a,D_b]$ on S is $R_{ab\lambda}{}^{\mu}\,\frac{1}{4}\gamma^\lambda\gamma_\mu$ and doesn't contain h-factors, so it is not an asymptotic operator.

$\boxed{\text{Thus something is wrong with the whole program.}}$

Let's check the $\frac{1}{2}$'s.

$$d\omega^\alpha = \tfrac{1}{2}\hat{\Gamma}^\alpha_{\lambda\mu}\,\omega^\lambda\omega^\mu \quad \in \Gamma(\Lambda^2 T^*)$$

$$D\omega^\alpha = \omega^\lambda \otimes \Gamma^\alpha_{\lambda\mu}\,\omega^\mu \quad \in \Gamma(T^*\otimes T^*)$$

Torsion zero says that under multiplication $D\omega \mapsto d\omega$ so that

$$d\omega^\alpha = \Gamma^\alpha_{\lambda\mu}\,\omega^\lambda\omega^\mu = \tfrac{1}{2}\left(\Gamma^\alpha_{\lambda\mu} - \Gamma^\alpha_{\mu\lambda}\right)\omega^\lambda\omega^\mu$$

Thus $\boxed{\Gamma^\alpha_{\lambda\mu} - \Gamma^\alpha_{\mu\lambda} = \hat{\Gamma}^\alpha_{\lambda\mu}}$

Levi-Civita connection is unique with torsion 0 and preserving metric.

$$\Gamma^{\alpha}_{\lambda\mu} = \tfrac{1}{2}\Big(\hat\Gamma^{\alpha}_{\lambda\mu} - \underbrace{\hat\Gamma^{\mu}_{\lambda\alpha} - \hat\Gamma^{\lambda}_{\mu\alpha}}_{\text{symm in } \lambda,\mu}\Big) \qquad \text{skew in } \mu, \alpha$$

$$\Longrightarrow \quad \Gamma^{\alpha}_{\lambda\mu} - \Gamma^{\alpha}_{\mu\lambda} = \hat\Gamma^{\alpha}_{\lambda\mu}$$

so the $\frac{1}{2}$ belongs.

The next thing I want to do is to pin down the connection on $T_{X/Y}$ in the transverse direction. Now one does have

$$\begin{array}{ccc} T^* & \xrightarrow{\quad d \quad} & \Lambda^2 T^* \\ \| & & \| \\ T^*_h \oplus T^*_{X/Y} & \longrightarrow & \Lambda^2 T^*_h \oplus \left(T^*_h \otimes T^*_{X/Y}\right) \oplus \Lambda^2 T^*_{X/Y} \end{array}$$

so that there is a canonical connection on $T^*_{X/Y}$ in the transverse direction. We then define the closest metric preserving connection. If ω^μ span $T^*_{X/Y}$, then

$$d\omega^\mu = \tfrac{1}{2}\hat\Gamma^{\mu}_{ab}\,\omega^a\omega^b + \hat\Gamma^{\mu}_{a\nu}\,\omega^a\omega^\nu + \tfrac{1}{2}\hat\Gamma^{\mu}_{\lambda\nu}\,\omega^\lambda\omega^\mu$$

so the canonical connection is

$$\hat D_a\,\omega^\mu = \hat\Gamma^{\mu}_{a\nu}\,\omega^\nu$$

and the closest metric-preserving connection is

$$D_a\,\omega^\mu = \underbrace{\tfrac{1}{2}\left(\hat\Gamma^{\mu}_{a\nu} - \hat\Gamma^{\nu}_{a\mu}\right)}_{\Gamma^{\nu}_{a\mu}}\omega^\nu$$

Thus $\Gamma^{\nu}_{a\mu}$ agrees with the Levi-Civita connection provided $\hat\Gamma^{a}_{\mu\nu} = 0$, and this is always the case when we have a foliation, no matter what transverse metric is used.

Prop: There is a unique connection on $T^*_{X/Y}$ which is induced from the LC connection for any choice of transverse metric.

We next want to consider the operator $[D_a, L]$ where $L = \gamma^\mu D_\mu$ is the Dirac operator. We have

$$[D_a, \gamma^\mu D_\mu] = \Gamma^\mu_{a\nu}\gamma^\nu D_\mu + \gamma^\mu [D_a, D_\mu]$$

and
$$[D_a, D_\mu] + \hat{\Gamma}^b_{a\mu} D_b + \hat{\Gamma}^\alpha_{a\mu} D_\alpha = F_{a\mu}$$

where $F_{a\mu}$ is the [redacted] mixed component of the curvature of the connection on $T^*_{X/Y}$. Let me assume $\omega^a = dy^a$ so that I know $\hat{\Gamma}^b_{a\mu} = 0$, i.e. $[X_a, X_\mu]$ is vertical. Then

$$[D_a, D_\mu] + \hat{\Gamma}^\alpha_{a\mu} D_\alpha = F_{a\mu}$$

where $F_{a\mu}$ is an endomorphism of the spinors associated to the curvature component $R_{a\mu}$. Specifically

$$F_{a\mu} = R_{a\mu\lambda}{}^\nu \tfrac{1}{4}\gamma^\lambda\gamma^\nu$$

What I really want to check is that when $R_{a\mu\lambda}{}^\nu$ is skew symmetrized over μ, λ, ν one gets zero, as otherwise $\gamma^\mu R_{a\mu\lambda}{}^\nu \frac{1}{4}\gamma^\lambda\gamma^\nu$ has order 3.

Let's try to prove this by identifying $R_{a\mu\lambda}{}^\nu$ with the corresponding curvature component for the LC connection on X associated to some choice of transverse metric. We shall do the computation at a point choosing a frame which is horizontal at the point, so that all Γ's are zero at the point. Start with

$$D(\omega^\nu) = (\omega^a \Gamma^\nu_{a\mu} + \omega^\rho \Gamma^\nu_{\rho\mu}) \otimes \omega^\mu$$

for the connection in $T^*_{X/Y}$ (note that in T^*_X there is also a horizontal ~~term~~). Then at the point where the Γ vanish

$$D(D\omega^\nu) = \omega^b(\omega^a X_b \Gamma^\nu_{a\mu} + \omega^\rho X_b \Gamma^\nu_{\rho\mu}) \otimes \omega^\mu$$
$$+ \omega^\lambda(\omega^a X_\lambda \Gamma^\nu_{a\mu} + \omega^\rho X_\lambda \Gamma^\nu_{\rho\mu}) \otimes \omega^\mu$$

So $$R_{a\lambda\mu}{}^{\nu} = X_a \Gamma_{\lambda\mu}^{\nu} - X_\lambda \Gamma_{a\mu}^{\nu}$$
at the point. But notice that this agrees with what we would calculate for the component of the curvature tensor in X. So if we skew symmetrize it over $\lambda\mu\nu$ we should get 0.

Curvature identity: $R_{ijkl} = R_{klij}$ follows from
$R_{ijkl} + R_{jkil} + R_{kijl} = 0$, R_{ijkl} skew $\begin{matrix} i \leftrightarrow j \\ k \leftrightarrow l \end{matrix}$ separately

Proof:

$$\begin{array}{ccccccc}
1234 & + & \cancel{2314} & + & \cancel{3124} & = 0 \\
-\cancel{2314} & - & 3412 & - & \cancel{4213} & = 0 \\
-\cancel{3124} & - & \cancel{1423} & - & 4321 & = 0 \\
\cancel{4213} & + & 2143 & + & \cancel{1423} & = 0 \\
\hline
\end{array}$$

$$2(1234) - 2(3412) = 0 \qquad \therefore 1234 = 3412$$

Better proof: Use only cyclic sum on first three indices

$$\begin{array}{ccccccc}
1234 & + & \cancel{2314} & + & \cancel{3124} & = 0 \\
\cancel{2341} & + & 3421 & + & \cancel{4231} & = 0 \\
\cancel{3142} & + & \cancel{1432} & + & 4312 & = 0 \\
\cancel{4213} & + & 2143 & + & \cancel{1423} & = 0 \\
\hline
\end{array}$$

$$\underbrace{1234 + 2143}_{2(1234)} + \underbrace{3421 + 4312}_{-2(3412)} = 0$$

Proof that $R_{ijkl} + R_{jkil} + R_{kijl} = 0$. Ultimately this results from

$$\begin{tikzcd}
T^* \arrow[r, "D"] \arrow[d, equal] & T^* \otimes T^* \arrow[r, "D"] \arrow[d] & \Lambda^2 T^* \otimes T^* \arrow[d] \\
T^* \arrow[r, "d"] & \Lambda^2 T^* \arrow[r, "d"] & \Lambda^3 T^*
\end{tikzcd}$$

but a component proof goes as follows. One choose a coframe ω^μ which is horizontal at the point of calculation:

$$D(\omega^\alpha) = \omega^\lambda \Gamma^\alpha_{\lambda\mu} \otimes \omega^\mu \qquad \text{where } \Gamma = 0 \text{ at } x_0.$$

$$D^2(\omega^\alpha) = \omega^\rho \omega^\lambda X_\rho \Gamma^\alpha_{\lambda\mu} \otimes \omega^\mu \qquad \text{at } x_0$$

$$\therefore \; R_{\rho\lambda\mu}{}^{\alpha} = X_\rho \Gamma^\alpha_{\lambda\mu} - X_\lambda \Gamma^\alpha_{\rho\mu} \qquad \text{"} .$$

So

$$\underset{\text{cyclic sum}}{R_{[\rho\lambda\mu]}{}^{\alpha}} = \begin{aligned} &X_\rho \Gamma^\alpha_{\lambda\mu} - X_\lambda \Gamma^\alpha_{\rho\mu} \\ &X_\lambda \Gamma^\alpha_{\mu\rho} - X_\mu \Gamma^\alpha_{\lambda\rho} \\ &X_\mu \Gamma^\alpha_{\rho\lambda} - X_\rho \Gamma_{\mu\lambda} \end{aligned}$$

$$= X_\rho \hat{\Gamma}^\alpha_{\lambda\mu} + X_\lambda \hat{\Gamma}^\alpha_{\mu\rho} + X_\mu \hat{\Gamma}^\alpha_{\rho\lambda}$$

and this is zero because $d\omega^\alpha = \frac{1}{2} \hat{\Gamma}^\alpha_{\lambda\mu} \omega^\lambda \omega^\mu$ and $d^2 \omega^\alpha = 0$.

$$D_a(\omega^\mu) = \Gamma^\mu_{a\nu} \omega^\nu \; (+ \Gamma^\mu_{ab} \omega^b)$$

$$D_\lambda(\omega^\mu) = (\Gamma^\mu_{\lambda a} \omega^a +) \; \Gamma^\mu_{\lambda\nu} \omega^\nu$$

$$\Gamma^\mu_{a\nu} = \frac{1}{2} \left(\hat{\Gamma}^\mu_{a\nu} - \hat{\Gamma}^\nu_{a\mu} - \cancel{\hat{\Gamma}^a_{\mu\nu}} \right) \qquad \text{skew-symm in } \mu\nu$$

$$\Gamma^\mu_{\nu a} = \frac{1}{2} \left(\hat{\Gamma}^\mu_{\nu a} - \cancel{\hat{\Gamma}^a_{\nu\mu}} - \hat{\Gamma}^\nu_{a\mu} \right)$$

$$= -\frac{1}{2} \left(\hat{\Gamma}^\mu_{a\nu} + \hat{\Gamma}^\nu_{a\mu} \right) \qquad \text{symm. in } \mu,\nu$$