July July 21 – Aug. 8, 1984 p. 107 – 209

Possible approach to computing the heat kernel in a Weyl algebra. ($\exists$ a basic role played by the Cayley transform to link a symplectic transformation with the Gaussian kernel representing the corresponding unitary operator as a Weyl transform.) p. 138

Itô equations. Example of image of Brownian motion under a map f. 111

Review of the supergrp $\mathbb{R}^{1,1}$ and Friedan - Windey 119-127

Bismut's factorization of $\parallel$ transport on $S \otimes E$ 128

Random walk on $\mathbb{C}^*$: $dz_t = z_t \, i \underset{\substack{\uparrow \\ \text{Brownian}}}{dw_t}$ doesn't stay on S^1 133

Fermion integrals and quantization ($tr_s \longleftrightarrow$ periodic b.c.) 143
Both Q.M. and field integrals are ambiguous for equal time Green's functions. Also (Cayley transform + Quant.)
Relation between fermion integral + character of Thom class 153, 154.

Bismut's form on Loop(M) 155 - 170
Interpretation of equivariant coh. of ΩM, $\Omega BU(k)$ 164
Case of $\Omega BU(1)$ - Bismut's form is not an equivariant form, but a Witten form. 167-168

Attempt at a local index formula when the metrics on the fibres vary 173 - 209.

Godbillon-Vey, characteristic classes for codim n foliation with trivial normal bundle: $W(gl_n)$ in Bott sense 177-179

Levi-Civita connection 176, curvature identities 208, 209

Connection on $T_{X/Y}$ preserving metric 188, 206

Bismut report after 172

July 21, 1984

Let's try to work out the steps of Bismut's proof of the index theorem in the case of a flat manifold, e.g. $\mathbb{R}^n/\Gamma$, for the Dirac operator $\gamma^\mu D_\mu$ with coefficients in an arbitrary hermitian bundle ξ with connection.

Brownian motion on $\mathbb{R}^n$ is a basic ingredient. It is a probability space W consisting of the space of continuous paths $w : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}^n$ equipped with the Wiener measure P with $P\{w_0 = 0\} = 1$. Part of its structure is the family of σ-fields:

$$F_t = \mathcal{B}(w_s \mid s \leq t)$$

(which describe questions in Mackey's sense about the Brownian motion for times $s \leq t$).

Two differentials: dw "Stratonovitch", and δw "Itô"

For me the first problem will be to define on W parallel translation operators with respect to the connection D_μ in ξ. What does this mean? According to Bismut, given an element w of W, and a point x_0 in $M = \mathbb{R}^n/\Gamma$ we get a continuous path $s \longrightarrow x_0 + w(s)$ in M. For almost all paths w it turns out that parallel translation along this path is defined. Thus really what one has associated to w is a flow of diffeomorphisms of the principal bundle of ξ.

Let's calculate an example. Take ξ to be the trivial line bundle over $\mathbb{R}^2$ with connection $D_\mu = \partial_\mu + A_\mu$. Then the parallel translation is $\exp\left\{-\int A dx\right\}$ along the path. Why should this makes sense for almost all paths? Why should $\int_0^t y\,dx$ makes sense for almost all Brownian paths in the plane?

This seems to be an interesting elementary question, namely how to define $\int_0^t y(s)\,dx(s)$ where $x(s), y(s)$ is a Brownian motion in the plane. It seems that I am trying to solve the stochastic DE

$$df = y\,dx$$

which should be simpler than the general parallel transport D.E.

$$dF = \left(-A_\mu(x)\,dx^\mu\right)F$$

One first needs to integrate before solving ODE's by the Picard method.

Let's admit the possibility of solving parallel transport equations over Brownian paths and go onto the other aspects of the Bismut proof. The first result is a formula for the operator $e^{+\frac{t}{2}\not{D}^2}$ as an integral over the space of paths.

First of all Bismut keeps W (= Brownian motion in $\mathbb{R}^n$) fixed but then puts in $\sqrt{t} = h$ in the form of weighting, so that the effective motion in $\mathbb{R}^n/\Gamma = M$ becomes narrower as $h \to 0$:

$$x_s^h = x_0 + h w_s$$

Now I think in the flat case the parallel transport operator $\tau_s^{0,h}$ from ξ_{x_0} to $\xi_{x_s^h}$ is independent of h - **NO** - the path itself depends on h. Then Bismut's first result (Thm. 2.5) expresses $e^{\frac{t}{2}\not{D}^2}$ as

$$\left(e^{\frac{1}{2}\not{D}^2}h\right)(x_0) = \text{Expectation}^P\left[\mathcal{U}_1\,\tau_0^1\,h(x_1)\right]$$

where $\mathcal{U}_s$ is a operator on spinors $\otimes\ \xi_{x_0}$

$$\frac{d\mathcal{U}_s}{ds} = \frac{1}{2}\mathcal{U}_s\left(\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}(x_s)\right)$$

What do I know? One has

$$\not{D} = \gamma^\mu D_\mu$$

$$\not{D}^2 = D_\mu^2 + \tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

Now I believe that I know the path integral representation for a heat operator $e^{\frac{1}{2}(D_\mu^2+V)}$ where V is a potential. It is

$$\int e^{-\int_0^1 \frac{\dot{x}^2}{2}ds}\, T\{e^{+\int_0^1 \frac{1}{2}V\,ds}\}$$

where the parallel transport operator has to be explained. Given a path x_s the potential $V(x_s)$ is an operator on the fibre of E at x_s, then it is to be transported back by the connection to E_{x_0} and then the little factor

$$1 + (\tfrac{1}{2}V)(x_s)\,ds$$

are to be multiplied together in order. For Bismut the order is backwards so that

$$\begin{cases} F(s) = T\{e^{\int_0^s \frac{1}{2}V\,ds}\} \\ \dfrac{dF}{ds} = F\cdot\tfrac{1}{2}V(x_s) \end{cases}$$

July 24, 1984

Itô's equation.

Let x_t denote the Brownian motion process on the line starting at $t=0$ at $x=0$. Let $f(x)$ be a real smooth function on the line and let's consider the process $f(x_t)$. This process is stationary and has independent increments. It seems more or less clear that the time-evolution operator should be given by

$$(*) \qquad K_t = \lim_{N\to\infty} \left(L_{t/N}\right)^N$$

where L_t denotes the operator with

$$dx\, L_t(x,y) = f_* \left\{ \begin{matrix} \text{Brownian motion probability after time } t \\ \text{starting at } f^{-1}(y) \end{matrix} \right\}$$

$$= f_* \left\{ dw \, \frac{e^{-\frac{(w-w_0)^2}{2t}}}{\sqrt{2\pi t}} \right\} \qquad f^{-1}(y) = w_0$$

In other words we run a random walk process as follows. Starting at a point y we lift back to $w_0 = f^{-1}(y)$, then use Brownian motion for a time t and project back.

Now from what I know about such processes, if the moments of L_t as $t \to 0$ are such that only the first + second are $O(t)$ and the rest are smaller, then K_t is the time-evolution operator for a second order parabolic equation. Let us now calculate the first + second moments of $dx\, L_t(x,y)$ as $t \to 0$:

$$\int dx\, L_t(x,y)\, e^{i\xi x} = \int dw \, \frac{e^{-\frac{(w-w_0)^2}{2t}}}{\sqrt{2\pi t}} \, e^{i\xi f(w)}$$

$$= \int du \, \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \, e^{i\xi f(w_0 + \sqrt{t}\, u)}$$

But $f(\omega_0 + \sqrt{t}\,u) = f(\omega_0) + f'(\omega_0)\sqrt{t}\,u + \frac{1}{2} f''(\omega_0)\, t\, u^2 + \cdots$

$$e^{i\xi f(\omega_0 + \sqrt{t}\,u)} = e^{i\xi y}\; e^{i\xi f'(\omega_0)\sqrt{t}\,u}\; e^{i\xi \frac{1}{2} f''(\omega_0) t u^2}$$

$$= e^{i\xi y}\left(1 + i\xi f'(\omega_0)\sqrt{t}\,u - \frac{\xi^2}{2} f'(\omega_0)^2 t\, u^2 \cdots\right)\left(1 + i\xi \tfrac{1}{2} f''(\omega_0) t u^2 + \cdots\right)$$

$$= e^{i\xi y}\left(1 + i\xi f'(\omega_0)\sqrt{t}\,u - \frac{\xi^2}{2} f'(\omega_0)^2 t\, u^2 + i\xi \tfrac{1}{2} f''(\omega_0) t u^2 + \cdots\right)$$

So now when the Gaussian integral is done one get

$$\boxed{\int dx\, L_t(x,y)\, e^{i\xi x} = e^{i\xi y + i\xi \frac{1}{2} f''(\omega_0) t - \frac{\xi^2}{2} f'(\omega_0)^2 t + o(t)}}$$

What this means is that for small t the probability distribution $dx\, L_t(x,y)$ has the first moment + second moment

$$M_1(y) = \tfrac{1}{2} f''(\omega_0)\, t + o(t) \qquad \omega_0 = f^{-1}(y)$$

$$M_2(y) = f'(\omega_0)^2 t + o(t) \qquad ''$$

So what we find is that the Brownian motion fluctuations seem to produce a drift proportional to f''.

July 25, 1984

A basic idea in Bismut's paper is to use ordinary Brownian motion as an input process which then generates other processes. I looked above at the simple example

$$x_t = f(w_t)$$

where w_t denotes Brownian motion. This is a global formula for the process - you take all the stuff you know about Brownian motion and push it forward under f. However one really wants a local description - how to compute x_{t+dt} knowing x_t. One has some sort of formula like

$$\begin{cases} dx_t = f'(w_t)\,dw_t + \frac{1}{2} f''(w_t)\underbrace{(dw_t)^2}_{dt} \\ (dx_t)^2 = f'(w_t)^2\,dt \end{cases}$$

whatever this might mean. Itô theory provides a rigorous way to deal with this stuff. A process x_t generated by Brownian motion is defined in practice by writing down a stochastic differential equation, which is an ordinary DE except that one has to incorporate these second order terms. In the above case the ODE should probably be

$$dx_t = f'(w_t)\,dw_t$$

or

$$x_t = x_0 + \int_0^t f'(w_t)\,dw_t .$$

Now the problem is how to interpret dw_t. One can't write $dw_t = \frac{dw_t}{dt}\,dt$ as w_t is not differentiable.

Let's leave the mysteries of the Itô theory for the moment and look at some further examples. I can keep track of what is going on by concentrating on the underlying heat equation, i.e. the infinitesimal generator of the process.

So let's start off with the problem of parallel translation. Let's consider a G-bundle over $\mathbb{R}^n$ with connection and the process which lifts $\omega_t \in \mathbb{R}^n$ horizontally.

Bismut's example: Consider Brownian motion ω_t in the Lie algebra $\mathfrak{g}$ of a compact Lie gp G. Then one wants to assign to each ω_t in $\mathfrak{g}$ the corresponding curve g_t defined by

$$dg_t = g_t \, d\omega_t \qquad\qquad g_0 = 1.$$

Thus

$$g_t = T\left\{ e^{\int_0^t d\omega_t} \right\}$$

where the time ordering takes place in the opposite order from the physicists convention.

The question is whether there is a 2nd order Itô term in the above process. I propose to find the associated heat equation on G corresponding to this random motion process. It clearly is a random motion in a non-commutative sense. If one has the position g_t at time t, then one makes a random jump $d\omega_t$ in the Lie algebra $\mathfrak{g}$ with a Gaussian distribution of "spread" dt, then exponentiates this into the group, and multiplies

g_t on the right so as to obtain the new position in the group:

$$g_t + dg_t = g_t \cdot e^{dw_t}$$

and so

$$dg_t = g_t\left(e^{dw_t} - 1\right)$$
$$= g_t\left(dw_t + \tfrac{1}{2}(dw_t)^2 + \cdots\right).$$

Thus it looks like one has a 2nd order Itô term of some sort.

What is the associated heat equation? Clearly the random process just described is left-invariant, so the heat kernel $K_t(g,g')$ is invariant under left translation

$$K_t(g,g') = K_t((g')^{-1}g, 1) = k_t((g')^{-1}g)$$

where $k_t(g) = K_t(g,1)$. Because the Brownian motion on $\mathfrak{g}$ is invariant under G-conjugation it is clear that the random walk process is in fact invariant under conjugation, and hence also right invariant.

In fact we get the same process on the group by using left translation, since

$$g_t\, e^{dw_t} = e^{Ad(g_t)\, dw_t}\, g_t$$

and $Ad(g_t)\, dw_t$ is distributed the same as dw_t. Thus the result of our random process has to be the usual heat equation on the compact Lie group G, that is, the infinitesimal generator is the Casimir operator up to a suitable normalization.

Itô stuff (after Stroock - Varadhan book)

Basic object: diffusion process on $\mathbb{R}^n$. Such things are described by a heat equation

$$\partial_t u = \Delta u$$

The backward differential equation is better for probabilistic purposes than the forward (or Fokker-Planck) equation. Let's write the Laplacean operator in the form

$$\frac{1}{2} a^{\mu\nu} \partial^2_{\mu\nu} + b^\mu \partial_\mu$$

Then one assumes a factorization $a = \sigma^* \sigma$.

Now the heat equation corresponds to a diffusion process $\xi(t)$ with variance $a^{\mu\nu}$ and drift b^μ. Then if we change variables:

$$d\xi = \sigma \cdot d\beta + b\,dt$$

it follows that β is Brownian motion. So one sees that it is natural to factor the diffusion problem.

My program now will be to tie up the index theorem with the path integrals. The problem will be to see how the physicists' fermion path integrals become on the probabilistic side. I have the feeling that the physicists' supersymmetry principle is worth something, so I start from their end.

The Dirac operator $\not{D} = \gamma^\mu D_\mu$ is a super-symmetric version of the covariant Laplacean D_μ^2, and the path integral expression for the Dirac Hamiltonian is supposed to result by applying the super-symmetry principle to the path integral for D_μ^2.

Let me consider then the operator $H = -\hbar^2 D_\mu^2 + V$ where V is a vector bundle endomorphism. The **path integral** expression for $\langle x | e^{-tH} | y \rangle$ results formally for the small-time asymptotics of this kernel. One only has to keep track of the moments to order t. [redacted]

Let's review this last statement. One supposes a process generated by a path L_t starting at 1 at $t=0$:

$$K_t = \lim_{n\to\infty} \left(L_{\frac{t}{n}} \right)^n$$

where the generating path L_t has kernel $L_t(x,y)$ with moments

$$\int (x-y)^n L_t(x,y)\, dx = \begin{cases} 1 + M_0(y)t + o(t) & n=0 \\ M_1(y)t + o(t) & 1 \\ M_2(y)t + o(t) & 2 \\ o(t) & >2 \end{cases}$$

For example take

$$L_t(x,y) = \frac{e^{-\frac{(x-y)^2}{2t\hbar^2}}}{(2\pi t \hbar^2)^{n/2}} \left(K_0(x,y) + K_1(x,y)t + \dots \right)$$

Take $y=0$ and suppose

$$K_0(x,0) = 1 + ax + bx^2 + \dots$$
$$K_1(x,0) = c + \dots$$

Then

$$\int x^n L_t(x,0)\, dx = \begin{cases} 1 + bt\hbar^2 + ct + o(t) \text{ [redacted]} & n=0 \\ at\hbar^2 + o(t) & n=1 \\ \text{[redacted]}\, t\hbar^2 + o(t) & n=2 \\ o(t) & n>2 \end{cases}$$

Thus as far as the process generated by L_t is concerned only $K_0(x,y)$ to second order and $K_1(x,x)$ is relevant to the heat equation and hense the path integral.

So now let's construct $L_t(x,0)$ to satisfy the heat equation:

$$\left(\partial_t - \tfrac{1}{2}h^2 D_\mu^2 + V\right) L_t(x) = 0$$

where
$$L_t(x) = \frac{e^{-\frac{(x)^2}{2th^2}}}{(2\pi t h^2)^{n/2}}\left(K_0(x) + tK_1(x) + \dots\right).$$

Then

$$\left\{\partial_t + \frac{x^2}{2t^2h^2} - \frac{n}{2t} - \frac{h^2}{2}\left(D_\mu - \frac{x^\mu}{th^2}\right)^2 + V\right\}\left(K_0 + tK_1 + \dots\right) = 0$$

$$\left\{\partial_t + \frac{1}{t}x^\mu D_\mu - \frac{h^2}{2}D_\mu^2 + V\right\}\left(K_0 + tK_1 + \dots\right) = 0$$

As usual one assumes a synchronous framing of the bundle, whence $x^\mu D_\mu 1 = x^\mu A_\mu = 0$ and $A_\mu(0) = 0$

Thus $\quad A_\mu(x) = -\frac{1}{2}F_{\mu\nu}(0)x^\nu + O(x^2)$

so
$$D_\mu^2 = \partial_\mu^2 + 2\underbrace{A_\mu}_{O(x)}\partial_\mu + \underbrace{\left(\partial_\mu A_\mu + A_\mu^2\right)}_{O(x)} = \partial_\mu^2 \text{ at } x=0$$

Then we have the equations

$$x^\mu D_\mu K_0 = 0 \implies K_0 = 1$$

$$K_1 + x^\mu D_\mu K_1 + \left(-\frac{h^2}{2}D_\mu^2 + V\right)K_0 = 0$$

$$\implies K_1(0) = -V(0).$$

Therefore we learn the following about the path integral for the kernel of $e^{-t\left(-\frac{h^2}{2}D_\mu^2 + V\right)}$. It should be built up out of parallel transport along curves over $[0,t]$ with the potential acting along the path.

The next question is how to incorporate this parallel transport with V into an actual Lagrangian. First suppose $V = 0$.

We know how to handle things in the case of [redacted] a $U(1)$ [redacted] (or perhaps more generally, an abelian) gauge field. Then one has a 1-form $A_\mu dx^\mu$ and the parallel transport operator is

$$e^{-\int A_\mu dx^\mu}$$

where the integral is along the path. So the path integral is

$$\int \mathcal{D}x(\tau) \quad e^{-\int_0^t \left(\frac{\dot{x}^2}{2h^2} + A_\mu \dot{x}^\mu\right) d\tau}$$

<u>July 26, 1984</u>

The immediate problem is to learn how the physicists do the index thm with path integrals involving fermions. I then want to compare this with Bismut's approach.

I need to learn how to set up the path integral with fermions. I find the physicists versions hard to understand. They start with a Lagrangian which is constructed from a standard Lagrangian by various recipes, namely, supersymmetry, inserting covariant derivatives and metrics. Having obtained the Lagrangian they "know" it can be quantized, and then they can identify the quantization with the Dirac operator.

In order to understand this I have to go back to more primitive ideas, namely, how a Lagrangian is related to a quantization, i.e. operators on some Hilbert space.

So will start with the Dirac operator $\not{D} = \gamma^\mu D_\mu$ the space $\mathcal{H}$ of spinors with values in a vector bundle E with connection D_μ. The path integral is supposed to represent the super heat operator

$$t, \theta \longmapsto e^{t\not{D}^2 + \theta \not{D}} = e^{t\not{D}^2}(1+\theta\not{D}),$$

which one can view as follows.

First we consider the Lie superalgebra generated by a single **odd** element Q. We have a repn. of this Lie superalgebra by sending Q to $\not{D}$. Exponentiating this representation we obtain a morphism of Lie supergps

$$\mathbb{R}^{1,1} \longrightarrow \underline{\mathrm{Aut}}(\mathcal{H})$$

described by the above formula.

So our problem is now to represent $e^{t\not{D}^2 + \theta\not{D}}$ as a path integral in some way. This should mean a way to write this operator as a product of small pieces.

It seems to me that $\not{D}$ is the infinitesimal generator of the process, so a good path integral should be a representation of $e^{t\not{D}^2+\theta\not{D}}$ as a product of things of the form

$$e^{\varepsilon\not{D}} = 1+\varepsilon\not{D}$$

They can't be the same as with usual case since

$$\left(e^{\varepsilon\not{D}}\right)^n = e^{n\varepsilon\not{D}}$$

doesn't involve $\not{D}^2$.

(Note the following formula:

$$\left(\partial_\theta - \theta\partial_t\right)\left(e^{t\not{D}^2+\theta\not{D}}\right) = \left(\partial_\theta - \theta\partial_t\right) e^{t\not{D}^2}(1+\theta\not{D})$$

$$= e^{t\not{D}^2}\left\{\partial_\theta - \theta(\partial_t + \not{D}^2)\right\}(1+\theta\not{D})$$

$$= e^{t\not{D}^2}\left(\not{D} - \theta\not{D}^2\right) = \not{D}\, e^{t\not{D}^2}(1+\theta\not{D})$$

or

$$\boxed{\left(\partial_\theta - \theta\partial_t\right) e^{t\not{D}^2+\theta\not{D}} = \not{D}\, e^{t\not{D}^2+\theta\not{D}}}$$

This shows that $\not{D}$ corresponds to the infinitesimal translation operator $\partial_\theta - \theta\partial_t$ in the supergroup, whose square is essentially the ordinary time derivative.

$$\left(\partial_\theta - \theta\partial_t\right)^2 = -\partial_t \quad . \quad \blacksquare$$

Let's work out the group law. This supergroup assigns to a comm. superalgebra A the group of pairs $(a,b) \in A^0 \times A^1$ with the group law obtained from multiplication of operators, where to (a,b) is assigned the operator $e^{a\not{D}^2 + b\not{D}}$. Thus as

$$\left(e^{a\not{D}^2+b\not{D}}\right)\left(e^{a'\not{D}^2+b'\not{D}}\right) = e^{(a+a')\not{D}^2+(b+b')\not{D} + \frac{1}{2}\underbrace{[b\not{D}, b'\not{D}]}_{-bb'\not{D}^2 + b'b\not{D}^2}}$$

= $e^{(a+a'-$[redacted]$bb')\not{D}^2 + (b+b')\not{D}}$

we see the group law is

$$\boxed{(a,b) * (a',b') = (a+a' - \text{[redacted]}\, bb',\ b+b')}$$

(Check: $(1+b\not{D})(1+b'\not{D}) = 1 + b\not{D} + b'\not{D} + b\not{D}b'\not{D}$

$= 1$ [redacted] $- bb'\not{D}^2 + (b+b')\not{D}$)

Let's now compute the effect of infinitesimal left-translation on a function $f(t,\theta) = a(t) + \theta b(t)$ on $\mathbb{R}^{1|1}$.

$$f[(\delta t, \delta\theta) * (t,\theta)] = f(\delta t + t - \delta\theta\,\theta,\ \delta\theta + \theta)$$

$$= a(\delta t + t - \delta\theta.\theta) + (\delta\theta + \theta) b(\delta t + t - \delta\theta.\theta)$$

$$= a(t) + a'(t)[\delta t - \delta\theta\cdot\theta] + (\delta\theta + \theta)[b(t) + b'(t)(\delta t - \delta\theta.\theta)]$$

$$= a(t) + \theta b(t) + \delta t\{a'(t) + \theta b'(t)\} + \delta\theta\{-a'(t)\theta + b(t)\}$$

$$= f(t,\theta) + \delta t\ \partial_t f(t,\theta) + \delta\theta\ (\partial_\theta - \theta\partial_t) f(t,\theta)$$

Thus we see that

$\partial_\theta - \theta\partial_t$ inf. left translation

probably $\partial_\theta + \theta\partial_t$ —— right ————.

and as a check note that

$$[\partial_\theta - \theta\partial_t,\ \partial_\theta + \theta\partial_t]_{\underset{(+)}{\uparrow}} = \partial_t - \partial_t = 0.$$

)

It looks like we can construct a general group element $e^{t\phi^2+\theta\phi}$ as the product of pieces of the form $e^{\varepsilon\phi} = (1+\varepsilon\phi)$. Specifically take a 1-parameter family of these and take the time-ordered product:

$$T\left\{e^{\int dt\, \varepsilon(t)\phi}\right\} = e^{\left(\int dt\, \varepsilon(t)\right)\phi + \frac{1}{2}\int_{t_1>t_2} dt_1\, dt_2\, [\varepsilon(t_1)\phi, \varepsilon(t_2)\phi]}$$

$$= e^{\left(\int dt\, \varepsilon(t)\right)\phi - \left(\int_{t_1>t_2} dt_1\, dt_2\, \varepsilon(t_1)\varepsilon(t_2)\right)\phi^2}$$

Now the problem is how to make sense of this as a path integral.

Review the notion of Gaussian process:

Suppose one is given a real vector space V with a positive definite quadratic form $Q(v)$. Then on V^* is a unique Gaussian measure $d\mu$ such that

$$\int_{V^*} d\mu(x)\, e^{i v(x)} = e^{-\frac{1}{2}Q(v)}$$

or equivalently Q gives the variance of the Gaussian measure, where we think of V as the space of linear functions on V^*. We know by writing V as an orthogonal direct sum of lines, better, by choosing an isom. of V with $\mathbb{R}^n$, that the polynomial functions $S(V)$ are dense in $L^2(V^*, d\mu)$. In fact $L^2(V^*, d\mu)$ is naturally isomorphic to $S(V)$ in the Hilbert space sense.

Now a Gaussian process is a family of random variables x_t such that the joint distributions of any finite set of these r.v. is Gaussian. Such a process is completely determined by giving the variance which in this case is the matrix of **inner** products $\langle x(t)\, x(t')\rangle$, which in the non degenerate case will be positive definite, and in general positive semi-definite. Clearly this is the same as giving a curve in a Hilbert space.

One can even assume the variance matrix $\langle x(t)\, x(t')\rangle$ is a distribution, because one can then define a **Hilbert space** by starting with smooth functions with

$$\|f\|^2 = \int dt\, dt'\ \overline{f(t)} f(t') \langle x(t)\ x(t')\rangle$$

provided this gives a non-negative inner product. One has to be a little careful with all this because, for example, if one takes $\langle x(t)\, x(t')\rangle = \delta(t-t')$, then $\|f\|^2$ = usual L^2 norm^2 of f, so that one is trying to put the Gaussian cylinder measure on Hilbert space, and this is not really a measure.

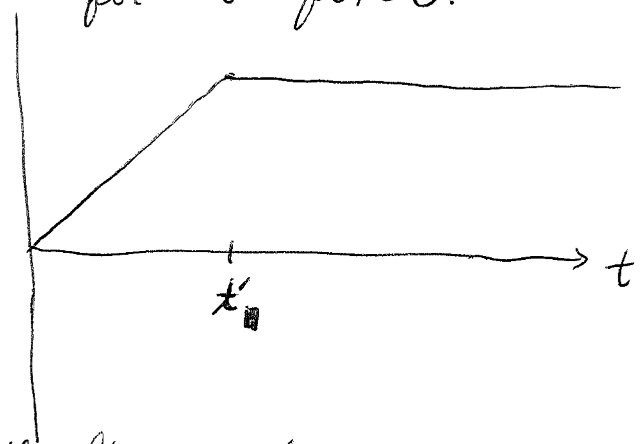
What is the variance for Brownian motion?

$$\langle x(t)\, x(t')\rangle = \underset{\overset{\|}{0}}{\langle (x(t) - x(t'))\, x(t')\rangle} + \underset{\overset{\|}{t'}}{\|x(t')\|^2}$$

if $t \geq t'$. Thus

$$\langle x(t)\, x(t')\rangle = \min\{t, t'\}$$

and here $t \geqslant 0$ since we start with $x(0) = 0$. Graph of $\langle x(t)\, x(t')\rangle$ for t' fixed.



Thus it is the Green's function for the operator $-\partial_t^2$ on $0 < t < \infty$.

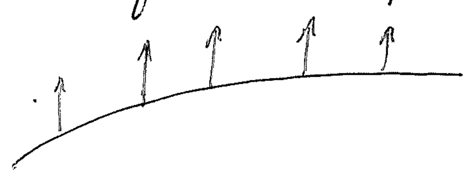
July 27, 1984:

Let's consider a connection $D_\mu = \partial_\mu + A_\mu$ over $M = \mathbb{R}^n$ and suppose we are given a "supercurve":

$$X^\mu = x^\mu(t) + \theta \psi^\mu(t) \quad : \mathbb{R}^{1,1} \longrightarrow M.$$

We should be able to pull back the connection via X and obtain a connection over $\mathbb{R}^{1,1}$. Our problem will be to describe this connection, and in particular to describe its horizontal sections.

X can be viewed as a curve $x(t)$ in M together with a vector field given along the curve. To simplify, let's suppose the situation non-degenerate, that is, $t \longrightarrow x(t)$ embeds $\mathbb{R}$ into M and $\psi(t)$ is independent of $\dot{x}(t)$. Thus we have an infinitesimal surface strip



embedded in $\mathbb{R}^n$. Given a function f on $\mathbb{R}^n$ it restricts to the supercurve

$$f(X) = f(x + \theta\psi) = f(x(t)) + \theta \partial_\mu f(x(t)) \psi^\mu(t)$$

or more concisely

$$f(X) = f(x) + \theta \partial_\mu f(x) \psi^\mu .$$

Because of the non-degeneracy hypothesis, every fn on $\mathbb{R}^{1,1}$ is the restriction of an f on $\mathbb{R}^n$.

The connection D_μ is a connection on the trivial bdle over $\mathbb{R}^n$ which pulls back to the trivial bundle over $\mathbb{R}^{1,1}$ with the same fibre v.s. (totally even). A section of the latter is of the form

$$N = \eta(t) + \theta \zeta(t)$$

where η, ζ are functions in the fibre v.s. We can assume N is the restriction of a vector fn. $\tilde{N}(x)$.

Assume $N = \eta(t) + \theta\xi(t)$ is the restriction of $\tilde{N}$:

$$N = \tilde{N}(X) = \tilde{N}(x + \theta\psi)$$

$$= \underbrace{\tilde{N}(x)}_{\eta} + \theta \underbrace{\partial_\mu \tilde{N}(x)\psi^\mu}_{\xi}$$

There are two covariant derivatives of $\tilde{N}$ at each point $x(t)$. Better, we can take the covariant derivative of $\tilde{N}$, namely $dx^\mu(\partial_\mu + A_\mu)\tilde{N}$ and restrict it to the two tangent directions $\dot{x}^\mu$ and ψ^μ we have at each t. We get

$$\dot{x}^\mu(\partial_\mu + A_\mu)\tilde{N} = \left[\partial_t + \dot{x}^\mu A_\mu(x)\right]\eta$$

$$\psi^\mu(\partial_\mu + A_\mu)\tilde{N} = \xi + \psi^\mu A_\mu(x)\eta$$

However this isn't the complete result, because $\mathbb{R}^{1,1}$ is two-dimensional; its module of diffl's is free of rank 2 generated by dt, $d\theta$. Consequently there should be two covariant derivative operators D_t, D_θ acting on the sections $N = \eta + \theta\xi$.

The only sensible thing to do is to pull back the differential form $dx^\mu A_\mu(x)$.

$$dX^\mu A_\mu(X) = d(x^\mu + \theta\psi^\mu)\; A_\mu(x + \theta\psi)$$

$$= \left(dt\,\dot{x}^\mu + d\theta\,\psi^\mu - \theta\,dt\,\dot{\psi}^\mu\right)\left(A_\mu(x) + \theta\psi^\nu\partial_\nu A_\mu(x)\right)$$

$$= dt\,(\dot{x}^\mu + \theta\dot{\psi}^\mu)(A_\mu + \theta\psi^\nu\partial_\nu A_\mu) + d\theta\,\psi^\mu(A_\mu + \theta\psi^\nu\partial_\nu A_\mu)$$

$$= dt\left[\dot{x}^\mu A_\mu + \theta(\dot{\psi}^\mu A_\mu + \dot{x}^\mu\psi^\nu\partial_\nu A_\mu)\right] + d\theta\left[\psi^\mu A_\mu + \theta\psi^\mu\psi^\nu\partial_\nu A_\mu\right]$$

which gives the covariant derivatives

$$D_t = \partial_t + \dot{x}^\mu A_\mu + \theta\left(\dot{\psi}^\mu A_\mu + \dot{x}^\mu \psi^\nu \partial_\nu A_\mu\right)$$

$$D_\theta = \partial_\theta + \psi^\mu A_\mu + \theta\, \psi^\mu \psi^\nu \partial_\mu A_\nu$$

Now the Friedan-Windey paper considers the action

$$\int dt\, d\theta \quad \bar{N}\left(D_\theta - \theta D_t\right) N,$$

so let's compute the θ coeff. in $\bar{N}(D_\theta - \theta D_t)N$.

$$(D_\theta - \theta D_t) N = \left\{\left(\partial_\theta + \psi^\mu A_\mu + \theta \psi^\mu \psi^\nu \partial_\mu A_\nu\right) - \theta\left(\partial_t + \dot{x}^\mu A_\mu\right)\right\}(\eta + \theta \xi)$$

$$= \left(\xi + \psi^\mu A_\mu \eta\right) + \theta\left[-\psi^\mu A_\mu \xi + \left(-\partial_t - \dot{x}^\mu A_\mu + \psi^\mu \psi^\nu \partial_\mu A_\nu\right)\eta\right]$$

Mult. by $\bar{N} = \bar{\eta} + \theta \bar{\xi}$ and take θ coeff.

$$\bar{\xi}\left(\xi + \psi^\mu A_\mu \eta\right) + \bar{\eta}\left[\psi^\mu A_\mu \xi + \left(\partial_t + \dot{x}^\mu A_\mu - \psi^\mu \psi^\nu \partial_\mu A_\nu\right)\eta\right]$$

Integrating w.r.t. t gives the action. Eliminate $\bar{\xi}$, ξ by the variational equations

$$(\star) \qquad \begin{aligned} \xi + \psi^\mu A_\mu \eta &= 0 \\ \bar{\xi} + \bar{\eta}\, \psi^\mu A_\mu &= 0 \end{aligned}$$

and we end up with the action

$$\bar{\eta}\left[\psi^\mu A_\mu\left(-\psi^\nu A_\nu \eta\right) + \left(\partial_t + \dot{x}^\mu A_\mu - \psi^\mu \psi^\nu \partial_\mu A_\nu\right)\eta\right]$$

$$= \boxed{\bar{\eta}\left[\partial_t + \dot{x}^\mu A_\mu - \tfrac{1}{2}\psi^\mu \psi^\nu F_{\mu\nu}\right]\eta}$$

This time around we know a little more about the meaning of things. In particular the condition $(\star)$ means that ξ is adjusted so the section N is horizontal in the transversal direction.

So the conclusion is that I still haven't managed to understand what is going on with the superfield business. Most mysterious is the action

$$\int dt\, d\theta\ \bar{N}\cdot(D_\theta - \theta D_t)N$$

which generalizes the action

$$\int dt\ \bar{\eta}\, D_t \eta$$

which one adds to get the parallel transport term in the path integral for $e^{tD_\mu^2}$.

Notice that Planck's constant is absent, hence this formalism should be classical or geometric.

At this point we have reconstructed the physicists' Lagrangian for the operator $e^{t\not{D}^2}$. So we should be able to make sense of their computation of the index density, and in particular to see the critical point evaluation as $h\to 0$. Then we can compare ~~this~~ with Bismut's procedure.

Bismut's approach differs from the physics approach because he doesn't use fermion integrals at all. He does not use $\bar{\eta},\eta$ integration to get at parallel transport in E, but rather has a trick for getting at parallel transport in $S\otimes E$. This goes as follows.

Consider two vector spaces S, E and endos $L_a(t)$ of S, $M_a(t)$ of E for $1\le a\le n$, $0\le t\le 1$. Let $w^a(t)$ be n-diml Brownian motion and consider the stochastic parallel transport operator

$$V_t = T\left\{ e^{\int_0^t L_a(t)\,\delta w_t^a} \right\}$$

defined by the Itô DE

$$dV_t = V_t \, L_a(t)\,\delta w_t^a \qquad\qquad V_0 = I.$$

Similarly define

$$W_t = T\left\{ e^{\int_0^t M_a(t)\,\delta w_t^a} \right\}$$

<u>Proposition</u>: The expectation of $V_t \otimes W_t$ is the parallel transport operator of $\sum_a L_a \otimes M_a$ on $S \otimes E$:

$$T\left\{ e^{\int_0^t (L_a \otimes M_a)(t)\,dt} \right\}$$

Proof: One has Itô equations

$$dV_t = V_t \, L_a(t)\,\delta w_t^a$$
$$dW_t = W_t \, M_a(t)\,\delta w_t^a$$

Then by the Itô calculus

$$d\left[V_t \otimes W_t\right] = (V_t \otimes W_t)\big(L_a(t) \otimes I + I \otimes M_a(t)\big)\,\delta w_t^a$$
$$+ (V_t \otimes W_t)\big(L_a(t) \otimes M_a(t)\big)\,dt$$

so taking expectations

$$d\langle V_t \otimes W_t \rangle = \langle V_t \otimes W_t \rangle (L_a \otimes M_a)(t)\,dt$$

whence the result.

Here is how this is applied. We want the heat operator $e^{h^2 \not{D}^2}$, where $\not{D}^2 = D_\mu^2 + \frac{1}{2}\gamma\gamma F$.
' ording to standard path integral lore, this be represented as a standard Wiener type

integral

$$\int \mathcal{D}x(t)\, e^{-\int_0^1 \frac{1}{4h^2}\dot{x}^2\, dt}\; T\left\{ e^{\int_0^1 [-\dot{x}^\mu A_\mu + \frac{h^2}{2}\gamma\gamma F(x)]\, dt} \right\}$$

By standard lore I mean from operators of the form $-D_\mu^2 + V$. Now this path integral involves parallel transport in the bundle $S \otimes E$. If I fix the curve I can use the connection in E to trivialize $S \otimes E$, whence $A_\mu = 0$ and we have the setup of the proposition with

$$L_a = \frac{h^2}{2}\gamma^\mu\gamma^\nu \qquad M_a = F_{\mu\nu}(x(t))$$

This gives

$$T\left\{ e^{\int_0^t [-\dot{x}^\mu A_\mu + \frac{h^2}{2}\gamma\gamma F]\, dt} \right\} = \int \mathcal{D}\omega_{\mu\nu}\; e^{-\int_0^t \frac{1}{2}\dot{\omega}_{\mu\nu}^2\, dt}$$

$$\times\, T\left\{ e^{\int_0^t \frac{h^2}{2}\gamma^\mu\gamma^\nu\, \delta\omega_{\mu\nu}} \right\} \otimes T\left\{ e^{\int_0^t [-\dot{x}^\mu A_\mu\, dt + F_{\mu\nu}(x)\,\delta\omega^{\mu\nu}]} \right\}$$

Now set $t = 1$, whence assuming we are integrating over loops we can take the supertrace on the spinors and the trace on E. Then we let $h \to 0$

But first I ought to let $x_t = x_0 + h\omega_t$ where ω_t is a standard Brownian bridge. Notice that only the last $\parallel$ transport operator depends on the path.

At this point we have to do a little computation to evaluate the function

$$\frac{1}{h^n}\, \mathrm{tr}_S\, T\left\{ e^{\int_0^t \frac{h^2}{2}\gamma^\mu\gamma^\nu\, \delta\omega_{\mu\nu}} \right\}$$

in the limit as $h \to 0$ and also

$$\mathrm{tr}\, T\left\{ e^{\int_0^1 F_{\mu\nu}\, \delta\omega^{\mu\nu}} \right\}.$$

July 28, 1984

Let's review the construction of $\langle x_0 | e^{h^2 \not{D}^2} | x_0 \rangle$. This is expressed as an integral over the Wiener probability space of Brownian loops ω_t, $\omega_0 = \omega_1 = 0$. To each loop ω_t we assign the path $x_t = x_0 + h\omega_t$ and then the operator on $(S \otimes E)_{x_0}$

$$T\left\{ e^{\int [-\dot{x}^\mu A_\nu + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}(x)]} \right\}.$$

Thus the ultimate object of interest for the index is the number

$$\mathrm{tr}_s \langle x_0 | e^{h^2 \not{D}^2} | x_0 \rangle = \int \mathcal{D}\omega \; e^{-\frac{1}{4}\int_0^1 \dot{\omega}^2 dt} \times$$

☆

$$\frac{1}{(4\pi)^{n/2}} \frac{1}{h^n} \mathrm{tr}_s \; T\left\{ e^{\int_0^1 [-\dot{x}^\mu A_\nu(x) + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}(x)]dt} \right\}$$

We now let $h \to 0$. Notice that we have a family of functions depending on h on the Wiener probability space of Brownian loops. The function is obtained by solving a stochastic DE depending on the loop. The stochastic DE is an Itô DE which means roughly the following.

Let's consider the problem of $\parallel$ transport along a path x, which means trying to solve an equation

$$dy_t = \left(-A_\mu(x_t)\, dx_t\right) y_t$$

if the path is smooth. But the smooth or even C' paths are negligible in Wiener space; in general one must deal with curves where $dx_t \sim \sqrt{dt}$. This means there are second order contributions to dy_t, and this is what the Itô theory is all about. Thus in passing from the level of differentiable curves to Brownian paths the [redacted] calculus takes a different form. The usual answers for smooth curves are perhaps going to be misleading.

<u>Example</u>: For Brownian motion [redacted] on the line one has the formula

$$d\,w_t^2 = 2w_t\,\delta w_t + dt$$

instead of what one expects from the usual calculus rules.

Now let's return to the formula $\star$ above. The problem is to understand what happens to the integrand as $h \longrightarrow 0$. It is natural to look for the power series expansion in h. There are contributions from the following: [redacted] First the path $x_t = x_0 + h w_t$ depends on h. Secondly, one has ~~the~~ $h^2 \gamma^\mu \gamma^\nu$. It would seem that my usual feeling about this is correct, namely that the leading term has to be the same as if one were to replace $h\gamma^\mu$ by dx^μ and take the [redacted] component of degree n, ending up with

$$\left(\frac{i}{2\pi}\right)^m tr\left\{e^{\frac{1}{2}dx^\mu dx^\nu F_{\mu\nu}(x_0)}\right\}.$$

However the only way I could a way of proving this works is to use the perturbation expansion in the F term. Bismut has a better way, namely, by introducing an auxiliary Brownian motion $w_t^{\mu\nu}$ he is able to separate the spinors and the bundle

E and so write

$$T\left\{ e^{\int[-\dot{x}^\mu A_\mu(x) + \frac{h^2}{2}\gamma\gamma F(x)]dt} \right\} = \int dW(\omega_t^{\mu\nu}) \times$$

$$T\left\{ e^{\int \frac{h^2}{2}\gamma^\mu\gamma^\nu \delta\omega^{\mu\nu}} \right\} \otimes T\left\{ e^{\int -\dot{x}^\mu A_\mu(x) dt + F_{\mu\nu}(x)\delta\omega^{\mu\nu}} \right\}$$

The point will be now that

$$\frac{1}{h^n} \operatorname{tr}_S T\left\{ e^{\int \frac{h^2}{2}\gamma^\mu\gamma^\nu \delta\omega^{\mu\nu}} \right\}$$

can be analyzed separately and seen to have a limit which is essentially the nth degree component of

(1) $$T\left\{ e^{\int_0^1 \frac{1}{2} dx^\mu dx^\nu \delta\omega_t^{\mu\nu}} \right\} \qquad \text{time } (2i)^m$$

The second function $\operatorname{tr}\left\{ e^{\int_0^1 [-\dot{x}^\mu A_\mu(x)\, dt + F_{\mu\nu}(x)\delta\omega^{\mu\nu}]} \right\}$ has the limit

(2) $$\operatorname{tr} T\left\{ e^{\int_0^1 F_{\mu\nu}(x_0)\delta\omega_t^{\mu\nu}} \right\}$$

as $h \to 0$. Now if we undo the process of separating the S and the E part, then upon taking the expectation wrt $\omega_t^{\mu\nu}$, we get

$$\operatorname{tr} T\left\{ e^{\int_0^1 \frac{1}{2} dx^\mu dx^\nu F_{\mu\nu}(x_0)} \right\}_{(n)} = \operatorname{tr}\left\{ e^{F(x_0)} \right\}_{(n)}$$

Now I have many questions about the functions (1), (2) above

I now want to explore the Itô calculus to get a feeling for how it works. I want to start with the physical idea of a random walk and Brownian motion as its continuous limit. Motion is tricky to describe in mathematical terms. Ordinary motion is described by calculus; Brownian motion is described by Itô calculus.

To specify a random walk we must give the probability distribution of how to jump at each stage. So one must specify the dx_t corresponding to the time interval $t, t+dt$. Think of the process as being geometric and the problem of describing it, or calculating it, as being mathematical.

Let's consider an example which is an example of a random walk on the multiplicative group $\mathbb{C}^*$. We must specify the jump dz_t and we will use group invariance to give the jump in the Lie algebra. Suppose the jump in the Lie algebra is Brownian motion in the imaginary direction so that our equation is tentatively

$$dz_t = z_t \, i\, d\omega_t$$

But $d\omega_t$ is a quantity of order $\sqrt{dt}$, so that we must worry about how to identify the Lie algebra with the group to second order at the identity.

In the Itô calculus the equation

$$dz_t = z_t \, i\, \delta\omega_t$$

is different from

$$d(\log z_t) = i\, \delta\omega_t$$

In the former we have

$$d|z_t|^2 = \bar{z}_t\, dz_t + d\bar{z}_t \cdot z_t + d\bar{z}_t\, dz_t$$

$$= \bar{z}_t\, i\,\delta w_t + (-i)\,\delta w_t\, z_t + |z_t|^2 (\delta w_t)^2$$

Taking expectations we get

$$d\langle |z_t|^2 \rangle = \langle |z_t|^2 \rangle\, dt$$

which means that although the motion in the Lie algebra is purely imaginary, the second order effects cause the absolute value to grow exponentially.

<u>July 29, 1984</u>

It is clear that a basic problem is to precisely understand what is mean by a fermion integral

$$\int \mathcal{D}\psi(t)\, e^{\int (\psi\dot{\psi} - \psi A \psi)\,dt}$$

where $\psi(t) = \psi^{\mu}(t)$ is defined on an interval $0 \leq t \leq 1$. My feeling is that this integral must represent the actual element in the spinor group obtained by integrating the path $A(t)$ in the Lie algebra.

Let's look at the symplectic situation. Here one has a path integral

$$\int \mathcal{D}q(t)\mathcal{D}p(t)\; e^{i\int (p\dot{q} - H)\,dt}.$$

Supposing H quadratic this integral is essentially given by evaluating at the critical points which are determined by Hamilton's equations. We thus have a symplectic transformation from (q,p)-space at $t=0$ to (q,p)-space at $t=1$.

Except one has some more information given by the specific choice of Lagrangian $p\dot{q} - H$. We could add a total time derivative to this without changing the equations of motion, e.g. $(pq)^{\cdot} = \dot{p}q + p\dot{q}$. The action of a solution of the equations of motion is a function S which comes out of this particular Lagrangian. If we view S as a function of the endpts $S = S(q, q')$, then the symplectic transf. is given by

$$p = \frac{\partial S}{\partial q} \qquad p' = -\frac{\partial S}{\partial q'}$$

In the fermion setup I don't have the breakup of space into q's and p's, i.e. the fundamental quadratic form is not hyperbolic. Hence I would like some picture in [redacted] the symplectic setup that is independent of the q, p splitting.

We have a symplectic transformation, [redacted] and slightly more, namely, a lift into the metaplectic group, thought of as explicitly embedded in the Weyl algebra.

In the fermion case I will have an orthogonal transformation and a lift of it into the spinor group, which is explicitly realized inside the Clifford algebra.

Here is the problem: One has a [redacted] special orthogonal transformation T of $V = \mathbb{R}^n$. This induces a transformation of the spinors defined up to sign. It would be better to start with an element $\tilde{T}$ of $\mathrm{Spin}(n)$ which sits inside of C_n. Now use the known additive isomorphism $C_n \simeq \Lambda(\mathbb{R}^n)$ and the problem is to give a formula for the image of $\tilde{T}$ in $\Lambda(\mathbb{R}^n)$. It should be a Gaussian type formula involving the [redacted] skew-symmetric [redacted] transformation corresponding to the orthogonal transformation T by Cayley transform.

Let's calculate for $n = 2$. One has

$$\left[\tfrac{1}{2}\gamma^1\gamma^2 \,,\; a_1\gamma^1 + a_2\gamma^2\right] = -a_1\gamma^2 + a_2\gamma^1$$

so that the isom $\quad \mathrm{Lie}\ \mathrm{Spin}(2) \xrightarrow{\sim} \mathrm{Lie}\ SO(2)$

is given by $\quad \tfrac{1}{2}\gamma^1\gamma^2 \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

In the spinor repsn. $\quad \gamma_1\gamma_2 = i\varepsilon = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and so

$$e^{t\frac{1}{2}\gamma^1\gamma^2} = \begin{pmatrix} e^{i\frac{t}{2}} & 0 \\ 0 & e^{-i\frac{t}{2}} \end{pmatrix} = \cos\left(\tfrac{t}{2}\right) + \sin\left(\tfrac{t}{2}\right)\gamma^1\gamma^2$$

Thus I can identify Spin(2) with the elements $a + b\gamma^1\gamma^2$ with $a^2+b^2=1$. The corresponding transformation of $\mathbb{R}^2$ is

$$\cos\left(\tfrac{t}{2}\right) + \sin\left(\tfrac{t}{2}\right)\gamma^1\gamma^2 \longmapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

(Check this in the spinor rep.

$$\begin{pmatrix} e^{i\frac{t}{2}} & 0 \\ 0 & e^{-i\frac{t}{2}} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & a_1 - ia_2 \\ a_1 + ia_2 & 0 \end{pmatrix}}_{a_1\gamma^1 + a_2\gamma^2} \begin{pmatrix} e^{-i\frac{t}{2}} & 0 \\ 0 & e^{it/2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{it}(a_1 - ia_2) \\ e^{-it}(a_1 + ia_2) & 0 \end{pmatrix}$$

so that conjugation with $e^{t(\frac{1}{2}\gamma^1\gamma^2)}$ on $\mathbb{R}^2$ coincide with rotation through the angle $-t$.)

Thus in this case $T = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, $\tilde{T} = \cos\left(\tfrac{t}{2}\right) + \sin\left(\tfrac{t}{2}\right)\gamma^1\gamma^2$

Now we use the isomorphism $\Lambda\mathbb{R}^2 \rightleftarrows C_2$ under which $\tilde{T}$ goes into

$$\cos\left(\tfrac{t}{2}\right) + \sin\left(\tfrac{t}{2}\right)\psi^1\psi^2 = \cos\left(\tfrac{t}{2}\right) e^{\tan\left(\frac{t}{2}\right)\psi^1\psi^2}$$

Next we relate this to the Cayley transform:

$$T = \frac{1+K}{1-K} \qquad K = \frac{T-1}{T+1}$$

$$K = \begin{pmatrix} \cos t - 1 & \sin t \\ -\sin t & \cos t - 1 \end{pmatrix} \underbrace{\begin{pmatrix} \cos t + 1 & \sin t \\ -\sin t & \cos t + 1 \end{pmatrix}^{-1}}_{\begin{pmatrix} \cos t + 1 & -\sin t \\ \sin t & \cos t + 1 \end{pmatrix} \frac{1}{(\cos t + 1)^2 + \sin^2 t}}$$

$$K = \underbrace{\frac{\cancel{2} \sin t}{\cancel{2}(1+\cos t)}}_{\frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\cos^2\left(\frac{t}{2}\right)} = \tan\left(\frac{t}{2}\right)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let us consider a Weyl algebra generated by operators T_v, $v \in V$ satisfying

$$T_{v+v'} = T_v T_{v'} e^{\text{[redacted]} A(v,v')}$$

where A is a skew-form. Let σ denote a symplectic transformation of V. We try to construct an operator implementing σ of the form

$$L = \int dv\, f(v)\, T_v$$

This means that

$$L\, T_v\, L^{-1} = T_{\sigma(v)}$$

or that

$$L = T_{\sigma(v)}\, L\, T_v^{-1}$$

$$= \int dw\, f(w)\, \underbrace{T_{\sigma(v)} \underbrace{T_w T_{-v}}_{T_{w-v}\, e^{\text{[redacted]} A(w,v)}}}_{T_{w+\sigma(v)-v}\, e^{-\text{[redacted]} A(\sigma(v),\, w-v) + \text{[redacted]} A(w,v)}}$$

$$= \int dw\, f(w-\sigma(v)+v)\, T_w\, e^{-\text{[redacted]} A(\sigma(v),\, w-\sigma(v)) + \text{[redacted]} A(w-\sigma(v)+v,\, v)}$$

So we want f to satisfy

$$f(w) = f(w - \sigma v + v)\, e^{A(w,\sigma v) + A(w,v) - A(\sigma v, v)}$$

Let's start again:

$$L = \int d\omega\, f(\omega)\, T_\omega$$

$$L T_v = T_{\sigma v} L$$

$$\int d\omega\, f(\omega)\, T_{\omega+v}\, e^{-A(\omega,v)} = \int d\omega\, f(\omega)\, T_{\sigma v+\omega}\, e^{-A(\sigma v,\omega)}$$

so we want

$$f(\omega-v)\, e^{-A(\omega-v,v)} = f(\omega-\sigma v)\, e^{-A(\sigma v,\omega-\sigma v)}$$

$$f(\omega-v)\, e^{-A(\omega,v)} = f(\omega-\sigma v)\, e^{A(\omega,\sigma v)}$$

$$f(\omega+\sigma v-v)\, e^{-A(\omega+\sigma v,v)} = f(\omega)\, e^{A(\omega,\sigma v)}$$

$$\frac{f(\omega+\sigma v-v)}{f(\omega)} = e^{A(\omega,\sigma v+v)+A(\sigma v,v)}$$

Now let $(\sigma-1)v = u$ or $v = (\sigma-1)^{-1} u$

$$\frac{f(\omega+u)}{f(\omega)} = e^{A(\omega,\frac{\sigma+1}{\sigma-1}u) + A(\frac{\sigma}{\sigma-1}u,\frac{1}{\sigma-1}u)}$$

Now because σ is symplectic relative to A we have $A(\sigma x, y) = A(x, \sigma^{-1}y)$

so that

$$A\left(\omega, \frac{\sigma+1}{\sigma-1}u\right) = A\left(\frac{\sigma^{-1}+1}{\sigma^{-1}-1}\omega, u\right)$$

$$= A\left(\frac{1+\sigma}{1-\sigma}\omega, u\right) = A\left(u, \frac{\sigma+1}{\sigma-1}\omega\right)$$

is symmetric in ω, u. So the obvious candidate for f is

$$\boxed{f(\omega) = e^{\frac{1}{2}A(\omega,\frac{\sigma+1}{\sigma-1}\omega)}}$$

To see this works we need

$$\frac{1}{2} A\left(u, \frac{\sigma+1}{\sigma-1} u\right) \overset{?}{=} A\left(\frac{\sigma}{\sigma-1} u, \frac{1}{\sigma-1} u\right)$$

$$\begin{array}{ccc} \| & & \| \\ \frac{1}{2} A\left(u, \frac{\sigma-1+2}{\sigma-1} u\right) & & A\left(\left(1+\frac{1}{\sigma-1}\right)u, \frac{1}{\sigma-1} u\right) \\ \| & & \| \\ A\left(u, \frac{1}{\sigma-1} u\right) & & A\left(u, \frac{1}{\sigma-1} u\right) \end{array}$$

so it works.

I should go back and see if I can now construct the heat kernel for D_μ^2 with a constant magnetic field.

In order to referree Bismut's paper, I should go over the proof of Dustermaat-Heckman via equivariant cohomology. Given an S^1-action on M, let X denote the vector field belonging to this action. (I need to fix a basis for the Lie algebra of S^1 in order for this to be defined). The Weil algebra of $G = S^1$ will have generators θ, u where $u = d\theta$ is the curvature. The complex of equivariant forms is

$$[W(\mathfrak{g}) \otimes \Omega(M)]_{basic} = k[u] \otimes \Omega(M)^{S^1}$$

with

$$d_{tot} = d_M - u\, i_X$$

The next point is the fixpt formula which tells one that for $\alpha \in H_G^*(M)$ one has

$$\pi_*(\alpha) = q_* \left\{ \frac{i^*\alpha}{e(\nu_i)} \right\}$$

$$\begin{tikzcd} M^G \arrow[r, "i"] \arrow[dr, "q"'] & M \arrow[d, "\pi"] \\ & pt \end{tikzcd}$$

in localized cohomology.

To get the DH result we take $\alpha = e^{\gamma}$ where γ is an equivariant 2-form obtained as follows. By assumption M has a closed 2-form ω which is G-invariant, and H is a function on M with $i_X \omega = dH$. Then

$$d_{tot}(\omega + uH) = (d - u i_X)(\omega + uH)$$

$$= d\omega + u(-i_X\omega + dH) - u^2 i_X H = 0$$

so $\omega + uH$ is a closed equivariant form.

Take $\alpha = e^{uH + \omega} = e^{uH} e^{\omega}$. Then

$$\pi_* \alpha = \int_M e^{uH} \frac{\omega^m}{m!} \qquad \dim M = 2m$$

and the other side of the formula is

$$\sum_P \frac{e^{uH(P)}}{u^m e_P}$$

where e_P is an integer essentially giving the product of the characters in the normal ~~space~~ to the fixpoint. I have left out 2π-factors.

What I need to do next is to go over how this formally is supposed to yield the index thm. The steps are as follows. The symplectic manifold is the free loop space of M with the natural action of S^1. Then one needs an equivariant diff'l form.

Actually for the Dirac operator with no twisting one uses the DH formula - the symplectic volume is a Pfaffian times the Riemannian volume, and this Pfaffian can be identified with the fermion integral.

Start again to get the steps right. One has to begin with the index as $\mathrm{tr}_s(e^{tD^2})$. Then, by what is called canonical quantization, this supertrace is identified with with functional integral involving free loops in M and fermion variables along the path. So far one is in the realm of path integrals and there is no mention of differential forms.

Now, however, one does the fermion integral for a given path $x(t)$ and gets a function on the path space, which is a kind of Pfaffian, and which can be identified with the analogue of $\frac{\omega^n}{n!}$ divided by the Riemannian volume in the finite diml. case. At this point it follows that the physicists functional integral for $\mathrm{tr}_s(e^{tD^2})$ is an integral of DH type.

One can then check that the stationary phase approx. side of the formula is the usual formula for the index as the integral of the $\hat{A}$-genus.

July 30, 1984

Let's consider a fermion integral

$$\int \mathcal{D}\bar\eta\, \mathcal{D}\eta\; e^{-\int_0^1 (\bar\eta \dot\eta - \bar\eta A \eta)\,dt} = \det(\partial_t - A)$$

where $\bar\eta, \eta$ are over $S^1 = [0,1]/0=1$. Two cases

1) anti-periodic boundary conditions: In this case the integral gives

$$\det(1+e^A) = tr(\Lambda e^A)$$

2) periodic boundary conditions. In this case the integral gives

$$\det(1-e^A) = tr_s(\Lambda e^A)$$

The above has an obvious generalization when A depends on t. One can see the signs are correct as follows: With periodic b.c. $\det(\partial_t - A)$ vanishes when A ~~has~~ $2\pi i n$ for eigenvalue, i.e. iff e^A has the eigenvalue 1.

Let's try to explain the physicists formula

$$\sum_{k\geq 0} e^{k\mu}\, tr_s\left(e^{D\!\!\!/_k^2}\right) = \int \mathcal{D}x\, \mathcal{D}\psi\, \mathcal{D}\bar\eta\, \mathcal{D}\eta\; e^{-\int L\,dt}$$

where

$$L = \frac{1}{4}\dot x^2 + \frac{1}{4}\psi\dot\psi + \bar\eta\left(\partial_t + \dot x^\mu A_\mu(x) - \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}(x)\right)\eta - \mu\bar\eta\eta$$

and $D\!\!\!/_k$ refers to the Dirac operator on $S\otimes\Lambda^k E$.

The following claims seem to be a correct interpretation of the formalism.

<u>Claim</u> : If we do the $\bar{\eta}, \eta$ integration and restrict to the $\bar{\eta}^* \eta = 1$ section we obtain

$$\mathrm{tr}_s(e^{\not{D}^2}) = \int \mathcal{D}x\, \mathcal{D}\psi \; \mathrm{tr}_E\, T\{e^{-\int L\,dt}\}$$

$$L = \frac{1}{4}\dot{x}^2 + \frac{1}{4}\psi\dot{\psi} + \dot{x}^\mu A_\mu(x) - \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}(x).$$

The time-ordered exponential is needed as for a bundle E of rank > 1, one has that $A_\mu, F_{\mu\nu}$ need not commute.

<u>Claim:</u> If we first do the ψ-integration we obtain

$$\sum e^{k\mu}\, \mathrm{tr}_s(e^{\not{D}_k^2}) = \int \mathcal{D}x\, \mathcal{D}\bar{\eta}\, \mathcal{D}\eta \; \mathrm{tr}_{s,\text{spinors}}\, T\{e^{-\int L\,dt}\}$$

where $L = \frac{1}{4}\dot{x}^2 + \bar{\eta}[\partial_t + \dot{x}^\mu A_\mu(x) - \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}(x) - \mu]\eta$

and the time ordered expenential is needed as the γ^μ don't commute.

<u>Claim</u>: If we do both of the above processes we get

$$\mathrm{tr}_s(e^{\not{D}^2}) = \int \mathcal{D}x \; \mathrm{tr}_{s,S\otimes E}\, T\{e^{-\int[\frac{1}{4}\dot{x}^2 + \dot{x}^\mu A_\mu(x) - \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}(x)]\,dt}\}$$

In order to justify the above one has to do integrals

$$\int \mathcal{D}\psi \; e^{-\int(\psi\dot{\psi})\,dt}\; \psi^{\mu_1}(t_1)\ldots\psi^{\mu_k}(t_k) \qquad t_1 \geq \cdots \geq t_k$$

and it is the same as $\mathrm{tr}_s(\gamma^{\mu_1}\ldots\gamma^{\mu_k})$ roughly.

Let us now try to establish the meaning of fermion functional integrals

$$\int \mathcal{D}\psi \; e^{-\int \psi\dot{\psi}\,dt} \quad \psi(t_1)\psi(t_2)\cdots\cdots\psi(t_n).$$

Especially I have to worry about equal times. I want to proceed formally.

Let's start with the generating function

$$\int \mathcal{D}\psi \quad e^{-\int(\psi\dot{\psi} \; \psi J)\,dt}$$

and recall that ψ is a collection $\psi^\mu(t)$ of anti-commuting quantities. To evaluate this integral formally, notice that it is Gaussian. So we find the critical point:

$$\delta\int(\psi\dot{\psi} - \psi J)\,dt \;=\; \int \delta\psi\,(2\dot{\psi} - J)\,dt \;=0$$

$$\Longrightarrow \quad \dot{\psi} = \tfrac{1}{2}J \quad \Longrightarrow \quad \psi = \tfrac{1}{2}\partial_t^{-1}J$$

The critical value of the exponents is

$$-\int(\psi\dot{\psi} - \psi J)\,dt \;=\; -\int\psi\left(\tfrac{1}{2}J - J\right)dt = \int\tfrac{1}{2}\psi J\,dt$$

$$= -\int\frac{1}{4}J\cdot\partial_t^{-1}J\;dt$$

Thus we seem to have the formula

$$\frac{\int \mathcal{D}\psi\; e^{-\int\psi\dot{\psi}\,dt \;+\; \int\psi J\,dt}}{\int\mathcal{D}\psi\; e^{-\int\psi\dot{\psi}\,dt}} \;=\; e^{-\frac{1}{4}\int J\,\partial_t^{-1}J\,dt}$$

so in particular taking quadratic parts in J, we get

$$\frac{1}{2}\int dt\,dt'\,\frac{\int\mathcal{D}\psi\; e^{-\int\psi\dot{\psi}\,dt}\quad\psi(t)\psi(t')}{\int\mathcal{D}\psi\, e^{-\int\psi\dot{\psi}\,dt}}\; J(t)J(t') \;=\; +\frac{1}{4}\int dt\,J\,\partial_t^{-1}J$$

or

$$\frac{\int \mathcal{D}\psi \, e^{-\int \psi\dot\psi\, dt} \psi^\mu(t)\psi^{\mu'}(t')}{\int \mathcal{D}\psi \, e^{-\int \psi\dot\psi\, dt}} = \frac{1}{8}\left[G_0^{\mu\mu'}(t,t') - G_0^{\mu'\mu}(t',t)\right]$$

where $G_0^{\mu\nu}(t,t')$ is a Green's fn. for ∂_t. A possible choice for G_0 is

$$G_0^{\mu\nu}(t,t') = \delta^{\mu\nu}\,\theta(t-t') \qquad \theta = \text{Heaviside}$$

whence the above is

$$\frac{1}{4}\cdot\frac{1}{2}\left(\theta(t-t') - \theta(t'-t)\right)\,\delta^{\mu\nu}.$$

Next let's turn to the problem of the same integral but where $t=t'$. This time we want to use the formula

$$\int \mathcal{D}\psi \, e^{-\int(\psi\dot\psi - \psi A\psi)dt} = \text{the element of}$$

the spinor group $T\{e^{\frac{1}{4}\int \gamma^\mu\gamma^\nu A_{\mu\nu}\,dt}\}$.

To first order in A we get

$$\int dt\, A(t)\left(\int \mathcal{D}\psi \, e^{-\int\psi\dot\psi\,dt}\,\psi(t)\psi(t)\right) = \int dt\, A_{\mu\nu}(t)\,\frac{1}{4}\gamma^\mu\gamma^\nu$$

i.e.

$$\int \mathcal{D}\psi \, e^{-\int\psi\dot\psi\,dt}\,\psi^\mu(t)\psi^\nu(t) = \frac{1}{4}\gamma^\mu\gamma^\nu$$

Actually we need some bdry conditions before this becomes meaningful. Let's use anti periodic boundary conditions, whence

$$\int \mathcal{D}\psi \, e^{-\int(\psi\dot\psi - A\psi\psi)\,dt} = \operatorname{tr} T\{e^{\frac{1}{4}\int \gamma^\mu\gamma^\nu A_{\mu\nu}\,dt}\}$$

whence we get

$$\int \mathcal{D}\psi \; e^{-\int \psi\dot{\psi}\,dt} \, \psi^\mu(t)\psi^\nu(t) = \mathrm{tr}\, \tfrac{1}{4}\gamma^\mu\gamma^\nu$$
$$= 0$$

since we specify $\mu \neq \nu$: otherwise $\frac{1}{2}[\gamma^\mu, \gamma^\nu]_-$ would be more appropriate.

If I were to take periodic conditions, then ∂_t^{-1} doesn't exist, and I have to produce a definition of the Green's function

$$\int \mathcal{D}\psi \; e^{-\int \psi\dot{\psi}} \, \psi^\mu(t)\,\psi^\nu(t')$$

in order to get somewhere.

July 31, 1984

The problem is to give a precise meaning or mode of calculation, to fermion integrals of the form

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \quad e^{-\int \bar{\psi}\dot{\psi}\,dt} \quad \psi^{\mu_1}(t_1)\cdots\bar{\psi}^{\mu_n}(t_n)$$

where $\{\bar{\psi}^{\mu}(t), \psi^{\mu}(t)\}$ is a family of anti-commuting variables.

We start with an example, where we know what the fermion integrals are supposed to be because they are supposed to give quantum mechanical answers. Consider 2-diml. spinors with creation + annihilation operators

$$a^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and the Hamiltonian $H = \omega a^* a$. We use the [redacted] following kind of average on the Clifford algebra

$$\langle A \rangle = \frac{tr_s(e^{-H}A)}{tr_s(e^{-H})}$$

(Normally, one takes the trace and that leads to fields ψ which are anti-periodic; here we take the super-trace which will lead to periodic ψ.)

Introduce the [redacted] two-point function

$$\langle T[a(t)\,a^*(t)]\rangle \qquad \text{where} \quad a(t) = e^{tH} a e^{-tH}$$

For $t > t'$ this is

$$\frac{tr_s(e^{-H} e^{+tH} a\, e^{-(t-t')H} a^*\, e^{-t'H})}{(1-e^{-\omega})}$$

and
$$\mathrm{tr}_s\left(e^{-H} e^{\Delta t H} a e^{-\Delta t H} a^*\right) = e^{-\Delta t \omega}$$

For $t < t'$ the numerator becomes
$$- \mathrm{tr}_s\left(e^{-H} e^{t'H} a^* e^{(t-t')H} a e^{-tH}\right)$$
$$= -\mathrm{tr}\left(e^{-H} e^{(t'-t)H} a^* e^{(t-t')H} a\right) = e^{-\omega} e^{(t'-t)\omega}$$

and so we have
$$\langle T[a(t) a^*(t')] \rangle = \begin{cases} \dfrac{e^{-(t-t')\omega}}{1-e^{-\omega}} & 0 \leq t' < t \leq 1 \\[2em] \dfrac{e^{-\omega}}{1-e^{-\omega}} e^{-(t-t')\omega} & 0 \leq t < t' \leq 1. \end{cases}$$

We note that this is the Green's function $G(t,t')$ for $\partial_t + \omega$ on $[0,1]$ with periodic b.c.

Now the standard correspondence between Q.M. and path integrals dictates that
$$\frac{\int \mathcal{D}\bar\psi \mathcal{D}\psi \, e^{-\int (\bar\psi \dot\psi + \omega \bar\psi \psi) dt} \psi(t) \bar\psi(t')}{\int \mathcal{D}\bar\psi \mathcal{D}\psi \, e^{-\int (\bar\psi \dot\psi + \omega\bar\psi\psi) dt}} = \langle T[a(t) a^*(t')] \rangle$$

and more generally for higher Green's functions.

(Idea: A baby approach to Wiener measure is to observe that a Gaussian measure on a vector space is specified by its variance and that L^2 of the Gaussian measure is obtained from the symmetric algebra of the dual vector space. Hence one ought to be able to mimic this approach in the fermion setup.)

Let me recapitulate. I am starting with the Hamiltonian picture of a single fermion system. This means a Hilbert space, namely two-diml spin space, equipped with the operators a^*, a. And, of course, a Hamiltonian operator $H = \omega a^* a$.

Corresponding to this Hamiltonian picture is a Lagrangian picture which is going to involve a path integral

$$\int \mathcal{D}\bar{\psi}(t)\,\mathcal{D}\psi(t)\; e^{-\int(\bar{\psi}\dot{\psi} + \bar{\psi}\omega\psi)\,dt}$$

The correspondence is specified by the fact that certain quantum mechanical quantities such as $\langle T[a(t)\,a^*(t)]\rangle$ coincide with fermion integrals.

It seems that one has to make some extra choices. The fermion integral I have written is a completely classical quantity; (all I did was to write down the Lagrangian $\bar{\psi}\dot{\psi} + \bar{\psi}\omega\psi$.)

One has to make a choice of boundary condition, possibly more, before it becomes defined. On the Hamiltonian side, starting from the classical data (the Heisenberg equations), one has to choose the averaging process $\langle\ \ \rangle$ and also the arbitrary constant which could be added to the Hamiltonian.

It would be useful to explore this further. Let's pin down the fermion integral. Formally we expect the formula

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; e^{-\int\bar{\psi}(\dot{\psi} + \omega\psi)\,dt + \int(\bar{J}\psi + \bar{\psi}J)\,dt} = \det(\partial_t + \omega)\; e^{\int \bar{J}(\partial_t+\omega)^{-1}J\,dt}$$

so the first step toward specifying the fermion integral is to give $(\partial_t + \omega)^{-1}$. This is a matter of boundary conditions.

It seems clear that the [redacted] choice of $\langle \; \rangle$ is equivalent to the boundary conditions on the [redacted] fields $\bar{\psi}(t)$, $\psi(t)$. [redacted] It's automatically true that $\langle T[a(t)\, a^*(t')] \rangle$ is a Green's function for $\partial_t + \omega$, independent of the nature of $\langle \; \rangle$.

But even when the boundary conditions are specified one doesn't know the diagonal values of the Green's function.

[redacted] Let's review the program. I am trying to pin down fermion integrals

$$\int D\bar{\psi} D\psi \; e^{-\int (\bar{\psi}\dot{\psi} + \bar{\psi}\omega\psi)\,dt} \; \psi(t_1) \cdots \psi(t_k)\, \bar{\psi}(t_k') \cdots \bar{\psi}(t_1')$$

by means of a specific [redacted] Hamiltonian model. The [redacted] attempt is [redacted] not successful when the times coincide, e.g. $t_1 = t_1'$ because the quantum mechanics doesn't specify $\langle T[a(t)\, a^*(t')] \rangle$ when $t = t'$.

[redacted] On the Hamiltonian side we took $H = \omega a^* a$ instead of $-\omega a a^*$ which would have given the same Heisenberg equations of motion. So now let ω became a variable function of time. The obvious thing to do is set

$$\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\; e^{-\int_0^1(\bar\psi\dot\psi+\omega\bar\psi\psi)dt} = tr_s\left(T\{e^{-\int_0^1\omega a^*a\,dt}\}\right)$$

$$= 1 - e^{-\int_0^1 \omega(t)dt}$$

In other words we propose to define the equal time fermion integral

$$\frac{\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\; e^{-\int(\bar\psi\dot\psi+\omega\bar\psi\psi)dt}\,\psi(t)\bar\psi(t)}{\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\; e^{-\int(\bar\psi\dot\psi+\omega\bar\psi\psi)dt}}$$

as $\displaystyle\frac{\delta}{\delta\omega(t)}\log tr_s\big(\underbrace{T\{e^{-\int_0^1\omega a^*a\,dt}\}}_{\text{Call this } U(1,0)}\big).$

$$\delta U(1,0) = -\int dt\; U(1,t)\,\delta\omega\, a^*a\, U(t,0)$$

$$\frac{tr_s\,\delta U(1,0)}{tr_s\,U(1,0)} = \frac{tr_s\big(U(1,0)\;U(0,1)\,\delta U(1,0)\big)}{tr_s\,U(1,0)}$$

$$= -\int dt\;\delta\omega(t)\,\langle\, U(0,t)\,a^*a\,U(t,0)\,\rangle$$

$$= -\int dt\;\delta\omega(t)\,\langle\,(a^*a)(t)\,\rangle$$

Therefore it seems we have

$$\boxed{\begin{aligned}\frac{\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\; e^{-\int\bar\psi(\dot\psi+\omega\psi)dt}\,\psi(t)\bar\psi(t)}{\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\; e^{-\int(\bar\psi\dot\psi+\omega\bar\psi\psi)dt}} &= -\langle (a^*a)(t)\rangle \\ &= G(t,t^+) \\ &= G(t^-,t)\end{aligned}}$$

The idea I had is to treat the fermion integral in analogy with Brownian motion. This means that we have an algebra generated by anti-commuting quantities $\bar{\psi}^\mu(t)$, $\psi^\mu(t)$ of square 0. The problem is to define the integral which is a linear functional on this algebra. We know how to integrate products of these variables at different times, and then we adopt a special way to handle equal times. In fact it seems that the way to generate all these Green's functions is to use a formula like

$$\boxed{\begin{aligned} &\int \mathcal{D}\bar{\psi}(t)\,\mathcal{D}\psi(t)\; e^{-\int_0^1 (\bar{\psi}\dot{\psi} + \bar{\psi}\omega\psi - \bar{\psi}J - \bar{J}\psi)\,dt} \\ &= \underbrace{\det(\partial_t + \omega)}_{\mathrm{tr}_s\, T\{e^{-\int_0^1 \omega\, dt\, a^*a}\} \;=\; \det\left(1 - T\{e^{-\int_0^1 \omega\, dt}\}\right)}\; e^{\int \bar{J}(\partial_t+\omega)^{-1} J\, dt} \end{aligned}}$$

So in particular

$$\int \mathcal{D}\bar{\psi}(t)\,\mathcal{D}\psi(t)\; e^{-\int_0^1 (\bar{\psi}\dot{\psi} + \bar{\psi}\omega\psi)\,dt}$$

$$= \mathrm{tr}_s\, T\{e^{-\int_0^1 \omega\, dt\, a^*a}\} = \det\left(1 - T\{e^{-\int_0^1 \omega\, dt}\}\right)$$

is the generating function for

$$\mathrm{tr}_s\; T\,[a^*a(t_1)\cdots a^*a(t_n)].$$

This is pretty complicated when there are more than one a. However, suppose we again have only an a, whence

$$\mathrm{tr}_s\; T\,[a^*a(t_1)\cdots a^*a(t_n)] = \begin{cases} 0 & n=0 \\ -1 & n \geq 1 \end{cases}$$

since here $H=0$, so $a^*a(t) = a^*a$.

I don't think there is much more to be added to the above discussion except to mention Feynman's diagram proof of

$$\int D\bar{\psi} D\psi \, e^{-\int(\bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi - \bar{\psi}J - \bar{J}\psi)dt} = \det(\partial_t+\omega)\, e^{\int \bar{J}(\partial_t+\omega)^{-1}J\,dt}$$

which I have been through many times.

Recall in connection with the Thom class in K-theory and its Chern character we proved

$$tr_s\left(e^{-\omega a^* a + \bar{J}a + a^* J}\right) = \det(1-e^{-\omega})\, e^{\bar{J}\frac{1}{\omega}J}.$$

This is obviously the special case where $\omega, \bar{J}, J$ are constant, of the formula

$$\boxed{tr_s\, T\{e^{-\int_0^1(\omega a^* a + \bar{J}a + a^* J)dt}\} = \underbrace{\det(\partial_t+\omega)}_{\det\left(1-T\{e^{-\int_0^1 \omega\, dt}\}\right)}\, e^{\int \bar{J}(\partial_t+\omega)^{-1}J\,dt}.}$$

So now I know a little more about the super connection calculation.