July 12, 1984.

Let's consider a connection D on $pr_2^*(E)$ over $Y \times M$. It can be written

$$D = d' + \Theta + D''$$

where D'' is a family of connections in E parametrized by Y, and where Θ is a 1-form on Y with values in $\operatorname{End} E$. The curvature is

$$D^2 = \underbrace{d'\Theta + \Theta^2}_{\Omega} + [d'+\Theta, D''] + (D'')^2$$

I want to consider now the case where $\boxed{M = S^1}$. In this case $(D'')^2 = 0$ and so the odd character forms on Y obtained by integrating $tr(e^{D^2})$ over M are very simple, namely

$$\int_M tr\left(e^{\Omega}\,[d'+\Theta, D'']\right)$$

I know I can write this as an equivariant form for $\mathcal{G}$ acting on $\mathcal{A}$, $\mathcal{G}$ and $\mathcal{A}$ being associated to the trivial rank r bundle E over S^1.

I now want to relate these odd character forms to two different things. First of all there is the family of left-invariant even forms on $\mathcal{G}$ one obtains by the transgression process. Secondly there is the map $\mathcal{A} \longrightarrow U$ given by monodromy which we know is equivariant wrt $\mathcal{G} \longrightarrow U$ given by evaluation at the basepoint, where U acts on itself by conjugation. On (U, U) are odd equivariant forms, it seems. These can be pulled back to $(\mathcal{G}, \mathcal{A})$

Lets describe the transgression process. We consider the $\mathcal{G}$-map $\mathcal{G} \longrightarrow \mathcal{A}$ given by a point D_0 of $\mathcal{A}$. Thus we have the family $g \longmapsto g D_0 g^{-1}$ of connections on $\mathcal{A}$ param. by elts of $\mathcal{G}$. The transgression of an equivariant form is computed as follows. One first lifts to $\mathcal{A}$, this means forget the $\mathcal{G}$-part, write as a coboundary and restrict to $\mathcal{G}$. One also restricts to $\mathcal{G}$ taking the universal Θ to be the Θ on $\mathcal{G}$ that descends; this means $\theta = g d' g^{-1}$ where $d' = d_g$. In the present case, the equivariant forms come from connections on $pr_2^*(E)$, so we might as well look at the connections on $pr_2^*(E)$ over $\mathcal{G} \times M$.

So the connection which corresponds to lifting to $\mathcal{A}$ and restricting to $\mathcal{G}$ is

$$d' + g D_0 g^{-1} = g(d' + g^{-1} d' g + D_0) g^{-1}$$

The connection which descends is

$$d' + g d' g^{-1} + g D_0 g^{-1} = g(d' + D_0) g^{-1}$$

To express the fact that both connections compute the same classes we use the linear path between them:

$$g(d' + t\omega + D_0) g^{-1} \qquad \omega = g^{-1} d' g$$

and the standard formula (need curvature

$$(d' + t\omega + D_0)^2 = D_0^2 + t[D_0, \omega] + (t^2 - t)\omega^2$$

and the fact that $D_0^2 = 0$ over S^1:

$$\operatorname{tr} e^{[D_0,\omega]} - \operatorname{tr} 1 = d\int_0^1 dt \operatorname{tr}\left(e^{t[D_0,\omega]+(t^2-t)\omega^2}\,\omega\right)$$

Now $[D_0,\omega]^2 = 0$ and ω^2 commutes with ω so taking the part of degree 1 in M on both sides we have

$$\boxed{\begin{aligned}\operatorname{tr}[D_0,\omega] &= d'\int_0^1 dt \operatorname{tr}\left(e^{(t^2-t)\omega^2}\,\omega\, t[D_0,\omega]\right)\\ &\quad + d''\int_0^1 dt \operatorname{tr}\left(e^{(t^2-t)\omega^2}\,\omega\right)\end{aligned}}$$

Now integrate over S^1 and one gets

$$d'\left\{\int_M \int_0^1 dt \operatorname{tr}\left(e^{(t^2-t)\omega^2}\,\omega\, t[D_0,\omega]\right)\right\} = 0$$

because

$$\int_M \operatorname{tr}[D_0,\omega] = \int_M d'' \operatorname{tr}(\omega) = 0.$$

Thus we find that the good left invariant $2k$-forms on $\mathcal{G}$ are

$$\boxed{\text{Const} \int_M \operatorname{tr}\left(\omega^{2k-1}[D_0,\omega]\right)}$$

for D_0 any point in $\mathcal{A}$. The constant is

$$\int_0^1 dt\, \frac{(t^2-t)^{k-1}}{(k-1)!}\, t = \frac{(-1)^{k-1}}{(k-1)!} \underbrace{\int_0^1 t^k (1-t)^{k-1} dt}_{\beta(k+1,\,k)}$$

$$= \frac{(-1)^{k-1}}{(k-1)!}\,\frac{k!\,(k-1)!}{(2k)!} = \boxed{(-1)^{k-1}\frac{k!}{(2k)!}}$$

Thus for $k=1$ and $D_0 = d$ we get

$$\frac{1}{2}\int_{S^1} tr(\omega\, d\omega)$$

Recall the constant for the basic forms on GL_n

$$\int_0^1 dt\ tr\left(e^{(t^2-t)\omega^2}\omega\right)$$

In degree $2k-1$ we get

$$tr(\omega^{2k-1}) \underbrace{\int_0^1 dt \frac{(t^2-t)^{k-1}}{(k-1)!}}_{\frac{(-1)^{k-1}}{(k-1)!}\beta(k,k)} = (-1)^{k-1}\frac{(k-1)!}{(2k-1)!}$$

These two constants differ by a factor of 2.

Digression: The van Est formalism suggests there is a multiplicative K-theory fitting into a long exact sequence

$$\longrightarrow K_n^{top}(A) \longrightarrow HC_n(A) \longrightarrow K_n^{\times}(A) \longrightarrow$$

which should somehow be related to Borel cohomology of $GL(A)$. The above should be viewed as a kind of analogue of the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^{\times} \longrightarrow 0$$

Also there should be a map

$$K_n^{alg}(A) \longrightarrow K_{n-1}^{\times}(A).$$

What should the relation be with complex coefficients. Taking primitives in the van Est spectral sequence gives a long exact sequence

$$\mathrm{Hom}(K^{top}_{n+2}A, \mathbb{C}) \longleftarrow HC^n(A) \longleftarrow \mathrm{Hom}(K^{\times}_n A, \mathbb{C})$$

(Curious: $HC^0(A)$ is related to K_2 directly rather that K_0. A trace on A determines a closed 1-form on $GL(A)$, hence a linear functional on $\pi_1 GL(A) = \pi_2 BGL(A) = K_2 A$.)

Problem: Find the best way to construct the regulator maps $K_{2n-1}\mathbb{C} \longrightarrow \mathbb{C}^{\times}$.

Method of Deligne cohomology.

Karoubi's method using relative K-theory

$$\begin{array}{ccccccc} K^{top}_{n+1}A & \longrightarrow & K^{rel}_n A & \longrightarrow & K^{alg}_n A & \longrightarrow & K^{top}_n A \\ & & \downarrow & & & & \\ & & HC_{n-1}A & & & & \end{array}$$

Check dimensions.

$$K^{rel}_n = \pi_n \text{ Fiber } \{BGL(A)^+ \longrightarrow B^{top}GL(A)\}$$

$$HC_{n-1}(A) = \mathrm{Prim}\{H_n(\mathfrak{gl}_0(A))\}$$

$$\begin{tikzcd}
K^{top}_{n+2} \arrow[r] \arrow[d, phantom, "\wr"] & K^{rel}_{n+1} \arrow[r] \arrow[d] & K^{alg}_{n+1} \arrow[r] \arrow[d] & K^{top}_{n+1} \\
K^{top}_{n} \arrow[r, "\alpha"] & HC_n \arrow[r] & ? \arrow[r] & {}
\end{tikzcd}$$

One conjectures that the map α coincides with the map defined by Connes. Connes' definition involves directly defining sequences stable under S in the cyclic cohomology attached to generators for K^{top}_0, K^{top}_1. The map α defined by the above diagram is transpose ~~associate~~ to the process of taking a cyclic cocycle, interpreting it as a diffl form on GL, and then integrating over spherical cycles.

Possibility: One might define the accessible algebraic K-theory K^{acc} as the fibre of the map α. The correct degrees are

$$K^{acc}_n = \text{Fibre theory of } \{K^{top}_n \longrightarrow HC_{n-2}\}$$

This is in the spirit of flat bundles — an element of K^{acc}_n will be a element in K^{top}_n with a trivialization of "order" depending on n.

To see what this means, recall

$$HC_{n-2} = \Omega^{n-2}/d \oplus H^{n-4} \oplus H^{n-6} \oplus$$

and the map α associates to an element of K its Chern character. Thus K^{acc}_n consists of K-elements ξ equipped with a reason why the classes $ch_0(\xi), ch_1(\xi), \ldots, ch_{\frac{n}{2}-1}(\xi)$ are zero, assuming n is even.

Discussion of some difficulties with defining regulator maps $K^{alg}_{2m+1}\mathbb{C} \longrightarrow \mathbb{C}^\times$ and various generalizations.

These regulator maps can be defined simply on the K-group level as follows. Use Max's relative sequence (n even)

$$\begin{tikzcd}
K^{top}_{2m+2}\mathbb{C} \arrow[r] \arrow[d, phantom, "\shortparallel"] & K^{rel}_{2m+1}\mathbb{C} \arrow[r] \arrow[d] & K^{alg}_{2m+1}\mathbb{C} \arrow[r] & K^{top}_{2m+1}\mathbb{C} \arrow[d, phantom, "\shortparallel"] \\
\mathbb{Z} & HC_{2m}(\mathbb{C}) \simeq \mathbb{C}. & & 0
\end{tikzcd}$$

In more elementary terms, an element of $K^{alg}_{2m+1}\mathbb{C}$ can be realized by a flat vector bundle over a homotopy S^{2m+1}. An invariant closed form on $GL_n\mathbb{C}$ then induces via the flat connection a closed form on the principal bundle of the vector bundle. The bundle is topologically trivial as $\pi_{2m+1}BU = 0$, and choosing a section we can pull back the form on the principal bundle to the homotopy sphere where it can be integrated to get a $\mathbb{C}$ number. The ambiguity in the trivialization means the no. is well-defined in $\mathbb{C}^\times$.

One sees that using the sphere in the above way is analogous to the way Witten constructs the Wess–Zumino action

$$\mathrm{Map}(S^{2n}, U) \longrightarrow S^1$$

However we know that there is a generalization of the regulator map just defined on K-groups. (Although the generalization might be off by the usual integers: $(-1)^{k-1}(k-1)!$)

Given a flat vector bundle ξ over M we know that it is possible to define characteristic classes in $H^{2k-1}(M, \mathbb{C}^\times)$. These are the Chern classes defined via Deligne cohomology. Hence there is a universal classes in

$$(*) \qquad \bar{c}_k \in H^{2k-1}(GL(\mathbb{C})_\delta, \mathbb{C}^\times)$$

which maps under Bockstein to c_k in integral cohomology.

The idea behind the Deligne cohomology is that Chern classes are integral classes which when viewed in DR cohomology have a certain filtration. Thus given a flat bundle ξ over M one builds some sort of associated fibre bundle and does constructions with differential forms inside, somehow tied to the holomorphic structure of the projective bundle.

Presumably it is possible to define the class $(*)$ in the continuous or Borel cochain cohomology

$$H^{2k-1}_{Borel}(GL(\mathbb{C}), \mathbb{C}^\times)$$

I tried to recall Graeme's construction of these. From the exponential sequence

$$\begin{tikzcd} H^{2k-1}_{Bor}(G, \mathbb{Z}) \arrow[r] \arrow[d, equal] & H^{2k-1}_{Borel}(G, \mathbb{C}) \arrow[r] \arrow[d, equal] & H^{2k-1}_{Bor}(G, \mathbb{C}^\times) \arrow[r] & H^{2k}_{Bor}(G, \mathbb{Z}) \arrow[d, equal] \\ H^{2k-1}_{Bor}(BG, \mathbb{Z}) = 0 & H^{2k-1}(\mathfrak{g}, k; \mathbb{C}) & & H^{2k}(BG, \mathbb{Z}) \end{tikzcd}$$

one can't produce the desired class on $GL(\mathbb{C})$, but if $G = U$, then we get an isomorphism

$$H^{2k-1}_{Bor}(U, \mathbb{C}) \xrightarrow{\sim} H^{2k-1}_{Bor}(U, \mathbb{C}^\times) \xrightarrow{\sim} H^{2k}(BU, \mathbb{Z})$$

So it looks like one might easily define S^1-valued odd characteristic classes for a flat unitary bundle.

Possible procedure: I want to use the idea that emerged when I looked at the Witten construction, namely, to replace the even sphere by any manifold M and then to use a Dirac operator on M as realizing the basic homology class which gives the ultimate number. [redacted]

[redacted]

In this case we have a flat unitary bdle E over a odd [redacted] dim manifold M and instead of getting a number out of a typical odd [redacted] homology class, we will want to get a number out of a Dirac operator on M. The number should be independent of the connection used in the Dirac operator. So we should get an [redacted] actual function on the space of connections $\mathcal{A}$ which is constant. The thing to take then is the difference of the η-invariant for the operator tensored with the flat bundle and the η-invariant of the operator tensored with a trivial bundle of the same rank.

Let's return to the map which goes from the space $\mathcal{a}$ of connections on the trivial bundle over S^1 to the unitary gp $U = U_r$ otained by taking parallel transport around the circle ~~starting~~ at the basepoint 0

$$A \longmapsto T\left\{e^{-\int_0^{2\pi} A\,dx}\right\}$$

This map identifies $\mathcal{a}/\Omega U$ with U. [redacted] Now I want to descend the trivial bundle over $\mathcal{a} \times S^1$ to a bundle over $U \times S^1$. This means that given a point ξ [redacted] of $U \times S^1$ I have to give an [redacted] r-dimensional vector space together with an identification of this vector space with the fibre of $pr_2^*(E)$ over any $A, x \in A \times S^1$ mapping to ξ and the whole business should be $\mathcal{G}_0$-equivariant.

It would be better to think of E as a given vector bundle over S^1 without a trivialization. But one is given a trivialization over the basepoint $0 \in S^1$, [redacted] so that the monodromy is defined for any connection. Now what is needed is a vector space attached to each $(g, x) \in U \times S^1$. Let's think of g as the class of connections with monodromy g. Given such a connection I can follow it back to the basepoint and so identify the fibre of $pr_2^*(E)$ at (A, x) with the fibre E_0 equivariantly under $\Omega U = \mathcal{G}_0$. There is a problem as x crosses 2π but then the given g tells how to make the identification smoothly.

So it is fairly clear how the bundle over $\mathcal{a} \times S^1$ is to be descended. One uses the connection to push backward to the basepoint and one obtains the same clutching construction needed to define the descended bundle $\bar{E}$ over $U \times S^1$.

The next point is to take the connection defined on $\bar{E}$ lift it back to $pr_2^*(E)$ over $\mathcal{U} \times S^1$. In terms of the trivialization of $pr_2^*(E)$ which we have given over $\mathcal{U} \times (0, 2\pi)$, it should be easy to write this connection down. So now I have a connection that descends and I can go thru the transgression process by joining this connection to the tautological connection on $pr_2^*(E)$. By the time one restricts to a G-orbit, the results ought to be the same as before.

Another project: Compare the invariant forms found on the loop group with the character forms on the Grassmannian under the "scattering map".

$$t^2 - t = (t - \tfrac{1}{2})^2 - \tfrac{1}{4}$$

$$\int_0^1 e^{(t^2-t)x^2} x \, dt = e^{-\frac{1}{4}x^2} \int_0^1 e^{(t-\frac{1}{2})^2 x^2} x \, dt$$

$$= e^{-\frac{1}{4}x^2} \int_{-1/2}^{1/2} e^{t^2 x^2} x \, dt = e^{-\frac{x^2}{4}} \int_{-x/2}^{x/2} e^{t^2} dt$$

$$\sim \begin{cases} e^{\frac{y^2}{4}} (i\sqrt{\pi}) & \text{if } x = iy \quad y \longrightarrow +\infty \\ e^{\frac{y^2}{4}} (-i\sqrt{\pi}) & \text{———} \quad y \longrightarrow -\infty \end{cases}$$

<u>July 14, 1984</u>

Let's consider $M = S^1$ and $E = S^1 \times \mathbb{C}^n$ over M, $\mathcal{A}$ = connections on E, $\mathcal{G}$ = gauge transf., $\mathcal{G}_0$ = gauge transf = id at $x = 0$, whence $\mathcal{G}_0 = \Omega U_n$. Make $\mathcal{G}$ act to the right on $\mathcal{A}$ by

$$g * A = g^{-1}dg + g^{-1}Ag$$

Then we know that there is an isomorphism given by monodromy

$$\mathcal{A}/\mathcal{G}_0 \xrightarrow{\sim} U_n$$

$$A \longmapsto T\{e^{-\int_0^{2\pi} A}\}$$

Now consider the action of $\mathcal{G}$ on the bundle $pr_2^*(E)$ over $\mathcal{A} \times M$. A section of this bundle is smooth vector function $\Phi(A,x)$ on $\mathcal{A} \times M$; the action of $g \in \mathcal{G}$ on Φ is

$$(g * \Phi)(A,x) = g(x)\,\Phi(g * A, x)$$

This $\mathcal{G}$-bundle descends to a bundle $\bar{E}$ over $\mathcal{A}/\mathcal{G}_0 \times M = U \times M$ (put U for U_n) which should coincide with the bundle over $U \times S^1$ associated to the canonical automorphism of the trivial bundle $U \times \mathbb{C}^n / U$. Thus there is a way to put a connection on $\bar{E}$ and use it to define odd forms on U. I want to <u>lift</u> this all back to $\mathcal{A}$

We are considering the trivial bundle over $\mathcal{A} \times S^1$ with fibre $\mathbb{C}^n$. We cover S^1 by the interval $I = [0, 2\pi]$. Over $\mathcal{A} \times I$ we have two trivializations of the trivial bundle defined as follows. The first uses the parallel transport in the S^1-direction from x back to zero to identify the fibre at A, x with the fibre at $A, 0$ where we use the given bases. Thus everywhere constant sections are defined by the function (invertible matrix function)

$$\varphi(A,x) = T\left\{ e^{-\int_0^x A\,dx} \right\}$$

which satisfies

$$\varphi(A,0) = I$$

$$(\partial_x + A)\,\varphi(A,x) = 0$$

The second trivialization is obtained by parallel transporting from x to 2π:

$$\psi(A,x) = T\left\{ e^{-\int_{2\pi}^x A\,dx} \right\}$$

(Think intrinsically: Suppose E is a v.b. over S^1 equipped with a fixed frame at $0 \in S^1$. Then over $\mathcal{Q} \times I$ we construct these two trivializations. The trivializations are evidently equivariant under the action of $\mathcal{G}_0$.) ∎

The connection on the trivial bundle over $\mathcal{Q} \times I$ associated to the φ-trivialization (resp. ψ) is $\varphi . d . \varphi^{-1}$ (resp $\psi . d . \psi^{-1}$) where $d = d_{\mathcal{Q}} + d_I$. We define a connection on the trivial bundle over $\mathcal{Q} \times I$:

$$\left(1 - \frac{x}{2\pi}\right) \varphi . d . \varphi^{-1} + \frac{x}{2\pi} \psi . d . \psi^{-1}$$

<u>Example:</u> Let us take the trivial line bundle over the circle and the 1-parameter family of connections

$$dx\,(\partial_x + iy)$$

Then

$$\varphi(y,x) = e^{-iyx}$$

$$\psi(y,x) = e^{iy(2\pi - x)}$$

Then the connection form I am interested in is

$$\left(1-\frac{x}{2\pi}\right) d(iyx) + \frac{x}{2\pi} d\big(iy(x-2\pi)\big)$$

$$= i\big(d(yx) - x\,dy\big) = i\,y\,dx$$

Notice that this agrees with the tautological connection $d + A dx = dy\,\partial_y + dx(\partial_x + iy)$, probably because the gauge transformation group is discrete in this case.

What we've done so far is to define a connection on the trivial bundle over $\mathcal{Q} \times [0, 2\pi]$ which is $\mathcal{G}_0$-invariant. The connection is

$$\left(1-\frac{x}{2\pi}\right) \varphi . d . \varphi^{-1} + \frac{x}{2\pi} \psi . d . \psi^{-1}$$

where $\varphi(A, x) = T\{e^{-\int_0^x A dx}\}$ $\qquad \psi(A,x) = T\{e^{-\int_{2\pi}^x A dx}\}$

so that $\qquad \varphi^{-1}\psi = T\{e^{-\int_{2\pi}^0 A dx}\}$ = parallel transport from 2π to 0. Let's denote this τ^{-1} so that

$$\psi = \varphi\tau^{-1}$$

Then our connection is

$$\varphi\left\{\left(1-\frac{x}{2\pi}\right) d + \frac{x}{2\pi}\, \tau^{-1} . d . \tau\right\} \varphi^{-1}$$

$$= \varphi\left\{ d + \frac{x}{2\pi} \tau^{-1} d(\tau)\right\} \varphi^{-1}$$

which shows more or less that the connection we have is the inverse of the one

$$d + \frac{x}{2\pi} g^{-1} dg \qquad \text{over} \qquad \mathcal{U} \times [0, \pi]$$

under the quotient by $\mathcal{G}_0$.

Now that we have our descendable connection we compare it to the tautological one

$$d + A\,dx$$

except that we now want to restrict to a $\mathcal{G}_0$-orbit, or rather pull-back by the map

$$\mathcal{G}_0 \longrightarrow \mathcal{A} \qquad g \longmapsto g\partial_x g^{-1} + gA_0g^{-1}$$

What is $\varphi(g,x)$? $\qquad g(\partial_x + A_0)g^{-1}\varphi = (\partial_x + A)\varphi = 0$

$$\varphi(g,x) = g\,\varphi_0(x) \qquad \varphi_0(x) = T\{e^{-\int_0^x A_0 dx}\}$$

so that $(\partial_x + A_0)\varphi_0 = 0$ or $\varphi_0 \partial_x \varphi_0^{-1} = A_0$.

What is $\varphi(d + \frac{x}{2\pi}\tau d\tau^{-1})\varphi^{-1}$ restricted to $\mathcal{G}_0$? The monodromy is constant on the $\mathcal{G}_0$-orbit $\Rightarrow d\tau = 0$. So we have the connection

$$\begin{aligned} \varphi . d . \varphi^{-1} &= g . \varphi_0 . d . \varphi_0^{-1} g^{-1} \\ &= g(d + \varphi_0 \partial_x \varphi_0^{-1})g^{-1} \qquad \text{as } \varphi_0 \text{ is ind. of } g \\ &= g.(d + A_0)g^{-1} \end{aligned}$$

The tautological connection restricted to $\mathcal{G}_0$ is

$$\begin{aligned} &d + dx(g\partial_x g^{-1} + gA_0g^{-1}) \\ &= d'_{\mathcal{G}} + g(dx.\partial_x + A_0)g^{-1} \\ &= g(d + g^{-1}dg + A_0)g^{-1} \end{aligned}$$

and so we are in the usual transgression situation.

Thus I have checked the fact, which one knew for general reasons, that upon restricting the descendable connection to the $\mathcal{G}_0$ orbit, one gets the connection $g(d+A_0)g^{-1}$, so the end result of transgressing the odd forms on U to ΩU is the same as calculated earlier.

Now one thing that is striking about this calculation is that we go to a lot of work to pass from the character forms on BU_n to forms on ΩU_n. The obvious method is to use the evaluation map

$$\begin{array}{ccc} \Omega^2 BU \times S^2 & \longrightarrow & BU \\ \cup & & \\ \Omega U & & \end{array}$$

pull back and integrate over S^2. Let's break this into 2 steps.

$$U_n \times S^1 \longrightarrow BU_n$$

$$\Omega U_n \times S^1 \xrightarrow{\text{eval}} U_n$$

The first map comes from assigning to $g \in U$ the graph $\binom{1}{tg} V^0 \subset V^0 \oplus V^1$ $(V^0 = C_n)$ and then letting t go from 0 to ∞. Strictly speaking I then [redacted] get paths from V^0 to V^1 in the Grassmannian, and I [redacted] have to pick a fixed path back.

The connection form for the graph embedding is
$$\frac{1}{1+T^*T} T^* dT$$
so that if I take $T = tg$ I get
$$\frac{1}{1+|t|^2}\bar{t}\,dt + \frac{|t|^2}{1+|t|^2} g^{-1}dg$$

for the connection form over this family. Let's just concentrate on the 1-parameter family of connections on the trivial bundle over the unitary group. Then [redacted] we get the family

$$\frac{t^2}{1+t^2}\, g^{-1}dg \qquad\qquad 0 \le t \le \infty \implies 0 \le \frac{t^2}{1+t^2} \le 1.$$

and so we get the usual odd forms on the unitary group. (Note that

$$\int \mathrm{tr}\left(e^{(u^2-u)\omega^2}\,\omega\right) du$$

is independent of the function u.)

<u>Remark</u>: This description of [redacted] how to obtain the odd forms on U suggests that the parameter t in the connection family $d + t\omega$ is sort of a [redacted] "moment map" associated to a circle action on the Grassmannian. So it is very natural to think of it in terms of convex linear combinations.

<u>Important idea</u>: Take $G^n_{2n}(\mathbb{C})$ and introduce a [redacted] circle action acting on the last n coordinates. Think in terms of the moment map for this circle action.

The next idea will be to look at the evaluation map $\Omega^1 U_n \times S^1 \longrightarrow U_n$. I know the forms defined on $\Omega^1 U_n$ [redacted] by pulling back and integrating are not left invariant. My guess is that these forms are ones obtained by the following naive process. One has the

map
$$\begin{array}{ccc} \mathcal{A} \times [0, 2\pi] & \longrightarrow & \mathcal{U} \\ A\ ,\ x & \longmapsto & \varphi(A, x) \end{array}$$

which is like taking a path ~~starting~~ at the identity and ~~evaluating~~ at x. If we pull-back a closed form ω and integrate over $[0, 2\pi]$, this gives a cobounding form for the pull-back of ω under monodromy. So if we restrict to $\mathcal{G}_0 \subset \mathcal{A}$ thought of as closed loops, we are transgressing ω on $\mathcal{U}$ to a form on $\Omega\mathcal{U}$.

My feeling is that if this is phrased in terms of the connections over $\mathcal{U} \times I$, then one is using two connections such as d and $g\,dg^{-1}$ over $\mathcal{G}$, not the restriction of the tautological connection.

The next project is to find how the character forms on the Grassmannian look on the loop group.

July 15, 1984

Let's work out the Dirac operator and the Weitzenbock formula, etc. using the principal frame bundle.

Let M be a Riemannian n-manifold and let P be the principal bundle of its tangent bundle. A point u of P over $m \in M$ is an isomorphism $\mathbb{R}^n \xrightarrow{\sim} T_{M,m}$ compatible with inner products. Let $\omega^\mu \in \Omega^1(P)$ be the one form whose effect on a tangent vector at u projects this vector down to M and gives the μ^{th} coord. of the projected vector relative to the framing u.

Alternatively we have

$$0 \longrightarrow \pi^*(T_M^*) \longrightarrow T_P^* \longrightarrow T_\pi^* \longrightarrow 0$$

and a canonical trivialization $\pi^*(T_M^*) \simeq P \times (\mathbb{R}^n)^*$, so the basis sections of $\pi^*(T_M^*)$ relative to this trivialization can be viewed as 1-forms on P.

Now suppose we have a connection in T_M^*. Then when lifted back to $\pi^*(T_M^*)$ it can be written relative to the trivialization in the form

$$D = d + \theta$$

or equivalently

$$\boxed{D\omega^\alpha = \theta_\nu^\alpha \otimes \omega^\nu}$$

where $\theta_\nu^\alpha \in \Omega^1(P)$. Recall

$$D : \Gamma(\pi^* T_M^*) \longrightarrow \Gamma(T_P^* \otimes \pi^* T_M^*)$$

so $D\omega^\alpha$ is indeed uniquely expressible in the above form.

Because the connection $D = d + \theta$ on $\pi^* T_M^*$ descends it follows that $\theta_\nu^\alpha = -\theta_\alpha^\nu$ and that the θ_ν^α with $\alpha < \nu$ form a basis for the vertical 1-forms. We can easily work out $g_* = (R_g)^*$ on the ω^μ and θ_ν^α, and also i_X for $X \in \mathrm{Lie}\, SO(n)$ if we want.

So we obtain a framing of $\Omega^1(P)$. Define vector fields X_μ on P to be the horizontal vector fields which project to the μ-th basis vector of the frame one is at: $i_{X_\mu}\omega^\nu = \delta_\mu^\nu$, $i_{X_\mu}\theta_\nu^\alpha = 0$.

Next let's consider the process of covariant differentiation on tensors or spinors, i.e. sections of a vector bundle associated to the principal frame bundle. This means we have a representation ρ of $G = SO(n)$ in a vector space V and the vector bundle is $P \times^G V$. Its sections are invariant elements of $\Omega^0(P) \otimes V$ and the connection will be the operator

$$D = d + \rho(\theta) \quad \text{on} \quad \left(\Omega^*(P) \otimes V\right)_{basic}.$$

<u>July 16, 1984</u>

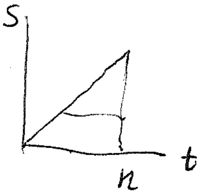
Recall the check that the curvature of the det line bundle given by superconnection formalism agrees with that given by the ζ formalism.

$$L = i\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$ Then the degree 2 component of $\operatorname{tr}_s e^{u(L^2+dL)}$ give a form which represents $u \cdot ch_1$. Compute the degree 2 component by the perturbation series.

$$\int_0^u dt_1 \int_0^{t_1} dt_2 \operatorname{tr}_s\left\{e^{(u-t_1)L^2} dL\, e^{(t_1-t_2)L^2} dL\, e^{t_2L^2}\right\}$$

$$= \int_0^u dt_1 \int_0^{t_1} dt_2 \quad \operatorname{tr}_s\left\{e^{(u-t_1+t_2)L^2} dL\, e^{(t_1-t_2)L^2} dL\right\} \qquad t_1 - t_2 = s$$

$$= \int_0^u dt_1 \int_0^{t_1} ds \quad \operatorname{tr}_s\left\{e^{(u-s)L^2} dL\, e^{sL^2} dL\right\}$$



$$= \int_0^u ds \; (u-s) \; \operatorname{tr}_s\{ \text{———} \}$$

By symmetry the supertrace term is invariant under $s \leftrightarrow u - s$. So we get

$$u \frac{1}{2}\int_0^u ds \; \operatorname{tr}_s\left\{e^{(u-s)L^2} dL\, e^{sL^2} dL\right\}$$

and so the ch_1 is represented by

$$\boxed{\int_0^t dt_1 \; \frac{1}{2} \operatorname{tr}_s\left\{e^{(t-t_1)L^2} dL\, e^{t_1L^2} dL\right\}}$$

for any t. Actually we have proved this:

$$\boxed{\operatorname{tr}_s\left(e^{tL^2 + \sqrt{t}\,dL}\right)_{(2)} = \int_0^t dt_1 \; \frac{1}{2} \operatorname{tr}_s\left\{e^{(t-t_1)L^2} dL\, e^{t_1L^2} dL\right\}}$$

Next consider the ζ-approach. The curvature is $-\bar{\delta}\delta\zeta'(0)$ and we have

$$-\delta\zeta(s) = s\,\mathrm{Tr}\left((D^*D)^{-s} D^{-1}\delta D\right)$$

$$= s\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{tr}\left(e^{-tD^*D} D^{-1}\delta D\right) t^s\frac{dt}{t}$$

So $$+\bar{\delta}\delta\zeta(s) = \frac{s}{\Gamma(s)}\int_0^\infty \mathrm{tr}\left\{\int_0^t dt_1\, e^{-(t-t_1)D^*D}\,\bar{\delta}D^*\,D\,e^{-t_1D^*D}\,D^{-1}\delta D\right\} t^s\frac{dt}{t}$$

$$= \frac{s}{\Gamma(s)}\int_0^\infty \mathrm{tr}\left\{\int_0^t dt_1\, e^{-(t-t_1)D^*D}\,\bar{\delta}D^*\,e^{-t_1DD^*}\,\delta D\right\} t^s\frac{dt}{t}$$

Assuming the limit exists we get

$$\boxed{-\text{curvature} = \bar{\delta}\delta\zeta'(0) = \lim_{t\to 0}\int_0^t dt_1\,\mathrm{tr}\left(e^{-(t-t_1)D^*D}\,\bar{\delta}D^*\,e^{-t_1DD^*}\,\delta D\right)}$$

which is clearly consistent with the previous formula.

Now in the ζ-approach, one requires not the limit as $t\to 0$, but only the zero-th coefficient in the asymptotic expansion. However if the local index formula for families works, then the actual limit as $t\to 0$ should exist.

I want to do the odd version of Grassmannian graph. Take a skew-adjoint operator T and put $L = T\sigma$, then the super-connection formalism gives the odd forms

$$\mathrm{tr}_\sigma\left(e^{\kappa(L^2+[D,L]+D^2)}\right).$$

However I don't know yet how these are to be normalized. So let's compute in the case of the Clifford multiplication $T = i\hat{\gamma}^\mu x^\mu$ over $\mathbb{R}^{n-1}$, $n=2m$.

According to my conventions $T\sigma = i\gamma^\mu x^\mu$, where γ^μ are the generators of C_n for $\mu < n$, and $\mathrm{tr}_\sigma(\cdots)$ becomes $\frac{1}{2i}\mathrm{tr}_s(\cdots\gamma^n)$. So with $D = d$ we have

$$\frac{1}{2i}\,\mathrm{tr}_s\left(e^{\kappa(-|x|^2 - i\gamma^\mu dx^\mu)}\gamma^n\right)$$

$$= \frac{1}{2i}\, e^{-\kappa|x|^2}(-i\kappa)^{n-1}\underbrace{\mathrm{tr}_s\left(\gamma^1 dx^1 \cdots \gamma^{n-1}dx^{n-1}\gamma^n\right)}_{\underbrace{\mathrm{tr}_s(\gamma^n\cdots\gamma^1)}_{(-1)^{\frac{n(n-1)}{2}}(2i)^m \;=\; (-2i)^m}\, dx^1\cdots dx^{n-1}}$$

$$= \frac{1}{\cancel{2i}}\cancel{i}\,\kappa^{n-1}\, e^{-\kappa|x|^2}(2i)^m\, dx^1\cdots dx^{n-1}$$

Now the convention about integrating an odd form over a $(n-1)$-cycle is to multiply by the factor

$$\left(\frac{i}{2\pi}\right)^m 2\sqrt{\pi}$$

which gives

$$\frac{1}{2i}\int\left(\frac{-\cancel{i}}{\cancel{2\pi}}\right)^m 2\sqrt{\pi}\; i\,\kappa^{n-1}e^{-\kappa|x|^2}(\cancel{2i})^m\,\frac{dx^1\cdots dx^{n-1}}{(\sqrt{\pi})^{n-1}}$$

$$= \cancel{2}i\,(-1)^m\,\kappa^{\frac{n-1}{2}} \qquad ?$$

Subexample: $T = ix$

$$\mathrm{tr}_\sigma\left(e^{\kappa(-x^2 + i\,dx\,\sigma)}\right) = e^{-\kappa x^2}\, i\kappa\, dx$$

$$\int \frac{i}{2\pi}\,2\sqrt{\pi}\; e^{-\kappa x^2}\, i\kappa\, dx = -\kappa^{1/2}$$

Anyway up to signs we should have

$$\boxed{\mathrm{tr}_\sigma\left(e^{\kappa(L^2 + [D,L] + D^2)}\right) \sim \sum_{m\geq 1}\kappa^{m-\frac{1}{2}}\, ch'_m}$$

So

$$\int_0^\infty \mathrm{tr}_\sigma\left(e^{u(D+L)^2}\right) e^{-\lambda u}\,\frac{du}{u} \sim \sum_{m\geq 1} \frac{\Gamma(m-\frac{1}{2})}{\lambda^{m-1/2}}\, ch'_m$$

Instead of ch'_m put e_{2k+1} and the above becomes

$$\sim \sum_{k\geq 0} \frac{\Gamma(k+\frac{1}{2})}{\lambda^{k+1/2}}\, e_{2k+1}$$

Now consider $D = d$, whence

$$\int_0^\infty \mathrm{tr}_\sigma\left(e^{u(L^2+dL)}\right) e^{-\lambda u}\,\frac{du}{u}$$

$$= -\,\mathrm{tr}_\sigma \log\left(1 - \frac{1}{\lambda-L^2}\,dL\right)$$

$$= \sum_{k\geq 1} \frac{1}{k}\,\mathrm{tr}_\sigma\left(\frac{1}{\lambda-L^2}\,dL\right)^k$$

~~$L = iT\sigma$~~
~~$dL = i\,dT\sigma$~~
~~and the i's factor cancel in pairs~~

$$\cancel{= \sum_{k\geq 0} \frac{1}{2k+1}\,\mathrm{tr}\left(\frac{1}{\lambda+T^2}\,dT\right)^{2k+1}}$$

Let's recall $L = T\sigma$, so

$$\left(\frac{1}{\lambda-L^2}\,dL\right)^2 = \frac{1}{\lambda-T^2}(dT)\sigma\,\frac{1}{\lambda-T^2}(dT)\sigma$$

$$= -\left(\frac{1}{\lambda-T^2}\,dT\right)^2.$$

Thus

$$\boxed{\int_0^\infty \mathrm{tr}_\sigma\left\{e^{u(L^2+dL)}\right\} e^{-\lambda u}\,\frac{du}{u} = \sum_{k\geq 1} \frac{(-1)^k}{2k+1}\,\mathrm{tr}\left(\frac{1}{\lambda-T^2}\,dT\right)^{2k+1}}$$

Now consider the map

$$T \longmapsto g = (1-T)^{-1}(1+T)$$

from skew-**adjoint** to unitary operators. Then

$$g^{-1}dg = (1+T)^{-1}(1-T)\left\{+(1-T)^{-1}dT(1-T)^{-1}(1+T) + (1-T)^{-1}dT\right\}$$

$$= (1+T)^{-1}\left\{dT\,(1-T)^{-1}(1+T) + dT\right\}$$

$$= (1+T)^{-1}\,dT\,(1-T)^{-1}\left\{(1+T)+(1-T)\right\}$$

$$= 2\,\frac{1}{1+T}\,dT\,\frac{1}{1-T}$$

and so

$$\mathrm{tr}\,(g^{-1}dg)^{2k+1} = 2^{2k+1}\,\mathrm{tr}\left(\frac{1}{1-T^2}\,dT\right)^{2k+1}$$

Rescaling $T \mapsto \lambda^{-1/2}T$ gives the forms

$$2^{2k+1}\,\mathrm{tr}\left(\frac{\lambda^{1/2}}{\lambda-T^2}\,dT\right)^{2k+1}$$

But recall that

$$e_{2k+1} = \underbrace{\int_0^1 dt\,\frac{(t^2-t)^k}{k!}}_{(-1)^k\frac{\beta(k+1,k+1)}{k!} = (-1)^k\frac{k!}{(2k+1)!}}\,\mathrm{tr}(\omega^{2k+1}) \qquad \omega = g^{-1}dg$$

Thus

$$\boxed{e_{2k+1} = (-1)^k\,\frac{k!\,2^{2k+1}}{(2k+1)!}\,\lambda^{k+\frac{1}{2}}\,\mathrm{tr}\left(\frac{1}{\lambda-T^2}\,dT\right)^{2k+1}}$$

But

$$\Gamma(k+\tfrac{1}{2}) = \Gamma(\tfrac{1}{2})\,\tfrac{1}{2}\,\tfrac{3}{2}\cdots\tfrac{2k-1}{2} = \Gamma(\tfrac{1}{2})\,\frac{(2k)!}{k!\,2^k\,2^k}$$

$$= \sqrt{\pi}\,\frac{(2k)!}{k!\,2^{2k}}$$

Hence

$$\boxed{\frac{\Gamma(k+\frac{1}{2})}{\lambda^{k+\frac{1}{2}}}\, e_{2k+1} = 2\sqrt{\pi}\ \frac{(-1)^k}{2k+1}\, \mathrm{tr}\left(\frac{1}{\lambda-T^2}\, dT\right)^{2k+1}}$$

and everything cross-checks very nicely.

For e_1 the above is as follows:

$$\boxed{\begin{aligned} e_1 &= \mathrm{tr}\,(g^{-1}dg) = 2\,\mathrm{tr}_\sigma\left(\frac{\sqrt{\lambda}}{\lambda-L^2}\, dL\right) \\ &= 2\ \mathrm{tr}\left(\frac{\sqrt{\lambda}}{\lambda-T^2}\, dT\right) \end{aligned}}$$

and

$$\mathrm{tr}_\sigma(e^{uL^2}\, u^{1/2} dL) \sim c\, e_1$$

for some constant c. Apply $\int e^{-\lambda u}\, u^{1/2} \frac{du}{u}$?

$$\frac{e_1}{2\sqrt{\lambda}} = \mathrm{tr}_\sigma\left(\frac{1}{\lambda-L^2}\, dL\right) \sim c\, e_1 \cdot \frac{\sqrt{\pi}}{\sqrt{\lambda}} \qquad \therefore\ 2c\sqrt{\pi} = 1$$

so

$$\boxed{\begin{aligned} e_1 &= 2\sqrt{\pi}\ \mathrm{tr}_\sigma\left(e^{uL^2}\, u^{1/2} dL\right) \\ &= 2\sqrt{\pi}\ \mathrm{tr}\left(e^{uT^2}\, u^{1/2} dT\right) \end{aligned}}$$

<u>July 17, 1984</u>

The transgression process for the e_1 class: (This is somewhat degenerate and has to be handled carefully so as to gain experience with the calculations, but yet not to get the wrong intuition).

We have a 1-diml class e_1 on $B\mathcal{G} = \mathcal{A}/\mathcal{G}$ which transgresses to a 0-diml reduced class on $\mathcal{G}$. (Recall transgression goes from $\operatorname{Ker}\{H^*(B) \to H^*(E)\} \to \operatorname{Coker}\{H^{*-1}(E) \to H^{*-1}(F)\}$, so [redacted] in the present case from

$$H^*(B\mathcal{G}) \longrightarrow \operatorname{Cok}\{H^0(\mathcal{A}) \to H^0(\mathcal{G})\} = \tilde{H}^0(\mathcal{G}).\ )$$

Geometrically one takes a 1-cocycle ω on $\mathcal{A}/\mathcal{G}$ lifts it up to $\mathcal{A}$ and writes it δf; then given a [redacted] basepoint $A_0 \in \mathcal{A}$ and $g \in \mathcal{G}$ one assigns to g the number $f(gA_0) - f(A_0)$. This gives one a function on $\mathcal{G}$ vanishing at the identity. If γ [redacted] is a path from A_0 to gA_0, then

$$f(gA_0) - f(A_0) = \int_{\partial\gamma} f = \int_\gamma \delta f = \int_\gamma \pi^*(\omega)$$

$$= \int_{\pi(\gamma)} \omega$$

and $\pi(\gamma)$ is a loop in $\mathcal{A}/\mathcal{G}$, whose homotopy class depends only on the component of $\mathcal{G}$ to which g belongs.

So this is the topology of the situation. Now we want to place ourselves in the situation where ~~when~~ we take the form ω representing e_1 on $\mathcal{A}/\mathcal{G}$, it happens that $\pi^*\omega = df$ within equivariant forms. This means in this case that f is constant on $\mathcal{G}$-orbits.

Note that this implies that $e_1 = 0$ and so the setup is very degenerate topologically, but nevertheless we might learn something from looking at what happens.

An equivariant 1-form on $\mathcal{A}$ is the same thing as a basic 1-form. Let's consider the possible forms representing e_1, or rather $(2\sqrt{\pi})^{-1} e_1$. There are the forms

$$\alpha_t = \operatorname{tr}(e^{t^2T^2} t\, dT)$$

for different t. If

$$\beta_t = \operatorname{tr}(e^{t^2T^2} T)$$

then

$$d\beta_t = \operatorname{tr}(e^{t^2T^2} t^2 \overbrace{d(T^2)}^{dT\cdot T + T\cdot dT} T) + \operatorname{tr}(e^{t^2T^2} dT) \qquad \text{as } [T, e^{t^2T^2}] = 0$$

$$= \operatorname{tr}(e^{t^2T^2}\, 2t^2T^2\, dT) + \operatorname{tr}(e^{t^2T^2} dT)$$

$$\partial_t \alpha = \operatorname{tr}(e^{t^2T^2}\, 2tT^2\, t\, dT) + \operatorname{tr}(e^{t^2T^2} dT)$$

so we have

$$\boxed{\partial_t \alpha = d\beta_t}$$

We can also combine the forms α_t by integration. Thus $\int f(t)\, \alpha_t\, dt$ should be cohomologous to $\left(\int f(t)\,dt\right) \alpha_s$ for any s. Specifically if

$$g(t) = \int_s^t f(t)\, dt$$

then

$$\int_a^b f(t)\, \alpha_t\, dt = \int_a^b g'(t)\, \alpha_t\, dt$$

$$= \left[g(t)\,\alpha_t\right]_a^b - \int_a^b g(t)\,\underbrace{\partial_t\alpha}_{d\beta}\,dt$$

so taking $s = 1$, $a=0$, $b=1$ gives

$$\int_0^1 f(t)\,\alpha_t\,dt = \left(\int_0^1 f(t)\,dt\right)\alpha_0 - d\int_0^1 g(t)\,\beta_t\,dt$$

One thing I want to use is that $\alpha_0 = 0$, as this corresponds to what we know about the classical limit of Dirac operators. Then

$$\alpha_1 = \int_0^1 \partial_t\alpha\,dt = d\int_0^1 \beta_t\,dt$$

$$= d\int_0^1 tr\left(e^{t^2T^2}T\right)dt$$

So if we take $\pi^*\omega$ to be α_1, then we get the G-invariant function on T

$$f(T) = \int_0^1 tr\left(e^{t^2T^2}T\right)dt$$

Natural question is how this compares to $\eta(A)$ where $A = iT$?

$$\eta_A(s) = tr\left(\frac{A}{|A|}\,|A|^{-s}\right)$$

$$= tr\left(A\,(A^2)^{-\frac{s+1}{2}}\right) = \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty tr\left(Ae^{-tA^2}\right)t^{\frac{s+1}{2}}\,\frac{dt}{t}$$

$$= \frac{2}{\Gamma(\frac{s+1}{2})}\int_0^\infty tr\left(e^{-t^2A^2}A\right)t^s\,dt$$

$$\eta(A) = \frac{2}{\sqrt{\pi}}\int_0^\infty tr\left(e^{-t^2A^2}A\right)dt$$

$$\boxed{i\pi\,\eta(A) = 2\sqrt{\pi}\int_0^\infty tr\left(e^{-t^2T^2}T\right)dt}$$

So we see again, without understanding why, that the form of interest is [redacted]

$$\int_0^\infty \beta_t \, dt .$$

This form has for boundary $\alpha_\infty - \alpha_0$ which is zero at least when T is invertible.

The fundamental problem therefore seems to be why the deformation $t \longrightarrow \infty$ should be equivalent topologically to the trivialization of the character form over $\mathcal{D}$ which results from the character form being an equivariant form.

Bott periodicity theorem from the differential form viewpoint.

Let's consider the Bott map which associates to a subspace a path from I to $-I$ in the unitary group. Let $F = 1$ on the subspace and -1 on the orthogonal complement, whence $F = 2e - 1$, where e is the projector on the subspace. The path will then be

$$g = \frac{1 + itF}{1 - itF} \qquad 0 \leq t \leq \infty$$

and we have

$$g^{-1}dg \quad \text{conj to} \quad 2 \, \frac{1}{1+t^2} \, d(itF)$$

so

$$\mathrm{tr}\,(g^{-1}dg)^{2k+1} = 2^{2k+1} \frac{(i)^{2k+1}}{(1+t^2)^{2k+1}} \, \mathrm{tr}\,(dt\,F + t\,dF)^{2k+1}$$

$$= \left(\frac{2i}{1+t^2}\right)^{2k+1} (2k+1)\, dt\, t^{2k} \;\; \mathrm{tr}\; F(dF)^{2k}$$
$$+ \text{ term not involving } dt$$

Check the constants. I need

$$\int_0^\infty \frac{t^{2k}\,dt}{(1+t^2)^{2k+1}} = \frac{1}{\Gamma(2k+1)}\int_0^\infty t^{2k}\,dt\int_0^\infty e^{-(1+t^2)u}\,u^{2k+1}\frac{du}{u}$$

$$= \frac{1}{\Gamma(2k+1)}\int_0^\infty e^{-u}\,u^{2k+1}\frac{du}{u}\int_0^\infty e^{-ut^2}\,t^{2k+1/2}\frac{dt}{2t}$$

$$= \frac{1}{2\Gamma(2k+1)}\int_0^\infty e^{-u}\,u^{2k+1/2}\frac{du}{u}\;\frac{\Gamma(k+\frac{1}{2})}{\cancel{u^{k+1/2}}}$$

$$= \frac{\Gamma(k+\frac{1}{2})^2}{2\Gamma(2k+1)} \qquad \Gamma(k+\tfrac{1}{2}) = \sqrt{\pi}\,\frac{1\cdot 3\cdots 2k-1}{2^k} = \sqrt{\pi}\,\frac{2k!}{2^{2k}\,k!}$$

Next recall $e_k = \dfrac{(-1)^k\,k!}{(2k+1)!}\operatorname{tr}(g^{-1}dg)^{2k+1}$ and

$$ch_k = \frac{1}{k!}\operatorname{tr} e(de)^{2k} = \frac{1}{k!\,2^{2k+1}}\operatorname{tr} F(dF)^{2k}$$

Thus e_k pulls back to

$$\frac{\cancel{(-1)^k}\,\cancel{k!}}{\cancel{(2k+1)!}\ \ \cancel{2k!}}\,2^{\cancel{2k}+1}\,i^{\cancel{2k}+1}\,\cancel{(2k+1)}\;\underbrace{\frac{\Gamma(k+\frac{1}{2})^2}{2\Gamma(2k+1)}}_{\dfrac{\pi\,((2k)!)^2}{(2^{2k}\,k!)^2\;2(2k)!}}\;\cancel{k!}\,\cancel{2^{2k+1}}\cdot ch_k$$

$$= (2i\pi)\,ch_k$$

Consider now the maps occurring in the above version of periodicity, but let's avoid assuming $L^2 = I$. Consider [redacted] invertible $L = i\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ and the [redacted] pull-back of ch_1 under the map $(t,T) \longmapsto$ graph tT. One gets the form

$$\frac{1}{2}\,\mathrm{tr}_s\left(\frac{1}{1-t^2L^2}\,d(tL)\right)^2.$$

Then

$$\int_0^\infty \frac{1}{2}\,\mathrm{tr}_s\left(\frac{1}{1-t^2L^2}(dt\,L + t\,dL)\frac{1}{1-t^2L^2}(dt\,L + t\,dL)\right)$$

$$= \int_0^\infty dt\ \mathrm{tr}_s\left\{\frac{1}{(1-t^2L^2)^2}\,tL\,dL\right\}$$

$$= \int_0^\infty dt\,\frac{1}{2}\,\mathrm{tr}_s\Big\{\underbrace{\frac{1}{(1-t^2L^2)^2}2tL^2}_{\partial_t \frac{1}{1-t^2L^2}}\ L^{-1}dL\Big\} = \left[\frac{1}{2}\,\mathrm{tr}_s\left\{\frac{1}{1-t^2L^2}L^{-1}dL\right\}\right]_0^\infty$$

$$= \boxed{-\frac{1}{2}\,\mathrm{tr}_s\left(L^{-1}dL\right).}$$

The minus sign is correct because

$$L = i\,(Ta^* + T^*a) \qquad L^{-1} = (-i)\big((T^*)^{-1}a^* + T^{-1}a\big)$$

$$dL = i\,(dT\,a^* + dT^*\,a)$$

$$L^{-1}dL = -\,T^{-1}dT\,aa^* - (T^*)^{-1}a^*a = -\begin{pmatrix} T^{-1}dT & 0 \\ 0 & (T^*)^{-1}dT^* \end{pmatrix}$$

so

$-\frac{1}{2}\,\mathrm{tr}_s\left(L^{-1}dL\right) = \frac{1}{2}\Big(\mathrm{tr}\left(T^{-1}dT\right)$ [redacted] $\mathrm{tr}\left((T^*)^{-1}dT^*\right)\Big)$

Here's the same calculation done with the Gaussian version

$$\mathrm{tr}_s\left(e^{t^2L^2 + t\,dL + dt\,L}\right)_{(2)} = \underbrace{\mathrm{tr}_s\left(e^{t^2L^2 + t\,dL}\right)_{(2)}}_{\alpha_t} + dt\ \underbrace{\mathrm{tr}_s\left(e^{t^2L^2}L\,t\,dL\right)}_{\beta_t}$$

Then

$$\int_0^\infty dt\,\beta_t = \int_0^\infty dt\ tr_s\{\underbrace{e^{t^2L^2}tL^2}_{\frac{1}{2}\partial_t e^{t^2L^2}}L^{-1}dL\} = -\tfrac{1}{2}tr_s(L^{-1}dL)$$

Here's the odd version treated above in Gaussian fashion, but now using the Cayley map:

$$t, T \longmapsto \frac{1+tT}{1-tT} = g$$

~~[illegible]~~ The pull-back of $tr(g^{-1}dg)$ is

$$tr\left\{2\frac{1}{1-t^2T^2}\,d(tT)\right\}$$

So $$\int_0^\infty tr\left\{2\frac{1}{1-t^2T^2}\,dt\,T\right\} = \int_0^\infty dt\ tr\left\{\frac{1}{1+tT}T + \frac{1}{1-tT}T\right\}$$

$$= \left[tr\left\{\log\left(\frac{1+tT}{1-tT}\right)\right\}\right]_0^\infty$$

July 18, 1984

So far we understand somewhat the behavior of periodicity and differential forms provided we stick to models of U and $\mathbb{Z}\times BU$ using involutions and projectors. This is why Connes theory reduces to F satisfying $F^2 = 1$.

But it is natural to expect that if we enlarge attention from unitaries to invertibles, i.e. from U to $GL(\mathbb{C})$, then perhaps the continuous cohomology of $GL(\mathbb{C})$ enters into the behavior of differential forms. This in turn suggests looking at groups of complex gauge transformations, and to get started, with complex gauge transformations over the circle and a Riemann surface.

Let's consider the example of $\bar\partial$ operators of index 0 on S^2. The group $\mathcal{G}_c$ of complex gauge transformations acts transitively on the open set of invertible $\bar\partial$-operators. Since $\mathcal{G}_c$ is connected questions about determinants should be answerable using the anomaly formulas and then integrating.

Let's restrict gauge transformations to be fixed at ∞, and then consider the following setup. We consider operators $\partial_{\bar z} + \alpha(z)$ over $\mathbb{C}$ with α decaying rapidly at ∞. $\mathcal{G}_c$ consists of invertible matrix functions $\varphi(z)$ which approach 1 rapidly at ∞. Then the operators we consider are those which are gauge-equivalent to $\partial_{\bar z}$:

$$\partial_{\bar z} + \alpha = \varphi\, \partial_{\bar z}\, \varphi^{-1}$$

What are the general considerations? Let's go back to S^2 with $\mathcal{A}$ = all Dirac ops of index zero and $\mathcal{G}_c$ = all complex gauge transformations fixing the fibre at ∞. We then know that $\mathcal{G}_c$ acts simply-transitively on the invertible Dirac operators.

Now we consider the determinant line bundle $\mathcal{L}$ over $\mathcal{A}$ with its canonical section and $\mathcal{G}_c$-action. Topologically this line bundle represents an element of $H^2(B\mathcal{G}_c, \mathbb{Z}) \simeq H^1(\mathcal{G}_c, \mathbb{Z})$. To realize this class on $\mathcal{G}_c$ we trivialize the line bundle $\mathcal{L}$ over $\mathcal{A}$; then over a $\mathcal{G}_c$ orbit one has two trivializations - more precisely given $A_0 \in \mathcal{A}$ and $g \in \mathcal{G}_c$ one has two ways to go from $\mathcal{L}_{A_0}$ to $\mathcal{L}_{gA_0}$, and hence an element $f(g) \in \mathbb{C}^\times$. Thus one gets a map $f : \mathcal{G}_c \longrightarrow \mathbb{C}^*$ realizing the class in $H^1(\mathcal{G}_c, \mathbb{Z})$. If A_0 is in the invertible set, then the trivialization of $\mathcal{L}$ maps the canonical section σ into a determinant function $\det(A)$, and $f(g) = \dfrac{\det(gA_0)}{\det(A_0)}$.

Question: How do we produce a differential 1-form on $\mathcal{G}_c$ realizing the class in $H^1(\mathcal{G}_c, \mathbb{C})$?

To answer this we can use the Atiyah-Singer announcement. Consider over $\mathcal{G} \times M$ the connection on $pr_2^*(E)$ obtained from the map $\mathcal{G} \to \mathcal{A}$, $g \longmapsto g D_{A^0} g^{-1}$ and the connection over $\mathcal{A} \times M$ which descends. This connection is

$$\delta + g\delta g^{-1} + g D_{A^0} g^{-1} = g(\delta + D_{A^0}) g^{-1}$$

and its curvature is $g \cdot F_{A^0} \cdot g^{-1}$. Better put $D_{A^0} = d_M + A^0$, so that we are dealing with the connection $\delta + d_M + A = \delta + d_M + g\delta g^{-1} + g d_M g^{-1} + g A^0 g^{-1}$

We want to join this to the connection

$$d = \delta + d_M$$

which will give the following transgression formula

$$\underbrace{\tfrac{1}{2}\,\mathrm{tr}\left(F_A^2\right)}_{0 \text{ because } F_A^2 = g F_{A_0} g^{-1} \text{ has degree } > \dim M.} = d \int_0^1 dt \ \mathrm{tr}\left(F_{tA}\, A\right)$$

~~Instead of joining d to $g(d+A_0)g^{-1}$ it might be easier to join $g^{-1}.d.g$ to $d+A_0$, whence~~

$$\cancel{F_t = t F_{A_0} + (t^2-t)(A_0 - g^{-1}dg)^2}$$

Now

$$F_{tA} = t F_A + (t^2-t)A^2 \qquad A = g\,dg^{-1} + g A_0 g^{-1}$$

so

$$\int_0^1 dt \ \mathrm{tr}\left(F_{tA}\, A\right) = \frac{1}{2}\,\mathrm{tr}\left(F_A A\right) - \frac{1}{6}\,\mathrm{tr}\left(A^3\right)$$

However it seems to me that I want to take $A_0 = 0$ in which case we are looking at $A = g\,dg^{-1} = -dg\,g^{-1}$ and the three form over $G \times M$ is $\frac{1}{6}\,\mathrm{tr}\,(dg\,g^{-1})^3$. This is then integrated over $M = S^2$ to get the 1-form on G.

But what confuses me is the calculations which produce the σ-model kinetic energy in the determinant. This is present even in the $U(1)$ case where the topology is trivial

July 19, 1984

General problem: Continuous cohomology + periodicity. One knows that the continuous cohomology of $Gl_n \mathbb{C}$ is periodic and given by the familiar differential forms of cyclic theory. The differential form theory should be more than just the topological K-theory.

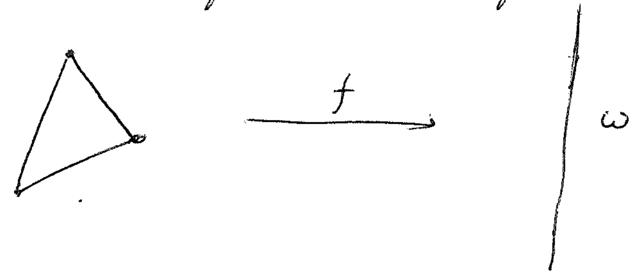
Let's try to unify what we know about the dilogarithm. It occurs in two forms described to me by Atiyah. The imaginary part of the dilog gives the volume of a tetrahedron in hyperbolic 3-space. The real part has to do with η-invariants and Chern-Simons terms for 3 manifolds.

Also in Gelfand-MacPherson the dilog occurs relative to $G_4^2(\mathbb{R})$.

Let's start with the simplest case we understand. Let us consider a flat $G = SL(2,\mathbb{C})$-bundle over M. Let P be the principal bundle and form the hyperbolic space bundle $P \times^G \mathcal{H}^3 = X$, $\mathcal{H}^3 = SL(2,\mathbb{C})/SU(2)$. Because of the flat connection, the volume form on $\mathcal{H}^3$ gives rise under the Weil homomorphism to a closed 3-form on X. As the fibre is contractible X has a section over M and pulling back this 3 form gives a closed 3-form on M, whose class is independent of the choice of section.

If we think of M as a simplicial complex, then a natural way to construct a section is to first lift the vertices of M. Over any simplex σ the bundle X becomes trivial $X|\sigma = \sigma \times \mathcal{H}^3$ and then using the convex structure of $\mathcal{H}^3$ there is a natural way to map σ to $\mathcal{H}^3$ given where the vertices are supposed to go.

The actual 3-cochain on M we get this way assigns to a 3-simplex $\sigma = (v_0, v_1, v_2, v_3)$, the volume of the tetrahedron in hyperbolic space with the vertices where v_0, v_1, v_2, v_3 go under the trivialization of X over σ. The cocycle condition will follow from the invariance of the volume and the fact the form is closed:



$$\int_{\partial\sigma} f^*\omega = \int_{\sigma} df^*\omega = \int_{\sigma} f^* d\omega = 0$$

Now the volume of a tetrahedron with vertices at the 2-sphere boundary of $\mathcal{H}^3$ is defined. One should start by saying that $\mathcal{H}^3$ has a compactification by $\mathbb{P}^1(\mathbb{C}) = S^2$ just as the upper half plane is compactified by adding $\mathbb{P}^1(\mathbb{R})$. Hence if we started constructing our section by choosing the vertices to lie at the boundary, then we get a much simpler formula for the cocycle.

What is happening I think is the following. The group G acts on the 3-disk $\overline{\mathcal{H}^3}$ obtained by adding S^2 to $\mathcal{H}^3$. (This action should be continuous at least.) We are constructing a section of the disk bundle with fibre $\overline{\mathcal{H}^3}$. Now we get a 3-cocycle which assigns to each 3-simplex $(v_0, \dots, v_3)$, the volume of the tetrahedron with vertices on the boundary of $\mathcal{H}^3$ associated to the trivalization of $\overline{X}|\sigma$.

Recall that give 4 distinct points $(z_0, \cdots, z_4)$ in $S^2 = \mathbb{C}P^1$ they have a cross-ratio which determines completely the orbit of $SL(2, \mathbb{C})$ [redacted] acting on 4-tuples of distinct points. The volume of the tetrahedron in $\mathcal{H}^3$[redacted] with the vertices $(z_0, \cdots, z_4)$ must therefore be a function of the cross-ratio. This is supposedly the Lobachevsky function.

July 20, 1984

Connes description of the multiplicative map on $K_1^{alg}(\mathcal{a})$ associated to a trace τ on $\mathcal{a}$:

$$K_2^{top}(\mathcal{a}) \longrightarrow K_1^{rel}(\mathcal{a}) \longrightarrow K_1^{alg}(\mathcal{a}) \longrightarrow K_1^{top}(\mathcal{a})$$

A trace induced a homomorphism $K_1^{rel}\mathcal{a} \rightarrow \mathbb{C}$ as follows. Given an element of $K_1^{rel}\mathcal{a}$ it is represented by a unit $u \in GL(\mathcal{a})$ together with a path u_t joining 1 to u. Then one obtains a number

$$\int_0^1 \tau(u_t^{-1}\dot{u}_t)\,dt \in \mathbb{C}.$$

If we compare the effect of two different paths joining 1 to u, i.e. take a loop u_t in $GL(\mathcal{a})$, then we obtain the ambiguity

$$\oint \tau(u^{-1}du).$$

By Bott periodicity this number is $2\pi i\,\tau(\alpha)$ where $\alpha \in K_0\mathcal{a}$ is the element which corresponds to the loop $u = (u_t)$.

(Here we see again how a trace determines an element of $(K_2^{top}(\mathcal{a}))^*$ in the van Est picture, rather than an element of $(K_0^{top}\mathcal{a})^*$ as one would expect.)

On the Novikov conjecture: Mischenko's basic idea is to generalize the concept of f.d. representation of a discrete group Γ as follows. One considers a pair of Hilbert space representations $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ together with a Fredholm $F: \mathcal{H}^+ \longrightarrow \mathcal{H}^-$ which commutes with the Γ action modulo compacts. Such data (ρ^+, ρ^-, F) define an element α of $K^0(B\Gamma)$ for which the Novikov property holds: Given a map $M \to B\Gamma$, M compact orientable, then the image of $\mathcal{L}(M)$ in $H_*(M)$ under the map paired with $ch(\alpha)$ is a homotopy invariant of M. I still don't see this last property clearly.

If π is the fundamental group of a negatively curved manifold M, then Mischenko proves that all of $K^0(B\Gamma)$ can be so represented. The idea is to use the Hilbert space of L^2-forms on $\tilde{M}$ and the "Gaussian de Rham" complex.

So let's try to understand this in the case of a torus $\mathbb{R}^n/\Gamma$, say $n=2$. Then in order to define the Gaussian de Rham complex I have to choose a base point of $\mathbb{C} = \mathbb{R}^2$. Any translation commutes with the operator up to a bounded function, so we clearly get the Mischenko situation, although unbounded.

Now it is supposedly possible to show that any element of K^0(torus) arises from such a Mischenko representation. It's clear that the ring of functions on the torus acts on the Hilbert space of L^2-forms upstairs and commutes with the Gaussian part of the operator. A function commutes with the de Rham part $d+\delta$ modulo a bounded function, which should be compact. So there is a Kasparov cup product setup - Given a bundle E on the torus one can tensor with the Gaussian DR complex to obtain a Mischenko representation.

It must be then easy to see that upon going back to $K^0(B\Gamma)$ we get the class of E.