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Attempts to construct e^{-tH} as a kernel on the tangent groupoid to a manifold. See formal construction using functions on T^*, $K(t,x,p)$, with $p_j * = p_j + \frac{h}{i}\partial_j$ to express the multiplication on p. 816 - 820.

May 12, 1984

Yesterday we learned a lot about the "Volterra method" for constructing e^{-tH}. One starts with a path $E(t)$, $t \geq 0$ with $E(0) = I$ and such that

$$(\partial_t + H) E(t) = -K(t)$$

is nice enough to allow the "Volterra series"

$$E + E*K + E*K*K + \cdots$$

$$= E(t) + \int_0^t dt_1\, E(t-t_1)K(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2\, E(t-t_1)K(t_1-t_2)K(t_2) + \cdots$$

to converge. Since

$$(\partial_t + H) \int_0^t dt_1\, E(t-t_1)\varphi(t_1) = \varphi(t) + \int_0^t dt_1\, \underbrace{(\partial_t + H)E(t-t_1)}_{-K(t-t_1)}\, \varphi(t_1)$$

one has

$$(\partial_t + H)(E + E*K + \cdots) = -K + (K - K*K) + (K*K - K*K*K) + \cdots$$
$$= 0$$

For example, if $K(t)$ is bounded in some norm. In particular if we have $E(t) = e^{-tH_0}$, then

$$-K(t) = (\partial_t + H)e^{-tH_0} = (-H_0 + H)e^{-tH_0}$$

will be bounded if $-H_0 + H$ is bounded and e^{-tH_0} is a contraction semi-group.

This enables one to construct e^{-tH} where $H = -\Delta + V$ on $\mathbb{R}^n$ when V is a bounded potential, starting from $e^{+t\Delta}$ which we know. However we can't yet handle the harmonic oscillator.

Let us now try to appreciate the idea that the $E(t)$ we need is a local gadget over the manifold where H operates. Hence the H_0 one uses can vary from point to point. I also want to get to the case where there is a Planck's constant.

So let us start now with the problem of constructing $E(t,x)$ satisfying

$$\Big[\partial_t \underbrace{-\partial_\mu^2 + V(x)}_{H}\Big] E(t,x) = \delta(t)\,\delta(x).$$

We want to use the kernel for $\partial_t + H_0$, $H_0 = -\partial_\mu^2$, namely

$$E_0(t,x,y) = \frac{e^{-(x-y)^2/4t}}{(4\pi t)^{n/2}}$$

Now we go thru the same sort of ~~iteration~~ process:

$$(\partial_t + H)\,E = \delta(t)\,\delta_0$$

$$(\partial_t + H_0)\,E = \delta(t)\,\delta_0 + (H_0 - H)\,E$$

$$E = E_0 * \big(\delta(t)\delta_0\big) + E_0 * (H_0 - H)\,E$$

or

$$E(t) = E_0(t)\,\delta_0 + \int_0^t dt_1\; E_0(t-t_1)\,(H_0 - H)\,E(t_1)$$

and now you iterate this equation. What this gives me is a series

$$E = E_0\,\delta_0 + (E_0 *)(-V)(E_0\delta_0) + \big[E_0 *)(-V)\big]^2 E_0\delta_0 + \cdots$$

and we should understand how the terms look.

To simplify replace $-\partial_\mu^2 = H_0$ by $-\frac{1}{4}\partial_\mu^2$ and V by $-V$. Then

$$E(t,x) = \frac{e^{-x^2/t}}{(\pi t)^{n/2}} + \int_0^t dt_1 \int dy\; \frac{e^{-\frac{(x-y)^2}{t-t_1}}}{(\pi(t-t_1))^{n/2}}\, V(y)\, \frac{e^{-\frac{y^2}{t_1}}}{(\pi t_1)^{n/2}} + \cdots$$

We are interested in what happens as $t \to 0$ and look for the critical points of $\dfrac{(x-y)^2}{t-t_1} + \dfrac{y^2}{t_1}$ as a function of y:

$$-2\frac{x-y}{t-t_1} + 2\frac{y}{t_1} = 0 \quad , \qquad t_1(x-y) = (t$$ [redacted] $$-t_1)y$$

$$t_1 x = t y \qquad \text{or} \qquad \boxed{y = \frac{t_1}{t}x}$$ is the critical point

The critical value is

$$\frac{1}{(t-t_1)}\left(1-\frac{t_1}{t}\right)^2 x^2 + \frac{1}{t_1}\left(\frac{t_1}{t}\right)^2 x^2 = \frac{t-t_1+t_1}{t^2}x^2 = \frac{x^2}{t}$$

and this is independent of t_1.

Thus if [redacted] we are interested in the asymptotic expansion as $t \to 0$ we seem to get the leading behavior $e^{-x^2/t}$ together with various integrals of V and its derivatives along the line from 0 [redacted] to x.

Project: I want to prove the existence of the heat [redacted] operator for the harmonic oscillator by deformation from the classical case $h=0$. I want to set up Connes' algebra for R^n. It is a crossed product of the Schwartz space of $f(p)$ with the Schwartz space of $f(q)$ over functions in h. What I think I get is a Weyl algebra over the functions in h. Perhaps it would be useful to think in terms of the convolution algebra of the Heisenberg group.

The first step will be to work out the product of two functions $f(q,p)$, $g(q,p)$ in this algebra. It will clearly be a twisted convolution as we have already learned

$$(f*g)(q,p) = \int dq_1\,dp_1\; f(q-q_1, p-p_1)\, g(q_1,p_1)\, e^{?}$$

The actual phase will depend on the way we propose to identify $f(q, p)$ with an operator. Here f is a smooth function with compact support, but we can think of f approaching a δ-function, and it is enough to assign an operator to the function $\delta(q-x)\,\delta(p-h\xi)$.

What is the operator belonging to $g(p)$?

$$\big(g(p)\,u\big)(x) = \int \frac{d\xi}{2\pi}\, e^{-i\xi x}\, g(h\xi)\, \underbrace{\hat{u}(\xi)}_{\int dy\, e^{-i\xi y} u(y)}$$

$$\therefore \quad \big(\delta(p-h\xi_0)\,u\big)(x) = \int \frac{d\xi}{2\pi}\, e^{i\xi x}\, \delta(h\xi - h\xi_0)\, \hat{u}(\xi)$$

$$= \frac{1}{2\pi h}\, e^{i\xi_0 x}\, \hat{u}(\xi_0)$$

$$e^{i\xi x}\,\hat{u}(\xi) = \int dy\, e^{i\xi(x-y)}\, u(y) \qquad ?$$

$$\therefore \quad \boxed{\delta(p-h\xi)\, u = |h\xi\rangle\langle h\xi | u\rangle}$$

is the "projection" onto the momentum state with momentum $h\xi$.

But I seem to be thinking the wrong way. We want to think of functions on the cotangent bundle as being operators, and then the basic operator are attached to the exponential functions

$$e^{i(aq+bp)}$$

rather than to the δ-functions.

So unfortunately there is some kind of dualization at the beginning.

Here's the explanation: We are looking at the Weyl algebra generated by the vector space of $aq+bp$. I wanted originally to think of such a Weyl algebra geometrically in terms of a line bundle + connection over phase space with constant curvature. The space of parallel translation operators D_v can be identified with the vector space of $aq+bp$, and the smooth Weyl algebra appears as a twisted convolution algebra of smooth rapidly decreasing fns. on phase space $= \{aq+bp\}$.

Now the problem is that this is a convolution algebra (when $h=0$) and so the heat kernel will look awkward in this form. The convolution emphasizes translations or differentiation, whereas we want to emphasize the multiplication. So instead of the basic representation of the Weyl algebra on functions on $V = \{aq+bp\}$ we used before

$$v \longmapsto D_v = \partial_v + \lambda_v$$

we want to represent the Weyl algebra on functions on V^* by

$$v \longmapsto v + i(\lambda_v)$$

where $i(\lambda_v)$ denotes differentiation on functions on V^*.

Let V be a real vector space with coords x^μ and skew-symmetric form $\frac{1}{2} dx^\mu dx^\nu F_{\mu\nu}$, $F_{\mu\nu}$ constant. Then we form the Weyl algebra generated by elements D_μ satisfying

$$[D_\mu, D_\nu] = F_{\mu\nu},$$

and we can interpret this Weyl algebra as an algebra of operators on sections on the line bundle L with connection having curvature F. Namely e^{D_v} is the parallel transport in the direction v. Then the convolution alg. consists of the operators

$$\int dv\, f(v)\, e^{D_v} \qquad f \in \mathcal{S}(V)$$

If we use the synchronous trivialization of L at 0, then D_μ becomes

$$D_\mu = \partial_\mu - \tfrac{1}{2} F_{\mu\nu} x^\nu$$

and the Weyl algebra becomes an algebra of operators on $\mathcal{S}(V)$. We can recover $f(v)$ by acting on $\delta(x) =$

$$\left(\int dv\, f(v)\, e^{D_v}\right) \delta(x) = f(x)$$

As mentioned above this picture is modelled on convolution, and we want the dual picture. So let V^* be the dual of V with the dual coordinates ξ_μ. Then we can also realize, or represent, the Weyl algebra on the functions on V^* by

$$D_\mu = i\left(\xi_\mu - \tfrac{1}{2} F_{\mu\alpha} \frac{\partial}{\partial \xi_\alpha}\right)$$

$$\left[i\left(\xi_\mu - \tfrac{1}{2} F_{\mu\alpha} \partial/\partial\xi_\alpha\right), i\left(\xi_\nu - \tfrac{1}{2} F_{\nu\beta} \partial/\partial\xi_\beta\right)\right]$$

$$= -\left(-\tfrac{1}{2} \underbrace{F_{\mu\alpha}\delta_{\alpha\nu}}_{F_{\mu\nu}} + \tfrac{1}{2} F_{\nu\beta}\delta_{\mu\beta}\right) = F_{\mu\nu}$$

All we are doing is to use the F.T. $\mathcal{S}(V) \simeq \mathcal{S}(V^*)$.

Then given $\int d\sigma\, f(\sigma)\, e^{D_\sigma}$ we can let it act on $1 \in S(V^*)$ corresp. to $\delta(x)$ in $S(V)$.

$$e^{i(\xi_\mu - \frac{1}{2}F_{\mu\alpha}\partial/\partial\xi_\alpha)} = e^{i\xi_\mu}\, e^{-\frac{i}{2}F_{\mu\alpha}\partial/\partial\xi_\alpha}$$

so
$$\int d\sigma\, f(\sigma)\, e^{D_\sigma} 1 = \int d^n x\, f(x)\, e^{x^\mu D_\mu} 1$$
$$= \int d^n x\, f(x)\, e^{i x^\mu \xi_\mu}$$

is just the F.T. of $f(x)$.

Lastly we compute the composition of two function of ξ, where we identify $g(\xi)$, with

$$\int d\sigma\, \hat{g}(\sigma)\, e^{D_\sigma}$$

May 13, 1984

I want to consider a vector bundle E with connection D (preserving inner product on E) over $\mathbb{R}^n$. Then we want to produce the heat operator e^{-tH}, where $H = -h^2 D_\mu^2$ is the covariant derivative Laplacean. The method will proceed formally in powers of h first. I need an algebra in which the heat operator will live and which contains an element h.

This algebra will be related to the algebra of differential operators on E, in analogy with the relation of the convolution algebra of Schwartz functions is related to polynomials. Let $R = \mathrm{Diff}(E,E)$; then R has a natural filtration given by the order of a diff'l operator and $\mathrm{gr}\, R = \Gamma(\mathrm{End}\, E \otimes S(T^*))$. I want to view R as a twisted polynomial ring "over" $\Gamma(\mathrm{End}\, E)$ generated by the D_μ, and then form $\tilde{R} \in R[h]$ generated by $\Gamma(\mathrm{End}\, E)$ and the hD_μ. In this algebra $\tilde{R}$ we can view the generators as the $p_\mu = hD_\mu$, and the bundle endomorphisms, and h.

Now the problem is to enlarge $\tilde{R}$ to an algebra of pseudo-differential operators, which means roughly that we have to allow more general functions of the p_μ. In particular one wants smooth rapidly decreasing functions of the p_μ. This step supposedly can be done via Connes tangent groupoid business. (Here I lose control.)

I want to construct the desired algebra formally i.e. (mod h^n). The problem is that I want to think of the algebra as a kind of crossed product of Schwartz functions of the p's with $\Gamma(\mathrm{End}\,E)$, and hence I have to have some idea of extending $[p,\ ]$ on $\Gamma(\mathrm{End}\,E)$ to a derivation $[f(p),\ ]$. It is not clear how to do this, in fact, it is probably a bad idea to think in crossed product terms.

Let's proceed formally.

New idea. Treat pseudo-diffl operators from the filtered algebra viewpoint. Normally one calculates with a series of homogeneous functions of the p's - this is working in the associated graded algebra. Instead introduce an algebra of smooth functions $f(h,q,p)$ where $h \neq 0$, and then a ~~filtration~~ on this algebra to describe the $h \to 0$ behavior. The associated graded algebra is to be the algebra of ψDO symbols.

What does one do when one constructs a ψDO symbol inverting an elliptic operator? Take $L = p^2 + V(x)$ and construct $(p^2+V)^{-1}$ on the symbol level:

$$L^{-1} = \sum_{n=-2}^{-\infty} L_n$$

$$1 = (p^2+V)L^{-1} = \sum_{n\leq -2} p^2 L_n(x,p) + \sum_{n\leq -2} V(x) L_n(x,p)$$

and $$p^2 * f(x,p) = f(x,p)p^2 + 2\frac{h}{i}\partial_x f(x,p)\,p + \left(\frac{h}{i}\right)^2 \partial_x^2 f(x,p)$$

Normally $h = 1$. We get recursion relations

$$1 = L_2(x,p)p^2$$

$$0 = L_n(x,p)p^2 + 2\frac{h}{i}\partial_x L_{n+1}(x,p)\,p + \left(\frac{h}{i}\right)^2 \partial_x^2 L_{n+2}(x,p) + V(x) L_{n+2}(x,p)$$

for $n < -2$

This is the term which is homogeneous of degree $n+2$.

The first thing to notice is that it is all a formal calculation at a given point x.

May 14, 1984

Yesterday I reached the point where I believe that to [redacted] work in the algebra of operators depending on h, where h is a formal quantity, is equivalent to the usual calculus of ψDO symbols. But there remains a great deal to get this all clear.

The basic problem is notational, namely, how to denote the algebra of operators depending on h. [redacted] Let's illustrate this difficulty with an example. Consider a vector bundle E and let's restrict attention to the differential operators. The problem then is how to describe the algebra R of differential operators on E. It is naturally filtered and so we can form the graded algebra $\tilde{R} = \bigoplus_{n\geq 0} (F_n R) h^n \subset R[h]$.

The problem is that one would like to describe R as a deformation of $gr(R) = \Gamma(\mathrm{End}\,E \otimes S(T)) =$ ring of polynomial functions on T^* with values in endos. of E. Thus we need a way to go from $gr(R)$ to R, i.e. a quantization process. [redacted] Such a quantization process gives us a new algebra structure on $gr(R)$.

There is a [redacted] general way to treat these deformations, namely one defines an action of the algebra R on $gr\,R$ which corresponds to left multiplication under the quantization process $R \simeq gr\,R$. Sometimes this is possible because the generators and relations for R are so simple.

I really want to carry out the existence of the heat operator in the simplest situation. So take either $\mathbb{R}^n$ or $\mathbb{R}^n$/lattice and the trivial bundle. The former should be simpler for describing the algebra of operators, but the latter might be better for existence questions. Let's start with $\mathbb{R}^n$.

The algebra in the formal situation consists of power series $\sum_{n\geq 0} a_n(q,p)h^n$ where the a_n are smooth functions on phase space and the product $\circ$ is defined by requiring h to be central and

$$f(q,p)\circ g(q,p) = \sum_\alpha \left(\frac{h}{i}\right)^{|\alpha|}\frac{1}{\alpha!}\frac{\partial^\alpha}{\partial p^\alpha}f(q,p)\cdot\frac{\partial^\alpha}{\partial q^\alpha}g(q,p).$$

Let me call this algebra $\mathcal{A}$. By construction it is a deformation of the algebra $\mathcal{A}_0 \simeq \mathcal{A}/h\mathcal{A}$ which consists of smooth functions on phase space. Notice that this algebra can be setup as a sheaf of algebras over phase space, since differentiation is a local operation.

Now we start with a Hamiltonian, say $H = p^2 + V(q)$, and wish to construct e^{-tH} in this algebra. I should worry a little bit about the topology on this algebra. We start with a topology on the functions on phase space, say the C^∞ topology, and then note that the above "$\circ$" product is obviously continuous. One uses the inverse limit topology wrt. the ideals generated by h^n.

It is perhaps simple to work with the Laplace transform $\frac{1}{\lambda - H}$. Here λ is a parameter such that $\frac{1}{\lambda-H}$ exists in the algebra $\mathcal{A}_0$. Then by algebra it exists in the algebra $\mathcal{A}$.

Anyway we now come to the real question: lifting from $\mathcal{A}$, the formal algebra, to the actual operators. It's here that the positivity of p^2 has to come in. I think I should write out what actually occurs in the iteration program. Let $K(t,q,p,h)$ denote the path e^{-tH} in the formal algebra. It is a series in h

$$K = \sum K_n(t,q,p)h^n$$

starting with $K_0(t,q,p) = e^{-tp^2-tV(q)}$ such that

$$\left[\partial_t + p^2 + V(q)\right] \underset{\substack{\uparrow \\ \text{composition product}}}{*} K(t,q,p,h) = 0$$

$$V * K = VK$$

Need

$$p^2 * K = p^2 K + \left(\frac{h}{i}\right) 2p\, \partial_q K + \left(\frac{h}{i}\right)^2 \frac{1}{2!} 2\partial_q^2 K$$

whence

$$\left[\partial_t + p^2 + V(q)\right] K_0 = 0$$

$$\left[\partial_t + p^2 + V(q)\right] K_1 + \frac{1}{i} 2p\, \partial_q K_0 = 0$$

$$\left[\partial_t + p^2 + V(q)\right] K_2 + \frac{1}{i} 2p\, \partial_q K_1 + \left(\frac{1}{i}\right)^2 \partial_q^2 K_0 = 0$$

from which we get

$$K_0 = e^{-t(p^2+V(q))}$$

$$K_1 = -\int_0^t dt_1\, e^{-(t-t_1)(p^2+V)} \frac{1}{i} 2p \frac{\partial}{\partial q} K_0(t_1,q,p)$$

Start again. We want to find

$$K(t,q,p,h) = \sum_{n\geq 0} K_n(t,q,p)\, h^n$$

satisfying

$$\left[\partial_t + p^2 + V(q)\right] * K = 0$$

$$K(0,q,p,h) = 1.$$

where $*$ is the product in $\mathcal{A}$ denoted $\circ$ above.

So

$$\left(\partial_t + p^2 + V(q)\right)K + \left(\frac{h}{i}\right)2p\,\partial_q K + \left(\frac{h}{i}\right)^2 \partial_q^2 K = 0$$

which leads to the recursion relations

$$\left(\partial_t + p^2 + V(q)\right)K_n + \left(\frac{1}{i}\right)2p\,\partial_q K_{n-1} + \left(\frac{1}{i}\right)^2 \partial_q^2 K_{n-2} = 0$$

$$K_n(0,q,p) = \begin{cases} 1 & n=0 \\ 0 & n>0. \end{cases}$$

These are all ordinary DE's in t. Since $p^2+V(q)$ is a scalar operator on functions, we can simplify by writing

$$K_n(t,q,p) = e^{-tp^2 - tV(q)} L_n(t,q,p).$$

and using

$$e^{t(p^2+V(q))}\begin{pmatrix}\partial_t \\ \partial_q\end{pmatrix} e^{-t(p^2+V(q))} = \begin{pmatrix}\partial_t - p^2 - V \\ \partial_q - t\,\partial_q V\end{pmatrix}$$

and so L satisfies

$$\partial_t L + \frac{h}{i}2p\left(\partial_q - t\partial_q V\right)L + \left(\frac{h}{i}\right)^2\left(\partial_q - t\partial_q V\right)^2 L = 0$$

or

$$\boxed{\begin{gathered}\partial_t L_n + \frac{1}{i}2p\left(\partial_q - t\partial_q V\right)L_{n-1} + \left(\frac{1}{i}\right)^2\left(\partial_q - t\partial_q V\right)^2 L_{n-2} = 0 \\ L_n(0,q,p) = \begin{cases} 1 & n=0 \\ 0 & n>0\end{cases}\end{gathered}}$$

From this we see that L_n is a polynomial in t, p with coefficients involving the derivatives of V.

From the recursion relations one sees that L_n is of degree $\leq 2n$ in t and of degree $\leq n$ in p. ~~It~~ seems actually that L_n includes terms from $t^{n/2}$ to t^{2n}. The lower $t^{n/2}$ comes from the fact that $(\partial_q - t\partial_q V)^2 L_{n-2}$ keeps the lowest t term the same, and then one integrates once to get L_n.

So what we have at this point is a formula for the heat operator as a formal series in h

$$K(t,q,p,h) = e^{-t(p^2+V(q))}\{1 + L_1(t,q,p)h + \cdots\}$$

where L_n is a poly in t, p, and the derivatives of V.

May 15, 1984:

Yesterday I constructed the heat operator belonging to $H = p^2 + V(x)$ in the algebra $\mathcal{A}$ consisting of power series in h

$$f(x,p,h) = \sum_{n\geq 0} f_n(x,p)h^n$$

whose coefficients are smooth functions on phase space. The algebra structure is determined by the composition product

$$f(x,p) * g(x,p) = \sum_{\alpha} \frac{1}{\alpha!}\left(\frac{h}{i}\right)^{|\alpha|} \partial_p^\alpha f(x,p)\cdot \partial_x^\alpha g(x,p)$$

The heat operator $\exp^*(-tH)$, $*$ denoting this composition product, is an element of $\mathcal{A}$ depending on t which we write $K(t,x,p,h)$. It satisfies

$$(\partial_t + H) * K = 0 \quad , \quad K(0,t,x,p,h) = 1$$

$$(\partial_t + p^2 + V) * K = 0$$

$$(\partial_t + p^2 + V)K + \frac{h}{i}2p\,\partial_x K + \left(\frac{h}{i}\right)^2 \partial_x^2 K = 0$$

If we put $K = e^{-t(p^2+V(x))}L$, then L satisfies

$$\left\{\partial_t + \frac{h}{i}2p(\partial_x - t\partial_x V) + \left(\frac{h}{i}\right)^2(\partial_x - t\partial_x V)^2\right\}L = 0$$

and we can grind out $L = \sum_{n\geq 0} L_n(t,x,p)t^n$ recursively with L_n a polynomial in t,p having coefficients constructed from V and its derivatives. We start with $L_0 = 1$.

Let's find L_1 two ways:

$$\partial_t L_1 + \frac{1}{i}2p(\partial_x - t\partial_x V)L_0 = 0$$

$$\partial_t L_1 = \frac{1}{i}2p\,t\,V' \qquad\qquad L_1 = \frac{1}{i}p\,t^2 V' = \frac{1}{i}t^2 V'(x)p$$

hence

$$K(t,x,p,h) = e^{-t(p^2+V(x))}\left\{1 + \frac{h}{i}t^2V'(x)p + \cdots\right\}$$

On the other hand working with operators (mod h^2)

$$e^{-t(p^2+V)} = e^{-tV}e^{-tp^2}e^{-\frac{1}{2}[tV,tp^2]}$$

$$\frac{1}{2}[tp^2,tV] = \frac{t^2}{2}\left(\underbrace{p^2V}_{} - Vp^2\right) \qquad = \frac{t^2}{\cancel{2}}\frac{h}{i}V'\cancel{2}p$$

$$Vp^2 + \frac{h}{i}V'2p + \cancel{\left(\frac{h}{i}\right)^2} \qquad = \frac{h}{i}t^2V'(x)p$$

so you end up again with the same formula. In n-dimensions

$$V'(x)p = \partial_\mu V(x)\cdot p_\mu .$$

So much for the formal theory. Now we must convert our formal series into an actual operator. Of course we want to use

$$\tilde{E} = \int\frac{dp}{(2\pi h)^n}e^{\frac{ip}{h}(x-y)}e^{-t(p^2+V(x))}\tilde{L}(t,x,p,h)$$

where $\tilde{L}$ represents some sort of truncation of the series for L, and now h is an actual no.

Now $\tilde{E}$ is supposed to be a forward parametrix for $\partial_t + H$. Hence we are concerned with the error which the analogue of the above but with

$$(\partial_t + H) * \tilde{L} = M(t,x,p,h)$$

instead of $\tilde{L}$. Here M will be a linear combination of $t^a p^b h^c$ with a coefficient poly $(V, V', \ldots)$. Hence we need to see the behavior as $t \to 0$ of

$$\int\frac{dp}{(2\pi h)^n}e^{\frac{ip}{h}(x-y) - tp^2}t^a p^b h^c$$

and we might as well suppose $h = 1$.

To simplify change tp^2 to $t\frac{p^2}{2}$ and complete the square:

$$\underbrace{-\cancel{t}\frac{p^2}{2} + \cancel{t}ip\frac{x-y}{t} - \frac{t}{2}\left(-i\frac{x-y}{t}\right)^2}_{-\frac{t}{2}\left(p - i\frac{x-y}{t}\right)^2} - \frac{(x-y)^2}{2t}$$

$$\int \frac{dp}{(2\pi)^n}\, e^{-\frac{t}{2}\left(p - i\frac{x-y}{t}\right)^2 - \frac{(x-y)^2}{2t}}\, t^a p^b$$

$$= \int \frac{dp}{(2\pi)^n}\, e^{-\frac{(x-y)^2}{2t}}\, e^{-\frac{t}{2}p^2}\, t^a \left(p + i\frac{x-y}{t}\right)^b$$

$$= \frac{e^{\cancel{-\frac{(x-y)^2}{2t}}}}{(2\pi)^n} \int \frac{dp}{t^{n/2}}\, e^{-\frac{p^2}{2}}\, t^a \left(\frac{p}{\sqrt{t}} + i\frac{x-y}{t}\right)^b$$

$$= \left\{ e^{-\frac{(x-y)^2}{2t}} \int dp\, e^{-\frac{p^2}{2}} \left(p + i\frac{x-y}{\sqrt{t}}\right)^b \right\} t^{a - \frac{b}{2} - \frac{n}{2}}$$

Now use what Melrose told me: That $e^{-u^2} u^k$ is bounded for $u \geq 0$. Hence the term in the braces is bounded for all x, y, t with $t \geq 0$.

Yesterday I learned a bit about constructing heat kernels. First of all, h is not the important parameter. Secondly, the calculations all take place in the algebra of polynomials in t, p_μ whose coefficients are polynomials in the various derivatives of V. Thirdly Seeley's method works even though the asymptotic expansion involves some "quadratic" exponent not the actual distance squared. I would now like to go over all these points so in order to make them clearer.

Let's consider a second order operator

$$H = \frac{1}{2} a_{ij}\, p_i p_j + b_j p_j + c$$

where a_{ij}, b_j, c are smooth functions of x. We wish to find $K(t, x, p, h)$ such that

$$(\partial_t + H*)K = 0$$

$$K(0, x, p, h) = 1.$$

(Leave out h from the notation). We seek a K of the form

$$e^{-t\frac{1}{2}a_{\mu\nu}p_\mu p_\nu} \sum c_{a,\alpha}(x)\, t^a p^\alpha$$

where the sum is a formal series. Thus we have to transform the operator $\partial_t + H*$ onto the formal series. This gets to be pretty messy since we have to write out $H*$.

Wait! One can use the Clifford multiplication idea maybe. Under the $*$ multiplication

$$p_j \longrightarrow p_j + \frac{h}{i}\partial_j$$

so that $H*$ operating on $f(x, p)$ is the operator

$$H* = \frac{1}{2} a_{ij}(x)\left(p_i + \frac{h}{i}\partial_i\right)\left(p_j + \frac{h}{i}\partial_j\right) + b_j(x)\left(p_j + \frac{h}{i}\partial_j\right) + c(x)$$

Now it should be easy to conjugate this with $e^{-t\frac{1}{2}a_{ij}p_ip_j}$. ∂_μ gets replaced by

$$e^{+\frac{1}{2}tapp}\,\partial_\mu\, e^{-t\frac{1}{2}app} = \partial_\mu - t\,\frac{1}{2}(\partial_\mu a_{ij})p_ip_j$$

and so

$$e^{\frac{1}{2}tapp}(H*)\,e^{-\frac{1}{2}tapp} = \frac{1}{2} a_{\mu\nu}(x)\left(p_\mu + \frac{h}{i}\partial_\mu - \frac{h}{i}\frac{t}{2}\,\partial_\mu a_{ij}\,p_ip_j\right)\times$$

$$\left(p_\nu + \frac{h}{i}\partial_\nu - \frac{h}{i}\frac{t}{2}\partial_\nu a_{ij}(x)\,p_ip_j\right) + b_\mu(x)\left(\longrightarrow\right) + c(x)$$

I only care about this as a polynomial in t, p_μ. This will be operating on such polynomials. What are the degrees. List

$0,2$: $\frac{1}{2}a_{\mu\nu}p_\mu p_\nu$

p

tp^3

1

tp^2

t^2p^4

Now when we take $e^{+\frac{t}{2}app}(\partial_t + H*)\,e^{-\frac{t}{2}app}$ the term $\frac{1}{2}app$ is killed and we get an operator of the form

$$\partial_t + Q = \partial_t + r\,t^2pppp + s\,tppp + u\,tpp + vp + w$$

where r, s, u, v, w are differential operators in x, ∂_x.

Now one can solve

$$(\partial_t + Q)L = 0$$

$$L(0,x,p) = 1$$

as a power series in t:

$$L(t,x,p) = \sum t^a L_a(x,p)$$

where $L_n(x,p)$ is a polynomial in p with coeffs. which are smooth functions of x

What happens next is we try to convert the formal heat operator

$$K(t,x,p) = e^{-\frac{t}{2}a_{\mu\nu}p_\mu p_\nu} \sum_{a,\alpha} c_{a\alpha}(x)\, t^a p^\alpha$$

just constructed into an actual operator. The idea is to truncate the above series and show the resulting error can be eliminated by a Volterra process. (The other possibility is to sum the above series by a Borel argument.)

In any case we have to have some idea of when a term $t^a p^\alpha$ is going to be small, or just what sort of operator it is. So one needs to associate to $e^{-\frac{t}{2}a_{\mu\nu}p_\mu p_\nu} c(x)\, t^a p^\alpha$ an operator on functions of x, and then look at the kernel of this operator. The kernel is

$$\int \frac{dp^n}{(2\pi h)^n}\, e^{\frac{ip}{h}(x-y)}\, e^{-\frac{t}{2}a_{\mu\nu}(x)p_\mu p_\nu}\, c(x)\, t^a p^\alpha$$

and it is a Gaussian integral; here we assume $a_{\mu\nu}(x)$ is positive-definite.

I did this integral yesterday and found that the kernel is smooth except for $x=y$, $t=0$ and that for all x,y it is bounded by a constant times $t^{a-\frac{|\alpha|}{2}-\frac{n}{2}}$. One can see this by putting absolute values in, whence the $e^{i\frac{p}{h}(x-y)}$ disappears and

One is left with

$$|c(\alpha)| \int \frac{d^n p}{2\pi h^n} e^{-\frac{t}{2} a_{\mu\nu}(x) p_\mu p_\nu} t^a |p|^\alpha$$

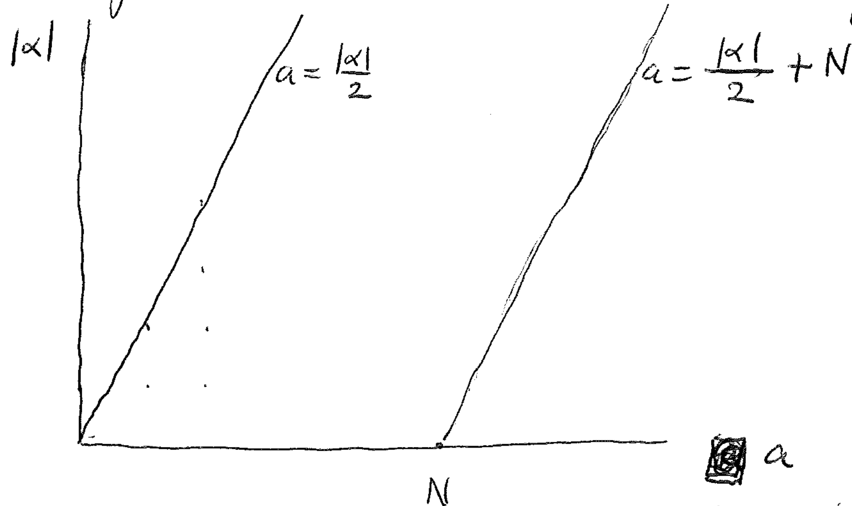
which is homogeneous of degree $\left(a - \frac{|\alpha|}{2} - \frac{n}{2}\right)$ in t.

Thus the singularity of the kernel becomes more benign as $a - \frac{|\alpha|}{2} \longrightarrow \infty$, and so the thing to see about the formal series

$$K(t,x,p) = e^{-\frac{t}{2} a_{\mu\nu} p_\mu p_\nu} \sum c_{a,\alpha}(x)\, t^a p^\alpha$$

is that there are only finitely many terms with $a - \frac{|\alpha|}{2}$ less than a given bound.

Let's plot the monomials in question



What we need to know is that there are only finitely many terms to the left of the line $a = \frac{|\alpha|}{2} + N$

So we look at the equation defining L

$$\begin{cases} (\partial_t + Q)\, L(t,x,p) = 0 \\ L(0,x,p) = 1 \end{cases}$$

write it as an integral equation

$$L(t) = 1 - \int_0^t dt'\, Q(t') L(t') .$$

Thus we want to look at the operator on polynomials of p with smooth coeffs in x given by multiplying

by $\quad Q = r t^2 pppp + s t ppp + u t pp + v p + w$
and then integrating from 0 to t. Call this operator S. Then the five terms of S do the following to the weight $a - \frac{|\alpha|}{2}$ we assign to a monomial $t^a p^\alpha$:

$$t^a p^\alpha \longmapsto \int_0^t r\, t^{a+2}\, ppppp^\alpha = r\, \frac{t^{a+3}}{a+3}\, ppppp^\alpha$$

$$\text{new weight} \quad a + 3 - \frac{|\alpha| + 4}{2} = a - \frac{|\alpha|}{2} + 1$$

$$t^a \alpha \longmapsto s\, \frac{t^{a+2}}{a+2}\, pppp^\alpha$$

$$\text{new weight} = a + 2 - \frac{3 + |\alpha|}{2} = \left(a - \frac{|\alpha|}{2}\right) + \frac{1}{2}$$

$$t^a \alpha \longmapsto u\, \frac{t^{a+2}}{a+2}\, ppp^\alpha$$

$$\text{new weight} = a + 2 - \frac{2 + |\alpha|}{2} = \left(a - \frac{|\alpha|}{2}\right) + 1$$

$$t^a \alpha \longmapsto v\, \frac{t^{a+1}}{a+1}\, pp^\alpha$$

$$\text{new weight} = a + 1 - \frac{1 + |\alpha|}{2} = \left(a - \frac{|\alpha|}{2}\right) + \frac{1}{2}$$

$$t^a \alpha \longmapsto w\, \frac{t^{a+1}}{a+1}\, p^\alpha$$

$$\text{new weight} = a + 1 - \frac{|\alpha|}{2} = \left(a - \frac{|\alpha|}{2}\right) + 1$$

Hence the operator S increases the weight and therefore it is nilpotent modulo the space of monomials of weight $\geq N$ for any N. It is then clear that

$$L = \left(1 + S + S^2 + S^3 + \cdots\right) 1$$

has only finitely many terms of weight $\leq N$ for any N.

Let us now try to describe the argument being presented without using the p-picture. I will work in 1-dimension to keep the notation simple. The heat kernel will be $K(t,x,y)$ and will satisfy

$$(\partial_t + H)K = 0$$
$$K(0,x,y) = \delta(x-y)$$

where $H = -\frac{\hbar^2}{2}a(x)\partial_x^2 + b(x)\frac{\hbar}{i}\partial_x + c(x)$. Before K was a function of t, x, p; denote this function $\hat{K}(t,x,p)$. The relation then is

$$K(t,x,y) = \int \frac{dp}{2\pi\hbar} e^{i\frac{p}{\hbar}(x-y)} \hat{K}(t,x,p).$$

Moreover $\hat{K}(t,x,p) = e^{-\frac{t}{2}a(x)p^2} \sum c_{a\alpha}(x) t^a p^\alpha$

so that upon doing the p integral one obtains a Gaussian expression

$$K(t,x,y) = \frac{e^{-\frac{(x-y)^2}{2\hbar^2 t\, a(x)}}}{\sqrt{2\hbar^2 t a(x)}}\left(1 + \cdots\cdots\right)$$

This reminds me of a point that bothered me about Seeley's construction of the heat kernel, namely instead of the exponent $\frac{(x-y)^2}{2a(x)}$, I expected to see the geodesic distance squared essentially. The key point that I didn't know then is the fact that because

$$e^{-\frac{(x-y)^2}{\alpha t}}\left(\frac{|x-y|}{\sqrt{t}}\right)^k$$

is bounded, a series with powers of t in the denominator can still be a good asymptotic series. Here's an example. Suppose for instance we want to compare

$$e^{-\frac{(x-y)^2}{2t\,a(x)}} \quad \text{and} \quad e^{-\frac{(x-y)^2}{2t\,a(y)}} .$$

Then

$$e^{-\frac{(x-y)^2}{2t\,a(x)}} = e^{-\frac{(x-y)^2}{2t\,a(y)}} \, e^{-\frac{(x-y)^2}{2t}\left[a^{-1}(x) - a^{-1}(y)\right]}$$

Now

$$a^{-1}(x) - a^{-1}(y) = \sum_{n \geq 1} c_n(y)(x-y)^n$$

and

$$e^{-\frac{1}{2t}\sum_{n\geq 1} c_n(y)(x-y)^{n+2}} \quad \text{will be a series}$$

of terms $\varphi_{\ell,m}(y)\dfrac{(x-y)^m}{t^{\ell}}$ where $m \geq 3\ell$

Now recalling that $e^{-\frac{(x-y)^2}{2t\,a(y)}} \dfrac{|x-y|^m}{t^{m/2}}$ is bounded, we see that

$$\varphi_{\ell,m}(y)\frac{(x-y)^m}{t^{\ell}} = O(t^{m/2-\ell})$$

and as $m/2 - \ell \geq \frac{3}{2}\ell - \ell = \frac{1}{2}\ell$, it is clear that we get an asymptotic expansion as $t \to 0$.

So now we do the calculus with y fixed looking for a formal series

$$K(t,x,y) = \frac{e^{-\frac{(x-y)^2}{2\cancel{h^2}t\,a(y)}}}{\sqrt{\cancel{2\pi h^2}t\,a(y)}} \sum c_{a,\alpha}(y)\, t^a (x-y)^\alpha$$

satisfying

$$(\partial_t + H)K = 0$$
$$K(0,x,y) = \delta(x-y).$$

Let me check that it works. Put $y=0$, $a=a(0)$, $h=1$

$$\frac{e^{\frac{x^2}{2at}}}{(2\pi t a)^{-1/2}}(\partial_t+H)\frac{e^{-\frac{x^2}{2at}}}{\sqrt{(2\pi t a)}} = \partial_t + \frac{x^2}{2at^2} - \frac{1}{2t} - \frac{1}{2}a(x)\left(\partial_x - \frac{x}{at}\right)^2 + \frac{1}{i}b(x)\left(\partial_x - \frac{x}{at}\right) + c(x)$$

$$= \partial_t + \frac{x^2}{2at^2} - \frac{1}{2t} - \frac{1}{2}a(x)\left[\partial_x^2 - 2\frac{x}{at}\partial_x + \frac{x^2}{a^2t^2} - \frac{1}{at}\right] + \frac{1}{i}b(x)\left(\partial_x - \frac{x}{at}\right) + c(x)$$

We write this in the form $\partial_t + Q$. We have to see what all the terms of $S = -\int_0^t Q\,dt$ do to a monomial $t^a x^\alpha$. What we are interested in is the weight $a + \frac{|\alpha|}{2}$, I think. It seems I have to worry about a $\log t$?

$$\partial_t + \frac{x^2}{2t^2}(a(x)-a) - \frac{1}{2t}\left(1 - \frac{a(x)}{a}\right) - \frac{1}{2}a(x)\partial_x^2 + \frac{a(x)}{a}\frac{1}{t}x\partial_x + \frac{1}{i}b(x)\partial_x - \frac{1}{i}\frac{b(x)}{a}\frac{x}{t} + c(x)$$

Operators of interest are $\frac{x^3}{t^2}$, $\frac{x}{t}$, ∂_x^2, $\frac{x}{t}\partial_x$, ∂_x

Not clear that this will work.

May 17, 1984

Let's see if I can put together a complete proof of the existence of the heat operator depending on Planck's constant together with the asymptotic evaluation of the trace as $h \to 0$.

I want to work over a $\underbrace{\text{torus}}_{M = \mathbb{R}^n/\Gamma}$ and with operators on smooth functions. Then one has Fourier series

$$\hat{u}(\xi) = \int_M dx\, e^{-i\xi x} u(x) \qquad \xi \in \hat{\Gamma}\cdot 2\pi$$

$$u(x) = \frac{1}{v}\sum_\xi e^{i\xi x}\hat{u}(\xi) \qquad v = \int_M dx$$

and a function $f(x,p)$ can be interpreted as an operator on functions

$$f(x,p) * u(x) = \frac{1}{v}\sum_\xi e^{i\xi x} f(x, h\xi)\, \hat{u}(\xi).$$

(provided this makes sense.) What this means is that we are assigning to $f(x,p)$ the operator with kernel

$$\langle x | f(x,p) | y\rangle = \frac{1}{v}\sum_\xi e^{i\xi(x-y)} f(x, h\xi)$$

The trace of this operator is

$$\operatorname{tr} f(x,p) = \int_M dx \langle x|f(x,p)|x\rangle = \int_M dx \frac{1}{v}\sum_\xi f(x, h\xi)$$

$$= \int_M dx\, \underbrace{\frac{1}{v}\sum_{p \in h\hat{\Gamma}\cdot 2\pi} f(x,p)}_{\sim \int \frac{dp}{(2\pi h)^n} f(x,p) \quad \text{as } h \to 0}$$

Now consider a Hamiltonian

$$H = \frac{1}{2} a_{ij} p_i p_j + b_j p_j + c$$

where a_{ij}, b_j, c are fns. of $x \in M$, and $(a_{ij}) > 0$. We now want to construct e^{-tH} as an operator on functions on M, and evaluate its trace asymptotically as $h \to 0$. e^{-tH} is to be given by a function $K(t,x,p,h)$ satisfying

$$(\partial_t + H) * K = 0$$

$$K(0,x,p,h) = 1.$$

(Notice that this K is defined for all p, not just $p \in 2\pi h \hat{\Gamma}$. So really it is being constructed over $\mathbb{R}^n$ and then pushed down to M.)

Yesterday we found a formal expansion for K:

$$K(t,x,p,h) = e^{-\frac{t}{2} a_{ij}(x) p_i p_j} \sum c_{a,\alpha} t^a p^\alpha$$

where $c_{a,\alpha}(x,h)$ is a polynomial in h with coeffs. smooth functions of M. The series has only finitely many terms with the weight $a - \frac{|\alpha|}{2} \leq$ a given constant.

Furthermore this weighting came from the following: Consider the kernel belonging to a Gaussian polynomial such as $e^{-\frac{t}{2} a_{ij}(x) p_i p_j} t^a p^\alpha$. I will work this out when $a_{ij} = \delta_{ij}$. First we do it on $\mathbb{R}^n$

$$\cancel{\int \frac{dp}{(2\pi h)^n} e^{-\frac{t}{2}p^2}} \qquad \int \frac{d^n \xi}{(2\pi)^n} e^{i\xi(x-y) - \frac{t}{2}h^2\xi^2} t^a (h\xi)^\alpha$$

$$= \int \frac{d^n\xi}{(2\pi)^n (h\sqrt{t})^n} \left(e^{-\frac{\xi^2}{2} + i \frac{\xi}{h\sqrt{t}}(x-y)} \, t^a \left(\frac{\xi}{\sqrt{t}}\right)^\alpha \right)$$

$$= \int \frac{d^n\xi}{(2\pi h)^n} \, e^{-\frac{1}{2}\left(\xi - \frac{i(x-y)}{h\sqrt{t}}\right)^2 - \frac{1}{2}\frac{(x-y)^2}{h^2 t}} \, t^{a - \frac{|\alpha|}{2} - \frac{n}{2}} \, \xi^\alpha$$

$$= e^{-\frac{1}{2}\frac{(x-y)^2}{h^2 t}} \int \frac{d^n\xi}{(2\pi h)^n} \, e^{-\frac{1}{2}\xi^2} \left(\xi + \frac{i(x-y)}{h\sqrt{t}}\right)^\alpha \, t^{a - \frac{|\alpha|}{2} - \frac{n}{2}}$$

This kernel has to be made periodic (replace y by $y+\gamma$ and sum over $\gamma \in \Gamma$).

What you want to conclude is that

$$K_{a,\alpha}(t,x,y) = \sum_{\xi \in 2\pi\hat{\Gamma}} e^{i\xi(x-y) - \frac{t}{2}h^2\xi^2} \, t^a (h\xi)^\alpha$$

is smooth outside $x=y$, $t=0$ and that it is uniformly-in-(x,y) $O(t^{a - \frac{|\alpha|}{2} - \frac{n}{2}})$.

May 19, 1984

After [redacted] all I did [redacted] toward understanding the analysis, I still don't have a very convincing proof. It is still not clear how the [redacted] construction is uniform as $h \to 0$, and how the traces work.

Let's consider again the simplest cases. [redacted] These are $H = p^2 + V(x)$, and motion in a magnetic field. What I really want to do is to find a Connes algebra and to construct e^{-tH} inside this algebra. The way I was lecturing, h is not the essential parameter. This may indicate there is a better approach.

Suppose we [redacted] consider the case $H = p^2 + V(x)$ with V bounded (say even periodic) on $\mathbb{R}^n$. Let's use perturbation theory

$$e^{-tH} = e^{-tp^2} + [redacted] \int_0^t dt_1\, e^{-(t-t_1)p^2}(-V)e^{-t_1 p^2} + \cdots$$

This series will converge. Where? If $h \neq 0$ one can think of this series taking place in the algebra of operators on smooth functions.

[redacted]

Here seems to be a good question. Take a torus $M = \mathbb{R}^n/\Gamma$ and form the crossed product algebra of $\mathcal{S}(\mathbb{R}^n)$ with $\mathcal{S}(M)$: $\mathcal{S}(M) \otimes \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{F}(\mathbb{R})$. This is a topological tensor product, and the last factor represents functions of h. I am not sure what I want the functions of h to be — maybe $\mathcal{F}(\mathbb{R})$ should be replaced by $C^\infty([0,1])$. The question then is to show that the above perturbation series converges in this crossed product algebra.

The idea I want to emphasize is that the above series makes sense in an algebra, but which is not the usual algebra of operators as fns. of M. Notice that if we have a function $f(x, p, h)$, then the corresponding operator [redacted] has the kernel

$$\langle x | f(x,p,h) | y \rangle = \frac{1}{v} \sum_{p \in 2\pi h \hat{\Gamma}} e^{i \frac{p}{h}(x-y)} f(x, p, h)$$

so a function $f(p)$ vanishing on $2\pi h \hat{\Gamma}$ will give the zero operator.

In order to show the perturbation series converges, we want to have an algebra norm such that

$$\| e^{-tp^2} \| \leq M e^{at}$$
$$\| V \| \leq C \quad .$$

Then

$$M e^{at} + \int_0^t dt_1\, M^2 C e^{at} + \int_0^t dt_1 \int_0^{t_1} dt_2\, M^3 C^2 e^{at} + \cdots\cdots$$

$$= M e^{at} e^{MCt}$$

Also notice that if $\| e^{-tp^2} \| \leq M'$ for $0 \leq t \leq \varepsilon$, then clearly its norm grows at most exponentially.

Here **is** what I want to do: I want to produce **an algebra** which is going to contain elements e^{-tp^2}, $t > 0$ and $V(x)$, h and which will have a topology such that **the** above perturbation series converges. The problem is to produce the algebra. The model will be some kind of ΨDO-type symbols; hence we want some kind of functions of x, p, h probably subject to growth conditions which define the topology we want. The second idea perhaps will be the left multiplication trick – ultimately our algebra will appear as an algebra of operators on itself, and hopefully the topology will just be an operator topology.

Let us then realize our operators as left multiplication somehow. The key step will be how to make e^{-tp^2} acts, and the first thing we will have to know is how to write

$$e^{-tp^2} * f(x,p,h)$$

as a function of x, p, h. I will follow the ΨDO model as much as possible. One first F.T.'s the x-variable:

$$e^{-tp^2} * \int \frac{d\xi}{2\pi} e^{i\xi x} \int dy\, e^{-i\xi y} f(y,p,h)$$

not correct as x is in $\mathbb{R}^4/\Gamma$

$$= \int \frac{d\xi}{2\pi} e^{i\xi x} e^{-t(p+h\xi)^2} \int dy\, e^{-i\xi y} f(y,p,h)$$

where I use

$$e^{-tp^2} * e^{i\xi x} = e^{i\xi x} \cdot e^{-t(p+h\xi)^2}$$

Let us consider smooth functions $f(x,p,h)$ for $x \in M = \mathbb{R}^n/\Gamma$, $p \in \mathbb{R}^n$, $h \in [0,1]$. It is clear how $V(x)$ operates on such functions, and so the real question is how e^{-tp^2} acts on these functions. Let's use the Fourier series

$$f(x,p,h) = \frac{1}{v} \sum_{\xi \in 2\pi\hat{\Gamma}} e^{-i\xi x} \hat{f}(\xi,p,h).$$

Here $\hat{f}(\xi,p,h)$ is a smooth function of p, h

depending on the lattice point ξ. The idea is that $\hat{f}(\xi, p, h)$ is rapidly decreasing in ξ and this is equivalent to $f(x,p,h)$ being smooth. Then

$$e^{-tp^2} * e^{i\xi x} = e^{i\xi x}\, e^{-t(p+h\xi)^2}$$

and so by the Fourier series transform, we have that $e^{-tp^2} *$ multiplies $\hat{f}(\xi,p,h)$ by $e^{-t(p+h\xi)^2}$. This, of course, decreases norms, so it should be clear that the operator $e^{-tp^2}*$ operating on $f(x,p,h)$ is bounded independently of t. (Note that h has to be real, more accurately $\mathrm{Re}(h^2) > 0$.)

Let's recapitulate and make the whole thing a bit more precise. I need still to find a careful definition of the functions $f(x,p,h)$ to be considered. The basic idea is to just have those which are needed to construct e^{-tH}.

Let's begin again: F. T. formulas

$$u(x) = \frac{1}{v} \sum_{\xi \in 2\pi\hat{\Gamma}} e^{-i\xi x}\, \hat{u}(\xi)$$

$$\hat{u}(\xi) = \int_M dx\, e^{-i\xi x}\, u(x)$$

From this follows that u is $C^\infty \iff \hat{u}$ is rapidly decreasing i.e. $\hat{u}(\xi)\,|\xi|^k \longrightarrow 0$ as $|\xi| \longrightarrow \infty$ for any k. This holds for $u(x)$ having values in any top v.s. defined by a family of norms.

We want to apply this to smooth functions

$f(x,p,h)$ which we can think of as a smooth map from $x \in M$ to smooth functions of p, h. Then f is smooth $\iff \xi \mapsto \hat{f}(\xi, p, h)$ is rapidly decreasing in the space of smooth functions of p, h.

I think I want $f(x,p,h)$ to belong to the Schwartz space $\mathcal{S}$ as a fn. of p. This means for fixed α, β

$$p^\alpha \frac{\partial^\beta f}{\partial p^\beta} \longrightarrow 0 \qquad \text{uniformly in } x, h \text{ as } |p| \to \infty.$$

OK. At this point I more or less have the norms on the $f(x,p,h)$ under control:

$$(|p|^2)^\ell \frac{\partial}{\partial h^k} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial p^\beta} f(x,p,h) \qquad \text{take sup of this summed over } \ell, k, |\alpha|, |\beta| \leq \text{const.}$$

and if we fourier transform h it becomes

$$(|\xi|^2)^{\ell'} (|p|^2)^\ell \frac{\partial}{\partial h^k} \frac{\partial}{\partial p^\beta} \hat{f}(\xi, p, h) \qquad \text{take sup here summed over } \ell', \ell, k, \beta \leq \text{const.}$$

Next we define operators on this space. Multiplication by a smooth function $V(x)$ is clear. The operator $e^{-tp^2}{}_*$ sends $\hat{f}(\xi,p,h)$ to $e^{-t(p+h\xi)^2}\hat{f}(\xi,p,h)$. Now we want to see this is bounded as $t \to 0$, so we have to apply $\frac{\partial^\ell}{\partial h^k}\frac{\partial^\beta}{\partial p^\beta}$ and so to $\hat{f}$ we are applying

$$e^{-t(p+h\xi)^2}\left(\frac{\partial}{\partial h} + (-t)2(p+h\xi)\xi\right)^k \left(\frac{\partial}{\partial p} + (-t)\,2(p+h\xi)p\right)^\beta.$$

So what one has brought down from the exponent is a polynomial in $p, \xi, h, \frac{\partial}{\partial h}, \frac{\partial}{\partial p}$?

May 19, 1984

Recall the program: I work over a torus $M = \mathbb{R}^n/\Gamma$ with certain functions of x, p, h which are smooth; here $x \in M$, $p \in \mathbb{R}^n$, $h \in [0,1]$, and the functions $f(x,p,h)$ should be rapidly decreasing in p. The problem is to describe the topology on this space by a sequence of norms and then show that for any of these norms the operator $e^{-tp^2}*$ is bounded uniformly in t for $t \geq 0$.

First we keep h fixed to keep things simple. By using the F.T. on the torus we transform from $f(x,p,h)$ to $\hat{f}(\xi,p,h)$, where $\xi \in 2\pi\hat{\Gamma}$, and here the operator e^{-tp^2} is

$$\hat{f}(\xi,p,h) \longmapsto e^{-t(p+h\xi)^2}\hat{f}(\xi,p,h)$$

Now what norms are used to describe the topology of these fns $\hat{f}(\xi,p)$? These are rapidly decreasing in both ξ and p, which means

$$|\xi|^k\, p^\alpha \frac{\partial^\beta}{\partial p^\beta}\hat{f}(\xi,p) \longrightarrow 0$$

as $|\xi|^2 + |p|^2 \longrightarrow \infty$. The way this is usually ~~stated~~ is in terms of something like.

$$\left|\partial_x^k\, \partial_p^l f(x,p)\right| \leq C_{k,l}\,(1+|p|)^{d-l}$$

At this point I am beginning to think that the usual symbol classes give the norms I must use. In other words lets look first at constant coeff operators and get the norms straight.

So we consider smooth functions $f(p)$ which are rapidly decreasing. This means that

$$p^\alpha \partial_p^\beta f(p)$$

is bounded for any α, β. Normally one thinks of functions $f(p)$ as giving rise to operators

$$f(p) * u(x) = \frac{1}{v}\sum_{\xi} e^{i\xi x} f(h\xi)\,\hat{u}(\xi)$$

with h fixed. The k-norm in the Sobolev sense for $u(x)$ is

$$\|u\|_k^2 = \frac{1}{v}\sum_{\xi} (1+|\xi|^2)^k |\hat{u}(\xi)|^2$$

so what is interesting is what sort of Sobolev norm shifting is caused by f. If

$$|f(h\xi)| \leq C|\xi|^d$$

then $\|f(p) * u(x)\|_k \leq C\|u\|_{k+d}$.

Thus one has a primary interest in the "degree" of $f(p)$ as $|p| \to \infty$, where degree means the order of growth. One says that f is a symbol of order d if for any β

$$|\partial^\beta f(p)| \leq C_\beta (1+|p|)^{d-|\beta|}$$

This implies

$$|p^\alpha \partial^\beta f(p)| \leq C_\beta (1+|p|)^{d+|\alpha|-|\beta|}$$

which doesn't mean much more.

I still haven't found the sequence of norms I want. What I have learned is that as an operator I am mainly interested in the size of $f(p)$

Start again: First take $\hbar = 1$. I consider the space of smooth functions $f(x,p)$ on $\mathbb{R}^n/\Gamma \times \mathbb{R}^n$ such that $\forall a,b,c$, $\exists C_{abc} \ni \forall x,p$

$$\left| p^b \partial_x^a \partial_p^c f(x,p) \right| \le C_{abc}$$

Now I propose to define an operator $e^{-tp^2}*$ on this space $\mathcal{F}$ and prove that the perturbation series for $e^{-t(p^2+V(x))}$ converges.

Use Fourier transform

$$f(x,p) = \frac{1}{v} \sum_{\xi} e^{i\xi x} \hat{f}(\xi,p)$$

$$e^{-tp^2} * f(x,p) = \frac{1}{v} \sum_{\xi} e^{i\xi x} e^{-t(p+\xi)^2} \hat{f}(\xi,p)$$

Thus I first have to describe the space of $f(x,p)$ in terms of the $\hat{f}(\xi,p)$. The $\hat{f}(\xi,p)$ consist of sequences of smooth functions of p indexed by $\xi \in 2\pi\hat{\Gamma}$ such that

$$\left| |\xi|^a p^b \partial_p^c \hat{f}(\xi,p) \right| \le C_{a,b,c}$$

Now I want to show bddness of $e^{-t(p+\xi)^2}$.

$$\partial_p^c \, e^{-t(p+\xi)^2} \hat{f}(\xi,p)$$

$$= e^{-t(p+\xi)^2} \left[\partial_p - 2t(p+\xi) \right]^c \hat{f}(\xi,p)$$

Better, let's use Leibnitz fmla.

$$\partial_p^c \left\{ e^{-t(p+\xi)^2} \hat{f}(\xi,p) \right\} = \sum_{k+l=c} \frac{c!}{k!\,l!} \partial_p^k \left(e^{-t(p+\xi)^2} \right) \partial_p^l \hat{f}(\xi,p)$$

It seems that the first thing I have to understand is why is multiplication by e^{-tp^2} a bounded operator on the Schwartz space of $f(p)$. It is clearly bounded with respect to any of the norms

$$\sup \; (1+|p|)^k f(p),$$

so we have to ask if it is bounded with respect to $\sup|\partial_p f|$. Now

$$\partial_p\{e^{-tp^2}f(p)\} = e^{-tp^2}(-2tp)f(p) + e^{-tp^2}\partial_p f(p)$$

so that

$$\sup\left|\partial_p\{e^{-tp^2}f(p)\}\right| \leq \sup\left|e^{-tp^2}(-2tp)f(p)\right| + \sup\left|e^{-tp^2}\partial_p f(p)\right|$$

Since we want estimates independent of t, the best we can do in the second term is $\sup|\partial_p f|$. There are two ways of handling the first term. On one hand we can argue that

$$e^{-tp^2}\,t^{1/2}|p| \quad \text{is bounded}$$

whence we get

$$\sup\left|\partial_p\{e^{-tp^2}f(p)\}\right| \leq C\left(\sup|f| + \sup|\partial_p f|\right)$$

hence e^{-tp^2} is bounded wrt $\|f\| = \sup|f| + \sup|\partial_p f|$.

Or we can argue

$$\sup\left|\partial_p\{e^{-tp^2}f(p)\}\right| \leq 2t\sup|pf(p)| + \sup|\partial_p f|$$

and so e^{-tp^2} is bounded wrt $\sup|pf(p)| + \sup|\partial_p f|$.

Discussion: I have been trying to construct $e^{-t(p^2+V(x))}$ by using the perturbation expansion relative to e^{-tp^2}. Equivalently this is the Volterra expansion associated to the approximate fundamental solution

$$(\partial_t + H)\, e^{-tH_0} = (H-H_0)e^{-tH_0}$$

so the problem is to show that $(H-H_0)e^{-tH_0}$ is bounded independent of t for $0<t<\varepsilon$. In the present case where $H-H_0$ = multiplication by $V(x)$, this operator is bounded, so it suffices to see e^{-tH_0} is bounded independent of t. The only way I can think of proving such boundedness is to find norms such that the operator is bounded relative to each individual norm.

It occurs to me that this is too hard especially since the Fourier transform mucks up the sup norms. So it would appear then that the parametrix method has some intrinsic advantages.

So let us go over the parametrix method. What this means is that we construct an approximate fundamental solution for $\partial_t + H$:

$$(\partial_t + H)\, E(t) = \delta(t) + \text{nice error}$$
$$E(t) = 0 \qquad t<0$$

In fact we should think of $E(t-t')$ as a kernel $K(xt, x't')$ on $M\times\mathbb{R}$. This kernel is the end result of the ψDO calculation. Now the question is why can the Volterra method be applied. Suppose you just know that the nice error is an $L(t,x,x')$ which is completely smooth and vanishes for $t<0$. Why can I solve the Volterra integral equation

$$u(x,t) - \int dt'\,dx'\, L(t-t', x, x')\, u(x',t') = f(x,t)$$

uniquely given $f=0$ for $t<0$ and that u is required to vanish for $t<0$.

One answer is that the Fredholm theory applies: One knows the resolvent $(1-\lambda L)^{-1}$ is a ratio of two entire functions of λ, the denominator being $\det(1-\lambda L) = \sum (-1)^k \lambda^k \operatorname{tr}(\Lambda^k L)$. Finally the form of the kernel $K(xt, x't') = L(t-t', x, x') = 0$ for $t \leq t'$, shows that this determinant is 1.

I somehow feel that it should be possible to get a version of this argument which is slightly more elementary. After all the Fredholm formulas lead to the same exponential bounds. (Recall that the convergence of the series for $\det(1-\lambda L)$ results from the Hadamard inequality for determinants + the $\frac{1}{k!}$ that results when one tries to evaluate $\operatorname{tr}(\Lambda^k L)$ as an integral of $\det\left(K(x_i, x_j)\right)$.)

May 20, 1984

Let's review the situation. The goal is to obtain a really clean and clear proof of the existence of e^{-tH} where $H = p^2 + V(x)$. The method is to find a forward parametrix for $\partial_t + H$, i.e. a $E(t)$ satisfying

$$(\partial_t + H)E(t) = \delta(t) - K(t)$$
$$E(t) = 0 \qquad t < 0$$

where $K(t) = K(t,x,x')$ is nice enough so that the Volterra integral equation

$$u(t,x) - \int K(t-t', x, x')\, u(t',x')\, dt'dx' = f(t,x)$$

can be solved on the space of $f(t,x)$ vanishing for $t<0$.

It therefore appears necessary to understand [redacted] why Volterra equations can be solved. The parametrix idea, where one constructs an inverse modulo smoothing operators, fits naturally with the Fredholm theory. The Fredholm theory attaches to a trace class operator K in a general sense a [redacted] determinant function $\det(1-\lambda K)$ which is an entire function of λ, and a cofactor operator $C(1-\lambda K)$, also entire, such that

$$(1-\lambda K)\,C(1-\lambda K) = \det(1-\lambda K)\cdot Id.$$

When K is a Volterra kernel, the operator is essentially upper triangular and so one should be able to see directly that $\det(1-\lambda K) = 1$. Hence it follows that $(1-\lambda K)^{-1}$ is an entire function of λ.

We now will [redacted] review the Fredholm formulas. What is interesting is the general convergence techniques, which somehow are related to Hadamard's determinant inequality.

Let's suppose we have a trace class operator represented by a kernel $K(x,x')$; here x, x' are indices which we think of as real variables. Then

$$\det(1-\lambda K) = 1 - \lambda\, \mathrm{tr}(K) + \lambda^2 \mathrm{tr}(\Lambda^2 K) - \cdots$$

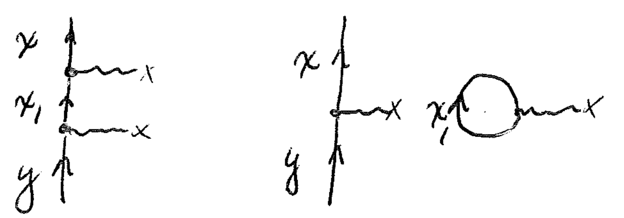
where
$$\mathrm{tr}(\Lambda^n K) = \frac{1}{n!}\int dx_1 \cdots dx_n \begin{vmatrix} K(x_1,x_1) & \cdots & K(x_1,x_n) \\ \vdots & & \vdots \\ K(x_n,x_1) & \cdots & K(x_n,x_n) \end{vmatrix}$$

Now suppose we have a partial ordering such that $K(x,x') \neq 0 \Rightarrow$ '$x > x'$ (and $\neq x'$). Then the above trace is zero because each $\det K(x_i,x_j)$ is zero. (Proof. Expand the determinant over the permutations, and write a permutation as a product of cycles. Each cycle product gives zero.) Recall

$$C(1-\lambda K) = \sum_{n \geq 0} (-\lambda)^n C_{n+1}$$

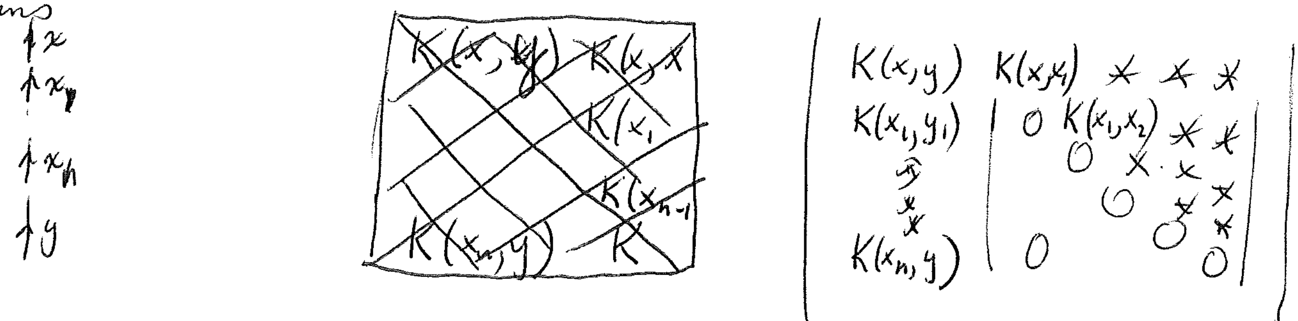
where
$$C_{n+1}(x,y) = \frac{1}{n!}\int dx_1 \cdots dx_n \begin{vmatrix} K(x,y) & K(x,x_1) & \cdots & K(x,x_n) \\ K(x_1,y) & & & \vdots \\ & & & \vdots \\ K(x_n,y) & \cdots & & K(x_n,x_n) \end{vmatrix}$$

(This comes from diagrams with two external lines labelled x, y: 2nd order diagram



$$K(x,x_1)K(x_1,y) - K(x,y)K(x_1,x_1)$$
$$= (-1)\begin{vmatrix} K(x,y) & K(x,x_1) \\ K(x_1,y) & K(x_1,x_1) \end{vmatrix} \quad)$$

Now let's assume $K(x,x') \neq 0 \Rightarrow x > x'$ and see what contributes to $C_{n+1}(x,y)$. Again break the determinant down as a sum of permutations. The cycles give zero, so the contributions are the diagrams



and similar ones with the x_i's in different order. Thus

$$C_{n+1}(x,y) = \pm \int_{x > x_1 > x_2 > \cdots > x_n > y} dx_1 \cdots dx_n \quad K(x,x_1)K(x_1,x_2)\cdots\cdots K(x_n,y)$$

and we are getting out the geometric series for $(1-\lambda K)^{-1}$.

What is interesting now is the convergence proof. Thus in the Fredholm theory is a convergence proof which I want to understand, which shows quite generally that $\operatorname{tr}(\Lambda^n K)$ is of the order of $\frac{1}{n!}$ as $n \to \infty$.

It seems that the correct point is that if K is of size M, then $K^{\otimes n}$ is of size M^n, and $\Lambda^n K$ is of size $\frac{M^n}{n!}$.

Let's now consider integral equations on the line, or perhaps better, on a large circle, of convolution type

$$u(x) - \lambda \int dx' \, K(x-x') u(x') = f(x)$$

What exactly does the Fredholm theory mean in this case?

Let's proceed formally. The natural way to describe convolution operators is to use the

Fourier transform:

$$\hat{u}(\xi) - \lambda \hat{K}(\xi)\hat{u}(\xi) = \hat{f}(\xi)$$

where $\hat{u}(\xi) = \int e^{-i\xi x} u(x)\,dx$ and ξ runs over a lattice in the case of the circle. [redacted]

The spectrum of $\hat{K}$ will be continuous in the case of the line so the Fredholm method can't work, it seems, since K is not of trace class.

On a finite interval, or rather circle, the determinant will be

$$\prod_{\xi} \left(1 - \lambda \hat{K}(\xi)\right)$$

and for this to converge, one needs to know that $\hat{K}(\xi) \longrightarrow 0$ sufficiently fast.

If we are working on the circle, then for $\hat{K}(\xi)$ to be an L^1 sequence implies $K(x)$ is continuous, and we run into Wiener's theory of continuous fns. with absolutely convergent Fourier series.

May 21, 1984

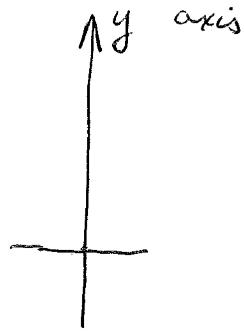
Review the local index theorem for families of Dirac operators. Suppose that X/Y is a differentiable fibre bundle whose fibres are compact Riemannian manifolds. Choose a connection in X/Y, that is, at each $x \in X$ we are given a "horizontal" subspace transversal to the fibre. Then we have the concept of parallel transport in X over curves in Y. Given a curve in Y starting from the image of x, there is a unique horizontal curve over it starting at x. (In fact we pull back X over this curve, and so obtain a fibre bundle over $\mathbb{R}$. The connection gives a vector field lying over $\frac{d}{dt}$ and the integral curves of the vector field give the required trivialization.)

In other words a connection in X/Y associates to each curve in Y a trivialization of X pulled back over this curve, hence a diffeomorphism between the fibres over two pts of the curve. This diffeom doesn't necessarily preserve the metric, so we don't immediately obtain a way of transporting tangent vectors on the fibres preserving the metric.

However suppose Y is equipped with a metric and we equip the horizontal tangent spaces in X with metrics so that the projection is isometric. Then X gets a metric. Now take a curve in Y and lift it horizontally to X and consider parallel transport of tangent vectors in X along this curve. Better pull back X over the curve and suppose Y is $\mathbb{R}$. The horizontal curves in X are then geodesics, because we are assuming the metric is $dt^2 +$ vertical metric. But parallel transport along a geodesic preserves the normal spaces to the curve and it is isometric. Hence we see that along any horizontal curve in X we have a way of transporting the fibre tangent vectors which preserves the metric.

Ex. Suppose X is a vector bundle over Y with both a connection and a metric. Then there seems to be an associated unitary connection to the given connection and metric. ~~Namely, make X a [illegible] [illegible] the connection trivializes the bundle~~ Namely we equip the vector bundle with a metric agreeing with what is given on the fibre, a given metric on Y, and such that horizontal & vertical are perpendicular. Then parallel transport normal to the zero section for the LC connection gives a metric preserving connection.

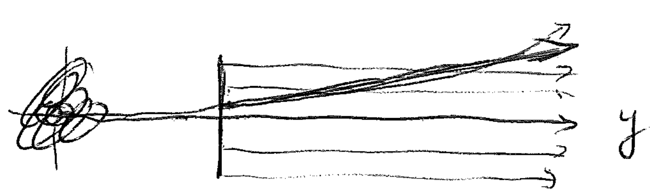
UHP.



$$ds^2 = \frac{dx^2+dy^2}{y^2}$$

Think of the UHP as a line bundle over the y-axis which is trivial with fibre coord. x.

Rotate



Horizontal lines are flat relative to connection

Parallel transport relative to the metric are straight lines which must expand.

May 22, 1984

Proof of the index theorem in the case $M = \mathbb{R}^n/\Gamma$.

$$\not{D} = h\gamma^\mu D_\mu \qquad \text{acting on} \qquad \Gamma(S\otimes E)$$

$$\begin{array}{ccccc} \not{D}^2 = & h^2 D_\mu^2 & + & \frac{1}{2} h\gamma^\mu h\gamma^\nu F_{\mu\nu} & \\ & \downarrow & & \downarrow & \text{as } h\to 0 \\ & -p_\mu^2 & + & \frac{1}{2} dx^\mu dx^\nu F_{\mu\nu} & \end{array}$$

$$\text{Index} = \operatorname{tr} e^{\not{D}^2} \underset{\text{as } h\to 0}{\sim} \int_{T^*M} \frac{d^nx\, d^np}{(2\pi h)^n} \operatorname{tr}_E\left(e^{-p_\mu^2 + F}\right)_{(n)} (2ih^2)^{n/2}$$

$$= \int_M \cancel{d^nx} \left(\frac{i}{2\pi}\right)^{n/2} \operatorname{tr}_E\left(e^F\right)_{(n)}$$

(One point which isn't convincing is [redacted] the way the symplectic volume is split up.)

Recall the structure of the associated graded algebra for the algebra of differential operators on $S\otimes E$. The graded algebra is generated by $\varphi \in \Gamma(\operatorname{End} E)$, $X \in \Gamma(T)$, $\omega \in \Gamma(T^*)$. The ω comes from the Clifford algebra $\underline{C(T^*)}$ action, specifically ω is identified with $\overline{c(h\omega)}$. Similarly X is identified with $\overline{(-i)h\nabla_X}$.

Now what about the trace?

First suppose there is zero curvature. Then the associated graded algebra is the algebra of polynomial functions on T^* with values in $\operatorname{End} E$ valued forms, i.e. $\Gamma(S(T)\otimes \operatorname{End} E \otimes \Lambda T^*)$. I can identify it with the algebra of horizontal forms on T^* which have polynomial dependence along each cotangent space.

Now the trace we are after [redacted] sees only the highest degree part of ΛT^*, hence at $x \in M$, it produces on [redacted] the cotangent space T^*_x a function with values in $\mathrm{End}(E_x) \otimes \Lambda^n T^*_x$. Taking E_x-trace we get a function on [redacted] T^*_x with values in $\Lambda^n T^*_x$. I want ultimately an [redacted] n-form on X, but there is no natural arrangement for integrating functions on T^*_x; one needs a choice of Haar measure on T^*_x.

So what seems clear is that the metric on T^* must survive somewhere in the $h \to 0$ limit.

Let's review the formulas for the super trace on the spinors. Let T^* be a [redacted] vector space with quadratic form; suppose T^* of even dimension $n = 2m$. Then we form $C(T^*)$ and let it act on ΛT^* by assigning to $v \in T^*$, the operator

$$e(v) + i(\lambda_v) \qquad\qquad \text{Put } V = T^*$$

where $\lambda : V \to V^*$ satisfies $\lambda_v(v) = Q(v)$. The symmetric [redacted] choice for λ_v is $\lambda_v(w) = \frac{1}{2}\left[Q(v+w) - Q(v) - Q(w)\right]$. By acting on $1 \in \Lambda(V)$ we get an isomorphism

$$C(V) \xrightarrow{\sim} \Lambda(V)$$

of vector spaces.

When the quadratic form Q on V is non-degenerate, $C(V)$ is [redacted] a simple algebra. In fact we have an algebra isomorphism

$$C(V) = \mathrm{End}(\Lambda W)$$

when V is isomorphic to the hyperbolic space $W \oplus W^*$, and this can be reached by base extension. Thus $C(V)$ has a (reduced) trace which is intrinsically defined. It is determined up to a scalar by requiring it to vanish on commutators. Direct calculation in a suitable case shows that what corresponds to $\Lambda^+(V)$ in $C(V)$ is the commutator space. And on 1 the trace is 2^m.

Thus one sees that the reduced trace on $C(V)$ has a picture in $\Lambda(V)$ which is independent of the quadratic form.

Next consider the super-trace. This depends upon the involution ε defining the grading on the spinors. In fact $tr_s(\alpha) = tr(\varepsilon\alpha)$. If $\gamma^1 \ldots \gamma^n$ [redacted] $\in C(V)$ are the generators belonging to an orthonormal basis, then

$$(\gamma^1 \ldots \gamma^n)^2 = (-1)^{\frac{n(n-1)}{2}} = (-1)^m$$

so that [redacted]

$$\varepsilon = i^{-m}\gamma^1 \ldots \gamma^n$$

[redacted] is an involution. (I use this convention so that $\gamma^1, \gamma^2, \varepsilon$ are the three Pauli matrices in order.) It follows that

$$tr_s(\gamma^1 \ldots \gamma^n) = tr(\varepsilon\gamma^1 \ldots \gamma^n) = (2i)^m$$

where $\gamma^1, \ldots, \gamma^n$ are an orthonormal basis of V with the 'correct' orientation.

Now what I want to do is to consider the Clifford algebra $C(V, h^2Q)$. I can identify this with $C(V,Q)$ by letting $\underset{h}{c}(v) = h c(v)$. So we have

$$\begin{tikzcd} C(V,Q) \arrow[r, "\simeq"] \arrow[d, "s"] & C(V, h^2Q) \arrow[d, "s"] \\ \Lambda V & \Lambda V \end{tikzcd} \qquad \begin{tikzcd} \gamma^{i_1} \ldots \gamma^{i_p} \arrow[r, mapsto] \arrow[d, mapsto] & h^p \gamma^{i_1} \ldots \gamma^{i_p} \arrow[d, mapsto] \\ e^{i_1} \ldots e^{i_p} & h^p e^{i_1} \ldots e^{i_p} \end{tikzcd}$$

This isn't very clear.

You want to start with the algebra $\tilde{C}$ over $k[h]$, generated by the hv in $C(V)\otimes k[h]$, and which specializes at $h=0$ to $\tilde{C}/h\tilde{C} = \operatorname{gr} C(V) = \Lambda V$. Inside this algebra will live $e^{\frac{1}{2}h\gamma^\mu h\gamma^\nu F_{\mu\nu}}$, and it will specialize to $e^{\frac{1}{2}\omega^\mu\omega^\nu F_{\mu\nu}}$, where $\omega_\mu = \overline{h\gamma^\mu}$. Now the algebra $\tilde{C}$ has a specialization $\simeq$ to $C(V)$ for each $h_0 \neq 0$, namely $hv \mapsto h_0 v$. This defines a "s-trace at h_0" function on $\tilde{C}$. These functions at different h_0 values fit together

$$\begin{tikzcd}
\tilde{C} \arrow[r, phantom, "\subset"] \arrow[d, "tr_s \text{ at } h_0"] & C(V)\otimes k[h] \arrow[d, "tr_s\otimes 1"] \\
k & k[h] \arrow[l, "h_0 \leftarrow\!\shortmid\, h"]
\end{tikzcd}$$

Thus we get

$$\begin{tikzcd}
\tilde{C} \arrow[r, phantom, "\subset"] \arrow[d, "s"] & C(V)\otimes k[h] \arrow[d, "s"] \arrow[ddr, "tr_s\otimes 1"] & \\
\tilde{\Lambda} = \bigoplus_{p=0}^{n} \Lambda^p(V)\, h^p k[h] \arrow[r, phantom, "\subset"] \arrow[d] & \Lambda(V)\otimes k[h] \arrow[d] & \\
\Lambda^n(V)\, h^n k[h] \arrow[r, phantom, "\subset"] & \Lambda^n(V)\otimes k[h] \arrow[r, "\text{volume elt in } \Lambda^n V"'] & k[h]
\end{tikzcd}$$

?

May 24, 1984

I have decided to become familiar with the sort of operators which occur in Kasparov K-theory, namely, invertible operators and projections, rather than Dirac operators. The operators motivate Connes' approach to cyclic theory.

Let's consider the Dirac operator on the line, or circle, or plane, or torus as giving the simplest description of the basic K-homology classes. We then have to establish the link between these operators and the operators used to describe these K-classes in the Kasparov theory.

~~For example, we want to take the Dirac on the circle d_x and an invertible matrix function f on the circle and link the following two operators obtained from f. The first uses f to construct a family of connections parametrized by S^1 on the trivial bundle. The Dirac operator with coefficients in this family~~

For example, let f be an invertible matrix function over S^1. Take the Dirac operator $\frac{1}{i}\partial_x$ on S^1 transform it by f and join by the linear path between these operators. One obtains a family of Dirac operators on the circle parametrized by the circle, whence a 2-dim'l Dirac operator on the torus. The index should be $\pm \deg(f)$.

On the other hand we can consider the Toeplitz operator belonging to f defined relative to the Hardy space of functions holomorphic in the disk. This operator also has index $= \pm \deg(f)$.

A first problem is to link these operators.

Consider a Dirac operator $D = d + A$ on the circle. We know that on the Lie algebra of infinitesimal gauge transformations is a 2-cocycle given by

$$X, Y \longmapsto \int tr(X\, DY).$$

I would like to believe that **this** 2-cocycles is attached to the Dirac operator by some process.

How is the fundamental K-homology class of the circle represented in the Kasparov theory?

My first idea was to think of functions on the circle as **0-degree** pseudo-differential operators relative to the Weyl calculus. Then one obtains a BDF extension

$$0 \longrightarrow \Psi^{-1} \longrightarrow \Psi^{0} \longrightarrow C(S^1) \longrightarrow 0$$

I reviewed what Connes told me about Kasparov's achievement: He constructs a positive lifting, which by GNS corresponds to a $C(S^1)$ action on a larger Hilbert space, and then the lifted action of $C(S^1)$ is a compression. Thus one obtains an idempotent operator e on a $C(S^1)$ Hilbert module such that $[f, e]$ is compact.

I concluded that the Kasparov way to think of a Dirac operator on S^1 is in terms of the projection onto its negative eigenspace. All these projections are congruent mod compacts, so one gets the same K-homology class.

So at this point I have to understand what Connes' odd theory amounts to. The first point is the standard cocycles attached to an idempotent. Let's review this.

The space of all idempotents is to be thought of as a Grassmannian, and the curvature of the Grassmannian connection is $ede de$. The n-th Chern character form is

$$tr \frac{1}{n!}(ededе)^n = \frac{1}{n!} tr\left(e(de)^{2n}\right).$$

Rewrite this in terms of the involution $F = 2e-1$, and one gets

$$\frac{1}{n!}\frac{1}{2^{2n+1}} tr\left(F(dF)^{2n}\right)$$

since $tr(de)^{2n} = 0$. So one gets the cyclic $(2n-1)$-cocycle

$$\varphi(a_0,\dots,a_{2n-1}) = \frac{1}{n!}\frac{1}{2^{2n+1}} tr\left(F[F,a_0]\cdots[F,a_{2n-1}]\right).$$

At this point I want to handle the ψDO extension pretending that it somehow comes from an idempotent.