January 1, 1984

Let $E = E^+ \oplus E^-$ be a $\mathbb{Z}_2$-graded vector bundle and let $\Omega(M,E) = \bigoplus_p \left[\Omega^p(M,E^+) + \Omega^p(M,E^-)\right]$ be $\mathbb{Z}_2$-graded in the obvious way. Let us define a connection (better: super-connection) in E to be an operator $\tilde{D}$ on $\Omega(M,E)$ of odd degree such that

$$\tilde{D}(\omega\alpha) = d\omega\cdot\alpha + (-1)^{\deg\omega}\,\omega\cdot\tilde{D}\alpha$$

for $\omega \in \Omega^{\pm}(M)$ and $\alpha \in \Omega$[redacted](M,E). An example of such a $\tilde{D}$ is furnished by a connection in E preserving the grading. It is clear that the set of super-connections is a torsor under the space of odd endomorphisms of $\Omega(M,E)$ over $\Omega(M)$, i.e. maps $L : \Omega(M,E) \circlearrowleft$ of odd degree satisfying

$$L(\omega\alpha) = (-1)^{\deg\omega}\,\omega\cdot L\alpha \qquad \begin{array}{l}\alpha \in \Omega(M,E)\\ \omega \in \Omega^{\pm}(M).\end{array}$$

What are such L? As $\Omega(M,E) = \Omega(M) \otimes_{\Omega^0(M)} \Omega^0(E)$, it follows that an odd degree endo of $\Omega(M,E)$ is given by a vector bundle morphism

$$\Omega^0(E) \longrightarrow \Omega(M,E) \qquad\qquad E \longrightarrow \Lambda T^* \otimes E$$

of odd degree, i.e. $\Omega^0(E^+)$ goes to $\Omega(M,E)^- = \Omega^{\text{even}}(M,E^-) \oplus \Omega^{\text{odd}}(M,E^+)$, and similarly for E^-. Now if one puts on a natural filtration condition to the effect that [redacted] $\tilde{D} = D_0 + L$ should be of first order, this restricts L to two pieces

$$E^+ \longrightarrow \text{[redacted]}\; E^- \oplus T^* \otimes E^+$$
$$E^- \longrightarrow E^+ \oplus T^* \otimes E^-$$

The components $E^{\pm} \longrightarrow T^{*} \otimes E^{\mp}$ can be combined with the connection D_0 to get a connection D on E preserving the grading. Thus our super-connection is of the form

$$\tilde{D} = D + L$$

where D is a connection and L is an odd-degree endomorphism of E extended to $\Omega(M,E)$ by the rule

$$L(\omega\alpha) = (-1)^{\deg\omega}\omega L(\alpha) \qquad \omega \in \Omega^{\pm}(M)$$

Let's now compute the curvature:

$$\tilde{D}^2 = D^2 + DL + LD + L^2$$

Each of these terms ~~is~~ $\Omega(M)$-linear:

$$(DL + LD)(\omega\alpha) = D\left[(-1)^{\deg\omega}\omega L\alpha\right] + L\left[d\omega.\alpha + (-1)^{\deg\omega}\omega D\alpha\right]$$

$$= \cancel{(-1)^{\deg\omega} d\omega L\alpha} + \omega DL\alpha + \cancel{(-1)^{\deg\omega+1} d\omega L\alpha}$$
$$+ \cancel{(-1)^{\deg\omega}}\omega LD\alpha$$

$$= \omega(DL + LD)\alpha$$

Suppose the bundles E^{+}, E^{-} are trivial whence

$$\tilde{D} = d + \tilde{A}$$

where $\tilde{A} : \Omega(M,E) \longrightarrow \Omega(M,E)$ is an odd degree $\Omega(M)$-linear map. I want to think of elements of $\Omega(M,E)$ as vector forms whence $\tilde{A}$ is given by a matrix of the form

$$\tilde{A} = A + L = \left(\begin{array}{c|c} A^{+} & L'' \\ \hline L' & A^{-} \end{array}\right)$$

where $A^{\pm}$ are matrices of 1-forms and L is a matrix of functions. We have to be careful that when we interpret L as an operator on vector forms we put in the signs so that

$$L(\omega\alpha) = (-1)^{\deg\omega} L\alpha .$$

Now let's calculate the curvature.

$$DL + LD = (d+A)L + L(d+A)$$
$$= d(L) + (AL + LA)$$

~~Notice that if α_i is a vector function, then~~

$$\cancel{(A\alpha)_i = A_{ij}\alpha_j}$$

~~is a vector 1-form so~~

$$\cancel{[L(A\alpha)]_i = L_{ij}(A\alpha)_j}$$

$$\cancel{=}$$

Now $A = dx^\mu A_\mu$ and so

$$AL + LA = dx^\mu A_\mu L + L\,dx^\mu A_\mu$$
$$= dx^\mu (A_\mu L - L A_\mu)$$

Thus

$$DL + LD = dx^\mu \left(\partial_\mu L + [A_\mu, L]\right)$$
$$= dx^\mu [D_\mu, L]$$

Perhaps here is the point. We are given a graded vector bundle $E = E^+ \oplus E^-$ over M, a connection on it D preserving the grading, and an endomorphism L of odd degree of E. Then we extend D and L to operators on $\Omega(M,E)$ as follows. D extends in an obvious way but L we extend so that

$$L(\omega\alpha) = (-1)^{\deg\omega}\omega \cdot L(\alpha)$$

Then $\tilde{D} = D + L$ is an operator on $\Omega(M,E)$ satisfying

$$\tilde{D}(\omega\alpha) = d\omega \cdot \alpha + (-1)^{\deg\omega}\omega \cdot \tilde{D}\alpha$$

(Actually where do we use the grading?)

Now we take the curvature

$$\tilde{D}^2 = D^2 + DL + LD + L^2$$

which is an $\Omega(M)$-linear endomorphism of $\Omega(M,E)$. Then we form $e^{\tilde{D}^2}$ and take its trace in some sense.

$$\delta e^{\tilde{D}^2} = \int_0^1 dt\, e^{(1-t)\tilde{D}^2}\left(\delta\tilde{D}\cdot\tilde{D} + \tilde{D}\cdot\delta\tilde{D}\right)e^{t\tilde{D}^2}$$

$$= \left\{\tilde{D}, \int_0^1 dt\, e^{(1-t)\tilde{D}^2}\,\delta\tilde{D}\, e^{t\tilde{D}^2}\right\}$$

In order to have the cohomology class unchanged, I will need certain things. But first I want the form to be closed. The standard proof is

$$d\,\mathrm{tr}\, e^{\tilde{D}^2} = \mathrm{tr}\left[\tilde{D}, e^{\tilde{D}^2}\right] = 0.$$

Let's go over the proof that $d\,tr(e^{D^2}) = 0$ in order to see what we need to make it generalize. The proof is

$$d\,tr(e^{D^2}) = tr\,[D, e^{D^2}] = 0.$$

and uses that $d\,tr(X) = tr\,[D,X]$ if X is an endomorphism of E, more generally an $End(E)$-valued form. [redacted] Clearly one can reduce to $X \in \Omega^0(M, End\,E)$ and probably to the case where $X = |s\rangle\langle\lambda|$, whence it should come out from the defn. of the trace.

But if one argues by local triviality one can assume that $D = d + \theta$ and then one needs

$$d\,tr(X) = tr(dX) = tr\,[d,X] = tr\,[d+\theta, X]$$

It doesn't seem impossible for θ to be an endomorphism of E extended to $\Omega(M,E)$ by the rule

$$\theta(\omega\alpha) = (-1)^{\deg\omega}\omega\theta\alpha$$

Example: Consider the operator on $\Omega(M)$

$$(d+L)\omega = d\omega + (-1)^{\deg\omega} f\omega \qquad \text{where } f \text{ is a function.}$$

Then
$$(d+L)(\omega\alpha) = d\omega.\alpha + (-1)^{\deg\omega}\omega\, d\alpha + (-1)^{\deg\omega+\deg\alpha}\omega f\alpha$$
$$= d\omega.\alpha + (-1)^{\deg\omega}\omega\underbrace{\left[d\alpha + (-1)^{\deg\alpha} f\alpha\right]}_{(d+L)\alpha.}$$

Then $(d+L)^2(\omega\alpha) = \omega\,(d+L)^2(\alpha)$ should be true:

$$(d+L)(d+L)(\alpha) = (d+L)\left(d\alpha + (-1)^{\deg\alpha}\alpha f\right)$$
$$= (-1)^{\deg\alpha}\left[d\alpha.f + (-1)^{\deg\alpha}\alpha.df\right]$$
$$+ (-1)^{\deg\alpha+1} d\alpha.f + (-1)^{2\deg\alpha}\alpha f^2 \quad = \alpha.(df + f^2)$$

Thus $(d+L)^2$ is right multiplication by $df+f^2$ hence it commutes with left-multiplication.

Now we would like to conclude that $\mathrm{tr}(e^{D^2})$ or say even simpler, that $\mathrm{tr}(D^2)$ is closed. What we have is an endo of a free module with one generator. The trace takes values in $\Omega(M)$ mod commutators <u>not</u> <u>graded</u> commutators. In this case we get $\Omega^0(M)\oplus\Omega^1(M)\oplus 0$. and so the trace is just df.

In the case of $\Omega(M,E)$ with the $\mathbb{Z}_2$-grading we have the super-trace for endos. which takes its values in $\Omega(M)$.

Let's discuss carefully the trace and super-trace. Let A be a ring and P a finitely generated projective right A-module. Then we have an isom. for any right A-module

$$\begin{array}{ccc} M \otimes_A \mathrm{Hom}_{A^\circ}(P,A) & \xrightarrow{\sim} & \mathrm{Hom}_{A^\circ}(P,M) \\ m \otimes \lambda & \longmapsto & (x \mapsto m\lambda(x)) \end{array}$$

Then we define the trace on $\mathrm{Hom}_{A^\circ}(P,P)$ by the following composition

$$\begin{tikzcd} P \otimes_A \mathrm{Hom}_{A^\circ}(P,A) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{A^\circ}(P,P) \\ A/[A,A]. & \end{tikzcd}$$

$$x\otimes\lambda \longmapsto \lambda(x)$$

One must divide out by commutators since

$$\begin{array}{ccc} xa \otimes \lambda & = & x \otimes a\lambda \\ \downarrow & & \downarrow \\ \lambda(xa) & & (a\lambda)(x) \\ \parallel & & \parallel \\ \lambda(x)a & & a\lambda(x) \end{array}$$

Now suppose that $A = A^+ \oplus A^-$ is a $\mathbb{Z}_2$-graded ring and $P = P^+ \oplus P^-$ is a $\mathbb{Z}_2$-graded module$/A$. [redacted] Think of a $\mathbb{Z}_2$-grading as an action of $\{\pm 1\}$. Then it is clear that $\mathrm{Hom}_{A^0}(P, P)$ is also a $\mathbb{Z}_2$-graded ring and that

1) $$P \otimes_A \mathrm{Hom}_{A^0}(P, A) \xrightarrow{\sim} \mathrm{Hom}_{A^0}(P, P)$$

is compatible with the $\{\pm 1\}$ action. But now consider the map

2) $$\begin{array}{ccc} P \otimes \mathrm{Hom}_{A^0}(P, A) & \longrightarrow & A \\ x \otimes \lambda & \longmapsto & [redacted]\ (-1)^{\deg x \cdot \deg \lambda} \lambda(x). \end{array}$$

Then what do we need for it to pass ~~to~~ the tensor product over A?

$$\begin{array}{ccc} xa \otimes \lambda & \longmapsto & (-1)^{(\deg x + \deg a)\deg \lambda} \lambda(xa) \\ \parallel & & \\ x \otimes a\lambda & \longmapsto & (-1)^{\deg x (\deg a + \deg \lambda)} a\lambda(x) \end{array}$$

So we need

$$\lambda(x)a = (-1)^{(\deg a)(\deg \lambda + \deg x)} a \lambda(x)$$

which means that we must [redacted] divide by the graded commutators.

Thus we define the super trace on $\mathrm{Hom}_{A^o}(P,P)$ to be the composition

$$\mathrm{Hom}_{A^o}(P,P) \xleftarrow{\sim} P \otimes_A \mathrm{Hom}_{A^o}(P,A) \longrightarrow A/[A,A]_{gr}$$

Let's see what this amounts to when P is free say

$$P = A^r \oplus A(1)^s$$

where $A(1)$ denotes A with the opposite grading.
Then $\mathrm{Hom}_{A^o}(P,P)$ can be identified with $(r+s)\times(r+s)$ matrices over A which we write in block form

$$\left(\begin{array}{c|c} + & - \\ \hline - & + \end{array}\right)$$

so as to see the grading.

P therefore has a basis $|i\rangle$ and the dual basis $\langle i|$ where these have a degree. [redacted]
An endomorphism is a sum of [redacted] endomorphisms of the form $|i\rangle a \langle j|$, which has the matrix with the entry a in the ith row + jth column.
We have

$$tr_s\, |i\rangle a \langle j| = (-1)^{\deg i(\deg a + \deg j)}\, a\langle j|i\rangle$$

which is zero if $i \neq j$. If $i = j$ we get

$$\begin{array}{lll} [redacted]\ a & \text{in } A/[A,A]_{gr} & \text{if } \deg|i\rangle = + \\ (-1)^{\deg a+1} a & \text{———} & \text{if } \deg|i\rangle = - \end{array}$$

So, for example, if $a \in A^+$, then we get a when it is part of an endo of P^+ and $-a$ when it is part of an endo of P^-.

I need some examples. Consider the Clifford algebra C_n and calculate its graded commutator quotient. We have

$$\begin{aligned}[\gamma^1, \gamma^1 \cdots \gamma^k] &= [\gamma^1, \gamma^1]\gamma^2 \cdots \gamma^k \\ &= 2\gamma^2 \cdots \gamma^k\end{aligned}$$

where we are calculating with graded commutators. (Check

$$\begin{aligned}[\gamma^1, \gamma^1 \cdots \gamma^k] &= \gamma^1\gamma^1 \cdots \gamma^k - (-1)^k \gamma^1 \cdots \gamma^k \gamma^1 \\ &= \gamma^2 \cdots \gamma^k - (-1)^k \gamma^1 (-1)^{k-1} \gamma^1 \gamma^2 \cdots \gamma^k \\ &= 2\gamma^2 \cdots \gamma^k \quad)\end{aligned}$$

This shows that all the basis elements of C_n except the top one $\gamma^1 \cdots \gamma^n$ are graded commutators. So we learn that

$$C_n / [C_n, C_n] \quad \text{is one dimensional of degree } (-1)^n$$

January 2, 1984

Source of confusion: When working with connections and curvature we tend to identify $\mathrm{End}(E)$-valued forms with operators on $\Omega(M,E)$ which are $\Omega(M)$-linear. Thus we have a natural product

$$\Omega^p(M, \mathrm{End}\, E) \times \Omega^q(M,E) \longrightarrow \Omega^{p+q}(M,E)$$
$$(\theta \quad , \quad \alpha) \longmapsto \theta\alpha$$

which allows us to [redacted] associate to θ the operator $\alpha \longmapsto \theta\alpha$. This operator is $\Omega(M)$-linear in the sense that

$$\theta(\omega\alpha) = (-1)^{\deg\theta\cdot\deg\omega}\,\omega\cdot\theta\alpha$$

or

$$\theta(\alpha\omega) = [redacted] \quad \theta\alpha\cdot\omega$$

A connection is an operator D on $\Omega(M,E)$ satisfying

$$D(\omega\alpha) = d\omega\cdot\alpha + (-1)^{\deg\omega}\omega D\alpha$$

or

$$D(\alpha\omega) = D\alpha\cdot\omega + (-1)^{\deg\alpha}\alpha\cdot d\omega$$

We know that D^2 and the difference of two connections is $\Omega(M)$-linear, and hence can be identified with elements of $\Omega^*(M, \mathrm{End}(E))$.

$$\boxed{\Omega(M, \mathrm{End}\, E) \xrightarrow{\sim} \mathrm{Hom}_{\Omega(M)^{op}}(\Omega(M,E), \Omega(M,E))}$$

Now [redacted] we have the trace map

$$\mathrm{tr}: \mathrm{End}(E) \longrightarrow \mathbb{1}$$

which induces

$$\mathrm{tr}: \Omega(M, \mathrm{End}\, E) \longrightarrow \Omega(M)$$

This trace map should be the super-trace of the endomorphism of $\Omega(M,E)$ over $\Omega(M)$.

$\Omega(M,E)$ is a graded $\Omega(M)$-module which ~~The~~ is finite type projective, so the super-trace of an[redacted] endom. is defined. Let's compute a bit more generally.

Let $A = A^+ \oplus A^-$ be a $\mathbb{Z}_2$-graded ring, or rather k-algebra. Let

$$P = E \otimes_k A$$

where E is a $\mathbb{Z}_2$-graded [redacted] finite diml' v.s. Then

$$\mathrm{Hom}_{A^\circ}(P, A) = \mathrm{Hom}_k(E, A) = A \otimes E^*$$

and so

$$P \otimes_A \mathrm{Hom}_{A^\circ}(P,A) = E \otimes A \otimes_A A \otimes E^* \xrightarrow{\sim} \mathrm{Hom}_{A^\circ}(P,P)$$

$$\qquad\qquad\qquad\qquad\qquad\quad \| $$

$$E \otimes A \otimes E^*.$$

In other words an endomorphism of P a sum of endos of the form

$$e a \lambda : \quad E \otimes A \longrightarrow E \otimes A$$

$$x \otimes \alpha \longmapsto e\, a\, \lambda(x)\, \alpha$$

The super-trace of $e a \lambda$ is

$$(-1)^{\deg e(\deg a + \deg \lambda)} \quad a\,\lambda(e)$$

In the case of $\Omega(M,E)$ with E of degree 0 $(E = E^+)$ then the super trace is

$$e a \lambda \longmapsto a\,\lambda(e).$$

In other words we are doing the following

$$\begin{array}{ccccc} A\otimes E\otimes E^* & \simeq & E\otimes A\otimes E^* & \xrightarrow{\sim} & \mathrm{Hom}_{A^\circ}(P,P) \\ \big\downarrow{\scriptstyle \text{trace on } E} & & & & \\ A \quad \text{mod graded commutators} & & & & \end{array}$$

However if $E^- \neq 0$, then there are some non-trivial signs.

Let's work with forms so as to see what is going on. I've already decided that working in the algebra [redacted]

$$\mathrm{End}_{\Omega(M)^{op}}(\Omega(M,E)) \subset \mathrm{End}_k(\Omega(M,E))$$

is the correct thing to do. [redacted] For it is within this algebra that one has the formulas

$$[\tilde{D}, e^{\tilde{D}^2}] = 0$$

$$\delta e^{\tilde{D}^2} = \int_0^1 dt\, e^{(1-t)\tilde{D}^2}\left(\tilde{D}\delta\tilde{D} + \delta\tilde{D}\,\tilde{D}\right)e^{t\tilde{D}^2}$$

$$= \left\{ \tilde{D},\ \int_0^1 dt\, e^{(1-t)\tilde{D}^2}\delta\tilde{D}\, e^{t\tilde{D}^2} \right\}$$

Now I want to use the identification given by left mult.

$$\Omega(M, \mathrm{End}\, E) \xrightarrow{\sim} \mathrm{Hom}_{\Omega(M)^{op}}(\Omega(M,E), \Omega(M,E))$$

and I think now that my sign difficulties arise from incorrectly calculating the super-trace.

Let's do the calculation carefully.

We start with the connection D on E preserving the grading and the odd degree endo. L of E. D is extended to $\Omega(M,E)$ in the obvious way so that we have

$$D(\alpha\omega) = D\alpha\cdot\omega + (-1)^{\deg_\Omega \alpha}\alpha\cdot d\omega$$

| wrong see p. 404 and p. 369

L is extended to $\Omega(M,E)$ by

$$L(\alpha\omega) = L(\alpha)\,\omega.$$

Let ε_Ω be the map of $\Omega(M,E)$ to itself given by

$$\varepsilon_\Omega(\alpha) = (-1)^{\deg_\Omega \alpha}\alpha.$$

We want to write down a <u>super-connection</u> $\tilde{D}$ which is supposed to satisfy

$$\tilde{D}(\alpha\omega) = \tilde{D}\alpha\cdot\omega + \underbrace{\varepsilon_E\varepsilon_\Omega}_{(-1)^{\deg_E\alpha + \deg_\Omega\alpha}}\alpha\cdot d\omega$$

For example

$$(\varepsilon_E D)(\alpha\omega) = \varepsilon_E\left[D\alpha\cdot\omega + \varepsilon_\Omega\alpha\cdot d\omega\right]$$
$$= (\varepsilon_E D\alpha)\,\omega + (\varepsilon_E\varepsilon_\Omega\alpha)\cdot d\omega$$

Then we can add L to it to get

$$\tilde{D} = \varepsilon_E D + L.$$

Then

$$\tilde{D}^2 = \varepsilon_E D\varepsilon_E D + \varepsilon_E D L + L\varepsilon_E D + L^2$$
$$= D^2 + \varepsilon_E[D,L] + L^2$$

Now the problem is to prove that

$$d\, tr_s\left(e^{\tilde{D}^2}\right) = 0.$$

Enough to prove

$$d\, tr_s(X) = tr_s\left([\tilde{D}, X]\right)$$

for X an $\Omega(M)^{op}$ endo of $\Omega(M,E)$. Presumably the commutator is the graded one with respect to the $\mathbb{Z}_2$-grading of endomorphisms. Let's work locally so we can assume E trivial, and I can compare the connection D with d. Then

$$\tilde{D} = \varepsilon_E d + (\varepsilon_E A + L)$$

where $\varepsilon_E A + L$ is an $\Omega(M)^{op}$ endo. of $\Omega(M,E)$ which is odd for the total grading $\varepsilon_E \varepsilon_\Omega$. So we need the formula

$$tr_s([Y, X]) = 0$$

for two endos. of $\Omega(M,E)$ where the graded commutator is understood. We also need

$$d\, tr_s X = tr_s [\varepsilon_E d, X]$$

and if we break X into its four pieces then it is enough to worry about the case where E is one-dimensional.

Let us take $X = v\,\omega\,\lambda$ with $v \in E^-$, $\omega \in \Omega^p(M)$, $\lambda \in (E^-)^*$. Recall that E is trivial.

$$\left(\varepsilon_E d X\right)(v_1 \alpha) = \varepsilon_E\, d\, v\, \omega\, \lambda(v_1)\alpha \qquad v_1 \in E^-$$
$$= -v\, d(\omega\alpha)\, \lambda(v_1)$$

$$(X\,\varepsilon_E d)(\sigma_1 \alpha) = v\,\omega\,\lambda(-\sigma_1)\,d\alpha$$
$$= -v\,\omega\,d\alpha\,\lambda(\sigma_1)$$

$$[\varepsilon_E d, \overset{p}{X}](\sigma_1 \alpha) = \left(\varepsilon_E d X - (-1)^p X\,\varepsilon_E d\right)(\sigma_1 \alpha)$$
$$= -v\left[d(\omega\alpha) - (-1)^p\,\omega\,d\alpha\right]\lambda(\sigma_1)$$
$$= -v\;d\omega\cdot\alpha\cdot\lambda(\sigma_1)$$
$$= (-v\,d\omega\,\lambda)(\sigma_1 \alpha)$$

$$\therefore\ \mathrm{tr}_s\,[\varepsilon_E d, X] = \mathrm{tr}_s\left(-v\,d\omega\,\lambda\right)$$
$$= -(-1)^{\deg\omega + 1 + 1}\,d\omega\,\lambda(v)$$

$$d\,\mathrm{tr}_s X = d\,\mathrm{tr}(v\,\omega\,\lambda)$$
$$= d\,(-1)^{\deg\omega + 1}\,\omega\,\lambda(v)$$

Thus in this case we have

$$d\,\mathrm{tr}_s X = \mathrm{tr}_s\,[\varepsilon_E d, X]$$

Sign error: A connection D on $E^+ \oplus E^-$ should be extended to $\Omega(M,E) = \Omega(M) \otimes_{\Omega^0(M)} \Omega^0(M,E)$ by

$$D(\omega\alpha) = d\omega\cdot\alpha + (-1)^{\deg\omega}\,\omega\,D\alpha.$$

Then if one defines the right module structure on $\Omega(M,E)$ over $\Omega(M)$ correctly, one has

$$D(\alpha\omega) = D\alpha\cdot\omega + (-1)^{\deg\alpha}\,\alpha\cdot d\omega$$

where $\deg\alpha \in \mathbb{Z}_2$ is the total degree.

<u>January 4, 1984</u>

Think out principal bundles under super-groups.

Let's first describe the foundations of super-think. I want to adopt an operator algebra viewpoint. A $\mathbb{Z}_2$ graded algebra [redacted] A [redacted] can be defined as an algebra A on which the group $\{\pm 1\}$ operates.

(Check: Normally a $\mathbb{Z}_2$-graded algebra is an alg. $A = A^+ \oplus A^-$ such that $A^+A^+ + A^-A^- \subset A^+$, $A^+A^- + A^-A^+ \subset A^-$. If one defines $\varepsilon(a^+ + a^-) = a^+ - a^-$, then we have

$$\begin{aligned}
\varepsilon(a^+ + a^-)\,\varepsilon(b^+ + b^-) &= (a^+ - a^-)(b^+ - b^-) \\
&= a^+b^+ + a^-b^- - (a^+b^- + a^-b^+) \\
\varepsilon[(a^+ + a^-)(b^+ + b^-)] &= \varepsilon\,[\underbrace{(a^+b^+ + a^-b^-)}_{\in A^+} + \underbrace{(a^+b^- + a^-b^+)}_{A^-}] \\
&= (a^+b^+ + a^-b^-) - (a^+b^- + a^-b^+)
\end{aligned}$$

so we see ε is an automorphism of A of order 2.)

I want to think of A as an algebra of operators, by which I mean that we have a graded vector space $V = V^+ \oplus V^-$ which is a faithful left-A-module such that if $\varepsilon_V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on $V^+ \oplus V$, then

$$\varepsilon(a) = \varepsilon_V\, a\, \varepsilon_V^{-1} \qquad \text{i.e.} \qquad \begin{aligned} \varepsilon_V(av) &= (\varepsilon_V a \varepsilon_V^{-1})(\varepsilon_V v) \\ &= \varepsilon(a)\,\varepsilon_V v \end{aligned}$$

So we now think of A as an alg. of endos. of a graded vector space [redacted] V which is closed under conjugation by the involution ε_V of V.

Now one of the things one does with [redacted] algebras is to form the tensor product $A \otimes B$. If $A \subset \mathrm{End}(V)$, $B \subset \mathrm{End}(W)$, then $A \otimes B \subset \mathrm{End}(V \otimes W)$ is the algebra generated by $A \otimes id_W$, $id_V \otimes B$.

In the super theory we have a different kind of tensor product $A \hat{\otimes} B$ which arises as follows. Observe first that $id_V \otimes b$ can be written as

$$T^{-1}(b \otimes id_V)\,T \qquad\qquad T : V \otimes W \xrightarrow{\sim} W \otimes V$$
$$v \otimes w \longmapsto w \otimes v$$

In the super theory we use a different T namely

$$\hat{T} : V \otimes W \longrightarrow W \otimes V$$
$$\hat{T}(v \otimes w) = (-1)^{\deg v \cdot \deg w}\, w \otimes v$$

for v, w homogeneous elements. (Here we use the fact that $\mathbb{Z}_2$ is a field, or at least, has a multiplication: $\deg v \in \mathbb{Z}/2$). Then $A \hat{\otimes} B \subset \mathrm{End}(V \otimes W)$ is the algebra generated by the elements $a \otimes id_W$ and $\hat{T}^{-1}(b \otimes id_V)\hat{T}$.

Let's check that this works: Let [redacted] a, b, v, w be homogeneous. Then

$$\hat{T}^{-1}(b \otimes id_V)\hat{T}\;(v \otimes w)$$
$$= \hat{T}^{-1}(b \otimes id_V)\;(-1)^{\deg v \cdot \deg w}(w \otimes v)$$
$$= \hat{T}^{-1}\,(-1)^{\deg v \cdot \deg w}(bw \otimes v)$$
$$= (-1)^{\deg v \cdot \deg w \,+\, \deg v \cdot \deg(bw)}\; v \otimes bw$$
$$= (-1)^{\deg v \cdot \deg b}\; v \otimes bw.$$

so

$$(a \otimes id_W)\,\hat{T}^{-1}(b \otimes id_V)\hat{T}\,(v \otimes w) = (-1)^{\deg v \cdot \deg b}(av \otimes bw)$$
$$\hat{T}^{-1}(b \otimes id_V)\hat{T}(a \otimes id_W)\;(v \otimes w) = (-1)^{\deg av \cdot \deg b}(av \otimes bw).$$

Hence if we put

$$1 \hat{\otimes} b = \hat{T}^{-1}(b \otimes id_V)\hat{T}$$

then we have

$$(a \otimes 1)(1 \hat{\otimes} b) = (-1)^{\deg a \cdot \deg b}(1 \hat{\otimes} b)(a \otimes 1).$$

More generally we can define

$$(a \hat{\otimes} b)(v \otimes w) = (-1)^{\deg b \cdot \deg v}(av \otimes bw)$$

so that

$$a \hat{\otimes} b = (a \otimes 1)(1 \hat{\otimes} b).$$

[redacted] Notice that one can also express [redacted] $1 \hat{\otimes} b$ as the composition

$$V \otimes W \xrightarrow{\eta} V \otimes W \xrightarrow{1 \otimes b} V \otimes W \xrightarrow{\eta} V \otimes W$$

$$v \otimes w \longmapsto (-1)^{\deg v \cdot \deg w} v \otimes w.$$

but that η is not $\varepsilon_V \varepsilon_W$.

The ultimate thing I want to understand is super-symmetry. Super-symmetry must be described by a super-group. Moreover [redacted] if one has an object with super-symmetry, then hopefully one can form a fibre bundle with this object as fibre. One should therefore have a notion of principal bundle under a super-group.

The next point is that I already have examples of super groups from quantum mechanics, in fact the quantum mechanics of the forced harmonic oscillator. Let's review this in outline.

A harmonic oscillator is described by a Hamiltonian

$$H = \omega a^* a$$

where a^*, a are creation and annihilation operators:

$$[a, a^*] = 1. \qquad [a,a] = [a^*, a^*] = 0$$

The forced oscillator is described by

$$H = \omega a^* a + a^* J + \tilde{J} a$$

where $J, \tilde{J}$ can depend on t. Associated to this operator is a scattering matrix

$$S = T\left\{ e^{-i\int_{-\infty}^{\infty}\left(a^*(t)J(t) + \tilde{J}(t)a(t)\right)dt} \right\}$$

where $a^*(t) = e^{iH_0 t} a^* e^{-iH_0 t}$ etc. $\qquad H_0 = \omega a^* a$

$$= e^{i\omega t} a^* \qquad [H_0, a^*] = \omega$$

Thus we have the path of operators

$$a^*(t)J(t) + \tilde{J}(t)a(t) = a^*\left(e^{i\omega t}J(t)\right) + a\left(\tilde{J}(t)e^{-i\omega t}\right)$$

which we are integrating in a way analogous to integrating a path in a Lie alg. so as to get a path in the group. The Lie algebra in this situation is spanned by a^*, a and 1; it is the Heisenberg algebra. The corresponding Lie group is the Heisenberg group.

Next consider the fermion oscillator:

$$H = \omega b^* b \qquad \{b, b^*\} = bb^* + b^*b = 1$$

$$\{b, b\} = 0 \qquad \{b^*, b^*\} = 0.$$

The appropriate forced oscillator has

$$H = \omega b^*b + b^*J + \tilde{J}b$$

where now $J(t)$ and $\tilde{J}(t)$ are anti-commuting numbers of square zero. The corresponding S-matrix is now going to involve these quantities. For example we could take $J(t)$ and $\tilde{J}(t)$ to be linear combinations

$$c_i(t)\,\varepsilon_i$$

where ε_i are elements in a Grassmann algebra.

The way to interpret this is to say that $b^*, b, 1$ generate a Lie superalgebra, with b^*, b of odd degree. The S-matrix is a point of the corresponding supergroup with values in the exterior algebra generated by the ε_i. At this point I think I should work out the formulas.

First eliminate time from consideration. Thus all I care about are the operators generated by products of

$$e^{b^*J_i} \qquad e^{\tilde{J}_j b}$$

where $J_i, \tilde{J}_j$ are anti-commuting quantities of square zero anti-commuting with b, b^*. Then we use

$$e^{\tilde{J}b}\, e^{b^*J} = \underbrace{e^{[\tilde{J}b,\, b^*J]}}_{e^{\tilde{J}\{b,b^*\}J} \,=\, e^{\tilde{J}J}}\, e^{b^*J}\, e^{\tilde{J}b}$$

Digression: Should J anti-commute with itself? We certainly want

$$e^{\tilde{J}_1 b_1} e^{\tilde{J}_2 b_2} = e^{\tilde{J}_1 b_1 + \tilde{J}_2 b_2}$$

and hence we want $[\tilde{J}_1 b_1, \tilde{J}_2 b_2] = 0$. But

$$\tilde{J}_1 b_1 \tilde{J}_2 b_2 = \tilde{J}_1 \tilde{J}_2 b_2 b_1 \qquad \tilde{J}_2 b_2 \tilde{J}_1 b_1 = -\tilde{J}_2 \tilde{J}_1 b_2 b_1$$

so we must have $\{\tilde{J}_1, \tilde{J}_2\} = 0$. So if one has a $\tilde{J}(t)\, b(t)$, then [redacted] $\tilde{J}(t)$ must anti-commute with $\tilde{J}(t')$ for $t \neq t'$ and hence with itself if $\tilde{J}(t)$ is continuous [redacted] in t.

Review: I am now treating the Heisenberg Lie super algebra with generators b, b^* of degree $+1$ and z of degree 0 satisfying

$$b^2 = (b^*)^2 = 0 \qquad \{b, b^*\} = z$$
$$[z, b] = [z, b^*] = 0.$$

We have a particular representation of this algebra with $z \longmapsto 1$.

We should now be in a position to describe the group of points of the corresponding super-group with values in a [redacted] commutative super-algebra R. How does this work? One way to construct a Lie group is to consider the one parameter subgroups. It seems that in the super situation a 1-parameter subgroup should be of the form

$$e^{t r b}$$

where r, b have the same parity, e.g. in this case $r \in R^-$.

January 5, 1984

Review super-groups and Lie superalgebras in the graded situation that I encountered in rational homotopy theory. In this case superalgebras are augmented and the augmentation ideal is effectively nilpotent. Everything should be given by the Lie algebra, the group law coming from the Baker Campbell Hausdorff formula.

Recall that a cocommutative Hopf algebra is of the form $U(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of primitive elements. How do we get a functor from commutative algebras to groups? It is the functor represented by the dual $U(\mathfrak{g})^*$ with group law given by the map

$$U(\mathfrak{g})^* \longrightarrow U(\mathfrak{g})^* \otimes U(\mathfrak{g})^*$$

transposed to the multiplication in $U(\mathfrak{g})$. But now we have the canonical PBW isom.

$$S(\mathfrak{g}) \simeq U(\mathfrak{g})$$

which is an isomorphism of coalgebras, and hence gives an algebra isomorphism

$$U(\mathfrak{g})^* \simeq S(\mathfrak{g})^* = S(\mathfrak{g}^*)$$

(This I think amounts to the fact that the exponential map $\mathfrak{g} \to G$ is bijective for a nilpotent grp and hence functions on the group can be identified with functions on the Lie algebra)

So there is an isom

$$\mathrm{Hom}_{alg}(U(\mathfrak{g})^*, A) = \mathrm{Hom}_k^{(0)}(\mathfrak{g}^*, A) = (A \otimes \mathfrak{g})_{(0)}$$

and probably we take the group law on $(A \otimes \mathfrak{g})_{(0)}$ given by the BCH formula in terms of its bracket.

Example: suppose $\mathfrak{g}$ consists of an X of degree 1 and X^2 of degree 2. (This is $\pi_*(\Omega S^2)$ rationally.) Then an element of $(A \otimes \mathfrak{g})_{(0)}$ is of the form $a_1 X + a_2 X^2$ with $a_1 \in A^1$, $a_2 \in A^2$. The bracket is

$$[a_1 X + a_2 X^2, a_1' X + a_2' X^2] = [a_1 X, a_1' X] = -2 a_1 a_1' X^2.$$

Hence the group law is

$$e^{a_1 X + a_2 X^2} \, e^{a_1' X + a_2' X^2} = e^{(a_1 X + a_2 X^2) + (a_1' X + a_2' X^2) + \frac{1}{2}[\ ,\ ]}$$

$$= e^{(a_1 + a_1') X + (a_2 + a_2' \text{[redacted]} - a_1 a_1') X^2}$$

Notice that this is non-commutative as $a_1' a_1 \neq a_1 a_1'$.

Example: The Heisenberg fermion algebra: This will be 3 diml generated by ψ, ψ^*, z such that ψ, ψ^* have degree 1, z has degree 0, and

$$\psi^2 = (\psi^*)^2 = 0 \qquad \{\psi, \psi^*\} = z \qquad [z, \psi] = [z, \psi^*] = 0$$

Then the points of the corresponding group with values in A can be written

$$e^{a \text{[redacted]} \psi + \text{[redacted]} b \psi^* + c z} \qquad a, b \in A^-, \quad c \in A^+.$$

and I know how to calculate the product.

A generalization of these two examples occurs when one has a vector space with quadratic form. There is an analogy with [redacted] extra special 2 groups.

What I learn from the nilpotent theory is that the group of points with values in A is the [redacted] group belonging to the Lie algebra [redacted] $(A\otimes\mathfrak{g})_{(0)}$. It would appear that, modulo the difficulty of the group-Liealgebra correspondence, the Lie superalgebra $A\otimes\mathfrak{g}$ is the internal $\underline{\mathrm{Map}}(\hat{A}, G)$:

$\hat{A} = Sp(A)$.

$$\mathrm{Map}(\hat{B}, \underline{\mathrm{Map}}(\hat{A},G)) = \mathrm{Map}(\hat{B}\times\hat{A}, G)$$

So unless I am mistaken, given a [redacted] Lie superalgebra $\mathfrak{g}$, then the corresponding super-group should be have as the group of points over k a Lie group with Liealg. $\mathfrak{g}^+$.

[redacted] At the moment understand the infinitesimal theory of supergroups in terms of Lie superalgebras. These Lie superalgebra correspond to connected cocommutative Hopf algebras, or by duality, linearly compact Hopf algs. which are commutative and whose aug. ideal is topologically nilpotent.

Now I want to discuss alg. supergroups. These which will be described by finitely generated commutative Hopf algebras. Such ~~things~~ should be described à la Grothendieck by a Tannakian category.

[redacted] I also feel that I should be able to describe [redacted] algebraic subgroups somehow as an alg. group (corresponding to the even part of the supergroup) together with an infinitesimal part (corresponding to the odd part of the supergroup.) [redacted]

What I really want is a complete description of representations of the supergroup. Somehow I feel that one should have a pair of representations of the even algebraic group together with supersymmetry transformations between the pieces. But it is necessary to get this precise.

So let me begin with the idea of a supersymmetry transformation. This should be the same as an element X of $\mathfrak{g}^-$. We have seen that such an element generates a 2-diml Lie subalgebra in general, hence X doesn't belong to a "1-parameter subgroup". It's a slightly different situation.

But in certain case, like $\mathfrak{gl}_{p,q} = \mathrm{End}(V)$ where $V = V^+ \oplus V^-$ and $\dim(V^+) = p$, $\dim(V^-) = q$, we can split $\mathfrak{g}^-$ into parts $\mathrm{Hom}(V^+, V^-)$ and $\mathrm{Hom}(V^-, V^+)$ which are commutative.

Now consider a manifold M and try to describe a principal bundle for the supergroup $GL_{p,q}$ over M. First we must know the group of points of $GL_{p,q}$ with values in the ring $C^\infty(M)$. This ring has no odd part, so clearly we should get just a pair of maps $M \longrightarrow GL_p$, $M \longrightarrow GL_q$. It follows that a principal bundle for $GL_{p,q}$ over M should be given by a pair consisting of a GL_p bundle and a GL_q bundle.

So the category of principal bundles for $GL_{p,q}$ over M should be equivalent to the category of pairs (E^+, E^-) of vector bundles of dims p, q, resp. Except this leaves out the supersymmetries. The way this probably ought to be handled is to assign an appropriate superalgebra corresponding to the "true" ring of functions on the principal bundle.

January 6, 1984

What is a supermanifold? It a manifold equipped with a bundle of $\mathbb{Z}_2$-graded algebras such that each fibre is an exterior algebra. Let $\mathcal{A}$ be this bundle of algebras, let $\mathcal{I}$ be the ideal of nilpotent elements (equivalently the ideal generated by $\mathcal{A}^-$ where $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$). Then $\mathcal{I}/\mathcal{I}^2$ is a vector bundle over the manifold, and if we choose a splitting of the exact sequence

$$0 \longrightarrow (\mathcal{I}^2)^- \longrightarrow \mathcal{I}^- \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow 0$$

we will get ~~an~~ $\mathbb{Z}_2$-graded algebra isomorphism

$$\Lambda(\mathcal{I}/\mathcal{I}^2) \xrightarrow{\sim} \mathcal{A}.$$

Thus any supermanifold is isomorphic to a manifold M equipped with $\mathcal{A} = \Lambda(E)$ for some vector bundle E. However one can not think of a super manifold as (M,E), since, as $\mathbb{Z}_2$-graded algebras, there are more morphisms $\Lambda(E) \longrightarrow \Lambda(F)$ than come from vector bdle morphisms.

The next topic I want to discuss is that of principal bundles in a super-context.

Let's restrict attention to the super group $GL_{p,q}$ associated to a graded vector space $V = V^+ \oplus V^-$ where V^+ has dim p (V^- has dim q). We can define a vector bundle over a superman. $(M, \mathcal{A})$ to be a graded vector bundle E on M with an action of $\mathcal{A}$ such that each fiber is free of type p,q over $\mathcal{A}$. Notice that when $\mathcal{A}$ = functions on M we just get a graded vector bundle of dim p,q.

We obvious need the group of points of $GL_{p,q}$ with values in a commutative superalgebra A. It should be a group belonging to the Lie algebra $(A \otimes \mathfrak{gl}_{p,q})_0 = (A \otimes \mathrm{End}(V))_0$, hopefully the group of autos of $A \otimes V$ of degree 0.

We have an identification to make.

Let $V = V^+ \oplus V^-$ be a graded vector space. Then $\mathrm{End}(V)$ is a $\mathbb{Z}_2$ graded algebra, hence it determines a Lie superalgebra in which the bracket is

$$[X, Y] = XY - (-1)^{\deg A \cdot \deg B}\, YX .$$

On the other hand if $\mathfrak{g}$ is a Lie superalg and A is a comm. superalg., then $A \otimes \mathfrak{g}$ is a Lie superalg. with bracket defined by

$$\begin{aligned}
[aX, bY] &= aXbY - (-1)^{(\deg a + \deg X)(\deg b + \deg Y)}\, bYaX \\
&= (-1)^{\deg X \cdot \deg b}\, ab\, XY - (-1)^{(\deg X)(\deg b + \deg Y)}\, ab\, YX \\
&= (-1)^{\deg X \cdot \deg b}\, ab\, [X, Y].
\end{aligned}$$

Next I want to identify $A \otimes \mathfrak{g}$ in the case where $\mathfrak{g} = \mathrm{End}(V)$. In this case $\mathfrak{g}$ is the Lie superalg. belonging to ~~as~~ superalg, so we have an alg. structure on $A \otimes \mathfrak{g}$, namely

$$aX\cdot bY = (-1)^{\deg X\cdot\deg b}\, ab\, XY.$$

Hence $A\otimes\mathfrak{g}$ has a Lie super structure given by

$$[aX, bY] = aX\cdot bY - (-1)^{(\deg a+\deg X)(\deg b+\deg Y)}\, bY\, aX.$$

Actually I took this as the way to calculate the bracket in on $A\otimes\mathfrak{g}$ in general.

So what I know is that in the case $\mathfrak{g} = End(V)$, the Lie superalg. $A\otimes\mathfrak{g}$ is the one belonging to the superalgebra $A\hat{\otimes} End(V)$. The last step will be to identify:

$$A\hat{\otimes} End(V) \xrightarrow{\sim} Hom_A(A\otimes V, A\otimes V)$$

$$a\otimes\varphi \longmapsto \left(a'\otimes v \longmapsto (-1)^{\deg a'\cdot\deg\varphi} aa'\otimes\varphi(v)\right)$$

<u>Check:</u>

$$aX(bY(a'v)) = aX(ba'\,Yv)(-1)^{\deg a'\deg Y}$$

$$= aba'\,XYv\,(-1)^{\deg X(\deg b+\deg a')+\deg a'\deg Y}$$

$$(-1)^{\deg X\cdot\deg b}\, ab\, XY a'v = (-1)^{\deg X\cdot\deg b}\, aba'\, XYv\; (-1)^{\deg(XY)\cdot\deg a'}$$

The conclusion is that the group of points of $Gl_{p,q}$ with values in a commutative A can be identified with the group of degree zero **autos** of $A\otimes V$. I just checked it on the Lie alg. level, and the remaining part should be a matter of how to define $Gl_{p,q}$. From this it should follow that I can identify $Gl_{p,q}$ torsors **over** a supermanifold $(M, \mathcal{a})$ with graded modules over $\mathcal{a}$ which are projective of rank p,q.

<u>Idea:</u> obtained when working with $A \otimes \mathfrak{g}$. This is a Lie algebra when A is commutative. However, when A is non-commutative and $\mathfrak{g} = \mathrm{End}(V)$ (more generally, the Lie algebra of an [redacted] algebra) then $A \otimes \mathfrak{g}$ is still a Lie algebra in a definite way. Perhaps this is significant.

<u>January 7, 1984</u>

$$E = E^+ \oplus E^-$$

$$\Omega(M,E) = \Omega(M) \otimes_{\Omega^0(M)} \Omega^0(M,E) \qquad \mathbb{Z}\times\mathbb{Z}_2 \text{ grading}$$

$$\Omega(M,\operatorname{End}E) = \Omega(M) \otimes_{\Omega^0(M)} \Omega^0(M,\operatorname{End}E) \qquad ''$$

We make $\Omega(M,\operatorname{End}E)$ into an algebra and $\Omega(M,E)$ into a module over this algebra by the rules

$$\omega X \cdot \eta Y = (-1)^{\deg X \cdot \deg \eta}\, \omega\eta\, XY$$

$$\omega X \cdot \xi v = (-1)^{\deg X \deg \xi}\, \omega\xi\, Xv.$$

This is a special case of what I discussed earlier, namely given A acting on V and B acting on W, both $\mathbb{Z}_2$ graded, we then have the algebra $A \hat{\otimes} B$ acting on $V \otimes W$.

We have the super trace map

$$tr_s : \operatorname{End}(E) \longrightarrow \mathbb{1}$$

which is zero on the odd degree endos and $tr_{E^+} - tr_{E^-}$ for even degree endos. Simplest formula:

$$tr_s(X) = tr_E(\varepsilon_E X).$$

We extend this to $\Omega(M,\operatorname{End}E)$ in the obvious way

$$tr_s(\omega X) = tr_E(\varepsilon_E\, \omega X) = \omega\, tr_E(\varepsilon_E X).$$

Now the thing to check is that $tr_s(\omega X)$ vanishes on graded commutators. But this follows from

$$[\omega X, \eta Y] = (-1)^{\deg X \cdot \deg \eta}\, \omega\eta \left(XY - (-1)^{\deg X \cdot \deg Y} YX \right)$$

and the fact that tr_s vanishes on graded commutators in

$\mathrm{End}(E)$.

So now we want to do curvature calculations entirely in the algebra $\Omega(M, \mathrm{End}\, E)$. Define a super-connection $\tilde{D}$ to be an odd degree (for the total $\mathbb{Z}_2$-grading) operator on $\Omega(M,E)$ satisfying

$$\tilde{D}(\omega\alpha) = d\omega\cdot\alpha + (-1)^{\deg\omega}\,\omega\,\tilde{D}\alpha$$

For example a connection in E preserving the grading gives rise to such a super-connection. The difference of two super connections is an odd degree operator B on $\Omega(M,E)$ satisfying

$$B(\omega\alpha) = (-1)^{\deg\omega}\,\omega\cdot B\alpha.$$

The curvature $\tilde{D}^2$ is an even degree operator satisfying

$$\begin{aligned}
\tilde{D}^2(\omega\alpha) &= \tilde{D}\left(d\omega\cdot\alpha + (-1)^{\deg\omega}\,\omega\,\tilde{D}\alpha\right)\\
&= d^2\omega\cdot\alpha + (-1)^{\deg d\omega}\, d\omega\cdot\tilde{D}\alpha\\
&\quad + (-1)^{\deg\omega}\left(d\omega\cdot\tilde{D}\alpha + (-1)^{\deg\omega}\,\omega\,\tilde{D}^2\alpha\right)\\
&= \omega\,\tilde{D}^2\alpha
\end{aligned}$$

Such operators $B, \tilde{D}^2$ come from unique elements of $\Omega(M,\mathrm{End}\,E)$ which one finds by

$$\begin{aligned}
&\mathrm{Hom}^{\pm}_{\Omega(M)}\left(\Omega(M,E),\ \Omega(M,E)\right)\\
&= \mathrm{Hom}^{\pm}_{\Omega^0(M)}\left(\Omega^0(E),\ \Omega(M,E)\right)\\
&= \Omega^{\pm}(M, \mathrm{End}\,E).
\end{aligned}$$

Now we need the Bianchi identity for the curvature. For this we assume that the bundle E is trivialized, and then

$$\tilde{D} = d + B \qquad B \in \Omega(M, \mathrm{End}\, E)^{\text{odd}}$$

$$\tilde{D}^2 = dB + Bd + B^2 \qquad \text{as operators}$$
$$= d(B) + B^2 \qquad \text{in } \Omega(M, \mathrm{End}\, E)^{\text{even}}$$

so

$$d\tilde{D}^2 = d(B^2) = dB \cdot B - B \cdot dB$$
$$= [\tilde{D}^2, B]$$

It would appear that I can do these calculations in the principal bundle for E inside the algebra $\Omega(P) \hat{\otimes} \mathrm{End}(V)$.

So let's check the calculations carefully. I work in the algebra $\Omega(M) \hat{\otimes} \mathrm{End}(V)$ on which I have the derivation d. This algebra operates on $\Omega(M) \otimes V$.

Let $\theta \in [\Omega(M) \otimes \mathrm{End}(V)]^{\text{odd}}$ be given and put $\Omega = d\theta + \theta^2$. Then

$$d\Omega = d\theta \cdot \theta - \theta \cdot d\theta = [\Omega, \theta]$$

so

$$d e^{\Omega} = \int_0^1 dt\, e^{(1-t)\Omega}\, d\Omega\, e^{t\Omega}$$
$$= \int_0^1 dt\, e^{(1-t)\Omega} [\Omega, \theta] e^{t\Omega}$$
$$= [e^{\Omega}, \theta]$$

Hence

$$d(\mathrm{tr}_s e^{\Omega}) = \mathrm{tr}_s(d e^{\Omega}) = \mathrm{tr}_s [e^{\Omega}, \theta] = 0$$

Independence of the connection goes as follows. One supposes given a 1-parameter family $\{\theta_t\}$ and considers over $\mathbb{R}\times M$ the connection

$$d_{\mathbb{R}\times M}+\theta = dt\,\partial_t + d + \theta \qquad d=d_M$$

whose curvature is

$$d\theta+\theta^2 + dt\,\dot\theta = \Omega + dt\,\dot\theta$$

Then we know that

$$tr_s\left(e^{\Omega+dt\,\dot\theta}\right) = tr_s\left(e^{\Omega}\right) + dt\; tr_s\left(e^{\Omega}\dot\theta\right)$$

is closed on $\mathbb{R}\times M$. This implies

$$\partial_t\, tr_s\left(e^{\Omega}\right) = d\, tr_s\left(e^{\Omega}\dot\theta\right)$$

and guarantees that the coh. class of $tr_s(e^{\Omega})$ is ind. of t.

So now we consider the case of the superconnection $\tilde D = D+L$ where D is an ordinary connection on E preserving the grading and $L\in\Omega^0(M,\mathrm{End}\,E)$ is odd. It is important to remember that

$$\Omega(M,E) = \Omega(M)\otimes_{\Omega^0(M)}\Omega^0(M,E)$$

is a tensor product of graded spaces and that L acts as $1\hat\otimes L = \varepsilon_\Omega\otimes L$. It is like taking the total D in a tensor product. The curvature is

$$\tilde D^2 = D^2 + (DL+LD) + L^2$$

where if we write $D = dx^\mu D_\mu$, then

$$(DL+LD) = dx^\mu D_\mu L + \underbrace{L\,dx^\mu D_\mu}_{-dx^\mu L} = dx^\mu D_\mu(L).$$

This can be identified with $D(L) \in \Omega^1(M, \mathrm{End}E)$. Thus the curvature is the sum of three terms

$$D^2 \in \Omega^2(M, (\mathrm{End}E)^+)$$
$$D(L) \in \Omega^1(M, (\mathrm{End}E)^-)$$
$$L^2 \in \Omega^0(M, (\mathrm{End}E)^+)$$

and we now know that the form

$$\mathrm{tr}_s\left(e^{D^2 + D(L) + L^2}\right)$$

is closed and its cohomology class is independent of the choice of D and L.

Notice that this form is even. First look at the ch_n class:

$$* \qquad \mathrm{tr}_s\left(\left(D^2 + D(L) + L^2\right)^n\right).$$

This form is even since an odd number of occurrences of $D(L)$ will be odd wrt the ε_E grading and hence will give zero trace. But now if we deform L to zero we get $\mathrm{tr}\,(D^2)^n$ which is homogeneous of degree $2n$. It follows that the components of $*$ in even degrees $< 2n$ must be exact.

The next project will be to show that if L is invertible then the cohomology class of

$$\mathrm{tr}_s\left(e^{D^2 + D(L) + L^2}\right)$$

is zero. We know this is true because if L is invertible, then the bundles E^+, E^- are isomorphic, so I can choose D, ~~so~~ that $\mathrm{tr}_s(e^{D^2}) = 0$. But I would like a direct argument.

Introduce a parameter z and replase L by zL. Then we have a form

$$tr_s\left(\left(D^2 + zD(L) + z^2L^2\right)^n\right)$$

which is a polynomial of degree $2n$ in z and which is closed for each z. Hence each coefficient of this polynomial in Z is a closed form and so we get a polynomial function of z with values in DR cohomology. We know this is independent of z hence each coefficient[redacted] cohomology class is zero, [redacted] except for the constant term. Still no contradiction.

So instead let us consider

$$tr_s\left(e^{D^2 + zD(L) + z^2L^2}\right)$$

and let us assume that $L^2 > 0$. Put $z = it$ [redacted] and let $t \longrightarrow +\infty$ in

$$tr_s\left(e^{D^2 + itD(L) - t^2L^2}\right)$$

Now $e^{D^2+itD(L)-t^2L^2}$ is an element of the [redacted] algebra $\Omega(M)\hat{\otimes}\,End(V)$. At each point of M we have a finite-diml algebra, so would seem that as $t \longrightarrow \infty$ this [redacted] matrix form goes to zero. Thus it must represent the zero chomology class.

January 8, 1984

Problem: Odd K-elements.

The question is how one should represent odd K-elements. Originally I believed that this was the point of introducing C_n-bundles, where C_n is the Clifford algebra. Somehow K^n is the K-theory of bundles of graded C_n-modules.

But the picture is more subtle. What happens is that $K^{-n}(X)$ is the relative K-theory of graded C_n-bundles modulo graded C_{n-1}+bundles. One can see this as follows. First of all a graded C_{n-1}-bundle is the same as an ungraded C_n-bundle. For n even, C_n is simple and for n odd C_n is the product of two simple algs. Thus for n even there is one irred C_n module which is ungraded and two irreducible graded C_n-modules. So

$$\begin{array}{ccc} K(X, \text{graded}\, C_{n+1}) & \longrightarrow & K(X, \text{graded}\, C_n) \\ \| & & \| \end{array}$$

$$\begin{cases} K(X) \xrightarrow{\ \Delta\ } K(X)^2 & n \text{ even} \\[2em] K(X)^2 \xrightarrow{\ +\ } K(X) & n \text{ odd} \end{cases}$$

and we see the relative or "cokernel" theory coincides with $K^{-n}(X)$.

However, notice that $+: K(X)^2 \longrightarrow K(X)$ is always onto so that an element of $K^{-1}(X)$ is not realized by a C_1-bundle. Now somehow this state of affairs improves in the infinite-diml.

theory. [redacted]

In the infinite diml version of $K^0(X)$, [redacted] a K-element [redacted] is represented by a graded Hilbert bundle $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ together with a bundle map F of degree 1 such that $F^2 - I$ is compact. This amounts to giving a Fredholm operator $\mathcal{H}^+ \longrightarrow \mathcal{H}^-$ and [redacted] an inverse modulo compacts. If we use the Kuiper thm. to trivialized the Hilbert bundle, then we get a map from X to Fredholm operators.

For $K^{odd}(X)$ one requires in addition a C_1-structure, more precisely a graded C_1-module structure, on $\mathcal{H}$ such that F anti-commutes with the generator γ^1 of C_1. Thus $\mathcal{H}^+ = \mathcal{H}^-$ and we have

$$\varepsilon = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad F = \begin{pmatrix} & iQ \\ iP & \end{pmatrix}$$

where $\gamma^1 F \gamma^1 = -F \iff P = -Q$. Thus

$$F = i\begin{pmatrix} 0 & -P \\ P & 0 \end{pmatrix} \qquad \text{where } P^2 - I \text{ is compact}$$

$$F = F^* \iff P = P^*$$

Thus we have the self-adjoint Fredholm operators with essential spectrum $\subset \{\pm 1\}$.

Note that if F can be homotoped to an F of square I, then we obtain a graded C_2-structure. Thus the obstruction of lifting F from $F^2 - I \in$ comp. to $F^2 = I$ is the important thing, if we think of the relative theory of graded C_n-modules modulo graded C_{n+1}-modules.

So we seem to be able to conclude that there are interesting (topologically) F which do not come from a finite dimensional graded C_1-bundle.

The relative theory of graded C_n-bundles modulo graded C_{n+1}-bundles over X seems to be described by maps of X to the space $\mathcal{F}_n$ consisting of odd endos. F of $C_n \otimes \mathcal{H}$ such that $F^2 - I$ is compact.

The only interesting question I see is whether one could obtain something for the differential forms. We can do something interesting for $n = 0$, whence a graded C_0-bundle is a pair E^+, E^- and our F is an odd endo L of E.

January 9, 1984

It is necessary to check very carefully your claims about

$$tr_E\left(\varepsilon_E\, e^{D^2+[D,L]+L^2}\right)$$

being closed and having cohomology class independent of L. Let $E = E^+ \oplus E^-$ consist of two trivial line bundles, let $D = d + A$ $\quad A = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix}$ and let $L = t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$[D,L] = [A,L] = t\begin{pmatrix} 0 & A^+-A^- \\ A^--A^+ & 0 \end{pmatrix} \qquad D^2 = \begin{pmatrix} F^+ & 0 \\ 0 & F^- \end{pmatrix}$$

where $F^{\pm} = dA^{\pm} + (A^{\pm})^2 = dA^{\pm}$ in the case of line bundles. When we compute the exponential we work in the algebra $\Omega(M)\hat{\otimes}\operatorname{End}(V)$. Put $B = A^+-A^-$ $\in \Omega^1(M)$. Now

$$[D,L] = tB\cdot\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the odd endom. $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of V becomes the operator $\varepsilon_\Omega \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\Omega(M)\otimes V$. I really want to work in the algebra of matrices over $\Omega(M)$, which is $\Omega(M)\otimes\operatorname{End}(V)$, not $\Omega(M)\hat{\otimes}\operatorname{End}(V)$. Thus it seems that

$$D^2+L^2 = \begin{pmatrix} F^++t^2 & 0 \\ 0 & F^-+t^2 \end{pmatrix} \qquad [D,L] = tB\varepsilon_\Omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

So when we exponentiate

$$e^{(D^2+t^2)+[D,L]} = e^{D^2+t^2}$$

$$+ \int_0^1 ds\, e^{(1-s)(D^2+t^2)}[D,L]e^{s(D^2+t^2)}$$

$$+ \int_0^1 ds_1 \int_0^{s_1} ds_2\, e^{(1-s_1)(D^2+t^2)} \underbrace{[D,L]}_{tB\varepsilon_\Omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}} e^{(s_1-s_2)(D^2+t^2)} \underbrace{[D,L]}_{tB\varepsilon_{-\Omega}\begin{pmatrix}0&1\\-1&0\end{pmatrix}} e^{s_2(D^2+t^2)}$$

$$+ \cdots$$

It seems that all 2nd order + higher terms are zero because D^2+t^2 commutes with B, hence we will be be able to move B thru to get a $B^2=0$ factor. So we get

$$tr_E\left(\varepsilon_E\, e^{(D^2+t^2)+[D,L]}\right) = e^{F^++t^2} - e^{F^-+t^2}$$

$$= e^{t^2}\left[e^{F^+} - e^{F^-}\right]$$

which is certainly closed as e^{F^+}, e^{F^-} are. Also we know in this case that the cohomology class is zero, hence independent of t.

However if my theory is correct, then it should give the derivative of $e^{t^2}[e^{F^+}-e^{F^-}]$ as d of something explicit, namely

$$tr_s\left(e^{\Omega}\dot{L}\right) \qquad \Omega = D^2+L^2+[D,L]$$

In this case $L = t\varepsilon_\Omega\begin{pmatrix}0&1\\1&0\end{pmatrix}$, so

$$\operatorname{tr}_s\left(e^{\Omega}\mathring{L}\right) = \operatorname{tr}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\int_0^1 e^{(1-s)\begin{pmatrix}F^++t^2&\\&F^-+t^2\end{pmatrix}} tB\varepsilon_\Omega\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$\times\, e^{s\begin{pmatrix}F^++t^2&\\&F^-+t^2\end{pmatrix}}\,\varepsilon_\Omega\begin{pmatrix}0&1\\1&0\end{pmatrix}\, ds$$

$$= \operatorname{tr}\int_0^1 ds\, e^{t^2}t\; e^{(1-s)\begin{pmatrix}F^+&\\&F^-\end{pmatrix}}\, e^{s\begin{pmatrix}F^-&\\&F^+\end{pmatrix}}\, B\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= e^{t^2}t \int_0^1 ds\left(e^{(1-s)F^++sF^-} + e^{(1-s)F^-+sF^+}\right)B$$

$$= 2e^{t^2}t\;\underbrace{\int_0^1 ds\; e^{(1-s)F^-+sF^+}\, B}$$

This I recognize as $\int_0^1 dt\left\{e^{(1-t)F_{A^-}+tF_{A^+}+(t^2-t)(A^+-A^-)^2}(A^+-A^-)\right\}$

Thus the formulas check out the way they are supposed to.