Review goal of the formulas: The idea is to associate cyclic cocycles to the ring $\text{End}\,\mathcal{E} = \Omega^0(M, \text{End}\,E)$. So I want to ~~define~~ construct a differential algebra with a trace on it to which $\text{End}\,\mathcal{E}$ maps. Now ~~the~~ general idea is that I can use a d on the algebra with $d^2 = 0$. So I am permitted to add any θ to the existing d satisfying $d\theta + \theta^2 = 0$ in the big algebra.

What are the requirements for this trace? Is there some way I can normalize it? Connes amazing formula that all you do is to use the S-operator in a simple way.

My analysis: Adjoin to $\Omega(M, \text{End}\,E) \subset \Omega(P) \otimes \text{End}\,V$ the connection form θ which satisfies $D = d + [\theta, \ ]$ and

$$d\theta + \theta^2 = K.$$

$$\iota_X(d\theta + \theta^2) = -d\iota_X\theta + L_X\theta + X\theta - \theta X = 0.$$

~~fo still wh~~ Also when I take $\Omega(M, \text{End}\,E) = e(\Omega(M) \otimes \text{End}\,V)e$ and adjoin de I find that I get

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Basically I see two constructions.

1) adjoin to $\Omega(M, \text{End}\,E) \subset \Omega(P) \otimes \text{End}\,V$ the connection form θ which satisfies $d\theta + \theta^2 = K$. This algebra is unital. ~~I believe~~ It is also independent of the choice of θ, ~~$d(\theta+\eta) + (\theta+\eta)^2 = d\theta + \theta^2 + d\eta + [\theta, \eta] + \eta^2$~~ since two connections differ by an element of $\Omega^1(M, \text{End}\,E)$. The actual connection should enter with the trace on this algebra.

2) Adjoin to $\Omega(M, \text{End}\,E) \subset \Omega(M) \otimes \text{End}\,V$ elements $X^+ = de \cdot e$ and $X^- = e\,de$

$$DX = DX \cdot e + eDX$$

$$eDX \cdot Xe = eX \cdot DXe$$

$$eDX(1-e)Xe = eX(1-e)DXe$$

$$0 = eDX \cdot Xe \mp eX \cdot DXe$$

At this point I have written something about the Lie cocycles I can produce with F.

Now how about Connes version of $\mathcal{S}$?

~~He~~ He introduces an idempotent e and he wants to tensor the given diff'l algebra

$$(\mathrm{End}\,\mathcal{H})^0 \longrightarrow (\mathrm{End}\,\mathcal{H}^1) \longrightarrow \cdots$$

with the diff'l forms on $\mathbb{C}e$. Can I realize that algebra in the above form? 2×2 matrices

Take $\mathcal{H} = \mathbb{C}e \oplus \mathbb{C}(1-e)$

$F = \sigma$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$de = [F,e] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$ede = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$[F,[F,X]] = [\underbrace{[F,F]}_{2F^2 = I},X] - [F,[F,X]]$$

$$\overset{\shortparallel}{0}$$

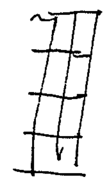
$$\cancel{dX = [X,F]}$$

$$\cancel{d(XY) = [XY,F] = (-1)^{\deg Y}[X,F]\,Y +}$$

Other point. Alg. gen. by θ, Ω with

$d\theta + \theta^2 = \Omega \qquad d\Omega = [\Omega, \theta]$

$$\theta = ede$$
$$\Omega = \begin{matrix} d\theta \\ +\theta^2 \end{matrix} = dede$$

$$\theta = ede$$
$$\Omega = d\theta + \theta^2 =$$ 

$$\varprojlim_{\dim V \to \infty} \left(\bigwedge V \otimes W^* \right)^{GL(V)} \xleftarrow{\sim} S(W \otimes W^*)$$

Complex dual which means that

$$V \otimes W^* + V^* \otimes W$$

standard invariant theory says that it's enough to look at possible permutations

$$(V \otimes W^*)^{\otimes p} \otimes (V^* \otimes W)^{\otimes p}$$

$$k[\Sigma_p] \otimes (W^* \otimes W)^{\otimes p} \qquad \Sigma_p \times \Sigma_p$$

~~so one wants the Σ_p - covariants~~ OKAY.

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Can check that lee

$$d(e\cdot de - de\cdot e) = 2\,de^2$$

$$(ede - de\cdot e)(e\cdot de - de\cdot e) = -\,e\cdot de^2\cdot e - de\cdot e\cdot de$$

$$= -ede^2 - (1-e)\,de^2 = -de^2$$

$\therefore\ dY + Y^2 = de^2$ but the point is that we only want $[dY+Y^2, \varphi]$ where $e\varphi e = \varphi$.

$[e.d.e, \varphi] = e.d.\varphi - \varphi.d.e$ (φ even)

$= e\,d\varphi\,e$

There seem to be lots of possible Y.

e.g. $X = ede + de\,e = de$

$$dX + X^2 = 0 + (de)^2$$

or $d(ede) + (ede)^2 = (de)^2$

or $d(-de\,e) + (-de\,e)^2 = (de)^2$

So in general what works?

$$d\left[a\ ede + b\ de\,e\right] = (a-b)\,de^2$$

$$(a\ ede + b\,de\ e)^2 = ab\ e\,de^2 + ba\ \underbrace{de\ e\ de}_{(1-e)\,de^2}$$

$$= ba\ de^2$$

So the condition is that $a - b + ba = 1$

or $1 - a + b - ab = 0$

or $(1-a)(1+b) = 0$ $\quad a = 1$ or $b = -1$ or both

~~3 possibilities~~.

$$F = \begin{bmatrix} 0 & \tilde{Q} \\ \tilde{P} & 0 \end{bmatrix} \qquad \tilde{P} = \begin{bmatrix} S_1 & -Q \\ P+S_0P & S_0 \end{bmatrix} \qquad \tilde{Q} = \begin{bmatrix} S_1 & Q \\ -P-S_0Q & S_0 \end{bmatrix}$$

$$S_0 = 1-PQ \;, \quad S_1 = 1-QP$$

$$a \rightarrow \begin{bmatrix} a & & & \\ & a & & \\ & & a & \\ & & & 0 \end{bmatrix} \qquad e = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}$$

the old action is $a \rightarrow ae$. Then any $a \in \mathcal{a}$ commutes with F, so $da = Fa - aF = a\,de = de\,a$.

$$\operatorname{Tr}\left(\varepsilon\,(f^0 \cdots f^{2n})\, e\, de \cdots de\right) = \varphi(f^0 f^1 \cdots f^{2n})$$ where

$$\varphi(f) = \operatorname{tr}(\varepsilon f \; e\,de \cdots de)$$ One has

$$[F,e] = \left[\begin{array}{cc|cc} & & -S_1 & \\ & & & S_0 \\ \hline S_1 & & & \\ & -S_0 & & \end{array}\right] \qquad [F,e]^2 = \begin{bmatrix} -S_1^2 & & & \\ & -S_0^2 & & \\ & & -S_1^2 & \\ & & & -S_0^2 \end{bmatrix}$$

so $$\varepsilon\, e\, [F,e]^{2n} = \begin{bmatrix} (-1)^n S_1^{2n} & & & \\ & & & \\ & & & -(-1)^n S_0^{2n} \end{bmatrix}$$

so $$\varphi(f) = (-1)^{n+1} \operatorname{Trace}\left(f S_0^{2n} - f S_1^{2n}\right)$$

Lemma: Let $\mathcal{A}$ be an algebra, φ a cocycle of dim $2n$, $\tilde{\varphi}$ the extension to $\tilde{\mathcal{A}}$. Let $e_0, e \in \mathrm{Proj}(M(\tilde{\mathcal{A}}))$ with $e - e_0 \in \mathcal{A}$ and $[(e, e_0)]$ the corresp. elt. of $K_0\mathcal{A}$

Assume $1\, de_0\, 1 = 0$ i.e. that $d(xe_0)dy = dx\, d(e_0 y)$ $\forall x, y \in \mathcal{A}$, then

$$\langle [e, e_0], [\varphi] \rangle = \frac{1}{n!} \frac{1}{(2\pi i)^n} \varphi(e - e_0, \ldots, e - e_0)$$

Let's consider a Dirac op over a point. This means an odd endo L of $V = V^0 \oplus V^1$. The curvature is L^2 and the Chern character is

$$\operatorname{tr}_s e^{L^2} = \dim V^0 - \dim V^1.$$

Now however I want the cyclic cocycles attached to L on the super algebra $\operatorname{End}(V)$, whatever these are. The best I can do to make this precise is to proceed as in the case of a <u>connection</u>?

One thing I can do now is to do the superconnection game on the Lie algebra cohomology. So I suppose that $\operatorname{End}(V)$?

Take a geometric ~~situation~~: E vector bundle over M $\tilde{\mathfrak{g}} = \operatorname{End}(E)$. In order to define cyclic homology of $\operatorname{End}(E)$ I worked in the bigraded algebra

$$C\big(\tilde{\mathfrak{g}}, \ \Omega(M, \operatorname{End} E)\big)$$

with the connections $\delta + D + t\theta$

When I replace E by a super bundle, then $\Omega(M, \operatorname{End} E)$ is just a super algebra.

A first project would be to construct the basic classes on $\operatorname{End}(V)$. So take MC form

$$\theta \in C^1(\tilde{\mathfrak{g}}) \otimes \operatorname{End}(V)$$

so what we want to do is to work in the algebra

$$\Lambda(\tilde{\mathfrak{g}}^*) = \Lambda(\mathfrak{g}^{ev})^* \otimes S(\mathfrak{g}^{odd})^*$$

and have θ as above. $\theta \in \left[\Lambda(\tilde{\mathfrak{g}}^*) \hat{\otimes} \operatorname{End}(V)\right]^1$ $\delta\theta + \theta^2 = 0$

and then you construct the family of connections !!

$$\delta + t\theta \quad \Big| \quad (\delta + t\theta)^2 = (t^2 - t)\,\theta^2$$

and so I end up with my cocycles namely

$$d \int_0^1 \mathrm{tr}\, e^{D^2 + tD\theta + (t^2-t)\theta^2}\theta = \mathrm{tr}\, e^{D^2+D\theta} - \mathrm{tr}\, e^{D^2}$$

actually you have to watch the range:

$$\int_0^1 \mathrm{tr}\left(D^2 + t[D,\theta] + (t^2-t)\theta^2\right)^n \theta$$

This is all very messy!! So what next?

The idea I had was to work in $\Omega(M)\otimes \mathrm{End}\, V$ with a more complicated d, namely

$$d + 2(ede - de\, e)$$

$$2\, d\left(ede - (de)e\right) = 2\left(dede + dede\right) = 4(de)^2$$

$$4 \quad \left(ede - (de)e\right)\left(ede - (de)e\right) = -e(de)^2 e - de\, e\, de$$

$$= -e(de)^2 - (1-e)de^2$$

$$= -4\,(de)^2$$

$$[Y,[Y,\varphi]] = Y(Y\varphi - \varphi Y) + (Y\varphi - \varphi Y)Y = [Y^2,\varphi]$$

Let me try to understand his system

$$d\varphi = d(e\varphi e) = de\,\varphi + ed\varphi e + (-1)^{\deg\varphi}\varphi de$$

$$= D\varphi + [de.e, \varphi] - [e\,de, \varphi]$$

$$D\varphi = d\varphi + [e.de - de.e, \varphi]$$

$D\varphi = d\varphi + [Y,\varphi]$. Then $D^2\varphi = d^2\varphi + d[Y,\varphi] + [Y, d\varphi + [Y,\varphi]])$

$$= d^2\varphi + [dY,\varphi] - \cancel{[Y,d\varphi]} + \cancel{[Y,d\varphi]} + [Y^2,\varphi]$$

So I start off with a Ξ, D and then I get a sequence of cyclic cocycles associated to the even form

$$\operatorname{tr}(e^{D^2})\cdot\hat{A}(M) \quad \text{on } M.$$

One of my first problems is to represent ~~this form~~ the cyclic cocycles belonging to this form. ~~Something~~ ~~Obvious~~ Somehow this will involve the S-operator. For example what happens for the Dirac operator on a torus. In this case the even form is just $\hat{A}(M)$. So the cyclic cocycle is obvious going to be $\int_{M^{2n}} a^0 da^1 \dots da^{2n}$. Now one of the things I am going to need is $ch(E\otimes F) = ch(E)\cdot ch(F)$ for product connections.

$$D(e\otimes f) = De\otimes f + e\otimes Df$$

$$D^2(e\otimes f) = D^2e\otimes f - De\otimes Df + De\otimes Df + e\otimes D^2 f$$

so this seems to work alright.

Now what do I do in the case of Dirac ops. The problem seems to be simply to extend an even form on M to a cyclic cocycle.