Witten Some QCD inequalities 12/14/83 1

At $\theta = 0$

(1) $m_\pi \leq m_\rho, m_N, m_\Delta$ (Weingarten) N = nucleon

(2) In vector like theories (like QCD) vector symmetries (like isospin + baryon no.) are unbroken (Vafa, E. Witten)

(3) $m^2_{\pi^+} > m^2_{\pi^0}$ (Witten)

(4) (Notation: A, B two flavors of quarks $m_{\bar{A}B}$ = mass of lightest state of $A\bar{B}$ qtm. nos.)

$m_{A\bar{B}} \geqslant \frac{1}{2}(m_{A\bar{A}} + m_{B\bar{B}})$ (Nussinov & Witten)

(based on no mixing with gluons - large N limit)

(5) if quarks are massless, there is a massless particle in spectrum provided $N_f \geq 4$ (Vafa, Witten) t'Hooft proved stronger result

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If we assume then according to t'Hooft there is a massless π or N in $m_q = 0$ limit. So Weingarten + t'Hooft $\Rightarrow$ ($m_q = 0 \Rightarrow$ massless pion $\exists$)

Plausible that this means chiral symm. is broken.

(3) applies to "vacuum alignment" problem in vector-like hyper (techni-) color.

(4) $\Longrightarrow 2M_D \geqslant m_\pi + m_\eta$

Idea behind all these

QCD path integral (at $\theta = 0$)

$$\int dA^a_\mu \, d\psi \, d\bar\psi \, \exp - \frac{1}{g^2}\int \mathrm{Tr}\, F_{\mu\nu}^2 \; \exp \int \bar\psi(\not{D}+M)\psi$$

$$= \int dA^a_\mu \, \exp\left(-\underline{\frac{1}{g^2}\int \mathrm{Tr}\, F_{\mu\nu}^2}\right) \; \det(\not{D}+M).$$

Everything comes from fact that $\det(\not{D}+M) \geq 0$ in vector-like theories

Pf: $\quad -i\not{D}\psi = \lambda\psi \qquad \lambda$ real

$\qquad i\not{D}\gamma_5\psi = -\lambda\gamma_5\psi$

so $\det(\not{D}+M) = \prod_{\lambda_i > 0}(-i\lambda+M)(i\lambda+M) \times \prod_{\lambda_i = 0} M = \prod \lambda^2+M^2 \times \prod M$

i.e. we suppose all eigenvalues of M are > 0

If some were < 0, then $\theta = \pi$

type of fermions	determinant
real	[redacted] pos
pseudo real	real
complex	cx

Effective measure

$$d\mu = \frac{1}{Z} \prod dA^a_\mu \; e^{-\int F^2} \det(\not{D}+M)$$

normal positive measure

Most primitive consequence of having pos. measure
Let X be an operator or product thereof

$$\langle x\rangle^A = \int d\psi d\bar\psi \; e^{\bar\psi i \not{D} \psi} \; x(\psi,\bar\psi,A) \qquad A \text{ fixed}$$

$$\langle x\rangle = \int dA \; \langle x\rangle^A$$

If $\langle x\rangle^A \leq N$ for any A
then $\langle x\rangle \leq N$.

Pf of $m_\pi < m_\rho$ if $m_{up} = m_{down}$

$$\langle \bar u \gamma_5 d(x) \; \bar d \gamma_5 u(0)\rangle \sim e^{-m_\pi |x|}$$

def. of m_π

$$\langle \bar u \gamma_\mu d(x) \; \bar d \gamma_\mu u(0)\rangle \sim e^{-m_\rho |x|}$$

call these $M_\pi(x)$, $M_\rho(x)$ resp. Weingarten shows that $M_\pi(x) \geq M_\rho(x)$ for all x.

Consider quantities $M_\pi^A(x) \geq M_\rho^A(x)$ — A fixed background gauge field.

$$M_\pi^A(x) = \mathrm{Tr}\, \gamma_5 S(x,0)^A \gamma_5 \, S(0,x)^A$$

$$M_\rho^A(x) = \mathrm{Tr}\, \gamma_\mu S(x,0)^A \gamma_\mu \, S(0,x)^A$$

where $S(x,0) = \langle x | \frac{1}{\not{D} + M} | x\rangle$

<u>Lemma</u>: $S(x,0) = \gamma_5 \underbrace{S(0,x)^*}_{} \gamma_5$

take adjoint of the 12×12 (4 spins, 3 colors) finite matrix $S(0,x)$.

true since $\quad \gamma_5 (\not{D} + M) \gamma_5 = -(\not{D}) + M = (\not{D}+M)^*$

$$\therefore \gamma_5 \frac{1}{\not{D}+M} \gamma_5 = \frac{1}{(\not{D}+M)^*}$$

$$M_\pi^A(x) = \mathrm{Tr}\left(S(x,0)^A S(x,0)^{*A} \right)$$

$$M_\rho^A(x) = \mathrm{Tr}\left(S(x,0)^A \gamma_\mu \gamma_5 \, S(x,0)^{*A} \gamma_\mu \gamma_5 \right)$$

no sum over μ

Now $(\gamma_\mu \gamma_5)^2 = 1$ so $M_\pi^A(x) \geq M_\rho^A(x)$ by linear algebra.

Lemma (simplified; see Vafa, W. paper)

$S(x,0;m)^A \leq C e^{-m|x|}$ true for spin 0

If one works with smeared fields this becomes true.

Proof of $m_\pi \leq m_N$ + 2).

Def $\quad |M_B^A(x)| = |\langle qqq(x)\, \bar{q}\bar{q}\bar{q}(0)\rangle^A|$

$$= |\langle qq(x)\, \bar{q}\bar{q}(0)\rangle^A \langle q(x)\, \bar{q}(0)\rangle^A|$$

$$\overset{\text{Lemma}}{\leq} C e^{-m(x)} \langle qq(x)\, \bar{q}\bar{q}(0)\rangle^A$$

$$= C e^{-m(x)} S(x,0)^A_{ik} \, S(x,0)^A_{jl}$$

$$\leq C e^{-m(x)} \underbrace{\mathrm{Tr}\, S(x,0)^A S(x,0)^{*A}}_{\text{pion propagator.}} \quad M_\pi(x)$$

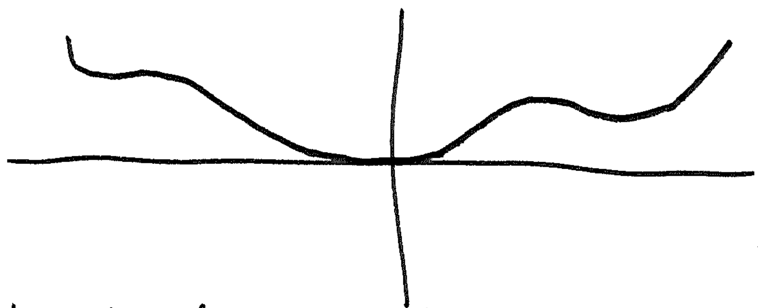
$\therefore \quad m_B \geqslant m_\pi + m_q^{(0)}$ quark bare mass

As cutoff $\Lambda \to \infty$ get $m_B \geqslant m_\pi$

Pf of 2) : vector symm. unbroken.

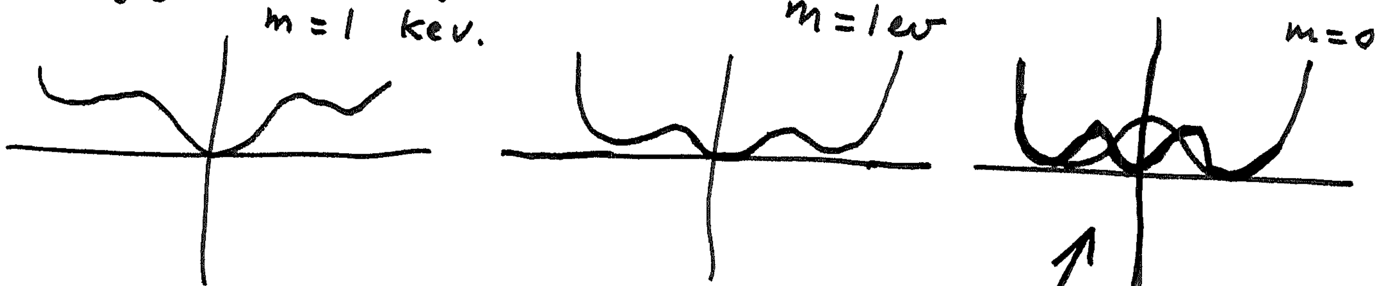
For any $m_u = m_d \neq 0$ $\quad \Lambda < \infty$

isospin unbroken



$\langle u\bar{u} - \bar{d}d \rangle$

Will show state of unbroken isospin is lower in energy that any broken



should occur for quark

Pf of $m_q \neq 0 \Rightarrow$ isospin isn't broken. Must show no Goldstone boson $\exists$. Current 2-pt fn. will contain "if there is one"

$$\underbrace{\langle \bar{u}\, \gamma_\mu\, d(x)\, \bar{d}\, \gamma_\mu\, u(0) \rangle^A}_{\text{need to show exp. decay}} = Tr\left(\gamma_\mu\, S(x,0)^A \gamma_\mu\, S(0,x)^A\right)$$

$$\leqslant 12 \cdot C^2 \cdot e^{-2m|x|}$$

(proof works ~~[illegible]~~ in cutoff theory as $\Lambda \to \infty$ $m \to 0$ so it doesn't work) classical current alg.

$$\left(m_{\pi^+}^2 - m_{\pi^0}^2\right)_{O(\alpha)} = \int d\mu \frac{e^2}{F_\pi^2} \int \frac{d^4k}{(2\pi)^4}$$

$$\times \left(\langle V_\mu^3(k)\, V_\mu^3(-k) \rangle - \langle A_\mu^3(k)\, A_\mu^3(-k) \rangle \right)$$

$V_\mu^3 = \bar{q}\, i \gamma_\mu \gamma_5 T^3 q$ vector current

$A_\mu^3 = \bar{q}\, i \gamma_\mu \gamma_5 T^3 q$ axial vector "

Finite Vol.

$$\text{Bracket} = \frac{1}{V} \int d^4x\, d^4y\, e^{ik(x-y)}$$

$$\left[\operatorname{Tr} \gamma_\mu S(x,y) \gamma_\mu S(y,x) - \operatorname{Tr} \gamma_\mu \gamma_5 S(x,y) \gamma_\mu \gamma_5 S(y,x) \right]$$

what appears is $S + \gamma_5 S \gamma_5$ which projects out the part of S that commutes with γ_5.

$$S(x,y) = \langle x | \frac{1}{\not{D} + m} | y \rangle$$

$$\text{Bracket} = \frac{2}{V} \cdot \int d^4x\, d^4y\, e^{ik\cdot(x-y)}$$

$$\times \operatorname{Tr} E(x,y) \gamma_\mu E(y,x) \gamma_\mu$$

E = even part

$$E = \frac{m}{(-\not{D})^2 + m^2} \geq 0$$

$$= \frac{2}{V} \operatorname{Tr} E M_\mu E M_\mu^*$$

$$= \sum \underbrace{\lambda_i \lambda_j}_{\text{as } E \geq 0} |\langle i | M_\mu | j \rangle|^2 \geq 0$$