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Review:

$$M = S^{2n} \qquad \not{D}_A = \gamma^\mu(\partial_\mu + A_\mu) \quad \text{on} \quad \underset{\substack{\uparrow \\ \text{spinors}}}{S} \otimes \mathbb{C}^N$$

$$A = A_\mu dx^\mu = \underset{\substack{\uparrow \\ \text{basis for } \mathfrak{g}.}}{\lambda_a} A^a_\mu dx^\mu \qquad \text{gauge field for} \quad G \subset U_N$$

$\mathcal{A}$ = space of A = $\Omega^1(M, \mathfrak{g})$

$\mathcal{G}$ = gauge transf. group = $\mathrm{Maps}_{pt}(M, G) = \Omega^{2n} G$

$\mathcal{G}$ acts on $\mathcal{A}$: $\qquad A_g = g^{-1}dg + g^{-1}Ag$

The action is free, so we get a principal $\mathcal{G}$-bundle

$$\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}.$$

Family of elliptic operators on M param. by $\mathcal{A}$:

$$\not{D}_A = \gamma^\mu(\partial_\mu + A_\mu)\left(\frac{1+\gamma^5}{2}\right) : S^+ \otimes \mathbb{C}^N \longrightarrow S^- \otimes \mathbb{C}^N$$

$$g^{-1} \cdot \not{D}_A \cdot g = \not{D}_{A_g}$$

Determinant line bundle $\mathcal{L}$ over $\mathcal{A}$ belonging to this family

$$\mathcal{L}_A = \Lambda^{max}(\mathrm{Ker}\, \not{D}_A) \otimes \Lambda^{max}(\mathrm{Ker}\, \not{D}_A^\dagger)^*$$

$\mathcal{L}$ is a $\mathcal{G}$-equivariant line bundle over $\mathcal{A}$, so get a line bdle $\bar{\mathcal{L}} = \mathcal{L}/\mathcal{G}$ over $\mathcal{A}/\mathcal{G}$.

Assertion 1: An acceptable gauge-invariant way of defining

$$\int \mathcal{D}\bar\psi \, \mathcal{D}\psi \, e^{-\bar\psi \not{D}_A \psi} \psi(x_1) \cdots \psi(x_n) \bar\psi(y_n) \cdots \bar\psi(y_1)$$

furnishes a non-vanishing section of $\bar{\mathcal{L}}$.

Top. problem $c_1(\mathcal{L}) \in H^2(\mathcal{A}/\mathcal{G}, \mathbb{Z})$.

Atiyah - Singer : In $H^2(\mathcal{A}/\mathcal{G}, \mathbb{C})$

$$c_1(\mathcal{L}) = \int_M ch_{n+1}(\bar{E})$$

where $\bar{E} = \cancel{(\mathcal{A} \times M \times \mathbb{C}^N)/\mathcal{G}}$ over $\mathcal{A}/\mathcal{G} \times M$.

$\mathcal{A} \times^{\mathcal{G}} (M \times \mathbb{C}^N)$ $\quad (A_g; x, z) \sim (A; x, g(x)z)$

and $ch_{n+1}(\bar{E})$ is represented by the form

$$\left(\frac{i}{2\pi}\right)^{n+1} \operatorname{tr} \frac{(\bar{D}^2)^{n+1}}{(n+1)!} \quad \text{on} \quad \mathcal{A}/\mathcal{G} \times M$$

$\bar{D}$ = a connection on $\bar{E}$

$\bar{D}^2$ = its curvature.

~~Transgression:~~ $\cancel{H^2(\mathcal{A}/\mathcal{G}) \longrightarrow H^1(\mathcal{G})}$ ~~isom. if~~ $\cancel{\pi_0 \mathcal{G} = 0}$ ~~is defined as follows:~~

$$\cancel{\mathcal{G} \xhookrightarrow{i} \mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{G}}$$
$$\cancel{g \longmapsto A^0_g}$$

Transgression in the fibre bundle

$$\mathcal{G} \xhookrightarrow{i} \mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{G} \qquad i(g) = A^0_g$$

$i^*\beta \qquad d\beta = \pi^*\alpha \qquad \alpha$

is the map

$$H^*(\mathcal{A}/\mathcal{G}) \longrightarrow H^*(\mathcal{G}) \qquad \text{deg } -1,$$

class of $\alpha \longmapsto$ class of $i_*\beta$ where $d\beta = \pi^*\alpha$

This map is an isom $H^2(\mathcal{A}/\mathcal{G}) \xrightarrow{\sim} H^1(\mathcal{G})$ if $\pi_0 \mathcal{G} = 0$.

Transgression in the fibre bundle

$$\mathcal{G} \overset{i}{\hookrightarrow} \mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{G} \qquad i(g) = A^0_g$$

$$\begin{tikzcd} & \pi^*\alpha & \alpha \arrow[l, "\pi^*"'] \\ i^*\beta & \beta \arrow[l, "i^*"'] \arrow[u, "d"] & \end{tikzcd}$$

gives a map $\quad H^*(\mathcal{A}/\mathcal{G}) \longrightarrow H^{*-1}(\mathcal{G})$

$\text{class}(\alpha) \longmapsto \text{class}(i^*\beta)$

If $\pi_0 \mathcal{G} = 0$, then $\quad H^2(\mathcal{A}/\mathcal{G}) \xrightarrow{\sim} H^1(\mathcal{G})$.

To compute image of $c_1(\mathcal{L})$ in $H^1(\mathcal{G})$: $\quad d+\bar{A}$ on $\mathcal{A}\times M\times\mathbb{C}^N \quad \bar{D}$ on $\bar{E}$

$$\mathcal{G}\times M \xrightarrow{i} \mathcal{A}\times M \xrightarrow{\pi} \mathcal{A}/\mathcal{G}\times M$$

$$\begin{tikzcd} & \frac{1}{(n+1)!}\operatorname{tr}(F_{\bar{A}}^{n+1}) & \operatorname{tr}\frac{(\bar{D}^2)^{n+1}}{(n+1)!} \arrow[l, "\pi^*"'] \\ t_{2n+1} & \underbrace{\int_0^1 dt\, \frac{1}{n!}\operatorname{tr}(\bar{A}\, F_{\bar{A},t}^n)}_{\text{Chern-Simons form}} \arrow[l, "i^*"'] \arrow[u, "d"] & \end{tikzcd}$$

$\times\left(\frac{i}{2\pi}\right)^{n+1}$

$$F_{\bar{A},t} = t\,d\bar{A} + t^2\bar{A}^2$$

$\therefore \int_M t_{2n+1} \in \Omega^1(\mathcal{G})$ rep. the image of $c_1(\mathcal{L})$ under transg: $H^2(\mathcal{A}/\mathcal{G}) \to H^1(\mathcal{G})$.

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Construction of $\bar{A}$, $\bar{D}$

$$\tilde{\mathfrak{g}} = \mathrm{Lie}(\mathcal{G}) = \{ v \in \Omega^0(M,\mathfrak{g}) \mid v(\infty) = 0 \}$$

$$g * A = g^{-1} dg + g^{-1} A g$$

$$v * A = dv + [A, v]$$

Then $A \to A + v * A$ is a v.f. on $\mathcal{A}$ whose effect on functions is

$$X_v \Phi(A) = \Phi(A + v * A) - \Phi(A)$$

$$= - \int dx \; v(x) \left(\partial_\mu + [A_\mu, \;] \right) \frac{\delta}{\delta A_\mu(x)} \Phi(A)$$

Next consider the vector bundles

$$\begin{tikzcd} E = \mathcal{A} \times M \times \mathbb{C}^N \arrow[d] & & \bar{E} \arrow[d] \\ \mathcal{A} \times M \arrow[rr, "\mathcal{G}\text{ orbits}"] & & \mathcal{A}/\mathcal{G} \times M \end{tikzcd}$$

On sections of E $\Phi(A, x) \in \mathbb{C}^N$ we have the taut. conn.

$$d_{\mathcal{A} \times M} + \tilde{A} \qquad \text{where } \tilde{A} \Phi(A,x) = A(x) \Phi(A,x)$$

and the $\mathcal{G}$-action

$$g * \Phi(A, x) = g(x) \Phi(g * A, x)$$

$$v * \Phi(A,x) = (X_v + v) \Phi(A, x).$$

The $\tilde{A}$ connection is $\mathcal{G}$-invariant:

$$[X_v + v, d + \tilde{A}] = 0.$$

so at first sight you might expect it to descend to a connection on $\bar{E}$. But

An invariant connection descends to the orbit space $\Longleftrightarrow$ the Higgs field is zero:

$$v * \Phi = i_{X_v}(d+\tilde{A})\Phi + \varphi_v \Phi$$

action on E — lift by connection of action on $\mathcal{A}\times M$ — Higgs field

We modify our connection to kill the Higgs field.

An invariant connection descends to the orbit space $\Longleftrightarrow$ the Higgs field is zero.

$$\iota_v \left(d_{\mathcal{A}\times M} + \tilde{A}\right) = X_v$$

$$v^* = X_v + v$$

The Higgs field is v in our case

We ~~must~~ modify our connection to kill the Higgs field ~~i.e. find a Θ such that~~ as follows:

Let Θ be a conn. form in $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\quad} & T_A(\mathcal{A}) \\ \| & & \| \\ \Omega^0(M,\mathfrak{g}) & \xrightarrow{D_A} & \Omega^1(M,\mathfrak{g}) \end{array} \qquad \begin{array}{l} v \longmapsto \delta_v A = [D_A, v] \\ \\ \text{put } D_A v = [D_A, v] \end{array}$$

$$\Theta_A : T_A(\mathcal{A}) \dashrightarrow \tilde{\mathfrak{g}}$$

Want $\Theta D_A v = v$. Simplest $\mathcal{G}$-inv. choice is

$$\Theta_A = \left(D_A^* D_A\right)^{-1} D_A^* \qquad \Theta \in \Omega^1(\mathcal{A}, \tilde{\mathfrak{g}})$$

New connection is

$$d_{\mathcal{A}\times M} + \bar{A} = d_{\mathcal{A}\times M} + \tilde{A} + \Theta$$

~~$\iota_v\left(d_{\mathcal{A}\times M} + \bar{A}\right)\Phi = (X_v + \underbrace{\iota_v\Theta}_{v})\Phi = v^*\Phi.$~~

~~(New curvature?)~~