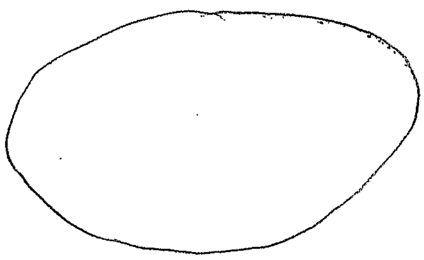
F. Kirwan Feb 28, 1983

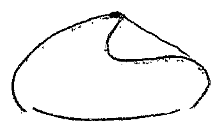
How can use link between convex bodies and algebraic varieties to prove thms. about convex bodies.

Isoperimetric inequalities

In R^2



$$L^2 \geqslant 4\pi A$$



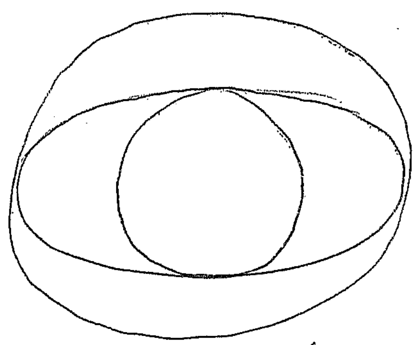
shows suffices for convex

In $\mathbb{R}^n$ K bounded convex

$$(\mathrm{vol}\ \partial K)^n \geqslant n^n (\mathrm{vol}\ B_n)(\mathrm{vol}\ K)^{n-1}$$

B_n: unit ball in $\mathbb{R}^n$.

In and out radius inequalities

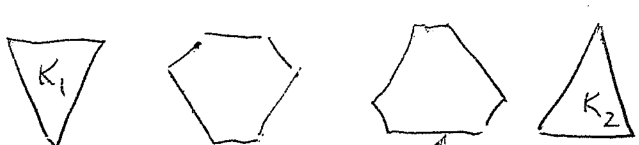


All these can be proved by methods in algebraic geometry.

① Mixed Volumes $K_1\ K_2 \subset \mathbb{R}^2$ convex

$$\lambda_1 K_1 + \lambda_2 K_2 = \{\lambda_1 x_1 + \lambda_2 x_2\ ,\ x_i \in K_i\}$$

e.g.



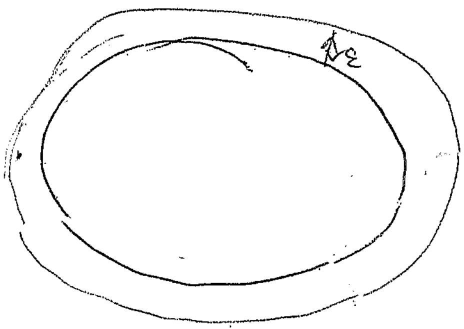
$\frac{2}{3}K_1 + \frac{1}{3}K_2$ $\qquad$ $\frac{1}{3}K_1 + \frac{2}{3}K_2$

[redacted] Fact

$$\mathrm{vol}(\lambda_1 K_1 + \lambda_2 K_2) = \lambda_1^2 \mathrm{vol}(K_1) + \lambda_1\lambda_2 \mathrm{vol}(K_1, K_2) + \lambda_2^2 \mathrm{vol}(K$$

Now look at

$$\mathrm{vol}(K + \varepsilon B_2) = \mathrm{vol}(K) + \varepsilon\, \mathrm{vol}(K, B_2) + O(\varepsilon^2)$$



$$\Longrightarrow \mathrm{vol}(K, B_2) = \mathrm{vol}(\partial K)$$

In $\mathbb{R}^n$

$$\mathrm{vol}(\lambda_1 K_1 + \lambda_2 K_2) = \lambda_1^n \mathrm{vol}(K_1) + n\lambda_1^{n-1}\lambda_2 \mathrm{vol}_1(K_1, K_2) + \binom{n}{2}\lambda_1^{n-2}\lambda_2^2 \mathrm{vol}_2(K_1, K_2) + \cdots\cdots$$

Then

$$n\, v_1(K, B_n) = \mathrm{vol}(\partial K)$$

Hence iso-perimetric inequality is equivalent to

$$v_1(K, B_n)^n \geqslant \mathrm{vol}(B_n)\, \mathrm{vol}(K)^{n-1}$$

Shall prove $\forall\, K_1, K_2 \subset \mathbb{R}^n$, convex, we have if $v_j(K_1, K_2)$, $0 \leqslant j \leqslant n$ $v_0/v_1 \leqslant v_1/v_2 \leqslant \cdots \leqslant v_{n-1}/v_n$ (Alexandroff-Fenchel inequalities 1930's)

These imply $(v_0/v_1)^n \leqslant v_0/v_n$ i.e. $v_0^{n-1} v_n \leqslant v_1^n$

Proof: Enough to prove for convex polyhedra with integral vertices, i.e. $K_i = \hat{S}_i$ S_i finite

$\alpha \in \mathbb{Z}^n \longleftrightarrow$ character of $T_{\mathbb{C}}^n = (\mathbb{C}^*)^n$

$$V_S = \bigoplus_{\alpha \in S} V_\alpha \qquad T_{\mathbb{C}}^n \text{ acts on } \mathbb{P}_S = \mathbb{P}(V_S)$$

We can assume orbit of $1, \dots, 1$ is faithful; ~~otherwise $\hat{S}$ contained~~ otherwise $\hat{S}$ has 0 vol.
Assume differences $\alpha - \beta$ $\alpha, \beta \in S$ span $\mathbb{Z}^n$.

Get $\quad T_{\mathbb{C}}^n \hookrightarrow \mathbb{P}_{S_1} \times \mathbb{P}_{S_2}$ $\qquad \omega_i$ Kahler forms on $\mathbb{P}_{S_i}$

call the image X; $\bar{X}$ is proj. var. Moment maps

$$\mu_i : \bar{X} \longrightarrow \mathbb{R}^n$$

$$\mathrm{Im}(\mu_i) = \hat{S}_i = K_i$$

Used this to get

$$\mathrm{vol}(\lambda_1 K_1 + \lambda_2 K_2) = \frac{1}{n!} (\lambda_1 \omega_1 + \lambda_2 \omega_2)^n [X]$$

So

$$\Longrightarrow \qquad v_j(K_1, K_2) = \frac{1}{n!} \omega_1^{n-j} \omega_2^j [\bar{X}]$$

AF inequals $\Longleftrightarrow \quad v_j^2 \geq v_{j+1} v_{j-1}$

Can reduce to a surface

$$S = \bar{X} \cap n-j-1 \text{ hyperplanes generic in } \mathbb{P}_{S_1}$$
$$\cap \ j-1 \text{ ——— } \mathbb{P}_{S_2}$$

Will assume for the moment that $\bar{X}$ is non-sing.
Bertini $\Longrightarrow$ S non-singular surface.

$$[S] = \omega_1^{n-j-1} \omega_2^{j-1} [\bar{X}]$$

$$v_j = \omega_1 \omega_2 [S], \quad v_{j-1} = \omega_1^2 [S], \quad v_{j+1} = \omega_2^2 [S].$$

Take $\quad C_i = S \cap$ gen. hyperplane of $\mathbb{P}_{S_i}$ $\qquad$ curves on S

Then $v_j = C_1 C_2 \qquad v_{j-1} = C_1^2 \qquad v_{j+1} = C_2^2$

and so we want $(C_1 \cdot C_2)^2 \geq C_1^2 C_2^2$.

Consider quadratic form $(C_1 + \lambda C_2)^2 = C_1^2 + 2\lambda C_1 C_2 + \lambda^2 C_2^2$

Desired ineq. $\iff$ form is indefinite

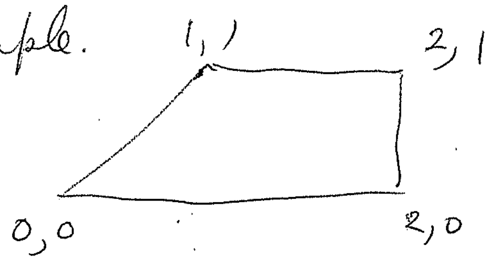
$\iff$ form not pos. def. (since $C_1^2 > 0$)

Precisely the Hodge index thm implies this.

Can desingularize S. But actually the singularities of $\bar{X}$ can be resolved by a combinatorial process. However can chop off corners to reduce to a non-singular case.

When is $\bar{X}$ non-singular? $n = 2$

Example.



orbit $(1, st, s^2 t, s^2)$ $\qquad s, t \in \mathbb{C}^*$

$zx^2 = y^2$ defines $\bar{X}$

If you add in 1,0 then non-sing.

Faces of $K = \hat{S} \iff$ orbits in $\bar{X}$

vertices $\longleftrightarrow$ fixpts.

Enough to show all fixpoints are non-singular