January 22, 1983

First lecture - sentences for:

A $\bar\partial$-operator on E is a first differential operator $D : \Gamma(E) \longrightarrow \Gamma(E \otimes T^{0,1})$ whose symbol is $id_E \otimes \{pr : T^* \to T^{0,1}\}$. Locally, if we choose a coordinate z and a trivialization of E so that sections of E become vector fns. we have

$$Df = (\partial_{\bar z} + \alpha) f \, d\bar z$$

where α is an $r \times r$ matrix function.

Local existence thm. says that locally we can find linearly independent sections in $\operatorname{Ker} D$. Using such a holomorphic trivialization one has $\alpha = 0$.

A parametrix for D is an operator $P : \Gamma(E \otimes T^{0,1}) \to \Gamma(E)$ such that $PD, DP \equiv I$ mod smooth kernel operators. The Schwartz kernel for P is a smooth section of a v.b over $M \times M - \Delta M$:

$$\langle z | P | z' \rangle \in E_z \otimes E^*_{z'} \otimes T^{0,1}_{z'}$$

and in a local holom. trivialization has the form:

$$\langle z | P | z' \rangle = (i/2\pi) \left\{ \frac{1}{z - z'} + \text{smooth} \right\} dz' .$$

Choosing 1) a metric on M, 2) a connection on E extending D we can construct a parametrix as follows: For (z, z') near the diagonal in $M \times M$ take

$$\left(\frac{i}{2\pi} \right) (-d'_{z'} \log r(z,z')^2) \cdot \underbrace{F(z,z')}_{\text{isom of } E_{z'} \xrightarrow{\sim} E_z \text{ given by radial parallel transport}}$$

and then extend smoothly to the rest of $M \times M$.

In the first lecture I show how to regularize $\operatorname{Tr}(D^{-1} dD)$, and so I define a connection on the determinant line bundle over the invertible set. Next I have to define the connection where D is not invertible. First one must describe the determinant line bundle.

Fix $\quad D_0 : \underbrace{\Gamma(E)}_{W} \longrightarrow \underbrace{\Gamma(E \otimes T^{0,1})}_{V}$.

Actually it seems better to work with Hilbert spaces W, V and the space $\mathcal{F}$ of all Fredholm operators $T : W \longrightarrow V$. To each T associate the 1-diml space

$$\mathcal{L}_T = \lambda(\mathrm{Cok}\, T) \otimes \lambda(\mathrm{Ker}\, T)^*.$$

I claim these are the fibres of a (holomorphic) line bundle over $\mathcal{F}$. To see this take a finite dimensional subspace $F \subset V$. The set $\mathcal{U}_F$ of $T \in \mathcal{F}$ which are transversal to F is open, and by choosing F suff. large we can make $\mathcal{U}_F$ contain any element of $\mathcal{F}$. Over $\mathcal{U}_F$ we have exact sequences

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Ker}\, T & \longrightarrow & W & \xrightarrow{T} & V & \longrightarrow & \mathrm{Cok}\, T & \longrightarrow 0 \\
& \| & & \cup & & \cup & & \| & \\
0 \longrightarrow & \mathrm{Ker}\, T & \longrightarrow & T^{-1}F & \longrightarrow & F & \longrightarrow & \mathrm{Cok}\, T & \longrightarrow 0
\end{array}$$

and we get a vector bundle with fibre $T^{-1}F$ at T. First say we have a canonical isom.

$$\lambda(\mathrm{Cok}\, T) \otimes \lambda(\mathrm{Ker}\, T)^* \simeq \lambda(F) \otimes \lambda(T^{-1}F)^*$$

then say that $T \longmapsto T^{-1}F$ is a vector bundle over $\mathcal{U}_F$; then conclude $T \longmapsto \mathcal{L}_T$ is a line bundle over $\mathcal{U}_F$.