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September 20, 1983

The problem is to construct the character forms in equivariant cohomology associated to an invariant connection D on any equivariant bundle E/M. Equivariant forms are basic elements of $W(\mathfrak{g})\otimes\Omega(M)$. Over the weekend I learned that I should think of $W(\mathfrak{g})$ in terms of its universal properties of representing connections. Thus there is a canonical connection form

$$\theta = \lambda_a\theta^a \in \mathfrak{g}\otimes W^1(\mathfrak{g})$$

such that $W(\mathfrak{g})$ is generated by the components $\{\theta^a\}$ of θ and of the curvature

$$\Omega = d\theta + \theta^2 \in \mathfrak{g}\otimes W^2(\mathfrak{g}).$$

The equivariant character forms will be obtained by using the connection

$$D + \varphi\theta = D + \theta^a\varphi_a$$

which is 'basic', unlike D which is just invariant. Now I calculated:

$$(D+\varphi\theta)^2 = D^2 - \theta^a\iota_a D^2 + \frac{1}{2}\theta^a\theta^b\iota_b\iota_a D^2 + \Omega^a\varphi_a$$

The first three terms on the right represent the image of D^2 under the isom.

$$\Omega(M) \xrightarrow{\sim} \left[(\Lambda\mathfrak{g}^*)\otimes\Omega(M)\right]_{horiz}$$

This isomorphism is the present obstruction to doing the calculation of the character forms. So I really should understand it a lot better. Let's start with some geometric arguments. The idea will be that $W(\mathfrak{g})$ is needed for arbitrary connections, ~~but~~ that $\Lambda(\mathfrak{g}^*)$ is all you need for flat connections.

Now if we have a flat G-bundle P over Y and a G-map from P to M, then I have two ways to go from $\Omega(M)^G$ to $\Omega(Y)$. The first uses the flat connection and the G-maps to define

$$\Lambda\mathfrak{g}^* \otimes \Omega(M) \longrightarrow \Omega(P)$$

and then one takes basic elements, using the isom.

$$\left[\Lambda\mathfrak{g}^* \otimes \Omega(M)\right]_{\text{horiz}} \simeq \Omega(M)$$

The second uses the associated fibre bundle $P\times^G M$, and the flat connection to define a map

$$\Omega(M)^G \longrightarrow \Omega(P\times^G M)$$

and then one composes with the section of $P\times^G M/Y$ given by the G-map $P\longrightarrow M$. These two ideas are obviously the same — consider the diagram

$$\begin{array}{ccccc}
\Lambda\mathfrak{g}^* \otimes \Omega(M) & \longrightarrow & \Omega(P\times M) & \xrightarrow{(\text{graph})^*} & \Omega(P) \\
\uparrow & & \uparrow & & \uparrow \\
\Omega(M)^G & \longrightarrow & \Omega(P\times^G M) & \longrightarrow & \Omega(Y)
\end{array}$$

What I see is that the map

$$\Omega(M)^G \xrightarrow{\sim} [\Lambda \mathfrak{g}^* \otimes \Omega(M)]_{basic}$$

which I want to describe algebraically, corresponds to the geometric map $\Omega(M)^G \longrightarrow \Omega(P \times^G M)$ defined by a flat connection. This doesn't seem to help.

Let us review the proof of the isom.

$$\Omega(M) \simeq [\Lambda \mathfrak{g}^* \otimes \Omega(M)]_{horiz.}$$

Take a basis λ_a for $\mathfrak{g}$ and let $\theta^a \in \mathfrak{g}^*$ be the dual basis. Put $\iota_a = \iota_{\lambda_a}$. Then we claim $\theta^a i_a$ (no summation convention) are commuting projectors.

$$\theta^a \iota_a \theta^b \iota_b = \theta^a(\delta_{ab} - \theta^b \iota_a)\iota_b$$

$$= \begin{cases} \theta^a \iota_a & \text{if } a = b \\ \theta^a \theta^b \iota_b \iota_a & \text{if } a \neq b. \end{cases}$$

and the last product is symmetric in a, b. An element of $\Lambda \mathfrak{g}^* \otimes \Omega(M)$ is horizontal when it is killed by all the operators i_a. Now

$$1 - \theta^a i_a = \iota_a \theta^a,$$

so $\quad \iota_a \alpha = 0 \implies \alpha = \iota_a \theta^a \alpha \implies \iota_a \alpha = \iota_a^2 \theta^a \alpha = 0.$

Hence the horizontal elements are those α in the image of the projection operator

$$\prod_a \iota_a \theta^a = \prod_a (1 - \theta^a \iota_a)$$

Next note that $(\Lambda \mathfrak{g}^*)^+ \otimes \Omega(M)$ is killed by this projection operator. I can write it

$$\iota_1 \theta^1 \cdots \iota_n \theta^n = \iota_1 \cdots \iota_n \theta^n \cdots \theta^1$$

and $\theta^n \cdots \theta^1$ kills $(\Lambda \mathfrak{g}^*)^+$. So the basic elements are of the form

$$\prod_a (1 - \theta^a \iota_a) \alpha = \sum_I (-1)^{|I|} \theta^I \iota_I \alpha \qquad \alpha \in \Omega(M)$$

where $I = \{j_1, \ldots, j_p\}$ runs over subsets of $\{1, \ldots, n\}$ and

$$\theta^I = \theta^{j_1} \cdots \theta^{j_p}, \quad \iota_I = \iota_{j_p} \cdots \iota_{j_1}.$$

We can see what α is by the augmentation map

$(*)$ $$(\Lambda \mathfrak{g}^*) \otimes \Omega(M) \longrightarrow \Omega(M)$$

killing $(\Lambda \mathfrak{g}^*)^+ \otimes \Omega(M)$. So we can formulate the result as follows.

Prop: The augmentation $(*)$ induces an isom.

$$\left[(\Lambda \mathfrak{g}^*) \otimes \Omega(M) \right]_{horiz} \xrightarrow{\sim} \Omega(M)$$

This is an algebra isomorphism.

The horizontal elements form an algebra because ι_X are derivations, and the kernel of a derivation is an algebra.

Next I want to verify that

$$S(\mathfrak{g}^*) = [W(\mathfrak{g})]_{horiz.}$$

I first check that $i_X \Omega = 0$ for any $X \in \mathfrak{g}$. (I am taking the viewpoint that $W(\mathfrak{g})$ has a universal connection; this means that I have to check $i_X \Omega = 0$ for any connection θ.)

So I start with

$$\theta \in \mathfrak{g} \otimes \Omega^1(P)$$

which is invariant:

$$[X, \theta] + \mathcal{L}_X \theta = 0$$

and s.t. $\quad i_X \theta = X.$ Then

$$\begin{aligned} i_X \Omega &= i_X\left(d\theta + \tfrac{1}{2}[\theta,\theta]\right) \\ &= i_X d\theta + \tfrac{1}{2}\left([X,\theta] - [\theta, X]\right) \\ &= \mathcal{L}_X\theta - \underbrace{d\, i_X \theta}_{=0 \text{ as } X \text{ is a const. fn. on } P} + [X,\theta] \\ &= 0 \quad \text{by invariance.} \end{aligned}$$

QED

The fact that $S(\mathfrak{g}^*) = W(\mathfrak{g})_{horiz}$ is then clear because we know the horizontal elements are given by the image of $\prod_a i_a \theta^a = i_1 \cdots i_n \theta^n \cdots \theta^1$ and this kills $(\Lambda \mathfrak{g}^*)^+$, etc.

The grand conclusion now is that to do computations in equivariant cohomology, I can take advantage of the algebra isomorphisms:

$$\circledast \qquad [W(\mathfrak{g}) \otimes \Omega(M)]_{horiz} \xrightarrow{\sim} S(\mathfrak{g}^*) \otimes \Omega(M)$$

$$\cup \qquad\qquad\qquad \cup$$

$$(\ast\ast) \qquad [W(\mathfrak{g}) \otimes \Omega(M)]_{basic} \xrightarrow{\sim} [S(\mathfrak{g}^*) \otimes \Omega(M)]^G$$

where the map is induced by the augmentation in $\Lambda \mathfrak{g}^*$, i.e. all $\theta^a \mapsto 0$.

So now go back to the equivariant curvature of E/M:

$$(D+\varphi\theta)^2 = D^2 - \theta^a \iota_a D^2 + \tfrac{1}{2}\theta^a\theta^b \iota_b \iota_a D^2 + \Omega^a \varphi_a .$$

This is a matrix of horizontal forms so that if I want to compute the character classes from it I can use the above isomorphism and work with the much simpler expression

$$D^2 + \Omega^a \varphi_a \quad \in S(\mathfrak{g}^*) \otimes \Omega(M, \mathrm{End}\, E).$$

New question: Consider the isomorphism $(\ast\ast)$. We know d on $W(\mathfrak{g}) \otimes \Omega(M)$ induces a d on the basic subcomplex (but not the horizontal subcomplex). Hence it is natural to ask what the formula for d is if we use this isomorphism. More precisely, what is the operator on $[S(\mathfrak{g}^*) \otimes \Omega(M)]^G$ which corresp. to d on the LHS of $(\ast\ast)$. Since the RHS has only things like d_M, Ω^a, ι_a the natural guess

is
$$d_M - \Omega^a \iota_a \quad \text{on} \quad [S(\mathfrak{g}^*) \otimes \Omega(M)]^G .$$
$\uparrow$

Start with an element of $\uparrow$. It is a linear combination of terms $p_j \alpha_j$ with $p_i \in S(\mathfrak{g}^*)$ a poly in the Ω^a and $\alpha_j \in \Omega(M)$. We lift $\sum p_j \alpha_j$ back to the horiz. element, which is basic

$$\sum_j p_j \cdot \left(\alpha_j - \theta^a \iota_a \alpha_j + \tfrac{1}{2} \theta^a \theta^b \iota_b \iota_a \alpha_j - \cdots \right)$$

apply d - the sum over j will ~~still~~ be basic. Then we apply the augmentation ε to get back in $S(\mathfrak{g}^*) \otimes \Omega(M)$. We get

$$\sum_j \varepsilon(dp_j) \cdot \alpha_j + p_j \, d\alpha_j - \varepsilon(d\theta^a) \iota_a \alpha_j .$$

Now
$$dp_j = \frac{\partial p_i}{\partial \Omega^a} d\Omega^a$$
and we know from the Bianchi identity:
$$d\Omega = [\Omega, \Theta] \quad \text{or} \quad d\Omega^a = f^a_{bc} \Omega^b \theta^c$$
that $\varepsilon(d\Omega^a) = 0$. Also from $\Omega = d\Theta + \theta^2$ or
$$\Omega^a = d\theta^a + \tfrac{1}{2} f^a_{bc} \theta^b \theta^c$$
we know that $\varepsilon(\Omega^a) = \varepsilon(d\theta^a)$. Hence we get

$$\sum_j p_j \left(d\alpha_j - \Omega^a \iota_a \alpha_j \right)$$

which shows that
$$d = d_M - \Omega^a \iota_a \quad \text{on} \quad [S(\mathfrak{g}^*) \otimes \Omega(M)]^G .$$

But I haven't used invariance anywhere in this computation. Now we know from the case of a circle action that for

$$D = d - u\iota_X \quad \text{on} \quad k[u] \otimes \Omega(M)$$

we don't get $D^2 = 0$ unless we restrict to invariants. Let's check this in general.

$$(d_M - \Omega^a \iota_a)^2 = \cancel{d_M^2}^{=0} - d_M \Omega^a \iota_a - \Omega^a \iota_a d_M + \cancel{\Omega^a \iota_a \Omega^b \iota_b} = 0 \text{ because } \Omega^a\Omega^b \text{ symm. } \iota_a \iota_b \text{ skew}$$

$$= -\Omega^a \underbrace{(d_M \iota_a + \iota_a d_M)}_{\mathcal{L}_{\lambda_a}}$$

so it works.

Extrapolating further, we expect an isom.

$$[W(\mathfrak{g}) \otimes \Omega(M, \mathrm{End}\,E)]_{basic} \xrightarrow{\sim} [S(\mathfrak{g}^*) \otimes \Omega(M, \mathrm{End}\,E)]^G$$

such that covariant diffn D on the LHS corresponds to

$$D_M - \Omega^a \iota_a \quad \text{on the RHS.}$$

Check:
$$(D_M - \Omega^a \iota_a)^2 = D_M^2 - D_M \Omega^a \iota_a - \Omega^a \iota_a D + \Omega^a \iota_a \Omega^b \iota_b$$

$$= D_M^2 - \Omega^a (D_M \iota_a + \iota_a D_M) + \cancel{\Omega^a \Omega^b \iota_a \iota_b} \quad (\text{symm} \cdot \text{skew})$$

$$= D_M^2 + \Omega^a \varphi_a \quad \text{on invariants because}$$

in general $\mathcal{L}_a = [\iota_a, D] + \varphi_a$.

<u>September 22, 1983</u>

Problem: Let's consider a bundle $E / Y \times M$ with a connection D which we split into horizontal and vertical connections

$$D = D' + D''.$$

The curvature is

$$D^2 = D'^2 + D'D'' + D''D' + (D'')^2$$

Let's work locally on $Y \times M$ and trivialize E. Then

$$D = d + A = d' + d'' + A' + A''$$
$$= \underbrace{d' + A'}_{D'} + \underbrace{d'' + A''}_{D''}$$

and
$$D^2 = \begin{pmatrix} d'A' \\ + (A')^2 \end{pmatrix} + \begin{pmatrix} d'A'' + d''A' \\ A'A'' + A'A'' \end{pmatrix} + \begin{pmatrix} d''A'' \\ + \\ A''^2 \end{pmatrix}.$$

The term $d''A'$ is strange in the sense that I don't see it appearing in the equivariant cohomology.

The equivariant picture goes a follows. Let P be the principal G-bundle over Y whose fibre at y is the set of isos. of $E / y \times M$ with E_0. Then if $\pi : P \to M$ is the projection

$$\pi^*(E) \simeq pr_2^*(E_0) = P \times E_0 \;/\; P \times M.$$

Now D'' is equivariant to a G-map $P \longrightarrow \mathcal{A}$, and D' is equivalent to a connection in P/Y, i.e.

a map $W(\tilde{\mathfrak{g}}) \longrightarrow \Omega(P)$. The idea is that over $\mathcal{A} \times M$, $pr_2^*(E_0)$ has a tautological invariant connection, hence the $\mathcal{G}$-map $P \to \mathcal{A}$ gives an invariant connection on $pr_2^*(E_0)$ over [redacted] $P \times M$. This will be of the form

$$D_{P\times M} = d' + D'' = d' + d'' + A''$$

and has curvature

$$D^2_{P\times M} = d'D'' + D''d' + (D'')^2 = d'A'' + \left(d''A'' + (A'')^2\right).$$

What next [redacted]? I have this invariant connection over $P \times M$, so I have to make an equivariant connection:

$$D_{P\times M} [redacted] \theta^a \iota_a D_{P\times M} \qquad ?$$

First understand $M = pt$. Given E/Y, let $f : P \longrightarrow Y$ be the associated principal bundle. Then $f^*(E) \simeq P \times E_0$ with $G = U_n$ acting diagonally. The invariant connection on $f^*(E) \simeq P \times E_0$ is just $D = d_P$ and the [redacted] Higgs field is the action $\varphi : \mathfrak{g} \longrightarrow \mathrm{End}(E_0) = \mathfrak{gl}_n$ if $E_0 = \mathbb{C}^n$. Given a connection $\theta \in \mathfrak{g} \otimes \Omega^1(P)$, the connection on $f^*(E)$ that descends is

$$d_P + \varphi\theta = d_P + \theta$$

(since φ is essentially the identity.) [redacted] If we have a trivialization of E/Y, i.e. a section s of P, then $E \simeq s^*(P \times E_0) = Y \times E_0$ with the connection

$$d_Y + s^*\theta .$$

Now I want to do exactly the same sort of thing when M is around. Absolutely nothing should change provided we restrict attention to the horizontal connection:

Given $E/Y \times M$, let $f: P \longrightarrow Y$ be the associated principal $\mathcal{G}$-bundle, so that $f^*(E)$ is canonically isomorphic to $P \times E_0$ with $\mathcal{G}$-acting diagonally. The invariant horizontal connection is $d'_P = d'$ on $P \times E = pr_2^*(E_0)$ over $P \times M$, and the Higgs field is the natural action

$$\varphi : \tilde{g} \longrightarrow \Gamma(\mathrm{End}(E_0)),$$

i.e. essentially the identity. Then if I am given a connection in P/Y, i.e.

$$\theta \in \tilde{g} \otimes \Omega^1(P) = \Omega^1(P, \tilde{g})$$

$$\varphi\theta \in \Omega^1(P, \Omega^0(M, \mathrm{End}\, E_0)) = \Omega^{1,0}(P \times M, \mathrm{End}(P \times E_0))$$

the horiz. connection that descends is

$$d'_P + \varphi\theta .$$

So if I have a horizontal trivialization of E, that is $E \simeq Y \times E_0$, or a section of S, then the hor. connection on E is

$$d'_Y + \varphi s^*\theta .$$

September 24, 1983

I want to review the construction of character classes on $B\mathcal{G} \times M$ with the goal of obtaining formulas in the different pictures.

The basic problem is the construction of cohomology classes for $B\mathcal{G}$, or equivalently the construction of characteristic classes for principal $\mathcal{G}$-bundles. Here $\mathcal{G}$ is the group of autos of E_0/M, so that a principal $\mathcal{G}$-bundle over Y is equivalent to a vector bundle $E/Y \times M$ with $E_y = E|y \times M$ isom. to E_0 for all $y \in M$.

Picture 1: Given $E/Y \times M$ we construct its characteristic classes as diff'l. forms on $Y \times M$ by choosing a connection D on E. Then we integrate over cycles of M to obtain diff'l forms on Y.

The problem here is how to find a formula for this construction. What does this mean? What would it mean for the construction of the characteristic classes for a bundle E/Y? The answer is provided by the Chern-Weil formalism.

One uses the fact that bundles can be obtained from universal examples, so it suffices to construct a characteristic class for the universal example. The formula is then the actual cochain or form in the universal case. It then gives the characteristic form for the bundle, once a classifying map is chosen.

Algebraically the classifying map corresponds to the chosen connection.

(Idea: Concerning physics, the goal is to find the mathematics behind the new physics. A gauge field might become a classifying map, or equivalently, a sort of rigidification.)

To obtain characteristic classes for E/Y here is what happens. One [redacted] introduces the principal bundle P/Y. A connection in E defines a map $W(\mathfrak{g}) \longrightarrow \Omega(P)$ which induces

$$\begin{array}{ccc} W(\mathfrak{g}) & \longrightarrow & \Omega(P) \\ \cup & & \cup \\ S(\mathfrak{g}^*)^G = W(\mathfrak{g})_{basic} & \longrightarrow & \Omega(P)_{basic} \cong \Omega(Y). \end{array}$$

As said above, the "formula" for the characteristic class is an element of $W(\mathfrak{g})_{basic}$, and under the above map it goes to [redacted] a form on Y. In practice the last isomorphism $\Omega(P)_{basic} \cong \Omega(Y)$ is achieved by a local section of P.

Summarizing, [redacted] I have the following idea of a formula for a characteristic class. It is an element α of $W(\mathfrak{g})_{basic} = S(\mathfrak{g}^*)^G$. To compute it for E/Y, one introduces P, chooses a connection, and then the characteristic class is the image of α under

$$W(\mathfrak{g})_{basic} \subset W(\mathfrak{g}) \longrightarrow \Omega(P) \longrightarrow \Omega(Y)$$

where the [redacted] last map comes from a local section.

Now go back to a bundle $E/Y \times M$, or equivalently, a principal G-bundle $\mathcal{P}/Y$. I have this process for construction a form on $Y \times M$, namely [redacted] choose a connection on E, then take the corresponding characteristic form on $Y \times M$. I want a formula for this form in the same spirit as the above. This means that I want to [redacted] construct the form in a universal situation, then pull it back by a classifying map.

Before I get to $Y \times M$, I should first get the formulas straight when M is a point. Start with a connection on E over Y. This is described by a differential operator D on $\Omega^{\cdot}(M, E)$ [redacted] of degree $+1$ satisfying a derivation property. The curvature is D^2 which is multiplication by an element $K \in \Omega^2(M, \operatorname{End} E)$. The m-th character form is

$$\operatorname{tr}(K^m/m!) \in \Omega^{2m}(M).$$

This is the Bott-Chern approach.

Now for the Weil approach. I have to introduce the principal bundle P/Y and explain the [redacted] relation between D above and a map

$$W(\mathfrak{g}) \longrightarrow \Omega(P) \qquad G \subset \operatorname{Aut}(E_0),\ \mathfrak{g} \subset \operatorname{End}(E_0)$$

compatible with $\mathfrak{g}$-operations. We pull the bundle E back to P to get a bundle $\pi^*(E)$ which is canonically trivial $\pi^*(E) = P \times E_0$, with E_0 a vector space. So $\Omega(P, \pi^*(E)) = E_0$-valued forms on P, and D pulls back to a connection on $\pi^*(E)$ which can be written

$$\tilde{D} = d + \theta \qquad \theta \in \Omega^1(P) \otimes \mathfrak{g}$$

I should have mentioned, when discussing the Bott-Chern approach, that one does the computations in a local trivialization, so that $E \simeq Y \times E_0$

$$D = d + A \qquad A \in \Omega^1(Y) \otimes \mathrm{End}(E_0)$$

Also I should consider a slightly more general situation where P is a principal G-bundle over Y, and $E = P \times^G E_0$, where G acts on E_0 via a representation $\rho : G \to \mathrm{Aut}(E_0)$. Then

$$\tilde{D} = d + \rho\theta \quad , \theta \in \Omega^1(P) \otimes \mathfrak{g}$$

where $\tilde{D}$ = lift of D to $\pi^* E \cong P \times E_0$. Given a trivialization $s : Y \to P$

$$D = d + A \qquad A = s^*(\rho\theta) = \rho(s^*\theta).$$

Now I should compute curvature

$$\begin{aligned} \tilde{D}^2 &= d\rho\theta + \rho\theta\rho\theta \\ &= \rho\left(d\theta + \tfrac{1}{2}[\theta,\theta]\right) \\ &= \rho(\Omega) \end{aligned}$$

where

$$\Omega = d\theta + \tfrac{1}{2}[\theta,\theta] \in \Omega^2(P) \otimes \mathfrak{g}.$$

Thus under any trivialization $s : Y \to P$ we have

$$\rho(s^*\Omega) = s^* \rho(\Omega) = dA + A^2.$$

(What I am doing here is to relate curvature define by D^2 for v.b. with curvature defined by $d\theta + \frac{1}{2}[\theta,\theta]$ for principal bundles.

Now we come to defining $W(\mathfrak{g})$. It is the universal DG algebra with G-action having a canon. element

$$\Theta \in W^1(\mathfrak{g}) \otimes \mathfrak{g}$$

invariant under G and such that

$$\iota_X \Theta = (1 \otimes) X \quad \text{in} \quad W^0(\mathfrak{g}) \otimes \mathfrak{g}.$$

One has to show that $W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$ with all the other relations.

Now from the connection form Θ on P, and the universal nature of $W(\mathfrak{g})$ we get a homom.

$$W(\mathfrak{g}) \longrightarrow \Omega(P)$$

Summary: Given a principal G-bundle P/Y with connection and a repn. $\rho : G \to \mathrm{Aut}(E_0)$, let the induced connection on the associated V. bundle $E = P \times^G E_0$ be denoted D. If $\Theta \in \Omega^1(P) \otimes \mathfrak{g}$ is the connection form, then D, Θ are related by the formula

$$\tilde{D} = d + \rho(\Theta) \qquad \text{on} \quad \pi^*(E) \cong P \times E_0 .$$

This implies that if we have a trivialization $E \cong Y \times E_0$ corresp. to $s : Y \to P$, then

$$D = d + s^*(\rho\Theta)$$

Hence we have

$$\tilde{D}^2 = \rho(\Omega) \qquad \Omega = d\Theta + \tfrac{1}{2}[\Theta, \Theta] \in \Omega^2(P) \otimes \mathfrak{g}$$

$$D^2 = s^* \rho(\Omega).$$

Hence given $p \in S((\mathfrak{gl}E_0)^*)^{GL(E_0)}$ we have the following

$$p \in S((\mathfrak{gl}E_0)^*)^{GL(E_0)} \xrightarrow{\rho^*} S(\mathfrak{g}^*)^G = W(\mathfrak{g})_{basic} \subset W(\mathfrak{g}) \xrightarrow[\theta^{univ} \to \theta]{} \Omega(P) \xrightarrow{s^*} \Omega(Y).$$

So $p \longmapsto p(\rho\Omega^{univ}) \longmapsto p(\rho\Omega) \longmapsto p(s^*\rho\Omega) = p(D^2)$, which expresses the fact that the characteristic class is computed in $W(\mathfrak{g})_{basic}$, then obtained for E/Y using the connection. YUCK.

Next bring in M. Given $E/Y\times M$, let $\mathcal{P}$ be the associated principal $\mathcal{G} = Aut(E_0/M)$ bundle, so that $\pi^*(E) \simeq \mathcal{P}\times E_0$ over $\mathcal{P}\times M$. Given a connection D in E, it splits into horizontal and vertical parts $D' + D''$. Now D' is equivalent to a connection in $\mathcal{P}$: We have

$$\tilde{D}' = d' + \rho\theta$$

where $\theta \in \Omega^1(\mathcal{P})\otimes\tilde{\mathfrak{g}}$ and ρ is the obvious representation of $\mathcal{G}$ on E_0. There is an identification here:

$$\Omega(\mathcal{P}\times M, \mathcal{P}\times E_0) = \Omega(\mathcal{P}, \Omega(M, E_0))$$

So

$$\tilde{D} = d' + \rho\theta + \tilde{D}''$$

where $\tilde{D}''$ is an invariant choice of connection on E_0/M

for each pt. of P. Thus we have a $\mathcal{G}$-map $P \longrightarrow \mathcal{A}$ and $\tilde{\hat{E}} \cong$ pull-back of $\mathcal{A} \times E_0$ with $\tilde{D}'' =$ pull-back of the tautological vertical connection on $\mathcal{A} \times E_0 / \mathcal{A} \times M$.

So we reach the following universal situation. Let $P\mathcal{G} \to B\mathcal{G}$ be a universal $\mathcal{G}$-bundle and suppose it is equipped with a connection which is universal - such things exist according to Chern-Simons. Then $W(\tilde{\mathfrak{g}}) \subset \Omega(P\mathcal{G})$, and from a practical viewpoint these are equal.

Over $P\mathcal{G} \times \mathcal{A} \times M$ we have the v. bundle $P\mathcal{G} \times \mathcal{A} \times E_0$ with the connection

$$D_{tot} = d_{P\mathcal{G} \times \mathcal{A}} + D'' + \rho\,\Theta$$

where Θ is the canonical connection on $P\mathcal{G}$, D'' is the obvious tautological vertical connection on $pr_2^*(E_0)$ over $\mathcal{A} \times M$, and $\rho = \varphi$ is the obvious repn. of $\tilde{\mathfrak{g}}$ on $pr_2^*(E_0)$.

Now when one is given $E / Y \times M$ with a D one gets a map

$$(\star) \qquad Y \times M \longrightarrow P\mathcal{G} \times \mathcal{A} \times M$$

from the composition

$$Y \xrightarrow[\text{triv.}]{\text{local}} P \xrightarrow[\text{hor + vert connection}]{} P\mathcal{G} \times \mathcal{A}$$

and (E, D) is induced from $(P\mathcal{G} \times \mathcal{A} \times E_0,\ D_{tot})$.

September 25, 1983

Yesterday I was concerned with the philosophy behind the formalism of Chern-Weil. The problem is to determine $H^*(BG)$, or equivalently all characteristic classes. The formalism of connections and curvature defines a map

$$\underbrace{H^*\left(W(\mathfrak{g})_{basic}\right)}_{\simeq\ S(\mathfrak{g}^*)^G} \longrightarrow H^*(BG)$$

Finally one has to prove for G a compact Lie group that the map is an isomorphism. (Bott has perhaps the best way to do this, because he interprets the map as an edge homomorphism.)

Now I am concerned with the problem of determining $H^*(B\mathcal{G})$, $\mathcal{G} = Aut(P_0/M)$. The formalism of connections and curvature defines a map

$$H^*\left(\left[W(\tilde{\mathfrak{g}}) \otimes \Omega(\mathfrak{a})\right]_{basic}\right) \longrightarrow H^*(B\mathcal{G})$$

which I believe to be an isomorphism because $\mathfrak{a}$ is contractible with compact isotropy groups. What's missing in this situation is a formula for the LHS. I am involved with producing certain elements from the tautological bundle $\tilde{E}$ over $B\mathcal{G} \times M$.

So I should review the construction of equivariant Chern class for an equivariant bundle E/M having an invariant connection D. D is an operator on $\Omega^\cdot(M,E)$ and the Higgs field $\varphi : \mathfrak{g} \to \Omega^0(M, End\, E)$ is defined by

$$\mathcal{L}_X = [i_X, D] + \varphi_X$$

Let's suppose G acts freely on M and that $\theta \in \mathfrak{g} \otimes \Omega^1(M)$ is a connection form, which means

$$\iota_X \theta = X$$

$$0 = \mathcal{L}_X(\theta) = [X, \theta] + \underbrace{[d, \iota_X]\theta}_{i_X d\theta}$$

Then we form the connection

$$\tilde{D} = D + \theta\varphi$$

(where $\theta\varphi$ is the contraction $\theta^a \varphi_a$ of $\theta \in \mathfrak{g} \otimes \Omega^1(M)$ and $\varphi \in \mathfrak{g}^* \otimes \Omega^0(M, \mathrm{End}\, E)$.) Then

$$[i_X, \tilde{D}] = [i_X, D] + \varphi_X = \mathcal{L}_X$$

which implies $\tilde{D}$ descends to $\bar{D}$ on $\bar{E}/\bar{M}$, where $\bar{M} = G \backslash M$.

We have

$$\begin{array}{ccccccc}
 & & & \theta \longleftarrow\!\!\shortmid\ \theta_{univ} & & & \\
\Omega(\bar{M}, \bar{E}) & \xrightarrow{\sim} & \Omega(M,E)_{basic} & \longleftarrow & [W(\mathfrak{g}) \otimes \Omega(M,E)]_{basic} & \xrightarrow{\sim} & [S(\mathfrak{g}^*) \otimes \Omega(M,E)]^G \\
\bar{D} & \longmapsto & \tilde{D} & \longleftarrow\!\!\shortmid & D + \overset{univ}{\theta}\varphi & \longmapsto & D - \Omega^a \iota_a \\
\bar{D}^2 & & \longleftarrow & & (D + \overset{univ}{\theta}\varphi)^2 & \longmapsto & D^2 + \Omega^a \varphi_a .
\end{array}$$

The point of this diagram is that to compute we can work with $D^2 + \Omega^a \varphi_a$ where the Ω^a are free commuting degree 2 variables. We don't have to work with $(D + \theta^a \varphi_a)^2$.

Actually if we use a nice flat local trivialization of M over $\bar{M}$ at the point we are computed $(\not{D} + \Theta\varphi)^2$ should equal $D^2 + \Omega\varphi$, since $\Theta^a = 0$ at this point.

Next we want to consider the case of the equivariant bundle $pr_2^*(E_0) = \mathcal{A} \times E_0$ over $\mathcal{A} \times M$, and the tautological invariant connection: $D = d' + D''$. Assume E_0 is trivial. Then a point of $\mathcal{A}$ is a matrix [redacted] 1-form $A_\mu dx^\mu$ on M, and

$$D = d'_{\mathcal{A}} + d''_M + A_\mu dx^\mu$$

$$= \sum_\nu \int_M dy \frac{\delta}{\delta A_\nu(y)} \delta A_\nu(y) + \frac{\partial}{\partial x^\mu} dx^\mu + A_\mu dx^\mu$$

Thus $\mathcal{A}$ has the coordinate fns. $A_\mu(y)$ for each μ, $y \in M$.

$$D^2 = d'A + d''A + A^2$$

where $(d'A)(x) = \int dy \sum_\nu \underbrace{\frac{\delta A_\mu(x)}{\delta A_\nu(y)}}_{\delta_{\mu\nu}\delta(x-y)} \delta A_\nu(y)\, dx^\mu = \delta A_\mu(x) \cdot dx^\mu$.

$$d''A + A^2 = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$$

I am interested in the [redacted] Chern class $c_1(\mathcal{L})$ of the determinant line bundle for the family of Dirac operators over M. Suppose M has trivial $\hat{A}$-genus, then

$$c_1(\mathcal{L}) = \left[(pr_1)_* \, ch(\tilde{E})\right]_{\text{deg 2 component}}$$

Suppose $\dim M = 2$, whence I want to look at $ch_2(\tilde{E}) = \kappa^2 \frac{1}{2} tr\{(D^2+\Omega\varphi)^2\}$, $\kappa = \frac{i}{2\pi}$. Since I am integrating over M I need a form of degree 2 in dx^μ. Recall

$$D^2 = d'A + F = \delta A_\mu \cdot dx^\mu + \tfrac{1}{2} F_{\mu\nu}\, dx^\mu dx^\nu$$

$$ch_2(\tilde{E}) = \frac{\kappa^2}{2} tr((D^2)^2) + \kappa^2 \Omega\, tr(D^2\varphi) + \frac{\kappa^2}{2}\Omega^a\Omega^b tr(\varphi_a\varphi_b)$$

The only thing that survives the $\int_M$ is $(d'A)^2$ [in the first term] and F [in the second term]

so

$$(pr_1)_* ch(\tilde{E}) = \frac{\kappa^2}{2}\int_M tr(d'A)^2 + \kappa^2\Omega^a \int_M tr(F\varphi_a).$$

Assume that this is the equivariant $c_1(\mathcal{L})$ relative to invariant connection on $\mathcal{L}/\mathcal{a}$. Then

$$\begin{cases} curv(\mathcal{L}) = \dfrac{\kappa}{2}\displaystyle\int_M tr\,\underbrace{(\delta A_\mu\, dx^\mu)^2}_{\delta A_\mu dx^\mu \cdot \delta A_\nu dx^\nu} \\ \qquad = -\dfrac{\kappa}{2}\displaystyle\int_M [tr(\delta A_\mu \delta A_\nu)\,\varepsilon^{\mu\nu}]\, d^2x \\ \varphi_X = \kappa \displaystyle\int_M tr(F X) \end{cases}$$

I should now check this calculation by verifying that

$$\iota_X(curv) = d\varphi_X.$$

September 27, 1983

Suppose we try to understand the canonical map $\mathcal{B}(G) \longrightarrow BG$ in terms of the Chern-Weil formalism. G is the space of autos of the trivial G-bundle. More precisely a map $Y \to G$ can be viewed as an autom. of the principal bundle $Y \times G/Y$. In general given a G-bundle P over Y and an auto of it I can form a bundle over $S^1 \times Y$. If P is provided with a connection, then using the linear structure on the space of connections we can define a connection on this bundle over $S^1 \times Y$.

Take $Y = G$ and the trivial bundle over G. Then we get a bundle over $S^1 \times G$ which has a fairly canonical connection, hence by Chern-Weil we have a map

$$S(\mathfrak{g}^*)^G = W(\mathfrak{g})_{basic} \longrightarrow \Omega(S^1 \times G).$$

I should be able to describe this map in detail.

Let's go over the formulas for vector bundles. Given E over Y and an automorphism g of E, then to get the bundle over $S^1 \times Y$, we start with $pr_2^*(E)$ over $[0,1] \times Y$ and use g to identify the bundle over $0 \times Y$ and $1 \times Y$. Given D_0 a connection on E/Y, let $D_1 = g \circ D_0 \circ g^{-1}$, and then consider over $I \times Y$ the connection

$$D = \partial_t dt + D_t$$

$$D_t = (1-t)D_0 + tD_1 = D_0 + t \cdot \underbrace{gD_0(g^{-1})}_{\theta}$$

Then $D^2 = dt \cdot \theta + D_t^2$ so

$$\operatorname{tr}\left(e^{D^2}\right) = \operatorname{tr}\left(e^{dt\,\theta + D_t^2}\right)$$
$$= dt \cdot \operatorname{tr}\left(\theta\, e^{D_t^2}\right)$$

where I have used that $(dt)^2 = 0$ and the formula

$$\delta \operatorname{tr}\left(e^{A}\right) = \operatorname{tr}\left(e^{A}\,\delta A\right).$$

For a general $\varphi \in S(\mathfrak{g}^*)^G$ I get

$$\varphi(D^2) = dt \cdot \varphi'(D_t^2; \theta).$$

Take the case E trivial, $D_0 = d$. Then

$$D_t^2 = \left(d + t\, g\, dg^{-1}\right)^2 \qquad \theta = g\, dg^{-1}$$
$$= t\, d\theta + t^2 \theta^2$$
$$= (t^2 - t)\,\theta^2.$$

Actually I want to consider the map

$$S(\mathfrak{g}^*)^G \longrightarrow \Omega(S^1 \times G) \xrightarrow[\text{degree } -1.]{\int_{S^1}} \Omega(G)$$

$$\operatorname{tr}\left(e^{D^2}\right) \longmapsto \int_0^1 dt\, \operatorname{tr}\left(\theta\, e^{(t^2 - t)\theta^2}\right)$$

For a general G we will have probably

$$\varphi \longmapsto \int_0^1 dt\; \varphi'\left((t^2 - t)\theta^2; \theta\right)$$

where θ is the M-C form on G.

Next consider the case of a gauge group $\mathcal{G}$. What we have learned about gauge groups is that we have to select a vertical connection in order to get at the characteristic classes. Thus even I take the trivial $\mathcal{G}$-bundle over Y, which corresponds to $pr_2^*(E_0)$ over $Y \times M$, I need to have vertical connection D_0 on E_0 chosen. Otherwise all I have is a trivial horizontal connection.

Now consider the canonical map $\Sigma\mathcal{G} \to B\mathcal{G}$. I know that over $\Sigma(\mathcal{G})$ or $S^1 \times \mathcal{G}$ is a canonical principal $\mathcal{G}$-bundle obtained from $I \times \mathcal{G} \times E_0$ over $I \times \mathcal{G} \times M$ by identifying the ends. $I \times \mathcal{G} \times E_0$ carries a canonical trivial horizontal connection - ? To make this clearer think of having an auto g of $Y \times E_0 / Y \times M$. The trivial Y-connection on $Y \times E_0$ transformed by g and connected linearly back to itself gives a canonical horizontal connection on $(I \times Y) \times E_0$ over $(I \times Y) \times M$ which descends to the glued bundle over $(S^1 \times Y) \times M$.

But I also need a vertical connection, and I can choose it for E_0/M pull it back to $Y \times E_0 / Y \times M$ and combine with the horizontal connection.

What's important: If I choose a basepoint $D_0 \in \mathcal{A}$, then I get a canonical map

$$I \times \mathcal{G} \longrightarrow \mathcal{A}$$

sending $0 \times \mathcal{G}$ to D_0 and $(1,g)$ to $g * D_0$ and (t,g) to $(1-t)D_0 + t \cdot g * D_0$. This induces a canonical map

$$\Sigma(\mathcal{G}) \longrightarrow \bar{\mathcal{A}} = \mathcal{G} \backslash \mathcal{A} .$$

In fact over $\Sigma(\mathcal{G}) \times M$ is a canonical bundle with a fairly canonical connection depending only on the choice of O_0. The horizontal part I have seen, and I just explained the vertical side.

The conclusion is that there ought to be a way to go from equivariant forms on $\mathfrak{g}$, or $\mathfrak{g} \times M$, to forms on $\mathcal{G}$ or $\mathcal{G} \times M$ depending only upon the choice of O_0.

September 29, 1983

First review the formula for the odd character classes belonging to a bundle automorphism. Let E/Y, let g be an auto, and let D be a connection on E. Then we use the 1-parameter family of connections

$$D_t = (1-t)D + t\,g^{-1}Dg$$
$$= D + t\theta \qquad \boxed{\theta = g^{-1}[D,g]}$$

Now
$$[D,\theta] = -g^{-1}[D,g]g^{-1}[D,g] + g^{-1}[D^2,g]$$
$$= -\theta^2 + g^{-1}D^2g - D^2$$

so
$$D_t^2 = D^2 + t[D,\theta] + t^2\theta^2$$
$$\boxed{D_t^2 = (1-t)D^2 + t\,g^{-1}D^2g + (t^2-t)\theta^2}$$

The odd character class is then

$$\int_0^1 dt\ \mathrm{tr}\left(\theta e^{D_t^2}\right)$$

with θ and D_t^2 given by the above formulas.

The problem goes as follows. I have found nice formulas for the [redacted] generators of $H^*(B\mathcal{G})$ using equivariant forms [redacted] on $\mathcal{a}$ for the $\mathcal{G}$-action. [redacted] The character classes

$$ch(\tilde{E}) \in H^*(B\mathcal{G}\times M)$$

are represented by canonical universal formulas which are in accordance with the Weil viewpoint.

Now I want to do the same thing for $H^*(\mathcal{G})$, but have run into the problem that I seem to have to choose a connection in E_0, i.e. a sort of basepoint in $\mathcal{A}$. In other words to construct the odd character classes

$$ch'(\tilde{g}) \in H^*(\mathcal{G} \times M)$$

as differential forms, I have found it necessary to have a connection on $pr_2^*(E_0)$ over $\mathcal{G} \times M$.

<u>Idea:</u> The situation should become canonical over $\mathcal{G} \times \mathcal{A} \times M$.

The problem is to construct differential forms representing the odd character classes

$$ch'(\tilde{g}) \in H^{odd}(\mathcal{G} \times M).$$

The idea is that these classes can be represented by canonical differential forms on $\mathcal{G} \times \mathcal{A} \times M$ which has the same cohomology as $\mathcal{G} \times M$, since $\mathcal{A}$ is contractible.

Here is how to construct the classes. A map $Y \to \mathcal{G}$ is the same thing as an automorphism g of $pr_2^*(E_0)$ over $Y \times M$. $pr_2^*(E_0)$ comes with a canonical trivial horizontal connection. If I am also given a vertical connection, which is the same as a map $Y \to \mathcal{A}$, then I put the two together to get a connection D on $pr_2^*(E_0)$ over $Y \times M$. Then I can form the odd character of g w.r.t. D to get odd forms on $Y \times M$.

Thus to any $Y \longrightarrow \mathcal{G} \times \mathcal{A}$ I have assigned closed forms on $Y \times M$, which means that I have defined forms on $\mathcal{G} \times \mathcal{A} \times M$. Are they left-invariant?

Let's get the formulas so I can check this. Over $Y \times M$ I have the bundle $pr_2^*(E_0)$, the autom. g, and the vertical connection D''. Then I take the full connection $D = d' + D''$ and compute

$$ch'(g, D) = \int_0^1 dt \, tr\left(\theta e^{D_t^2}\right) \qquad \begin{aligned} \theta &= g^{-1}[D, g] \\ D_t &= D + t\theta \end{aligned}$$

Let $g_0 \in \mathcal{G}$ and think of it as an autom of $pr_2^*(E_0)$ constant in the Y-direction. Use

$$\theta = g^{-1}[D, g]$$

$$D_t^2 = (1-t)D^2 + t g^{-1} D^2 g + (t^2 - t)\theta^2$$

and see what happens if we make the replacements

$$g \longmapsto g_0^{-1} g, \qquad D \longmapsto g_0^{-1} D g_0. \quad \text{Then}$$

$$\theta \longmapsto g^{-1} g_0 \underbrace{[g_0^{-1} D g_0, g_0^{-1} g]}_{g_0^{-1} D g_0 g_0^{-1} g - g_0^{-1} g g_0^{-1} D g_0}$$

$$g^{-1} D g - g_0^{-1} D g_0$$

$$= g_0^{-1}\left(g_0 g^{-1} D g g_0^{-1} - D\right) g_0 \quad ?$$

??

This sort of argument can't work, because the natural thing to do is to conjugate g by g_0.

Different viewpoint: Recall that if X is say the symmetric space of G, then $\Omega(X)^G$ give characteristic classes for flat bundles P/Y via

$$\Omega(X)^G \xrightarrow[\text{conn.}]{\text{flat}} \Omega(P\times^G X) \xrightarrow{s^*} \Omega(Y),$$ where

s a section of $P\times^G X/Y$, which exists as $X \sim pt$.

Also we interpreted $\Omega(G)^G = \Lambda \mathfrak{g}^*$ as characteristic forms for flat connections on the trivial G-bundle:

$$\Omega(G)^G \xrightarrow[\text{conn.}]{\text{flat}} \Omega(P) \xrightarrow{s^*} \Omega(Y)$$

The point now is that $\Omega(G\times X)^G$ gives rise to characteristic forms for pairs (θ, λ) where θ is a flat connection on the trivial G-bundle and $\lambda : Y \to X$ is a map:

$$\Omega(G\times X)^G \xrightarrow[\text{conn.}]{\text{flat}} \Omega(P\times^G (G\times X)) = \Omega(P\times X) \xrightarrow{(s,\lambda)^*} \Omega(Y).$$

Could the complex $\Omega(\mathcal{G}\times \mathcal{A})^{\mathcal{G}}$ have the same homology as $\Omega(\mathcal{G})$, so that elements of $H^*(\mathcal{G})$ are represented by invariant forms on $\mathcal{G}\times\mathcal{A}$?

NO: We have $\mathcal{G}\times\mathcal{A} \xrightarrow{\sim} \mathcal{G}\times\mathcal{A}$, $(g,a)\mapsto(g,ga)$ which shows $\mathcal{G}\times\mathcal{A}$ with diagonal action is equivalent to $\mathcal{G}\times\mathcal{A}$ with just action on the first factor. But then

$$\Omega(\mathcal{G}\times\mathcal{A}) \simeq \Omega(\mathcal{G})\otimes \underbrace{\Omega(\mathcal{A})}_{\text{trivial } \mathcal{G}\text{-action}}$$

hence the $\mathcal{G}$-homotopy equivalence $\Omega(\mathcal{A}) \to \mathbb{C}$, and $\Omega(\mathcal{G}\times\mathcal{A}) \to \Omega(\mathcal{G})$, so one only gets the invariant forms.

Alternatively

$$\Omega(G \times a)^{G} \simeq \Lambda \tilde{g}^* \otimes \Omega(a)$$

where the action has disappeared, so one gets just the Lie algebra cohomology.

Finally, think of the van Est spectral sequence:

$$\left[\Omega(G) \otimes \Omega(a)\right]^{G} = \Omega(G \times a)^{G}$$

is ~ the bicomplex of continuous cochains with values in $\Omega(G)$. So one gets a spectral sequence

$$E_2^{pq} = H^p_{cont}(G, H^q_{top.}(G)) \Rightarrow H^*(\Omega(G \times a)^{G})$$

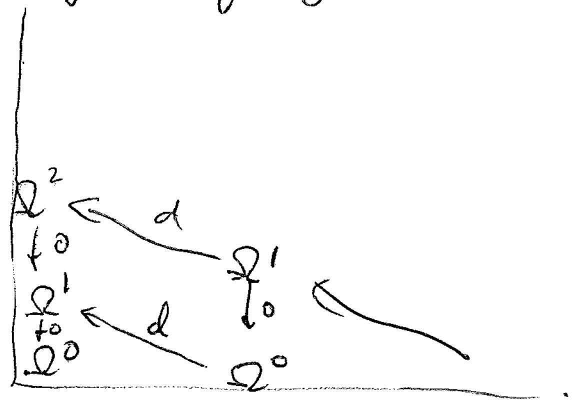
and the abutment is supposed to be the Lie algebra cohomology.

October 2, 1983

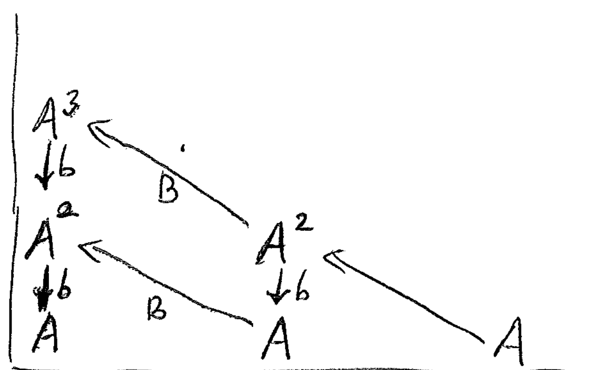
The problem: suppose $\mathfrak{g} = \mathfrak{gl}_n(A)$, or more general that we are given a repn. $\rho: \mathfrak{g} \to \mathfrak{gl}_n(A)$. Then we have seen how the formula

$$\int_0^1 dt \; \mathrm{tr}\left(\theta e^{td''\theta + (t^2-t)\theta^2}\right)$$

defines a cocycle of $\mathfrak{g}$ with values in the complex



The problem is to lift this cocycle to one with values in $\mathcal{B}(A)$:



in a natural way.

Let's try to get the formulas to work around Lie alg. cohomology degree = 3.

First of all we know from the non-commutative differential form approach, that we should be able to do curvature calculations using $C^*(\mathfrak{g}, \Omega^*_{nc}(A))$.

Now $\quad \theta \in C^1(\mathfrak{g}, gl_n(A)) \subset gl_n\{C^*(\mathfrak{g}, A)\}$

and things like θ^2 are defined in the latter ring. We need also a d" operation, say a differential algebra

$$A \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow \cdots$$

satisfying $\quad d^2 = 0 \quad , \quad d(\eta\omega) = d\eta \cdot \omega + (-1)^{\deg \eta} \eta . d\omega$

Here is an example: Take the Amitsur complex

$$A \longrightarrow A \otimes A \longrightarrow A \otimes A \otimes A \longrightarrow \cdots\cdots$$

where $\quad d(a_0, \ldots, a_p) = (1, a_0, \ldots, a_p) - (a_0, 1, a_1, \ldots, a_p) + (a_0, a_1, 1, \ldots) \\ - \cdots$

and with the product

$$(a_0, \ldots, a_p)(b_0, \ldots, b_q) = (a_0, \ldots, a_{p-1}, a_p b_0, b_1, \ldots, b_q)$$

We should check that d is a derivation:

$$[d(a_0, \ldots, a_p)] \cdot (b_0, \ldots, b_q) = (1, a_0, \ldots, a_{p-1}, a_p b_0, b_1, \ldots, b_q)$$
$$- (a_0, 1, a_1, \ldots, a_p b_0, \ldots, b_q) + \cdots + (-1)^{p+1}(a_0, \ldots, a_p, b_0, \ldots, b_q)$$

$$(-1)^p (a_0, \ldots, a_p)[d(b_0, \ldots, b_q)] = (-1)^p(a_0, \ldots, a_p, b_0, \ldots, b_q) + (-1)^p(a_0, \ldots, a_p b_0, 1, b_1, \ldots, b_q)$$
$$+ (-1)^p(-1)^{q+1}(a_0, \ldots, a_p b_0, \ldots, b_q, 1)$$

$$d(a_0, \ldots, a_p b_0, \ldots, b_q) = (1, a_0, \ldots, a_p b_0, \ldots, b_q) - \cdots + (-1)^p(\quad 1, a_p b_0, b_1 \cdots)$$
$$+ (-1)^{p+1}(\cdots, a_p b_0, 1, b_1, \ldots) + \cdots + (-1)^{p+q+1}(\cdots b_q, 1)$$

It is reasonably clear that the non-commutative DR complex sits inside the Amitsur complex as the non-degenerate part. There is such a thing as the tensor [redacted] algebra of a bimodule M over a non-commutative A:

$$A \oplus M \oplus M \otimes_A M \oplus \cdots$$

The Amitsur algebra is the case $M = A \otimes_k A$ and the NCDR algebra is the case where $M = \mathrm{Ker}\{A \otimes A \longrightarrow A\}$.

Remark: The embedding

$$\begin{array}{ccc} A \otimes \bar{A} & \longrightarrow & A \otimes A \\ a\,dx & \longmapsto & a \otimes x - ax \otimes 1 \end{array}$$

is fairly intricate and things might appear much more natural in the Amitsur picture

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So now let's go back to the algebra $C^*(\mathfrak{g}, \Omega^*)$ which is bigraded and has two differentials. Now we can consider the MC form

$$\theta \in \mathfrak{gl}_n\{C^*(\mathfrak{g}, \Omega^*)\}$$

of type $(1,0)$, and form $d'\theta = -\theta^2$, $d''\theta$ etc.

October 4, 1983

Today I read some physics (Witten's Baryons in the $1/N$ expansion). The reason is that I wanted to understand mesons in the quark model. Mesons are somehow quark bilinears i.e. a combination $q\bar{q}$ of a quark and anti-quark. So a meson field $\varphi(x)$ should be related to the Lie algebra of infinitesimal gauge transformations.

On the other hand I believe that the σ-model, which is supposed to eventually appear as a low-energy version of QCD, is made up of the π-octet of mesons $(3\pi, 2K, 2\bar{K}, \eta)$. Also we know this has been described in terms of fields $U(x)$ with values in $SU(3)_R \times SU(3)_L / \Delta SU(3) = SU(3)$.

Some observations of Witten's paper:

There is an attempt to integrate out the gauge fields and to get to an interacting fermion theory. Examples:

p.85 - baryons made of heavy quarks are to be understood as N-bosons with attractive Coulomb potentials.

p107. - in two dimensions one can eliminate the gluon fields in favor of Coulomb interactions

σ-trick for eliminating $(\bar{\psi}\psi)^2$ in favor of something quadratic in the fermi field. p.108.

Herman-Weyl's book. He mentions the idea of replacing the group $GL(V)$ by the ring it generates in $End(V^{\otimes p})$ namely

$$End(V^{\otimes p})^{\Sigma_p}.$$

So what we get is an algebra of operators on $V^{\otimes p}$ commuting with Σ_p, hence which gives operators on $\Lambda^p V$ and $S^p V$.

Now I want to let $\dim(V) = N \to \infty$ and also $p \to \infty$. I should be getting something like the universal enveloping algebra of gl_∞.

I can look at the map

$$U(gl(V)) \longrightarrow End(\Lambda V)$$

whose image lands in the endos of degree zero. Do I get all endos of degree 0? Rough argument is to use the filtration on both sides:

$$S(g) \longrightarrow [\Lambda V \otimes V^*]^{(0)}$$

Certainly looks onto. Not an isom:

$$S^2(V \otimes V^*) \longrightarrow \Lambda^2 V \otimes \Lambda^2 V^*$$

$$\frac{N^2(N^2+1)}{2} \qquad\qquad \left(\frac{N(N-1)}{2}\right)^2$$

I am interested in the Lie algebra $gl_N(A)$ which acts on $A^N = V \otimes A$, hence on

$$(V \otimes A)^{\otimes p} = V^{\otimes p} \otimes A^{\otimes p}$$