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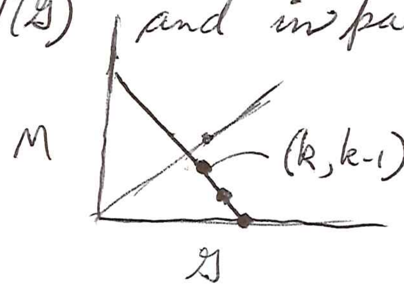

for the Lie classes

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August 7, 1983

E_0 fixed v.b. over M with hermitian metric
$\mathcal{G}$ = gauge gp of autos. of E_0, $\tilde{g}$ its Lie algebra.

The problem is to determine $H^*(B\mathcal{G})$.

More generally start with a principal G-bundle P_0 over M with G compact and let $\mathcal{G}$ = gauge gp.

<u>Fact</u>: principal $\mathcal{G}$-bundles $\mathcal{P}/Y$ are the same as principal G-bundles P over $Y \times M$ which are isomorphic to P_0 over each $\{y\} \times M$.

It's clear how to go from $\mathcal{P}$ to P, namely $P = \mathcal{P} \times^{\mathcal{G}} P_0$. Conversely given P we need first to know $P \simeq Y \times P_0$ locally. This we can do by choosing a connection in P, then parallel translate along horizontal geodesics issuing from points of $y \times M$ to construct such an isomorphism in a nbd of y. On the over-laps we need

$$\mathrm{Aut}(Y \times P_0 / Y \times M) = \mathrm{Map}(Y, \mathcal{G})$$

e.g. if P_0 is trivial

$$\mathrm{Map}(Y \times M, G) = \mathrm{Map}(Y, \mathrm{Map}(M, G))$$

The above fact is related to the fact that

$$(*) \qquad B\mathcal{G} \longrightarrow \mathrm{Map}(M, BG)$$

identifies the former up to h.eq. with the component of the latter corresponding to P_0.

Now we can construct elements of $H^*(B\mathcal{G})$ as follows. Corresponding to the universal bundle $P\mathcal{G}$ over $B\mathcal{G}$ is a principal G-bundle P_{univ} over $B\mathcal{G} \times M$, hence a map

$$B\mathcal{G} \times M \longrightarrow BG$$

which should be the same as the one obtained from $(*)$. Thus we have

$$\begin{array}{ccc} \varphi & \longmapsto & \varphi(P_{univ}) \\ H^*(BG) & \longrightarrow & H^*(B\mathcal{G} \times M) \\ \| & & \| \\ S(\mathfrak{g}^*)^G & & H^*(B\mathcal{G}) \otimes H^*(M) \end{array}$$

and so associated to $\varphi \in S(\mathfrak{g}^*)^G$ and $\gamma \in H_*(M)$ we have a class

$$\boxed{\int_\gamma \varphi(P_{univ}) \in H^*(B\mathcal{G}).}$$

In certain cases I know that $H^*(B\mathcal{G})$ is freely generated by these classes as φ runs over the primitive generators for $S(\mathfrak{g}^*)^G$ and γ runs over a basis for the homology. The question is whether this might be true in general. This is a problem in rational homotopy theory. The good formula would be

$$\boxed{\begin{aligned} \mathrm{Prim}\{H^*(B\mathcal{G})\} &= \mathrm{Hom}(\pi_* BG, H_* M) \\ &= \bigoplus H_*(M)\,[\text{shift}] \end{aligned}}.$$

The next project is to realize these classes by differential forms, that is, given P over $Y\times M$, to construct differential forms on Y. Here there are two apparently different methods one can use:

1) Choose a connection on P over $Y\times M$ and use the curvature to compute $\varphi(P)\in H^*(Y\times M)$.

2) Choose a connection on P, [redacted] and think of it as a choice of horizontal connection D' and vertical connection D''. If we pull P back to $\mathcal{P}\times M$, then it becomes canonically isomorphic to $pr_2^*(P_0)=\mathcal{P}\times P_0$ and D'' is the same as an invariant [redacted] vertical connection in $pr_2^*(E_0)$, i.e. an equivariant map $\mathcal{P}\to\mathcal{A}$.

$$\begin{tikzcd}
P \arrow[d] & \mathcal{P}\times P_0 \arrow[l] \arrow[r] \arrow[d] & \mathcal{A}\times P_0 \arrow[r] \arrow[d] & P_0 \arrow[d] \\
Y\times M & \mathcal{P}\times M \arrow[l] \arrow[r] & \mathcal{A}\times M \arrow[r] & M
\end{tikzcd}$$

$$Y \longleftarrow \mathcal{P} \longrightarrow \mathcal{A}$$

~~Better~~: D'' is the same as an invariant vertical connection on $pr_2^*(P_0)=\mathcal{P}\times P_0$ over $\mathcal{P}\times M$. But $pr_2^*(P_0)$ has an obvious flat horizontal connection. Combining these we get an invariant connection on $pr_2^*(P_0)$ over $\mathcal{P}\times M$, Hence equivariant char. classes

Finally the horizontal connection on P over $Y\times M$ is the same as a connection on $\mathcal{P}/Y$, and this can be to descend the equivariant classes over $\mathcal{P}\times M$ down to $Y \times M$.

Now I believe the two methods yield the same forms on $Y \times M$. To see this I would need to understand the descent process for equivariant forms a bit better.

<u>Logical point</u>: For compact group principal bundles one only has to choose a connection to construct the char. classes. Then all the char. classes are given by invariant polynomials on the Lie algebra:

$$S(\mathfrak{g}^*)^G \simeq H^*(BG)$$

Another way of putting this is that the equivariant forms for G acting on a point give the cohomology of BG.

Now it seems that for a gauge group $\mathcal{G}$, a choice of connection in P is not enough to enable us to construct the classes. We also must choose an equivariant map $P \longrightarrow \mathcal{A}$. So we could hope to prove that the equivariant forms for $\mathcal{G}$ acting on $\mathcal{A}$ give the cohomology of $B\mathcal{G}$.

Let's return to the "fibration

$$\begin{array}{ccccccccc}
 & & G & \longrightarrow & \mathfrak{g} & \longrightarrow & B_cG & \longrightarrow & BG \\
 & & \uparrow\wr & & \uparrow & & \uparrow & & \uparrow\wr \\
G^d & \longrightarrow & G & \longrightarrow & B\bar{G} & \longrightarrow & BG^d & \longrightarrow & BG.
\end{array}$$

We have been struggling with $H^*(B\mathcal{G})$, but things might be simpler with $H^*(B_c \mathcal{G})$. This is roughly $H^*(B\mathcal{G}^d)$ which contains characteristic classes for bundles P over $Y \times M$ equipped with a flat Y-connection.

August 8, 1983

Continuous cohomology - I find this mysterious, and the only real way I have to think of it is the cohomology of BG^d which one can write down.

Recall the calculation of $H^*(B_c G)$ when G is a connected Lie group with maximal compact K.

$$(*) \qquad G \longrightarrow \mathfrak{g} \longrightarrow B_c G$$

With complex coefficients $H^*(\mathfrak{g}) = H^*(\mathfrak{u})$ where $\mathfrak{u}$ is the Lie algebra of the maximal compact U of G_c. Thus $(*)$ is cohomologically the same as

$$K \longrightarrow U \longrightarrow B_c G$$

and so we have the formula for the continuous coh:

$$H^*(B_c G) = H^*(U/K) = H^*(\mathfrak{u}, \mathfrak{k})$$

$$= H^*(\mathfrak{g}, k) = I(G/K) = \left\{ \begin{matrix} \text{inv. forms} \\ \text{on } G/K \end{matrix} \right\}$$

(Recall $I(G/K)$ has zero d.)

Geometrically a $\varphi \in I(G/K)$ can be interpreted as a characteristic class for flat G-bundles as follows. Given P/Y with a flat connection we can form the associated fibre bundle with fibre G/K. Because the connection is flat we have a map

$$I(G/K) \longrightarrow \Omega(P \times^G (G/K))$$

which commutes with d. Thus φ gives rise to a closed form on $P \times^G (G/K)$. Since G/K is contractible, we can choose a section s of $P \times^G (G/K)$ over Y and pull-back this form.

General picture: Let G act on F and let $X = P \times^G F$ be the associated fibre space over Y. Then each fibre of X/Y is isomorphic to F and the isom. is unique up to an elt of G. This means that there is a well-defined map

$$\Omega^{\cdot}(F)^G \longrightarrow \Omega^{\cdot}_{X/Y}$$

compatible with d. On the other hand a connection in P/Y induces a connection in X/Y and gives a map

$$\Omega^{\cdot}_{X/Y} \longrightarrow \Omega^{\cdot}_X$$

of algebras which is compatible with d if the connection has zero curvature.

Another way: We have associated to the connection in P a map

$$W(\mathfrak{g}) \longrightarrow \Omega(P)$$

which we use to form the map

$$W(\mathfrak{g}) \otimes \Omega(F) \longrightarrow \Omega(P) \otimes \Omega(F) \longrightarrow \Omega(P \times F).$$

This is a DG alg map compatible with the L_v, i_v operators for $v \in \mathfrak{g}$. It induces

$$\{W(\mathfrak{g}) \otimes \Omega(F)\}_{basic} \longrightarrow \{\Omega(P \times F)\}_{basic} = \Omega(P \times^G F).$$

Let's now turn to the case of the gauge group $\mathcal{G}$. Given a principal $\mathcal{G}$-bundle $\mathcal{P}/Y$ we have seen that in order to construct its characteristic classes we wanted to choose a connection in $\mathcal{P}/Y$ and a equivariant map $\mathcal{P} \to \mathcal{a}$. The latter is equivalent to a section of the fibre space $\mathcal{P} \times^{\mathcal{G}} \mathcal{a}$ over Y with the contractible fibre $\mathcal{a}$.

I want to proceed by analogy with the above but using $\mathcal{a}$ instead of the symmetric space. So all we choose is the connection in $\mathcal{P}/Y$ and then we get

$$\begin{array}{ccc} \{W(\tilde{\mathfrak{g}}) \otimes \Omega(\mathcal{a})\}_{basic} & \longrightarrow & \{\Omega(\mathcal{P} \times \mathcal{a})\}_{basic} \\ \| & & \| \\ [S(\tilde{\mathfrak{g}}^*) \otimes \Omega(\mathcal{a})]^{\mathcal{G}} & & \Omega(\mathcal{P} \times^{\mathcal{G}} \mathcal{a}) \end{array}$$

If the connection is flat then we get a map

$$\Omega(\mathcal{a})^{\mathcal{G}} \longrightarrow \Omega(\mathcal{P} \times^{\mathcal{G}} \mathcal{a}).$$

Question: Is the complex

$$\left[W(\tilde{\mathfrak{g}}) \otimes \Omega(a)\right]_{basic} = \left[S(\tilde{\mathfrak{g}}^*) \otimes \Omega(a)\right]^{\mathcal{G}}$$

going to give the cohomology of $B\mathcal{G}$?

The answer should be yes. Possible proof: We know $\mathcal{G}$ acts nearly freely on $\Omega(a)$, and in fact the isotropy groups are compact. Also a is contractible, hence it is reasonable to expect that $\Omega(a)$ can be used to compute continuous cohomology:

$$\circledast \qquad H_c^*(\mathcal{G}, m) = H^*\left((m \otimes \Omega^{\cdot}(a))^{\mathcal{G}}\right)$$

On the other hand we have Bott's spectral sequence

$$H_c^*(\mathcal{G}, S(\tilde{\mathfrak{g}}^*)) \Longrightarrow H^*(B\mathcal{G}),$$

so it seems to all work out.

Question: Is the complex $\Omega(a)^{\mathcal{G}}$ going to give the continuous cohomology $H^*(B_c\mathcal{G})$?

This should be true by the same fact $\circledast$ used above.

August 9, 1983

Why $B\mathcal{G}$ is the component of $\mathrm{Map}(M, BG)$ corresponding to P_0/M: There is a fibration

$$(*) \qquad \begin{array}{c} \mathrm{Map}_G(P_0, PG) \\ \downarrow \\ \mathrm{Map}(M, BG) \end{array}$$

with image the maps classifying P_0. Since the square

$$\begin{tikzcd} P_0 \arrow[r, "f"] \arrow[d] & PG \arrow[d] \\ M \arrow[r, "\bar{f}"] & BG \end{tikzcd}$$

is cartesian, another G-map f' with the same $\bar{f}$ will correspond to an automorphism of P_0. Thus $(*)$ is a principal fibration with group $\mathcal{G}$. Finally $\mathrm{Map}_G(P_0, PG)$ is contractible, so the result is clear.

Idea: So far we have spent much time on $H^*(B\mathcal{G})$ and $H^*(B_c\mathcal{G})$, which has necessitated using forms on $\mathcal{A}$. However primitive classes for $H^*(B\mathcal{G})$ and $H^*(\mathcal{G})$ ought to correspond under transgression. Hence it might be simpler to try to produce the generators for $H^*(\mathcal{G})$.

For example, the first Chern class of the det. line bdl in $H^2(B\mathcal{G})$ corresponds to a class $H^1(\mathcal{G})$ which Singer describes as a ζ fn. determinant map

$$\mathcal{G} \longrightarrow \mathbb{C}^* .$$

Hence the problem of trivializing $\mathcal{L}$ can be seen already in $H^1(\mathcal{G})$. Hence the anomaly formula should be an explicit 1-form on $\mathcal{G}$.

Why not all cohomology in $B\mathcal{G}$ comes from invariant polynomials on $\tilde{\mathfrak{g}}$: I think I should be able to prove that the cohomology of $B_c\mathcal{G}$ and $B\mathcal{G}$ are given resp. by the complexes

$$\Omega(a)^{\mathcal{G}} \qquad\qquad \left[W(\tilde{\mathfrak{g}})\otimes\Omega(a)\right]_{basic} = \left[S(\tilde{\mathfrak{g}}^*)\otimes\Omega(a)\right]^{\mathcal{G}}.$$

If true, it follows that the classes in $H^*(B\mathcal{G})$ coming from $S(\tilde{\mathfrak{g}}^*)^{\mathcal{G}}$ go to zero in $H^*(B_c\mathcal{G})$. So if $H^*(B\mathcal{G})$ is generated by $S(\tilde{\mathfrak{g}}^*)^{\mathcal{G}}$, then $H^*(B\mathcal{G})\longrightarrow H^*(B_c\mathcal{G})$ would be zero which we know is not the case.

Next let's turn to the question of describing $H^*(\mathcal{G})$. A map $Y\longrightarrow\mathcal{G}$ can be interpreted as an automorphism g of $E=pr_2^*(E_0)$ over $Y\times M$. In general to an automorphism of a vector bundle we can define odd character classes.

Suppose g is an automorphism of E. Pick a connection D in E. Then $g^{-1}Dg = D + g^{-1}D(g)$ is another conn. and we have the linear path

$$D_t = D + t\, g^{-1}D(g)$$

leading to the formula

$$\mathrm{tr}\left(e^{D_1^2}\right) - \mathrm{tr}\left(e^{D^2}\right) = d\int_0^1 \mathrm{tr}\left(e^{D_t^2}\, g^{-1}D(g)\right) dt$$

Since $D_1^2 = g^{-1} D^2 g$ [redacted] the LHS $= 0$, so the integral gives closed forms.

So given an auto g of $E = pr_2^*(E_0)$ over $Y \times M$ in order to construct forms on Y we choose a connection D on E. The simplest is

$$D = d' + D''$$

where D'' is the pull-back of a connection on E_0/M.

To make things even simpler suppose E_0 trivial and take

$$D = d' + d'' = d.$$

Then our path of connections is

$$D_t = d + t(g^{-1}dg).$$

What's happening is that we have the evaluation map

$$\mathcal{G} \times M \longrightarrow G$$

in this case and we are pulling back the odd character forms on G to $\mathcal{G} \times M$.

Now [redacted] contrast this with what I did for the Lie algebra cohomology. There I considered over $Y \times M$ a [redacted] flat horizontal conn. on $pr_2^*(E_0)$. In the universal case $Y = \mathcal{G}$ I considered the path of horizontal connections [redacted]

$$D_t' = d' + t(g^{-1}d'g).$$

Extend to full connections by choosing a D'', say [redacted] d'' assuming E_0 trivial. Thus I consider the family

$$D_t = d + t(g^{-1}d'g).$$

The problem arises to show consistency of the two kinds of classes under the map $H^*(\tilde{\mathcal{G}}) \to H^*(\mathcal{G})$. They are represented by the forms

$$\int_0^1 tr(e^{td''\theta + (t^2-t)\theta^2}\theta)\,dt, \qquad \int_0^1 tr(e^{(t^2-t)\omega^2}\omega)\,dt$$

where $\theta = g^{-1}d'g$ and $\omega = g^{-1}dg$.

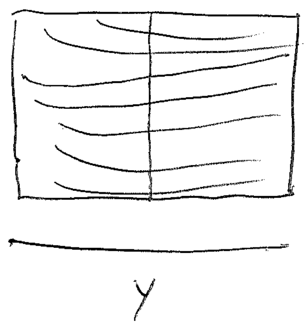
August 10, 1983

Perhaps I can get some new insights into the Lie algebra cohomology of gauge groups by looking at Gelfand-Fuks cohomology.

$\mathcal{G}$ is now Diff(M), $\tilde{\mathfrak{g}}$ = vector fields on M.

A map $Y \longrightarrow B\mathcal{G}$ corresponds to a differentiable fibre bundle over Y with fibre M. (I suppose M is compact.)

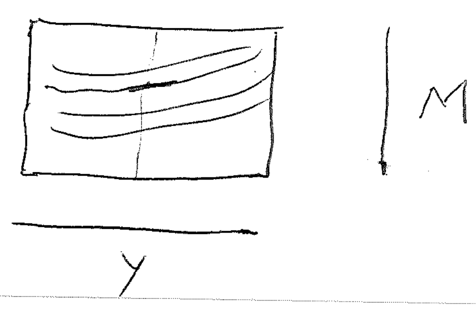
A map $Y \longrightarrow B\mathcal{G}^d$ corresponds to a differentiable fibre bundle over Y with fibre M equipped with a transverse foliation



Standard diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d, equal] & B\bar{\mathcal{G}} \arrow[r] \arrow[d] & B\mathcal{G}^d \arrow[r] \arrow[d] & B\mathcal{G} \arrow[d, equal] \\ \mathcal{G} \arrow[r] & \mathfrak{g} \arrow[r] & B_c\mathcal{G} \arrow[r] & B\mathcal{G} \end{tikzcd}$$

A map $Y \longrightarrow B\bar{\mathcal{G}}$ corresponds to a transverse foliation on $Y \times M / Y$.



There are now two theorems which describe $B\bar{\mathcal{G}}$ and $\mathcal{G}$ as spaces of sections over M. Mather-Thurston; and Bott-Segal, Haefliger. [redacted]

[redacted]

Is it possible to [redacted] produce cohomology in $\tilde{\mathcal{G}}$ by a variant of the procedure used for gauge groups?

Let's consider then a transeverse foliation [redacted] in $Y \times M / Y$. Then the normal bundle ν to the foliation is isomorphic canonically to $pr_2^*(T_M)$. I want to produce two connections on ν. Fix a connection [redacted] on T_M and use it to define a vertical connection on $\nu = pr_2^*(T_M)$ which we denote

$$D'' : \nu \longrightarrow \nu \otimes pr_2^*(T_M^*).$$

The pull-back of the connection [redacted] on T_M is $d' + D''$.

But [redacted] because ν is the normal bundle to a foliation $F \subset T_{Y\times M}$, we know that ν has a canonical partial F-connection

$$D' : \nu \longrightarrow \nu \otimes F^*$$

which is flat. Now

$$T^*_{Y\times M} \xrightarrow{\sim} F^* \oplus pr_2^*(T_M^*)$$

so if we combine the partial F connection with D'' we obtain a connection D_1 extending D'.

The difference of D_1 and $d' + D''$ is a 1-form Θ with values in $\mathrm{End}(pr_2^* T_M)$. By construction Θ vanishes on vertical tangent vectors so

$$\Theta \in \Omega^{1,0}(Y \times M, pr_2^*(\mathrm{End}\, T_M))$$
$$= \Omega^1(Y, \Omega^0(M, \mathrm{End}\, T_M))$$

On the other hand F is the graph of a map from $pr_1^* T_Y$ to $pr_2^* T_M$, hence there is a canonical element

$$\theta \in \Omega^1(Y, \Omega^0(M, T_M)).$$

One could guess that $\Theta = [\theta, \cdot\,]$.

So we have two connections D_1, $d' + D''$ on V. $d' + D''$ has curvature of type $(0,2)$, so the Chern forms computed with this curvature are of type $(0, 2p)$. D_1 has curvature ?

Is it possible at this point that we can forget F, that we have a connection D_1 on $pr_2^*(T_M)$ which is Y-flat?

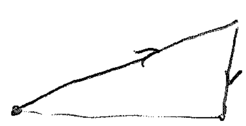
<u>August 11, 1983</u>

Yesterday I wanted to construct Gelfand-Fuks cohomology classes by a variant of the method used for gauge Lie algebras.

We consider a transverse foliation ~~on~~ $Y \times M / Y$ and want to produce forms on Y. The normal bundle to the foliation, call it Q, is isomorphic to $pr_2^*(T_M)$. Q has a canonical flat connection along the leaves of the foliation; $pr_2^*(T_M)$ has a canon. flat connection in the Y-direction. We apply Bott's procedure to each of these partial connections: Extend to full connections, then compute char. classes using the curvature. The hope was that the two Chern forms would be of high filtration, so the difference form constructed from the linear homotopy between the two connections would give closed forms of odd degree modulo the filtration, etc.

The problem is that there are two filtrations on $\Omega(Y \times M)$ one coming from the $Y \times \{m\}$ and the other from the given foliation F. These filtrations are powers of the ideals generated by $\Omega^{0,1}(Y \times M)$ and $F^\perp \subset \Omega^1(Y \times M)$ respectively.

If M is parallelizable what does seem to work is to define a flat Y-connection on Q by parallel translating along a leaf and then coming back thru M:



Consider the map

$$H^*(\tilde{\mathfrak{g}}) \longrightarrow H^*(\mathcal{G}).$$

Can I really calculate the latter?

A map $Y \to \mathcal{G}$ is the same as an automorphism g of $E = pr_2^*(E_0)$ over $Y \times M$. In general given an auto. of a bundle E one has attached odd diml. char. classes. One chooses a connection D_0 in E and uses the linear path between $g^{-1}(D_0)g$ and D_0, plus the fact that $\phi[(g^{-1}(D_0)g)^2] = \phi(D_0^2)$. If ϕ is of degree m this construction will give us a class in $H^{2m-1}(Y \times M) = H^{2m-1}(Y, \Omega_M^\cdot)$

On the other hand we have associated to ϕ a class in

$$H^{2m-1}(Y, \Omega_M^{<m}).$$

I would like to relate these classes. Notice that there doesn't seem to be an ~~[illegible]~~ obvious map because $\Omega^{<m} = \Omega^\cdot / F_m \Omega$ is a quotient complex.

Let's consider the case where E_0 is trivial and we take $D_0 = d$ over $Y \times M$. Then

$$g^{-1}(d)g = d + g^{-1}dg$$

so

$$D_t = d + t\,\underbrace{g^{-1}dg}_{\omega}$$

$$D_t^2 = \cancel{d^2} + t\,d\omega + t^2\omega^2 = (t^2 - t)\omega^2$$

The odd char. form is

$$\int_0^1 tr\left(e^{(t^2-t)\omega^2}\omega\right)dt$$

which in degree $2m-1$ is

$$\frac{1}{(m-1)!}\,tr\left(\omega^{2m-1}\right)\int_0^1 (t^2-t)^{m-1}dt$$

What one has done is to take the map

$$g : Y\times M \longrightarrow G$$

and to pull-back the primitive forms on G.

Next consider the Lie alg. classes. Start with a flat Y-connection on $pr_2^*(E_0)$ over $Y\times M$; this has the form $d'+\Theta$. To obtain forms on $Y\times M$ we use a vertical connection D'' coming from a connection on E_0, and the linear homotopy

$$D_t = (d'+D'') + t\,\Theta$$

Let's again consider the case where E_0 is trivial and $D''=d''$:

$$D_t = d + t\Theta$$

$$D_t^2 = t\,d\Theta + t^2\Theta^2 \qquad\qquad d'\Theta = -\Theta^2$$

$$= t\,d''\Theta + (t^2-t)\Theta^2$$

The odd char. form on $Y\times M$ is

$$\int_0^1 tr\left(e^{t d''\Theta + (t^2-t)\Theta^2}\,\Theta\right)dt.$$

and the degree $2m-1$ part is closed in $\Omega^{\cdot}(Y, \Omega_M^{<m})$.

How do we interpret the map $H^*(\tilde{\mathfrak{g}}) \longrightarrow H^*(\mathfrak{g})$?

When we have an auto g of $pr_2^*(E_0)$ over $Y \times M$ we have in particular a flat Y-connection

$$d' + g^{-1}d'g$$

So we have the problem of comparing the forms over $Y \times M$:

$$\operatorname{tr}(\omega^{2m-1}) \quad \frac{1}{(m-1)!} \int_0^1 (t^2-t)^{m-1}\,dt \qquad\qquad \omega = g^{-1}dg$$

$$\int_0^1 \operatorname{tr}\left(\frac{(td''\theta + (t^2-t)\theta^2)^{m-1}}{(m-1)!}\,\theta\right) dt \qquad\qquad \theta = g^{-1}d'g$$

Where do these forms come from? Over $Y \times M$ we have one-parameter families of connections

$$d + t\theta\,, \quad d + t\omega$$

and we just integrate:

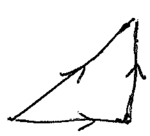
$$(*) \qquad \int_0^1 \phi'(D_t^2, \dot{D}_t)\,dt$$

Here

$$\omega = g^{-1}dg = \underbrace{g^{-1}d'g}_{\theta} + g^{-1}d''g\,,$$

so the thing to do is to consider the two-parameter family

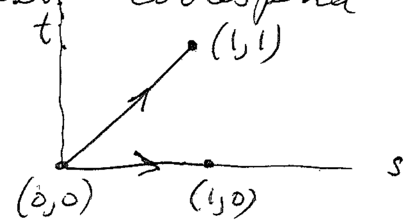
$$D_{s,t} = d + s\omega' + t\omega''$$

of connections. It should be true that the forms of type $(*)$ for the paths 

differ by the boundary of something like:

$$\phi''(D_{st}, \partial_s D, \partial_t D).$$

[redacted] The two forms of interest correspond to the paths

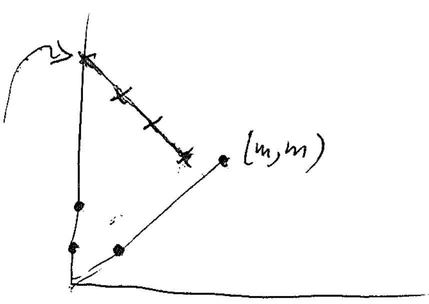


and I really want to show that these forms represent the same class in $H^{2m-1}(Y, \Omega_M^{<m})$, i.e. modulo $F_m \Omega_{Y\times M}$. So it is enough to look at the vertical line $D_{1,t} = d + \omega' + t\omega''$ from $(1,0)$ to $(1,1)$.

$$\begin{aligned}(D_{1,t})^2 &= d(\omega' + t\omega'') + (\omega' + t\omega'')^2 \\ &= \cancel{d'\omega'} + \overset{1,1}{d''\omega'} + \overset{1,1}{t d'\omega''} + \overset{0,2}{t d''\omega''} \\ &\quad + \cancel{\omega'^2} + \overset{1,1}{t(\omega'\omega'' + \omega''\omega')} + \overset{0,2}{t^2(\omega'')^2}\end{aligned}$$

$$\dot{D}_{1,t} = \omega'' \qquad 0,1$$

so $\phi(D_{1,t}^2 ; \dot{D}_{1,t})$ lives on



and it [redacted] has filtration $\geq m$, so dies in $\Omega^\cdot(Y, \Omega_M^{<m})$.

Let's consider now the problem of computing the rational cohomology of $B\mathcal{G}$ and $\mathcal{G}$. Recall that $B\mathcal{G}$ is a component of $\mathrm{Map}(M, BG)$ where G is a connected compact Lie group. We think of M as a connected finite complex and want to proceed by induction on the number of cells. Let us then suppose that

$$M = M' \cup e^d$$

for some attaching map $S^{d-1} = \dot{e}^d \xrightarrow{\alpha} M'$. We have a fibration

$$(*) \qquad \mathrm{Map}(M, BG) \longrightarrow \mathrm{Map}(M', BG)$$

and try first to understand the image and the fibre.

A map $M' \xrightarrow{f'} BG$ can be restricted to the $S^{d-1} \xrightarrow{\alpha} M'$ to get a map $S^{d-1} \xrightarrow{f'\alpha} BG$. If this is null-homotopic, then f' can be extended to M and conversely. Thus the image of $(*)$ consists exactly of those components of $\mathrm{Map}(M', BG)$ corresponding to G-bundles over M' whose restriction to α is trivial.

Suppose then we have an $f' : M' \longrightarrow BG$ which extends to M, how can we describe the possible extensions M. The actual fibre can be identified with the space all maps $e^d \longrightarrow BG$ whose restriction to S^{d-1} is a given map $f'\alpha : S^{d-1} \longrightarrow BG$ which is null-homotopic.

$$\mathrm{Map}(e^d, BG) \longrightarrow \mathrm{Map}(S^{d-1}, BG)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\cup\!\!\!|}{f'\alpha}$$

Now G connected $\Longrightarrow$ BG simply-connected. Looks like the fibre is $\Omega^d BG = \Omega^{d-1} G$.

August 12, 1983

G compact connected Lie group, M finite complex $\mathcal{G}$ is the gauge group of a principal G-bundle P_0/M, and we identify

$$B\mathcal{G} = \text{component of } \mathrm{Map}(M, BG) \text{ corresponding to } P_0$$

I am after the rational cohomology of $B\mathcal{G}$ and $\mathcal{G}_{(1)}$. Recall the Milnor-Moore thm.

$$\pi_*(\mathcal{G}_{(1)}) \otimes \mathbb{Q} = \mathrm{Prim}\{H_*(\mathcal{G}_{(1)}, \mathbb{Q})\}$$

So I want to formulate a conjecture as to what $\pi_*(\mathcal{G}_{(1)}) \otimes \mathbb{Q}$ is. I need a canonical map. Start with the canonical map

$$B\mathcal{G} \times M \longrightarrow BG$$

which will give us

$$\begin{tikzcd}
H_*(B\mathcal{G}) \otimes H_*(M) \arrow[r] & H_*(BG) \\
\pi_*(B\mathcal{G}) \otimes H_*(M) \arrow[u] & \pi_*(BG) \otimes \mathbb{Q} \arrow[u, hook]
\end{tikzcd}$$

It seems likely that this can be completed to a square. In any case we can pick a projection $H_*(BG) \to \pi_*(BG) \otimes \mathbb{Q}$ i.e. a set of primitive generators for $H^*(BG)$. So we get a map

$$\pi_*(B\mathcal{G}) \longrightarrow \mathrm{Hom}(H_*(M), \pi_*(BG)\otimes\mathbb{Q})$$
$$= H^*(M, \pi_*(BG)\otimes\mathbb{Q}).$$

Another description:

$$H^*(BG) \longrightarrow H^*(B\mathcal{G}\times M) = H^*(B\mathcal{G}; H^*(M))$$
$$\longrightarrow \mathrm{Hom}(\pi_* B\mathcal{G}, H^*(M)).$$

Notice that the map factors through the Hurewicz homomorphism for $B\mathcal{G}$. Actually we have

$$H_*(\mathrm{Map}(M, BG)\times M) \longrightarrow H_*(BG) \xrightarrow{\text{choice}} \pi_*(BG)\otimes\mathbb{Q}$$

$$H_*(\mathrm{Map}(M, BG)) \longrightarrow H^*(M, \pi_*(BG)\otimes\mathbb{Q})$$

In degree 0, this assigns to each G-bundle P/M its primitive characteristic classes.

Anyway we have a functorial in M map

$$\pi_*(B\mathcal{G})^{\otimes\mathbb{Q}} \longrightarrow H^*(M, \pi_*(BG)\otimes\mathbb{Q})$$

and we want to show that it is an isomorphism in positive degrees.

We try to prove this by induction on the number of cells in M, hence consider the case where $M = M' \cup e_d$. $\mathcal{G}$ is the space of sections of the bundle of group $P\times^G(G)$, where G acts on itself by the adjoint action. Let $\mathcal{G}'$ be the space of sections over M'; it is the gauge grp of P_0 restricted to M'. Then we have a restriction homomorphism

$$\mathcal{G} \longrightarrow \mathcal{G}'$$

which is a fibration. The kernel or fibre is the space of maps of e_d to G which are the identity on $\dot{e}_d$, hence it is $\Omega^d G$.

Put $L_* = \pi_*(G) \otimes \mathbb{Q}$ so that we have maps

$$\pi_* \mathcal{G} \otimes \mathbb{Q} \longrightarrow H^{-*}(M, L_*)$$

functorial in M. Consider now the map of long exact sequences

$$\begin{tikzcd}
\arrow[r] & \pi_k \Omega^d G \otimes \mathbb{Q} \arrow[r] \arrow[d, "\cong"] & \pi_k \mathcal{G} \otimes \mathbb{Q} \arrow[r] \arrow[d] & \pi_k \mathcal{G}' \otimes \mathbb{Q} \arrow[r] \arrow[d] & \pi_{k-1} \Omega^d G \otimes \mathbb{Q} \arrow[r] \arrow[d, "\cong"] & {} \\
\arrow[r] & H^{-k}(e_d, \dot{e}_d; L_*) \arrow[r] & H^{-k}(M, L_*) \arrow[r] & H^{-k}(M', L_*) \arrow[r] & H^{-k+1}(e_d, \dot{e}_d; L_*) \arrow[r, dash] & {}
\end{tikzcd}$$

where the top row is the long exact homotopy sequence of the fibration

$$\Omega^d G \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'$$

Now using induction and five lemma we can conclude that

$$\pi_k \mathcal{G} \otimes \mathbb{Q} \xrightarrow{\sim} H^{-k}(M, L_*) \qquad \text{for } k \geq 1.$$

For $k = 0$ we have to be more careful because we don't know that $\pi_0 \mathcal{G}$ is an abelian group. We do have the diagram

$$[\dot{e}_d, G] = \pi_{d-1}(G) \text{ as } \pi_0 G = 0.$$

$$\begin{tikzcd}
\pi_1 \mathcal{G}' \arrow[r] \arrow[d] & \pi_0(\Omega^d G) \arrow[r] \arrow[d] & \pi_0 \mathcal{G} \arrow[r] \arrow[d] & \pi_0 \mathcal{G}' \arrow[r] \arrow[d] & {[\dot{e}_d, G]} \arrow[d] \\
H^{-1}(M', L_*) \arrow[r] & H^0(e_d, \dot{e}_d, L_*) \arrow[r] & H^0(M, L_*) \arrow[r] & H^0(M', L_*) \arrow[r] & H^0(\dot{e}_d, L_*)
\end{tikzcd}$$

I would like to prove that $\pi_0 \mathcal{G}$ is nilpotent, that the kernel and cokernel of

$$\pi_0 \mathcal{G} \longrightarrow H^0(M, L_*)$$

are torsion. It seems that diagram-chasing plus induction allows me to prove that the kernel and cokernel are torsion groups. In order to see that $\pi_0 \mathcal{G}$ is nilpotent I would need to know that

$$\mathrm{Im}\{\pi_0(\Omega^d G) \longrightarrow \pi_0 \mathcal{G}\}$$

is contained in the center of $\pi_0 \mathcal{G}$. This is obvious: $\Omega^d G$ is a normal subgroup of $\mathcal{G}$, so this gives the action. But $\mathcal{G}$ acts through the sections over e_d which is connected since e_d is contractible and G is connected.

Summary: At this point I think that I can compute the rational cohomology of $B\mathcal{G}$ and $\mathcal{G}_{(0)}$ for any gauge group. Borel's thm. will take care of $H^*(B\mathcal{G}_{(0)})$, but we may need a separate argument to get the last bit.

Example: $G = U(1) =$ circle. Here $\tilde{\mathfrak{g}}$ is abelian so I know that $H^*(\tilde{\mathfrak{g}})$ is an exterior algebra on $\tilde{\mathfrak{g}}^*$. Take $P_0 =$ trivial bundle. Then

$$\tilde{\mathfrak{g}} = \mathrm{Map}(M, \mathfrak{g}) = \mathrm{Map}(M, \mathbb{R}) = \text{real-valued smooth fns. on } M$$

so $\tilde{\mathfrak{g}}^* =$ distributions on M

$=$ 0-diml currents.

$\mathcal{G} = \mathrm{Map}(M, S^1)$. Use exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow S^1 \longrightarrow 0$$

$$0 \longrightarrow \underset{\substack{\| \\ \mathbb{Z}}}{\mathrm{Map}(M,\mathbb{Z})} \longrightarrow \mathrm{Map}(M,\mathbb{R}) \longrightarrow \underset{\substack{\| \\ \mathcal{G}}}{\mathrm{Map}(M,S^1)} \longrightarrow H^1(M,\mathbb{Z}) \longrightarrow 0$$

(M connected)

In fact the circle is a $K(\mathbb{Z},1)$, hence

$$\pi_0\,\mathcal{G} = [M, S^1] = H^1(M,\mathbb{Z}).$$

The identity component $\mathcal{G}_{(0)} = \mathrm{Map}(M,\mathbb{R})/\mathbb{Z}$ has the homotopy type of S^1. In fact

$$\mathcal{G}_{(0)} = S^1 \times \big(\mathrm{Map}(M,\mathbb{R})/\mathbb{R}\big).$$

Let us assume that continuous cohomology for a product is the tensor product. Then

$$H_c^*(\mathcal{G}_{(0)}) = \underbrace{H_c^*(S^1)}_{\text{trivial}} \otimes \underbrace{H_c^*\big(\mathrm{Map}(M,\mathbb{R})/\mathbb{R}\big)}_{\Lambda\left\{\substack{0\text{ diml currents} \\ \text{of integral } 0}\right\}}$$

So corresponding to

$$\mathcal{G}_{(0)} \longrightarrow \tilde{g} \longrightarrow B_c\,\mathcal{G}_{(0)}$$

we get primitive cohomology entirely in dimension 1:

$$\mathbb{C} \longleftarrow \left\{\substack{0\text{-diml} \\ \text{currents}}\right\} \longleftarrow \left\{\substack{0\text{-diml currents} \\ \text{of integral } 0}\right\}$$

August 13, 1983

The program is the determination of the continuous and Lie algebra cohomology of gauge groups. I want to use the van Est spectral sequence to relate:

$$\mathcal{G} \longrightarrow \mathfrak{g} \longrightarrow B_c\mathcal{G} \longrightarrow B\mathcal{G}$$

Yesterday I convinced myself that there should be no real problems with the cohomology of $\mathcal{G}$ and $B\mathcal{G}$. The assertion will be that the cohomology of $\mathcal{G}_{(0)}$ and $B\mathcal{G}$ is freely generated by explicit classes.

The hope will be that the cohomology of $\mathfrak{g}$ and $B_c\mathcal{G}$ will also be freely generated by the explicit classes that I have already described. I have the ~~potential~~ generators for the cohomology; the problem is now to find an <u>upper bound</u> argument.

Notice that in the van Est spectral sequence it is possible for cohomology of $\mathfrak{g}$ and $B_c\mathcal{G}$ to cancel, hence a good upper bound argument is absolutely essential. This reminds me of Kazhdan's stuff about Riemann surfaces, where he used stuff about loop groups and Verma modules to pin down the continuous cohomology for a disk.

The first stage will be to get a precise hold on the conjectural formulas for $H^*(\mathfrak{g})$, $H_c^*(\mathcal{G})$. Let's start with $H^*(\mathcal{G}_{(0)})$ and $H^*(B\mathcal{G})$.

We have a canonical map $B\mathcal{G} \times M \longrightarrow BG$

which gives $H^*(BG) \longrightarrow H^*(B\mathcal{G}) \otimes H^*(M)$.
So if $\phi_j \in H^{2d_j}(BG)$ is a generator we get

$$\phi_j \longmapsto \phi_j \in \bigoplus_p H^p(B\mathcal{G}) \otimes H^{2d_j - p}(M)$$

and hence a map

$$\bigoplus_{j=1}^{\ell} H_{2d_j - p}(M) \longrightarrow H^p(B\mathcal{G}).$$

The conjecture which I think I can prove is that $H^*(B\mathcal{G})$ is freely generated and

$$\boxed{\bigoplus_{j=1}^{\ell} H_{2d_j - p}(M) \xrightarrow{\sim} Q H^p(B\mathcal{G}) \qquad \text{for } p \geq 1.}$$

Also we should have

$$Q H^p(B\mathcal{G}) \xrightarrow{\sim} P H^{p-1}(\mathcal{G}_{(0)}) \qquad p \geq 2$$

so that

$$\boxed{\bigoplus_{j=1}^{\ell} H_{2d_j - 1 - p}(M) \xrightarrow{\sim} P H^p(\mathcal{G}_{(0)}) \qquad \text{for } p \geq 1}$$

Now I want to compute the composition

$$\bigoplus_{j=1}^{\ell} H_{2d_j - p}(M) \longrightarrow H^p(B\mathcal{G}) \longrightarrow H^p(B_c\mathcal{G}).$$

Fix a $\phi = \phi_j$ of degree $m = d_j$. The image of ϕ under the map

$$H^{2m}(BG) \longrightarrow H^{2m}(B\mathcal{G} \times M)$$

can be computed using equivariant forms on $\mathcal{G} \times M$. The point is that over $\mathfrak{a}\times M$, the bundle $pr_2^*(P_0)$ has a canonical invariant connection. Applying ϕ to the equivariant curvature gives a closed form in

$$\left[S(\tilde{\mathfrak{g}})^* \otimes \Omega(\mathfrak{a}\times M)\right]^{G}$$

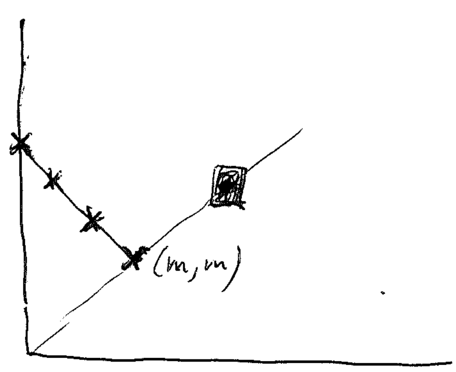
representing the ϕ-class in $H^{2m}(BG\times M)$. Now I have seen that to get the image in the continuous cohomology one passes from the equivariant forms to the invariant forms.

Lie alg cohomology	:	invariant forms	on	G
continuous "	:	"	"	" $\mathfrak{a}$

So this means that the ϕ-class in $H^{2m}(B_c G\times M)$ is represented by the form in

$$\Omega(\mathfrak{a}\times M)^{G} = \Omega(\mathfrak{a})^{G}\otimes\Omega(M)$$

which is obtained by applying ϕ to the curvature of the canonical connection of $pr_2^*(P_0)$ over $\mathfrak{a}\times M$. Now this connection is flat in the $\mathfrak{a}$-direction, so the curvature is of filtration ≥ 1. Hence ϕ of the curvature is of degree $2m$ and filtration $\geq m$.



In fact we should get a class

$$\mathrm{Im}(\phi) \in H^{2m}(B_c\mathcal{G}; F_m\Omega^{\cdot}_M)$$

$$= H^m(B_c\mathcal{G})\otimes\{\mathrm{Ker}\, d \text{ on } \Omega^m\} \oplus \bigoplus_{p>m} H^p(B_c\mathcal{G})\otimes H^{2m-p}(M)$$

This leads to the conjecture that $H^*(B_c\mathcal{G})$ is freely generated with generators

$$\boxed{\bigoplus_{j=1}^{l} \left\{\begin{array}{ll} H_{2d_j-p}(M) & d_j>p \\ C_p/\partial C_{p+1} & d_j=p \\ 0 & d_j<p \end{array}\right\} \xrightarrow{\sim} QH^p(B_c\mathcal{G}) \quad \text{for } p\geq 1}$$

where C_p are the currents on M of degree p.

The conjecture for the Lie algebra says that $H^*(\tilde{\mathfrak{g}})$ is freely-generated with generators:

$$\boxed{\bigoplus_{j=1}^{l} \left\{\begin{array}{ll} H_{2d_j-1-p}(M) & p>d_j \\ \mathrm{Ker}\,\partial \text{ on } C_{p-1} & p=d_j \\ 0 & p<d_j \end{array}\right\} \xrightarrow{\sim} QH^p(\tilde{\mathfrak{g}})}$$

Picture:

$$\begin{array}{lcccccccc}
 & H^p(\mathcal{G}) & \longleftarrow & H^p(\tilde{\mathfrak{g}}) & \longleftarrow & H^p(B_c\mathcal{G}) & \longleftarrow & H^p(B\mathcal{G}) \\
d_j=p+1 & H_{p+1}(M) & \longleftarrow & 0 & \longleftarrow & H_{p+2}(M) & \longleftarrow & H_{p+2}(M) \\
d_j=p & H_{p-1}(M) & \longleftarrow & \mathrm{Ker}\,\partial \text{ on } C_{p-1} & \longleftarrow & C_p/\partial C_{p+1} & \longleftarrow & H_p(M) \\
d_j=p-1 & H_{p-3}(M) & \longleftarrow & H_{p-3}(M) & \longleftarrow & 0 & \longleftarrow & H_{p-2}(M)
\end{array}$$

Better picture is to expect a long exact sequence of primitive cohomology even for a single ϕ of degree m. Thus one has

$$
\begin{array}{ccccc}
PH^{m+1}(\tilde{\mathfrak{g}}) & \longleftrightarrow & PH^{m+1}(B_c\mathcal{G}) & \longleftrightarrow & PH^{m+1}(B\mathcal{G}) \\
H_{m-2}(M) & & 0 & & H_{m-1}(M) \\
PH^{m}(\tilde{\mathfrak{g}}) & \longleftarrow & PH^{m}(B_c\mathcal{G}) & \longleftarrow & PH^{m}(B\mathcal{G}) \\
\mathrm{Ker}\,\partial \text{ on } \mathcal{C}_{m-1} & & \mathcal{C}_m/\partial\mathcal{C}_{m+1} & & H_m(M) \\
PH^{m-1}(\tilde{\mathfrak{g}}) & \longleftarrow & PH^{m-1}(B_c\mathcal{G}) & \longleftarrow & PH^{m-1}(B\mathcal{G}) \\
0 & & H_{m+1}(M) & & H_{m+1}(M)
\end{array}
$$

which works <u>very</u> nicely.

Special case: Take $G = SU(2)$ so that there is a single ϕ of degree 4. According to the conjectures

$$
PH^*(\tilde{\mathfrak{g}}) = \begin{cases} \mathrm{Ker}\,\partial \text{ on } C_1 & * = 2 \\ H_0(M) & * = 3 \end{cases}
$$

and the rest of the primitive cohomology is trivial. So

August 17, 1983

Jackiw told me yesterday that Polyakov had found a nice formula for $\det(\partial\!\!\!/ + A\!\!\!/)$ in two dimensions in the non-abelian setting. The formula involves adding to the Schwinger term a type of Chern-Simons term which is trivial in the $U(1)$ case.

As usual I use

$$\partial\!\!\!/ = \begin{pmatrix} 0 & -\partial_t + \frac{1}{i}\partial_x \\ \partial_t + \frac{1}{i}\partial_x & 0 \end{pmatrix}$$

which is essentially

$$\begin{pmatrix} & \partial_{\bar{z}}^* \\ \partial_{\bar{z}} & \end{pmatrix}$$

where $z = x + it$.

I want to do as much as possible for $\log\det(\partial_{\bar{z}} + \alpha)$. We work over the (x,t)-plane and suppose that $\alpha(x,t)$ decays as $(x,t) \to \infty$. Actually we will suppose that there is a function

$$\varphi : z\text{-plane} \longrightarrow GL_n$$

such that

$$\alpha = \varphi^{-1}\partial_{\bar{z}}\varphi \quad \left(\iff \varphi^{-1}\circ\partial_{\bar{z}}\circ\varphi = \partial_{\bar{z}} + \alpha \right)$$

For example if $n = 1$, then we can solve

$$\partial_{\bar{z}} \log\varphi = \alpha$$

by

$$\log\varphi = \int \frac{d^2z'}{\pi} \frac{\alpha(z')}{z - z'}$$

Now we want to define

$$W(\alpha) = \log\det(\partial_{\bar z}+\alpha)$$

by the variational formula

$$\delta W = \mathrm{Tr}_{reg}\,(\partial_{\bar z}+\alpha)^{-1}\delta\alpha = \int \mathrm{tr}\,(J\,\delta\alpha)$$

where J is the finite part of the Green's fn. for $(\partial_{\bar z}+\alpha)^{-1}$ on the diagonal. This I define using the connection

$$\nabla = \partial_z\,dz + (\partial_{\bar z}+\alpha)\,d\bar z$$

so that I get an analytic determinant function.

$$D = \varphi^{-1}\circ\partial_{\bar z}\circ\varphi$$

$$\langle z|D^{-1}|z'\rangle = \varphi^{-1}(z)\frac{1}{\pi(z-z')}\varphi(z') \qquad \text{set } \tilde\varphi = \varphi^{-1}$$

$$= \frac{1}{\pi(z-z')}\left\{\tilde\varphi(z') + \partial_z\tilde\varphi(z')(z-z') + \partial_{\bar z}\tilde\varphi(z')(\overline{z-z'}) + \dots\right\}\varphi(z').$$

So we find (as I did roughly 1 year ago) that

$$J = \frac{1}{\pi}\partial_z\tilde\varphi\cdot\varphi = -\frac{1}{\pi}\varphi^{-1}\partial_z\varphi$$

The curvature of ∇ is

$$\nabla^2 = (\partial_z\alpha)\,dz\,d\bar z$$

and the anomaly formula is

$$[\partial_{\bar z}+\alpha, J] = -\frac{1}{\pi}(\partial_z\alpha)$$

To find W I have to integrate the differential equation

$$\frac{\delta W}{\delta \alpha} = J.$$

This means we pick a path starting from a basepoint 0 in the space of α and ending with α and then integrate $\delta W = \int tr(J \cdot \delta\alpha)$ along this path.

So I will assume given φ_t, a deformation of φ to the identity. Then we want to integrate

$$\frac{d}{dt} W(\varphi_t) = \int tr\,(J\ \partial_t \alpha)$$

$$= -\frac{1}{\pi}\int tr\left(\varphi^{-1}\partial_z\varphi\ \ \partial_t(\varphi^{-1}\partial_{\bar z}\varphi)\right) d^2z$$

In the 1-diml case this is

$$= -\frac{1}{\pi}\int \left(\partial_z \log\varphi \cdot \partial_t(\partial_{\bar z}\log\varphi)\right) d^2z$$

$$= \frac{d}{dt}\left\{-\frac{1}{2\pi}\int\left(\partial_z\log\varphi\cdot\partial_{\bar z}\log\varphi\right) d^2z\right\}$$

where one uses $\int \partial_z\psi\cdot\partial_{\bar z}\psi'\ d^2z$ is symmetric in ψ,ψ' by integration by parts.

In the non-abelian case compare the expressions $\frac{d}{dt}W(\varphi_t)$ and

$$\frac{d}{dt}\frac{-1}{2\pi}\int tr\left(\varphi^{-1}\partial_z\varphi\cdot\varphi^{-1}\partial_{\bar z}\varphi\right) d^2z \qquad =$$

$$\left(\frac{-1}{2\pi}\right)\int d^2z \;\mathrm{tr}\left\{ -\varphi^{-1}\partial_t\varphi\cdot\varphi^{-1}\partial_z\varphi\cdot\varphi^{-1}\partial_{\bar z}\varphi + \varphi^{-1}\partial_z\varphi\left(-\varphi^{-1}\partial_t\varphi\,\varphi^{-1}\right)\partial_{\bar z}\varphi \right.$$
$$\left. \varphi^{-1}\partial^2_{tz}\varphi\cdot\varphi^{-1}\partial_{\bar z}\varphi + \varphi^{-1}\partial_z\varphi\cdot\varphi^{-1}\partial^2_{t\bar z}\varphi \right\}$$

$$= \frac{1}{2\pi}\int d^2z\;\mathrm{tr}\left\{ \varphi^{-1}\partial_t\varphi\cdot\left[\varphi^{-1}\partial_z\varphi\,\varphi^{-1}\partial_{\bar z}\varphi + \varphi^{-1}\partial_{\bar z}\varphi\,\varphi^{-1}\partial_z\varphi\right] \right.$$
$$\left. \partial_t\varphi\cdot\partial_z\left(\varphi^{-1}\partial_{\bar z}\varphi\varphi^{-1}\right) + \partial_t\varphi\cdot\partial_{\bar z}\left(\varphi^{-1}\partial_z\varphi\,\varphi^{-1}\right) \right\}$$

$$\frac{d}{dt}W(\varphi_t) = -\frac{1}{\pi}\int \mathrm{tr}\left\{\varphi^{-1}\partial_z\varphi \;\; \partial_t\left(\varphi^{-1}\partial_{\bar z}\varphi\right)\right\} d^2z$$

$$= -\frac{1}{\pi}\int \mathrm{tr}\left\{ \begin{array}{l} \varphi^{-1}\partial_z\varphi\left(-\varphi^{-1}\partial_t\varphi\,\varphi^{-1}\right)\partial_{\bar z}\varphi \\ +\varphi^{-1}\partial_z\varphi\cdot\varphi^{-1}\cdot\partial^2_{t\bar z}\varphi \end{array}\right\} d^2z$$

$$= \frac{1}{\pi}\int \mathrm{tr}\left\{ \begin{array}{l} \varphi^{-1}\partial_t\varphi \;\; \varphi^{-1}\partial_{\bar z}\varphi \;\; \varphi^{-1}\partial_z\varphi \\ +\; \partial_{\bar z}\left(\varphi^{-1}\partial_z\varphi\,\varphi^{-1}\right)\partial_t\varphi \end{array}\right\} d^2z$$

Now use $\varphi^{-1}\partial_z\varphi\,\varphi^{-1} = -\partial_z\varphi^{-1}$ and then the lower terms in the $\{\;\}$ braces are $\partial_t\varphi\cdot\partial^2_{z\bar z}(\varphi^{-1})$

$$\frac{d}{dt}W(\varphi_t) = \frac{d}{dt}\left(-\frac{1}{2\pi}\int \mathrm{tr}\left(\varphi^{-1}\partial_z\varphi\,\varphi^{-1}\partial_{\bar z}\varphi\right)d^2z\right)$$
$$+\frac{1}{2\pi}\int d^2z\;\mathrm{tr}\left(\varphi^{-1}\partial_t\varphi\left[\varphi^{-1}\partial_z\varphi\cdot\varphi^{-1}\partial_{\bar z}\varphi - \varphi^{-1}\partial_{\bar z}\varphi\cdot\varphi^{-1}\partial_z\varphi\right]\right)$$

Now

$$\frac{i}{2}\,\frac{1}{3}\,\mathrm{tr}\left[(\varphi^{-1}d\varphi)^3\right] = \mathrm{tr}\left(\varphi^{-1}\partial_t\varphi\left[\varphi^{-1}\partial_z\varphi\,\varphi^{-1}\partial_{\bar z}\varphi - \varphi^{-1}\partial_{\bar z}\varphi\,\varphi^{-1}\partial_z\varphi\right]\right)$$
$$\times\, dt\,\underbrace{dz\,d\bar z\,\tfrac{i}{2}}$$

and $d^2z = \frac{i}{2}\,dz\,d\bar z$

Final formula is

$$\boxed{\begin{aligned} W(\varphi) = &-\frac{1}{2\pi}\int \mathrm{tr}\left(\varphi^{-1}\partial_z\varphi \cdot \varphi^{-1}\partial_{\bar z}\varphi\right) d^2z \\ &+ \frac{i}{12\pi}\int \mathrm{tr}\left(\varphi^{-1}d\varphi\right)^3 \end{aligned}}$$

Here the second integral is taken over $[0,1]\times\mathbb{R}^2$ as I derived it. But Jackiw implied it could be taken over a suitable 3 manifold with boundary $\mathbb{R}^2$.

August 19, 1983

Problem: To understand renormalization.

Let's first review the effective potential idea in the case of a 0-diml field theory. Consider a particle on the x-axis in the potential field $V(x)$ at inverse temperature β. If a force J is applied the particle is subject to the potential $V(x) - Jx$, and it is found at the (average) position

$$\bar{x} = \frac{1}{\beta}\frac{\partial}{\partial J}\log Z$$

where

$$Z(J) = \int e^{-\beta(V(x)-Jx)}\,dx.$$

If $W(x)$ is defined to be the Legendre transform:

$$W(x) = Jx - \frac{1}{\beta}\log Z \qquad \text{where} \qquad x = \frac{\partial}{\partial J}\left(\frac{1}{\beta}\log Z\right)$$

then we have

$$W'(x) = J \iff x = \bar{x}(J).$$

In other words the particle is found as if it were in the potential field W, that is, W is the effective potential.

The simplest physical quantities of interest are the position when the force $J = 0$:

$$\bar{x}(0) = \frac{\partial}{\partial J}\left.\frac{\log Z}{\beta}\right|_{J=0}$$

and the effective "spring constant", namely the value of $W''(x)$ at this equilibrium point. Now

$$W''(x) = \frac{dJ}{dx} = \left(\frac{dx}{dJ}\right)^{-1}$$

and
$$\frac{dx}{dJ} = \frac{\partial^2}{\partial J^2}\frac{\log Z}{\beta}$$
$$= \frac{\partial}{\partial J}\frac{1}{\beta}\frac{Z'}{Z} = \frac{1}{\beta}\frac{Z'' - Z'^2}{Z}$$
$$= \frac{1}{\beta}\left(\frac{Z''}{Z} - \left(\frac{Z'}{Z}\right)^2\right) = \beta\left(\langle x^2\rangle - \langle x\rangle^2\right)$$
is essentially the second moment of the Boltzmann distribution $e^{-\beta V(x)}/Z$.

Now recall the expansion
$$W(x) = \text{const} + (-\Gamma_1)x + (a-\Gamma_2)\frac{x^2}{2} + (-\Gamma_3)\frac{x^3}{3!} + \cdots$$
where the coefficients are given by irreducible diagrams. If the odd Γ_n are 0, then $\bar{x} = 0$ when $J = 0$, and the spring constant is
$$a - \Gamma_2.$$
So $-\Gamma_2$ is a correction to the bare spring constant a due to the 'interaction'.

Next we consider a field theory where instead of a single degree of freedom x we have an infinite number $\varphi(x)$ indexed by points x of space-time. This time the analogue of the moments $\langle x^n\rangle$ are the n-point functions
$$\langle 0 | T[\varphi(x_1)\cdots\varphi(x_n)] | 0\rangle.$$
So the second moment corresponds to the two-point function. Assume $\langle\varphi(x)\rangle = 0$ which should be

the case when the vertices are of even multiplicity. Then

$$G(x, x') = \langle 0 | T[\varphi(x)\, \varphi(x')] | 0 \rangle$$

is the second moment $\dfrac{\delta^2}{\delta J(x)\, \delta J(x')} \log Z(J)$ where

$$Z(J) = \int [\mathcal{D}\varphi]\, e^{-\int (L(\varphi) - J\varphi)\, dx}$$

Hence this 2-point function is the inverse of $(a - \Gamma_2)$, where a is the free propagator.

Now at this point I need suitable examples. I would like ultimately to have fermion examples. For example I could take a gas of fermions on the line with a pair-wise interaction. This leads to a kind of fermion integral over fields over space-time. Now I understand the free situation very well. I know exactly how the Green's functions can be computed in terms of the inverse of the Dirac operator.

Feynman's formula for $\frac{1}{ab}$:

$$\frac{1}{ab} = \int_0^\infty e^{-ta}\,dt \int_0^\infty e^{-ub}\,du = \iint e^{-ta-ub}\,dt\,du$$

Put $s = t+u$, or $u = s-t$

$$\frac{1}{ab} = \int_0^\infty ds \int_0^s dt \; e^{-ta-(s-t)b} \qquad \bar{t} = \frac{t}{s}$$

$$= \int_0^\infty ds \int_0^1 s\,d\bar{t}\; e^{-s[\bar{t}a+(1-\bar{t})b]}$$

$$= \int_0^1 d\bar{t} \int_0^\infty e^{-s[\bar{t}a+(1-\bar{t})b]}\, s\,ds$$

$$= \int_0^1 d\bar{t}\; \frac{1}{[\bar{t}a+(1-\bar{t})b]^2}$$

In general

$$\frac{1}{a_1 \cdots a_n} = \int_0^\infty dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \; e^{-[(t_n-t_{n-1})a_1 + \cdots + (t_1)a_n]}$$

because if we set $t_i = x_i + x_{i-1} + \cdots + x_1$, then we are integrating over $x_i \geq 0$, $i = 1, \ldots, n$ and

$$dt_i\, dt_{i-1} \cdots dt_1 \overset{\text{ind}}{=} \underbrace{dt_i}_{dx_i + dx_{i-1} + \cdots + dx_1}(dx_{i-1} \cdots dx_1) = dx_i \cdots dx_1$$

Finally putting $t = t_n$ and rescaling $t_i = t\bar{t}_i$ we get

$$\frac{1}{a_1 \cdots a_n} = \int_0^1 d\bar{t}_{n-1} \int_0^{\bar{t}_{n-1}} d\bar{t}_{n-2} \cdots \int_0^{\bar{t}_2} d\bar{t}_1 \, \frac{1}{[(1-\bar{t}_{n-1})a_1 + \cdots + \bar{t}_1 a_n]^n}$$

I want next to consider the field theory of a real-valued field $\phi(x)$, $x \in \mathbb{R}^n$ given by the action

$$S(\phi) = \int d^n x \left\{ \frac{1}{2}\phi(-\Delta + m^2)\phi + \frac{\lambda}{4!}\phi^4 \right\}.$$

The theory ultimately consists of Green's functions which are conveniently obtained from a generating function

$$Z(J) = \int \mathcal{D}\phi \, e^{-S(\phi) + \int d^n x \, J(x)\phi(x)}$$

$$= \int \mathcal{D}\phi \, e^{-S_0(\phi) + \int J\phi - \frac{\lambda}{4!}\int \phi^4}$$

which one formally evaluates using Feynman diagrams.

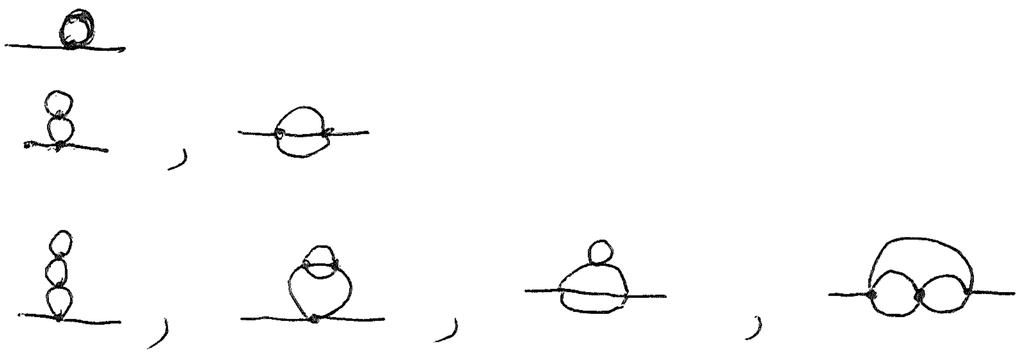
Particularly interesting is the 2-point function

$$G(x, x') = \langle \phi(x)\phi(x') \rangle$$

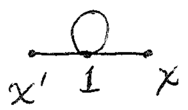
which is the analogue of the inverse spring constants in the present ~~situation~~. I know that

$$G = \frac{1}{(-\Delta + m^2) - \Gamma_2}$$

where Γ_2 is a sum over 1-particle irred. graphs with two external edges. I have already made a list of the graphs:



so now I should work out the contribution of each graph.



$$-\frac{\lambda}{2}\int G_0(x';1)\,G_0(1,1)\,G_0(1,x)\quad d1$$

Where did this come from?

$$\langle\phi(x)\phi(x')\rangle = \frac{\int \mathcal{D}\phi\; e^{-S_0(\phi)-\frac{\lambda}{4!}\int\phi^4(y)\,dy}\;\phi(x)\phi(x')}{\int\mathcal{D}\phi\; e^{-S_0(\phi)-\frac{\lambda}{4!}\int\phi^4(y)\,dy}}$$

~~The term we are interested in is~~

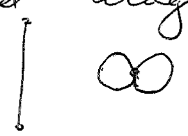
$$\cancel{\int dy\;\frac{\int\mathcal{D}\phi\; e^{-S_0(\phi)}\;-\frac{\lambda}{4!}\phi^4(y)\,\phi(x)\phi(x')}{\int\mathcal{D}\phi\; e^{-S_0(\phi)}}}$$

$$= \frac{\int\mathcal{D}\phi\; e^{-S_0(\phi)}\left[1-\frac{\lambda}{4!}\int dy\;\phi^4(y)+\dots\right]\phi(x)\phi(x')}{\int\mathcal{D}\phi\; e^{-S_0(\phi)}\left[1-\frac{\lambda}{4!}\int dy\;\phi^4(y)+\dots\right]}$$

Assume $\mathcal{D}\phi$ normalized so that $\int \mathcal{D}\phi\, e^{-S_0(\phi)} = 1$.
Then this becomes

$$\langle \phi(x)\phi(x') \rangle = \frac{\langle \phi(x)\phi(x')\rangle_0 - \frac{\lambda}{4!}\int dy\, \langle \phi^4(y)\phi(x)\phi(x')\rangle_0 + \cdots}{1 - \frac{\lambda}{4!}\int dy\, \langle \phi^4(y)\rangle_0 + \cdots}$$

In $\langle \phi^4(y)\,\phi(x)\,\phi(x')\rangle_0$ we have a mess of possible contractions. The denominator cancels the $\phi(x)\,\phi(x')$ contraction, which corresponds to the disconnected diagram

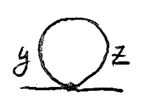


So the term we are after is

$$-\frac{\lambda}{4!}\, 4\cdot 3 \cdot \int dy\; G_0(x,y)\, G_0(y,y)\, G_0(x',y).$$

as claimed.

From the viewpoint of the general theory one should not think of labelling a vertex by a space point, but rather one [redacted] labels the edges coming into the vertex with space points. Hence the first order contribution to $\Gamma_2(x,x')$ is



$$-\frac{\lambda}{2}\int dy\,dz\; G_0(y,z)\, \delta(x-y)\, \delta(y-z)\, \delta(z-x')$$

$$= -\frac{\lambda}{2}\int dy\; G_0(y,y)\;\delta(x-y)\,\delta(y-x')$$

$$= -\frac{\lambda}{2}\, G_0(x,x)\,\delta(x-x')$$

It would have been better if I hadn't used x, x' but rather something neutral:

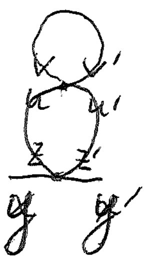
$$\Gamma_2(y, y') = -\frac{\lambda}{2}\, G_0(y,y)\, \delta(y-y') + O(\lambda^2)$$

Hence

$$\langle \phi(x)\,\phi(x')\rangle = G_0(x,x') + \int dy\, dy'\; G_0(x,y)\,\Gamma_2(y,y')\,G_0(y',x) + \cdots$$

$$= G_0(x,x') - \frac{\lambda}{2}\int dy\; G_0(x,y)\, G_0(y,y)\, G_0(y,x') + \cdots$$

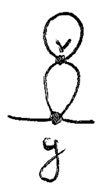
Next let's work out the contributions belonging to the second order diagrams.



$$\frac{(-\lambda)^2}{4}\int d\{\cancel{z}\,\cancel{z'}\,\cancel{u}\,\cancel{u'}\; v\,\cancel{v'}\}\;\delta(\cancel{y-z},\ \cancel{z-z'},\ \cancel{z'-y'},\ \cancel{u-v},\ \cancel{v-v'},\ \cancel{u-u'})\; G_0(\cancel{z}\,y, \cancel{u}\,v)\, G_0(v, \cancel{v'}\,v)\, G_0(\cancel{v'}\,v, \cancel{z'}\,y)$$

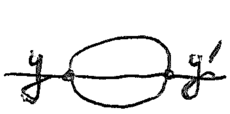
$$= \frac{(-\lambda)^2}{4}\int dv\; G_0(y,v)\, G_0(v,v)\, G_0(v,y)\;\; \delta(y-y')$$

In this calculation I could have labelled the vertices provided I multiply by $\delta(y-y')$



$$+\frac{(-\lambda)^2}{4}\int dv\; G_0(y,v)\, G_0(v,v)\, G_0(v,y)\;\; \delta(y,y').$$

Next consider the other 2nd order diagram



$$\frac{(-\lambda)^2}{3!}\; G_0(y,y')^3$$

Next we do the calculations in momentum space i.e. using the Fourier transform

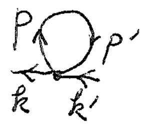
$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{k} e^{ikx} \phi_k$$

Then

$$L_{int}(\phi) = \frac{\lambda}{4!} \int_V \phi^4(x)\, dx = \frac{\lambda}{4!} \frac{1}{V} \sum_{k+l+p+q=0} \phi_k \phi_l \phi_p \phi_q$$

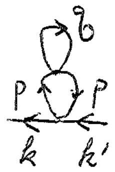
and

$$\langle \phi_k \phi_l \rangle_0 = \langle \phi_k | \frac{1}{-\Delta + m^2} | \phi_l \rangle = \frac{1}{k^2+m^2} \delta_{k,l}$$

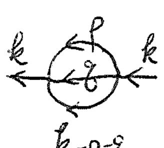
<u>Diagram:</u>  has contribution

$$\frac{(-\lambda)}{V \cdot 2} \sum_{p,p'} \delta(p+k-p'-k') \frac{\delta(p-p')}{p^2+m^2}$$

$$= \frac{-\lambda}{2} \left(\frac{1}{V} \sum_p \frac{1}{p^2+m^2} \right) \delta(k-k') \longrightarrow \frac{-\lambda}{2} \left(\int \frac{d^n p}{(2\pi)^n} \frac{1}{p^2+m^2} \right) \delta(k-k')$$



$$\frac{(-\lambda)^2}{4} \frac{1}{V^2} \sum_{p,q} \frac{1}{(p^2+m^2)^2} \frac{1}{q^2+m^2} \delta(k-k')$$



$$\frac{(-\lambda)^2}{6} \int \frac{d^n p}{(2\pi)^n} \frac{d^n q}{(2\pi)^n} \frac{1}{p^2+m^2} \frac{1}{q^2+m^2} \frac{1}{(p+q-k)^2+m^2}$$

At this point we have to analyze the nature of the divergences. The first one occurring in <u>Ω</u>

involves
$$\int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2+m^2} = G_0(x,x).$$

This is logarithmically divergent for $n=2$, and is worse for $n>2$.

Let us next try to estimate the degree of divergence of a graph. A graph which is connected [redacted] satisfies:

[redacted]

$$\begin{Bmatrix}\text{no. of}\\ \text{vertices}\end{Bmatrix} - \begin{Bmatrix}\text{no. of}\\ \text{edges}\end{Bmatrix} = 1 - \begin{Bmatrix}\text{no. of}\\ \text{loops}\end{Bmatrix}$$

$$v - e = 1 - l$$

One has [redacted] a momentum variable for each edge, but one has momentum conservation at each vertex. What is the total number of independent momenta? A choice of momenta is a 1-chain which is in fact a cycle. So we want the dimension of the space of cycles which is l.

Each edge contributes a quadratic factor to the denominator. So a rough estimate would be

$$\int \frac{r^{nl-1}\,dr}{r^{2e}} = \int r^{nl-2e}\,\frac{dr}{r}$$

$n =$ dim of space

which will converge for

$$nl - 2e < 0.$$

For example if $n=2$ we have $2(l-e) = 2(1-v)$ which is <0 as soon as [redacted] $v \geqslant 2$. Notice that this is independent of the type of interaction, which may be related to the fact we can do $P(\varphi)_2$ for arbitrary polys. P.

Let's now consider a simple quartic interaction. In this case if we have 2 external edges, then we have

$$2e+2 = 4v \implies e = 2v-1$$

and hence $\quad l = e - v + 1 = 2v-1-v+1 = v.$

So if $n=3$ one has

$$\underbrace{3l - 2e}_{3v-2(2v-1) \,=\, 2-v} < 0$$

and so we see all diagrams with $v \geq 3$ should be convergent.

Finally if $n=4$ one has for a simple quartic interaction

$$4l - 2e = 4v - 2(2v-1) = 2$$

for all diagrams contributing to Γ_2.

Supposedly the ϕ^4-theory in 4 dims. is renormalizable. If so, then a ϕ^6-theory in 3-dims should be renormalizable:

$$2e+2 \leq 6v \implies e \leq 3v-1$$

$$\implies 3l-2e = 3(e-v+1)-2e = e-3v+3$$

$$\leq 3v-1-3v+3 = 2.$$

The next project will be to carry out the renormalization. In the case of dimension 2 the only infinite diagram is the first one  and I think one should be able to remove this one by mass renormalization. This means that we put in a cutoff into the momentum integrals, then remove the cutoff and vary m_0 at the same time to achieve a finite limit.

I first should understand dimension 1 where there are no divergences. The field theory with action

$$S(\phi) = \int \left[\phi(-\Delta + m^2)\phi + \frac{\lambda}{4!}\phi^4\right] dx$$

where ϕ is a real-valued function on R should be the imaginary-time version of the anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2} + \frac{m^2}{2}q^2 + \frac{\lambda}{4!}q^4$$

I want to understand this much better. In particular, it would be a good idea to work out the 2-point function and its Fourier transform $G(k)$; here k is frequency.