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May 10, 1983

Let me begin with the general idea that there ought to exist for a complex manifold a good way to describe the holomorphic bundle K-theory along Kasparov lines using operators in some way. Two reasons: 1) Connes defines the K-theory for foliated manifolds along Kasparov lines, and foliated manifolds are very similar to complex manifolds. 2) If I want an analytical GRR theorem, I will need analytical gadgets for which I can take $f_!$.

Question: Is there a quantum field theory that we can attach to a complex manifold, say a Riemann surface? If so, then maybe the quantum theory will suggest the type of K-theory of holomorphic bundles along Kasparov lines that I am after.

In physics we have the fields ψ, A, g where ψ describes the matter, A the gauge field, g the metric on space-time. Geometrically we have a vector bundle E over M and ψ is a section of E, A is a connection on E, and g is a metric on M. (I work with imaginary time, so that g is > 0, not Lorentzian.) Although the fields interact, the first case to study is the behavior of matter with A, g fixed, and then the behavior of ψ, A with g fixed. This will be the order I propose to follow.

To be specific start with 1-dimensional space which is the circle $\mathbb{R}/\mathbb{Z}\cdot L$ and imaginary time also the circle $\mathbb{R}/\mathbb{Z}\beta$, so that M is a 2-torus. So we are considering particles on the circle at the inverse temperature β. Let's now go over the physics, with a view toward a generalization to an arbitrary Riemann surface.

First we describe the Hilbert space on which we have the field operators. We start with the single particle Hilbert space and Hamiltonian, denote them $\mathcal{H}$ and H. In our case

$$H = \frac{1}{i}\partial_x \quad \text{on} \quad \mathcal{H} = L^2(S^1).$$

Then the field theory will describe a fermion gas of these particles. The Hilbert space will be the fermion Fock space of $\mathcal{H}$:

$$\mathcal{F} = \Lambda \mathcal{H} \qquad (\text{suitably interpreted})$$

with Hamiltonian $\hat{H}$ = derivation of $\Lambda\mathcal{H}$ extending H. (I am going to write things as if $\mathcal{H}$ were finite dimensional.)

Let $|k\rangle$ be an orthonormal basis for $\mathcal{H}$ consisting of eigenfunctions for H:

$$H|k\rangle = \varepsilon_k |k\rangle,$$

and let a_k, a_k^* be the corresponding destruction and creation operators on $\mathcal{F}$.

Let's begin again with the physics. I wish to consider the fermion gas made of non-interacting particles described by $\mathcal{H} = L^2(S^1)$ and the Hamiltonian $H = \frac{1}{i}\partial_x$. The gas is then described by the Fock space $\mathcal{F} = \Lambda\mathcal{H}$ (suitably interpreted) and the Hamiltonian $\hat{H}$ = the derivation of $\mathcal{F}$ extending H.

Part of the physics of the gas are the operators $\psi(x)$ = interior multiplication by $\langle x|$, and its adjoint $\psi^*(x)$. We need these to describe operators like the density at x operator

$$\rho(x) = \psi^*(x)\psi(x).$$

In general an operator A on $\mathcal{H}$ extends to a derivation, or one-particle operator, on $\mathcal{F}$ given by

$$\hat{A} = \int dx\,dy\, \psi^*(x)\langle x|A|y\rangle\psi(y)$$

Now we form the cylinder which is the product of S^1 and the (imaginary) time axis $\mathbb{R}$. In fact let us restrict to $0 \leq t \leq \beta$. Recall that sometime in the fall I discovered that if I had a holom. v.b. over a Riemann surface with two boundary components, then there is a canonical way to map half-spaces at one component to half spaces at the other, by looking at boundary values of holomorphic sections. In the case of the cylinder over $0 \leq t \leq \beta$ this map from half spaces in H at $t=0$ to half-spaces at $t=\beta$ corresponds to the operator

$$e^{-\beta \hat{H}} : \mathcal{F} \longrightarrow \mathcal{F}.$$

May 11, 1983

The problem is to find a generalization of the Fredholm operator picture of topological K-theory which applies to holomorphic bundles. Maybe one can find this by looking at Connes theory for foliated manifolds. Let us consider the Kronecker foliation and list the objects which we can play with.

The smooth algebra belonging to the foliation is the crossed product of the alg. $C^\infty(\mathbb{T}^2)$ with $\mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is a convolution algebra, and the Kronecker foliation is thought of as an $\mathbb{R}$-action on $\mathbb{T}^2$. This has a completion which is a C^*-algebra. The infinite dimensional spaces to be considered are Hilbert C^*-modules over this C^*-algebra. One should think of being given a family of Hilbert spaces for each point of $\mathbb{T}^2$ and an action of $\mathbb{R}$ of some sort on this field.

I know I can replace $C^\infty(\mathbb{T}^2) \tilde{\otimes} \mathcal{S}(\mathbb{R})$ with $C^\infty(\mathbb{T}) \tilde{\otimes} \mathcal{S}(\mathbb{Z})$ up to Morita equivalence. This may not work for a complex structure on the torus.

Here is a possibility: Let's go over the known projective modules over the smooth algebra.

A module over $C^\infty(\mathbb{T}^2) \tilde{\otimes} \mathcal{S}(\mathbb{R})$ will be a module over $C^\infty(\mathbb{T}^2)$, for example, the space of sections of a vector bundle over $\mathbb{T}^2$. It will also be a module over $\mathcal{S}(\mathbb{R})$ which roughly means that there is an $\mathbb{R}$-action. This $\mathbb{R}$-action has to be "unitary" in order to extend to $\mathcal{S}(\mathbb{R})$. The action in very good cases will be given by the action of the Lie algebra. Hence it would seem that the modules we are looking at can be vieived as a module M over $C^\infty(\mathbb{T}^2)$ together with an operator $\partial_t : M \longrightarrow M$ consistent with the ∂_t operator on $C^\infty(\mathbb{T}^2)$ given by the Kronecker flow. Examples are given by the space of smooth sections of a vector bundle over $\mathbb{T}^2$ with inner product and with unitary connection along the leaves of the foliation.

Other examples should come from transversals to the foliation. Now one of the perpetually-perplexing problems is how to deal with these rings without unit. Let's look at this problem. Two examples come to mind:

(i) $\varinjlim_n M_n(A)$ $\qquad M_n = n \times n$ matrices

More generally one can consider all compact operators in a Hilbert space

(ii) Continuous functions in a locally compact space vanishing at ∞. Denote this $C(X \cup \infty, \infty)$.

I need a way to think of the K-theory attached to these rings without unit.

Consider $M_\infty(A)$ where A has a unit. Now we know the K-theory of $M_n(A)$ is the same as that of A. This is because the categories of projective modules are equivalent. In general if P is an object in an additive category $\mathcal{A}$, then we have a functor

$$\begin{array}{ccc} \mathcal{P}_R & \longrightarrow & \mathcal{P} \\ N & \longmapsto & P \otimes_R N \end{array} \qquad R = \operatorname{End}_{\mathcal{P}}(P)^{op}$$

which is an equivalence when every object of $\mathcal{P}$ is a direct summand of $P^{\oplus n}$ for some n. This leads to Morita equivalence between

$$\mathcal{P}_R \qquad \mathcal{P}_S$$

provided we are given ${}_S P_R$ in $\mathcal{P}_S$ and ${}_R Q_S$ in $\mathcal{P}_R$ such that

$${}_R Q_S \underset{S}{\otimes} {}_S P_R = R \quad , \quad {}_S P_R \otimes_R {}_R Q_S = S$$

So we have an equivalence

$$\mathcal{P}_A \xrightarrow{\sim} \mathcal{P}_{M_n(A)}$$

defined by $P = A^n$ which is a left A and right $M_n(A)$-module. (Think of A^n as row vectors). It is via this equivalence that one identifies

$$K_0(A) \quad \text{and} \quad K_0(M_n(A)).$$

(the other equiv. is given by $A^n \otimes_A$? (where $Q = A^n$ is the column vectors))

Question: Where does the ring $M_\infty(A)$ come in? This is the limit of the embeddings

$$< M_n(A) < M_{n+1}(A) < \cdots$$

which do not <u>preserve the identity</u>.

It seems that the K-theory for a non-unital C^*-algebra A is defined by

$$K_0(A) = \operatorname{Ker} \{ K_0(\tilde{A}) \twoheadrightarrow K_0(\mathbb{C}) \}$$

where $\tilde{A} = A \oplus \mathbb{C}1$ is obtained by adjoining a unit.

This certainly gives the right thing for the algebra $C_0(X) = C(X \cup \infty, \infty)$ of continuous functions on a locally compact X vanishing at ∞. In effect

$$C_0(X)^{\sim} = C(X \cup \infty)$$

so

$$K_0(C_0(X)) = \mathrm{Ker}\{K(X \cup \infty) \longrightarrow K(\infty)\} = K_c(X).$$

So the next thing to do is to see if it works for $M_n(A)$. But first one should observe that if A has already an identity element, call it e, then

$$\tilde{A} \simeq A \times \mathbb{C}$$

$$a + c1 \longmapsto (a + ce, c)$$

is a ring isomorphism. ($\tilde{A}$ is universal for maps of A into a ring with identity.) So

$$K_0(\tilde{A}) = K_0(A) \times K_0(\mathbb{C})$$

and so we do obtain an extension of K_0 to non-unital rings.

But now we should check that for the (non-unital) homomorphism $M_m(A) \longrightarrow M_n(A)$ the induced map on K_0 is an isomorphism. Take $m = 1$. More generally suppose we have an idempotent e in a ring B and that $A = eBe$, and suppose B has an identity e'. Let's compute $K_0(A) \to K_0(B)$. This will be induced by the ring homomorphism $\tilde{A} \longrightarrow \tilde{B}$. Since A has the identity e, we have $\tilde{A} = A \times k$ hence

$$P_{\tilde{A}} = P_A \times P_k. \qquad\qquad k = \mathbb{C}.$$

Let $M \in P_A$. It is regarded as an $\tilde{A}$-module with $1, e$ acting as the identity, hence $(1-e)M = 0$. Then the induced $\tilde{B}$ module is

$$\tilde{B} \otimes_{\tilde{A}} M = B \otimes_{\tilde{A}} M \oplus k \otimes_{\tilde{A}} M$$

The element e in $\tilde{A}$ goes to zero in k and acts as 1 on M, hence the second term is 0. We have

$$B \otimes_{\tilde{A}} M = Be \otimes_{\tilde{A}} M \oplus B(1-e) \otimes_{\tilde{A}} M$$

and the second term is 0 for the same reason. Hence it is clear we have

$$(*) \qquad \tilde{B} \otimes_{\tilde{A}} M = Be \otimes_{(eBe)=A} M$$

with the obvious $\tilde{B}$-action.

Summary: Suppose $A = eBe$ where e is an idempotent in the ring B with unit e'. Then I am computing the following map

$$K_0(A) \longrightarrow K_0(\tilde{A}) \longrightarrow K_0(\tilde{B}) \longrightarrow K_0(B).$$

Better is to draw the diagram

$$\begin{tikzcd}
K_0(A) \arrow[r, dashed] \arrow[d] & K_0(B) \arrow[d] \\
K_0(\tilde{A}) \arrow[r, "\tilde{B} \otimes_{\tilde{A}}"] \arrow[d] & K_0(\tilde{B}) \arrow[d] \\
K_0(k) \arrow[r, equal] & K_0(k)
\end{tikzcd}$$

and say I am computing the top arrow. The upper vertical arrows come from regarding an A module as an $\tilde{A}$ module with e acting as 1 (resp. for B), i.e. the Gysin map for $\tilde{A} \longrightarrow A$. The formula $(*)$ shows that we do get a $\tilde{B}$-module with e' acting as 1.

Finally look at the case where $B = M_n(A)$ and $e = e_{11}$. Then $Be = A^n$ considered as column vectors. Thus it really does work.

Maybe the point of all this is that there is this extra kind of homomorphism around for rings where there are idempotents. Even if you restrict to unital rings there are these extra non-unital homomorphisms.

Connes paper: On $K_j(A) = K_{j+1}(A \rtimes_\alpha \mathbb{R})$ in the Advances in Math 1981. The following I find very interesting, namely, the Thom map

$$K_0(A) \longrightarrow K_1(A \rtimes_\alpha \mathbb{R})$$

One adds an identity to A so that an element of $K_0(A)$ is represented by a f.t. proj. A-module $\mathcal{E}$. The alg $A \rtimes_\alpha \mathbb{R}$ is essentially $A \otimes_\alpha \mathcal{S}(\mathbb{R})$, so it doesn't have a unit, however, one uses the exact sequence

$$0 \longrightarrow A \otimes_\alpha \mathcal{S}(\mathbb{R}) \longrightarrow A \otimes_\alpha \widetilde{\mathcal{S}(\mathbb{R})} \longrightarrow A \longrightarrow 0$$

and the K_1 is described as a relative K_1. Next take the $\mathcal{E}$ and choose a connection ∇ on it so that $\mathcal{E} \otimes \widetilde{\mathcal{S}(\mathbb{R})}$ becomes a module over $A \otimes_\alpha \widetilde{\mathcal{S}(\mathbb{R})}$. Finally one needs an element of $K_1(\mathcal{S}(\mathbb{R}))$, so one takes an $f \in \mathcal{S}(\mathbb{R})$ such that $b = 1 + \hat{f}$ is invertible with winding number 1. Then $b(i\nabla)$ is an automorphism of $\mathcal{E} \otimes \widetilde{\mathcal{S}(\mathbb{R})}$ which is the gadget one wants.

May 12, 1983

If A is not necessarily unital and $\tilde{A} = A + k1$, what are the idempotents in $\tilde{A}$? Since we have a homomorphism $\tilde{A} \longrightarrow k$ an idempotent goes to either 0 or 1 in k. If the latter, say $e = 1 - a$, with $a \in A$, then

$$e^2 = (1-a)^2 = 1 - 2a + a^2 = 1 - a \quad \Longrightarrow \quad a^2 = a.$$

Hence the idempotents in $\tilde{A}$ are of the form e or $1-e$ with e an idempotent in A.

My goal is to calculate $K_0(A)$ which is defined as the kernel of $K_0(\tilde{A}) \longrightarrow K_0(k)$. [illegible] I would like to see $K_0(A)$ described in terms of idempotent matrices over A. However, this is impossible for the following reason. Take $A = C_0(X) =$ cont. fns. vanishing at ∞ on the locally compact space X. An idempotent matrix $e \in M_n(A)$ is an idempotent matrix over $C(X \cup \infty)$ and so it will have constant rank over X, assuming $X \cup \infty$ is connected (i.e. X has no compact component). Since e is zero at ∞, it follows that e must be zero.

This tells me that if I can realize elements of $K_0(A)$ by idempotent matrices over A, then something special is going on. For example, although I know that

$$K_0(A) = K_0(M_\infty(A)),$$

I can't conclude that elements of $K_0(A)$ are all described by idempotents in $M_\infty(A)$. Suppose that A has an identity, so that [illegible] generators for $K_0(A)$ are obtained from idempotent matrices over A. An idempotent matrix $e \in M_n(A)$ determines an element of both $K_0(A)$ and $K_0(M_n(A))$, and one can ask whether these elements correspond under the canonical isomorphism

$$K_0(A) = K_0(M_n(A)).$$

Review yesterday: We have [illegible] $A = eBe$ where e is an idempotent in B. We have

$$\begin{tikzcd} K_0(A) \arrow[r, dashed] \arrow[d, hook] & K_0(B) \arrow[d] \\ K_0(\tilde{A}) \arrow[r] & K_0(\tilde{B}) \end{tikzcd} \quad (*)$$

where the dotted arrow is induced by taking $M \in P_A$ (notice A has the unit e) into the $\tilde{B}$-module

$$\tilde{B} \otimes_{\tilde{A}} M = Be \otimes_A M$$

which is killed under the augmentation $\tilde{B} \to k$ since

$$k \otimes_{\tilde{B}} (\tilde{B} \otimes_{\tilde{A}} M) = k \otimes_{\tilde{A}} M = 0 \qquad \begin{cases} e \mapsto 0 & \text{in } k \\ e = 1 & \text{on } M. \end{cases}$$

~~Now take an idempotent~~

We learned yesterday that the K-theory on unital rings is functorial for non-unital maps. Such a map $A \to B$ sends 1_A to e in B and so factors

$$A \longrightarrow eBe \subset B.$$

A natural question is whether we can interpret the map $K_0(A) \to K_0(B)$ as coming from a functor $P_A \to P_B$. Such a functor could be given by

$$M \longmapsto {}_BP_A \otimes_A M$$

where ${}_BP_A \in P(B)$ as a B-module.

Maybe there are two stages: First [illegible] treat unital rings, then extend to non-unital ones. Possibly the extension can be generalized. [illegible]

Repeat this for emphasis. First discuss the induced maps on K_0 [illegible] for unital rings. This means I should explain the maps in $(*)$ e.g.

$$K_0(A) \longrightarrow K_0(\tilde{A})$$

where $\tilde{A} = A \oplus k1$. Use e for the unit of A. This just restriction of scalars for the homomorphism $\tilde{A} \to A$

sending a to a and 1 to e. Thus

$$ {}_{\tilde{A}}P_A = A $$

Next the homomorphism $K_0(\tilde{A}) \longrightarrow K_0(\tilde{B})$ is base change relative to $\tilde{A} \longrightarrow \tilde{B}$, and is explained by

$$ {}_{\tilde{B}}P_{\tilde{A}} = \tilde{B}. $$

So the composition $K_0 A \to K_0 \tilde{A} \longrightarrow K_0 \tilde{B}$ is explained by, or better induced by the functor

$$ M \longmapsto \tilde{B} \otimes_{\tilde{A}} A \otimes_A M = \tilde{B} \otimes_{\tilde{A}} M = Be \otimes_A M. $$

Thus

$$ {}_{\tilde{B}}P_A = Be $$

and finally

$$ {}_B P_A = Be. $$

So we conclude that given a non-unital homomorphism $A \longrightarrow B$ with $1_A \longmapsto e$, then the induced [redacted] map on K_0 is given by the functor $M \longmapsto Be \otimes_A M$.

Now we can answer the following

<u>Question</u>: Given $\varepsilon \in M_n(A)$ idempotent, it defines elements of $K_0(A)$ and $K_0(M_n(A))$. Do these coincide under the canonical isomorphism?

Recall that the isomorphism is given by an equivalence

$$ \begin{array}{ccc} P_A & & P_{M_nA} \\ M & \longmapsto & (A^n)_{col} \otimes_A M \\ (A^n)_{row} \otimes_{M_nA} N & \longleftarrow\!\shortmid & N \end{array} $$

Now ~~to~~ ε we associate the A-module [redacted] endomorphism of $(A^n)_{row}$ of right mult. by ε. Thus we get in P_A the module $(A^n)_{row}\varepsilon$ and in P_{M_nA} the module $M_n(A)\varepsilon$. Clearly these coincide under the equivalence.

<u>Summarize:</u> One starts with $K_0(A)$ defined on unital rings via P_A. Then it has the functorality given by Morita maps. (These are the additive functors $P_A \longrightarrow P_B$ and are given by ${}_BP_A$ in P_B. Ultimately I want to replace these by some kind of "correspondences".)

Included among the Morita maps are non-unital ring homomorphisms. This is connected with the fact that K_0 can be extended to a functor on non-unital rings.

One can check that the Morita equivalence

$$K_0A \cong K_0(M_nA)$$

corresponds to the non-unital map $A \longrightarrow M_nA$ given by the first entry. Hence

$$K_0A \cong K_0(M_\infty A)$$

For a general non-unital ring the K_0 is not given by idempotent matrices. However idempotent matrices do define K-elements, and it appears that using these matrices is consistent with the Morita equivalence.

May 18, 1983

Goal: Prepare for the K-theory conference in Luminy by working out all of the details concerned with Connes's cyclic homology. Especially the exact sequence and the relation to Hodge + Deligne cohomology.

The first thing one can do is to make sense of the long exact sequence relating Hochschild and cyclic homology. Let $t = t_p$ denote the cyclic permutation, or better, just a generator of the cyclic group of order p. If M is a module over the cyclic group, then one has the complex (standard)

$$\longrightarrow M \xrightarrow{1-t} M \xrightarrow{1+t+\cdots+t^{p-1}} M \xrightarrow{1-t} M \longrightarrow 0$$

for computing $H_*(\mathbb{Z}/p, M)$. We will be working where p is invertible, so that the cohomology is zero in positive degrees. The best way to put this is perhaps that one has idempotents in the group ring $k[\mathbb{Z}/p] = k[t]/(t^p-1)$, namely

$$e = \frac{1}{p}(1+t+\cdots+t^{p-1})$$

which projects onto $\operatorname{Ker}(t-1)$.

$$1-e = \frac{1}{p}\left((1-t)+(1-t^2)+\cdots+(1-t^{p-1})\right) \in (t-1)k[\mathbb{Z}/p].$$

Now one takes $M = A^{\otimes p}$ but fits these together for different p using the Hochschild boundary, and there are two kinds of Hochschild boundary depending on whether one uses all the faces. Let's go over the proof that one can divide by the cyclic perms. to get a quotient complex.

$$\begin{tikzcd}
A^{\otimes 3} \arrow[r] \arrow[d] & A^{\otimes 2} \arrow[r] \arrow[d] & A \arrow[d] \\
A^{\otimes 3}/\sim & A^{\otimes 2}/\sim & A
\end{tikzcd}$$

How to penetrate this mess? Certainly you can't expect the formulas to explain what is going on. Thus if you expect to understand the exact sequence you have to find a different and more intelligent viewpoint.

One of the mysteries: There seem to be two ways to derive the exact sequence i) via cyclic combinatorics - somehow using the periodicity of the cyclic group. ii) via Connes double complex which in addition to the Hochschild boundary has the operator raising degree which resembles exterior differentiation.

Let us go over the analogy between circle actions and the double complex. Consider a circle S acting on M. Then the equivariant cohomology $H_S(M)$ fits into an exact sequence

$$\cdots \longrightarrow H^p_S(M) \xrightarrow{ue} H^{p+2}_S(M) \longrightarrow H^{p+2}(M) \longrightarrow H^{p+1}_S(M) \longrightarrow \cdots$$

Also there is the exact sequence of the fibration $PS\times^S M \longrightarrow BS$. with the E_2 term $\bigoplus_{p\geq 0} u^p H^*(M)$, $\deg u = 2$.

Suppose one calculates $H^*_S(M)$ using differential forms. This was described by Atiyah + Bott. One starts with the forms $\Omega^\cdot(M)$ on M and tensors with the Weil algebra of S which is $k[u]\otimes\Lambda[\theta]$ with $d\theta = \pm u$ and $i(X)\theta = 1$. Then one must take basic elements in

$$k[u]\otimes\Lambda[\theta]\otimes\Omega^\cdot(M).$$

Take the horizontal elements (these are killed by $i(X)$) we get something isom. to

$$k[u]\otimes\Omega^\cdot(M)$$

where now a form in $\Omega^\cdot(M)$, say α, is to be identified with the horizontal form

$$\alpha - \theta\cdot i(X)\alpha.$$

Next take invariants, which since θ, X are invariants, means

α is invariant. Finally

$$d(\alpha - \theta\, i(X)\alpha) = d\alpha - u\, i(X)\alpha + \theta\, \underbrace{d i(X)}_{\mathcal{L}(X) - i(X)d}\alpha$$

$$= [1 - \theta\, i(X)]\{d\alpha - u\, i(X)\alpha\}$$

Thus the complex of basic forms in $k[u]\otimes\Lambda[\theta]\otimes\Omega^{\cdot}(M)$ can be identified with $k[u]\otimes\Omega^{\cdot}(M)^S$ with the differential

$$d - u\, i(X). \quad \text{and} \quad du = 0$$

Note also that $\Omega^{\cdot}(M)^S$ comes with d and $\delta = i(X)$ satisfying $d\delta + \delta d = \mathcal{L}(X) = 0$.

The conclusion of this calculation is that if we are given a graded vector space $K^{\cdot}$ like $\Omega^{\cdot}(M)^S$ with d raising degree by 1, δ lowering degree by 1, satisfying $d^2 = 0$, $\delta^2 = 0$, $d\delta + \delta d = 0$, then we can form $\bigoplus_{p\geq 0} u^p K^{\cdot}$ with $\tilde{d} = d - u\delta$ and get all the structure of $H^*_S(M)$ and its exact + spectral sequences.

Analogy: Take $M = \Omega_f(N)$ where N is a manifold. Then one wants to think of the exact sequence for the circle acting on M:

$$\longrightarrow H^*_S(\Omega N) \xrightarrow{u} H^*_S(\Omega N) \longrightarrow H^*(\Omega N) \longrightarrow$$

as analogous to the cyclic cohomology sequence for $A = C^\infty(N)$

$$\longrightarrow HC(A) \longrightarrow HC(A) \longrightarrow Hoch(A) \longrightarrow$$

The limit cohomology (localizing wrt u) in the first case is roughly (very approximately)

$$H^*(\text{fixpt}) = H^*(N) \qquad * = \text{odd or even}$$

whereas in the second case it is $H^*_{DR}(N)$ $\quad * =$ odd or even

Any way of making this analogy precise should be good mathematics for the following reasons: 1) since M is infinite-dimensional, the localization thm. doesn't apply. 2) the degrees go wrong: In the second sequence $HC(A) \to HC(A)$ lowers degree by 2 whereas u raises degree by 2.

The analogy between [redacted] cyclic cohomology and the loop space I learned from Bott. It is related to Witten's ideas on ΩN and the index thm. in some way.

Here's how the Russians Feigin and Tsygan derive the exact sequence. One has on the graded vector space $\{A^{\otimes p}, p \geq 0\}$ [redacted] the Hochschild boundary d which gives the Hochschild homology $\mathrm{Tor}_*^{A\otimes A}(A,A)$, and the Hochschild boundary δ without the [redacted] last face which gives an acyclic complex. Then one has the exact sequence for the cyclic action:

$$\xrightarrow{N} A^{\otimes p} \xrightarrow{1-t} A^{\otimes p} \xrightarrow{N} A^{\otimes p} \xrightarrow{1-t}$$

One calculates that this gives an exact of complexes

[redacted]

$$\longrightarrow A^*_\delta \xrightarrow{1-t} A^*_d \xrightarrow{N} A^*_\delta \xrightarrow{1-t}$$

i.e. one proves by computation that

$$d(1-t) = (1-t)\delta$$
$$\delta N = N d$$

We are interested in the complex $C^\lambda(A) = \mathrm{Coker}(1-t) = \mathrm{Im}\, N$ [redacted] $=$ [redacted] $\mathrm{Ker}\{(1-t) \text{ on } A^*_\delta\}$. Then from the exact sequence and acyclicity of δ

$$0 \longrightarrow C_\lambda(A) \longrightarrow A^*_\delta \longrightarrow \mathrm{Ker}(1-t) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ker}(1-t) \longrightarrow A^*_d \longrightarrow C_\lambda(A) \longrightarrow 0$$

the long exact sequence results easily.

Hyperfunctions: First case to understand is the case of the circle S^1 which we can embed analytically in $\mathbb{C}$ as $|z|=1$. Then we take the "complex" neighborhood given by an annulus if we need to.

Because S^1 is compact we don't have to worry about supports. Hyperfunctions are linear functionals on the space of analytic functions. In the case of the circle analytic functions are described by Fourier series

$$f(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$$

where a_n decreases exponentially: $|a_n| \leqslant C r^{|n|}$ for some $0<r<1$. Hence hyperfunctions should be such series where $a_n r^{+|n|}$ is bounded for any such r.

Here are can see that any analytic function in the disk $|z|<1$ has a hyperfunction for boundary values. In effect

$$f(z) = \sum a_n z^n$$

will be analytic in the disk if $|a_n| r^n$ is bdd for any $0<r<1$.

The hard thing to see about hyperfunctions is that they can be localized, i.e. restricted to open sets. Perhaps that is why one uses a description as sheaf cohomology.

May 19, 1983

Let's suppose we have a holom. family of holom. curves $f: X \longrightarrow Y$ and a holom. v.b. E over X. Then we have over Y the determinant line bundle

$$\lambda\left(Rf_*(E)\right)$$

which is a holomorphic line bundle over Y. Next put a metric on E and on the fibres of f. Then analytic torsion gives us a metric in the determinant line bundle, and hence a connection and curvature. Suppose we now try to prove our conjecture that this curvature is given by the degree 2 component of

$$f_*\left(ch(E)\, Todd(X/Y)\right)$$

where these characteristic classes are interpreted as diffl forms using the canonical connections associated to the metrics + holom. structure.

It should be enough to suppose $\dim Y = 1$ and to suppose $Y =$ unit disk. The first problem will be how to handle the determinant line bundle.

I have concluded already that one ought to trivialize the family + bundle from the C^∞-viewpoint. In fact I have only to calculate the value of a 2-form at the origin in Y, so I should be able to use deformation theory on a fixed surface.

But let us not make such choices until forced to. It should be possible to discuss the det. line. bdl first without such choices.

The idea is that we have a family of elliptic operators on the fibres. (This forgets the holom. structure "over Y".) Specifically over Y we have an infinite dimensional bundle, whose fibre at y is the space of smooth (or H_s = Sobolev) sections of E over $f^{-1}(y)$. Similarly we get an infinite-diml bundle from $E \otimes T^{0,1}_{X/Y}$ and an operator between these bundles over Y given by the $\bar\partial$-operator:

$$E \xrightarrow{D} E \otimes T^{0,1}_{X/Y}$$

The problem is now to describe $\lambda(Rf_*(E))$ as a holomorphic line bundle over Y. [redacted]

There seems to be great difficulty in using the above $\bar\partial$ operator if one wants to keep track of the holom. structure on X, E. Perhaps the difficulty is related to the dictum that one can speak of smooth families of analytic gadgets, but not analytic families of smooth gadgets?

Nevertheless the perfect complex concept should do something. Let's see what it says. Let $\mathcal{E}$ = sheaf of holomorphic sections of E over X. Then

should be full Dolbeault cx.

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(E \otimes T^{0,1}_{X\text{[redacted]}}) \longrightarrow \text{[redacted]} \cdots$$

is [redacted] a resolution of $\mathcal{E}$ by acyclic sheaves for f_*. Then $Rf_*(\mathcal{E})$ is isomorphic in the derived category to the complex of sheaves on Y

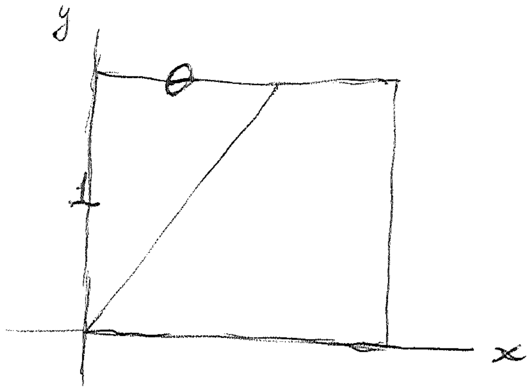
ditto

$$f_*[\mathcal{C}^\infty(E)] \longrightarrow f_*^{\text{[redacted]}}[\mathcal{C}^\infty(E \otimes T^{0,1}_{X\text{[redacted]}})] \longrightarrow \cdots$$

This is a perfect complex of $\mathcal{O}_Y$-modules. What does this mean? Locally quasi-isomorphic to a complex of loc. free f.t. $\mathcal{O}_Y$-modules. For example if D_y is onto for each y, then [redacted] the above complex is quis $f_*[\mathcal{E}]$ which is locally-free over $\mathcal{O}_Y$.

[redacted]

Let me review Connes algebra for the Kronecker foliation, and try to keep things as smooth as possible. [redacted] Start with $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and let the leaves be given by



$dx = \theta\, dy$

The algebras of the foliation of different types (C^∞, C^*, vN) is generated by functions on the torus and by $f(t)$ $C_0^\infty(\mathbb{R})$, where $\mathbb{R}$ acts on the torus via the vector field $\theta\partial_x + \partial_y$. Now the functions on the torus are generated by $e^{2\pi i x}$, $e^{2\pi i y}$. So in some sense the generators are $e^{2\pi i x}, e^{2\pi i y}, \theta\partial_x + \partial_y$.

Let us begin with something slightly simpler. Let us consider the $\mathbb{R}$-action on $\mathbb{R}/\mathbb{Z} = \mathbb{T}$ and form the algebra generated by the functions on $\mathbb{T}$ under multiplication and the convolution algebra of $\mathbb{R}$. This is

$$\mathcal{S}(\mathbb{T}) \,\tilde{\otimes}\, \mathcal{S}(\mathbb{R})$$

roughly. It should be Morita equivalent to the convolution alg $\mathcal{S}(\mathbb{Z})$.

What does one need to write down a projector? Actually we can ask this already for $\mathcal{S}(\mathbb{T}) \tilde{\otimes}$ convolution alg. of the circle. Supposedly the algebra generated by multiplication and translation operator contains all operators on $L^2(S^1)$, when generated is suitably interpreted. Hence one can try to write out a projector of rank 1. This is easy because the convolution algebra contains projectors, namely

$$(1 * f)(x) = \int 1(x-y)f(y)\,dy = \int f(y)\,dy$$

projects onto the constant functions. In fact the characters of $\mathbb{T}$ also give projectors. One can see this using the Fourier transform which converts the convolution algebra into fns. on $\hat{T} = \mathbb{Z}$ under multiplication which decay at ∞. So the minimal idempotents correspond to the points of $\hat{\pi}$.

Descent: Given a Galois covering $X \xrightarrow{G} Y$ belonging to rings $A \longrightarrow B$, then descent says that one has an equivalence between A-modules and equivariant B-modules. So this gives a Morita equivalence between A and $B \tilde{\otimes} k[G]$.

Is there some version of this that might work for $\mathbb{R} \longrightarrow S^1$?

But first how is the Morita equivalence between A and $B \tilde{\otimes} k[G]$ given? Corresponding to A as an A-module is the left $B\tilde{\otimes}k[G]$, right A-module B. What makes B projective. Since $1 \in B$ is a generator this means that $R = B \tilde{\otimes} k[G]$ contains an idempotent e such that $Re \cong B$.

Incidently if $A = k$, $B = k^G$, then $R = k^G \tilde{\otimes} k[G]$ acts on $B = k^G$ and is the full endo. ring over k. Here G is finite.

Next let's try to get something analogous for the infinite covering $\mathbb{R} \longrightarrow S^1$, where A is now functions (smooth or analytic) on the circle. B should perhaps be something like functions on $\mathbb{R}$.

In the situation $A = k$, $B = k^{\mathbb{Z}}$, then the Russians Feigin + Tsygan look at $R = k^{\mathbb{Z}} \tilde{\otimes} k[\mathbb{Z}]$ = ring of "finite band around diagonal" matrices, and shows its Lie K-theory is periodic with single generators in even degrees. Thus R is not Morita equivalent to k.

We can make $\mathbb{Z}\times\mathbb{Z}$ act on $\mathcal{S}(\mathbb{R})$ by letting one generator translate by 1 and the other multiply by $e^{2\pi i x}$. In this way the "smooth" convolution alg. of $\mathbb{Z}\times\mathbb{Z}$, which is isomorphic to the smooth fns. on $\mathbb{T}^2$, acts on $\mathcal{S}(\mathbb{R})$. In fact it is a projective module of rank 1 over $\mathcal{S}(\mathbb{T}^2)$ as can be seen as follows.

Given $f \in \mathcal{S}(\mathbb{R})$ let

$$g(x,y) = \sum_{n\in\mathbb{Z}}' f(x+n)e^{2\pi i n y}$$

Then g is smooth in x, y, is periodic in y and

$$g(x+1,y) = e^{-2\pi i y}\, g(x,y).$$

Hence g is a section of the line bundle over the torus obtained from the cylinder $\{0\le x\le 1\}\times S^1$ by using the clutching fn. $e^{-2\pi i y}$ at $x=1$. We know the sections of this line bundle form a rank 1 proj. $\mathcal{S}(\mathbb{T}^2)$-module. But given $g(x,y)$ with these periodicity conditions, put

$$f(x,n) = \int_0^1 g(x,y)\,e^{-2\pi i n y}\,dy$$

Then $f(x+1,n) = f(x,n+1)$, so that $f(x,n) = f(x+n)$ can be written. In fact f is given by

$$f(x) = \int_0^1 g(x,y)\,dy .$$

Finally from the first formula because g is smooth and periodic in y one has that $f(x,n)\to 0$ as $|n|\to\infty$ uniformly in x, etc. etc., so $f\in\mathcal{S}(\mathbb{R})$.

Thus $\mathcal{S}(\mathbb{R})$ becomes isomorphic to the sections of the line bundle over the torus.

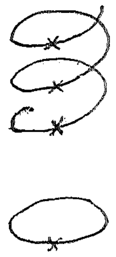
May 20, 1983

The problem is to see if $\mathcal{S}(\mathbb{R})\rtimes\mathbb{Z}$ is Morita equivalent to $\mathcal{S}(S^1)$ because of the covering $\mathbb{R}\longrightarrow S^1$.

Consider $\mathcal{S}(\mathbb{R})\rtimes\mathbb{Z}$ as operators à la Connes. To each point $x\in\mathbb{R}$ we assign the Hilbert space $\ell^2(x+\mathbb{Z})$ with $\mathcal{S}(\mathbb{R})$ acting by restriction $\mathcal{S}(\mathbb{R})\to\mathcal{S}(x+\mathbb{Z})$ and $\mathbb{Z}$ acting by translation. Thus I have a Hilbert bundle over $\mathbb{R}$, in fact, over $\mathbb{R}/\mathbb{Z} = S^1$ because the bundle is equivariant.

$\mathcal{S}(\mathbb{R})\rtimes\mathbb{Z}$ acts in each of the spaces $\ell^2(x+\mathbb{Z})$ and gives us the smooth kernel operators, i.e. we get the translations and the rapidly-decreasing diagonal matrices.

Here's the picture. One has a bundle E over the circle consisting at $x+\mathbb{Z}$ of say rapidly decreasing functions on the coset $x+\mathbb{Z}$. Then $\mathcal{S}(\mathbb{R})\rtimes\mathbb{Z}$ acts as endomorphisms of this bundle. In fact it is an algebra (without unit) over $\mathcal{S}(S^1)$ and at each point it maps onto the smooth-kernel operators on $\mathcal{S}(x+\mathbb{Z})$. Hence it seems that $\mathcal{S}(\mathbb{R})\rtimes\mathbb{Z}$ can be identified with the smooth kernel endos. of this bundle:



$$\mathcal{S}(\mathbb{R})\rtimes\mathbb{Z} = \text{End sm. ker.}(E \text{ over } S^1)$$

This is the sort of <u>Morita equivalence</u> I am after.

Next problem: Consider a complex structure on the torus.

I have identified $\mathcal{S}(\mathbb{R})$ with the space of smooth sections of a line bundle over $\mathbb{T}^2$. Then a $\bar\partial$-operator on the line bundle corresponds to some kind of operator on $\mathcal{S}(\mathbb{R})$. Let's find out which.

The correspondence is given by

$$f(x) \longmapsto g(x,y) = \sum_{n\in\mathbb{Z}} f(x+n)\,e^{2\pi i n y}$$

and here g is any smooth fn. periodic in y satisfying

(*) $$g(x+1,y) = e^{-2\pi i y}\,g(x,y).$$

Now a $\bar{\partial}$-operator on the line bundle will be of the form $\partial_x + i\partial_y + \alpha(x,y)$ on $g(x,y)$, where α must be consistent with (*). Put $D = \partial_x + i\partial_y$.

$$[(D+\alpha)g](x+1,y) = [D+\alpha(x+1,y)]\,g(x+1,y)$$

$$= e^{-2\pi i y}\,[D + \underbrace{i\,\partial_y(-2\pi i y)}_{2\pi} + \alpha(x+1,y)]\,g(x,y)$$

$$= e^{-2\pi i y}\,[D+\alpha(x,y)]\,g(x,y)$$

$$\Longrightarrow \quad \alpha(x+1,y) + 2\pi = \alpha(x,y)$$

Also $\alpha(x,y+1) = \alpha(x,y)$. Thus

$$\alpha(x,y) = -2\pi x + \text{periodic}$$

and so the simplest $\bar{\partial}$-operator is

$$\partial_x + i\partial_y - 2\pi x.$$

Try to find what this corresponds to on f.

$$(\partial_x + i\partial_y - 2\pi x)\sum f(x+n)e^{2\pi i n y} = \sum\left[f'(x+n) - (2\pi n + 2\pi x)f(x+n)\right]e^{2\pi i n y}$$

so it corresponds to

$$f \longmapsto (\partial_x - 2\pi x)f.$$

If we look for a holomorphic section, then we want

$$(\partial_x - 2\pi x)f = 0 \quad\Longrightarrow\quad f = \text{const}.\,e^{\pi x^2}$$

which is not in $\mathcal{S}(\mathbb{R})$. Therefore this line bundle is negative for the complex structure $\partial_x + i\partial_y$, and positive for the complex structure $\partial_x - i\partial_y$.

I want to manipulate Morita equivalences formally in order to calculate the K-theory of the Kronecker foliation. [redacted]

$$\mathcal{S}(\mathbb{T}^2) \rtimes \mathbb{R} \sim \underbrace{\mathcal{S}(\mathbb{C})}_{\mathcal{S}(\mathbb{R}_{\text{trans}}) \otimes \mathcal{S}(\mathbb{R}_{\text{long}})} \rtimes (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R})$$

$*$

$$\sim \mathcal{S}(\mathbb{R}_{\text{trans}}) \rtimes (\mathbb{Z} \times \mathbb{Z}) \sim \mathcal{S}(\mathbb{T}) \rtimes \mathbb{Z}$$

Here I use what seems to be quite general:

$$\mathcal{S}(G) \rtimes G = \text{compacts}$$

Next, I guess it is known that when $\mathbb{Z}$ acts trivially on the K-theory of A, then

$$K_*(A \rtimes \mathbb{Z}) = K_*(A) \otimes K_*(S^1)$$

Also there's Connes thm. $K_*(A \rtimes \mathbb{R}) = K_{*+1}(A)$. These isomorphisms seems to be consistent with the equivalences $*$. To be clearer, let us look at a simpler case namely

[redacted]

$$\mathcal{S}(\mathbb{R}) \rtimes \mathbb{Z} \sim \mathcal{S}(\mathbb{T})$$

$\mathcal{S}(\mathbb{R})$ has a $\mathbb{Z}$ for K_1, and the convolution alg. of $\mathbb{Z}$ is isom. to $\mathcal{S}(\mathbb{T})$. Another case might be

$$\mathcal{S}(\mathbb{T}) \rtimes \mathbb{R} \sim \text{conv. alg } \mathbb{Z} \cong \mathcal{S}(\mathbb{T})$$

Anyway one sort of sees the suspension process taking place, but it is not very clear.

Now, one thing I particularly want to look at, is whether the $\mathbb{R}$-action on $\mathbb{T}^2$ can be replaced by a [redacted] complex structure. I can ask for generalizations of the above equivalences.

One thing that does have an apparent analogue is the algebra $S(\mathbb{T}) \rtimes \mathbb{Z}$. I can replace $\mathbb{T}$ by $\mathbb{C}^*$ with $\mathbb{Z}$'s generator acting by $q = e^{2\pi i \tau}$.

I know from Connes theory that algebra $S(\mathbb{T}) \rtimes \mathbb{Z}$ has interesting projective modules. I want to see if these modules generalize to the $(\mathbb{C}^*, \mathbb{Z})$ situation. It is first necessary to describe the projective modules and see why they are projective.

These modules result from different transversals to the foliation. Let us start with the algebra

$$S(\mathbb{R}) \rtimes (\mathbb{Z} \times \mathbb{Z})$$

where $\mathbb{R} = \mathbb{C}/\text{flow}$ and $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R}$ by translating via the number $1, \theta$. Now if we take a direct factor $\mathbb{Z}(p,q)$ in $\mathbb{Z} \times \mathbb{Z}$, then we get an M-equivalence of the above algebra with

$$S(\mathbb{R}/\mathbb{Z}(p+q\theta)) \rtimes (\mathbb{Z}\times\mathbb{Z}/\mathbb{Z}(p,q)).$$

Hence for each (p,q) we get a corresponding projective module over $S(\mathbb{R}) \rtimes \mathbb{Z}$. Geometrically the M-equivalence corresponds to equivalences

$$(\mathbb{R}/\mathbb{Z}, \mathbb{Z}) \longleftarrow (\mathbb{R}, \mathbb{Z}\times\mathbb{Z}) \longrightarrow (\mathbb{R}/\mathbb{Z}(p+q\theta), \mathbb{Z}\times\mathbb{Z}/\mathbb{Z}(p,q)).$$

This is not clear.

Think of a transversal as being a covering of the final object of the topos. If $X \to e$ is a covering then one looks at $X \times X \rightrightarrows X$ which is a groupoid. Descent says objects over e are objects over X with action of the groupoid. Finally if $Y \to e$ is another covering then $X \times Y$ is something like the functors from the Y-groupoid to the X-groupoid. It is acted on one side by X and on the other by Y.

Let me consider two transversals:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & S^1 \arrow[d] \\ S^1 \arrow[r] & \text{leaves} \end{tikzcd}$$

Then the fibre product W is a covering of S^1 with fibre $\mathbb{Z}$. So one can get $W = S^1 \times \mathbb{Z}$ or $W = \mathbb{R}$; the former occurs if both transversals are the same. Here is a case where $W = \mathbb{R}$. Take the horizontal circle to be the x-circle and the vertical S^1 to be the y-circle. Now $W = \{ (x,y) \in S^1 \times S^1 \mid (x,0) \text{ and } (0,y) \text{ are in same leaf} \}$.

This means $(x,-y) = t(\theta,1) + (m,n)$ $\quad m,n \in \mathbb{Z}$

In other words letting $x = t\theta$, $y = -t$ as t ranges over $\mathbb{R}$ gives us a map from $\mathbb{R}$ to W which is onto. It is 1-1 because $t(\theta,1)$, $t'(\theta,1)$ can't be congruent mod $\mathbb{Z} \times \mathbb{Z}$ unless $t = t'$.

So we have

$$\begin{tikzcd} \mathbb{R} \arrow[r, "t \mapsto -t = y"] \arrow[d, "t \mapsto t\theta = x"] & S^1 \arrow[d] \\ S^1 \arrow[r] & \text{leaves} \end{tikzcd}$$

vertical arrows equalize $\mathbb{Z}\theta^{-1}$-orbits
horiz. arrows equalize $\mathbb{Z}$-orbits

$\mathcal{S}(\mathbb{R})$ will be a module over $\mathcal{S}(S^1) \rtimes \mathbb{Z}$ with $e^{2\pi i x} \in \mathcal{S}(S^1)$ acting as $U = e^{2\pi i \theta t}$ on $\mathcal{S}(\mathbb{R})$ and $\mathbb{Z}$-acting by

$$f(t) \overset{V}{\longmapsto} f(t+1).$$

Then $f(t) \overset{V^{-1}}{\longmapsto} f(t-1) \overset{U}{\longmapsto} e^{2\pi i \theta t} f(t-1) \overset{V}{\longmapsto} e^{2\pi i \theta} e^{2\pi i \theta t} f(t)$

so that $$VUV^{-1} = e^{2\pi i \theta} U$$

Now the goal will be to understand why this is a projective f.g. module over $\mathcal{S}(S^1) \rtimes \mathbb{Z}$.

There is a commuting action with $e^{2\pi i y}$ acting as $e^{-2\pi i t}$ on $\mathcal{S}(\mathbb{R})$ and with the $\mathbb{Z}$ acting by the translation through θ^{-1}!

March 21, 1983

Let's consider the problem of showing that the space of sections of the degree 1 line bundle over $\mathbb{T}^2$ is a projective module over the ring of functions on $\mathbb{T}^2$. Call the space of sections L and the ring A. The former can be identified with smooth $g(x,y)$ on $\mathbb{R}^2$ satisfying

$$g(x,y+1) = g(x,y) \quad , \quad g(x+1,y) = e^{-2\pi i y} g(x,y)$$

and the latter with doubly-periodic g. The dual module L^* has k with the $e^{-2\pi i y}$ factor replaced by $e^{2\pi i y}$. Then we have the pairing

$$\begin{array}{ccc} L \otimes L^* & \longrightarrow & A \\ g \quad h & \longmapsto & gh \end{array}$$

whose image is an ideal. We want to show 1 is in the ideal $1 = \sum_{j=1}^{n} g_j h_j$; this then embeds L as a direct factor of A^n. The usual proof shows that the ideal contains elements not vanishing at any point of $\mathbb{T}^2$ and then piecing these together with a partition of 1.

But let's try to see exactly what we need, and try to first see why we can't have $1 = gh$. Use the fact that g, h correspond to $f, k \in \mathcal{S}(\mathbb{R})$ by

$$g(x,y) = \sum f(x+n) e^{2\pi i n y}$$

$$h(x,y) = \sum k(x+n) e^{-2\pi i n y}$$

Then we want $1 = gh$ or

$$1 = \sum_{n,m} f(x+n)\, k(x+m)\, e^{2\pi i (n-m) y}$$

$$1 = \sum_{m,p} f(x+m+p)\, k(x+m)\, e^{2\pi i p y}$$

This can happen if and only if

$$\sum_m f(x+m+p)\, k(x+m) = \delta(p)$$

Put $f(x+m) = \int \hat{f}(x,\theta) e^{2\pi i m\theta}\, d\theta$; clearly this is just going back to $g(x,y)$, so we will just get the equation $gh = 1$ back.

Let's go back to the correspondence between $f \in \mathcal{S}(\mathbb{R})$ and smooth sections of the degree -1 line bundle over $\mathbb{T}^2$ given by

$$g(x,y) = \sum f(x+n) e^{2\pi i n y}$$

We saw that

$$(\partial_x + i\partial_y - 2\pi x) g = \sum e^{2\pi i n y}[\partial_x - 2\pi n - 2\pi x] f(x+n)$$

$$(\partial_x - i\partial_y + 2\pi x) g = \sum e^{2\pi i n y}[\partial_x + 2\pi n + 2\pi x] f(x+n)$$

hence

$$\left.\begin{array}{l} \partial_x + i\partial_y - 2\pi x \\ -\partial_x + i\partial_y - 2\pi x \end{array}\right\} \text{on } g \longleftrightarrow \left.\begin{array}{l} \partial_x - 2\pi x \\ -\partial_x - 2\pi x \end{array}\right\} \text{on } f$$

Notice that

$$\|g\|^2 = \int_0^1 dx \int_0^1 dy\, |g(x,y)|^2 = \int_0^1 dx \sum_{n\in\mathbb{Z}} |f(x+n)|^2$$

$$= \int_{-\infty}^{\infty} |f|^2 dx = \|f\|^2$$

so the above correspondence is unitary. Also the metric on the line bundle is given by $|g|^2$ since the transition function [redacted] $e^{-2\pi i y}$ has modulus 1.

So I conclude that the operators $\partial_x + i\partial_y - 2\pi x$ and its adjoint are essentially creation and annihilation operators. This will allow me to see the spectrum of the Laplacean in this bundle. So if we put

$$D = \partial_x + i\partial_y - 2\pi x \qquad\qquad \partial_x - 2\pi x$$

$$D^* = -\partial_x + i\partial_y - 2\pi x \qquad\qquad -\partial_x - 2\pi x$$

the Laplacean is $D^*D \longleftrightarrow (-\partial_x - 2\pi x)(\partial_x - 2\pi x) = -\partial_x^2 + (2\pi)^2 x^2 + 2\pi$. This is essentially a simple harm. osc. Hamiltonian with frequency 2π, so the eigenvalues are

$$2\left(n+\tfrac{1}{2}\right)2\pi + 2\pi = 2(n+1)2\pi = (n+1)4\pi \qquad n \geq 0.$$

Hence the ζ-function will be the Riemann zeta function.

One thing I know about the harmonic oscillator is that there are various interesting states in the Hilbert space, either the occupation number, or the coherent states.

Return to the problem of showing the $\mathcal{S}(\mathbb{R})$ with $U = e^{2\pi i x}$, V = translation by θ is projective over $A = \mathcal{S}(\mathbb{T}) \rtimes \mathbb{Z} = \{ \sum a_{mn} U^m V^n \mid a_{mn} \text{ rapid decrease} \}$. Let's describe A-linear maps $\varphi : \mathcal{S}(\mathbb{R}) \longrightarrow A$. Such a linear map φ has the form

$$\varphi(f) = \sum_p \varphi_p(f) V^p$$

where $\varphi_p(f)$ is a periodic function of x, such that

$$\varphi_p(e^{2\pi i x} f) = e^{2\pi i x} f(x).$$

An obvious way to product such a φ_p is

$$\varphi_p(f) = \sum_n f(x+n)\, k_p(x+n).$$

So let us see what condition on $\{k_p\}$ guarantees that

$$\varphi(f) = \sum_{n,p} f(x+n)\, k_p(x+n)\, V^p$$

commutes with the action of V.

$$\varphi(Vf) = \sum_{n,p} f(x+\theta+n)\, k_p(x+n)\, V^p$$

$$V\varphi(f) = \sum_p V\Big\{\sum_n f(x+n)\, k_p(x+n)\Big\} V^p$$

$$= \sum_p \Big\{\sum_n f(x+\theta+n)\, k_p(x+\theta+n)\Big\} V^{p+1}$$

So we want $k_{p+1}(x+n) = k_p(x+\theta+n)$ or $k_p(x) = k(x+p\theta)$. Thus

$$\varphi(f) = \sum_{n,p} f(x+n)\, k(x\overset{+n}{+}p\theta)\, V^p$$

where to get convergence we probably need $k \in \mathcal{S}(\mathbb{R})$.

Now to show $\mathcal{S}(\mathbb{R})$ is projective one wants f_j, k_j such that

$$\circledast \qquad \sum_j \sum_{n,p} f_j(x+n)\, k_j(x+n+p\theta)\, V^p = 1.$$

Now it is enough by partitions of 1 to produce f, k such that

$$\sum_{n,p} f(x+n)\, k(x+n+p\theta)\, V^p = \alpha(x)$$

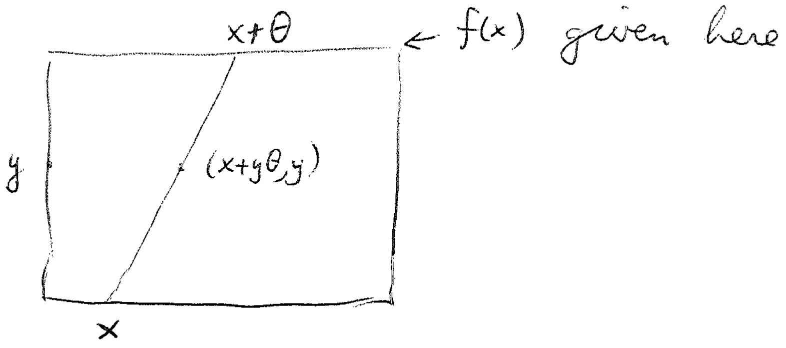
with $\alpha \neq 0$ at a specified point of S^1, say 0. Choose f to be a  so that if $|x| \le \frac{1}{2}$, then only $n=0$ contributes, and choose k similarly so that only $p=0$ contributes. Clear.

Problem: I want to take $\theta \in UHP$ and replace $\mathcal{S}(\mathbb{R})$ by some kind of analytic functions for which complex translation makes sense. Then the problem will be to see if we can construct $\circledast$.

May 22, 1983

I want to take the $\mathcal{S}(\mathbb{T})\rtimes\mathbb{Z}$ module resulting from a bundle flat along the leaves of the Kronecker foliation, and see if I can resolve it by projective modules. Thus I hope to understand to understand how to go from topos-type vector bundles to Connes vector bundles.

Let us restrict ~~attention~~ the line bundles over $\mathbb{T}^2$ with YM connection, and take the induced connection on the leaves. Start with the trivial line bundle with the connection $\quad \nabla = d + i(a\,dx + b\,dy) \quad a, b$ real constants. To get our $\mathcal{S}(\mathbb{T})\rtimes\mathbb{Z}$ module, take sections over $y=0$ with obvious $\mathcal{S}(\mathbb{T})$ structure. To define V use the identification of $y=0$ with $y=1$, to put a section over $y=0$ on $y=1$. Then parallel translate back along the leaves



We get the new section

$$f(x+\theta)\cdot e^{-\int_{y=0}^{y=1} i\,a(d(x+y\theta)+b\,dy)} = f(x+\theta)e^{-i(a\theta+b)}.$$

Thus we obtain the module

$$M = \mathcal{S}(\mathbb{T}) \qquad \text{with} \quad U = \text{mult. by } e^{2\pi i x}$$
$$\text{and} \quad V : f(x) \longmapsto \xi f(x+\theta).$$

It would seem that we have an exact sequence

$$0 \longrightarrow \mathcal{S}(\mathbb{T})\rtimes\mathbb{Z} \xrightarrow{\ \times(V-\xi)\ } \mathcal{S}(\mathbb{T})\rtimes\mathbb{Z} \xrightarrow[\text{on } 1\in M]{\text{action}} M \longrightarrow 0$$

so M represents 0 in the Grothendieck group.

May 23, 1983

I am trying to understand the map from topos type v. bundles to Connes vector bundles. Let us now take the line bundle of degree -1 over $\mathbb{T}^2$ whose sections are $g(x,y)$ smooth on $\mathbb{R}^2$ satisfying $g(x,y+1) = g(x,y)$ and $g(x+1,y) = e^{-2\pi i y} g(x,y)$. I claim that

$$\nabla = d + 2\pi i x\, dy$$

is a connection on this line bundle

$$(\nabla_x g)(x,y+1) = \partial_x g(x,y+1) = \partial_x g(x,y) = (\nabla_x g)(x,y)$$

$$(\nabla_x g)(x+1,y) = \partial_x g(x+1,y) = \partial_x e^{-2\pi i y} g(x,y)$$

$$= e^{-2\pi i y} (\nabla_x g)(x,y)$$

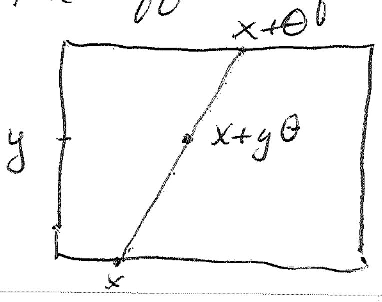
$$(\nabla_y g)(x,y+1) = (\partial_y + 2\pi i x) g(x,y+1) = (\nabla_y g)(x,y)$$

$$(\nabla_y g)(x+1,y) = [\partial_y + 2\pi i (x+1)] e^{-2\pi i y} g(x,y)$$

$$= e^{-2\pi i y} [\partial_y + 2\pi i x] g(x,y) = e^{-2\pi i y} (\nabla_y g)(x,y).$$

(Note the curvature is $[\nabla_x, \nabla_y]\, dx dy = 2\pi i\, dx dy$, so the degree is $\iint \frac{i}{2\pi} 2\pi i\, dx dy = -1$.)

Let's now define a module over $S(\mathbb{T}) \rtimes \mathbb{Z}$ corresponding to this line bundle over $\mathbb{T}^2$ with connection. Actually what I get is a line bundle over the x circle $\mathbb{T}$ with an action of $\mathbb{Z}$ coming from parallel translation along the leaves. Then I take the sections of this equivariant bundle to get my module. The line bundle over the x circle is the restriction of the line bundle over $\mathbb{T}^2$ to this circle. Its sections are $g(x,0)$ which will be periodic in x as $e^{-2\pi i 0} = 1$. Then to get the effect of $\mathbb{Z}$ one must parallel translate.



$$-\int_0^1 2\pi i\, (x + y\theta)\, dy$$

$$= -2\pi i x - \pi i \theta$$

Hence the operation is

$$f(x) \longmapsto \zeta e^{-2\pi i x} f(x+\theta)$$

where $\zeta = e^{-\pi i \theta}$ is a constant which we could change by tensoring with the trivial bundle with a flat connection.

Thus I get a $\mathcal{S}(\mathbb{T}) \rtimes \mathbb{Z}$ module given by

$$M = \mathcal{S}(\mathbb{T}) \quad \text{with} \quad \begin{array}{l} Uf(x) = e^{2\pi i x} f(x) \\ Vf(x) = \zeta e^{-2\pi i x} f(x+\theta) \end{array}$$

(In general we can get a new module by replacing V by $\tilde{V} = \zeta U^k V$.)

Interesting question: Put $R = \mathcal{S}(\mathbb{T}) \rtimes \mathbb{Z}$ $= k\{U, V\}$, $VUV^{-1} = e^{2\pi i \theta} U$ Is it true that $SL_2(\mathbb{Z})$ acts on the ring R? Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

and consider the elements

$\lambda = e^{2\pi i \theta}$

$$\tilde{U} = U^a V^b$$
$$\tilde{V} = U^c V^d.$$

Then

$$\tilde{U}\vec{V} = U^a V^b U^c V^d = U^{a+c} \lambda^{bc} V^{b+d}$$
$$\tilde{V}\tilde{U} = U^c V^d U^a V^b = U^{c+a} \underbrace{\lambda^{da}}_{\lambda\lambda^{bc}} V^{d+b}.$$

So

$$\tilde{V}\tilde{U} = \lambda \tilde{U}\tilde{V}$$

And so we do get an automorphism of R.

What is the effect of this automorphism on $K_0 R$? Supposedly $K_0 R = \mathbb{Z} \oplus \mathbb{Z}$, so how can $SL_2(\mathbb{Z})$ act on it. I know the action leaves $[R]$ fixed. In effect given $\alpha : R \to R$ we want the class of the

R-module $R \underset{\alpha\nwarrow R}{\otimes} R$. But we have the map

$$R \underset{\alpha\nwarrow R}{\otimes} R \longrightarrow R, \quad x\otimes y \longmapsto x\alpha(y).$$

I'm being stupid. Certainly given $\varphi : A \to B$ a homom. then the induced map $\varphi_* : K_0A \to K_0B$ sends $[A]$ to $[B\otimes_A A] = [B]$. On the other hand if φ is an isomorphism then we have **also** $\varphi^* : K_0 B \to K_0 A$ given by restriction of scalars. This should be inverse to φ_*; this is clear if you replace B by A with $\varphi = id$.

Here is a way to see that the $SL_2(\mathbb{Z})$ action on R has to fix the class of $\mathcal{S}(\mathbb{R})$ in $K_0(R)$. The point is that to any symplectic transformation of $\mathbb{R}^2$ we can associate an automorphism of $\mathcal{S}(\mathbb{R})$. Here we think of $\mathbb{R}^2$ as spanned by $p = \frac{1}{i}\partial_x$ and $q = x$. The operators U, V correspond to expenentiation of generators for a lattice in $\mathbb{R}^2$, so if the symplectic transformation preserves the lattice then it will induce an autom. of R compatible with the transformation on $\mathcal{S}(\mathbb{R})$.

For example lets start with $U = e^{2\pi i x}{}_{x}$ and $V : f(x) \mapsto f(x+\theta)$ on $\mathcal{S}(\mathbb{R})$, and consider the map

$$e^{i(a\frac{x^2}{2}+bx)} : \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{S}(\mathbb{R}).$$

Define $\tilde{U}, \tilde{V}$ on the right $\mathcal{S}(\mathbb{R})$ so that this map is compatible with U, V on the left:

$$\tilde{U} e^{i(a\frac{x^2}{2}+bx)} = e^{i(a\frac{x^2}{2}+bx)} U \implies \tilde{U} = U$$

$$\tilde{V} = e^{ia(\frac{x^2}{2}+bx)} V e^{-i(a\frac{x^2}{2}+bx)} = e^{ia[\frac{x^2}{2} - \frac{(x+\theta)^2}{2}] + bx - b(x+\theta)]} V$$

$$= e^{-ia\theta x - i(a\frac{\theta^2}{2} + b\theta)} V$$

So if $a\theta = 2\pi n$ and choose b appropriately we have

$$\tilde{V} = U^n V, \qquad \tilde{U} = U.$$

In other words if one transforms the standard $\mathcal{S}(\mathbb{R})$ by the automorphisms $U \mapsto U$, $V \mapsto \tilde{V} = U^n V$, then the result is still isomorphic to $\mathcal{S}(\mathbb{R})$. Similarly the Fourier transform will do a similar thing for the automorphism $U \mapsto V$, $V \mapsto U^{-1}$.

Let us pursue this symplectic idea more closely. We introduce the Heisenberg group G which is a central extension

1) $0 \longrightarrow \mathbb{T} \longrightarrow G \longrightarrow \mathbb{R}^2 \longrightarrow 0.$

G acts on $\mathcal{S}(\mathbb{R})$, and this is a nice way to identify it. It is generated by 1-parameter subgroups

$$e^{itq} = \text{mult by } e^{itx}$$
$$e^{itp} = f(x) \longmapsto f(x+t)$$

which satisfy

$$e^{itp} e^{it'q} e^{-itp} = e^{itt'} e^{it'q}$$

Now on the other hand we have the group Γ generated by U, V and $\mathbb{T}$ subject to $VUV^{-1} = e^{2\pi i\theta} U$. It fits into an exact sequence

2) $0 \longrightarrow \mathbb{T} \longrightarrow \Gamma \longrightarrow \mathbb{Z}^2 \longrightarrow 0.$

To get an action of $\mathcal{S}(T) \rtimes \mathbb{Z}$ on $\mathcal{S}(\mathbb{R})$ we need a homomorphism of the exact sequence 2) to 1) which is the identity on T. Think of $\mathbb{R}^2$ as consisting of $(aq + bp)$ with commutator pairing determined by taking the symplectic pairing $\Lambda^2 \mathbb{R}^2 \longrightarrow \mathbb{R}$, $p \wedge q \longmapsto 1$, followed by $\exp(i \times ?)$.

The point is that a homom. $\Gamma \longrightarrow G$ under $\mathbb{T}$ induces a map $\mathbb{Z}^2 \longrightarrow \mathbb{R}^2$ compatible with the commutator pairing. Thus

$U \longmapsto aq+bp$, $V \longmapsto cq+dp$ and we must have $\exp(i(bc-ad)) = e^{-2\pi i\theta}$. For example

$$U \longmapsto e^{2\pi i q} \qquad V \longmapsto e^{i\theta p} \qquad \Gamma \to G.$$

(Check: $(U,V) = UVU^{-1}V^{-1} = e^{-2\pi i\theta}$). Thus

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc \equiv 2\pi\theta \quad \mathrm{mod}(\mathbb{Z}).$$

So what I obtain is an integer invariant for a way to have Γ act on $\mathcal{S}(\mathbb{R})$. Now look at the automorphisms of Γ under Π. Such a thing induces an auto. $\mathbb{Z}^2 \to \mathbb{Z}^2$ which also must satisfy $2\pi\theta(ad-bc) \equiv 2\pi\theta$. Since $ad-bc=\pm 1$ this isn't possible unless $ad-bc=1$. In fact we can't even have maps $\Gamma \longrightarrow \Gamma$ under Π unless the quotient map on $\mathbb{Z}^2$ is in $SL_2(\mathbb{Z})$.

May 25, 1983

Notes on cyclic homology: Yesterday when preparing my talk I succeeded in understanding how Connes double complex is related to cyclic homology.

First one sets up a double complex

$$\tilde{C}(A) \qquad \begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "b'"] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & {} \arrow[l, "1-t"'] \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & {} \arrow[l, "1-t"'] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & {} \arrow[l, "1-t"']
\end{tikzcd}$$

Put − sign on b′ so that d^h and d^v anti-commute

in which the columns are alternately $(A^{\otimes *}, b)$ and $(A^{\otimes *}, b')$, and horizontally we have the standard complex for calculating the homology for the cyclic group. Use Feigin-Tsygan for formulas $b(1-t) = (1-t)b'$, $b'N = Nb$.

In char. 0 the rows are acyclic, hence one has a surjective quis

$$\tilde{C}(A) \twoheadrightarrow (A^{\otimes *}, b)/(1-t) = C(A)$$

showing $\tilde{C}(A)$ gives the cyclic homology.

In general char one has exact sequences

$$0 \longrightarrow \begin{pmatrix}\text{first two}\\ \text{columns}\end{pmatrix} \longrightarrow \tilde{C}(A) \longrightarrow \tilde{C}(A)[2] \longrightarrow 0$$

$$0 \longrightarrow (A^{\otimes *}, b) \longrightarrow \begin{pmatrix}\text{first two}\\ \text{columns}\end{pmatrix} \longrightarrow (A^{\otimes *}, b')[1] \longrightarrow 0$$

Since $(A^{\otimes *}, b')$ is acyclic, we deduce the long exact sequence

$$\longrightarrow \underbrace{H_p(A^{\otimes *}, b)}_{\substack{\parallel \\ \mathrm{Hoch}_{p-1}(A)}} \longrightarrow HC_p(A) \longrightarrow HC_{p-2}(A) \longrightarrow H_{p-1}(A^{\otimes *}, b) \longrightarrow$$

Because the odd columns are contractible one can find a subcomplex of $\tilde{C}(A)$ with the same homology. Choose a homotopy operator ξ for the complex $(A^{\otimes *}, b')$:

$$\xi b' + b' \xi = 1$$

Then consider the following double complex

$$X(A): \quad \begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & & \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[d, "b"'] \arrow[ul, dash] & A^{\otimes 3} \arrow[d, "b"] & \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[ul, "B"'] \arrow[d, "b"'] & A^{\otimes 2} \arrow[ul, "B"'] \arrow[d, "b"] & \\
A & A \arrow[ul, "B"'] & A \arrow[ul, "B"'] & {} \arrow[ul]
\end{tikzcd}$$

which I have written in a strange way so that one can see the way it will map to $\tilde{C}(A)$. Here

$$B = (1-t)\xi N : A^{\otimes p} \longrightarrow A^{\otimes (p+1)}.$$

Diagram chasing shows that $B^2 = 0$ and $bB + Bb = 0$ (this last following from $\xi b' + b'\xi = 1$).

To see the embedding of $X(A)$ in $\tilde{C}(A)$ one uses

$$\begin{tikzcd}
Bx & \xi N x \arrow[l, mapsto] \arrow[d, mapsto] \\
\begin{array}{c} -b'\xi N x \\ + N x \end{array} \arrow[d, equal] & x \arrow[l, mapsto] \arrow[d, mapsto] \\
\xi N (bx) & bx
\end{tikzcd}
\qquad
\begin{aligned} \xi N b x &= \xi b' N x \\ &= Nx - b'\xi N x \end{aligned}$$

The embedding sends $x \in X(A)_{pq} = A^{\otimes q}$, p even, into

$$x + \xi N x \in \tilde{C}(A)_{pq} + \tilde{C}(A)_{p-1,q+1}$$

Take the boundary of this in $\tilde{C}(A)$:

$$\begin{aligned} &(d^h + d^v)x + (d^h + d^v)\xi N x \\ = \; &N x + \underset{\checkmark}{bx} + \underset{\checkmark}{(1-t)\xi N x} - b'\xi N x \end{aligned}$$

On the other hand take the differential of x in $X(A)$

$$bx + Bx$$

and embed this in $\tilde{C}(A)$ to get

$$bx + Bx + \xi N(bx + Bx) \qquad\qquad NB = 0$$

$$= \underset{\checkmark}{bx} + \underset{\checkmark}{(1-t)\xi Nx} + \underbrace{\xi \underbrace{Nb}_{b'N} x}_{Nx - b'\xi Nx}$$

and so it works!

So now one turns to the case where A is a smooth commutative algebra over A where one knows that

$$\mathrm{Hoch}_p(A) = \mathrm{Tor}_p^{A\otimes A}(A,A) = \Omega^p_{A/k}.$$

In fact it seems one has a map

$$(A^{\otimes *}, b) \longrightarrow (\Omega_{A/k}, 0 \text{ differential})$$

which is a quis. Also that $B : A^{\otimes p} \longrightarrow A^{\otimes(p+1)}$ induces $d : \Omega^{p-1} \longrightarrow \Omega^p$ up to a constant factor $\neq 0$. Then $X(A)$ maps quasi-isomorphically to

$$\begin{tikzcd}
\Omega^3 \arrow[d] & {} \arrow[d] & \\
\Omega^2 \arrow[d, "0"] & \Omega^2 \arrow[d, "0"] \arrow[ul, "d"'] & \Omega^2 \arrow[d] \\
\Omega^1 \arrow[d, "0"] & \Omega^1 \arrow[d, "0"] \arrow[ul, "d"'] & \Omega^1 \arrow[d, "0"] \arrow[ul, "d"'] \\
\Omega^0 & \Omega^0 \arrow[ul, "d"'] & \Omega^0 \arrow[ul, "d"']
\end{tikzcd}$$

i.e. to $\quad \bigoplus_p \left(F_{\leq p}\Omega^{\cdot}\right)[2p+1]$

May 25, 1983 (cont.)

An important thing I have learned at the Marseilles conference is that for a non-singular variety X which is not complete, the naive idea of filtered de Rham cohomology and Deligne cohomology is wrong. Instead one uses

$$H^*(X, \mathbb{C}) = H^*(X', \underbrace{\Omega_{X'}\langle \log \rangle})$$

and the stupid filtration on ~~in~~ order to put a (mixed) Hodge structure on $H^*(X, \mathbb{C})$. Supposedly I should see the difference already for an affine curve, [redacted] where the Pic holomorphically and algebraically differ.

The idea I get is that there might be a good K-theory (for complex manifolds) with compact support. [redacted] One would want it to compare nicely with algebraic K-theory for an algebraic variety (GAGA type result) For example take [redacted] a closed Riemann surface with a finite number of points removed.

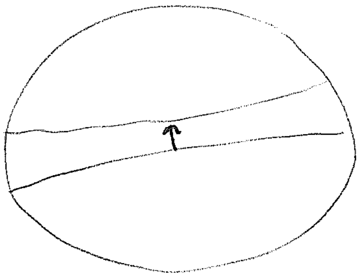
It seemed to be interesting to ask whether upon removing a point or a disk for a closed Riemann surface one obtains different open Riemann surfaces. [redacted]

Example: Take $\mathbb{P}^1$. Removing a point gives $\mathbb{C}$ and a disk gives the unit disk. These are different by Liouville's thm which implies there are no bounded non-constant analytic functions on $\mathbb{C}$.

Similarly if M is a closed R.S., then M-pt has no bounded non-analytic functions, [redacted] because [redacted] the singularity is removable. On the other hand M-disk has such a function, because we can find an analytic function on M-(point inside the disk) with pole at this point.

General arguments: If M' is an open R.S., look at $\widetilde{M'}$ with is either $\mathbb{C}$ or the disk. [redacted] $\pi_1 M'$ acts on $\widetilde{M'}$ without fixpoints. If $\widetilde{M'} = \mathbb{C}$, then $\pi_1 M'$ must be a translation group, hence either $\mathbb{Z}$ or $\mathbb{Z}^2$, so M' is either the cylinder $\mathbb{C}/\mathbb{Z} = \mathbb{C}^\times$ or an elliptic curve $\mathbb{C}/\mathbb{Z}^2$. In the other case where $\tilde{M}' =$ disk or UHP, we know $\pi_1 M \subset PSL_2(\mathbb{R})$ preserves the Poincaré metric. Thus M' has a constant curvature metric which is unique up to a constant factor. Then one can distinguish between $M -$ finite set and $M -$ some disks + points, because the former has finite area.

Simple case: Take an annulus $r_1 < |z| < r_2$. Then the deck transformation is supposed to be hyperbolic



and the amount moved is essentially its trace. This is the only moduli and turns out to be r_1/r_2 which lies in $[0,1]$. (If both $r_1 = 0$ and $r_2 = \infty$, then one has $\mathbb{C}^\times$).

May 26, 1983

KMS condition. Start with a state φ on a $*$ algebra $\mathcal{B}$. For example let $\mathcal{B}$ be the bounded operators on $\mathcal{H}$ and let $\varphi(A) = tr(\rho A)$ where ρ is a density matrix. Assume φ is faithful in the sense that $\varphi(A^*A) = 0 \Rightarrow A = 0$. In this case I want

$$\rho = \sum \lambda_i |\psi_i\rangle\langle\psi_i|$$

where ψ_i is an orthonormal basis for the Hilbert space and all $\lambda_i > 0$.

There is a general construction (highly non-trivial) due to Takesaki which associates to a faithful φ on a v.N. alg. $\mathcal{B}$ a 1-parameter group of autos. α_t of $\mathcal{B}$. This auto. group extends to the upper half plane and the key is the KMS condition

$$\varphi(A\alpha_i(B)) = \varphi(BA) \qquad \text{for } A, B \in \mathcal{B}.$$

Here $i = \sqrt{-1}$.

Let's see how this works for bounded operators with φ as above. Let H be the Hamiltonian such that

$$\rho = e^{-H}/Tr(e^{-H})$$

i.e. $$H = \sum \varepsilon_i |\psi_i\rangle\langle\psi_i| \qquad \frac{e^{-\varepsilon_i}}{\sum e^{-\varepsilon_i}} = \lambda_i .$$

Then $$\alpha_t(B) = e^{+itH} B e^{-itH} \qquad \alpha_i(B) = e^{-H} B e^{H}$$

$$\varphi(A\alpha_i(B)) = tr\left(e^{-H} A e^{-H} B e^{+H}\right)$$

$*$

$$= tr\left(e^{-H} B A\right) = \varphi(BA)$$

So it works for finite dimensions. What do you do in infinite dimensions, e.g. how does one make sense of $e^{-H}Be^{H}$? If $e^{-H}Be^{H}$ can be defined as a bounded operator, then the above calculation, which involves moving $e^{-H}Be^{H}$ past $e^{-H}A$ (trace class), makes sense. Maybe the KMS conditions holds on a dense set of $\mathcal{B}$.

The study of the different α_t was done by Connes and forms the basis for his classification of type III factors. The point is that type III algebras don't have a trace, and the KMS formula is a substitute.

Consider $M = \mathbb{C}/\Gamma$ Γ lattice e.g. $\mathbb{Z}+\mathbb{Z}\tau$

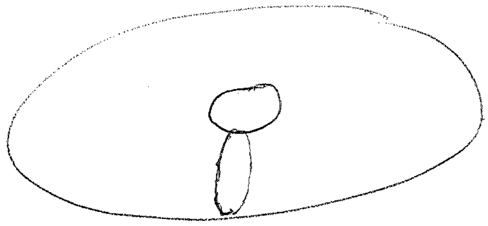
Take a direct factor $\mathbb{Z}\gamma \subset \Gamma$. Then we have

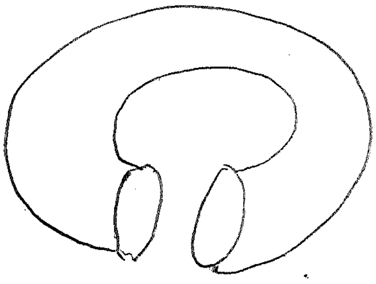
$$\mathbb{C}/\mathbb{Z}\gamma \longrightarrow M$$

covering with group $\Gamma/\mathbb{Z}\gamma \simeq \mathbb{Z}$, and also $\mathbb{C}/\mathbb{Z}\gamma \simeq \mathbb{C}^*$. In other words we get various ways of representing M as a quotient of $\mathbb{C}^*$

I want to think of these coverings $\mathbb{C}^* \longrightarrow M$ as analogous to circles transversal to the Kronecker foliation.

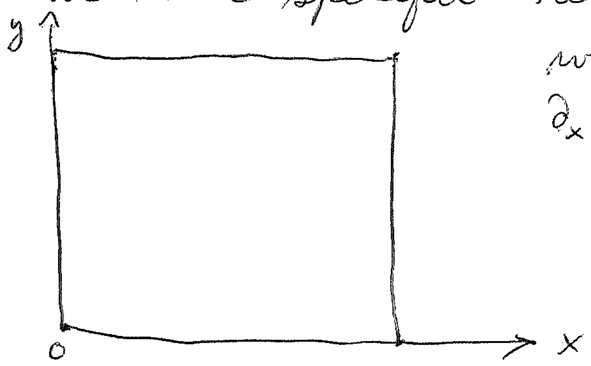
Another idea: If I take a embedded circle in a Riemann surface I know how to disconnect, or cut, the surface along the curve.





Then the $\bar{\partial}$-operator on the surface gives rise to an operator of the Fock space on the circle.

To be more specific take an example. Take $\mathbb{T}^2$ with the complex structure $\partial_x + i\partial_y$ to begin with.



Formula: $t : A^{\otimes p} \longrightarrow A^{\otimes p}$ is the forward shift

$$t[a_1, \dots, a_p] = (-1)^{p-1}[a_p, a_1, \dots, a_{p-1}]$$

$j : A^{\otimes p} \longrightarrow A^{\otimes(p-1)}$ is the last face of b:

$$j[a_1, \dots, a_p] = (-1)^{p-1}[a_p a_1, a_2, \dots, a_{p-1}]$$

Then calculate $t^{i-1} j t^{-i}$ for $1 \le i \le p-1$

$$[a_1, \dots, a_p] \xmapsto{t^{-i}} (-1)^{(p-1)i}[a_{i+1}, \dots, a_p, a_1, \dots, a_i]$$

$$\xmapsto{j} (-1)^{pi-i+p-1}[a_i a_{i+1}, \dots, a_p, a_1, \dots, a_{i-1}]$$

$$\xmapsto{t^{i-1}} (-1)^{pi-i+p-1}(-1)^{p(i-1)}[a_1, \dots, a_i a_{i+1}, \dots]$$

$$= (-1)^{i-1}[a_1, \dots, a_i a_{i+1}, \dots].$$

Thus $$b = \sum_{i=1}^{p} t^{i-1} j t^{-i}, \quad b' = \sum_{i=1}^{p-1} t^{i-1} j t^{-i}$$

Now
$$(1-t)b' = \sum_1^{p-1} t^{i-1} j t^{-i} - \sum_1^{p-1} t^{i} j t^{-i}$$

$$b(1-t) = \sum_1^{p} \underset{\substack{\uparrow \\ \text{extra } j}}{t^{i-1} j t^{-i}} - \sum_1^{p} \underset{\substack{\uparrow \\ \text{extra } j}}{t^{i-1} j t^{-i+1}}$$

so they are equal.

$$Nb = \sum_{j=1}^{p-1} t^j \sum_{i=1}^{p} t^{i-1} j t^{-i} = \sum_{j=1}^{p-1} \sum_{i=1}^{p} t^j j t^{-i} = NjN$$

$$b'N = \sum_{i=1}^{p-1} t^{i-1} j t^{-i} \sum_{j=1}^{p} t^j = \sum_{i=1}^{p-1} \sum_{j=1}^{p} t^{i-1} j t^j = NjN$$

May 29, 1983

Why $\prod_{n=1}^{\infty}(1-q^n) = \prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})$ is a modular function. Formally we take the product of the factors $m+n\tau$ for $n>0$ all m, or $n=0$ and all $m>0$, and use the formal equality

$$\prod_{m\in\mathbb{Z}}(x-m) = 1-e^{2\pi i x} .$$

Thus $\prod_{n=1}^{\infty}(1-q^n)$ is a way to make sense of the product of the $m+n\tau$, hence should depend only on the lattice $\mathbb{Z}+\mathbb{Z}\tau$.

According to Graeme the standard asymptotic formula for the partition function f

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^n)} = \sum_{n=0}^{\infty} p(n)q^n$$

shows that the boundary value [redacted] of this [redacted] on $|q|=1$ [redacted] is [redacted] a hyperfunction, not a distribution.

[redacted] Review Morita invariance: Start with the case of $S(\mathbb{R})\rtimes\mathbb{Z}$ being equivalent to $S(\mathbb{T})$. [redacted]

I begin ~~in finite dimensions~~ with a finite [redacted] covering $f: P\longrightarrow X$ with Galois group Γ. If $A\longrightarrow B$ is the map of rings of functions corresponding to f, we have a Morita equivalence between A and $B\rtimes\Gamma$. The equivalence is given by B with natural left $B\rtimes\Gamma$ action and right A action.

We have a Morita equivalence between A and $B\rtimes\Gamma$ by descent, which says that $M\longmapsto B\otimes_A M$ is an equivalence between A-modules and B-modules with equivariant Γ-action. [redacted]

May 30, 1983

Cyclic homology.

Let's introduce the Goodwilly idea where one enlarges the category Δ of ordered sets $[p] = \{0, \cdots, p\}$ to the category of cyclically ordered sets. An object is a cyclic graph

so there is one basic object for each $p \geq 1$. What are the morphisms? Let's use the notation $\langle p \rangle = \{1, 2, \cdots, p\}$ for the basic object of size p. We want $\langle p \rangle \longmapsto A^{\otimes p}$ for A an algebra with unit, or $\langle p \rangle \longmapsto G^p$ with G a monoid, to be a functor. (These are not the same perhaps because $G^0 = pt$ makes sense and maps to and from G whereas $A^{\otimes 0} = k$ maps to A but not from A).

So work with $A^{\otimes p}$ consisting of $[a_1, \cdots, a_p]$ which generate. We can multiply a segment which gives us face operators

$$[a_1, \cdots, a_p] \longmapsto [\ , a_i a_{i+1}, \cdots] \qquad i = 1, \cdots, p-1$$

$$[a_1, \cdots, a_p] \longmapsto [a_p a_1, \cdots\ a_{p-1}]$$

$$\text{or } [a_2, \cdots, a_p a_1]$$

where the last two are related by cyclic permutations. How should I think of these? In the case of the bar construction a vertex is a space between the elements

$$(\cdots, g_i, g_{i+1} \cdots) \longmapsto (\cdots, g_i g_{i+1}, \cdots)$$

Therefore the arrows are important.

Let's see if interpreting cyclic graphs as a category gives us the appropriate morphisms.

This doesn't seem to work because one would have a discrete groupoid.

Let's go back to our basic models $A^{\otimes p}$. We get face operators $A^{\otimes p} \longrightarrow A^{\otimes(p-1)}$ by multiplying two consecutive a_i, a_{i+1} and then possibly a cyclic permutation. Raising operators allow one to insert a 1.

Review the theory of cyclic homology.

Hochschild homology $H_n(A,M) = \mathrm{Tor}_n^{A\otimes A^{op}}(M,A)$, where M is an A-bimodule. Use standard resolution

$$\longrightarrow A\otimes A\otimes A \xrightarrow{b'} A\otimes A \xrightarrow{b'} A \longrightarrow 0$$

of A as an $A\otimes A^{op}$ module, where

$$b'[a_0, \dots, a_{p+1}] = \sum_{i=0}^{p} (-1)^i [\dots, a_i a_{i+1}, \dots]$$

and a contracting homotopy is

$$\xi[a_0, \dots, a_p] = [1, a_0, \dots, a_p].$$

When we tensor with M on the right, we use

$$\begin{aligned} M \otimes_{A\otimes A^{op}} (A\otimes A^p \otimes A) &\xleftarrow{\sim} M\otimes A^p \\ m\otimes[1, a_1, \dots, a_p, 1] &\longleftarrow\!\shortmid [m, a_1, \dots, a_p] \\ m\otimes[a_0, \dots, a_{p+1}] &\longmapsto [a_{p+1} m a_0, a_1, \dots, a_p] \end{aligned}$$

and we compute that the boundary ~~on~~ $M\otimes A^p$ is

$$b[m, a_1, \dots, a_p] = [ma_1, \dots, a_p] + \sum_1^{p-1} (-1)^i [m, \dots a_i a_{i+1}, \dots] + (-1)^p [a_p m, a_1, \dots, a_{p-1}]$$

Now let the generator of $\mathbb{Z}/\mathbb{Z}p$ act on $A^{\otimes p}$ by

$$t_p[a_1, \dots, a_p] = (-1)^{p-1}[a_p, a_1, \dots, a_{p-1}]$$

and put $N_p = \sum_{i=0}^{p-1} t_p^i$. Drop subscripts. Then the computation lemma says that we have a double complex

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
 & A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & {} \arrow[l, "1-t"'] \\
q=2 \arrow[r] & A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & {} \arrow[l, "1-t"'] \\
q=2 \arrow[r] & A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & {} \arrow[l, "1-t"'] \\
 & p=0 \arrow[u] & p=1 \arrow[u] & p=2 \arrow[u] &
\end{tikzcd}$$

A good definition of the cyclic homology is the homology of this double complex. Why:

1) In char 0 the rows are exact in positive degrees, hence the double complex is quis to H_0^h which is the complex $A^{\otimes *}/(1-t)$ with differential induced by b.

2) One has exact sequences

$$0 \longrightarrow \begin{Bmatrix}\text{first two} \\ \text{columns}\end{Bmatrix} \longrightarrow C_{..}(A) \longrightarrow C_{..}(A)[2] \longrightarrow 0$$

$$0 \longrightarrow (A^{\otimes *}, b) \longrightarrow \begin{Bmatrix}\text{first two} \\ \text{cols.}\end{Bmatrix} \longrightarrow \underset{\text{acyclic}}{(A^{\otimes *}, b')[1]} \longrightarrow 0$$

which gives the long exact sequence relating Hochschild and cyclic homology.

3) Relation with DR cohomology when A is a smooth commutative k-algebra. [redacted] I want to see whether this works in char. $\neq 0$.

We begin with the Hochschild, Kostant, Rosenberg thm. that

$$H_p(A, A) \cong \Omega^p_A$$

Before I get involved with commutative rings, I should see if I can remove the flab in the double complex. The flab comes from the odd columns which are acyclic and also from the fact that the Hochschild complex is simplicial, hence splits off an acyclic subcomplex, namely the degenerate subcomplex. This is the complex spanned by $[a_1, \ldots, a_p]$ in $A^{\otimes p}$ where some $a_j = 1, j > 1$.

Unfortunately the degenerate subcomplex is not stable under the cyclic permutation t.

June 1, 1983

On cyclic homology for rings without unit. Let's consider an augmented k-algebra $A = k \oplus I$. At least in characteristic zero I know that

$$HC(A) = HC(k) \oplus HC(I)$$

where $HC(I)$ is defined as either the homology of the complex $(I^{\otimes *}, b)/(1-t)$, or the equivalent double complex

$$\begin{tikzcd} \arrow[d] & \arrow[d] & \\ I^{\otimes 2} \arrow[d, "b"] & I^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & \arrow[l, "N"'] \\ I & I \arrow[l, "1-t"'] & \arrow[l, "N"'] \end{tikzcd}$$

ⓧ

I'd like to show that the double complex gives the good definition in all characteristics.

One idea is that Hochschild complex $(A^{\otimes *}, b)$ is quasi-isomorphic to its normalized complex which is isomorphic to $A \otimes I^{\otimes p}$. Then we have an exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & I^{\otimes(p+1)} \arrow[r] \arrow[d, "b"] & A \otimes I^{\otimes p} \arrow[r] \arrow[d, "b"] & I^{\otimes p} \arrow[r] \arrow[d, "\bar{b}"] & 0 \\ 0 \arrow[r] & I^{\otimes p} \arrow[r] & A \otimes I^{\otimes(p-1)} \arrow[r] & I^{\otimes(p-1)} \arrow[r] & 0 \end{tikzcd}$$

Let's compute $\bar{b}$: It is calculated by lifting $[x_1, \ldots, x_p] \in I^{\otimes p}$ back to $[1, x_1, \ldots, x_p] \in A \otimes I^{\otimes p}$ and applying b to get

$$b[1, x_1, \ldots, x_p] = [x_1, \ldots, x_p] + \sum_1^{p-1} (-1)^i [1, \ldots, x_i x_{i+1}, \ldots, x_p]$$
$$+ (-1)^p [x_p, x_1, \ldots, x_{p-1}].$$

then projecting back to $I^{\otimes(p-1)}$ to get

$$\bar{b}[x_1,\dots,x_p] = \sum_1^{p-1}(-1)^i[x_1,\dots,x_ix_{i+1},\dots,x_p]$$

which is $-b'$. Next notice that the boundary (or connecting) homomorphism in the long exact sequence belonging to this exact sequence has to be induced by the map $(1-t): I^{\otimes p} \to I^{\otimes p}$ on cycles by the above computation.

Hence we are led to the idea that the Hochschild complex $(A^{\otimes *}, b)$ is quis to the cone on the map $(1-t): (I^{\otimes *}, b') \longrightarrow (I^{\otimes *}, b)$. This should be true except at the very bottom, where for $*=1$ we get I instead of A.

It seems clear that the Hochschild complex with the bottom A replaced by I, and normalized is <u>isomorphic</u> to the above cone. The point is the cone will be spanned by $[x_1,\dots,x_p]$ and $[1,x_1,\dots,x_p]$ with the differential as given on the bottom of the preceding page.

This being the case it is clear that when we take the first two columns of the double complex ⊛ we get equivalent to

$$\begin{tikzcd}
 \arrow[d, "b"] & \arrow[d, "-b'"] \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] \\
I & A \arrow[l, "1-t"']
\end{tikzcd}$$

Thus if I define the cyclic homology of the ring I without unit to be the homology of the double complex ⊛ it is clear that I will indeed have $HC(A) = HC(I) \oplus HC(k)$.

Let's calculating cyclic homology for a ring of dual numbers $A = k \oplus I$, $I^2 = 0$. Then all the vertical arrows in the double complex are zero and so we get a direct sum of complexes

$$I^{\otimes p} \xleftarrow{1-t} I^{\otimes p} \xleftarrow{N} I^{\otimes p} \longleftarrow \cdots$$

shifted p-times. Thus we have a non-periodic part $I^{\otimes p}/(1-t)$ and then a periodic-of-period 2 part coming from the periodic homology of the cyclic group. Clearly the Connes periodicity is just the periodicity of [redacted] the homology of the cyclic group.

June 2, 1983

The problem is to find a version of the relative cyclic homology of A modulo k. I know how to do this for an augmented ring $A = k \oplus I$, namely the double complex

$$\begin{tikzcd}
I^{\otimes 3} \arrow[d] & I^{\otimes 3} \arrow[l, "1-t"'] \arrow[d] & \arrow[l, "N"'] \\
I^{\otimes 2} \arrow[d] & I^{\otimes 2} \arrow[l, "1-t"'] \arrow[d] & \arrow[l, "N"'] \\
I & I \arrow[l, "1-t"'] & \arrow[l, "N"']
\end{tikzcd}$$

Another piece of information is that the complex formed of the first two columns of this double complex is isomorphic to the normalized Hochschild complex $A \otimes I^{\otimes(p-1)}$, more precisely:

$$\longrightarrow A \otimes I^{\otimes 2} \xrightarrow{b} A \otimes I \xrightarrow{b} I$$

and this has a definite meaning for an arbitrary unital k-algebra A, [redacted] namely

(1) $$\longrightarrow A \otimes \bar{A}^{\otimes 2} \longrightarrow A \otimes \bar{A} \longrightarrow \bar{A}$$

Now what I want to do is to find a periodic "resolution" of the complex

(2) $$\xrightarrow{b} \bar{A}^{\otimes 3}/(1-t) \xrightarrow{b} \bar{A}^{\otimes 2}/(1-t) \xrightarrow{b} \bar{A}$$

which I know makes sense as a quotient of (1). I hope that the kernel of the map from (1) to (2) is quis to a complex

(3) $$\longrightarrow (\bar{A}^{\otimes 3})^t \xrightarrow{b'} (\bar{A}^{\otimes 2})^t \xrightarrow{b'} \bar{A} \, .$$

Then I get what I want by composing with the norm map from (2) to (3). Notice that the sort of thing I am after exists in the augmented case.

So the first thing to check is that the complex (3) makes sense, i.e. that the [redacted] operator b' on $(A^{\otimes *})^t$ induces an operator on $(\bar{A}^{\otimes *})^t$. In char. zero this follows from the isomorphism $N: (A^{\otimes *}/(1-t), b) \simeq ((A^{\otimes *})^t, b')$ and the fact we know the descent already for the first complex.

We have $b': A^{\otimes p} \longrightarrow A^{\otimes(p-1)}$ is given by

$$b'[a_1,\dots,a_p] = \text{[redacted]} \sum_1^{p-1} (-1)^{i-1} [a_1,\dots,a_i a_{i+1},\dots,a_p].$$

Consider the composition $A^{\otimes p} \xrightarrow{b'} A^{\otimes(p-1)} \xrightarrow{\pi} \bar{A}^{\otimes(p-1)}$.

Then it is clear that

$$\pi b'[a_1,\dots,a_p] = 0 \qquad \text{if} \quad a_2 \text{ or } a_3 \text{ or} \dots \text{or } a_{p-1} = 1.$$

$$\pi b'[1, a_2,\dots,a_p] = [\bar{a}_2,\dots,\bar{a}_p]$$

$$\pi b'[a_1,\dots,a_{p-1},1] = (-1)^p [\bar{a}_1,\dots,\bar{a}_{p-1}].$$

If I [redacted] pick a splitting $A = k \oplus \bar{A}$, then $A^{\otimes p}$ splits into 2^p pieces, so $\bar{A}^{\otimes p}$ is an equivariant direct summand of $A^{\otimes p}$ and $(A^{\otimes p})^t \longrightarrow (\bar{A}^{\otimes p})^t$ is onto. Pick a basis of $\bar{A}$ and combine it with 1 to get a basis for A. Then $(A^{\otimes p})^t$ will have a basis given by orbits of $\mathbb{Z}/p\mathbb{Z}$ on J^p where J is the set of indices for the basis, except that some orbits will give a zero (e.g. $\Lambda^2 A$). [redacted]

Now consider $\xi \in$ [redacted] $K_p = \operatorname{Ker}\{A^{\otimes p} \longrightarrow \bar{A}^{\otimes p}\}$ which is also cyclically invariant. Better I should look at the basis I have of $(A^{\otimes p})^t$ and [redacted] notice that these basis elements go to zero or are mapped 1-1 onto a basis of $(\bar{A}^{\otimes p})^t$. Thus I have a basis of K_p^t described by orbits [redacted] on J^p containing at least one 1. So it seems one can suppose ξ is the sum of

$$[1, x_2, \dots, x_k, 1, x_2, \dots, x_k, \dots\quad]$$

and its cyclic translates. If there is more than one 1, then each term will be killed by $\pi b'$. The only problem is

$$[1, x_2, \dots, x_p]$$

but then ξ also contains the translate $(-1)^{p-1}[x_2, \dots, x_p, 1]$ and so things cancel.

Simpler proof: Because $(A^{\otimes p})^t \longrightarrow (\bar{A}^{\otimes p})^t$ is onto we only have to show that any ξ in its kernel is killed by $\pi b' : A^{\otimes p} \longrightarrow A^{\otimes (p-1)} \longrightarrow \bar{A}^{\otimes (p-1)}$. We use the basis $[a_{j_1}, \dots, a_{j_p}]$ of $A^{\otimes p}$, where a_j is a basis for A containing 1, and write ξ as a linear combination of this basis. We know that $\pi b'$ kills the basis elements having 1 in an interior position or two 1's at the ends. I forgot to mention that because ξ is in the kernel only basis elements with at least one $a_j = 1$ occur. So we look at the term in ξ corresponding to the basis element $[1, a_2, \dots, a_p]$ and see that by cyclic invariance [redacted] there is present also $(-1)^{p-1}[a_2, \dots, a_p, 1]$ and that $\pi b'$ of the sum is zero.

Summarize the situation. I am trying to establish a triangle

$$\left(\bar{A}^{\otimes *}/(1-t), b\right) \longleftarrow \left(A \otimes \bar{A}^{\otimes (*-1)}, b\right) \longleftarrow \left((\bar{A}^{\otimes *})^t, b'\right)[\ ]$$

which generalizes what I know works for augmented rings. So we have the following exact sequence of complexes:

$$0 \longleftarrow \bar{A}^{\otimes p}/(1-t) \longleftarrow A \otimes \bar{A}^{\otimes (p-1)} \longleftarrow 1 \otimes \bar{A}^{\otimes (p-1)} \oplus (1-t)\bar{A}^{\otimes p} \longleftarrow 0$$

The notation is tricky: One must embed $\bar{A}^{\otimes p}$ into $A \otimes \bar{A}^{\otimes (p-1)}$ by choosing an embedding of $\bar{A}$ into A. Strictly speaking the Kernel has a filtration:

$$0 \longleftarrow (1-t)\bar{A}^{\otimes p} \longleftarrow \text{Ker} \longleftarrow 1 \otimes \bar{A}^{\otimes (p-1)} \longleftarrow 0$$

but these are not complexes. In fact

$$b[1, a_2, \dots, a_p] = [a_2, \dots, a_p] + \sum_{2}^{p-1} (-1)^{i-1} [1, a_i a_{i+1}, \dots a_p] + (-1)^p [a_p, a_2, \dots, a_{p-1}]$$

so within Ker the boundary [redacted] maps $1 \otimes \bar{A}^{\otimes(p-1)}$ onto the cokernel $(1-t) A^{\otimes(p-1)}$.

Now there are two possibilities for achieving the goal of showing Ker is quis $((\bar{A}^{\otimes *})^t, b')$. One is to embed this subcomplex in Ker and show contractibility of the ~~quotient.~~ The other might be to enlarge Ker to a contractible complex $\cong 1 \otimes \bar{A}^{\otimes(p-1)} + \bar{A}^{\otimes p}$ mapping onto Ker with kernel $\simeq (\bar{A}^{\otimes p})^t$.

June 3, 1983

I have been approaching cyclic homology from the viewpoint of Hochschild homology, but Connes and Karoubi arrive at it from the viewpoint of the ~~[illegible]~~ commutator quotient of the non-commutative de Rham complex

(1) $$A \xrightarrow{d} A\otimes\bar{A} \xrightarrow{d} A\otimes\bar{A}\otimes\bar{A} \xrightarrow{d}$$

$$d(a_1 da_2 \cdots da_k) = da_1 da_2 \cdots da_k$$

What would be nice is ~~to~~ show that d is equivalent to Connes B operator. In other words I would like to begin with either the NCDR cx or the Amitsur complex

(2) $$A \longrightarrow A\otimes A \longrightarrow A\otimes A\otimes A \longrightarrow \cdots\cdots$$

and then regard the Hochschild boundary as something in the same spirit as the Alexander-Spanier completion.

Take $A = k + I$, $I^2 = 0$. Then the normalized Hochschild complex for A is essentially the first 2 columns of the double complex

$$\begin{tikzcd}
I^{\otimes 3} \arrow[d, "b"] & I^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "-b'"] & I^{\otimes 3} \arrow[l, dotted, "N"'] \arrow[d, "b"] & I^{\otimes 3} \arrow[l] \arrow[d, "b"] \\
I^{\otimes 2} \arrow[d, "b"] & I^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & I^{\otimes 2} \arrow[l, dotted, "N"'] \arrow[d, "b"] & I^{\otimes 2} \arrow[l] \arrow[d, "b"] \\
I & I \arrow[l, "1-t"'] & I \arrow[l, dotted, "N"'] & I \arrow[l]
\end{tikzcd}$$

N induces the B-map. The only hope I have is that because of the $1-t$ the model for the Hochschild complex is somewhat flabby, i.e. differentials are $\neq 0$. So I might be able to deform N to the d in (1). But look at I^2 in the 3rd column. It consists of cycles and can only map to $I^{\otimes 3} + I^{\otimes 2}$, and as there is no natural map $I^{\otimes 2} \longrightarrow I^{\otimes 3}$ we must have a map $I^{\otimes 2}$ to $I^{\otimes 2}$. Similarly the $I^{\otimes 2}$ in the 2nd column contains no boundaries, so by these arguments one has no possibilities of deforming B.

The only other possibility is the identification of the part $A \otimes \bar{A}^{\otimes p}$ of the Hochschild and NCDR complexes.

My hope was to look at

$$\begin{tikzcd}
& & \\
A\otimes\bar{A}^{\otimes 2} \arrow[d, "b"] & A\otimes\bar{A}^{\otimes 2} \arrow[d, "b"] \arrow[ul, "B"'] & A\otimes\bar{A}^{\otimes 2} \arrow[d] \\
A\otimes\bar{A} \arrow[d, "b"] & A\otimes\bar{A} \arrow[d, "b"] \arrow[ul, "B"'] & A\otimes\bar{A} \arrow[d] \arrow[ul, "B"'] \\
A & A \arrow[ul, "B"'] & A \arrow[ul, "B"']
\end{tikzcd}$$

and to identify B with the NCDR differential. If this were possible, then B would be exact and so in the augmented case one would have

$$\begin{tikzcd}
I^{\otimes 3} & \oplus & I^{\otimes 2} & \simeq & A\otimes\bar{A}^{2} \\
I^{\otimes 2} \arrow[urr, "N"] & \oplus & I & \simeq & A\otimes\bar{A} \arrow[u, "d"'] \\
I \arrow[urr, "1"] & \oplus & k & \simeq & A \arrow[u, "d"']
\end{tikzcd}$$

.

But this is clearly impossible as N is not an isom.

Next let ~~me~~ try to prove the conjecture about the long exact sequence relating $HC(k) \to HC(A) \longrightarrow HC(A|k)$. I start with the double complex for A and then introduce degenerate subcomplexes.

$$\begin{tikzcd}
0 & A^{\otimes p}/(1-t) \arrow[l] & A^{\otimes p} \arrow[l] \arrow[d] & A^{\otimes p} \arrow[l, "1-t"'] \arrow[d] & A^{\otimes p} \arrow[l, "N"'] \\
0 & \bar{A}^{\otimes p}/(1-t) \arrow[l] & A\otimes\bar{A}^{\otimes p-1} \arrow[l] & A\otimes\bar{A}^{\otimes p-2}\otimes A \arrow[l, dashed, "1-t"'] &
\end{tikzcd}$$

Recall that $(A^{\otimes *}, b')$ contains a degenerate (acyclic) subcomplex spanned by $[a_1, \ldots, a_p]$ with one of the inside a_i $1<i<p$ equal to 1. And $(A^{\otimes *}, b)$ contains an acyclic subcomplex spanned by the $[a_1, \ldots, a_p]$ with some a_i, $1<i\leq p$ equal to 1. Now I want to enlarge the degenerate subcomplex of $(A^{\otimes *}, b')$ by adding the image under N of the degenerate complex of $(A^{\otimes *}, b)$. This will give me a further quotient complex of $(A^{\otimes *}, b')$ which I will denote by $(\widetilde{A^{\otimes *}}, b')$. Then

$$\widetilde{A^{\otimes p}} = A\otimes\bar{A}^{\otimes (p-2)}\otimes A \Big/ \left\{ N[a_1, \ldots, a_p] \,\Big|\, \begin{matrix}\text{some } a_i = 1\\ i>1\end{matrix} \right\}$$

So what I end up with is a double complex, a normalized quotient of the double complex for A, which looks as follows

$$A\otimes\bar{A}^{\otimes (p-1)} \xleftarrow{1-t} \widetilde{A^{\otimes p}} \xleftarrow{N} A\otimes\bar{A}^{\otimes (p-1)} \longleftarrow$$

The main thing to check is that $(\widetilde{A^{\otimes p}}, b')$ is acyclic except for a piece at the bottom which should contribute the $HC(k)$.

Let's take the standard contracting homotopy for $(A^{\otimes *}, b')$, namely

$$\xi[a_1, \ldots, a_p] = [1, a_1, \ldots, a_p].$$

Then let's see if it passes to $\widetilde{A^{\otimes *}}$.

First of all if one of the intermediate a_i in $[a_1, \ldots, a_p]$ is one, ~~then~~ this holds also for $[1, a_1, \ldots, a_p]$. Next consider

$$[1, a_1, \ldots, a_{p-1}] + (-1)^{p-1}[a_1, \ldots, a_{p-1}, 1]$$

which gets carried by ξ into

$$[1, 1, a_1, \ldots, a_{p-1}] + (-1)^{p-1}[1, a_1, \ldots, a_{p-1}, 1].$$

Modulo things with intermediate 1's this is congruent to

$$(-1)^{p-1}\Big\{\underbrace{[1, a_1, \ldots, a_{p-1}, 1]}_{p+1} + (-1)^p[a_1, \ldots, a_{p-1}, 1, 1]\Big\}$$

which is again in the subcomplex we divide out to get $\widetilde{A^{\otimes *}}$.

This ξ passes to $\widetilde{A^{\otimes *}}$ and shows this complex is contractible. Except something should go wrong in low degrees.

$$\begin{tikzcd}
\{[1,a,b]+[a,b,1]\} \arrow[r] & A\otimes\bar{A}\otimes A \arrow[r, two heads] \arrow[d, "b'"] & \widetilde{A^{\otimes 3}} \arrow[d] \\
\{[1,a]-[a,1]\} \arrow[r] & A\otimes A \arrow[r] \arrow[d, "b'"] & \widetilde{A^{\otimes 2}} \arrow[d] \\
 & A \arrow[r, phantom, "="] & A
\end{tikzcd}$$

We see this works down to the bottom.

$$\begin{tikzcd}
A\otimes\bar{A}^{\otimes 2} \arrow[d, "b"] & \widetilde{A^{\otimes 3}} \arrow[l, "1-t"'] \arrow[d, "-b'"] \\
A\otimes\bar{A} \arrow[d] & \widetilde{A^{\otimes 2}} \arrow[l, "1-t"'] \arrow[d, "-b'"] \\
\boxed{A} & A \arrow[l, "1-t"']
\end{tikzcd}$$

should be an $\bar{A}$ if we want to get $\bar{A}^{\otimes p}/(1-t)$.

The problem will now show up when we try to [illegible] prove that the normalized quotient of the double complex for A is a resolution of $\bar{A}^{\otimes *}/(1-t)$. It appears that $\widetilde{A}^{\otimes 2}$ as defined is 1-copy of k too big. Use splitting $A = k \oplus \bar{A}$. Then

$$A^{\otimes 2} = \bar{A} \otimes \bar{A} + 1 \otimes A + A \otimes 1 + (1 \otimes 1)k$$

and the subspace to be divided out by consists of

$$\{ [1,a] - [a,1] \}$$

that is the diagonal part of $1 \otimes A + A \otimes 1$. Note the $(1 \otimes 1) \cdot k$ remains. However for

$$A \otimes \bar{A} \otimes A = \bar{A}^{\otimes 3} + 1 \otimes \bar{A} \otimes \bar{A} + \bar{A} \otimes \bar{A} \otimes 1 + 1 \otimes \bar{A} \otimes 1$$

and $[1,a,b] + [a,b,1]$, so we get the last factor by taking $b = 1$. So it seems that we want to modify the definition of $\widetilde{A}^{\otimes p}$ for $p = 1, 2$ as follows

$$\widetilde{A}^{\otimes 2} = A^{\otimes 2} / \{ [1,a] - [a,1] , 1 \otimes 1 \}$$

$$\tilde{A} = A/k .$$

Then it is still true that the ~ complex is acyclic. The rows should be a resolution of $\bar{A}^{\otimes *}/(1-t)$ in char. 0. This has nothing to do with the mult. on A, so ~~[illegible] to check in finite dimensions, and then it has to work by the fact that we have a [illegible]~~ it is enough to check this when A is augmented where it should be clear.

Summarizing we should get the following kind of double complex.

$$\begin{tikzcd}
A\otimes\bar{A}^{\otimes 2} \arrow[d, "b"] & \widetilde{A^{\otimes 3}} \arrow[l] \arrow[d] & \arrow[l] \\
A\otimes\bar{A} \arrow[d, "b"] & \widetilde{A^{\otimes 2}} \arrow[l] \arrow[d] & \arrow[l] \\
\bar{A} & \tilde{A} \arrow[l] & \arrow[l]
\end{tikzcd}$$

where $\widetilde{A^{\otimes p}}$ is isomorphic $A\otimes\bar{A}^{\otimes(p-1)}$ up to a filtration:

$$0 \longrightarrow 1\otimes\bar{A}^{\otimes(p-1)} \longrightarrow \widetilde{A^{\otimes p}} \longrightarrow \bar{A}^{\otimes p} \longrightarrow 0$$

and the $\widetilde{A^{\otimes *}}$ complex is acyclic whereas the $(A\otimes\bar{A}^{\otimes(p-1)}, b)$ complex is the normalized Hochschild complex, except for the $\bar{A}$ at the bottom.

In char 0 this resolves the complex $(\bar{A}^{\otimes p}/(1-t), b)$.

In general if the above is denoted $C(A,k)$, then we have maps

$$C(k) \longrightarrow C(A) \longrightarrow C(A,k)$$

which we can see will give a triangle.

Next Atiyah tells me that Connes has defined his cohomology using the functor $n \longmapsto A^{\otimes n}$ on cyclically ordered finite sets. The cohomology is $\mathrm{Ext}(A^{\otimes *}, k^{\otimes *})$ over this category. More generally $\mathrm{Ext}(A^{\otimes *}, B^{\otimes *})$ is to be a bivariant Kasparov version.

First step in the theory is to compute the homotopy type of the category of cyclically-ordered finite sets. Supposedly this turns out to be $BS^1 = \mathbb{C}P^\infty$.

There will be one object $[n] = \{1,2,\dots,n\}$ with the usual cyclic ordering for each $n \geq 1$. To this object we associate the vector space $A^{\otimes n}$. Using only the associative multiplication in A we have face maps $A^{\otimes n} \longrightarrow A^{\otimes (n-1)}$ which multiply consecutive factors. So there are a total of 3-maps

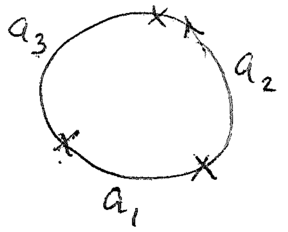
$$A^{\otimes 3} \longrightarrow A$$

$$[a_1, a_2, a_3] \longmapsto \begin{array}{l} a_1 a_2 a_3 \\ a_2 a_3 a_1 \\ a_3 a_1 a_2 \end{array}$$

So it seems most natural to have face morphisms correspond to embeddings going the other way:

$$\{1, 2, 3\} \longleftarrow \{1\}$$

It is clear that 1, 2, 3 should be viewed as the spaces between the $[a_1, a_2, a_3]$. Try to use the picture

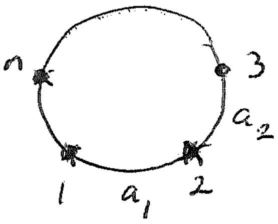


and then faces involve deleting the vertices.

Let's give this category the notation $\mathcal{C}$ and filter it by $F_n\mathcal{C}$ = full subcategory of $[m]$ with $m < n$.

June 4, 1983

Introduce the cyclic category consisting of the sets $[n] = \{1, \dots, n\}$ with their natural cyclic ordering and with injective maps for morphisms. Then $[n] \mapsto A^{\otimes n}$ is a contravariant functor on this category when A is an associative k-algebra (possible non-unital). One should think of the elements of $[n]$ as being the $\otimes$-signs in $A^{\otimes n}$. Thus $[a_1, \dots, a_n]$ can be written



and the face operators delete the vertices.

Let us denote by $\mathcal{C}$ the cyclic category, and then we can filter it by $F_n\mathcal{C}$ = full subcategory consisting of $[m]$ with $m \leq n$. Then one has cocartesian setup:

$$\begin{tikzcd}
(F_{n-1}\mathcal{C}/[n], \mathrm{Aut}[n]) \arrow[r] \arrow[d] & ([n], \mathrm{Aut}[n]) \arrow[d] \\
F_{n-1}\mathcal{C} \arrow[r, phantom, "\subset"] & F_n\mathcal{C}.
\end{tikzcd}$$

Now $F_{n-1}\mathcal{C}/[n]$ = poset of subsets S of $[n]$ with $S < [n]$, so it has the homotopy type of S^{n-2}. So it would seem that we have a long exact sequence

$$\longrightarrow H_i(F_{n-1}\mathcal{C}) \longrightarrow H_i(F_n\mathcal{C}) \longrightarrow H_i(\mathbb{Z}/n, \mathbb{Z}^{sgn}[n-1]) \rightarrow$$

which is going to be very messy. Rationally

$$H_i(\mathbb{Z}/n, \mathbb{Z}^{sgn}[n-1]) = \begin{cases} \mathbb{Q} & i = n-1, \ n \text{ odd} \\ 0 & \text{otherwise,} \end{cases}$$

so that $$H_i(\mathcal{C}) = \begin{cases} \mathbb{Q} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

Thus this filtration of C corresponds to filtering the double complex horizontally and is not the one I want to use. I want the horizontal filtration which begins with the Hochschild homology.

The double complex makes sense for any contravariant functor from $\mathcal{C}$ to Ab. It gives rise to a long exact sequence on homology, hence if we can prove ~~co~~effaceability, it follows that the double complex computes $L_* \varinjlim_{\mathcal{C}} (F) = H_*(\mathcal{C};F)$. The first column should perhaps be formed by [redacted] what computes the homology of the functor restricted to finite non-empty linear ordered sets and injective maps. This is the category Δ which leads to semi-simplicial gadgets.

Here's how we can construct a map $BC \longrightarrow BS^1$. To each $[n]$ in C [redacted] we associate the space of all embeddings of $[n]$ into the circle which are compatible with the cyclic ordering. This is a torsor- NO ?

The idea was to construct a circle bundle over C. To each $[n]$ I want to associate a space L_n having the homotopy type of a circle and to each $[n] \to [m]$ a map $L_m \longrightarrow L_n$ which is a homotopy equivalence and functorial. Thus if L_n = embeddings $[n] \subset S^1$ compatible with the cyclic ordering, then clearly L_n fibres over S^1 with fibre an open simplex, so L_n has the homotopy type of S^1, and is functorial, etc.

Should this be a homotopy equivalence of BC with BS^1? Given a circle S^1 we want to look at the category formed of objects $[n]$ of C [redacted] and embeddings $[n] \longrightarrow S^1$. This category is a poset it would seem. C is equivalent to finite cyclically ordered sets and embeddings, so the category of embeddings of objects of C into S^1 should be equivalent to the poset of non-empty subsets of S^1, which is a contractible category.

This argument allows one to work with the bigger C with degeneracies, i.e. all finite non-empty cyclically ordered sets and maps. We can assign to S all maps $S \longrightarrow S^1$ compatible with the cyclic ordering; again for fixed S, we can pick a base point of S and the image of this fibres the space of maps $S \to S^1$ over S^1 with fibre a simplex. On the other hand if we fix S^1 then the fibred category over C with fibre the set of $S \longrightarrow S^1$ is homotopy equivalent to non-empty finite subsets of S^1 so is contractible.

Let's go back to just the embeddings. The next idea will be to derive the long exact sequence. The main point will be to bring in the Hochschild part which is the total space of the circle bundle. I was trying to realize this by the linearly ordered category Δ.

Let's study Δ: The objects are non-empty linearly ordered finite sets and all injective monotone maps. Then given $M: \Delta^{op} \longrightarrow Ab$ we have a standard complex

$$(*) \qquad \xrightarrow{d} M_3 \xrightarrow{d} M_2 \xrightarrow{d} M_1$$

where $d = \sum_{i=1}^{p} (-1)^{i-1} d_i$: $M_p \longrightarrow M_{p-1}$ where d_i corresponds to the embedding of $\{1, .., p-1\}$ into $\{1, .., p\}$ omitting i.

I want to check that this standard complex gives the homology $H_*(\Delta, M)$ with the obvious degree shift.

Two approaches: One is to check the effaceability of the homology computed from (*). So one takes the functors

$$[n] \longrightarrow \bigoplus_{[n]\to[p]} \mathbb{Z}$$

for each $p \geq 1$ and checks that * gives an exact sequence. It is pretty clear that this should work because

$$\bigoplus_{[n]\to[p]} \mathbb{Z} = \bigoplus_{1\leq i_1<\cdots<i_n\leq p} \mathbb{Z} = \Lambda^n(\mathbb{Z}e_1\oplus\cdots\oplus\mathbb{Z}e_p)$$

and the differential d is clearly interior product with the linear function $\lambda(e_i) = 1$. This will be acyclic because of the identity

$$e(e_i)\, i(\lambda) + i(\lambda)\, e(e_i) = \lambda(e_i) = 1.$$

for any i.

2nd approach: Use the obvious filtration of the category Δ by $F_n\Delta$. Then

$$\begin{tikzcd} sk_{n-1}[n] \arrow[r] \arrow[d] & {[n]} \arrow[d] \\ F_{n-1}\Delta \arrow[r] & F_n\Delta \end{tikzcd}$$

should be cocartesian. This should give a spectral sequence

$$E^1_{pq} = \underbrace{H_{p+q}(F_p\Delta, F_{p-1}\Delta; M)}_{M_p \text{ in degree } p-1} \Longrightarrow H_*(\Delta, M)$$

which should degenerate at the E^1 level.

Why did I do this calculation? Because I know that in the double complex the Hochschild part is not just the first column, but the first two columns.

So therefore I can conclude that the ~~total~~ total space of the circle bundle over C is not Δ, but probably Δ together with something maybe with an initial object.

Think in terms of the circle bundle. Recall that a point of this bundle is an object $[n]$ of C together with an embedding $[n] \hookrightarrow S^1$ preserving the cyclic ordering, and a point ξ on the circles. There are two different kinds of objects:

i) $\xi \notin$ Image of $[n]$

ii) $\xi \in$ Image of $[n]$.

It is clear that the objects of type i) form a subcategory. Moreover up to homotopy, the effect of choosing ξ outside of $[n]$ is to break the cyclic ordering and make it into a linearly ordered set. So we should get the category Δ.

ii) will give a kind of open category. Because if we have $[n] \hookrightarrow [m]$ and $\xi \in [n]$, then we get $\xi \in [m]$.

So what I will try to do is to construct a category D over C which will be a fibred category with discrete fibres. It will assign to $[n]$ the set of vertices disjoint union the set of edges. Then if we have an embedding $[m] \hookrightarrow [n]$ it is clear what to do. Edges move backwards but a vertex becomes an edge when it is not in the image.

What are the isom classes of objects? For each $[n]$ there are two possibilities, depending on whether an edge or a vertex is chosen.

Review: I form the category C consisting of cyclic sets and their embeddings. The problem is to show that the double complex I can associate to a functor $M: C^{op} \longrightarrow ab$ computes $H_*(C, M)$. One possible way to do this is to prove the homology calculated via the double complex is effaceable.

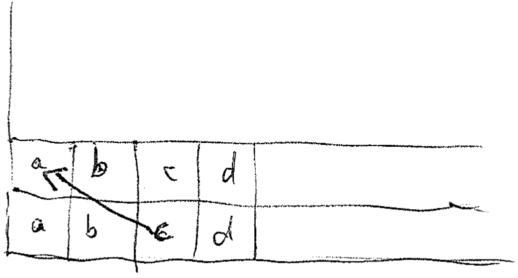
On the other hand we want to establish the long exact sequence, and we can perhaps do this by exhibiting a circle bundle which will take the form of a functor $f: C' \longrightarrow C$. We have $M: C^{op} \longrightarrow ab$, and we have a spectral sequence realized on the derived cat. level by

$$\mathbb{L}\varinjlim_{C} \mathbb{L}f_! \simeq \mathbb{L}\varinjlim_{C'}$$

This gives us

$$H_p\big(C, [n] \longmapsto \underbrace{L_q f_!(f^*M)[n]}_{H_q([n]\backslash f,\, f^*M)}\big) \Longrightarrow H_{p+q}(C', f^*M)$$

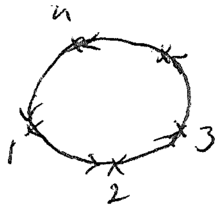
The hope would be that because the fibres of f are circles we get a spectral sequence



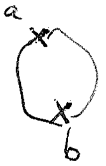
leading to the desired long exact sequence.

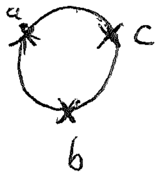
To carry this out I need to define C' carefully. I think the idea is to define C' to be the fibred category over C where the fibre over $[n]$ is the category given by the oriented graph 

Let's try to make this clearer. The objects of C' over [n] are the vertices and edges of the graph



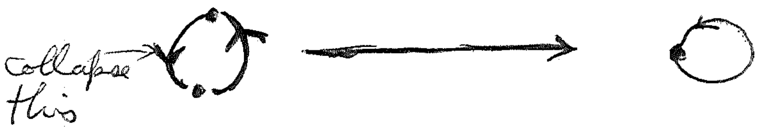
and the morphisms over [n] are maps between ~~an~~ edge and its vertices [redacted]. Now when I have a map $[m] \hookrightarrow [n]$, then it is clear how to construct a functor going the other way.





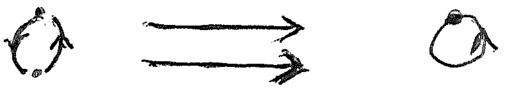
all these, the vertex c and ~~its~~ edges adjacent go to the edge a b

I think this works for all maps. Try



Nope it doesn't work.

One beautiful thing Connes does is to point out the duality of the category if degeneracies are allowed. [redacted] I think one must carefully use the graphs in order to define morphisms. For example there are two maps



which one won't see if one looks at the maps on vertices. Similarly there are two maps

which one can see on vertices but not on edges. There are two [redacted] maps. [redacted]

June 5, 1983

Hsiang & Staffeldt seem to prove that for a tensor algebra $A = T(V)$, the relative cyclic homology $HC_p(T(V), k)$ is zero for $p \neq 1$. For $p \neq 1$ it is $\bar{A}/[A,A] = \bar{T}(V)/[V, T(V)] = \bigoplus_{n \geq 1} V^{\otimes n}/(1-t)$ where t acts without the sign.

~~Lets check the last formulas.~~

Let's begin by calculating the Hochschild homology of $A = T(V)$. Suppose V has basis $x_1, \ldots, x_d$. Consider

$$(*) \qquad \begin{tikzcd} 0 \arrow[r] & I \arrow[r] & A \otimes A \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \qquad a \otimes b \longmapsto ab$$

Claim that for any A-bimodule M we have

$$\mathrm{Hom}_{A \otimes A^{op}}(I, M) = \mathrm{Der}(A, M)$$

where by a derivation on means a k-linear map $D: A \to M$ such that $D(ab) = a(Db) + (Da)b$. (This $\Rightarrow$ $D1 = 0$.)

To prove this I need first ~~to~~ exhibit a canonical derivation $d: A \to I$, namely

$$d(a) = a \otimes 1 - 1 \otimes a$$

Check:
$$\left.\begin{aligned} d(ab) &= ab \otimes 1 - 1 \otimes ab \\ a\,db + (da)b &= (a \otimes 1)(b \otimes 1 - 1 \otimes b) + (a \otimes 1 - 1 \otimes a)(1 \otimes b) \end{aligned}\right\}.$$

~~[illegible]~~

Next split the sequence $(*)$ using the section $a \longmapsto a \otimes 1$. This means we have the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & A \otimes A \arrow[r] & I \arrow[r] & 0 \end{tikzcd}$$
$$a \longmapsto a \otimes 1. \qquad a \otimes b \longmapsto a \otimes b - ab \otimes 1 = a\,db$$

Now suppose given a derivation $D: A \to M$ we can define $\varphi: I \to M$ by $\varphi(a\,db) = a\,Db$. The fact that D is linear over k and $D1 = 0$ implies that D is well-defined. It is clearly a left A-module map. Finally

$$\left.\begin{aligned} \varphi(a\,db\,x) &= \varphi(a\,d(bx) - ab\,dx) \\ &= a\,D(bx) - ab\,Dx \\ \varphi(a\,db)x &= (aDb)x \end{aligned}\right\}$$

it is a right A-module map.

The reason for showing that I represents derivations is that when $A = T(V)$ is free we know that

$$\mathrm{Der}(A, M) = \mathrm{Hom}_k(V, M)$$

that is any derivation D is completely determined by $Dx_i \in M$ and these can be assigned arbitrarily. It follows that

$$\begin{aligned} A\otimes V\otimes A &\xrightarrow{\sim} I \\ a\otimes x\otimes b &\longmapsto a\,dx\,b \end{aligned}$$

and in particular that I is a free $A\otimes A^{op}$-module. Thus $(*)$ can be used to calculate Hochschild homology. We have

$$H_p(A, A) = 0 \qquad p \geq 2.$$

Let's compute the effect of tensoring with A over $A\otimes A^{op}$

$$\begin{tikzcd}
 & & a\otimes x\otimes b \arrow[r, mapsto] \arrow[ddl, mapsto] & ax\otimes b - a\otimes xb \arrow[ddr, mapsto] & \\
A\otimes_{(A\otimes A^{op})}: & & A\otimes V\otimes A \arrow[r] \arrow[dd] & A\otimes A \arrow[dd] & \\
 & ba\otimes x & & & bax - xba \\
 & & A\otimes V \arrow[r, "{[\ ,\ ]}"] & A &
\end{tikzcd}$$

Be careful:

$$\begin{tikzcd}
\alpha\otimes(a\otimes x\otimes b) \arrow[r, mapsto] & b\alpha a\otimes x \\
A\otimes_{A\otimes A^{op}}(A\otimes V\otimes A) \arrow[r, "\sim"] & A\otimes V \\
1\otimes(a\otimes x\otimes 1) & a\otimes x \arrow[l, mapsto]
\end{tikzcd}$$

Therefore we obtain an exact sequence

$$0 \longrightarrow H_1(A,A) \longrightarrow A\otimes V \xrightarrow{[\,,\,]} A \longrightarrow H_0(A,A) \longrightarrow 0.$$

Let's look at this bracket map in a given degree n.

$$V^{\otimes(n-1)} \otimes V \longrightarrow V^{\otimes n}$$

$$[x_1, \ldots, x_{n-1}] \otimes x_n \longmapsto [x_1, \ldots, x_n] - [x_n, x_1, \ldots, x_{n-1}]$$

So it is $1-t : V^{\otimes n} \longrightarrow V^{\otimes n}$ where t is the forward shift without the sign. In char. 0 I know that the kernel and cokernel are isomorphic, so it is possible for Connes' B operator to give this isom.

Look at the double complex for an augmented ring:

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
\bar{A}^{\otimes 2} \arrow[d, "b"'] & \bar{A}^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "-b'"] & \bar{A}^{\otimes 2} \arrow[l] \arrow[d] & {} \arrow[l] \\
\bar{A} & \bar{A} \arrow[l, "(1-t)=0"] & \bar{A} \arrow[l, "N=1"] & {} \arrow[l]
\end{tikzcd}$$

Then the B operator at the bottom:

$$\bar{A} \xrightarrow{B} A\otimes\bar{A}$$

degree 1 in the reduced Hochschild complex

is

$$a \longmapsto [1, a] = 1\otimes a$$

So we have to compute for $A = T(V)$ the class of $[1, a]$ in $H_1(A,A)$.

$$\begin{tikzcd}
 & & {[1,a]} \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \\
 & A\otimes\bar{A}^{\otimes 2} \arrow[r, "b"] & A\otimes\bar{A} \arrow[r, "b = \text{bracket}"] \arrow[d, phantom, "\cup"] & \bar{A} \\
 & & A\otimes V \arrow[ur, "\text{bracket}"'] & \\
 & H_1(A,A) \arrow[ur] & & \\
0 \arrow[ur] & & &
\end{tikzcd}$$

Recall $$b[x,y,z] = [xy,z] - [x,yz] + [zx,y]$$

so that $$[x,yz] \equiv [xy,z] + [zx,y] \qquad \text{mod Im } b.$$

Then $$[1, x_1 \cdots x_n] \equiv \underbrace{[x_1, x_2 \cdots x_n]}_{\underbrace{[x_1x_2, x_3 \cdots x_n]}_{[x_1x_2x_3, x_4 \cdots x_n] + [x_4 \cdots x_n x_1 x_2, x_3]} + [x_3 \cdots x_n x_1, x_2]} + [x_2 \cdots x_n, x_1]$$

$$\equiv \sum_{i=1}^{n} [x_{i+1} \cdots x_n x_1 \cdots x_{i-1}, x_i].$$

Consequently we conclude that if we identify
$$(A \otimes V)_n = V^{\otimes(n-1)} \otimes V = V^{\otimes n}.$$

Then
$$B : \underset{\substack{\| \\ V^{\otimes n}/(1-t)}}{(\bar{A}/[\bar{A},\bar{A}])_n} \longrightarrow \underset{\substack{\| \\ (V^{\otimes n})^t}}{(H_1(A,A))_n}$$

is just given by N = sum over cyclic permutations.

Formula for $HC(\bar{T}(V))$:

$$HC_1(\bar{T}(V)) = \bigoplus_{n \geq 1} V^{\otimes n}/(1-t)$$

$$HC_{ev}(\qquad) = \bigoplus_{n \geq 1} \frac{(V^{\otimes n})^t}{N(V^{\otimes n})}$$

$$HC_{odd}(\qquad) = \bigoplus_{n \geq 1} \frac{\text{Ker } N \text{ on } V^{\otimes n}}{\text{Im } (1-t) \text{ on } V^{\otimes n}}$$

Next project: Comparison with DR cohomology. The problem is to relate the HKR formula

$$H_p(A,A) \cong \Omega^p_A$$

and Connes B operator. Connes B operator induces a map on Hochschild homology raising degree by 1, hence $B : \Omega^p \longrightarrow \Omega^{p+1}$ should be a multiple of d, and the point will be to compute the exact multiple.

On the other hand I really want to construct a quis between the cyclic double complex for A and the double complex

$$\begin{tikzcd}
{} & {} & {} & {} \\
\Omega^2 & \Omega^2 \arrow[ul] & {} & {} \\
\Omega^1 & \Omega^1 \arrow[ul, "d"] & \Omega^1 \arrow[ul] & {} \\
\Omega^0 & \Omega^0 \arrow[ul, "d"] & \Omega^0 \arrow[ul] & {} \arrow[ul]
\end{tikzcd}$$

So how can I ~~[illegible]~~ accomplish all of this? It is first necessary to see if the HKR isomorphism can be refined to a quis between $(A^{\otimes *}, b)$ and Ω^* with 0 differentials.

I know that $(A^{\otimes *})$ is a simplicial augmented commutative A-algebra, hence $(A^{\otimes *}, b)$ is a DG anti-commutative algebra. The product on the Hochschild homology should coincide with exterior product for general reasons. This necessitates describing the ring structure on $\mathrm{Tor}_*^{A\otimes A}(A,A)$, maybe.

I do have to go over the HKR thm. The idea is that if $I = \mathrm{Ker}\{A\otimes A \longrightarrow A\}$, then I is generated by a regular sequence, at least locally over $B = A\otimes A$. If $x_1, \dots, x_d$ is a regular sequence in B generating I, then we can compute the Tor using the Koszul complex

$$K(\underline{x}, B) = \Lambda_B^*(B^d)$$

which is a DG anti-comm. B-algebra. Then we get the isomorphism

$$\mathrm{Tor}_*^B(B/I, B/I) = \Lambda^*(B/I)^d = \Lambda^*_{B/I}(I/I^2)$$

So in order to use this theorem, it is necessary for me to relate $K(\underline{x}, B)$ with $(A^{\otimes *}, b')$. Perhaps it is possible to construct a ring map

$$(*) \qquad K(\underline{x}, B) \longrightarrow (A^{\otimes *}, b').$$

compatible with the augmentations to A. Then it should be possible to show the induced map on homology is an isom. after tensoring with A.

It seems clear that $(*)$ exists because $(A^{\otimes *}, b')$ is a DG anti-commutative ring.

$$\begin{tikzcd} A\otimes A^{\otimes 2}\otimes A \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & A\otimes A\otimes A \arrow[l, bend right, shift right=2] \arrow[l, bend right, shift right=4] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A\otimes A \arrow[l, bend right] \end{tikzcd}$$

simplicial commutative ring $\Rightarrow$ it becomes a DG ring under shuffle product. Now I need to find the elements in $A\otimes A\otimes A$ which are cycles and which will become the d-generators in I/I^2

$$\begin{tikzcd} A\otimes A \arrow[r, shift left] \arrow[r, shift right] & A \\ a, b \arrow[r, mapsto] & \begin{matrix} ab \\ ba \end{matrix} \end{tikzcd}$$

I choose maybe to work in the normalized complex, and take elements $x_1, \ldots, x_d$ in A and consider the elements $x_i \otimes 1 - 1 \otimes x_i$ in $A^{\otimes 2}$. Lift these back up to $A\otimes A\otimes A$ to $x_i \otimes 1 \otimes 1 - 1 \otimes$?

I am lacking some insight. Perhaps it would be useful to think in terms of a polynomial ring. It seems that the Hochschild $(A^{\otimes *}, b)$

is a free semi-simplicial commutative A-algebra with d generators of degree 1, namely $x_i \otimes 1 - 1 \otimes x_i$. The [redacted] idea will now be that if I take the shuffle product of p of these I get an alternating sum over the symmetric group Σ_p

$$(x_1 \otimes 1 \otimes 1 \otimes \cdots \; - 1 \otimes x_1 \otimes 1 \cdots)$$
$$\times \, (1 \otimes x_2 \otimes 1 \otimes \cdots \; - 1 \otimes 1 \otimes x_2 \cdots)$$

The point is that a form $a\,dx_1 \ldots dx_{p-1}$ in Ω^{p-1} is represented in the complex [redacted] $(A^{\otimes p}, b)$ by a sum over Σ_{p-1}. When I apply B I sum in addition over $\mathbb{Z}/p\mathbb{Z}$, thereby getting a sum over Σ_p. So it should be true that B yields d. Let's try to get this all straight.

The hope is as follows: I can define a map

$$\Omega^{p-1} \longrightarrow A^{\otimes p}$$

using shuffle product. (This should be OKAY. All you have to do is to pick a section of $I \twoheadrightarrow I/I^2 = \Omega^1$.) Then I hope that the image lands in cycles and [redacted] gives me the desired quis. Then I hope that B commutes with d.

Let us suppose to simplify that $A = S(V)$ is a polynomial ring, and consider the Hochschild algebra

$$A \otimes A \otimes A \qquad A \otimes A \qquad A$$

which is a simplicial commutative algebra over A considered as operating by multiplying on the left. I want to take out this constant simplicial ring, and effectively think of [redacted] having a simplicial polynomial ring over A. Then I take new generators $x\otimes 1 - 1\otimes x \in A^{\otimes 2}$ with $x \in V$. Then

$$s_0(x\otimes 1 - 1\otimes x) = x\otimes 1\otimes 1 - 1\otimes 1\otimes x$$
$$s_1(x\otimes 1 - 1\otimes x) = x\otimes 1\otimes 1 - 1\otimes x\otimes 1$$

give generators for $A^{\otimes 3}$ over A etc.

It seems that what I obtain is analogous to a "suspension" of a commutative k-algebra B, namely

$$B\otimes B\otimes B \qquad B\otimes B \qquad B \qquad k$$

with suitable face and degeneracy operators. Here $d_0, d_1 = 0$ on $\bar{B}$ and

$$d_0 s_0(b) = d_1 s_0(b) = b. \qquad ?$$

My problem is to take the Hochschild simplicial algebra

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A\otimes A \arrow[l, bend right, shift right=3] \arrow[l, bend right=50, shift right=5] \arrow[r, shift left] \arrow[r, shift right] & A \arrow[l, bend right] \end{tikzcd}$$

and make it ~~into~~ a differential anti-commutative algebra.

$$d_0[a_0, a_1, a_2] = [a_0a_1, a_2] \qquad s_0[a_0, a_1] = [a_0, 1, a_1]$$
$$d_1[\qquad\quad] = [a_0, a_1a_2] \qquad s_1[a_0, a_1] = [a_0, a_1, 1].$$
$$d_2[\qquad\quad] = [a_2a_0, a_1]$$

Let's try to make this work in dimension 2.

$$\begin{aligned}[][a_0, a_1][b_0, b_1] &= s_0[a_0, a_1]\, s_1[b_0, b_1] - s_1[a_0, a_1]\, s_0[b_0, b_1] \\ &= [a_0, 1, a_1][b_0, b_1, 1] - [a_0, a_1, 1][b_0, 1, b_1] \\ &= [a_0b_0, b_1, a_1] - [a_0b_0, a_1, b_1]\end{aligned}$$

or the opposite sign which would look a little better.

June 6, 1983

Cyclic and DR homology.

Start with the resolution of A over $A\otimes A$

(1) $$\xrightarrow{b'} A\otimes\bar{A}\otimes\bar{A}\otimes A \xrightarrow{b'} A\otimes\bar{A}\otimes A \xrightarrow{b'} A\otimes A$$

This is the normalization of a simplicial commutative ring, hence it is a DG anti-commutative (in the strict sense) algebra over $A\otimes A$. The product is the shuffle product

$$[a_0, a_1, \dots, a_p, a_{p+1}]\cdot[b_0, b_1, \dots, b_q, b_{q+1}]$$

$$= \sum_{\substack{p,q\\ \text{shuffles}}} \varepsilon(\mu,\nu)\ [a_0b_0, \quad a_i \quad b_j \quad , a_{p+1}b_{q+1}]$$

Better formula

$$[a, a_1, \dots, a_p, a']\cdot[b, a_{p+1}, \dots, a_{p+q}, b']$$

$$= \sum_{\sigma} (-1)^{\sigma}[ab, a_{\sigma^{-1}1}, \dots, a_{\sigma^{-1}(p+q)}, a'b']$$

where σ runs over perms of degree $p+q$ preserving the order of $\{1,\dots,p\}$ and $\{p+1,\dots,p+q\}$.

All you need to do is to check that

$$b'(\omega_1\,\omega_2) = (b'\omega_1)\,\omega_2 + (-1)^{\deg\omega_1}\omega_1(b'\omega_2)$$

$$\omega_2\omega_1 = (-1)^{pq}\omega_1\omega_2$$

$$\omega^2 = 0 \quad \text{if } \deg\omega \text{ is odd}$$

Next point: Map the Koszul complex into (1). Put $B = A\otimes A$ and regard B as an augmented A-algebra via $a \mapsto a\otimes 1$. Then

$$0 \longrightarrow I \longrightarrow B \longrightarrow A \longrightarrow 0$$

$$\begin{tikzcd} 0 \arrow[r] & I^2 \arrow[r] & I \arrow[r] & \underbrace{I/I^2}_{\text{projective over } A} \arrow[l, dashed, bend right] \arrow[r] & 0 \end{tikzcd}$$

hence $\exists$ A-splitting, the dotted arrow as above.

$$B \otimes \bigwedge_A^{*} I/I^2$$

is the Koszul complex. Assume $I/I^2 = A\bar{z}_1 + \cdots + A\bar{z}_d$ is free, where $z_i \in I$.

This is confusing and to get it straight let us assume A is a poly ring in $z_1, \cdots, z_d$. Then we know $I = \operatorname{Ker}\{A\otimes A \longrightarrow A\}$ is generated by $z_i \otimes 1 - 1\otimes z_i$. I lift these back to ~~[illegible]~~ $A\otimes\bar{A}\otimes A$ using ξ:

$$\begin{tikzcd} 1\otimes z_i \otimes 1 - \cancel{1\otimes 1\otimes z_i}. \arrow[d, "b'"] \\ z_i \otimes 1 - 1\otimes z_i \end{tikzcd}$$

Now by use of the multiplication I get a ~~[illegible]~~ alg. map

$$B \otimes \bigwedge_A^{p} A^d \longrightarrow (A\otimes\bar{A}^{\otimes p}\otimes A, b')$$

which is nothing other than

$$b\otimes b' \otimes a_1 \wedge \cdots \wedge a_p \longmapsto \sum_{\sigma\in\Sigma_p}{}' (-1)^{\sigma} [b, a_{\sigma^{-1}1}, \cdots, a_{\sigma^{-1}p}, b']$$

So we get a map

$K($ ~~[illegible]~~ $z_i\otimes 1 - 1\otimes z_i, B) \longrightarrow (1).$

and it induces an isom. on homology when tensored with A, giving

$$\Lambda^p(I/I^2) \xrightarrow{\sim} H_p(A, A)$$

So now we have to look at Connes B operator. In general we have $B : A^{\otimes p} \longrightarrow A^{\otimes(p+1)}$ is the composite $(1-t)sN$ where $s[a_1, \dots, a_p] = [1, a_1, \dots, a_p]$. [redacted] Let's check that B induces a map on normalized complexes

$$B : \ A \otimes \bar{A}^{\otimes(p-1)} \longrightarrow A \otimes \bar{A}^{\otimes(p)}$$

$$(1-t)sN[a_1, \dots, a_p] = (1-t)s \sum (-1)^{(p-1)i} [a_{i+1}, \dots, a_p, a_1, \dots, a_i]$$

$$= (1-t) \sum (-1)^{(p-1)i} [1, a_{i+1}, \dots, a_p, a_1, \dots, a_i]$$

Now $p \geq 1$, so t will move the 1 into the degenerate complex, so the above is

$$\equiv \sum (-1)^{(p-1)i} [1, a_{i+1}, \dots, a_p, a_1, \dots, a_i]$$

in $A \otimes \bar{A}^{\otimes p}$. [redacted] This will be zero if any of the $a_i = 1$.

So now compute

$$\Omega^{p-1} \xrightarrow{\alpha} A \otimes \bar{A}^{\otimes(p-1)} \xrightarrow{B} A \otimes \bar{A}^{\otimes p}$$

Let's first do things for polynomial rings. The first map α is given by choosing

$$\Omega^1 \longrightarrow A \otimes \bar{A} = I$$

which is an A-linear section of the obvious surjection $I \longrightarrow \Omega^1$. Assume that there is a basis $dx_1, \dots, dx_d$ for Ω^1 over A such that the lifting is

$$a_i dx_i \longmapsto \sum a_i \otimes \bar{x}_i$$

Then the map α is given by

$$a \, dx_{i_1} \cdots dx_{i_{p-1}} \longmapsto \sum (-1)^{\sigma} a \otimes x_{i_{\sigma^{-1}1}} \otimes \cdots \otimes x_{i_{\sigma^{-1}p}}$$

so it is essentially an anti symmetrization over Σ_{p-1}. So now when we apply N we further anti-symmetrize over Σ_p.

A good notation is

$$y_1\, dy_2 \cdots dy_p \longmapsto \sum_{\sigma\in\Sigma_{p-1}} (-1)^\sigma\, y_1 \otimes y_{\sigma^{-1}2} \otimes \cdots \otimes y_{\sigma^{-1}(p)}$$

$$dy_1 \cdots\cdots dy_p \xmapsto{N} \sum_{\Sigma_p} (-1)^\sigma\, y_{\sigma^{-1}1} \otimes y_{\sigma^{-1}2} \otimes \cdots \otimes y_{\sigma^{-1}p}$$

So therefore it seems you have the same problem as y_1 need not be a coordinate.

Prepare paper on cyclic homology. Purpose is to give an exposition of some aspects of the theory using the double complex

$\mathcal{C}(A)$:

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \arrow[d] & \\
A^3 \arrow[d, "b"'] & A^3 \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^3 \arrow[l, "N"'] \arrow[d, "b"] & \arrow[l, "1-t"'] \\
A^2 \arrow[d, "b"'] & A^2 \arrow[l, "1-t"'] \arrow[d, "-b'"] & A^2 \arrow[l, "N"'] \arrow[d, "b"] & \arrow[l, "1-t"'] \\
q=0 \quad A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & \arrow[l, "1-t"'] \\
p=0 & & &
\end{tikzcd}$$

§1 is to contain

the construction of $\mathcal{C}(A)$; defn of $HC_n(A)$.

Prop. char $k=0 \implies \mathcal{C}(A)$ quis $\left(A^{\otimes(*+1)}/(1-t), b\right)$

Prop. long exact sequence

$$\to \mathcal{H}_n(A) \to HC_n(A) \longrightarrow HC_{n-2}(A) \longrightarrow \mathcal{H}_{n-1}(A) \to \cdots\cdots$$

the quis of $\mathcal{C}(A)$ with

$\mathcal{B}(A)$:

$$\begin{tikzcd}
 & & \\
A^3 \arrow[d, "b"] & A^3 \arrow[d, "b"] \arrow[ul] & \\
A^2 \arrow[d, "b"] & A^2 \arrow[d, "b"] \arrow[ul, "B"'] & A^2 \arrow[d, "b"] \arrow[ul, "B"'] \\
A & A \arrow[ul, "B"'] & A \arrow[ul, "B"']
\end{tikzcd}$$

$B = (1-t)\,s\,N$

~~Could $\mathcal{B}(A)_{norm.}$~~

Spectral sequence: $E^2_{pq} = \begin{cases} \mathcal{H}_q(A) & p \text{ even} \geq 0 \\ 0 & p \text{ odd} \end{cases}$

with d^2 induced by B.

9/2

Second section to discuss normalized + reduced homology.

Assume $k \subset A$ and put $\bar{A} = A/k$.

Define normalized Hochschild complex $(A \otimes \bar{A}^n, b) = C^h(A)_{nor}$ and reduced Hochschild complex $C^h(A)_{red} = C^h(\bar{A})_{norm} / C^h(k)_{norm}$

$\mathcal{B}(A)_{norm}$:

$$\begin{tikzcd}
{} \arrow[d] & & \\
A \otimes \bar{A}^{2} \arrow[d] & {} \arrow[d] & \\
A \otimes \bar{A} \arrow[d] & A \otimes \bar{A} \arrow[d] \arrow[ul, "B"'] & {} \arrow[l, "B"'] \\
A & A \arrow[ul, "B"'] & \\
p=0 \arrow[u] & p=2 \arrow[u] &
\end{tikzcd}$$

$B(a_0, \ldots, a_n)$

$= \sum_{i=0}^{n} (-1)^{in} (1, a_i, \ldots a_n, a_0 \ldots a$

<u>Def</u>; $\bar{H}_n(A) = H_n(C^h(A)/C^h(k)) = H_n(C^h(A)_{red})$

$\overline{HC}_n(A) = H_n(\mathcal{B}(A)_{red})$

<u>Prop</u>.

$$\to HC_n(k) \to HC_n(A) \to \overline{HC}_n(A) \to HC_{n-1}(k) -$$
$$\to \bar{H}_n(A) \to \overline{HC}_n(A) \to \overline{HC}_{n-2}(A) \to \bar{H}_{n-1}(A) \to$$

Augmented k-algebras equivalent to non-unital algebras via $I \longmapsto A = k \oplus I$. Note that $C(I)$ makes sense for a non-unital ring.

<u>Prop</u>: $\mathcal{C}(I)$ is isomorphic to $\mathcal{B}(A)_{red}$. Hence $\overline{HC}(A) = HC(I)$ and $HC(A) = HC(k) \oplus HC(I)$.

<u>Prop</u>. In char. 0, $\overline{HC}(A) = H_*(\bar{A}^{\otimes(*+1)}/(1-t), b)$.

For the proof we need to construct a normalized and reduced version of $C(A)$.