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January 27, 1983

There are many things I don't understand very well even for $\dim M = 1$ or 2.

For $M =$ circle what does the explicit index theorem say, assuming it is true? In general it should be a version of GRR:

$$ch(f_! x) = f_*(\text{Todd} \cdot ch\, x)$$

Here f is something like $M \times B\mathcal{G} \longrightarrow B\mathcal{G}$, and x represents the tautological bundle $\tilde{E}$ over $M \times B\mathcal{G}$. I propose to calculate $ch\, x$ by lifting E over M to $M \times \mathcal{A}$ where it acquires a $\mathcal{G}$-invariant connection and then take the equivariant character forms of this connection. Thus the RHS should be easily computable.

I need formulas for the canonical connection over $M \times \mathcal{A}$ (p. 351-353, 435-437). How can I do this in a sensible way? One starts with

$$T_{M \times \mathcal{A}} = pr_1^* T_M \oplus pr_2^* T_{\mathcal{A}}$$

$$\text{and} \quad T_{\mathcal{A}} \cong \pi^*\left(\Gamma(\operatorname{End} E \otimes T_M^*)\right) \qquad \pi : \mathcal{A} \to pt$$

So the tangent space of $M \times \mathcal{A}$ at a point (m, A) is $T_M(m) \oplus \Gamma(\operatorname{End} E \otimes T_M^*)$. The connection on the bundle $\tilde{E} = pr_1^*(E)$ is the sum of the connection on E associated to A in the M-direction, and the flat connection in the $\mathcal{A}$-direction. If we assume E trivialized around the point m at which we are working we have

$$\nabla_A = d_M + A = dx(\partial_x + \alpha)$$

on M and so

$$\tilde{\nabla} = d_M + A + d_{\mathcal{A}}.$$

It is always awkward to write out $d_{\mathcal{A}}$.

$$d_{\mathcal{A}} = \int_M dx\, \delta A(x) \frac{\delta}{\delta A(x)}$$

is one possibility, except $\delta A(x) \in \mathrm{End}(E_x) \otimes T_x^*$ is a matrix. And so the integration means summing also over matrix indices. Another possibility is to assume one has a family $Y \longrightarrow \mathcal{A}$, $y \longmapsto A(y)$ and then

$$d_{\mathcal{A}} = dy^\mu \frac{\partial}{\partial y^\mu}$$

Then the curvature of the connection is

$$\tilde{\nabla}^2 = (d_M + A + d_{\mathcal{A}})^2$$

$$= \underbrace{d_M A + A d_M + A^2}_{F_A} + (d_{\mathcal{A}} A + A d_{\mathcal{A}}).$$

Let's try computing the last term: $A = dx \cdot \alpha$

$$d_{\mathcal{A}} A + A d_{\mathcal{A}} = dy^\mu \frac{\partial}{\partial y^\mu} dx\, \alpha + dx\, \alpha\, dy^\mu \frac{\partial}{\partial y^\mu}$$

$$= dy^\mu dx \frac{\partial \alpha}{\partial y^\mu}.$$

In the other notation

$$\int dy\, \delta\alpha(y) \frac{\delta}{\delta\alpha(y)} \overbrace{dx\, \alpha}^{A(x)} + dx\, \alpha \int dy\, \delta\alpha(y) \frac{\delta}{\delta\alpha(y)}$$

$$= \int dy\, \delta\alpha(y)\, \delta(x-y)\, dx = \delta\alpha \cdot dx.$$

This is a 2-form on $M \times \mathcal{A}$ with values in $pr_1^*(\mathrm{End}\, E)$. Namely $\delta\alpha$ is a tangent vector to $\mathcal{A}$, hence $\delta\alpha \in \Gamma(\mathrm{End}\, E \otimes T_M^*)$; dx is a tangent vector to M, hence $\delta\alpha . dx \in \mathrm{End}\, E$. Not very clear.

Another possibility:

$$T^*_{M\times\mathfrak{a}}(m,A) = T^*_M(m) \oplus \underbrace{T^*_{\mathfrak{a}}(A)}_{\substack{\text{canonical}\\ \text{isom to } \mathcal{B}^*}}$$

$$\mathcal{B} = \Gamma(\mathrm{End}\,E \otimes T^*_M)$$

Hence
$$\Lambda^2 T^*_{M\times\mathfrak{a}}(m,A) = \Lambda^2 T^*_M(m) \oplus T^*_M(m)\otimes\mathcal{B}^* \oplus \Lambda^2\mathcal{B}^*$$
or
$$\Lambda^2 T^*_{M\times\mathfrak{a}} = pr_1^*\Lambda^2 T^*_M \oplus pr_2^* T^*_M \underset{k}{\otimes} \mathcal{B}^* \oplus \underset{k}{\Lambda^2}\mathcal{B}^*$$

Now let us use a basis $dx\,\alpha_\mu$ for $\mathcal{B}$, and the corresponding coordinates y^μ. Then
$$\tilde{\nabla} = dx\,(\partial_x + y^\mu\alpha_\mu) + dy^\mu\,\partial/\partial y^\mu$$

where I am using an identification $\mathcal{B} \simeq \mathfrak{a}$ centered at a trivialization. So over the circle
$$\tilde{\nabla}^2 = dy^\mu\,dx\,\alpha_\mu .$$
This is clearly a 2-form on $M\times\mathfrak{a}$ with values in $pr_1^*(\mathrm{End}\,E)$; the $\alpha_\mu \in \Gamma(\mathrm{End}\,E)$ so can be evaluated.

Next I need the other part of the equivariant curvature, the big moment map. In general this is a mapping $\varphi : \tilde{g} \longrightarrow$ [redacted] $\Gamma(pr_1^*(\mathrm{End}\,E))$, and here it is the obvious one: $\tilde{g} = \Gamma(\mathrm{End}\,E)$, and an endo. of E over M induces an endo. of pr_1^*E over $M\times\mathfrak{a}$. [redacted]

Let's review the formulas for the equivariant DR complex. In general when g acts on M we form the complex
$$W(g)\otimes\Omega^\circ(M) = S(g^*)\otimes\Lambda(g^*)\otimes\Omega(M)$$
and take the basic elements. Let X_α be a basis for g and θ_α the dual basis for g^*. Use θ^α to denote the generators for the exterior algebra, and u^α for the corresponding

elements in the symmetric algebra, so that

$$W(\mathfrak{g}) = k[u^\alpha; \theta^\alpha]$$

Then the differential in $\Lambda(\mathfrak{g}^*)$ alone is something like

$$d\theta^\alpha = \sum' c^\alpha_{\beta\gamma}\,\theta^\beta\theta^\gamma.$$

Let's get this straight.

We should consider first the complex $\Lambda\mathfrak{g}^*$ of left-invariant forms on G, especially since I have to eventually write this all down for the gauge group. Right multiplication of G on itself will preserve the complex of left-invariant forms. Infinitesimal right multiplication on G gives a left-invariant vector field. Thus the left invariant vector fields on G are naturally identified with elements of $\mathfrak{g}$, and so we get operators $i(X)$, $\mathcal{L}(X)$ associated to each $X \in \mathfrak{g}$. These are derivations of $\Lambda(\mathfrak{g}^*)$ and hence are determined by what they do on $\mathfrak{g}^*$.

$$\mathcal{L}(X)\, i(Y)\omega = i(\mathcal{L}(X)Y)\omega + i(Y)\mathcal{L}(X)\omega$$

Thus if $\omega \in \mathfrak{g}^*$ we have

$$i(Y)\,\mathcal{L}(X)\omega = -\, i([X,Y])\omega$$

or $$(\mathcal{L}(X)\omega)(Y) = -\,\omega([X,Y]).$$

Also $$i(X)\,d\omega = \mathcal{L}(X)\omega - d\,i(X)\omega$$

or if $\omega \in \mathfrak{g}^*$ $$i(Y)i(X)\,d\omega = i(Y)\,\mathcal{L}(X)\omega = -\,\omega([X,Y]).$$

Now I am inclined to write

(*) $$(d\omega)(X,Y) = \langle X \wedge Y \mid d\omega\rangle = i(Y)i(X)\,d\omega = -\,\omega([X,Y]).$$

because of the more general formula

$$(d\omega)(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$$

but this is off by a sign from the usual formula. However let's persist using (*). Then let's introduce the basis X_α and the dual basis θ^α for $\mathfrak{g}^*$. Put

$$[X_\alpha, X_\beta] = c^\gamma_{\alpha\beta} X_\gamma$$

Then

$$i(X_\beta)\,i(X_\alpha)\,d\theta^\gamma = -\,\theta^\gamma([X_\alpha, X_\beta])$$
$$= -\,c^\gamma_{\alpha\beta}$$

and so

$$d\theta^\gamma = -\frac{1}{2} c^\gamma_{\alpha\beta}\,\theta^\alpha\theta^\beta$$

$$\boxed{\begin{aligned} &\text{Formula:} \quad [X_\alpha, X_\beta] = c^\gamma_{\alpha\beta} X_\gamma \\ &\iff \quad d\theta^\gamma = -\tfrac{1}{2} c^\gamma_{\alpha\beta}\,\theta^\alpha\theta^\beta \end{aligned}}$$

In the Weil algebra the formula becomes

$$\boxed{d^W\theta^\gamma + \frac{1}{2} c^\gamma_{\alpha\beta}\,\theta^\alpha\theta^\beta + u^\gamma = 0}$$

The formula for $d^W u^\gamma$ can now be derived by requiring $d^W u^\gamma = 0$.

$$0 = d^W u^\gamma + \underbrace{d\left(\frac{1}{2} c^\gamma_{\alpha\beta}\quad \theta^\alpha\theta^\beta\right)}_{0} + \frac{1}{2} c^\gamma_{\alpha\beta}\left(-u^\alpha\theta^\beta + \theta^\alpha(u^\beta)\right)$$

or

$$d^W u^\gamma + c^\gamma_{\alpha\beta}\,\theta^\alpha u^\beta = 0$$

or

$$\boxed{d^W u^\gamma = c^\gamma_{\alpha\beta}\,u^\alpha\theta^\beta}$$

Now what I want to do is to work out the basic forms in $W(\mathfrak{g}) \otimes \Omega^\bullet(M)$. One first takes the horizontal forms, which involves the

$i(X)$ operations on $\Lambda \mathfrak{g}^* \otimes \Omega^\cdot(M)$. Now I want to go over carefully why the horizontal forms in $\Lambda \mathfrak{g}^* \otimes \Omega^\cdot(M)$ can be identified with $\Omega^\cdot(M)$ embedded in a funny way.

Geometrically we look at the forms on $G \times M$ which are basic relative to the "diagonal" action. On one hand because they are invariant under G they are determined by their effect on tangent vectors along $\{1\} \times M$, just as a left-invariant form on G is determined by its effect on vectors at $1 \in G$. Thus

$$\Omega^\cdot(G \times M)^G \cong \Lambda \mathfrak{g}^* \otimes \Omega^\cdot(M).$$

But now you want to see there is an $i(X)$ action here, which requires some care because $i(X)$ is not defined on invariant forms. Very tricky. In any case we have the map

$$\begin{array}{ccc} G \times M & \longrightarrow & M \\ g \quad m & \longmapsto & g^{-1}m \end{array}$$

which is the quotient for diagonal action of G on $G \times M$. Hence I get an isomorphism

$$\Omega^\cdot(M) \xrightarrow{\sim} \text{basic forms in } \Omega^\cdot(G \times M).$$

Now
$$\Omega^\cdot(G \times M) = \Omega^0(G) \otimes \underbrace{\Lambda^\cdot \mathfrak{g}^*}_{\text{left-inv forms on } G} \otimes \Omega^\cdot(M)$$

Now take horizontal forms

$$\Omega^\cdot(G \times M)_{horiz} = \Omega^0(G) \otimes \left(\Lambda \mathfrak{g}^* \otimes \Omega^\cdot(M)\right)_{horiz}$$

Still too tricky to be convincing.

Get the impression that you want to take G-invariant first, then deal with the vector bundles on the base.

One of the causes of confusion is the fact that $i(X)$ on $\Lambda\mathfrak{g}^*$ comes from the <u>right</u> action of G on itself, because $\Lambda \mathfrak{g}^*$ = left-invariant diffl forms. ~~This makes it hard~~ Thus one is not in the position of taking invariants and then horizontal forms. But this maybe can be used.

Lets consider then the isom.

$$G \times M \xrightarrow{\sim} G \times M$$

$$(g, m) \longmapsto (g, g^{-1}m)$$

left G-action: diagonal action (g, gm) $\longleftarrow$ (g, m) left action on first factor

right G-action: messy $\qquad (g,m)g_1 = (gg_1, m)$

By looking on the side of the trivial action on M one sees that taking basics can be done by first taking left-invariants followed by horizontals for the right action. So the same works on the other side, however it's all still too tricky.

So what I want to do now is to write down a formula for

$$\Omega(M) \xrightarrow{\sim} \left(\Lambda\mathfrak{g}^* \otimes \Omega(M)\right)_{\text{horiz}}$$

The idea is to choose bases X_α, θ^α for $\mathfrak{g}, \mathfrak{g}^*$ resp. Then

$$e(\theta^\alpha) i(X_\alpha) \cdot e(\theta^\beta) i(X_\beta) = e(\theta^\alpha)\{\delta_{\alpha\beta} - e(\theta^\beta) i(X_\alpha)\} i(X_\beta)$$

which shows first of all that the operators $e(\theta^\alpha) i(X_\alpha)$ commute, secondly that they are projection operators. A form is horizontal $\iff$ reproduced by all

$$E_\alpha = 1 - e(\theta^\alpha) i(X_\alpha)$$

Thus the horizontal forms are the image of the projection operator

$$\prod_\alpha E_\alpha = \prod_\alpha (1 - e^\alpha i_\alpha) \qquad \begin{cases} i_\alpha = i(X_\alpha) \\ e^\alpha = e(\theta^\alpha) \end{cases}$$

$$= \prod_\alpha i_\alpha e^\alpha$$

(This suggests analogies: $a_k a_k^* + a_k^* a_k = 1$.

Now

$$\prod_k a_k a_k^* = a_1 a_1^* \, a_2 a_2^* \cdots a_n a_n^*$$

$$= (-1)^{1+2+\cdots(n-1)} a_1 \cdots a_n \, a_1^* \cdots a_n^*$$

$$= a_1 \cdots a_n \, a_n^* \cdots a_1^*$$

and this makes things clear in finite dimensions, since $a_n^* \cdots a_1^*$ <u>kills everything</u> but 1 in $\Lambda \mathfrak{g}^*$.)

The next step is to use the isomorphism

$$\left[S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*) \otimes \Omega(M) \right]_{\text{horiz}} = S(\mathfrak{g}^*) \otimes \Omega(M)$$

to write down the differential in the complex $\left[S(\mathfrak{g}^*) \otimes \Omega(M) \right]^{\mathfrak{g}}$. This might be hard because of the embedding, however we might be able to do the calculations in the ring $\mathbb{C}[u^\alpha, \theta^\alpha] \otimes \Omega(M)$.

January 28, 1983

Today's lecture, Possible topics: Example of elliptic curves, the determinant line bundle and the connection, why the connection is flat

Let's try writing an introduction to explain the use of the formula $\mathrm{Tr}_{(reg)}(D^{-1}\delta D)$. The point would be to correlate with the expansion

$$\log\det\left(D^{-1}(D+B)\right) = \sum_1^\infty \frac{(-1)^{n-1}}{n}\,\mathrm{Tr}\,(D^{-1}B)^n,$$

and also to make clear why the particular regularization you have chosen, based on a fixed choice of d' operator, leads to a coherently defined determinant locally.

Let's begin with the proof that the connection form is closed. Recall that on the space of invertible D we defined a 1-form θ:

$$i(\delta D)\theta = \mathrm{Tr}_{(reg)}(D^{-1}\delta D)$$

where the regularized trace is defined to be

$$\mathrm{Tr}_{(reg)}(D^{-1}\delta D) = \int_M \mathrm{tr}\left(\mathrm{F.P.}\,\langle z|D^{-1}|z\rangle\cdot\delta D(z)\right).$$

Here
$$\begin{aligned}\langle z|D^{-1}|z'\rangle &= G(z,z')\,dz' \\ &= \frac{i}{2\pi}\left\{\frac{1}{z-z'} + \text{smooth}\right\}dz'.\end{aligned}$$

A parametrix for D is given by

$$G_p(z,z')\,dz' = \frac{i}{2\pi}F(z,z')\left(-\partial_{z'}\{\log r(z,z')^2\}\,dz'\right)$$

where $F(z,z') : E_{z'} \xrightarrow{\sim} E_z$ is parallel transport along the geodesic from z' to z, with respect to a connection extending the operator D. Finally

$$\mathrm{F.P.}\,\langle z|D^{-1}|z\rangle = \left(G(z,z')\,dz' - G_p(z,z')\,dz'\right)\Big|_{z=z'}$$

Why is Θ closed? I wish to use the following principle: Given a 1-form $\omega = a_\mu(x)dx^\mu$ on a vector space, we differentiate it

$$\delta\omega = \delta a_\mu \, dx^\mu$$
$$= \frac{\partial a_\mu}{\partial x^\nu} \delta x^\nu dx^\mu$$

and it is closed iff this is a symmetric form in $\delta x, dx$. Here I have

$$\omega_D(\delta D) = \int tr(J_D \, \delta D)$$

and so what I must do is differentiate it as a function of D: i.e. consider the change

$$\delta_1 \omega(\delta D) = \int tr(\delta_1(J_D)\,\delta D)$$

corresponding to an another displacement $\delta_1 D$. Then I must see that the corresponding expression is symmetric in the vectors $\delta_1 D, \delta D$.

In general $\delta_1 D^{-1} = -D^{-1}\delta_1 D \, D^{-1}$

$$\langle z | \delta_1 D^{-1} | z' \rangle = -\left[\int G(z,w)\,dw \;\; \delta D_1(w)\,d\bar{w} \;\; G(\bar{w}, z')\right] dz'$$

Now what you want to do is to prove:

$$\int_M tr\left(\delta_1 J_D \cdot \delta D\right) = -\lim_{\varepsilon \to 0} \int_{|z-w|>\varepsilon} \langle z | D^{-1} | w \rangle \, \delta_1 D(w) \langle w | D^{-1} | z \rangle \, \delta D(z)$$

which you might call the regularized trace of $D^{-1}\delta_1 D \cdot D^{-1}\delta D$. So it is enough to show that

$$(\delta_1 J_D)(z) = \lim_{\varepsilon \to 0} -\int_{|w-z|>\varepsilon} \langle z | D^{-1} | w \rangle \, \delta_1 D(w) \langle w | D^{-1} | z \rangle$$

with some kind of uniform convergence.

What is the analytical point? Ultimately you must examine

$$\int \frac{i}{2\pi} \frac{1}{z-w} \, dw \cdot \alpha(w) \, d\bar{w} \; \frac{1}{w-z'}$$

and explain its singularities as a distribution in z. So what about

$$\int \frac{i}{2\pi} \frac{1}{z-w} \frac{1}{w-z'} \, dw \, d\bar{w} \quad ?$$

It doesn't converge over $\mathbb{C}$.

$$\frac{1}{w-z} - \frac{1}{w-z'} = \frac{z'-z}{(w-z)(w-z')}$$

$\therefore$

January 29, 1983

Transgression fmla in Chern-Simons's paper:

Let $P \in S(\mathfrak{g}^*)^G$. Then P gives rise to a characteristic class for principal G-bundles $G \to E \xrightarrow{\pi} M$; one chooses a connection θ, let's $\Omega = d\theta + \theta\theta = d\theta + \frac{1}{2}[\theta,\theta]$, and associates the form $P(\Omega)$ on M. One knows that $\pi^* P(\Omega) \in \operatorname{Im} d$, because if E is pullback to itself it becomes trivial. In fact there is an explicit form $TP(\theta, \Omega)$ constructed by Chern + Simons. The reason for the form is that the class of $\pi^* P(\Omega)$ is independent of the connection.

I want to do this for GL_n bundles using $P(X) = \operatorname{tr}(e^X)$. So we have 2-connections D_1, D_0 and form $D_t = D_0 + tB$. Then over $I \times M$ we have the connection

$$\nabla = dt\, \partial_t + D_t$$

$$\nabla^2 = D_t^2 + dt\, B$$

$$P(\nabla^2) = \operatorname{tr}\left(e^{D_t^2 + dt\, B}\right)$$

$$= \operatorname{tr}\left(e^{D_t^2}\right) + dt\, \operatorname{tr}\left(e^{D_t^2} B\right)$$

Now given $\alpha + dt\,\beta$, a closed form, on $I \times M$ we have

$$d\alpha = dt\, d\beta \quad \Rightarrow \quad \partial_t \alpha = d_M \beta$$

Hence

$$d_M\left(\int_0^1 dt\, \beta\right) = \int_0^1 dt\, d_M\beta = \int_0^1 dt\, \partial_t \alpha = \alpha\Big|_{t=1} - \alpha\Big|_{t=0}.$$

Thus

$$\operatorname{tr}\left\{e^{D_1^2}\right\} - \operatorname{tr}\left\{e^{D_0^2}\right\} = d\int_0^1 dt \left[\operatorname{tr}\left(e^{D_t^2} B\right)\right]$$

To derive the Chern-Simons formula one uses $D_1 =$ given connection $d + \theta$ on E pulled up to E, $D_0 =$ flat connection given by section. Then $B = \Theta$

$$D_t^2 = (d+t\Theta)^2 = t^2\Theta^2 + t\underbrace{d\Theta}_{\Omega - \Theta^2}$$
$$= t\Omega + (t^2-t)\Theta^2$$

Now I have to understand this formula better from the viewpoint of the Weil algebra. Especially I want to see if it relates [redacted] classes in $\Lambda \mathfrak{g}^*$ with ones in $S(\mathfrak{g}^*)$ the way I want.

I should look at the restriction of the transgression cochain [redacted] to a fibre.

Explanation of the transgression [redacted] cochain: When we pull EG back over itself we are classifying principal bundles [redacted] together with [redacted]. a trivialization. More specifically a map $X \longrightarrow EG$ should be viewed as a bundle over X together with a trivialization. So to understand the transgression cochain I should think of having a vector bundle E with connection and a trivialization. Thus

$$D = d + \Theta$$

where d is defined relative to the trivialization. We have from the above

$$\operatorname{tr}\{e^{D^2}\} = d\Big\{ \underbrace{\int_0^1 dt \; \operatorname{tr}\{e^{\overbrace{(d+t\Theta)^2}^{\substack{td\Theta + t^2\Theta^2 \\ = t\Omega + (t^2-t)\Theta^2}}}\,\Theta\}}_{\text{transgression form}} \Big\}$$

Now to restrict the transgression form to the fibre $G \subset EG$ means that we have a [redacted] map $X \xrightarrow{\cancel{f}} G$, i.e. the trivial vector bundle [redacted] $X \times \mathbb{C}^n$ together a trivialization $X \times \mathbb{C}^r \xleftarrow{g} X \times \mathbb{C}^r$. Thus we are comparing the connections

$$d = g\,d\,g^{-1} + \theta \qquad \therefore \ \theta = dg \cdot g^{-1}, \quad \Omega = 0$$

and so

$$\int_0^1 dt \, tr\{e^{(t^2-t)\theta}\,\theta\} \qquad \text{where } \theta = dg \cdot g^{-1}.$$

which gives the form const $tr(\theta^{2p-1})$ in degree $2p-1$. The constant is

$$\frac{1}{(p-1)!}\int_0^1 dt\,(t^2-t)^{p-1} = \frac{(-1)^{p-1}}{(p-1)!}\,\overbrace{\beta(p,p)}^{\frac{(p-1)!\,(p-1)!}{(2p-1)!}}$$

$$= (-1)^{p-1}\frac{(p-1)!}{(2p-1)!}$$

It seems that all of the Bott-Chern formulas become simpler for the character: Write $D = d + \theta$ so that $D^2 = \Omega = d\theta + \theta\theta$. Then

$$d\,tr(e^{D^2}) = tr(e^{\Omega}d\Omega) \qquad 0 = [D, \Omega] = d\Omega + [\theta, \Omega]$$

$$= tr(e^{\Omega}[\Omega, \theta])$$

$$= tr([\Omega, e^{\Omega}\theta]) = 0$$

(much simpler is $d\,tr(e^{D^2}) = tr([D, e^{D^2}]) = 0$)

Also if $D_t = D_0 + tB$, then $\dot{\Omega}_t = D_t\dot{D}_t + \dot{D}_t D_t = [D_t, B]$

$$\partial_t\,tr(e^{\Omega_t}) = tr(e^{\Omega_t}\dot{\Omega}_t) = tr(e^{\Omega_t}[D_t, B])$$

$$= \cancel{d}\,tr([D_t, e^{\Omega_t}B])$$

so that

$$tr(e^{\Omega_1}) - tr(e^{\Omega_0}) = d\int_0^1 dt\; tr(e^{\Omega_t}B)$$

$$(= d\,tr(e^{\Omega_t}B))$$

January 30, 1983

General problem of relating Lie K-theory to algebraic K-theory. In general one has this group G with Lie algebra $\mathfrak{g}$ and discrete group G_d. Then one has $\underline{G}$ = fibre of $G_d \to G$ and

$$
\begin{tikzcd}
\underline{G} \arrow[r] & G_d \arrow[r] & G \arrow[r] \arrow[d, equal] & B\underline{G} \arrow[r] \arrow[d] \arrow[r, phantom, "\sim G/G_d"', shift left=12] & BG_d \arrow[r] \arrow[d] & BG \\
 & & G \arrow[r] & \mathfrak{g} \arrow[r] & B_{cc}G &
\end{tikzcd}
$$

where the latter comes from van Est.

Thus there are two maps of interest:

1) $H^*(\mathfrak{g}) \longrightarrow H^*(B\underline{G}) = H^*(G/G_d)$. Invariant forms on G give characteristic classes for flat bundles which are smoothly trivialized.

2) $H^*(B_{cc}G) \longrightarrow H^*(BG_d)$. Invariant

$$\begin{array}{c} H^*(B_{cc}G) \\ \parallel \\ H^*(\mathfrak{g}, K) \end{array}$$

differential forms on the symmetric space G/K give characteristic classes for flat bundles.

If we take $G = GL_n\mathbb{C}$, then the Lie algebra coh. of G as a real Lie group is twice as big as the Lie algebra coh. of U_n, which is $H^*(G)$. ~~Hence~~ in this case $H^*(\mathfrak{g}, K) = H^*$ of $\mathfrak{gl}_n$ over $\mathbb{C}$.

One problem is that the Connes map of K_0A, K_1A to the Lie K-theory is of the wrong parity to be compatible with products. Thus one has K_0A mapping to $HC_{odd}(A)$.

I want ~~to~~ find a more sensible relationship.

General idea: Look at work on foliations and Gelfand-Fuks cohomology for ideas

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Consider $\mathcal{G} = GL_n(A)$, A = smooth $\mathbb{C}$-valued functions on M. Then the cohomology of $\mathcal{G}_d$ gives the algebraic K-theory of A, and hence the continuous cochain cohomology of $\mathcal{G}$ will give us some characteristic classes for representations of discrete groups over A. But via van Est, and the fact we can compute both the cohomology of $\mathcal{G}$ and $\tilde{\mathfrak{g}}$, we should be able to compute the continuous cochain cohomology of $\mathcal{G}$.

Now $\mathcal{G} = Maps(M, GL_n\mathbb{C})$ has the same homotopy type as the unitary gauge group $\mathcal{G}_u = Map(X, U_n)$, and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_u \otimes_{\mathbb{R}} \mathbb{C}$. So when we compute the Lie alg. cohomology of $\tilde{\mathfrak{g}}$ as a real Lie algebra we will get two copies of its cohomology as a complex Lie algebra. Look at van Est:

$$\begin{array}{ccccc} \mathcal{G} & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & B_{cc}\mathcal{G} \\ \downarrow & & \wr & & \downarrow \\ \tilde{\mathfrak{g}}_u & \longrightarrow & \tilde{\mathfrak{g}}_u \times \tilde{\mathfrak{g}}_u & \longrightarrow & \tilde{\mathfrak{g}}_u \end{array}$$

In other words one can expect to get characteristic classes for representations over A of discrete groups from the invariant differential forms on $\mathcal{G}/\mathcal{G}_u = Map(M, GL_n\mathbb{C}/U_n)$.

Moreover the cohomology of the invariant diffl forms on $\mathcal{G}/\mathcal{G}_u$, $H^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_u)$ should be isomorphic to the cohomology of $\tilde{\mathfrak{g}}$ over $\mathbb{C}$.

It seems that in this way we obtain a homomorphism

$$K_*A \longrightarrow PH_*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_u)$$

Let's review characteristic classes of representations. A characteristic class for representations is a natural transformation

$$\theta : \mathrm{Rep}_A(G) \longrightarrow H^0(G, R_\bullet) = \prod_{i\geq 0} H^i(G, R_i)$$

$$= \mathrm{Hom}^{(0)}(H_\bullet(G), R_\bullet) \qquad \left(\begin{array}{c}\text{coeffs}\\ \text{a field } k\end{array}\right)$$

The universal such has the coefficients

$$\mathcal{A} = \bigoplus_P H_\bullet(\mathrm{Aut}\, P)$$

where P runs over the isom. classes of $\mathcal{P}(A)$.
The map $\mathcal{A} \to R_\bullet$ corresponding to θ is a ring hom. $\Longleftrightarrow$ $\theta(\rho+\sigma) = \theta(\rho)\theta(\sigma)$, $\theta(0) = 1$. θ extends to $R_A(G) \Longleftrightarrow$ the homom. $\mathcal{A} \to R_\bullet$ extends to

$$\mathcal{A}[\sigma^{-1}] = \mathbb{Z}[K_0 A] \otimes H_\bullet(GL(A))$$

Finally θ is additive $\Longleftrightarrow$ $\mathcal{A} \to R$ factors thru

$$\mathcal{A} \longrightarrow Q(\mathcal{A}) = (K_0 A)\otimes k \oplus Q\{H_\bullet(GL(A))\}$$

so that the universal additive characteristic class for repns. has coefficients

$$(K_0 A)\otimes k \oplus Q\{H_\bullet(GL(A))\}$$

Now let's look at what this means for the theory of Chern classes in filtered DR cohomology. This gives a character

$$ch : R_A(G) \longrightarrow \prod_p H^{2p}(G\times \mathrm{Sp}A, F_p\Omega^\cdot)$$

$$= \prod_{i,p} H^i(G)\otimes H^{2p-i}(\mathrm{Sp}A, F_p\Omega^\cdot)$$

$$= \prod_i \mathrm{Hom}\Big(H_i G, \prod_p H^{2p-i}(\mathrm{Sp}A, F_p\Omega)\Big)$$

Thus the coefficients are

the bigraded ring $\prod_{i,p} H^{2p-i}(Sp\,A, F_p\Omega^\circ)$, and the character corresponds to a homomorphism

$$K_i(A) \longrightarrow \prod_p H^{2p-i}(SpA, F_p\Omega^\circ)$$

Similarly this is refined by

$$K_i(A) \longrightarrow \prod_p H^{2p-i}(SpA, \mathcal{D}(p))$$

where $\mathcal{D}(p)$ is the Deligne complex.

Now I want to see carefully in the case of $A = C^\infty(M)$, and when the De Rham complex is computed with the C^∞ topology, how this compares with the ~~[illegible]~~ homomorphism to the Connes homology:

$$K_i(A) \longrightarrow PH_i(\tilde{\mathcal{J}}, \tilde{\mathcal{J}}_u) = \Omega^{i-1}/d \oplus H^{i-3}(M) \oplus \cdots$$

Recall $\mathcal{D}(p)$ is the complex

$$\begin{array}{cccc} \mathbb{Z} & \Omega^0 & & \Omega^{p-1} \\ 0 & 1 & & p \end{array}$$

hence we have

$$0 \to \Sigma(\Omega^\circ/F_p\Omega^\circ) \longrightarrow \mathcal{D}(p) \longrightarrow \mathbb{Z} \to 0.$$

In the affine case all of the complexes Ω^q are acyclic hence one gets

$$\begin{aligned} H^q(\Omega^\circ/F_p\Omega^\circ) &= H^q(M, \mathbb{C}) && q < p-1 \\ &= \Omega_M^{p-1}/d\Omega_M^{p-2} && q = p-1 \\ &= 0 && q > p-1 \end{aligned}$$

which gives a long exact sequence

$$\to 0 \longrightarrow H^{p+1}(\mathcal{D}(p)) \to H^{p+1}(M,\mathbb{Z}) \to 0$$

$$\to \Omega^{p-1}_M / d\Omega^{p-2}_M \longrightarrow H^p(\mathcal{D}(p)) \longrightarrow H^p(M,\mathbb{Z}) \to$$

$$\to H^{p-2}(M,\mathbb{C}) \longrightarrow H^{p-1}(\mathcal{D}(p)) \longrightarrow H^{p-1}(M,\mathbb{Z}) \to$$

$$H^{p-2}(\mathcal{D}(p)) \longrightarrow H^{p-2}(M,\mathbb{Z}) \to$$

Since we know that $H^{p-1}(M,\mathbb{C}) \hookrightarrow \Omega^{p-1}/d\Omega^{p-1}$ it is clear that one ought to be able to prove

$$\boxed{\begin{aligned} H^q(\mathcal{D}(p)) &= H^{q-1}(M,\mathbb{C}^*) \qquad q<p \\ &= H^q(M,\mathbb{Z}) \qquad q>p \\ H^{p-1}(M,\mathbb{Z}) \to \Omega^{p-1}_M/d\Omega^{p-2}_M &\longrightarrow H^p(\mathcal{D}(p)) \longrightarrow H^p(M,\mathbb{Z}) \to 0 \end{aligned}}$$

Think of the weaker Deligne complex which replaces $\mathbb{Z}$ by $\mathbb{R}$; it measures the fact that Chern classes are real but become trivial as complex classes for flat bundles. Then $\mathcal{D}(p)$ is

$$\mathbb{R} \longrightarrow \Omega^0 \longrightarrow \cdots\cdots \longrightarrow \Omega^{p-1}$$

and really since $\mathbb{Z} \longrightarrow \mathbb{C}$ is $\mathbb{Z} \xrightarrow{2\pi i} \mathbb{C}$, the $\mathbb{R}$ should be $i\mathbb{R}$. Thus if we take the real part of $\mathcal{D}(p)$ we get simply

$$0 \longrightarrow \Omega^0_{\mathbb{R}} \longrightarrow \cdots\cdots \longrightarrow \Omega^{p-1}_{\mathbb{R}}$$

which is going to give

$$\begin{cases} H^q(M,\mathcal{D}(p)_{\mathbb{R}}) = H^{q-1}(M,\mathbb{R}) & q<p \\ H^p(M,\mathcal{D}(p)_{\mathbb{R}}) = \Omega^{p-1}_{\mathbb{R}}/d\Omega^{p-1}_{\mathbb{R}} & q=p \\ H^q(M,\mathcal{D}(p)_{\mathbb{R}}) = 0 & q>p \end{cases}$$

Finally look at

$$K_i(A) \longrightarrow \prod_p \underbrace{H^{2p-i}(M, \mathcal{D}(p))}$$

0 if $2p-i > p$ or $p > i$

The first case is $p = i$ which gives

$$H^i(\mathcal{D}(i)) = \Omega^{i-1}/d\Omega^{i-2}$$

then comes $p = i-1$ which gives

$$H^{i-2}(M, \mathcal{D}(i-1)) = H^{i-3}(M, \mathbb{R}).$$

So now it is clear that characteristic classes for representations [redacted] which one obtains from the rel. Lie algebra cohomology of $\mathfrak{gl}_n(A)$, $\mathfrak{u}_n(A)$ gives exactly the same sort of information as what one gets from the Deligne classes measuring the non-unitarity of the representation of the fundamental group. But actually these latter characteristic classes I can define by direct differential form methods.*

(~~This last sentence refers to~~ I had the idea of forming the contractible space of hermitian inner products on E which is G_c/G, and then using invariant differential forms on G_c/G. But this idea doesn't work as invariant forms on G_c/G are like biinvariant forms on [redacted] G.)

* This sentence refers to the idea of defining real characteristic classes of odd degree for flat bundles by [redacted] observing the [redacted] imaginary part of the complex-valued Chern class [redacted] vanishes for 2 reasons, because the Chern class can be defined in real cohomology using a unitary connection, and because of the flat connection.

February 1, 1983:

Bonora, Cotta-Ramasino: Remarks on BRS Transf, Anomalies and Lie cohomology of gauge tr. gp. Comm. Math. Phys. 87 589-603 (1983).

In this paper one identifies the ghost field with the Maurer-Cartan form on $\mathcal{G}$, i.e. the left-invariant form on $\mathcal{G}$ with values in $\tilde{\mathfrak{g}}$ whose value at 1 is the identity. The BRS transformation for ghosts is identified with exterior differentiation for forms on $\mathcal{G}$.

Mentioned is the Lie algebra cohomology of $\tilde{\mathfrak{g}}$ acting on $\Omega^\circ(\mathcal{A})$, in particular $\Omega^0(\mathcal{A})$. $H^0(\tilde{\mathfrak{g}}, \Omega^0(\mathcal{A}))$ is the space of gauge-invariant functionals on $\mathcal{A}$ with no ghosts. $H^p(\tilde{\mathfrak{g}}, \Omega^0(\mathcal{A}))$ consists of (integrated) anomalous terms with p ghosts.

Anomalies. Formula for the anomaly in $n=2$:

$$\psi_1^2(\xi) = \int \mathrm{tr}(\xi\, d\eta(A))$$

and for $n=4$

$$\psi_1^4(\xi) = \check{f}_3\big(\xi, d\eta(A), d\eta(A)\big) + \tfrac{1}{4}\check{f}_3\big(\xi, d\eta(A), [\eta(A),\eta(A)]\big)$$
$$-\tfrac{1}{2}\check{f}_3\big(\xi, \eta(A), [d\eta(A), \eta(A)]\big)$$

Here f_3 is a trilinear symmetric invariant map on $\mathfrak{gl}$ and $\vee$ means probably integration of the forms over M. $\eta(A) = A - A_0$ where A_0 is a basepoint, so that

$$\eta(A) \in \Gamma(\mathrm{End}\,E \otimes T^*) = \Omega^1(M, \mathrm{End}\,E)$$
$$d\eta(A) \in \Omega^2(M, \mathrm{End}\,E).$$

How do I think of the anomaly? I start with the determinant line bundle over $\mathcal{A}$ which is equivariant for $\mathcal{G}$ and has an invariant connection. Then the corresponding moment map is a *linear* function on $\tilde{g}$ associated to each connection, i.e. a map

$$\tilde{g} \longrightarrow \Omega^0(\mathcal{A})$$

That certainly is what the $\psi_1(\xi)$ are.

But the significance of $\eta(A)$ is not clear; for example it is not clear why I couldn't work with functions on $\mathcal{A}$. In fact it seems that all they want to do is to ~~restrict~~ to functions on $\mathcal{A}$ which are "local" in some sense. What does this mean? $\mathcal{A}$ is the space of sections of a vector bundle over M provided a basepoint A_0 is chosen, so a multilinear map of degree m is given by a distribution on M^m and one requires it to be supported in the diagonal. So one gets a class of poly. fns. on $\mathcal{A}$ whose homogeneous pieces have this "local" form. Now does the class of functions depend on the basepoint. This should come down being able to translate in a symmetric homogeneous form. Hence we want for ψ_m symmetric homogeneous that

$$\psi_m(A_0+B, A_0+B, \dots, A_0+B)$$

$$= \sum \quad \psi_m(A_0 \cdots A_0 \; B \cdots B)$$

be a sum of terms with the correct local description. Thus need $B \longmapsto \psi_m(A_0 \cdots A_0 \; B \cdots B)$ "$\psi_k(B)$" to have the correct form. Thus look at $M^m \longrightarrow M^k$ projecting on the last k factors, and if ψ_m is supported in Δ_m, then it is clear that the $\psi_k(B)$ should be supported in Δ_k. Seems OKAY.

So I will ignore $\eta(A)$, and treat it as just A. Then I go back to the anomaly formula. Introduce

$$S = \bigoplus S^{p,k}$$

where $S^{p,k}$ consists of p-cochains on $\tilde{\mathfrak{g}}$ with values in functions $\mathcal{a} \longrightarrow \Omega^k(M)$, i.e.

$$S^{p,k} = \Lambda^p \tilde{\mathfrak{g}}^* \otimes \Omega^0(\mathcal{a}, \Omega^k(M)).$$

For example start with the anomaly

$$\psi_1^2(\xi) \text{ [redacted] } \in \Lambda^1 \tilde{\mathfrak{g}}^* \otimes \Omega^0(\mathcal{a}) \otimes \Omega^2(M).$$

Now a lemma is that the double complex $S^{p,k}$ is acyclic in the k direction, so one can then chase in the double complex to get an element

$$\psi_3^0 \in \Lambda^3 \tilde{\mathfrak{g}}^* \otimes \Omega^0(\mathcal{a}) \otimes \Omega^0(M)$$

<u>Idea:</u> $\mathcal{a}$ is a principal bundle for $\mathcal{G}$, so if we [redacted] proceed as in the Bott-Segal paper:

$$\Omega^\cdot(\mathcal{a})_{horiz} \otimes \Lambda^\cdot(\tilde{\mathfrak{g}}^*) \xrightarrow{\sim} \Omega^\cdot(\mathcal{a})$$

leads to a spectral sequence

$$E_1^{pq} = H^q(\tilde{\mathfrak{g}}, \Omega^p(\mathcal{a})_{horiz}) \Longrightarrow H^*(B\mathcal{G}),$$ NO should be $\mathcal{a}$

we see that there is an edge homomorphism

$$E_1^{0,q} = H^q(\tilde{\mathfrak{g}}, \Omega^0(\mathcal{a})) \longleftarrow H^q(B\mathcal{G}) = 0$$ $\mathcal{a}$

February 2, 1983

Notation: E is a smooth complex vector bundle over M, $\mathcal{G} = \mathrm{Aut}(E) =$ gp. of complex gauge transf., $\mathcal{G}_u =$ grp. of unitary gauge transfs. relative to an inner product on E, $\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}_u$ the Lie algebras. If E is trivial of rank r, then $\mathcal{G} = GL_r(C^\infty(M))$, $\tilde{\mathfrak{g}} = gl_r(C^\infty(M))$.

The problem is to construct classes in $H^*(B\mathcal{G}_\delta)$ where $\mathcal{G}_\delta$ is the underlying discrete gp. of $\mathcal{G}$. Such classes come from continuous cochain cohomology of $\mathcal{G}$, which is related to the cohomology of $\mathcal{G}$ and $\tilde{\mathfrak{g}}$ by the van Est spectral sequence. If we pretend that the cohomology of $\mathfrak{g}$ and the continuous cochain cohomology of $\mathcal{G}$ are the cohomology of fictitious spaces $\mathfrak{g}$ and $B_{cc}\mathcal{G}$, then the van Est spectral sequence is just the spectral sequence of a fibration:

$$\mathcal{G} \longrightarrow \mathfrak{g} \longrightarrow B_{cc}\mathcal{G}$$

~~In the limit $r \to \infty$, where $\mathcal{G} = GL_r(C^\infty(M))$, the cohomology $H^*(\tilde{\mathfrak{g}})$ is a Hopf algebra~~

~~I now want to take the case $\mathcal{G} = GL_r(C^\infty(M))$ where $r \to \infty$. In this case the cohomologies $H^*(\mathcal{G}), H^*(\tilde{\mathfrak{g}}), H^*(B_{cc}\mathcal{G})$ should be Hopf algebras, and I can think of the primitive subspace as being dual to the homotopy groups of the (fictitious) spaces. So I am thinking of $\pi_*(\mathfrak{g})$ as the primitive part of $H_*(\tilde{\mathfrak{g}})$, and this is known from Connes + Loday Quillen.~~

Now look at

$$\begin{array}{ccccc} \mathcal{G}_u & \longrightarrow & \tilde{\mathfrak{g}}_u & \longrightarrow & B_{cc}\mathcal{G}_u \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & B_{cc}\mathcal{G} \end{array}$$

and use that $\mathcal{G}_u \to \mathcal{G}$ is a homotopy equivalence. [redacted] Also $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_u \otimes_{\mathbb{R}} \mathbb{C}$ so when we compute the cohomology of $\tilde{\mathfrak{g}}$ as a real Lie algebra with complex coefficients we get the cohomology of $\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}$ as a complex Lie algebra. $\therefore H^*(\tilde{\mathfrak{g}}; \mathbb{R}) = H^*(\tilde{\mathfrak{g}}_u) \otimes H^*(\tilde{\mathfrak{g}}_u)$

I conclude that there might be a fibration of the form

$$B_{cc}\mathcal{G}_u \longrightarrow B_{cc}\mathcal{G} \longrightarrow \tilde{\mathfrak{g}}_u .$$

This is to be interpreted precisely as a spectral sequence in cohomology. In any case since $H_*(\tilde{\mathfrak{g}}_u)$ is known by Connes work and Loday-Quillen, we can try to construct cohomology classes in $H^*(B\mathcal{G}_\delta)$ belonging to $\mathrm{Prim}\{H^*(\tilde{\mathfrak{g}}_u)\}$. We have

$$\mathrm{Prim}\{H_p(\tilde{\mathfrak{g}}_u)\} = \Omega^{p-1}/d\Omega^{p-2} \oplus H^{p-3}_{DR}(M) \oplus \cdots$$

and everywhere I am using real coefficients.

The idea will be to use the fact that for flat bundles the Chern classes computed as diff forms [redacted] are real for two reasons. Because one can pick a unitary connection in which case $\frac{i}{2\pi}K$ is real, and because one has the flat connection such that $K=0$.

Given a discrete group Γ acting on E we can [redacted] construct a vector bundle E_Γ over $B\Gamma \times M$ which is flat over the [redacted] leaves $B\Gamma \times \{m\}$. A better geometric model is to take a vector bundle E over $Y \times M$ equipped with a partial flat connection [redacted] along the leaves $Y \times \{m\}$. Now the idea is to understand the characteristic classes of E.

Suppose we have a line bundle L to simplify. I extend the partial connection in the horizontal direction to a connection on $Y \times M$. Locally L is generated by a section s which is flat for the given partial connection:

$$\nabla' : L \longrightarrow L \otimes pr_1^* T_Y^* \qquad \nabla' s = 0$$

The extension to a connection is an operator $\nabla' + \nabla'' = \nabla$ where

$$\nabla'' : L \longrightarrow L \otimes pr_2^* T_M^*$$

and we have

$$\nabla s = \nabla'' s = \theta s$$

where $\theta \in pr_2^* T_M^*$. The curvature is

$$K = d\theta \in pr_1^* T_Y^* \otimes pr_2^* T_M^* \oplus pr_2^* \Lambda^2 T_M^* .$$

The second component gives the curvature of L in the vertical direction.

I want to get some idea about this curvature form K. The first thing I want to look at is the case where I have just a line bundle L over $Y \times M$ and I pick a connection and hence get a curvature, or closed 2-form. Then I can restrict to each fibre $y \times M$ and I get a closed 2-form on M which varies smoothly in y, and whose cohomology class doesn't change. If we use Kunneth

$$H^2(Y \times M, \mathbb{Z}) = H^2(Y, \mathbb{Z}) \oplus H^1(Y, \mathbb{Z}) \otimes H^1(M, \mathbb{Z}) \oplus H^2(M, \mathbb{Z})$$

assuming no torsion, the middle term should associate to a path in Y a map of M to S^1 up to homotopy; presumably, this is effect of parallel translation along the curve $y \times m$ as y varies on the path.

So the next thing is to see what we expect when we have this given flat connection in the Y-directions.

How unique is θ? The section s is unique up to a non-vanishing function f which is constant in the Y-direction, so θ is unique up to $d\log f$?

What do we know about $c_1(L)$ in the Grothendieck picture? This time $c_1(L) \in$ $H^1(B\Gamma \times M, \underline{\underline{\Omega}}^0(M)^\times)$. When resolved into Kuenneth compOnents one gets the Chern class of L over M in $H^1(M, \underline{\Omega}^0(M)^\times) = H^2(M, \mathbb{Z})$, and an element of $H^1(B\Gamma, \Omega^0(M)^\times) = \mathrm{Hom}(\Gamma, \Omega^0(M)^\times)$. The latter via $\log|\;|$ gives rise to a homomorphism $\Gamma \longrightarrow \Omega^0(M)_{\mathbb{R}}$.

Philosophy: Start with a vector bundle E over $Y \times M$ which is flat in the Y-direction and then compute its character form using a pleasant connection. You also want the character form using a unitary connection, and the difference between these expressed as a coboundary. Then you take this differential form data restrict to a cycle in M and integrate to get forms on Y. The forms should be closed but might vary continuously with respect to the cycle in M.

Example: Go back to the line bundle case, and let K^f be the curvature calculated with the partially flat connection, and K^u the curvature connected with a unitary connection. Then K^u is purely imaginary and

$$K^f - K^u = dh$$

where h is a global 1-form on $Y \times M$. Let h' be its $(1,0)$ component i.e. in $pr_1^* T_Y$. Then

$$0 = \big[\mathrm{Re}(K^f) - \underset{\overset{\shortparallel}{0}}{\mathrm{Re}(K^u)}\big]^{(2,0)} = \big[d(\mathrm{Re}\,h')\big]^{2,0} = d_Y(\mathrm{Re}\,h')$$

which means that if $\mathrm{Re}\, h'$ is restricted to $Y \times m$ we get a closed 1-form on Y. (In fact fact we don't have to introduce h'. The point is that restriction $\overset{\text{to } Y\times m}{}$ + real part kill dh, so that we get the closed 1-form

$$i_m^* \{\mathrm{Re}\, h\} \quad \text{on } Y.$$

This will then [redacted] be an element of $H^1(Y, \mathbb{R}) = \mathrm{Hom}(\pi_1 Y, \mathbb{R})$ which is probably the $\log |\;|$ of the representation on E corresponding to m. It depends on m.

February 4, 1983

Third lecture: Determinant line bundle and the connection on it.

Consider all Fredholm operators $D : V' \longrightarrow V$ where V', V are Hilbert spaces. (e.g. Sobolev spaces of sections of vector bundles). Associate to D the 1-diml space

$$\mathcal{L}_D = \lambda(\operatorname{Cok} D) \otimes \lambda(\operatorname{Ker} D)^*$$

$$\text{where} \quad \lambda(V) = \Lambda^{max}(V).$$

I will show the family $\{\mathcal{L}_D\}$ is a line bundle over the space of Fredholm operators D.

If F is a finite-diml subspace of V put

$$\mathcal{U}(F) = \{ D \mid D(V') + F = V \}$$

$$\Updownarrow$$

$$D \text{ is transversal to } F.$$

It is known that $\mathcal{U}(F)$ is open in $\mathcal{D}$ and that the family $\{D^{-1}F\}$ is a vector bundle over $\mathcal{U}(F)$.

Start again: V', V are Hilbert spaces, $\mathcal{D} =$ space of Fredholm operators $D : V' \longrightarrow V$ of index p. Define

$$\mathcal{L}_D = \lambda(\operatorname{Cok} D) \otimes \lambda(\operatorname{Ker} D)^*$$

$$\lambda(V) = \Lambda^{max}(V).$$

Claim $\{\mathcal{L}_D\}$ forms a line bundle over $\mathcal{D}$, in fact a holomorphic line bundle in a certain sense.

Let $F \subset V$ be finite-diml. Put

$$\mathcal{U}(F) = \{ D \mid D(V') + F = V \} \qquad D \text{ trans. to } F.$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} D & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & \operatorname{Cok} D \longrightarrow 0 \\
 & & \| & & \cup & & \cup & & \| \\
0 & \longrightarrow & \operatorname{Ker} D & \longrightarrow & D^{-1}F & \longrightarrow & F & \longrightarrow & \operatorname{Cok} D \longrightarrow 0
\end{array}$$

$$\mathcal{L}_D = \lambda(\operatorname{Cok} D) \otimes \lambda(\operatorname{Ker} D)^* \underset{\text{can}}{\overset{\text{can}}{=\!=}} \lambda(F) \otimes \lambda(D^{-1}F)^*$$

Claim $\mathcal{U}(F)$ open in $\mathcal{D}$ and $\{D^{-1}F\}$ for $D \in \mathcal{U}(F)$ form a holom. vector bundle of rank dim. $F + p$.
This $\Rightarrow \{\mathcal{L}_D\}$ is a holom. line bundle over $\bigcup \mathcal{U}(F) = \mathcal{D}$.

Let $W' \subset V$ have cod $=$ dim $F + p$. Then have induced map by D

$$\bar{D}^+ : W' \longrightarrow V/F$$

and we put
$$\mathcal{U}(W', F) = \{ D \mid \bar{D}^+ \text{ is an isom.} \}$$

Certainly open in $\mathcal{D}$. Also

$$\bigcup_{W'} \mathcal{U}(W', F) = \mathcal{U}(F)$$

because given $D : V' \to V \to V/F$ onto, can choose $W' \oplus D^{-1}F = V'$ whence $\bar{D} : W' \xrightarrow{\sim} V/F$. Finally we have over $\mathcal{U}(W', F)$ the isomorphism

$$D^{-1}F \xrightarrow{\sim} V/W'$$

which shows the family $\{D^{-1}F\}$ over $\mathcal{U}(W', F)$ is a trivial vector bundle with fibre V/W'.

Next point: Over $\mathcal{U}(W', F)$ we have not only D^+ above which is invertible, but we also have the map

$$V'/W' \xleftarrow{\sim} D^{-1}F \xrightarrow{D} F$$

which we denote D^-.

Over $\mathcal{U}(W', F) \cap \mathcal{U}(V', 0)$ we have the isom.

$$\mathcal{L}_D = \lambda(F) \otimes \lambda(D^{-1}F)^* \xleftarrow{\sim} \lambda(F) \otimes \lambda(V'/W')^*$$

$$\lambda(0) \otimes \lambda(0)^*$$

February 16, 1983

The central problem is to define the character of the index as a differential form. Already there is something non-trivial, or at least something I don't understand, happening over a surface, and possibly over a circle.

Today I want to concentrate on the metric version. What this means is that I [redacted] want to think of the index as a virtual [redacted] holomorphic bundle over the space of Fredholm operators with a kind of ζ-function metric. Then I [redacted] expect the metric + holomorphic structures to give me definite character forms. I know how to do this in the case of the first Chern class over a surface.

Let's review how this works. [redacted] I have on the determinant line bundle $\mathcal{L}$ the analytic torsion metric, and as $\mathcal{L}$ is holom. I get a globally defined connection on $\mathcal{L}$. Then I compute the [redacted] connection forms belonging to various sections of $\mathcal{L}$. (Notice: a non-vanishing section of $\mathcal{L}$ seems to be related to a type of Green's function.)

Idea: Existence of $\operatorname{Tr}_{reg}(D^{-1}\delta D)$ is same sort of thing as the existence of the η-invariant. Picture:

$$\eta \qquad 0\text{-form on } \{\text{invertible } D\}$$

$$\theta = \operatorname{Tr}_{(reg)}(D^{-1}\delta D) \qquad 1\text{-form on } \{\text{------}\}.$$

and we can extend over the singular set in 2 ways: The [redacted] crudest is to take $d\eta$, $d\theta$. But the best way is to bring in line bundle stuff. For example [redacted] the map $e^{\pi i \eta}$ to S^1, or the connection on the determinant line bundle.

Let's pursue the analogy further. We have two spaces $\mathcal{F}$, $\mathcal{F}^{sa}$ of Fredholm and self-adjoint Fredholm operators resp. Inside are open dense sets of invertible operators and these open sets are contractible. Hence the character classes we are after will become trivial over these open sets. The open set in $\mathcal{F}$ is the set of invertible operators on which we have the hypothetical forms $\mathrm{Tr}\,(D^{-1}dD)^{odd}$, which in dim 1 can be precisely defined as

$$\mathrm{Tr}\left((D^*D)^{-s}D^{-1}dD\right)\Big|_{s=0}.$$

What are the corresponding even forms on invertible s.a. Fred. operators?

It is misleading to think of the **forms** $\mathrm{Tr}(D^{-1}dD)^{odd}$ on the open set of invertible operators, and at the same time think of this open set as contractible. It is probably better to think of the actual Dirac operators as being not-too-far-from-finite-dimensional-operators.

What is the basic problem? On one hand I have invertible D with the bi-invariant forms $\mathrm{Tr}\,(D^{-1}dD)^{odd}$. On the other hand I have invertible self-adjoint D and I need natural candidates for the even character classes. By requiring invariance under conjugation by unitaries we **don't** cut down the possibilities completely — there are still the eigenvalues of D. Connes procedure is to associate to D the polarization $J = D/|D|$ and then take the character forms

$$\mathrm{Tr}\left(J(dJ)^{even}\right)$$

on the space of polarizations. It is not yet clear whether this is the good thing to do.

In fact we know that in finite dimensions the different orbits of the unitary group on the space of

self-adjoint matrices have different symplectic forms.
There are lots of invariant forms and so the problem is to find the correct one.

Actually it seems to be bad strategy to use Connes forms. They are too algebraic and don't give any insight into how to regularize. It would be better to go back to the eigenvalues and eigenspaces.

Let's now concentrate on constructing the character forms by using the natural Grassmannian connection on the eigenspace bundles, and by making this converge via a suitable ζ-type argument. Notice that it is easier to get at $d\eta$ than η. Recall

$$\eta_A(s) = \operatorname{Tr}\left(\frac{A}{|A|}|A|^{-s}\right) = \operatorname{Tr}\left(A(A^2)^{-\frac{s+1}{2}}\right)$$

$$\delta\eta_A(s) = -s\operatorname{Tr}\left((A^2)^{-\frac{s+1}{2}}\delta A\right)$$

$$= -s\frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty \underbrace{\operatorname{Tr}\left(e^{-tA^2}\delta A\right)}_{\sim ct^{-1/2}} t^{\frac{s+1}{2}}\frac{dt}{t}$$

$$\sim -s\frac{1}{\Gamma(\frac{s+1}{2})}c\int_0^1 t^{s/2}\frac{dt}{t} \xrightarrow{s\to 0} -\frac{2c}{\Gamma(\frac{1}{2})}$$

Hence $d\eta_A$ is essentially the coeff of $t^{-1/2}$ in $\operatorname{Tr}(e^{-tA^2}dA)$, and so we see $d\eta_A$ is given by a local formula. This suggests to me that it ought to be possible to construct the curvature form for the determinant line bundle without treating the more subtle question of whether the connection form $\operatorname{Tr}((D^*D)^{-s}D^{-1}\delta D)\big|_{s=0}$ is defined.

February 17, 1983

The problem is to construct the character of the index of a family of operators $D: W \longrightarrow V$. Here W and V are spaces of sections of vector bundles. Let's work near a point D_0 in the family and choose $a > 0$ which is not an eigenvalue of $D_0^* D_0$. Then for D near D_0 we get vector bundles $F_a^D W$ (resp. $F_a^D V$) by taking the sum of the eigenspaces of eigenvalue $< a$ for D^*D (resp DD^*). Because $F_a^D W$ is naturally a direct summand of W, there is a Grassmannian connection on $F_a W = \{F_a^D W\}$ and similarly for $F_a V$. Notice that it is immaterial whether W is smooth or L^2 sections since all we need is the projection of W onto $F_a^D W$.

Now I want to see how $\omega_a = ch(F_a W) - ch(F_a V)$ depends on a. Presumably one can calculate $\omega_b - \omega_a$ for $b > a$ in the form $d v_{a,b}$. Then the idea will be to find an expression $v_{a,b} = u_b - u_a$, and then $\omega_a - du_a$ will be independent of a.

Now $v_{a,b}$ should involve the isomorphism

$$F_b W / F_a W \xrightarrow{\sim} F_b V / F_a V$$

induced by D. Also we have to compare $ch(F_b) - ch(F_a)$ with $ch(F_b / F_a)$, where the latter is defined using some connection on F_b / F_a.

This raises the following question: Given vector bundles $E = E' \oplus E''$ and a connection D on E, it induces connections D' and D'' on E' and E'', resp. Is

$$ch(D) = ch(D') + ch(D'') ?$$

In the case I am concerned with I can suppose that I am considering the flag manifold of pairs of subspaces $F_1 \subset F_2$ of fixed dimensions in the space W. Over this $D_{r_1,r_2}(W)$ we have the canonical exact sequence

$$0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{p} E'' \longrightarrow 0$$

of holomorphic vector bundles. Now from an inner product on W these bundles inherit metrics and hence canonical connections. Now I checked already that the connections D' and D'' obtained this way agree with what one gets from D using the decomposition $E \simeq E' \oplus E''$. In other words

$$D' = i^* D i \qquad D'' = p D p^*$$

So a better question might be whether for any exact sequence of holomorphic bundles with metric, does one have additivety for the character?

Special case where this is true: Suppose we are over the Grassmannian: $r_2 = \dim W$, so that E is a trivial bundle. Then we know $ch(D') + ch(D'')$ is an invariant form under the action of the unitary group of W. But the invariant forms give the cohomology, hence one concludes $ch(D') + ch(D'') = 0$.

Another check: Suppose $E' \oplus E'' \simeq E$ and D is a connection on E with $D^2 = 0$. Let $i : E' \to E$, $j : E'' \to E$ be the inclusions, so that $D' = i^* D i$, $D'' = j^* D j$. Then relative to this decomp.

$$D = \begin{pmatrix} D' & \beta \\ \alpha & D'' \end{pmatrix} \qquad \begin{aligned} \alpha &= j^* D i \\ \beta &= i^* D j \end{aligned}$$

and α, β are 0-th order operators. Also

$$0 = D^2 = \begin{pmatrix} (D')^2 + \beta\alpha & \\ & \alpha\beta + (D'')^2 \end{pmatrix}$$

hence $(D')^2 = -\beta\alpha$, $(D'')^2 = -\alpha\beta$. Thus

$$ch(D') + ch(D'') = tr(e^{-\beta\alpha}) + tr(e^{-\alpha\beta})$$

and the point is that

$$tr(\beta(\alpha\beta\alpha \cdots \beta\alpha)) = -tr((\alpha\beta \cdots \beta\alpha)\beta)$$

because β, α are 1-forms. Thus $ch(D') + ch(D'') = 0$

Applying this result formally, say with W and V finite dimensional we get

$$ch(F_a W) = -ch(W/F_a W)$$
$$ch(F_a V) = -ch(V/F_a V)$$

hence

$$ch(F_a W) - ch(F_a V) + d\,\tau\left(W/F_a W \xrightarrow[\sim]{D} V/F_a V\right) = 0$$

where τ is the transgression class. In infinite dimensions we don't expect to get zero, but we can hope to obtain something independent of a.

<u>February 18, 1983</u>

Problem: Computing the differential of the 1-form $\mathrm{Tr}_{reg}(D^{-1}dD)$ in the Riemann surface case. In particular I want to show that if I regularize using a fixed ∂-operator, then this 1-form is closed.

First why is this 1-form closed when things are of trace class, say in finite dimensions? We have

$$\delta(D^{-1}) = -D^{-1}\delta D\, D^{-1}$$

to first order. Thus if we make independent variations $\delta_1 D$ and $\delta_2 D$ we have

$$\delta_1 \mathrm{Tr}(D^{-1}\delta_2 D) = -\mathrm{Tr}(D^{-1}\delta_1 D . D^{-1}\delta_2 D).$$

This is symmetric in $\delta_1 D$, $\delta_2 D$. The skew-symmetrization is essentially the same as

$$d\,\mathrm{Tr}(D^{-1}dD),$$

so the latter is zero.

Another possible exposition:

$$d\,\mathrm{Tr}(D^{-1}dD) = \mathrm{Tr}(dD^{-1}dD)$$
$$= -\mathrm{Tr}(D^{-1}dD . D^{-1}dD)$$

This is zero because $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$ and on the other hand 1-forms anti-commute.

Down to earth: $\mathrm{Tr}(D^{-1}\delta D)$ is like $a_\mu dx^\mu$ and it is closed when

$$d(a_\mu dx^\mu) = (\partial_\nu a_\mu)\, dx^\nu dx^\mu = 0$$

i.e. when $\partial_\nu a_\mu$ is symmetric. In coord. free notation a_μ is like $\mathrm{Tr}(D^{-1}\delta_1 D)$ where $\delta_1 D$ is one tangent vector and then $\partial_\nu a_\mu$ is like

$$\delta_2 \, \mathrm{Tr}(D^{-1}\delta_1 D) = -\,\mathrm{Tr}(D^{-1}\delta_2 D \cdot D^{-1}\delta_1 D)$$

corresponding to another tangent vector $\delta_2 D$. Thus the form is closed because the trace is symmetric.

So now I want to look at the regularized trace

$$\mathrm{Tr}_{reg}(D^{-1}\delta_1 D) = \mathrm{Tr}\left[(D^{-1}-P_D)\delta_1 D\right]$$

where P_D is a parametrix depending on D.
Take the variation in another direction δD:

$$\mathrm{Tr}\left(\left(-D^{-1}\delta D\, D^{-1} - \delta P_D\right)\delta_1 D\right).$$

We want to show this is symmetric in $\delta D, \delta_1 D$.
We can $\delta D, \delta_1 D \in \Gamma(\mathrm{End}(E)\otimes T^{0,1})$ are supported in small coordinate patches.

Let's calculate δP_D: Recall that if

$$D = (\partial_{\bar z} + \alpha)\, d\bar z$$

then we lift it to a connection

$$\nabla = (\partial_z + \beta)\, dz + (\partial_{\bar z} + \alpha)\, d\bar z$$

and then

$$P_D(z,z') = \frac{i}{2\pi} F_\nabla(z,z') \underbrace{\left(-\partial_{z'} \log r(z,z')^2 \, dz'\right)}_{\left(\frac{1}{z-z'} + \text{smooth}\right) dz'}$$

where $F_\nabla(z,z')$ is radial parallel transport, hence

$$F_\nabla(z,z') = I - \beta(z')(z-z') - \alpha(z')\overline{(z-z')} + \text{second order in } z-z', \overline{z-z'}$$

$$P_D(z,z') = \frac{i}{2\pi}\left(\frac{1}{z-z'} - \beta(z') - \alpha(z')\frac{\overline{z-z'}}{z-z'} + \cdots\right) dz'$$

Assume now as D changes to $D+\delta D$ β remains fixed. Then

$$\delta P_D(z,z') = \frac{i}{2\pi}\left(-\delta\alpha(z')\frac{\overline{z-z'}}{z-z} + \text{terms}\,\frac{(\overline{z-z'})^n}{z-z'} + \text{smooth}\right)dz'$$

hence

$$\delta P_D(z,z') = \frac{i}{2\pi}\left(-\delta\alpha(z')\frac{\overline{z-z'}}{z-z'}\right)dz' + \text{continuous kernel}$$

For the remaining calculations I set $\alpha = 0$

February 19, 1983:

Problem: On the space of Fredholm operators $D: W \longrightarrow V$ I would like to define character forms at least formally. One possibility is to use the embedding of such operators into a Grassmannian of subspaces of $W \times V$ given by the graph $D \mapsto \Gamma_D$. Then one can pull-back the character forms on the Grassmannian. It would be interesting to know what these forms were over the set of $D: W \to V$ and the set of invertible D even in finite dimensions. These forms restricted to $\operatorname{Hom}(W,V)$ are necessarily boundaries.

So consider $G_p(W \times V)$ where $p = \dim W = \dim V$. The character forms are invariant for the action of $U(W \times V)$. If I consider the unitary isomorphisms $U(W;V)$ embedding in the Grassmannian via the graph, then $U(W;V)$ will be stable under $U(W) \times U(V)$ and hence the character forms will be biinvariant forms on $U(W;V)$. This implies ch_1 vanishes and probably the other components do also.

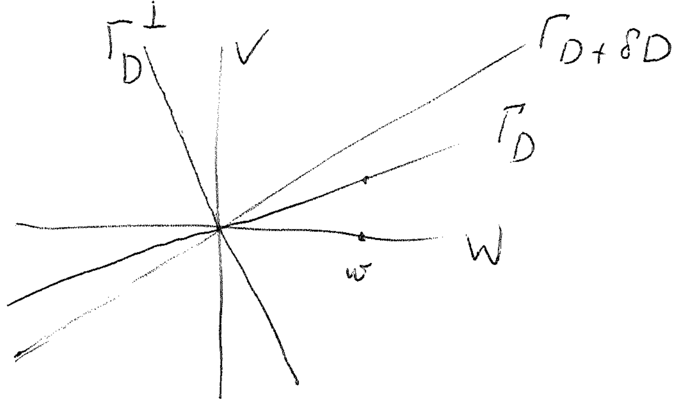
So now I should recall how the character forms look on the Grassmannian. Fix a point say the subspace Γ in $W \times V$. Then a tangent vector to Γ can be identified with a map $A: \Gamma \longrightarrow \Gamma^\perp$. The first form is probably

$$A, B \longmapsto \operatorname{Tr}(A^*B) - \operatorname{Tr}(B^*A).$$

and in general are probably anti-symmetrizes

$$A_1, \ldots, A_{2p} \longmapsto \operatorname{Tr}\left(A_1^* A_2 \cdots A_{2p-1}^* A_{2p}\right)$$

So now suppose $\Gamma = \Gamma_D = \{(w, Dw) \in W \times V\}$.
Then $\Gamma_D^{\perp} = \{(-D^* v, v) \mid v \in V\}$. So given a change δD what is the corresponding operator $A : \Gamma_D \longrightarrow \Gamma_D^{\perp}$?



Take $(w, Dw + \delta D w) \in \Gamma_{D+\delta D}$ and write it in $\Gamma_D \oplus \Gamma_D^{\perp}$.

$$(w, Dw + \delta D w) = (w, Dw) + \underbrace{(0, \delta D w)}_{(w_1, Dw_1) + (-D^* v_1, v_1)}$$

$$\Longrightarrow \quad w_1 = D^* v_1$$
$$\delta D w = DD^* v_1 + v_1 = (1 + DD^*) v_1$$
$$v_1 = (1 + DD^*)^{-1} \delta D w$$

so $\quad (w, Dw + \delta D w) = (w + w_1, D(w + w_1)) + (-D^* v_1, v_1)$
which means $A : \Gamma_D \to \Gamma_D^{\perp}$ satisfies

$$A(w + w_1, D(w + w_1)) = (-D^* v_1, v_1)$$

But now identify $\Gamma_D = W$ and $\Gamma_D^{\perp} = V$ in the obvious way. We get

$$A(w + w_1) = v_1$$
$$A\left[1 + D^*(1 + DD^*)^{-1}\delta D\right] = (1 + DD^*)^{-1}\delta D$$

or finally to first order in δD we get simply

$$\boxed{A = (1+DD^*)^{-1}\delta D.}$$

Now we have to compute A^* but must be careful to observe the difference in norms under $\Gamma_0 \sim W$. So let $\langle \omega | \omega \rangle$ denote the Γ_D norm of an elt. of ω. Precisely

$$\langle \omega | \omega \rangle$$

$$= \|(\omega, D\omega)\|^2 = \|\omega\|^2 + \|D\omega\|^2$$

$$= (\omega | (1+D^*D)\omega)$$

Then

$$\langle v | A\omega \rangle = (v | (1+DD^*)A\omega)$$

$$= (v | \delta D \omega)$$

$$= (\delta D^* v | \omega)$$

$$= ((1+D^*D)^{-1}\delta D^* v \,|\, (1+D^*D)\omega)$$

$$= \langle (1+D^*D)^{-1}\delta D^* v \,|\, \omega \rangle$$

and consequently

$$\boxed{A^* = (1+D^*D)^{-1}\delta D^*}$$

Past experience suggests this $(1+D^*D)^{-1}$ is apt to be the wrong sort of thing. Maybe it is not a good idea to use the graph construction to go from Fredholm operators to a Grassmannian.

Let's see if we can prove an explicit index theorem over the circle. Here $\mathcal{G}$ = maps from S to U_n acts on $\mathcal{A}$ = unitary connections in trivial n-diml. bundle over S.
Write a connection in the form $D = d + A$, $A = \alpha\, dt$ where α is a map from S^1 to $\mathrm{Lie}(U_n)$. Then we get a self-adjoint elliptic operator $\frac{1}{i}(\partial_t + \alpha)$ acting on the Hilbert space of sections of the trivial n-diml. bundle. So we have a family $\mathcal{A} \longrightarrow$ s.a. Fred. ops. which is equivariant under $\mathcal{G}$. The problem is to define the character of the index of this family.

February 20, 1983

Today I want to examine the circle case: $\mathcal{G} = \mathrm{Maps}(S, U_n)$, $\mathcal{A} =$ unitary connections $D = (\partial_t + \alpha)dt$ on trivial n-diml. v.b. over S. Associated to a connection D is the covariant differentiation operator $D/dt = \partial_t + \alpha$ on sections of the trivial bundle. This is skew adjoint hence $\frac{1}{i}(\partial_t + \alpha)$ is s.a. So we get an equivariant map $\mathcal{A} \longrightarrow$ s.a. Fred ops on $L^2(S)^n$ for the action of $\mathcal{G}$.

My goal remains to define the character of this family as an equivariant form over $\mathcal{A}$. This character should be an odd form.

Because we are over the circle we know that the $\mathcal{G}$ orbits on $\mathcal{A}$ are described by the monodromy which is a conjugacy class in $U = U_n$. Precisely if we pick a basepoint and let $\Omega U \subset \mathcal{G}$ be the based loops, then ΩU acts freely on $\mathcal{A}$ and

$$\Omega U \backslash \mathcal{A} \xrightarrow{\sim} U$$

where the map associates to a connection the monodromy transformation at the base point. Then the U action on the left corresponds to conjugation. Hence

$$H^{\cdot}_{\mathcal{G}}(\mathcal{A}) = H^{\cdot}_{U}(U)$$

Now we know from past work that in the fibration

$$U \longrightarrow PU \times^{U} (U) \longrightarrow BU$$

for the conjugation action, the fibre is ~~totally~~ non-homologous to zero. Hence $H_U(U) \longrightarrow\!\!\!\!\to H^{\cdot}(U)$ and upon lifting the primitive generators we obtain an isom $\quad H^{\cdot}_{U}(U) = H^{\cdot}(BU) \otimes H^{\cdot}(U).$

Let's review the proof: The diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & PU \times^{U} U \arrow[r] \arrow[d] & P(U\times U) \times^{(U\times U)} U \sim BU \arrow[d] \arrow[dl, "\Delta"'] \\
pt \arrow[r] & BU \arrow[r, "\Delta"'] & BU \times BU
\end{tikzcd}$$

(conjugation)

leads to maps of cohomology sequences:

$$\begin{tikzcd}
 & 0 \arrow[r] & H^{2i}(BU\times BU, BU) \arrow[r] \arrow[d] & H^{2i}(BU\times BU) \arrow[r, "\Delta^*"] \arrow[d, "\Delta^*"] & H^{2i}(BU) \\
0 \arrow[r] & H^{2i-1}(PU\times^{U} U) \arrow[r] \arrow[d] & H^{2i}(BU, PU\times^{U} U) \arrow[r] \arrow[d] & H^{2i}(BU) & \\
 & H^{2i-1}(U) \arrow[r, "\sim"] & H^{2i}(pt, U) & &
\end{tikzcd}$$

So if $\varphi \in H^{2i}(BU)$, then $\Delta^*(\varphi\otimes 1 - 1\otimes\varphi) = 0$ and so by the diagram we get a definite class $\tilde{\varphi} \in H^{2i-1}(PU\times^{U} U) = H^{2i-1}_{U}(U)$. The only thing I still want to see is that the image of $\tilde{\varphi}$ in $H^{2i-1}(U)$ is the suspension map of φ. This follows from the map

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & PU \times^{U} U \arrow[r] \arrow[d] & P(U\times U) \times^{(U\times U)} U \arrow[d] \\
pt \arrow[r] & BU \arrow[r] & BU\times BU.
\end{tikzcd}$$

(left action)

In other words we ~~start~~ with $U\times U$ acting on U and then restrict to the subgroups $pt \subset \begin{matrix} U\times pt \\ \Delta U \end{matrix} \subset U\times U$, etc.

Anyway we now know that the basic forms $\mathrm{Tr}(g^{-1}dg)^{\mathrm{odd}}$ on U can be refined to equivariant differential forms for the conjugation action. Notice that [redacted] with the exception of $\mathrm{Tr}(g^{-1}dg)$ these forms do not come from the space $U \backslash U$ of conjugacy classes. For example take SU_2; the volume form will be refineable to an equivariant 3-form, but it doesn't come from the space of conjugacy classes which is a 1-simplex.

Problem: Explicitly describe equivariant forms on U for the conjugation action refining $\mathrm{Tr}(g^{-1}dg)^{\mathrm{odd}}$.

Question: Does equivariant cohomology have an operational significance like characteristic classes? Can I describe an element α of $H_G(M)$ with mentioning the classifying space of G?

For example an element α of $H_G^*(pt)$ [redacted] is a natural way of assigning to any principal G-bundle $P \to X$ a cohomology class on X, namely

$$\alpha \in H_G^*(pt) \longrightarrow H_G^*(P) = H^*(X).$$

Generalizing, an elt $\alpha \in H_G^*(M)$ is a natural way of assigning to a principal G-bundle $P \to X$ plus G-map $P \xrightarrow{f} M$ an elt of $H^*(X)$, namely

$$\alpha \in H_G^*(M) \xrightarrow{f^*} H_G^*(P) = H^*(X)$$

Next pass to differential forms. It is my understanding that given a [redacted] connection in a principal bundle $P \to X$, then equivariant forms on P give rise to forms on X. Assuming this an equivariant form on M will then associate to a principal bundle $P \to X$ plus connection plus G-map $P \to M$ a form on X. This certainly is OK when $M = pt$, because then an equivariant form on M is something in $S(\mathfrak{g}^*)^G$ which can be applied to the curvature to get a form in X.

Now take $M = G$ with G acting on itself by conjugation. Then a G-map $P \to G$ is the same as a section of $P \times^G (G) = \underline{Aut}(P)$. Hence an equivariant form for G acting on itself by conjugation should assign to any $P \to X$ plus connection plus auto of P a form on X.

Specializing to $G = U_n = U$ we see that given a n-diml vector bundle E with metric and unitary connection over X, there should be a way to assign to a unitary automorphism g of E, odd forms on X which restrict to $\mathrm{Tr}(g^{-1}Dg)^{\text{odd}}$ over each fibre. This doesn't quite make senses. The idea is that $g^{-1}[D,g] = g^{-1}Dg - D \in \Gamma(\mathrm{End}(E) \otimes T^*)$ hence we can raise it to an odd power and take the trace to get odd forms on X. But the problem is that this form isn't closed:

$$d\ \mathrm{Tr}\left(g^{-1}[D,g]\right)^j = \mathrm{Tr}\left([D, (g^{-1}[D,g])^j]\right)$$

$$= j\ \mathrm{Tr}\left(\underbrace{[D, g^{-1}[D,g]]}_{-g^{-1}[D,g]\,g^{-1}[D,g] \;+\; g^{-1}[D^2,g]} (g^{-1}[D,g])^{j-1}\right)$$

Now $\quad \mathrm{Tr}\left(g^{-1}[D,g]\right)^k = 0 \quad$ for k even,

so for j odd

$$d\left(\mathrm{Tr}\left(g^{-1}[D,g]\right)^j\right) = j\, \mathrm{Tr}\left(g^{-1}[D^2,g]\left(g^{-1}[D,g]\right)^{j-1}\right)$$

For $j=1$, $\quad \mathrm{Tr}\left(g^{-1}[D^2,g]\right) = \mathrm{Tr}\left(g^{-1}D^2g - D^2\right) = 0$

but it doesn't seem to be zero in general.

Actually I solved this problem (see 558-560 for Chern's transg. formula, but the question was raised 434 and solved 439.) See also trans forml. 538

One first takes two bundles E, F with connection and any isomorphism $g: E \longrightarrow F$, not necessarily preserving the connection. (This is exactly what you get by a map $X \longrightarrow P(U\times U) \overset{(U\times U)}{\times} U$ where $BU \times BU$ represents pairs of bundles + connections)

Then we have

$$ch(E) - ch(F) = d$$ transgression form coming from joining the connection in E to g^{-1} of the connection in f.

So if $F = E$, then the transg. form is closed.

In down to earth terms

$$\mathrm{tr}\left(e^{(g^{-1}Dg)^2}\right) - \mathrm{tr}\left(e^{D^2}\right)$$

$$= d\int_0^1 dt\, \mathrm{tr}\left(e^{D_t^2} B\right)$$

where $\quad D_t = D + tB \qquad B = g^{-1}Dg - D = g^{-1}[D,g]$

But $e^{(g^{-1}Dg)^2} = e^{g^{-1}D^2g} = g^{-1}e^{D^2}g \quad$ so the two characters are $=$.

February 22, 1984

Consider $M =$ circle S^1. Here is where the first case of the explicit index theorem should be understood. Last night I [redacted] thought of getting away from the $(\mathcal{G}, \mathcal{a})$ situation and instead formulating the problem for a general family. For example, if I consider the standard family of operators $D: E \longrightarrow E \otimes T^*$ over S^1 parametrized by $\mathcal{a}$, then I can descend to the quotient $(\Omega U) \backslash \mathcal{a} = U$ and get a family of elliptic operators on S^1 parametrized by U. The difference is that by descending the bundle $\tilde{E}$ over $\mathcal{a} \times M$ I get a bundle over $U \times M$ which is not obviously trivial, even though $\tilde{E} = pr_2^*(E)$ is trivial. In fact the bundle $\tilde{E}$ over $U \times M$ represents the tautological bundle over $B(\Omega U) \times M$ obtained from the action of ΩU on E over M. Actually it would [redacted] be interesting to identify it, also to understand what sort of connection it might have, as well as the corresponding Hilbert bundle over U.

So take the trivial n-diml bundle E over $S^1 = M$ lift it to $\tilde{E} = pr_2^*(E)$ over $\mathcal{a} \times S$ which is again trivial, and has a natural action of $\mathcal{G}$ on it. Now I have a basepoint of S^1 chosen and I want to divide out by the action of ΩU using the isom. $(\Omega U) \backslash \mathcal{a} = U$ to identify $(\Omega U) \backslash \mathcal{a}$. Thus $\tilde{E}$ will descend to a vector bundle $\bar{E}$ over $U \times S$, which will be absolutely canonical. Hence it is fairly clear that $\bar{E}$ will be the bundle obtained by starting with the trivial bundle over $U \times [0,1]$ and identifying at the ends by the tautological isom.

How is the descent carried out? We have to take a point $(A, m) \in \mathcal{A} \times M$ and identify the fibre of $\tilde{E}$ there with the corresponding fibre of $\bar{E}$ over $U \times M$ at the point (T_A, m). The fibre of $\bar{E}$ at (T_A, m) is $\mathbb{C}^n$ if $0 \leq m < 1$, and it's clear we take the fibre of $\tilde{E}$ at (A,m), use parallel transport to go from m back to 0.

Summary: I start with $\tilde{E} = pr_2^*(E)$ on $\mathcal{A} \times M$ where $M = S$ and I descend for the action of ΩU to get a bundle $\bar{E}$ over $U \times S$ which comes equipped with a connection in the vertical direction with the obvious monodromy. So therefore I will have a family of elliptic s.a. operators on S parameterized by U. For this family the bundle $\bar{E}$ over $U \times S$ is not the pull-back of a bundle on S. We know this because we can compute the character, say.

To compute the character what I have to do is choose a connection on $\bar{E}$. The obvious one is to take the zero connection d on the trivial bundle over U and it's transform under the canonical auto. g and then take the linear path of connections from d to $g^{-1}dg$. Then $D = \frac{d}{dt}dt + d + t\, g^{-1}dg$

$$D^2 = dt \cdot g^{-1}dg + (t^2 - t)(g^{-1}dg)^2$$

$$\pi_* ch(\bar{E}) = \int_0^1 dt \ \mathrm{Tr}\left(e^{(t^2-t)(g^{-1}dg)^2} g^{-1}dg\right)$$

so we get the odd character forms on U.

The next question concerns the Hilbert bundle over U.

February 23, 1983

A central problem seems to be to [redacted] understand what an analytical proof of the index theorem for families might mean or be. [redacted] Let's try to get this idea clear.

Suppose we have a family of elliptic operators on the fibres of $\pi: X \to Y$, say $D: E \to F$. The index of this family is a K^0 element on Y. The symbol of the operator D is a K element [redacted] in $T^*_{X/Y}$, and then the index theorem for families is just a K-theory theorem. [redacted] The topological fmla for the index involves a pair of Gysin–Thom isoms. All [redacted] this stuff has been souped-up with the Kasparov[redacted] theory.

Index of a family $D: E \to F$ on the fibres of $\pi: X \to Y$ can be defined as follows. With metrics chosen we get Hilbert bundles $\pi_* E$, $\pi_* F$ [redacted] over Y; the fibre at y is a suitable Hilbert space (Sobolev space) of sections of E (or F) over the fibre X_y. Then D gives a Fredholm operator between these Hilbert bundles. By virtue of Keiper's thm these Hilbert bundles are trivial, hence if we pick trivializations (which are unique up to homotopy by Keiper's thm), then we get a map from Y to Fredholm operators. This is the index of the family.

Feb. 23

Question: What would be an analytical formula for this index?

Example: Consider the G-index theorem of Atiyah and Singer. One has a compact gp G acting on the compact manifold M and an equiv. operator $D : \Gamma(E) \to \Gamma(F)$ which is elliptic transversal to the orbits. In this case there is a distributional index which is a central distribution on G. Atiyah's version of the distribution is that $\operatorname{Ker} D$, $\operatorname{Cok} D$ have finite multiplicity for any irreducible representation of G; the index is thus an infinite integral linear combination of characters. But also he uses that for any convolution operator $f \in C^\infty(G)$, the trace on $\operatorname{Ker} D$ and $\operatorname{Cok} D$ are defined.

However let me consider the case where D is actually elliptic over M. Then over BG I get a family of elliptic operators. Actually this holds over the base Y of any principal G-bundle. So we actually see the Hilbert bundles, the operators, the curvature explicitly but decomposed as a direct sum of finite-dimensional bundles indexed by the different irreducible representations of G. In this case the character diff. form of the index will be the difference of the character diff. form of the representations $\operatorname{Ker} D$, $\operatorname{Cok} D$ of G. Hence we expect infinitely many components of the Chern character to be $\neq 0$.

The index thm. when D is an invariant ell. operator over $G/H = M$ was worked out by Bott. (Morse symposium)

Let's return to the circle S. Recall that we have a vector bundle E over $Y \times S$ with partial connection $D: E \longrightarrow E \otimes_S T_S^*$ in the vertical direction. If dt is the volume form on S, and $D = D_{/dt}\, dt$, then $\frac{1}{i} D/dt : E \longrightarrow E$ is the family of s.a. elliptic operators I am interested in.

To make sense of the right side of the index formula as a form on Y, I have to extend D to a connection of E over $Y \times S$. Then I should get a connection on the Hilbert bundle $\pi_* E$ over Y consisting of L^2-sections of E over the various fibres $\{y\} \times S$, $y \in Y$. I want to somehow combine this connection on $\pi_* E$ with the family of s.a. operators $\frac{1}{i} D/dt$ on $\pi_* E$ to get odd Chern character forms on Y. From the homotopy viewpoint, the bundle $\pi_* E$ is trivial in a unique-up-to-homotopy-way, hence we have a map from Y to s.a. Fred which does give odd Chern character classes on Y.

If this program is going to work it should work locally on Y. Hence I can suppose E is trivial over $Y \times S$ and that I am given a connection on it which is completely arbitrary. Here $\pi_* E$ is just the trivial bundle over Y which fibre is the L^2 vector functions on the circle $L^2(S^1)^n$. The vertical connection gives

$$D/dt = \partial_t + \alpha(y,t)$$

and the horizontal connection gives

$$\nabla' = \partial_{y^\mu} dy^\mu + \alpha_\mu(y,t)\, dy^\mu .$$

[redacted] <u>Review:</u> Consider a connection [redacted] on a vector bundle E over $Y \times S$, let $\pi_* E$ be the Hilbert bundle over Y whose fibres are L^2-sections of E along the fibres of $\pi : Y \times S \longrightarrow Y$. The horizontal part of the connection on E should give a connection on $\pi_* E$, and the vertical part should give [redacted] self-adjoint Fredholm operators on the fibres of $\pi_* E$. The problem is to extract from these two things odd forms on Y.

I can work locally on Y and suppose that E is trivial. The connection is then given by operators

$$D_0 = \frac{\partial}{\partial t} + A_0(y,t) \qquad \text{vertical}$$

$$D_\mu = \frac{\partial}{\partial y^\mu} + A_\mu(y,t) \qquad \text{horizontal}$$

The curvature of $\pi_*(E)$ for its connection is

$$[D_\mu, D_\nu] = \frac{\partial A_\nu}{\partial y^\mu} - \frac{\partial A_\mu}{\partial y^\nu} + [A_\mu, A_\nu]$$

which is a 2-form on Y with values in bounded skew-adjoint operators on $\pi_* E$. The operators $[D_\mu, D_\nu]$ are multiplication operators on the fibres of $\pi_*(E) = L^2(S^1)^n$, hence there is no possibility of taking traces. Also it is not just a question of regularizing the traces because I want odd diml. forms.

February 24, 1983 (still dizzy)

Let E be a hermitian vector bundle with connection over $Y \times M$ where M is a Riemann surface. Then we get a family of $\bar{\partial}$-operators on M parametrized by Y; this just uses the vertical part of the connection. The horizontal part of the connection induces a connection on the Hilbert bundles $\pi_*(E)$ and $\pi_*(E \otimes T_M^{0,1})$. Recall that $\pi_*(E)_y =$ L^2-sections of E over the fibre $\{y\} \times M$, and I suppose that I have chosen a metric on M consistent with the complex structure.

So over Y I have the Hilbert bundles and the $\bar{\partial}$-operator $D = \{D_y\} : \pi_*(E) \longrightarrow \pi_*(E \otimes T_M^{0,1})$ and the problem is now to define the Chern character of this triple.

Notice first that if I have two bundles E, F each with connection, and I give an isom $f : E \xrightarrow{\sim} F$ [redacted] not nec. compatible with the connection, then I get a formula

$$ch(E) - ch(F) = d\tau(\text{connections on } E, F; f).$$

In the present situation E, F are infinite dimensional and the Chern characters are not defined because [redacted] the curvature endomorphisms don't have traces. Also the operator D is not an isomorphism. The idea should be that the characters of $\pi_*(E)$ and $\pi_*(E \otimes T^{0,1})$ are divergent but that their difference can be regularized using the operator D where it is invertible.

The first case to understand is where D is invertible. Before, when working over a [redacted] where the bundles $\pi_*(E)$ and $\pi_*(E \otimes T_M^{0,1})$ are flat I was puzzled by the fact that the component ch_1 is non-trivial but that all the other ch_j, $j \geq 2$ are zero. But now I should be able to get a verifiable formula, independent of regularization in the range $j \geq 2$ where the traces are well-defined.

To be specific suppose we take a trivial bundle E over an elliptic curve M and pull-back to $\tilde{E}$ over $Y \times M$. Then $\pi_*(E) = \pi_*(E \otimes T^{0,1})$ = trivial bundle over Y with fibre $L^2(M, E)$. The connection on $\tilde{E}$ is then $\nabla = \nabla^h + \nabla^v$ and ∇^v induces the $\bar{\partial}$-operator. Put $T : \pi_* E \to \pi_* E$ for the $\bar{\partial}$-operator. I want to suppose T is invertible. Let D be the connection on $\pi_* E$ induced by ∇^h. Then I want to make sense of the formula

$$\operatorname{tr}\left(e^{D_1^2}\right) - \operatorname{tr}\left(e^{D_0^2}\right) = d \int_0^1 dt \ \operatorname{tr}\left(e^{D_t^2} B\right)$$

in the infinite diml. setting. D_0 = the connection D
$D_1 = T^{-1} D T$ and $D_t = D_0 + tB$ where

$$B = T^{-1} D T - D = T^{-1}[D, T]$$

is the standard 1-form.

$$\begin{aligned} D_t^2 = (D + tB)^2 &= D^2 + t(DB + BD) + t^2 B^2 \\ &= D^2 + t\,[D, T^{-1}[D,T]] + t^2 B^2 \\ &= D^2 + t\left(T^{-1}[D^2, T]\right) + (t^2 - t) B^2 \end{aligned}$$

Now D^2 is a multiplication operator, so are $[D^2, T]$, $[D,T]$ whereas T^{-1} is of order -1. Thus if we look at the ch_2 components

$$\tfrac{1}{2}\left[\operatorname{tr}(T^{-1} D T)^4 - \operatorname{tr}(D^4)\right] = d \int_0^1 dt \ \operatorname{tr}\left(D_t^2 B\right)$$

Now my idea is that $tr(e^{D_1^2}) - tr(e^{D^2})$ is meaningless in this situation but we can make sense out of the right side. But in the case of ch_2 this means that I have to make sense out of

$$\mathrm{Tr}\left[\left(\underbrace{D^2}_{0} + t\,(\underbrace{T^{-1}[D^2,T]}_{-1}) + (t^2-t)\underbrace{B^2}_{-2}\right)B\right]$$

Only the last term is free of regularization difficulties.

So I can conclude from this little calculation that if I proceed in the way I have been thinking, then every component of the character will require <u>regularization.</u>

It is time to go over the computation of the curvature of the determinant line bundle when there is cohomology. The first thing is to work around points where $D : \Gamma(E) \longrightarrow \Gamma(E\otimes T^{0,1})$ is surjective. In this case $\mathrm{Ker}(D)$ is a subbundle of $\Gamma(E)$ with an induced L^2-metric. So by orthogonal projection a connection on the Hilbert bundle, e.g. the trivial one, induces a connection on $\{\mathrm{Ker}\,D\}$. If ψ_α is an orthonormal frame for $\{\mathrm{Ker}(D)\}$, i.e. $D_y\psi_\alpha(y) = 0$, the orthogonal projection is $\sum_\alpha \psi_\alpha\langle\psi_\alpha|$ and the connection takes

$\psi(y) = \psi_\alpha(y)f_\alpha(y)$ into $d\psi(y) = \psi_\alpha(y)\,df_\alpha(y) + d\psi_\alpha(y)\,f_\alpha(y)$

projected into $\mathrm{Ker}\,D_y$, which gives

$$\psi_\alpha(y)\,df_\alpha(y) + \psi_\beta(y)\langle\psi_\beta(y)|d\psi_\alpha(y)\rangle f_\alpha(y)$$

Take the line bundle case to simplify the notation and I get $\quad \nabla\psi = \psi\langle\psi|d\psi\rangle$

where d denotes derivative in the y direction.

February 25, 1983 Lecture 5

ζ function determinants or analytic torsion.

A s.a. positive elliptic operator.

$$\zeta_A(s) = \operatorname{Tr}(A^{-s})$$ convergent for $Re(s) >> 0$ then it has a merom. cont.

$$\Gamma(s)\zeta(s) = \int_0^\infty \underbrace{\operatorname{Tr}(e^{-tA})}_{\sim \sum_{k=-\frac{\dim}{\operatorname{ord}}} a_k t^k} t^s \frac{dt}{t}$$

simple poles at $s = \frac{\dim}{\operatorname{ord}}, \frac{\dim - 1}{\operatorname{ord}} \dots\dots$

Known that ζ is analytic at $s=0$ so we can define

$$\text{zeta fn } \det(A) = e^{-\zeta'(0)}$$

Finite dims.
$$\zeta(s) = \sum \lambda^{-s}$$
$$\zeta'(s) = \sum -\lambda^{-s}\log\lambda$$
$$\zeta'(0) = -\sum\log\lambda = -\log\det A$$

What is $\delta \log \det_\zeta(A)$? I will assume that $\zeta_A(s)$ is smooth in both A, s. Then

$$\delta\zeta_A'(s) = (\delta\zeta_A(s))'$$

$$\delta\zeta_A'(0) = (\delta\zeta_A(s))'\Big|_{s=0} = \lim_{s\to 0}\frac{\delta\zeta_A(s) - \delta\zeta_A(0)}{s}$$

$$\delta\zeta_A(s) = -s\operatorname{Tr}(A^{-s-1}\delta A) = -s\frac{1}{\Gamma(s+1)}\int_0^\infty \operatorname{Tr}(e^{-tA}\delta A)\, t^{s+1}\frac{dt}{t}$$
$$= -s\frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}(e^{-tA}A^{-1}\delta A)\, t^s\frac{dt}{t}$$

TFAE [redacted] (i) $\delta\zeta_A(0) = 0$

(ii) In asymptotic expansion for $\operatorname{Tr}(e^{-tA}\delta A)$ the coeff of t^{-1} is zero

(iii) $\operatorname{Tr}(e^{-tA} A^{-1} \delta A)$ has an asymptotic expansion in powers of t through t^0.

Try again: Use that $A \longmapsto \zeta_A(s)$ is a smooth map into analytic fns. near $s=0$. Then

$$\delta[\zeta_A'(s)] = [\delta \zeta_A(s)]' \qquad ' = \frac{d}{ds}$$

$$\delta[\zeta_A'(0)] = \lim_{s\to 0} \frac{\delta\zeta_A(s) - \delta\zeta_A(0)}{s} \qquad \text{finite.}$$

But we also have

$$\delta\zeta_A(s) = -s \operatorname{Tr}(A^{-s-1}\delta A).$$

This is analytic at $s=0$. Hence

$$\delta\zeta_A(0) = 0 \iff \operatorname{Tr}(A^{-s-1}\delta A) \quad \text{finite at } s=0$$

$$\Downarrow$$

$$-\delta\zeta_A'(0) = \operatorname{Tr}(A^{-s}A^{-1}\delta A)\Big|_{s=0}$$

Further $$\operatorname{Tr}(A^{-s}A^{-1}\delta A) = \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}(e^{-tA}A^{-1}\delta A)\, t^s \frac{dt}{t}$$

It seems to be true that one can integrate asymptotic expansions. Thus if

$$f(t) = \operatorname{Tr}(e^{-tA}A^{-1}\delta A)$$

$$f'(t) = \operatorname{Tr}(e^{-tA}\delta A) \sim \cdots + b_{-1}t^{-1} + b_0 + b_1 t + \cdots\cdots$$

$$\Longrightarrow f(t) = \cdots + b_{-1}\log t + c + b_0 t + b_1 \frac{t^2}{2} + \cdots\cdots$$

So $b_{-1} = 0 \iff \operatorname{Tr}(A^{-s}A^{-1}\delta A)\big|_{s=0}$ finite

and in this case $\operatorname{Tr}(A^{-s}A^{-1}\delta A)\big|_{s=0} = c =$ constant term in asympt. exp. $\operatorname{Tr}(e^{-tA}A^{-1}\delta A)$

February 26, 1983

The basic problem is to find an analytical formula for the index of a family of elliptic operators. Such a formula should provide me with a means for proving the index theorem for families. There has to be a way to get from the analytical [redacted] index to the integration [redacted]-over-the-fibre formula for it.

My present idea is to define a Chern character form on the base, by suitable $\int$ fn. methods. [redacted]

A general principle is that in [redacted] asking for an analytical formula for the index we replace the integers by the complex numbers and interpret the trace. Hence the analytical index of a family should be in $K(\text{base}) \otimes \mathbb{C}$ and so we are replacing the index by its Chern character. So it is again [redacted] a kind of trace, $tr(e^{\nabla^2})$.

One way to obtain an analytical formula for the index of [redacted] a single ell. operator D is to [redacted] choose a parametrix P. Then $PD = I - K^0$, $DP = I - K^1$ where K^0, K^1 are smoothing operators. Then

$$\operatorname{Ind}(D) = tr(K^0) - tr(K^1).$$

Let us now use the principle that traces are to be replaced by Chern characters. Then if I have a family $K = \{K_y\}$ of smooth kernels it is natural to ask if there is associated a Chern character form on Y. It is clear that we can combine the operators $K, [D,K], [D^2,K], \ldots\ldots$ in many ways to get forms so we need a better idea of what we want.

Let us suppose K is a finite rank projection operator: $K^2 = K$. I am supposing K operates in a (Hilbert) bundle with connection D. I take the induced connection KDK on the image of K. Change K back to e with $e^2 = e$. Then the ~~curvature~~ connection is $\nabla = eDe$ and the curvature is

$$\nabla^2 = eDeDe$$

$$= e[D,e]\underbrace{De}_{[D,e]+eDe}e + eD^2e$$

And $[D,e^2] = [D,e]e + e[D,e] \implies e[D,e]e = 0$

Thus

$$\nabla^2 = e\left(D^2 + [D,e]^2\right)e$$

and so the Chern character is

$$tr\left(e^{\nabla^2}\right) = tr\left(e \cdot e^{D^2 + [D,e]^2}\right)$$

where one should avoid confusion with the e's!

The above is not very helpful. Perhaps what I should be doing is to think of these operator K^i as being in some C^*-algebra and we want to apply a kind of trace to this C^*-algebra. The actual C^*-algebra is fairly clear, namely continuous sections of the bundle of compact operators. We know that this C^*-algebra doesn't have interesting traces, even if we take smooth sections. Thanks to Connes theory this C^*-algebra has interesting n-traces.

So now what is suggested is that we must refine our calculation of the index slightly. It is more than **just** the operators K^0, K^1 in the same way that an idempotent operator e is more than an operator: it gives rise to cycles $e \otimes \cdots \otimes e$ (odd no. of times) in the Connes theory.

So the question arises as to whether we can define such cycles starting with just D, P ?

Further possibilities for defining the index of a family as stable Connes cycles ~~on~~ the ring of smooth functions on the base.

$$\mathrm{tr}(K \otimes K \otimes K) - \mathrm{tr}(L \otimes L \otimes L)$$

where $\quad PD = 1 - K, \quad DP = 1 - L. \qquad$ or

$$\int_{t_1+t_2+t_3=t} \mathrm{tr}\,[e^{-t_1 D^*D} \otimes e^{-t_2 D^*D} \otimes e^{-t_3 D^*D}) \quad - \quad \text{similar}$$

The trouble is I can't show that these are cycles, and possibly I have the wrong formula. In any case I don't have enough intuition.

February 27, 1983

Formal structure of the proof of the index thm: The difference

$$(*) \qquad \mathrm{Tr}\left(e^{-tD^*D}\right) - \mathrm{Tr}\left(e^{-tDD^*}\right)$$

is independent of t. Formally

$$\frac{d}{dt}\mathrm{Tr}\left(e^{-tD^*D}\right) = -\mathrm{Tr}\left(e^{-tD^*D}D^*D\right)$$
$$= -\mathrm{Tr}\left(D^*e^{-tDD^*}D\right)$$
$$= -\mathrm{Tr}\left(e^{-tDD^*}DD^*\right) = \frac{d}{dt}\mathrm{Tr}\left(e^{-tDD^*}\right)$$

As $t \to \infty$, $e^{-tD^*D} \longrightarrow P^+$ = projection on $\mathrm{Ker}\, D$ and $e^{-tDD^*} \longrightarrow P^-$ = projection on $\mathrm{Ker}\, D^*$. Thus the difference $(*)$ approaches $\mathrm{Ind}(D)$ as $t \to \infty$. On the other hand as $t \downarrow 0$ one can use the asymptotic expansion for the heat kernels to get an expression for $(*)$ as the integral of a density over M. By algebraic methods this density is shown to be the appropriate characteristic class.

Vague idea: In the case of a family generalize $\mathrm{Tr}\left(e^{-tD^*D}\right)$ to be a differential form on the base. Exploit the similarity of this with the Chern character $\mathrm{Tr}\left(e^{\Omega}\right)$. Quite possibly $\mathrm{Tr}\left(e^{-tD^*D}\right)$ generalized should be a path integral involving [redacted] fermion fields [redacted] representing the normal variation of D and D^*

February 28, 1983

Bott-Chern paper: If E is a holomorphic v.b. then any metric determines a connection and hence a character. Two different metrics lead to character forms which differ by something in the image of $d'd''$, so one gets the concept[redacted] of [redacted] refined Chern classes.

Consider first the case of a line bundle L. Then we know that the curvature form is

$$d''d' \log N(s)$$

where $N(s) = |s|^2$ for the norm s. [redacted] Review this:

Put $N_{ij} = (s_i|s_j)$, [redacted] $\nabla s_i = \theta_{ki} s_k$. Then

$$dN_{ij} = (\nabla s_i | s_j) + (s_i | \nabla s_j) = \overline{\theta_{ki}} N_{kj} + N_{ik}\theta_{kj}$$

whence if θ is of type $(1,0)$ we have

$$\theta [redacted] = N^{-1} d'N.$$

Then the curvature is

$$d\theta + \theta\theta = d''\theta + \overbrace{d'\theta + \theta\theta}^{d'(N^{-1}d'N) + (N^{-1}d'N)^2 = 0}$$

$$\boxed{\Omega = d''\theta = d''(N^{-1}d'N)}.$$

Now I want to consider an infinitesimal variation in the metric

March 1, 1983

Need Bott-Chern formulas for Chern forms in the holomorphic setting. Let E be a holomorphic vector bundle; then any metric on E determines a connection and hence character. If s_i is a holom. frame let $N_{ij} = (s_i|s_j)$ for the metric, [redacted] and let $\nabla s_i = \theta_{ki} s_k$ be the connection. It is characterized by $\theta_{ki} \in T^{1,0}$ and

$$d(s_i|s_j) = (\nabla s_i|s_j) + (s_i|\overbrace{\nabla s_j}^{\theta_{kj} s_k})$$

$$dN_{ij} = \overline{\theta_{ki}} N_{kj} + N_{ik}\theta_{kj}$$

$$\Longrightarrow \quad \theta = N^{-1}d'N$$

Relative to the holomorphic frame we have

$$[\text{redacted}] \quad \nabla = d + \theta = \underbrace{d' + \theta}_{} + d''$$
$$= \nabla' + d''$$

but this decomposition is true in general. Notice that [redacted] the curvature is

$$\Omega = \nabla^2 = \underbrace{(\nabla')^2}_{0} + \underbrace{[\nabla', d'']}_{d''\theta} + \underset{\overset{\shortparallel}{0}}{(d'')^2}$$

because
$$(\nabla')^2 = (d'+\theta)^2 = d'\theta + \theta\theta$$
$$= d'(N^{-1}d'N) + (N^{-1}d'N)^2 = 0.$$

Thus ∇' is an anti-holomorphic ~~structure~~ on E naturally defined in terms of the holomorphic ~~structure~~ and metric. Obviously what you get by using the metric to identify E with conjugate of $E^\vee$. One has

Prop: [redacted] Given a metric on a holom. v.b. E we can conjugate the $\bar\partial$ operator $D'': E \to E \otimes T^{0,1}$ defining the holomorphic structure into a ∂-operator $D' : E \longrightarrow E \otimes T^{1,0}$. Then $D = D' + D''$ is the connection on E determined by the metric + holom. structure. In particular the curvature

$$D^2 = (D' + D'')^2 = D'D'' + D''D'$$

is of type $(1,1)$.

Now let us look at how the Chern forms vary with a change in the metric. I can consider an infinitesimal change to be denoted by a dot. Then only D' changes so we have

$$\dot{D} = \dot{D}' \quad \text{of type } (1,0).$$

One has

$$\mathrm{Tr}\left(e^{D^2}\right)^{\bullet} = \mathrm{Tr}\left(e^{D^2}[D,\dot{D}]\right) = d\,\mathrm{Tr}\left(e^{D^2}\dot{D}\right)$$

in general. Since $\mathrm{Tr}(e^{D^2})$ is of type (p,p), and $\dot{D}$ is of type $(1,0)$, we have

$$\mathrm{Tr}(e^{D^2})^{\circ} = d''\,\mathrm{Tr}(e^{D^2}\dot{D}), \qquad d'\,\mathrm{Tr}(e^{D^2}\dot{D}) = 0.$$

Now I want to write $\mathrm{Tr}(e^{D^2}\dot{D})$ in the form $d'\,\mathrm{Tr}(e^{D^2}L)$, so I compute

$$d'\,\mathrm{Tr}(e^{D^2}L) = \mathrm{Tr}([D', e^{D^2}L]).$$

Now I know $[D, e^{D^2}] = 0$ and e^{D^2} is of type (p,p), hence I can conclude

$$[D', e^{D^2}] = 0 \quad \text{and} \quad [D'', e^{D^2}] = 0.$$

(Alternatively the second one is clear locally using a holom. frame where $D^2 = d''\theta$ and $D'' = d''$; by symmetry using an anti holom. frame we would have $D^2 = d'\theta$, $D' = d'$.)

Anyway

$$d' \operatorname{Tr}(e^{D^2}L) = \operatorname{Tr}([D', e^{D^2}L])$$
$$= \operatorname{Tr}(e^{D^2}[D', L]).$$

and so I want L to be such that

$$[D', L] = \dot{D}$$

The idea finally is that L is the operator on E which is hermitian and gives the change in the inner product (which is a hermitian form) relative to the metric. Locally

$$\dot{N}_{ij} = (s_i | s_j)^{\cdot} = (s_i | L s_j) = (s_i | L_{kj} s_k)$$
$$= N_{ik} L_{kj} \implies L = N^{-1}\dot{N}$$

Then

$$[D', L] = [d' + \theta, L]$$
$$= d'(N^{-1}\dot{N}) + [N^{-1}d'N, N^{-1}\dot{N}]$$
$$= \cancel{-N^{-1}d'N N^{-1}\dot{N}} + N^{-1}d'\dot{N} + \cancel{N^{-1}d'N N^{-1}\dot{N}}$$
$$- N^{-1}\dot{N} N^{-1} d'N$$

and

$$\dot{D} = \dot{\theta} = (N^{-1}d'N)^{\cdot} = -N^{-1}\dot{N}N^{-1}d'N + N^{-1}d'\dot{N}.$$

Thus we have

$$[D', L] = \dot{D}$$

as desired.

<u>Prop</u>: Let the infinitesimal change in the metric on E be given by the hermitian operator L. Then the change in the connection is given by

$$\dot{D} = [D', L]$$

and the change in the Chern character is

$$\mathrm{Tr}(e^{D^2})^{\bullet} = d''d'\{\mathrm{Tr}(e^{D^2}L)\}$$

<u>Remark</u>: Actually the formula $\dot{D} = [D', L]$ perhaps should be thought of as coming from an action of gauge transformations on the [redacted] space of [redacted] ∂-operators on E. We have fixed the $\bar{\partial}$-operator D'' and then can consider all possible ∂-operators D' such that $(D')^2 = 0$. [redacted] The group of complex gauge transformations acts on this [redacted] space of D'. The Chern forms will be of type (p,p) for the connection $D = D' + D''$ and the above formulas show that [redacted] in the direction of gauge orbits the character changes only by something in [redacted] the image of $d''d'$.

So now let us go on to the problem of an exact sequence of holomorphic vector bundles

$$0 \longrightarrow E_{\mathrm{I}} \longrightarrow E \longrightarrow E_{\mathrm{II}} \longrightarrow 0.$$

This time we will move around in the space of [redacted] d'-operators. Let's split the exact sequence on the C^∞-level; we can do this by a metric if we want. In any case we have

$$D'' = \begin{pmatrix} D''_{\mathrm{I}} & * \\ 0 & D''_{\mathrm{II}} \end{pmatrix}.$$

Now suppose I take

$$D' = \begin{pmatrix} D'_{\mathrm{I}} & 0 \\ 0 & D'_{\mathrm{II}} \end{pmatrix}$$

for say the d'-operators corresponding to the d''-operators with respect to the metrics in E_I, E_{II}. Then clearly we obtain a connection

$$D = D' + D'' = \begin{pmatrix} D_I & * \\ 0 & D_{II} \end{pmatrix}$$

for which we have

$$\operatorname{Tr} e^{D^2} = \operatorname{Tr}\left(e^{D_I^2}\right) + \operatorname{Tr}\left(e^{D_{II}^2}\right).$$

At this point it should be possible to work out the Bott-Chern refined Whitney product formula. The idea is to construct a suitable path in the space of D' operators on $E = E_I \oplus E_{II}$ which will carry us from the D' defined by the metric to $D'_I \oplus D'_{II}$ and always in directions with $\dot{D}'_t = [D'_t, L_t]$.

First question: Is D' gauge equivalent to $D'_I \oplus D'_{II}$? This should be equivalent to the question (analogous) for $\bar\partial$-operators. Thus given the exact sequence of holomorphic bundles

$$0 \to E_I \longrightarrow E \longrightarrow E_{II} \longrightarrow 0$$

is it true that the $\bar\partial$-operator on E and the $\bar\partial$-operator on $E_I \oplus E_{II}$ are isomorphic under an isomorphism $E \simeq E_I \oplus E_{II}$. This would imply $E \simeq E_I \oplus E_{II}$ as holomorphic bundles which is impossible.

So what Bott-Chern probably do is to use the linear structure on $H^1(M, \operatorname{Hom}(E_{II}, E_I))$. One considers the gauge transformation given by

$$g_t = \begin{pmatrix} t & \\ & 1 \end{pmatrix} \quad \text{on} \quad E_I \oplus E_{II}$$

whose effect on the holomorphic ~~structure~~ is

$$g_t D'' g_t^{-1} = \begin{pmatrix} D''_{I} & t\,* \\ 0 & D''_{II} \end{pmatrix}$$

Now as $t \to 0$ we get a path of character forms all differing by elements of $\operatorname{Im} d''d'$ going from the character of D to that of D_{I} + that of D_{II}.

Let's try this out. Relative to $E = E_{I} \oplus E_{II}$ we have the $\bar\partial$-operator

$$D'' = \begin{pmatrix} D''_{I} & V \\ 0 & D''_{II} \end{pmatrix}$$

where $V : E_{II} \longrightarrow T^{0,1}_{M} \otimes E_{I}$ is [redacted] of 0th-order and is such that $(D'')^2 = 0$. Because the splitting is orthogonal for the metric on E the corresponding d'-op has the form

$$D' = \begin{pmatrix} D'_{I} & 0 \\ W & D''_{I} \end{pmatrix}$$

where $W : E_{I} \longrightarrow T^{1,0}_{M} \otimes E_{II}$ is essentially the adjoint of V. I propose to use the gauge transf.

$$g_t = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

whence I get the family of d'-ops.

$$D'_t = g_t^{-1} D' g_t = \begin{pmatrix} D'_{I} & 0 \\ tW & D'_{II} \end{pmatrix}$$

satisfying $\frac{d}{dt} D'_t = [D'_t , g_t^{-1} g_t]$ $\qquad L_t = g_t^{-1}\dot g_t = \begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 0 \end{pmatrix}$

The family of connections is

$$D_t = D_t' + D'' = \begin{pmatrix} D_I & V \\ tW & D_{II} \end{pmatrix}$$

According to earlier calculations

$$\frac{d}{dt} \operatorname{Tr}\left(e^{D_t^2}\right) = d'd'' \operatorname{Tr}\left(e^{D_t^2} L_t\right)$$

and what I want to do is to integrate this from $t=0$ to $t=1$. The problem is that L_t has a simple pole at $t=0$, which means that we have something logarithmically divergent. Let's expand

$$\operatorname{Tr}\left(e^{D_t^2} L_t\right) = \frac{1}{t} \underbrace{\operatorname{Tr}\left\{e^{D_0^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right\}}_{\operatorname{Tr}\left(e^{D_I^2}\right).} + O(1) \ldots$$

$$D_0 = \begin{pmatrix} D_I & V \\ 0 & D_{II} \end{pmatrix}$$

Since $\operatorname{Tr}\left(e^{D_I^2}\right)$ is a closed form we can just throw it away.

Notice that $D_t = \begin{pmatrix} D_I & V \\ 0 & D_{II} \end{pmatrix} + t\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}$

so that D_t^2 is linear in t. Possibly one might be able to write

$$\operatorname{Tr}\left(e^{D_t^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

in a simple form. Let's compute the ch_1 component

$$\frac{d}{dt} \operatorname{Tr}\left(D_t^2\right) = d'd'' \operatorname{Tr}\left(D_t^2 L_t\right)$$

$$D_t^2 = \begin{pmatrix} D_I & V \\ tW & D_{II} \end{pmatrix}\begin{pmatrix} D_I & V \\ tW & D_{II} \end{pmatrix} = \begin{pmatrix} D_I^2 + tVW & D_I V + V V_{II} \\ t(D_{II}W + WD_I) & D_{II}^2 + tWV \end{pmatrix}$$

So:

$$\mathrm{Tr}(D_t^2) = \mathrm{Tr}(D_I^2) + \mathrm{Tr}(D_{II}^2) + t\{\mathrm{Tr}(VW) + \mathrm{Tr}(WV)\}.$$

By the symmetry of the trace this last sum should be zero.

Let's ~~state~~ this more generally (see 581-82).

Prop. Let D be any connection on $E = E_I \oplus E_{II}$ and write $D = \begin{pmatrix} D_I & \beta \\ \alpha & D_{II} \end{pmatrix}$. Then

$$\mathrm{Tr}(D^2) = \mathrm{Tr}(D_I^2) + \mathrm{Tr}(D_{II}^2)$$

Proof:

$$D^2 = \begin{pmatrix} D_I & \beta \\ \alpha & D_{II} \end{pmatrix}\begin{pmatrix} D_I & \beta \\ \alpha & D_{II} \end{pmatrix} = \begin{pmatrix} D_I^2 + \beta\alpha & D_I\beta + \beta D_{II} \\ \alpha D_I + D_{II}\alpha & \alpha\beta + D_{II}^2 \end{pmatrix}$$

$$\mathrm{Tr}(D^2) - \mathrm{Tr}(D_I^2) - \mathrm{Tr}(D_{II}^2) = \mathrm{Tr}(\beta\alpha) + \mathrm{Tr}(\alpha\beta)$$

and this is zero by symmetry of the trace

Return to the formula $\frac{d}{dt}\mathrm{Tr}(D_t^2) = d'd''\,\mathrm{Tr}(D_t^2 L_t)$.

$$\mathrm{Tr}(D_t^2 L_t) = \frac{1}{t}\mathrm{Tr}(D_I^2) + \mathrm{Tr}(VW)$$

so the natural question is why is

$$d'd''\{\mathrm{Tr}(VW)\} = 0 \ ?$$

It should follow from $D_{II}''V + VD_I'' = 0$ and the analogous thing for W.

Alternative proof of above prop. is to use the linear path D_t between D and $D_I \oplus D_{II}$. One has

$$\frac{d}{dt}\mathrm{Tr}(D_t^2) = d\,\mathrm{Tr}(\dot{D}_t), \quad \dot{D}_t = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} \Rightarrow \mathrm{Tr}(\dot{D}_t) = 0$$

Next I really want to understand the curvature of the determinant line bundle in the general case (over a Riemann surface, say, so that the operator D varies holomorphically). We [redacted] fix a point D_0 and let $a > 0$ be ***outside*** of the spectrum of $D_0^* D_0$. Then we have $W = F^D_{<a} W \oplus F^D_{>a} W$, $V = F^D_{<a} V \oplus F^D_{>a} V$ where $D : W \to V$ as usual, and $F_{<a} W$, $F_{<a} V$ are subbundles for D near D_0.

Now the bundles $F_{<a} V$ have curvature, and this I should know how to compute from perturbation theory. In fact this whole business is reminiscent of the partition function $Tr(e^{-\beta H})$ game. [redacted]

March 2, 1983

Let's return to the curvature calculation for the determinant line bundle belonging to the family of $\bar\partial$-operators over a Riemann surface. All these operators go from $V^0 = \Gamma(E)$ to $V^1 = \Gamma(E \otimes T^{0,1})$

I have constructed the determinant line bundle in two different ways; one exhibits the holomorphic structure, the other exhibits the metric. I should probably check that the two approaches define the same C^∞-bundle, and do this carefully without just checking the formulas fibre by fibre.

The difficulty is that if we want to compare the index bundle constructed from

$$\begin{array}{ccccccccc} & & V^0 & \xrightarrow{D} & V^1 & & \\ & & \cup & & \cup & & \\ 0 \longrightarrow \operatorname{Ker} D \longrightarrow & & D^{-1}F & \longrightarrow & F & \longrightarrow \operatorname{Cok} D \longrightarrow 0 \end{array}$$

where F is a finite diml subspace of V^1, with the index bundle constructed from

$$\begin{array}{ccc} V^0 & \xrightarrow{D} & V^1 \\ \cup & & \cup \\ F_a V^0 & \longrightarrow & F_a V^1 \end{array}$$

then we want to put F and $F_a V^1$ in the same finite dimensional subspace of V^1. The idea will be to allow F to vary with D smoothly. Then we get a smooth formula for the index bundle, namely

$$[D^{-1}F] - [F].$$

All I have to check is that any two choices for F can be embedded in a third; this should be a straightforward application of transversality. It should be possible to ignore this point until later. In fact it is not important to take F and F' and embed them in an F'', rather one describes things in terms of a (W, F) and then shows the independence of F by keeping W fixed.

So now we have the two descriptions of the determinant line bundle, one holomorphic, the other metric, and so there is a connection. So whenever I have a non-vanishing section of $\mathcal{L}$ I get a connection form. My problem is to compute this connection.

Special case: Suppose we work around a point where D_0 is invertible. Then corresponding to the canonical section is the connection form

$$i(\delta D)\theta = \operatorname{Tr}\left((D^*D)^{-s} D^{-1}\delta D\right)\Big|_{s=0}$$

$$= \text{constant term in the asymptotic expansion as } t \to 0 \text{ of } \operatorname{Tr}\left(e^{-tD^*D} D^{-1}\delta D\right)$$

Modulo the problem of whether $\operatorname{Tr}\left((D^*D)^{-s}D^{-1}\delta D\right)$ is regular at $s=0$, this makes sense for any <u>family of</u> invertible operators.

To compute the curvature we take $d''\theta$, i.e. look at how it is antiholomorphic in D:

$$\delta_1 \operatorname{Tr}\left(e^{-tD^*D} D^{-1}\delta D\right) = \operatorname{Tr}\Big(\underbrace{\delta_1 e^{-tD^*D}}_{\int_0^t e^{-(t-t_1)D^*D}\left[-(\delta_1 D^*)D\right]e^{-t_1 D^*D}\,dt_1} D^{-1}\delta D\Big) + \begin{smallmatrix}\text{linear}\\ \text{in } \delta_1 D\end{smallmatrix}$$

$$= -\int_0^t \operatorname{Tr}\left(e^{-(t-t_1)D^*D}\,\delta_1 D^*\, e^{-t_1 DD^*}\,\delta D\right) dt_1 \quad + \begin{smallmatrix}\text{linear}\\ \text{in } \delta_1 D\end{smallmatrix}$$

Thus the formula for the curvature is going to be

$$i(\delta_1 D^*)\, i(\delta D)\, d''\theta \;=\; \pm \text{ constant term in asymptotic expansion of}$$

$$\int_0^t dt_1 \; \mathrm{Tr}\left(e^{-(t-t_1)D^*D}\, \delta_1 D^*\, e^{-t_1 DD^*}\, \delta D\right)$$

Can we calculate this using eigenvalues?

Let us assume to simplify that D^*D has a simply spectrum with eigenvalues λ_n^2 and normalized eigenfunctions ψ_n; analogously DD^* has normalized eigenfunctions ψ_n' and the eigenvalues λ_n^2. Put 0 modes in later. We can suppose

$$\lambda_n \psi_n' = D\psi_n \qquad D^*\psi_n' = \lambda_n \psi_n .$$

Then

$$\mathrm{Tr}\left(e^{-(t-t_1)D^*D}\, \delta_1 D^*\, e^{-t_1 DD^*}\, \delta D\right)$$

$$= \sum_{n,m} e^{-(t-t_1)\lambda_m^2} \langle m|\delta_1 D^*|n\rangle e^{-t_1\lambda_n^2} \langle n|\delta D|m\rangle .$$

$$\int_0^t e^{-(t-t_1)\lambda_m^2 - t_1\lambda_n^2}\, dt_1 = e^{-t\lambda_m^2} \int_0^t e^{+t_1(\lambda_m^2-\lambda_n^2)}\, dt_1$$

$$= e^{-t\lambda_m^2}\, \frac{e^{t(\lambda_m^2-\lambda_n^2)} - 1}{\lambda_m^2 - \lambda_n^2}$$

$$= \frac{e^{-t\lambda_n^2} - e^{-t\lambda_m^2}}{\lambda_m^2 - \lambda_n^2}$$

So the general answer is

$$\text{curvature} = \pm \sum_{m,n} \frac{e^{-t\lambda_n^2} - e^{-t\lambda_m^2}}{\lambda_n^2 - \lambda_m^2} \langle m | \delta_1 D^* | n \rangle \langle n | \delta D | m \rangle$$

One sees that this expression is not independent of t.

March 3, 1983:

Good formalism:

$$\frac{d}{dt} \operatorname{Tr}\left(e^{-tD^*D}\right) = -\operatorname{Tr}\left(e^{-tD^*D} D^*D\right)$$
$$= -\operatorname{Tr}\left(D^* e^{-tDD^*} D\right)$$
$$= -\operatorname{Tr}\left(e^{-tDD^*} DD^*\right) = \frac{d}{dt} \operatorname{Tr}\left(e^{-tDD^*}\right)$$

I want to carry this argument thru in a [redacted] supersetting. [redacted] At this point I have some choices. I can review the Schwinger calculation first or I can try to push thru the general case. The first approach seems best as you could hope to digest the formulas Witten gave you.

First example - the harmonic oscillator with a source. The unperturbed operator is described by

$$H_0 = a^* \omega a \qquad (\text{short for } a_i^* \omega_{ij} a_j)$$

The perturbation is

$$H_{int} = a^* J + \tilde{J} a$$

where $J, \tilde{J}$ are time-dependent and of compact support in time. [redacted] The S-matrix is

$$S = T\left\{ e^{-i\int_{-\infty}^{\infty} dt \, [\hat{a}^*(t) J + \tilde{J} \hat{a}(t)]} \right\}$$

where $\hat{a}(t) = e^{iH_0 t} a e^{-iH_0 t} = e^{-i\omega t} a$

and $\qquad \widehat{a^*}(t) = a^* e^{i\omega t}$. $\quad$ Then one has

$$S = e^{-\iint_{t>t'} dt\,dt'\; \tilde{J}(t) e^{-i\omega(t-t')} J(t)} \; e^{a^*\left(-i\int dt\, e^{i\omega t} J\right)} \; e^{\left(-i\int dt\, \tilde{J} e^{-i\omega t}\right) a}$$

This formula can be explained via Feynman diagrams with the vertices:

for each time t (possibly also for the indices: J_μ).

In the above we were thinking of $J, \tilde{J}$ as functions of t, but in deriving the formula for S the only thing we really use is that the values $J(t)$, $\tilde{J}(t')$ of $J, \tilde{J}$ commute [redacted] mutually and with the operators a, a^*.

2nd example is the fermion analogue of the preceding one. To simplify I want to assume that there is a single a, so our initial Hilbert space is a 2-diml spin space. Now [redacted] the $J(t)$ and the $J(t')$ have to be quantities, (i.e. operators?), which mutually anti-commute and which anti-commute with a and a^*. (Note: a, a^* don't anti-commute: $aa^* + a^*a = 1$.)

[redacted] What is the simplest example? If I want to integrate the Schroedinger equation

$$\begin{aligned} i\partial_t \psi &= (H_0 + H_{int})\psi \\ &= (\omega a^* a + a^* J + \tilde{J} a)\psi \end{aligned}$$

then I have to be working in an algebra containing the operators $a, a^*, J(t), \tilde{J}(t')$. [redacted] One thing to do is to tensor spin space with a exterior algebra with generators ε_μ, then take $J(t) = f_\mu(t)\varepsilon_\mu$ and similarly for $\tilde{J}(t')$. The universal situation is to take the exterior alg. on the space of pairs of functions

$J, \tilde{J}$.

Next let us go back to $V^0 \underset{D^*}{\overset{D}{\rightleftarrows}} V^1$ and let us take variations $B = (\delta D)$, $\tilde{B} = (\delta_1 D)^*$. Introduce quantities $\varepsilon, \tilde{\varepsilon}$ which probably we want to anti-commute $\varepsilon^2 = \tilde{\varepsilon}^2 = 0$ $\varepsilon\tilde{\varepsilon} + \tilde{\varepsilon}\varepsilon = 0$. Put $\mathcal{D} = D + \varepsilon B$, $\tilde{\mathcal{D}} = D^* + \tilde{\varepsilon}\tilde{B}$. Can I evaluate

$$\mathrm{Tr}\left(e^{-t\tilde{\mathcal{D}}\mathcal{D}}\right) - \mathrm{Tr}\left(e^{-t\mathcal{D}\tilde{\mathcal{D}}}\right)$$

and do I get something interesting? We have

$$\tilde{\mathcal{D}}\mathcal{D} = (D^* + \tilde{\varepsilon}\tilde{B})(D + \varepsilon B) = D^*D + \tilde{\varepsilon}\tilde{B}D + D^*\varepsilon B + \tilde{\varepsilon}\tilde{B}\varepsilon B$$

and I haven't yet decided where $\varepsilon, \tilde{\varepsilon}$ are to commute or anticommute with operators of degree 1 like $B, \tilde{B}, D, D^*$.

Let's look at the first order terms

$$e^{-t\tilde{\mathcal{D}}\mathcal{D}} = e^{-tD^*D} - \int_0^t dt_1\, e^{-(t-t_1)D^*D}\left(\tilde{\varepsilon}\tilde{B}D + D^*\varepsilon B\right)e^{-t_1D^*D}$$

$$e^{-t\mathcal{D}\tilde{\mathcal{D}}} = e^{-tDD^*} - \int_0^t dt_1\, e^{-(t-t_1)DD^*}\left(\varepsilon BD^* + D\tilde{\varepsilon}\tilde{B}\right)e^{-t_1DD^*}$$

If I take the trace the first order terms cancel: The ε part of the first is

$$-\int_0^t dt_1 \quad D^*\varepsilon\left(e^{-(t-t_1)DD^*}Be^{-t_1D^*D}\right)$$

and the ε part of the second is

$$-\int_0^t dt_1 \quad \varepsilon\left(e^{-(t-t_1)DD^*}B\, e^{-t_1D^*D}\right)D^*$$

where I use that ε commutes with DD^*. Actually I want those formulas for future reference. Clearly one has

$$\mathrm{Tr}\left(-\int_0^t dt_1\, e^{-(t-t_1)D^*D}\, D^*\varepsilon B\, e^{-t_1D^*D}\right) = \mathrm{Tr}\left(D^*\varepsilon B\, e^{-tD^*D}(-t)\right)$$

$$\mathrm{Tr}\left(-\int_0^t dt_1\, e^{-(t-t_1)DD^*}\, \varepsilon BD^*e^{-t_1DD^*}\right) = \mathrm{Tr}\left(\varepsilon BD^*e^{-tDD^*}(-t)\right)$$

and these are equal for reasons given before.

Next I compute the second order terms and hope for something non-zero but constant in t. To simplify the calculations I change εB to B and $\tilde{\varepsilon}\tilde{B}$ to $\tilde{B}$.

$$\tilde{\mathcal{D}}\mathcal{D} = (D^* + \tilde{B})(D + B) = D^*D + D^*B + \tilde{B}D + \tilde{B}B$$

The second order order part of $e^{-t\tilde{\mathcal{D}}\mathcal{D}}$ is

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \quad e^{-(t-t_1)D^*D}\left(D^*B + \tilde{B}D\right) e^{-(t_1-t_2)D^*D}\left(D^*B + \tilde{B}D\right) e^{-t_2 D^*D}$$

$$- \int_0^t dt_1 \;\; e^{-(t-t_1)D^*D}\, \tilde{B}B\, e^{-t_1 D^*D}$$

March 6, 1983:

7th lecture - computation of the curvature of $\mathcal{L}$ for the analytic torsion metric.

Begin with generalities about positive self-adjoint elliptic operators A. Assume A is 2nd order on an even diml. M.

$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tA}\, t^s \frac{dt}{t}$$

$$\mathrm{Tr}(e^{-tA}) \approx \cdots + a_{-1}t^{-1} + a_0 + a_1 t + \cdots \qquad \text{as } t \to 0$$

$$\Gamma(s)\,\zeta(s) = \Gamma(s)\mathrm{Tr}(A^{-s}) = \cdots\cdots + \frac{a_{-1}}{s-1} + \frac{a_0}{s} + \cdots + \frac{a_k}{s+k} + \text{entire for } \mathrm{Re}(s) > -k-1$$

$$\Longrightarrow \begin{cases} \zeta(s) \text{ analytic at } s = 0 \\ \zeta(0) = a_0 . \end{cases}$$

Now suppose A depends smoothly on a parameter u, and let $\delta = \frac{\partial}{\partial u}$.

$$\delta\,\mathrm{Tr}(e^{-tA}) \sim \cdots + \delta a_{-1} t^{-1} + \delta a_0 + \delta a_1 t + \cdots$$
$$\| $$
$$\mathrm{Tr}(e^{-tA}(-t\delta A))$$

$$-\mathrm{Tr}(e^{-tA}\delta A) \sim \cdots + \frac{\delta a_{-1}}{t^2} + \frac{\delta a_0}{t} + \delta a_1 + \cdots$$

OK to integrate asymptotic expansions

$$\mathrm{Tr}(e^{-tA}A^{-1}\delta A) \sim \cdots + -\frac{\delta a_{-1}}{t} + \delta a_0 \log t + c + \delta a_1 t + \cdots$$

where c is a constant.

Prop: $\delta\zeta(0) = 0 \iff \mathrm{Tr}(e^{-tA}A^{-1}\delta A)$ has an asympt. expansion in powers of t (no $\log t$ term occurs).

Translate in terms of ζ:

$$\delta\zeta(s) = -s\,\mathrm{Tr}\,(A^{-s-1}\,\delta A)$$

$$-\frac{\delta\zeta(s)}{s} = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\,(e^{-tA}A^{-1}\delta A)\,t^s\,\frac{dt}{t}$$

$$\delta\zeta'(0) = \frac{d}{ds}(\delta\zeta(s))\Big|_{s=0} = \frac{\delta\zeta(s)-\delta\zeta(0)}{s}\Big|_{s=0}$$

Prop: If $\delta\zeta(0)=0$, then

$$-\delta\zeta'(0) = \mathrm{Tr}\,(A^{-s-1}\,\delta A)\Big|_{s=0}$$

= constant term in asymptotic expansion for $\mathrm{Tr}\,(e^{-tA}A^{-1}\delta A)$.

In other words if $\delta\zeta(0)=0$, then

$$\delta\,\cancel{\log}\,\det_\zeta(A) = \zeta\text{-regularization of } \mathrm{Tr}\,(A^{-1}\delta A).$$

Next $\log|s|^2 = -\zeta'_{D^*D}(0)$

$$\delta\log|s|^2 = \mathrm{Tr}\left(e^{-tD^*D}(D^*D)^{-1}\overbrace{\delta(D^*D)}^{D^*\delta D+\delta D^*D}\right)_{\text{const. term}}$$

$$= \underbrace{\mathrm{Tr}\left(e^{-tD^*D}D^{-1}\delta D\right)_{\text{const term}}}_{d'\log|s|^2\ =\ \text{connection form }\theta.} + \text{c.c.}$$

$$\therefore\ i(\delta D)\theta = \mathrm{Tr}\left(e^{-tD^*D}D^{-1}\delta D\right)_{\text{const term}}$$

March 5, 1983

Idea: Suppose E, F are vector bundles with connection and $D : E \longrightarrow F$ is a map of constant rank, then assuming metrics around so that we have induced connections, one has a formula

$$[ch(E) - ch(F)] - [ch(\mathrm{Ker}\, D) - ch(\mathrm{Cok}\, D)] = d\{ \quad \}.$$

Can the term in $\{ \quad \}$ be constructed out of the heat kernels e^{-tD^*D}, e^{-tDD^*} which are identity for $t=0$ and the projectors on $\mathrm{Ker}\, D$, $\mathrm{Cok}\, D$ at $t = \infty$.

Better idea to look at first: Given two holom. vector bundles E, F with metrics and an isomorphism between them, the Bott-Chern theory gives

$$ch(E) - ch(F) = d''d'\{ \quad \}$$

So in particular over $GL_n(\mathbb{C})$ one has the tautological isomorphism between trivial bundles of rank n to which this can be applied. This will replace the form

$$\mathrm{Tr}\,(g^{-1}dg)^{\mathrm{odd}}$$

which is holomorphic by something in the image of d'.

Go back over Bott-Chern. The basic idea is to look at a *fixed* holomorphic bundle E and the possible connections extending the $\bar{\partial}$-operator: $D = D' + D''$. One knows that an infinitesimal change in the metric produces a $\mathring{D} = \mathring{D}'$ of the form $[D', L]$. Then

$$(\mathrm{tr}\, e^{D^2})^{\circ} = \mathrm{tr}(e^{D^2}[D, \mathring{D}]) = d\, \mathrm{tr}\,(e^{D^2}\underset{1,0}{\mathring{D}}) = d''\, \mathrm{tr}(e^{D^2}\mathring{D})$$

Also $[D, e^{D^2}] = 0 \implies [D', e^{D^2}] = [D'', e^{D^2}] = 0$ so

$$d'\, \mathrm{tr}\,(e^{D^2}L) = \mathrm{tr}([D', e^{D^2}L]) = \mathrm{tr}(e^{D^2}[D', L]) = \mathrm{tr}(e^{D^2}\mathring{D})$$

Thus $$\left(\operatorname{tr} e^{D^2}\right)^{\circ} = d'' d' \operatorname{tr}\left(e^{D^2} L\right)$$

I [redacted] now want to apply this to go between the Chern characters belonging to the standard metric $|s|^2$ on the trivial bundle, and the metric [redacted] $|gs|^2$ where g is a holomorphic automorphism. Hence $N = g^* g$ so the connection form is $\theta = N^{-1} d' N = (g^* g)^{-1} g^* dg = g^{-1} dg$. The linear deformation $D = d + t g^{-1} dg$ leads to the forms $\operatorname{tr}(g^{-1} dg)^{\text{odd}}$. Here I am trying to understand a non-linear deformation which results from a path in the metrics. The obvious choice is to use $(g^* g)^s = N_s$, s going from 0 to 1.

Then
$$L = N_s^{-1} \dot{N}_s = \log(g^* g)$$
$$\operatorname{Tr}(\log g^* g) = \log \det(g^* g)$$
$$= \log |\det g|^2 = \log \det g + \text{c.c.}$$

and $$d' \operatorname{Tr}(\log g^* g) = \operatorname{Tr}(g^{-1} dg)$$ as required.

But for the higher forms I seem to get a mess:

$$\theta_s = N_s^{-1} d' N_s \qquad d' N_s = d' (g^* g)^s$$

Formally we have the form
$$\int_0^1 ds \operatorname{Tr}\left(e^{D_s^2} L\right) = \int_0^1 ds \operatorname{Tr}\left(e^{d''\left[(g^*g)^{-s} d'(g^*g)^s\right]} \log(g^* g)\right)$$

Nevertheless I think it ought to be possible to go from $\operatorname{Tr}(g^{-1} dg)^3$ which is a $(3,0)$-type form to a [redacted] form of the form

$$d' (\text{type } 1,1)$$

March 6, 1983

Yesterday I concluded that it is unlikely for the thickened version of $Tr(e^{-tD^*D}) - Tr(e^{-tDD^*})$ to be independent of t. More likely is for the cohomology class to be independent of t and for the $t \to \infty$ limit to have some interpretation as the character of $\operatorname{Ker} D - \operatorname{Cok} D$, assuming these are [redacted] vector bundles.

It is therefore imperative to understand precisely what is happening [redacted] with the 2-forms. The idea is that for each eigenvalue of D^*D I get a [redacted] 2-form on the base, and I should be able to calculate everything very easily.

The basic data should be the operator $D: V^0 \to V^1$ and two variations $\delta_i D$ of it corresponding to two tangent vectors in the base. I should be able to write down 2-forms corresponding to the subbundle of V^1 belonging to a given eigenvalue or range of eigenvalues.

Let's suppose we have a simple eigenvalue $\lambda > 0$ for D^*D, and let ψ be an eigenvector for it of norm 1. Then corresponding to a first order change δD is a change $\delta\lambda$ and change $\delta\psi$.

$$(D^*D - \lambda)\psi = 0$$

$$(\delta(D^*D) - \delta\lambda)\psi + (D^*D - \lambda)\delta\psi = 0$$

$$\delta\lambda = \frac{(\psi|\delta(D^*D)|\psi)}{(\psi|\psi)} \qquad \delta\psi = (D^*D-\lambda)^{-1}\left[(\delta(D^*D) - \delta\lambda)\psi\right]$$

Actually this calculation should be done for a self-adjoint operator H_0. Assuming H_0 has a simple eigenvalue λ, then for H near to H_0 we have a map which associates to H the [redacted] eigenline which is the deformation of $\operatorname{Ker}(H_0 - \lambda)$. This gives us a map to projective space and we can pull back the Kahler form. [redacted] There really ought to be a simple formula. Need to know the tangent map, i.e. how to associate to δH a map from $\mathbb{C}\psi = \operatorname{Ker}(H_0 - \lambda)$ to $(\mathbb{C}\psi)^\perp$. This is given by $\psi \mapsto \delta\psi$

$$\delta\psi = (H_0-\lambda)^{-1}(\delta H - \delta\lambda)\psi.$$

So one writes out δH as a block matrix relative to $V = \operatorname{Ker}(H_0-\lambda) \oplus \operatorname{Im}(H_0-\lambda)$, then $\delta\lambda = (\psi|\delta H|\psi)$ is the longitudinal part ~~and~~ and

$$\delta\psi = (H_0-\lambda)^{-1}(\delta H - \delta\lambda)\psi$$

is the transverse part. The Kähler form is then

$$\|\delta\psi\|^2 = \quad ?$$

Different viewpoint: I am considering a family of operators $D: V^0 \longrightarrow V^1$, a basepoint in this family D_0 and an eigenvalue $\lambda > 0$ of $D_0^* D_0$. For D near D_0 we obtain subbundles V^0_λ, V^1_λ of the trivial bdl with fibres V^0, V^1 such that $V^i_\lambda(D)$ is the gen eigenspace for D^*D, DD^*, which coincides at D_0 with the λ eigenspace. Each of these bundles inherits a Grassmann connection, and these bundles are isomorphic via $D: V^0_\lambda(D) \xrightarrow{\sim} V^1_\lambda(D)$, or D^* going the other way. Hence we know that the difference of the first Chern forms for these two bundles is the differential of a 1-form. In fact this 1-form is the difference of the connection form for the line bundles $\lambda(V^i_\lambda)$.

Let's do the computation in a simple case namely where λ is a simple eigenvalue. Then I ~~chor~~ choose a section of V^0_λ, call it ψ^0_D, and suppose it is of unit length. The connection form of V^0_λ relative to this section just gives the deviation of ψ^0_D from varying perpendicular (in the complex sense to itself). Thus the connection form is

$$i(\delta D)\,\theta^{\circ} = (\psi^{\circ}_D \mid \delta\psi^{\circ}_D)$$

and I can suppose this vanishes at the point D_0.

Next $D\psi^{\circ}_D$ is a section of V'_λ of norm squared $\|D\psi^{\circ}_D\|^2 = (\psi^{\circ}_D \mid D^*D\,\psi^{\circ}_D) = \lambda_D$, hence $\psi'_D = \frac{1}{\sqrt{\lambda_D}} D\psi^{\circ}_D$ is a section of norm 1. The connection form relative to ψ'_0 is $(\psi'_D \mid \delta\psi'_D)$, but what I want is the connection form relative to $D\psi^{\circ}_D$. Since $\psi'_D = f\, D\psi^{\circ}_D$ we have

$$\begin{aligned} \nabla \psi'_D &= \theta'\psi'_D = \theta' f\, D\psi^{\circ}_D \\ &\parallel \\ \nabla(f\, D\psi^{\circ}_D) &= df\, D\psi^{\circ}_D + f\, \underbrace{\nabla(D\psi^{\circ}_D)}_{\theta\, D\psi^{\circ}_D} \end{aligned}$$

$$\therefore \quad \theta = -\frac{df}{f} + \theta'$$

or

$$i(\delta D)\theta = -\delta \cancel{\log} \frac{1}{\sqrt{\lambda_D}} + \underbrace{(\psi'_D \mid \delta\psi'_D)}_{\left(\frac{1}{\sqrt{\lambda_D}} D\psi^{\circ}_D \,\middle|\, \cancel{\delta\left(\frac{1}{\sqrt{\lambda_D}}\right) D\psi^{\circ}_D} + \frac{1}{\sqrt{\lambda_D}}\, \delta(D\psi^{\circ}_D)\right)}$$

$$= \frac{1}{\lambda_D}\Big(D\psi^{\circ}_D \,\Big|\, \underbrace{\delta(D\psi^{\circ}_D)}_{D\,\delta\psi^{\circ}_D + (\delta D)\psi^{\circ}_D}\Big)$$

$$= \frac{1}{\lambda_D}\Big(\underbrace{(D\psi^{\circ}_D \mid D\,\delta\psi^{\circ}_D)}_{\lambda_D(\psi^{\circ}_D \mid \delta\psi^{\circ}_D)} + \underbrace{(D\psi^{\circ}_D \mid D D^{-1}\delta D\, \psi^{\circ}_D)}_{\lambda_D(\psi^{\circ}_D \mid D^{-1}\delta D\,\psi^{\circ}_D)}\Big)$$

So this calculation shows that the difference of the two connection forms is

$$(\psi^{\circ}_D \mid D^{-1}\delta D\, \psi^{\circ}_D)$$

at least when D is invertible. In general I get

$$\frac{1}{\lambda_D}\left(D\psi_D^0 \mid \delta D\,\psi_D^0\right)$$

which can be written

$$\Big(\psi_D^0 \,\Big|\, \underbrace{(D^*D)^{-1}D^*}\,\delta D\,\psi_D^0\Big)$$

Hodge inverse of D after projecting on $\operatorname{Im} D \overset{\perp}{=} (\operatorname{Ker} D^*)^\perp$.

I will assume the same result can be proved even when λ is not a simple eigenvalue. Thus we have.

<u>Prop</u>: Given a family of $D : V^0 \longrightarrow V^1$ an eigenvalue $\lambda > 0$ of $D_0^*D_0$ we consider the eigenbundles $\{V_\lambda^i(D)\}$ for D near D_0. Then the difference of the two first Chern forms for the $\{V_\lambda^i(D)\}$ is the differential of the form $\operatorname{Tr}_{V_\lambda^0(D)}(D^{-1}\delta D)$.

Consequence of this formula: It gives the significance of

$$d\operatorname{Tr}\left(e^{-tD^*D}\,D^{-1}\delta D\right)$$

when D is invertible say. Here we are taking the weighted sum $d\sum_\lambda e^{-t\lambda}\operatorname{Tr}_{V_\lambda^0(D)}(D^{-1}\delta D)$ of the differences of the first Chern forms?? Not quite because λ varies.

Discussion: The basic problem remains to prove a local index theorem for families of Dirac type operators, which would combine the Patodi type proofs with theory of determinant line bundles.

It is essential to concentrate on the analytical proof of such a theorem. Therefore you should get your hands dirty with a proof of the index theorem, understand its workings, and try to generalize.

Let us take the case of a Dirac operator over a torus. In other words I want the Riemannian geometry, hence the symbol of the operator, to be flat. Then I am prepared to bring in all possible connections and gauge transformations.

Let's try to ignore the gauge transformations first and work over the space $\mathcal{A}$ of all connections. In the Riemann surface case I seem to have found out the forms on $\mathcal{A}$ given by the RHS of the index formula were uninteresting in degrees > 2. Let's review this calculation.

We are dealing with a trivial bundle E over M and we pull it back to $\mathcal{A} \times M$, whence it acquires a canonical connection. Geometrically the connection is flat in the $\mathcal{A}$ direction and the obvious connection in the M direction. A typical connection on E is of the form $d + A = (\partial_i + A_i)dx^i$. Let's use the linear structure of $\mathcal{A}$ and choose a basis α_μ and let y^μ be the corresponding coordinates:

$$A = y^\mu \alpha_\mu = y^\mu \alpha_{\mu i} dx^i$$

where the $\alpha_{\mu i}$ are fixed matrix functions of x.

The canonical connection on $pr_2^*(E)$ over $\mathcal{a} \times M$ is given by

$$\nabla = d_{\mathcal{a}} + d_M + A$$

$$= \partial_{y^\mu} dy^\mu + (\partial_{x^i} + y^\mu \alpha_{\mu i}) dx^i$$

The curvature is

$$\nabla^2 = [d_{\mathcal{a}}, A] + (d_M + A)^2$$

$$= \alpha_{\mu i}\, dy^\mu dx^i + \underbrace{(d_M A + AA)}_{F_{ij}\, dx^i dx^j}$$

So now the right side of the index fmla gives

$$(pr_1)_* \operatorname{tr}(e^{\nabla^2})$$

and one sees that potentially there are lots of non-zero components. Now suppose that $\dim M = m$. Then $(\nabla^2)^{m+1} = 0$, because ∇^2 is of type $(1,1)+(0,2)$ over $\mathcal{a} \times M$, hence $(\nabla^2)^{m+1}$ is of type $(m+1, m+1), (m, m+2)$ etc. and these are zero.

Of particular interest to me now is the case of the degree 2 component which supposedly is the curvature of the determinant line bundle. Let's first do the degree 0 component. Suppose m even.

[redacted]

$$(pr_1)_* \operatorname{tr}\left(\frac{1}{(m/2)!} (\nabla^2)^{m/2}\right)$$

The only way to get all [redacted] m of the dx^i is to use all $m/2$ copies of the F-part of ∇^2.

Write
$$\nabla^2 = d'A + F$$
Then
$$\left[(pr_1)_*\left(\operatorname{tr} e^{\nabla^2}\right)\right]_0 = \frac{1}{(m/2)!}\, pr_{1*}\left(\operatorname{tr}\left(F^{m/2}\right)\right)$$
which should give the standard formula for the index. Next
$$\left[(pr_1)_*\left(\operatorname{tr} e^{\nabla^2}\right)\right]_2 = \frac{1}{(m/2+1)!}\, pr_{1*}\left(\operatorname{tr}\left(\nabla^2\right)^{m/2+1}\right)$$

$$\begin{aligned}
\operatorname{tr}\left(\nabla^2\right)^n &= \operatorname{tr}\left(F + d'A\right)^n \\
&= \operatorname{tr}\left(F^n\right) + n \operatorname{tr}\left(F^{n-1} d'A\right) \\
&\quad + \sum_{1 \le i,j \le n} \operatorname{tr}\big(F \ldots F \underset{\substack{\uparrow \\ i}}{d'A} F \ldots F \underset{\substack{\uparrow \\ j}}{d'A} F \ldots F\big) \\
&\quad + \ldots
\end{aligned}$$

When we take $(pr_1)_*$ we integrate out $dx^1 \ldots dx^m$ and hence we need exactly m dx's. F has 2 and $d'A$ has one. So if $n = \frac{m}{2}+1$, then $F^{n-1} d'A$ has $2\frac{m}{2}+1$ dx's which is too many. Hence we find
$$\left[(pr_1)_* \operatorname{tr}\left(e^{\nabla^2}\right)\right]_2 = \frac{1}{\left(\frac{m}{2}+1\right)!}\, pr_{1*}\left\{\sum_{1 \le i < j \le n} \operatorname{tr}\overbrace{\big(F \ldots \underset{\substack{\uparrow \\ i\text{th}}}{d'A} F \ldots \underset{\substack{\uparrow \\ j\text{th}}}{d'A} \ldots F\big)}^{\frac{m}{2}+1 \text{ factors},\ \frac{m}{2}-1 \text{ factors} = F.}\right\}$$

Our next project will be to find an analytical formula giving this answer.

Obviously we want to start with a proof of the index theorem. Introduce the total Dirac operator:
$$\not{D} = \gamma^\mu D_\mu \qquad D_\mu =. \partial_\mu + A_\mu$$
where the γ^μ are self-adjoint anti-commuting of square 1.

Then

$$+\not{D}^2 = \tfrac{1}{2}\left(\gamma^\mu D_\mu \gamma^\nu D_\nu + \gamma^\nu D_\nu \gamma^\mu D_\mu\right)$$
$$= \tfrac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) D_\mu D_\nu - \tfrac{1}{2}\gamma^\nu\gamma^\mu [D_\mu, D_\nu]$$
$$= D_\mu^2 + \tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}.$$

Note that $\not{D}$ is skew-adjoint and $\not{D}^2 \le 0$. Now

$$\frac{1}{\lambda + \not{D}^2} = \frac{1}{\lambda + D^2} + \frac{1}{\lambda + D^2}\left(-\tfrac{1}{2}\sigma F\right)\frac{1}{\lambda + D^2} + \cdots .$$

Using the basic lemma that $\mathrm{tr}_{sp}(?\,\gamma_5)$ kills any product of $\le m$ γ-matrices, one sees that

$$\mathrm{tr}_{sp}\left(\frac{1}{\lambda + \not{D}^2}\gamma_5\right) \sim \text{const}\ \underbrace{\frac{1}{\lambda + D^2} F \frac{1}{\lambda + D^2} F \cdots}_{m/2\ F\text{-factors}}.$$

This is not very clear.

March 9, 1983

Over $Y \times M$ we have a connection $D_\mu = \partial_\mu + A_\mu$. (Think of Y, M as Euclidean spases modulo lattices.) Then we can produce a Chern character form on Y by taking

$$(pr_1)_* \left\{ \mathrm{Tr}\,(e^F) \right\}.$$

The goal is to obtain this form analytically. One of the ideas of Atiyah is to use the transitivity of the Dirac operator. What might this mean?

When we take $(pr_1)_*(S \otimes E)$, $S =$ spinors over M, we get a connection in this infinite-diml bundle over Y. Then I could tensor with spinors on Y to get a Dirac operator on Y with coefficients in an infinite diml bundle. What remains is to bring in the actual 'vertical' Dirac operator in the M-direction.

The whole thing should be a simple tensor product business. One looks at functions on $Y \times M$ with coefficients in the constant spinor module:

$$S_{Y\times M} = S_Y \otimes S_M$$

Here I divide the coordinates on $Y \times M$ into the coordinates of Y and of M: y^i, x^μ. Then we have γ^i on S_Y and γ^μ on S_M. Our connection is given by

$$D = D' \oplus D'' \qquad \begin{aligned} D' &= \{\partial_i + A_i\} \\ D'' &= \{\partial_\mu + A_\mu\} \end{aligned}$$

The total Dirac operator over $Y \times M$ is

$$\gamma^i D_i + \gamma^\mu D_\mu$$

acting on functions on $Y \times M$ with values in $S_Y \otimes S_M$.

Now apply $(pr_1)_*$, and try to write this as as an operator on functions on Y with values in $Sp\,($functions on M values in $S_M)$. Now $\gamma^\mu D_\mu$ is an operator on $C^\infty(M, S_M)$, and D_i is an operator on $C^\infty(Y, C^\infty(M, S_M))$.

The situation then is as follows. Over Y we have the infinite-diml bundle H whose sections are

$$\Gamma(Y, pr_{1*} S_{\underline{M}}) = C^\infty(Y, C^\infty(M; S_M))$$

and on H we have a connection $\{D_i\}$ and a bundle map $\gamma^\mu D_\mu$. Then one forms the operator

$$\gamma^i D_i + \gamma^\mu D_\mu \qquad \text{on } S_Y \otimes H$$

with the understanding that the γ^i anti-commute with the operator $\gamma^\mu D_\mu$.

This raises the question of what should one know about generalizations of the Dirac operator of the form

$$\gamma^i D_i + L$$

where L anti-commutes with the γ^i. In general this takes place on $S \otimes E$, and D_i is a connection on E. L should be an endo. of E, but then what does it mean for L to anti-commute with the γ^i?

It is necessary to review spinors and Clifford algebras. In general when one works over the reals a module V over the Clifford algebra $C(\mathbb{R}^n)$ will give rise to a Dirac operator in V:

$$e^i \partial_i$$

which is self-adjoint and has square $-\partial_i^2$.

Thus the e^i are skew-adjoint anti-commute and have square -1. When one works over the complex numbers one prefers to write the Dirac operator

$$\frac{1}{i}\gamma^\mu \partial_\mu$$

so that the γ^μ are self-adjoint, anti commute, and have square $+1$.

Let C_n be the algebra over $\mathbb{C}$ with generators γ^μ $\mu = 1, \ldots, n$ satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu}$. Then C_n has dimension 2^n and is graded; also it has an increasing filtration $F_p C_n =$ span of monomials in the γ^μ of degree $\leq p$, and

$$gr(C_n) = \text{exterior alg. on } n \text{ generators.}$$

One can describe the structure of C_n as follows. If n is even, then C_n is a matrix ring of degree $2^{n/2}$; If n is odd it is a product of two matrix rings of degree $2^{\left(\frac{n-1}{2}\right)}$. Why: If $n = 2m$, then consider ΛV where $V = \mathbb{C}^m$ with usual inner product. We have the operators

$$i(v^*) + e(v) \qquad\qquad v^*(w) = (v|w).$$

which satisfy

$$\{i(v^*) + e(v),\ i(v_1^*) + e(v_1)\} = (v_1|v) + (v|v_1).$$

Hence if we put

$$\gamma^\mu = i((v^\mu)^*) + e(v^\mu)$$

where $v^1, \ldots, v^{2m}$ is an orthonormal basis for V as a real inner product space, then ΛV becomes a C_n-module. Thus we get a map of rings

$$C_n \longrightarrow \operatorname{End}(\Lambda V)$$

such that the image contains the operators $i(v^*)$, $e(v)$ for all $v \in V$. It's easy to see that the map is onto so the map must be an isomorphism.

If $n = 2m+1$ consider the element

$$\varepsilon = \gamma^1\gamma^2 \cdots \gamma^{2m+1} \in C_n.$$

Then ε commutes with each γ^μ, hence lies in the center. In general for $k \leq n$

$$(\gamma^1 \cdots \gamma^k)^2 = (-1)^{(k-1)+\cdots+1} = (-1)^{k(k-1)/2}$$

so that $\varepsilon^2 = (-1)^{m(2m+1)} = (-1)^m$. Thus we get two homomorphisms

$$C_n \longrightarrow C_{n-1} \qquad \varepsilon \rightarrow \pm i^m$$

which are obviously onto; hence $C_n \xrightarrow{\sim} C_{n-1} \times C_{n-1}$ because the center of C_n contains two central idempotents and the corresponding decomposition involves the rings C_{n-1}, C_{n-1}.

We have $C_p \hat{\otimes} C_n = C_{p+n}$ where $\hat{\otimes}$ denotes the tensor product in the graded sense. Consequently the tensor product of graded modules over C_p, C_n will be a graded module over C_{p+n}. From the above structure of the algebra C_n we have

$$K_0(C_n) = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}+\mathbb{Z} & n \text{ odd.} \end{cases}$$

Next I want to determine $K_0^{gr}(C_n)$. First suppose n is even: $n = 2m$. The element $\varepsilon = \gamma^1 \cdots \gamma^n$ anti-commutes with all the γ^μ and satisfies

$$\varepsilon^2 = (-1)^{2m(2m-1)/2} = (-1)^m.$$

If I have a graded module $M = M^+ \oplus M^-$, then I can decompose M^+ into eigenspaces under ε. Then the $\varepsilon = i^m$ eigenspace of M^+ together with the $\varepsilon = -i^m$ eigenspace of M^- will be a graded C_n-module, [redacted] whose grading is completely described by the eigenspace of ε. Thus a C_n module has two standard ways of becoming a graded C_n-module, and so

$$K_0^{gr}(C_n) = \mathbb{Z} \oplus \mathbb{Z} \qquad n \text{ even.}$$

[redacted]

Actually the whole business is much simpler. A grading on a C_n-module is just an operator of square 1 which anti-commutes with the γ^μ. Hence it is just a C_{n+1}-module structure. Thus

$$K_0^{gr}(C_n) = K_0(C_{n+1}) = \begin{cases} \mathbb{Z}+\mathbb{Z} & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases}$$

Next we can determine the ring structure on

$$\bigoplus_{n \geq 0} K_0^{gr}(C_n)$$

which is a commutative ring (* see below). Basic generators:

$$K_0^{gr}(C_0) = \mathbb{Z}\sigma^+ \oplus \mathbb{Z}\sigma^- \qquad \text{where} \quad \begin{array}{l} \sigma^+ = [\mathbb{C} \oplus 0] \\ \sigma^- = [0 \oplus \mathbb{C}] \end{array}$$

$$K_0^{gr}(C_1) = \mathbb{Z}\eta \qquad \text{where} \quad \eta = [\mathbb{C} \oplus \mathbb{C}]$$

with γ^1 giving the obvious shift between the two factors

Thus η = class of C_1 as a graded C_1 module.
So η^2 = class of C_2 as a graded C_2-module.
Let $\lambda = \Lambda\mathbb{C}$ be the basic C_2-module, in the way $\lambda^m = \Lambda(\mathbb{C}^m)$ is the basic C_{2m}-module. Then clearly we have

$$\eta^2 = \lambda(\sigma^+ + \sigma^-)$$

Formula:

$$\bigoplus_{n\geq 0} K_0^{gr}(C_n) = \mathbb{Z}[\lambda]\otimes\{\mathbb{Z}\sigma^+ \oplus \mathbb{Z}\sigma^- \oplus \mathbb{Z}\eta\}$$

where $\quad (\sigma^+)^2 = \sigma^+ \qquad (\sigma^-)^2 = \sigma^+ \qquad \sigma^+\sigma^- = \sigma^-$

$$\sigma^+\eta = \sigma^-\eta = \eta$$
$$\eta^2 = \lambda(\sigma^+ + \sigma^-)$$

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*. In general one has to be careful about $\bigoplus K^{gr}(C_n)$ being commutative. The point is that given M over C_p and N over C_q one gets two modules over C_{p+q}, namely $M\otimes N$ and $N\otimes M$. These are related via an automorphism of C_n given by a permutation of the γ^μ. Now $O(n)$ is the group of autos of C_n preserving the space spanned by the γ^μ, so it might happen that $\pi_0 O(n) = \mathbb{Z}/2\mathbb{Z}$ acts on $K^{gr}(C_n)$. The problem is whether changing the sign of a single γ can be realized by an inner automorphism of C_n as a graded algebra, so that it acts trivially on $K^{gr}(C_n)$. In the complex case there should be no problem, but let's check.

Consider

$$K^{gr}(C_n) = K(C_{n+1}) = \begin{cases} \mathbb{Z}+\mathbb{Z} & n \text{ even} \\ \mathbb{Z} & n \text{ odd}\end{cases}$$

and we want the effect of changing $\gamma^1 \mapsto -\gamma^1$. Recall for $n+1$ odd we have

$$\varepsilon = \gamma^1 \cdots \gamma^{n+1}$$

belongs to the center, and its eigenspaces give the the two $\mathbb{Z}$'s. It seems that $\gamma^1\mapsto -\gamma^1$ will act non-trivially on $K^{gr}(C_n)$, n even.

However it doesn't matter because η^2 is clearly the same element under the interchange of γ^1, γ^2. ∎

Next point is the homomorphism

$$\bigoplus_n K^{gr}(C_n) \longrightarrow \bigoplus_n K_c(\mathbb{R}^n)$$

which comes from taking a graded C_n-module $M = M^+ \oplus M^-$ and looking at the map of bundles over $\mathbb{R}^n$ with value

$$\gamma^\mu \xi_\mu : M^+ \longrightarrow M^-$$

at $\xi \in \mathbb{R}^n$. Since $(\gamma^\mu \xi_\mu)^2 = \xi_\mu^2$ this is an isomorphism away from 0, so we get a definite relative K-element for $\mathbb{R}^n$ mod $\mathbb{R}^n - \{0\}$. Under this map

$$\sigma^+ \longmapsto 1 \qquad \eta \longmapsto 0$$
$$\sigma^- \longmapsto -1$$
$$\lambda \longmapsto \text{Bott element } \beta \in K_c(\mathbb{R}^2).$$

One point from the Atiyah Bott Shapiro paper is that in general for a vector bundle: *

$$K^{gr}(C(V\oplus 1)) \longrightarrow K^{gr}(C(V)) \longrightarrow K_c(V)$$

NO see below

in particular

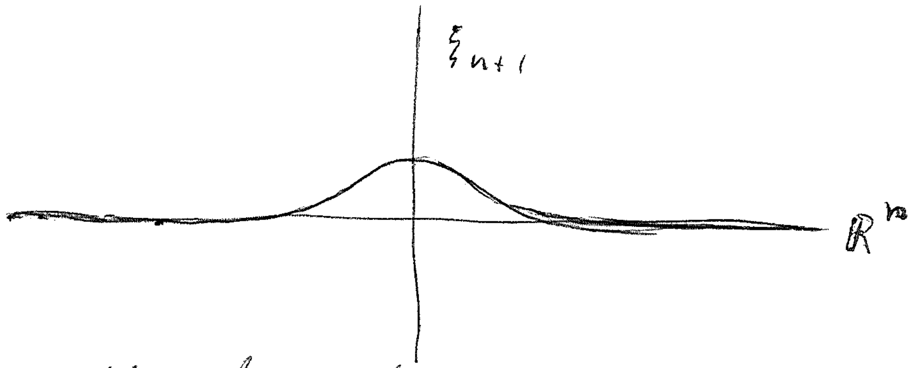
$$K(C_{n+1}) \longrightarrow K(C_n) \longrightarrow K_c(\mathbb{R}^n) \to 0.$$

This is a basic theorem in their paper (perhaps very important in general, e.g. Suslin + Severi-Brauer varieties.)

Let's go over why if M is a C_{n+1}-module, then the element of $K_c(\mathbb{R}^n)$ is trivial. This is because we have

$$\gamma^\mu \xi_\mu + \gamma^{n+1} \xi_{n+1}$$

defined over $\mathbb{R}^{n+1}$ and invertible off 0.



Use the above kind of deformation.

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* ABS do not ~~state~~ the good thm., which Karoubi does, namely that $K_c(V)$ is the relative K-group for the map

$$K^{gr}(C(V+1)) \longrightarrow K^{gr}(C(V)).$$

Thus it is not the cokernel. For example take $V = X \times \mathbb{R}$. Then a graded C_1-bundle over X is a graded bundle $E = E^+ \oplus E^-$ together with an isomorphism $\gamma' : E^+ \xrightarrow{\sim} E^-$. So

$$K^{gr}(C(V)) = K(X).$$

On the other hand a graded C_2 bundle over X is the same as a C_3-bundle, i.e. a pair of vector bundles. So the map above is the K-map for

$$\mathrm{Vect}(X)^2 \xrightarrow{+} \mathrm{Vect}(X)$$

and it is fairly clear that the relative group a version of $K'(X)$.

Dwyer-Kan: Classification thm. for diagrams of simplicial sets. D is a small category and we consider a D-diagram in spaces: $X : D \longrightarrow S$.

Fix a simplex $[n] \xrightarrow{J} D$ and consider the correspond simplex in S:

$$X_J : \quad X_{J(0)} \longrightarrow \cdots \longrightarrow X_{J(n)} .$$

I would like to talk about the topological monoid $\mathrm{haut}(X_J)$ of self-homotopy equivalences of X_J. Then we get a nice functor $J \longmapsto \mathrm{haut}(X_J)$, and so can form the homotopy inverse limit

$$\underset{\Delta/D}{\underleftarrow{\mathrm{holim}}} \left\{ J \longmapsto B\,\mathrm{haut}(X_J) \right\}$$

What should this be? There answer is that you take the category of all functors $D \xrightarrow{Y} S$ and all weak equivalences between such Y such that for any simplex J, Y_J is weakly equivalent to X_J.

In particular $B\,\mathrm{haut}(X)$ for a space is equivalent to the category of all weakly equivalent spaces to X and weak equivalences between them.

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