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after Berkeley trip I learned how to compute the transgression $H^*(B\mathcal{G}) \longrightarrow H^*(\mathcal{G})$. Also I learned to compute with connection forms

p. 217 left - versus right - invariant vector fields.

p. 232 equivariant curvature formulas in both D and A notations

p. 240 action of gauge transf. on $\mathcal{A}$:

$$g^{-1} \circ D \circ g \longleftrightarrow d + g^* A$$

on $\Omega(M,E)$ on $\Omega(P) \otimes V$

p. 227 connections in principal bundles.

p. 201 bundle autos: $(BG)^{S^1} \sim PG \times^G (G_c)$ where $G_c = G$ with G acting by conjugation.

<u>November 10, 1983</u> (after Berkeley trip)

The problem is to calculate the odd classes on $\mathcal{G}$ obtained from the even classes in $B\mathcal{G}$. According to Singer there are some simple formulas. Now how am I to understand this? First we look topologically. We have a canonical map

$$B\mathcal{G} \times M \longrightarrow BG$$

which classifies the tautological principal G-bundle over $B\mathcal{G} \times M$.

$$\begin{tikzcd}
H^*(BG) \arrow[r] \arrow[d] & H^*(B\mathcal{G} \times M) \arrow[r, "\int_\gamma"] \arrow[d] & H^*(B\mathcal{G}) \arrow[d] \\
H^{*-1}(G) \arrow[r] & H^{*-1}(\mathcal{G} \times M) \arrow[r, "\int_\gamma"] & H^{*-1}(\mathcal{G})
\end{tikzcd}$$

What seems to be the case is that we have a "map" $\mathcal{G} \times M \longrightarrow G$ and that the classes on $\mathcal{G}$ I am after are obtained by taking the secondary characteristic classes on G, pulling back, and integrating over cycles in M.

Here's how to describe $\mathcal{G} \times M \longrightarrow G$. Over the space $\mathcal{G} \times M$ we have the bundle $pr_2^*(P_0)$ with a canonical automorphism.

Let's consider a bundle P/X with a bundle automorphism α, and try to understand why there is a corresponding map $X \longrightarrow G$ in the homotopy category. Let's pick a classifying map for P

$$\begin{tikzcd} P \arrow[r, "\alpha", "\sim"'] \arrow[dr] & P \arrow[r, "u"] \arrow[d] & PG \arrow[d] \\ & X \arrow[r] & BG \end{tikzcd}$$

then $u\alpha$ is also a classifying map (i.e. G-equiv.), and, so we know there is a homotopy joining u to $u\alpha$ thru G-equivariant maps. This gives a self-homotopy of the map $X \xrightarrow{\bar{u}} BG$. Next we want to use the fibration

$$\mathrm{Map}(X, G) \longrightarrow \mathrm{Map}(X, PG) \longrightarrow \mathrm{Map}(X, BG).$$

But now we see the difficulty, namely that $\bar{u}$ is not in the image of this map when P_0 is not trivial. The fibration over $\mathrm{Map}(X, BG)$ whose image is the component belonging to P_0 is

$$\mathrm{Aut}(P/X) \longrightarrow \mathrm{Map}_G(P, PG) \longrightarrow \mathrm{Map}(X, BG)_{(\bar{u})}$$

So the conclusion is that we don't get a map $\mathcal{G} \times M \longrightarrow G$. So the question arises how do we get cohomology classes in $\mathcal{G} \times M$? Note that a map $X \longrightarrow \mathcal{G} \times M$ is the same thing as a map $X \xrightarrow{u} M$ together with an automorphism of the pull-back $u^*(P_0)$. Thus a characteristic class for bundle autos gives cohomology on $\mathcal{G} \times M$. Given a bundle P/X and an automorphism, one can form a Möbius bundle over $X \times S^1$. The characteristic classes of this Möbius bundle, when integrated over S^1, give characteristic classes for bundle autos.

We can describe what happens as follows. We have a map

$$(S^1 \times \mathcal{G}) \times M \longrightarrow B\mathcal{G} \times M \longrightarrow BG$$

and hence a map

$$\mathcal{G} \times M \longrightarrow (BG)^{S^1}$$

to the <u>free</u> <u>loop</u> <u>space</u>. [redacted]

<u>Proposition</u>: $(BG)^{S^1} \sim PG \times^G (G_c)$, where G_c denotes the G-space G with G acting by conjugation

<u>Proof</u>:

$$\begin{tikzcd} (BG)^{S^1} \arrow[r] \arrow[d] & BG^I \arrow[d, "\text{[redacted]}"] \\ BG \arrow[r, "\Delta"] & BG \times BG \end{tikzcd}$$

is cartesian

so $(BG)^{S^1}$ is the h-fibre product of $\Delta(BG)$ with itself [redacted] over $BG \times BG$. Thus we have

$$(BG)^{S^1} \sim BG \times_{(BG)^2} (PG)^2/\Delta G = PG \times^G (G_c).$$

In other words we replace Δ by the fibration $P(G\times G) \times^{G\times G} (G) \to B(G\times G)$ with $G \times G$ acting by left and right multiplication. Then we pull-back via $\Delta : BG \to B(G\times G)$ and get $PG \times^G (G_c)$. QED.

This proposition states that $(BG)^{S^1}$ classifies bundles & automorphisms. What is interesting is the fact that the free loop space occurs naturally in Waldhausen theory and also Witten - super-symmetric nonlinear σ-model in $1+0$ dimensions; better super-symmetric quantum mechanics.

So I now should review how to compute the cohomology of $PG \times^G (G_c)$, or $(BG)^{S^1}$. I can use the EM spectral sequence, but then I would have to worry about the differentials. The method I prefer is to use transgression. I know that in the fibration $G \longrightarrow PG \longrightarrow BG$, the cohomology of the fibre has a simple system of transgressive generators, hence by the map

$$\begin{tikzcd} G^2 \arrow[r] \arrow[d] & (PG)^2 \arrow[r] \arrow[d] & (BG)^2 \arrow[d, equal] \\ G \arrow[r] & (PG)^2 \times^{G^2} G \arrow[r] & (BG)^2 \end{tikzcd}$$

of fibrations. ? ?

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \overset{Q}{\overset{\parallel}{PG \times^G (G_c)}} \arrow[r] \arrow[d] & BG \arrow[d] \\ pt \arrow[r] & BG \arrow[r, "\Delta"] & BG^2 \end{tikzcd}$$

$$\begin{tikzcd} H^{2i}(Q) & H^{2i}(BG) \arrow[l] & 0 \\ H^{2i}(BG) \arrow[u] & H^{2i}(BG^2) \arrow[l] \arrow[u] & c\otimes 1 - 1\otimes c \\ H^{2i}(BG, Q) \arrow[u] & H^{2i}(BG^2, BG) \arrow[l] \arrow[u] & \\ H^{2i-1}(Q) \arrow[u] & 0 \arrow[u] & \\ H^{2i-1}(BG) = 0 \arrow[u] & & \end{tikzcd}$$

So $c\otimes 1 - 1\otimes c \in H^{2i}(BG^2)$ gives rise to $e \in H^{2i-1}(Q)$ which can be easily seen to restrict to the transgression of c in $H^{2i-1}(G)$ by factoring $pt \to BG \xrightarrow{\text{inj}} BG^2$.

Actually the nature of transgression should be made clear. One has a map

$$\mathrm{Ker}\left\{ H^*(B) \xrightarrow{\pi^*} H^*(E) \right\} \longrightarrow \mathrm{Coker}\left\{ H^{*-1}(E) \to H^{*-1}(F) \right\}$$

for which the transgression is a sort of inverse. Hence it is clear that for $PG^2 \times^{G^2} G \longrightarrow BG^2$ the cohomology of the fibre is generated by above map applied to $c \otimes 1 - 1 \otimes c$.

What perhaps I might try to understand is the exact relation between the transgression formulas for $PG \longrightarrow BG$ and the ones for $PG^2 \times^{G^2} G \longrightarrow BG^2$. On the diff'l form level I am comparing the formulas:

$$ch(gDg^{-1}) - ch(D) = d\int_0^1 dt \ldots$$

with $ch(d+\theta) - ch(d) = d\int_0^1 dt\, tr(\theta e^{td\theta + t^2\theta^2})$. The first formula might better be written

$$ch(D_1) - ch(D_0) = d\int_0^1 dt\, tr\left((D_1 - D_0)e^{D_t^2}\right).$$

The idea is that one should think of BG^2 as representing pairs of bundles with connection, and $PG^2 \times^{G^2} G$ as pairs (E_i, D_i) $i = 1,2$ together with an isomorphism $E_1 \cong E_2$, hence as a single bundle with two connections.

Summary: I have a way to produce cohomology classes on $B\mathcal{G}$, namely, by pulling back by the map

$$B\mathcal{G} \times M \longrightarrow BG$$

and integrating over cycles in M. I want to produce cohomology in $\mathcal{G}$ essentially by suspension from the classes I have on $B\mathcal{G}$. Hence I use the canonical map (Mobius)

$$S^1 \times \mathcal{G} \longrightarrow B\mathcal{G}$$

pull-back the classes on $B\mathcal{G}$ and integrate over S^1. So ultimately I start with $c \in H^{ev}(BG)$ pull-back via

$$S^1 \times \mathcal{G} \times M \longrightarrow B\mathcal{G} \times M \longrightarrow BG$$

then integrate over $S^1 \times$ cycle γ in M. But this map can be written

$$\mathcal{G} \times M \longrightarrow (BG)^{S^1} \sim PG \times^G (G_c).$$

represents bundles + autos.

So what I am claiming is that if I suspend the class in $H^*(B\mathcal{G})$ obtained from c, γ, then the class in $H^*(\mathcal{G})$ I obtain is obtained from the secondary characteristic class e associated to c applied to the canonical bundle + auto over $\mathcal{G} \times M$, and the cycle γ.

November 11, 1983

The problem concerns the cohomology of $\mathcal{G}$ the gauge group. I understand fairly well the cohomology of $B\mathcal{G}$, because of the equivariant form machinery. But I still haven't understood the cohomology of $\mathcal{G}$ as well as I would like.

Let's go back to the Riemann surface case where things should have been worked out already. We have a determinant line bundle $\mathcal{L}$ over the space $\mathcal{A}$ of connections, which is equivariant for $\mathcal{G}$, and an invariant connection D in $\mathcal{L}$. Then I know the curvature and Higgs map of this [redacted] connection. I know this situation fits the moment map setup, because the curvature of $\mathcal{L}$ is the symplectic 2-form on $\mathcal{A}$. [redacted]

Thus it seems reasonable to first consider the case of an equivariant line bundle (G, L, M) with equivariant connection D whose curvature is exact. In this case I can transgress or suspend $c_1(L) \in H^2_G(M)$ to get an element of $H^1(G)$. What does this process [redacted] look like?

First I have to get the formulas straight. I propose to start with the Higgs field defined by

$$\mathcal{L}_X = [\iota_X, D] + \varphi_X .$$

Then

$$\begin{aligned}[][\iota_X, D^2] &= (\mathcal{L}_X - \varphi_X)D - D(\mathcal{L}_X - \varphi_X) \\ &= [D, \varphi_X]\end{aligned}$$

which in the case of line bundles gives

$$\iota_X \omega = d\varphi_X .$$

This is the defining equation of the moment map.

Now I have to recall how one constructs the equivariant $c_1(L) \in H^2_G(M)$ as an equivariant form. I have a choice of introducing the Weil algebra with its canonical connection form Θ, or assuming that the G-action on M is free, and then picking a connection $\Theta \in \mathfrak{g} \otimes \Omega^1(M)$. The point of the Weil algebra approach is the isomorphism below

$$[\Omega(M,E)]_{hor} \longleftarrow [W(\mathfrak{g}) \otimes \Omega(M,E)]_{hor} \xrightarrow{\ \sim\ } S(\mathfrak{g}^*) \otimes \Omega(M,E)$$

which simplifies the formulas, e.g.

$$D + \varphi\Theta \longleftrightarrow D + \varphi\Theta \longleftrightarrow D - \Omega^a \iota_a$$

$$(D+\varphi\Theta)^2 \longleftrightarrow D^2 + \Omega^a \varphi_a$$

(Digression: Let's check this last step. On one hand we know that $D^2 + \Omega^a \varphi_a$ on the $S(\mathfrak{g}^*) \otimes \Omega(M, \operatorname{End} E)$ side lifts to

$$\left(1 - \Theta^a \iota_a + \tfrac{1}{2}\Theta^a\Theta^b \iota_b \iota_a \cdots\right)(D^2 + \Omega^a \varphi_a)$$

$$= D^2 + \Omega^a \varphi_a - \Theta^a [\iota_a, D^2] + \tfrac{1}{2}\Theta^a\Theta^b [\iota_b, [\iota_a, D^2]]$$

Here I have used that ι_a on $\Omega(M, \operatorname{End} E)$ is $[\iota_a, \ ]$ in the sense of operators on $\Omega(M,E)$. Now use

$$[\iota_a, D^2] = [D, \varphi_a]$$

so $$[\iota_b, [\iota_a, D^2]] = [\iota_b, [D, \varphi_a]]$$
$$= [[\iota_b, D], \varphi_a]$$

where $[\iota_b, \varphi_a] = 0$ as $\Omega(M, \mathrm{End}\, E) = \Omega(M) \otimes_{\Omega^0(M)} \Omega^0(M, \mathrm{End}\, E)$, so that ι_b and φ_a commute. The above

$$= [\mathcal{L}_b - \varphi_b, \varphi_a] = \underbrace{[\mathcal{L}_b, \varphi_a]}_{\varphi_{[b,a]} \,=\, f_{ba}^{c} \varphi_c} - [\varphi_b, \varphi_a]$$

Thus we get

$$D^2 + \Omega^a \varphi_a - \theta^a [D, \varphi_a] + \tfrac{1}{2} \theta^a \theta^b \left(f_{ba}^{c} \varphi_c - [\varphi_b, \varphi_a] \right)$$

On the other hand we should get

$$(D + \theta^a \varphi_a)^2 = D^2 + d\theta^a \varphi_a - \theta^a [D, \varphi_a] + \underbrace{\theta^a \varphi_a \theta^b \varphi_b}_{\frac{1}{2} \theta^a \theta^b [\varphi_a, \varphi_b]}$$

These two coincide as $\Omega^a = d\theta^a + \frac{1}{2} \theta^i \theta^j f_{ij}^{a}$.)

Summarize: We use the Weil algebra to simplify the curvature.

Repeat the program: 1) The problem is to calculate the classes on $\mathcal{G}$ corresponding to the classes described on $B\mathcal{G}$ by the equivariant forms.

2) To get insight I look at the surface case

which suggests that I consider in general the ~~situation~~ (G, M, L, D) of an equivariant line bundle with an invariant connection.

Let's look at what I want to do in this ~~situation~~. Assume G acts freely on M and a connection form Θ for $M \longrightarrow M/G$ given. Then $H^*_G(M) = H^*(M/G)$ and I know that $c_1(L)$ is represented by the curvature of the modified connection $\bar{D} = D + \varphi\Theta$ descended to the orbit space. Thus the 2-form in question is

$$\bar{D}^2 = (D + \varphi\Theta)^2 = D^2 + d(\varphi\Theta)$$

descended to M/G. To compute the transgression I lift this up to M whence I get $D^2 + d(\varphi\Theta)$ write it as d of something which I assume to be possible, then restrict to a G-orbit.

Notice at this point that thinking in terms of equivariant forms is lousy, so it seems, because I see that what I have to do is write D^2 as $d\alpha$, then ~~restrict~~ $\alpha + \varphi\Theta$ to a G-orbit whence the Θ disappears (i.e. it becomes the identity.)

Here's the picture: Over $\mathcal{A} \times M$ we have the $\mathcal{G}$ bundle $pr_2^*(E_0)$ with its canonical connection D. Then we modify it to $\bar{D} = D + \Theta\varphi$ which descends. I then compute $tr(e^{\bar{D}^2})$ over $(\mathcal{A}/\mathcal{G}) \times M$ and integrate over M so as to get cohomology on $\mathcal{A}/\mathcal{G}$. Now to compute the transgression of these classes I lift back to $\mathcal{A}$ and try to write the form as a coboundary. Since

$$\begin{tikzcd} \mathcal{A}\times M \arrow[r] \arrow[d] & (\mathcal{A}/\mathcal{G})\times M \arrow[d] \\ \mathcal{A} \arrow[r] & \mathcal{A}/\mathcal{G} \end{tikzcd}$$

is transversal cartesian, we see that it suffices to write $tr(e^{\bar{D}^2})$ over $\mathcal{A}\times M$ as a coboundary, then integrate over M. But we have

$$tr(e^{\bar{D}^2}) - tr(e^{D^2}) = d\int_0^1 dt\; tr\left((\theta\varphi)\, e^{(D+t\theta\varphi)^2}\right)$$

so this means we have to write $tr(e^{D^2})$ as a coboundary.

Recall now that the high degree components of $tr(e^{D^2})$ are zero because of filtration reasons, that is, D over $\mathcal{A}\times M$ is flat in the $\mathcal{A}$-direction, so that D^2 is of type $1,1 + 0,2$. It follows that $D^{2j}=0$ for $j>n=\dim M$. In other words for $j>n$ there are going <u>to be</u> canonical forms somewhere.

Let's try to understand the Atiyah-Singer viewpoint which avoids vector bundles and works with the principal bundles.

<u>November 12, 1983</u>

Yesterday I had difficulty understanding Iz's computation of the transgression, so now I want to work out carefully what happens over a surface. (Actually I will do the computations on $\mathbb{R}^2 = \mathbb{C}$.)

We are given the $\mathcal{G}$-equivariant line bundle $\mathcal{L}$ over $\mathcal{A}$ with its invariant connection D and Higgs map φ. If Θ is a connection form for $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$, then the connection $\bar{D} = D + \Theta\varphi$ descends to $\mathcal{A}/\mathcal{G}$ and can be used to compute $c_1(\bar{\mathcal{L}})$. The curvature of $\bar{\mathcal{L}}$ for $\bar{D}$ is $\bar{D}^2 = D^2 + d(\Theta\varphi)$ descended to $\mathcal{A}/\mathcal{G}$. To compute the transgression I lift up to $\mathcal{A}$, getting $D^2 + d(\Theta\varphi)$, and write as a coboundary. This means writing

$$D^2 = d\eta$$

or equivalently replacing the given connection D on $\mathcal{L}$ by the flat connection $D - \eta$. At this point we have

$$\bar{D}^2 = d(\eta + \Theta\varphi)$$

so the transgression is given by restricting the 1-form $\eta + \Theta\varphi$ on $\mathcal{A}$ to a <u>$\mathcal{G}$-orbit</u>.

Geometrically, here is what the transgression process amounts to. One takes $\bar{\mathcal{L}}$ over $\mathcal{A}/\mathcal{G}$ lifts it to $\mathcal{L}/\mathcal{A}$, then trivializes $\mathcal{L}$, then compares this trivialization with the canonical trivialization one has over a $\mathcal{G}$-orbit. Alternatively we construct a determinant function on $\mathcal{a}$, which divided into the canonical section is the trivialization of $\mathcal{L}$, then one restricts the determinant function to the $\mathcal{G}$-orbit, getting a map $\mathcal{G} \to \mathbb{C}^*$.

General Picture: Here is how to compute the transgression at least in principle. We have this bundle $pr_2^*(P)$ over $\mathcal{a}\times M$ with the $\mathcal{G}$-invariant connection D. (Note: if you think in terms of principal bundles D is just a $\mathfrak{g}$-valued form on $pr_2^*(P) = \mathcal{a}\times P$.). In order to compute the characteristic classes of the descended bundle $\mathcal{Q}$ over $(\mathcal{G}\backslash\mathcal{a})\times M$, I have to modify the connection $D \mapsto \bar{D} = D + \theta\varphi_{\mathcal{Q}}$ so it descends. Then the characteristic forms of $\mathcal{Q}$ are $p(\bar{D}^2)$ descended to $(\mathcal{G}\backslash\mathcal{a})\times M$. Then I integrate over M to obtain a form on $\mathcal{G}\backslash\mathcal{a}$.

It will be important to notice that if $\deg p = 2k$ (p is an invariant polynomial on $\mathfrak{g}$), then

$$q_{2k-m} = \int_M P_{2k}(\bar{D}^2) \quad \text{has degree} \quad 2k - \dim(M).$$

Next I want to compute the transgression of q_{2k-m}. I lift q back to $\mathcal{a}$ and want to write it as a coboundary. This can be done by lifting $P_{2k}(\bar{D}^2)$ to $\mathcal{a}\times M$ writing it as du and taking $\int_M u$.

Now $p_{2k}(\bar{D}^2)$ up on $\mathfrak{A}\times M$ is the p-class of the bundle $\mathfrak{A}\times P$, computed using the connection $\bar{D} = D+\theta\varphi$. Other connections we could use are D or the pull-back of a connection on P over M, call this D_B. Now

$$p_{2k}(D_B^2) = 0$$

because $2k > \dim(M)$. So the way to obtain the transgression is to use some path of connections from D_B to $D+\theta\varphi$.

Notice that if $k > \dim(M)$, then we have

$$p_{2k}(D^2) = 0$$

because D^2 has type $(1,1)+(0,2)$. Hence we can join $D+\theta\varphi$ to D which is a canonical choice and will lead to invariant differential forms on $\mathfrak{A}$.

November 13, 1983

I consider Iz's process of multiplying by $vol_B \in \Omega^d(P)$ and then integrating over the fibre. Here $d = \dim G$ and B is a connection on P and vol_B is the vertical form restricting to the Haar volume on each fibre. So if I denote B by the old notation θ I find that $vol_B = \prod_a \theta^a$ where $\theta = \lambda_a \theta^a$ and λ_a is an orthonormal basis for $\mathfrak{g}$.

Now integration over the fibre is the same as first taking the interior product with [redacted] a generator for [redacted] Λ^{max}(Tangent space to fibre) and then integrating. Thus it is $\prod_a \iota_a$ followed by averaging over the group.

So Iz's process is a map

$$(*) \qquad \Omega(P) \xrightarrow{\pm \prod_a \iota_a \prod_a \theta^a} \Omega(P)_{horiz} \xrightarrow{\int} \Omega(P)_{basic} = \Omega(M)$$

and the map

$$\pm \prod_a \iota_a \prod_a \theta^a = \prod_a i_a \theta^a = \prod (1 - \theta^a \iota_a)$$

is the thing you encountered with equivariant forms.

Another description:

$$(*) \qquad \underset{\substack{\uparrow \\ \text{connection}}}{\Omega(P) \cong} \Lambda \mathfrak{g}^* \otimes \Omega(P)_{horiz} \xrightarrow[\text{comp}]{\text{bottom}} \Omega(P)_{horiz} \xrightarrow{\int} \Omega(M).$$

This process is used as follows. Let A be another connection form. We can consider the Chern-Simons form:

$$tr(e^{F_A}) - tr(1) = d \int_0^1 dt \; tr\left(A e^{tF_A + (t^2-t)A^2}\right)$$

and we can consider the difference form for the characters of the two connections A, B.

$$\operatorname{tr}(e^{F_A}) - \operatorname{tr}(e^{F_B}) = d \int_0^1 dt \operatorname{tr}\left((A-B)\, e^{F_{B+t(A-B)}}\right)$$

The second form is basic, and Ezy's draft seems to suggest that applying the projector $(*)_B$ to the Chern–Simons form for A gives this basic form.

Let's check this: First rewrite the projector $(*)_B$ of $\Omega(P)$ on $\Omega(P)_{basic}$ as follows:

$$\Omega(P) \xleftarrow{\sim} \Omega(P)_{horiz} \otimes \Lambda \mathfrak{g}^* \xrightarrow{id \otimes \varepsilon} \Omega(P)_{horiz} \xrightarrow{H} \Omega(P)_{basic}$$

where ε is the augmentation of $\Lambda \mathfrak{g}^*$. Now we use that $id \otimes \varepsilon$ is a ring homomorphism which kills $B \overset{=}{\text{or}} \Theta$. Put $A = B + \eta$ or $A = \Theta + \eta$. Then

$$A^2 \longmapsto \eta^2 \qquad dA = d\Theta + d\eta \longmapsto F_B + d\eta \ ?$$

We have to be careful. We know that η is horizontal but we have to be careful about $d\eta$. Now $\Theta + t\eta$ is a family of connection forms, so

$$d(\Theta + t\eta) + (\Theta + t\eta)^2 = d\Theta + \Theta^2 + t\left(d\eta + [\Theta, \eta]\right) + t^2\eta^2$$

is horizontal, whence

$$d\eta + [\Theta, \eta] \quad \text{is horizontal.}$$

The calculation goes as follows. We start with the Chern-Simons form

$$\int_0^1 dt \operatorname{tr}\left(A e^{tF_A} + (t^2 - t) A^2\right) \qquad A = \Theta + \eta$$

and first compute its image under the projector $(*)_B : \Omega(P) \longrightarrow \Omega(P)_{hor,y}$ which we know is a ring homomorphism. Now $A \mapsto \eta$ and F_A is already horizontal. Hence

$$tF_A + (t^2-t)A^2 \longmapsto t(d\theta + d\eta + \theta^2 + [\theta,\eta] + \eta^2) + (t^2-t)\eta^2$$

$$\longmapsto t(\underbrace{d\theta + \theta^2}_{F_B} + d\eta + [\theta,\eta]) + t^2\eta^2$$

So the image of the Chern-Simons form under $(*)_B$ is

$$\int_0^1 dt\ \mathrm{tr}\left(\eta e^{t(F_B + d\eta + [\theta,\eta]) + t^2\eta^2}\right).$$

On the other hand what we want to get is

$$\int_0^1 dt\ \mathrm{tr}\left(\eta e^{F_{\theta + t\eta}}\right)$$

where $F_{\theta+t\eta} = d\theta + t\,d\eta + \theta^2 + t[\theta,\eta] + t^2\eta^2$, i.e.

$$\int_0^1 dt\ \mathrm{tr}\left(\eta e^{F_B + t(d\eta + [\theta,\eta]) + t^2\eta^2}\right).$$

So it appears that the two forms are different because of the tF_B instead of F_B.

(Better proof 11/20 : see p. 251)

November 13, 1983 (continued)

Let return to the point we reached yesterday. On $\mathcal{A}\times P$ over $\mathcal{A}\times M$ we have the tautological connection form $\hat{A}$ which is of type $(0,1)$ and on the fibre $\{A\}\times M$ restricts to A. We change it to $\hat{A}+\Theta\varphi$ so that it descends to $\mathcal{A}\times^{\mathcal{G}} P$ over $\mathcal{A}/\mathcal{G}\times M$. Upon taking $p(F_{\hat{A}+\Theta\varphi})$ we get a basic form which descends to $\mathcal{A}/\mathcal{G}\times M$; then integrating over M we get [redacted] a form on $\mathcal{A}/\mathcal{G}$.

To compute the transgression of the form on $\mathcal{A}/\mathcal{G}$ one pulls back to $\mathcal{A}$ and writes as a coboundary. Hence it [redacted] suffices to write $p(F_{\hat{A}+\Theta\varphi})$ on [redacted] $\mathcal{A}\times P$ as the coboundary of a basic form. This form is the p-class of the bundle $\mathcal{A}\times P$ over $\mathcal{A}\times M$ which vanishes as we can suppose $\deg(p) > \dim M$. Thus a path from $\hat{A}+\Theta\varphi$ to a connection [redacted] such that p gives zero when applied to the curvature will give a class u on $\mathcal{A}\times M$ with $du = p(F_{\hat{A}+\Theta\varphi})$. There are two choices: i) Pick a $B\in\mathcal{A}$ and then go from $\hat{A}+\Theta\varphi$ to B pulled up to $\mathcal{A}\times P$.

ii) If $\deg(p) = 2k$ (degree in $W(\mathfrak{g})$) and $k > \dim M$, then we know that $p(F_{\hat{A}}) = 0$.

Now my problem is to see these choices when I pull-back to the bundle $\mathcal{G}\times P$ over $\mathcal{G}\times M$ using an orbit $\mathcal{G}\longrightarrow\mathcal{A}$. I am then going to have three connections on $\mathcal{G}\times P$ over $\mathcal{G}\times M$, namely, the pull-backs of $\hat{A}$, $\hat{A} + \theta\varphi$, B.

Let's think in terms of horizontal and vertical connections on the vector bundle $\mathcal{A} \times E$ over $\mathcal{A} \times M$. Then this bundle has a tautological vertical connection and a canonical horizontal connection. Together they give $\hat{A}$. In my old notation I would write

$$d' + d'' + \hat{A} \qquad \text{operates on} \quad \Omega(\mathcal{A} \times P, V)$$

where $P \times_M E = P \times V$. Then the others are

$$d' + \theta\varphi + d'' + \hat{A} \qquad \text{and} \qquad d' + d'' + B$$

An important point is that when we restrict to $\mathcal{G}$, θ becomes the Maurer-Cartan form of $\mathcal{G}$, and hence $d' + \theta\varphi$ becomes the canonical flat horizontal connection on $\mathcal{G} \times P$ over $\mathcal{G} \times M$. On the other hand $d'' + \hat{A}$ should pull back to the vertical connection over $\mathcal{G} \times M$ which is $g(d_P + A_0)g^{-1}$ over $\{g\} \times M$. Thus my guess for $\hat{A} + \theta\varphi$ pulled back to $\mathcal{G} \times M$ is the connection

$$d_{\mathcal{G}} + g^{-1} d_{\mathcal{G}} g + g(d_P + A_0)g^{-1}$$

So it is clear that I have the signs all mixed up. And since these have to be straightened out for my class tomorrow, I should work this out now.

~~The first point is that we want to preserve the algebra of vector fields and differential forms. There is a sign problem caused by the fact that the Lie derivative $\mathcal{L}_X$ is supposed to satisfy the conditions~~

The first requirement is to preserve the usual algebra of operators on differential forms. If G acts on M, we then get for each $X \in \mathfrak{g}$ a vector field on M, again denoted by X. Here $\mathfrak{g}$ is the space of 1-parameter subgroups and given $X \in \mathfrak{g}$ one denotes the 1-parameter subgroup by e^{tX}, and the corresponding flow on M gives rise to the vector field X. Now the algebra of forms takes over.

$$\mathcal{L}_X = \iota_X d + d\,\iota_X$$

so
$$\mathcal{L}_X f = \iota_X df = Xf = \frac{d}{dt} f \circ e^{tX} \Big|_{t=0}.$$

The problem arises because if one wants G to act both on the left of M and $\Omega(M)$, one should have put in a minus sign in the last equality. So we therefore are forced to have G acting on M to the right.

<u>Ex</u>: G = group of matrices $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ acting on $\mathbb{R}$ by $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} ax+b \\ 1 \end{bmatrix}$. This is a left action. The 1-parameter subgrp $\begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}$ has generator $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and corresponds to the vector field $x\frac{\partial}{\partial x}$ as

$$\frac{d}{dt} f(e^t x)\Big|_{t=0} = x\frac{\partial f}{\partial x}$$

The 1-par. subgp $\begin{bmatrix} 1 & t \\ & 1 \end{bmatrix}$ has generator $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and corresponds to $\frac{\partial}{\partial x}$. But then

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ whereas

$$\left[x\frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right] = -\frac{\partial}{\partial x}$$

So in order to preserve the usual formulas we need to have G acting on the right. [redacted]

Next consider a principal bundle P over M for G. There are two kinds of vector fields along the fibres of P [redacted] over M which one can consider

(i) vector fields invariant under [redacted] the G action

(ii) vector fields generated by the flows e^{tX} with $X \in \mathfrak{g}$.

Already when $P = G$ these are different. Case (i) gives the right-invariant vector fields and these are the infinitesimal left multiplications. Also (i) is bad because in general one would have an infinite diml space of these.

So we consider the vector fields (ii) on P. Thus on G itself we consider left invariant vector fields, i.e. infinitesimal right multiplications. It should be clear from the fact we need a right G-action that the bracket of left-invariant differential vector fields corresponds to the bracket defined by commutators of 1-parameter subgroups.

Next connections

Remark: The vector fields (i) are the elements of $\tilde{\mathfrak{g}}$, i.e. the infinitesimal gauge transformations on P.

November 14, 1983

Work out the formulas before tomorrow in Jaffe's seminar.

Given two connections D_0, D_1 I join them by a linear path

$$D_t = (1-t)D_0 + tD_1 .$$

The curvature is

$$\begin{aligned} D_t^2 &= (1-t)^2 D_0^2 + (1-t)t[D_0,D_1] + t^2 D_1^2 \\ &= (1-t-t(1-t))D_0^2 + t(1-t)[D_0,D_1] + (t-t(1-t))D_1^2 \\ &= (1-t)D_0^2 + tD_1^2 - t(1-t)\{D_0^2 - [D_0,D_1] + D_1^2\} \end{aligned}$$

$$\boxed{D_t^2 = (1-t)D_0^2 + tD_1^2 - t(1-t)(D_1-D_0)^2}$$

This is a nice formula for it gives in the case of $D_0 = d+B$ and $D_1 = d+A$ the formula for the curvature at time t

$$F_t = (1-t)F_B + tF_A - t(1-t)(A-B)^2$$

Thus the transgression formula is

$$\boxed{\begin{aligned} &tr(e^{F_A}) - tr(e^{F_B}) = \\ &\quad d\int_0^1 dt\, tr\left((A-B)e^{(1-t)F_B + tF_A - t(1-t)(A-B)^2}\right) \end{aligned}}$$

Next I want to apply this in the case of $\mathcal{G}\times M$. I suppose given two connections A_0, B on the bundle E (or P) over M, let's write them in terms of a trivialization of E as

$$d_M + A_0, \quad d_M + B.$$

Pull them back to connections on $\mathcal{G} \times E$

$$d' + d'' + A_0 \quad , \quad d' + d'' + B$$

and transform the former by the canonical gauge transformation $\hat{g}$ of $\mathcal{G} \times E$ over $\mathcal{G} \times M$.

$$\begin{aligned} d' + d'' + A &= \hat{g}^{-1}(d' + d'' + A_0)\hat{g} \\ &= d' + d'' + \underbrace{\hat{g}^{-1}d'\hat{g}}_{\substack{\text{Maurer-Cartan} \\ \text{form } \theta \text{ on } \mathcal{G}}} + \underbrace{\hat{g}^{-1}d''\hat{g} + \hat{g}^{-1}A_0\hat{g}}_{\hat{g} * A_0} \end{aligned}$$

Now $\quad F_A = \hat{g}^{-1} F_{A_0} \hat{g} \quad$ is of type $(0,2)$ since F_{A_0} is. The same is true for F_B. Now

$$A - B = \underbrace{\theta}_{1,0} + \underbrace{\hat{g} * A_0 - B}_{0,1}$$

so the formula will be very messy.

Suppose now $B = 0$, i.e., we are thinking of E as trivial. Then we get the form on $\mathcal{G} \times M$

$$\int_0^1 dt \ \operatorname{tr}\left((\hat{g} * A_0 + \theta)\, e^{t\, \hat{g}^{-1} F_{A_0} \hat{g} + (t^2 - t)(\hat{g} * A_0 + \theta)^2}\right)$$

To simplify the notation let's fix a point g of $\mathcal{G}$ and put $A = g * A_0$, whence we get

$$\int_0^1 dt \ \operatorname{tr}\left((A + \theta)\, e^{t F_A + (t^2 - t)(A + \theta)^2}\right)$$

November 15, 1983

Let's review the formulas from yesterday morning. We have a bundle $\mathcal{G}\times E$ over $\mathcal{G}\times M$ and we are after the secondary character class associated to the tautological automorphism of this bundle. Normally we compute this as follows. We choose a connection D_0 on the bundle transform it by the automorphism ϕ to get $D_1 = \phi^{-1}D_0\phi$. Then we have two connections on the same bundle and their character forms differ by a coboundary

$$\operatorname{tr}\left(e^{D_1^2}\right) - \operatorname{tr}\left(e^{D_0^2}\right) = d\int_0^1 dt\, \operatorname{tr}\left((D_1 - D_0)\, e^{(1-t)D_0^2 + tD_1^2 + (t^2-t)(D_1-D_0)^2}\right)$$

But $D_1^2 = \phi^{-1}D_0^2\phi$ so the LHS is zero and we get a ~~well defined~~ closed form from the integral on the right.

In the present situation we join $D_1 = \phi^{-1}D_0\phi$ [redacted] not to D_0, but to D_B, and then [redacted] join D_B to D_0. Also we deal with just the components of high degree of the character, so in this case [redacted] $\operatorname{tr}\,(D_B^2)^k = 0$.

The bundle is $\mathcal{G}\times E$ over $\mathcal{G}\times M$, say E is trivialized. $D_B = d$, $D_0 = d + A_0$ where $A_0 \in \mathcal{A}$ is pulled back to $\mathcal{G}\times M$. The bundle auto is

$$\begin{aligned} \mathcal{G}\times E &\xrightarrow{\ \phi\ } \mathcal{G}\times E \\ g, e &\longmapsto g, ge \end{aligned}$$

and the real obstruction is computing $\phi^{-1}D_0\phi$. The point will be that it splits into two pieces

$$\phi^{-1} D_0 \phi = d + \theta + A$$

where $\theta = \phi^{-1} d' \phi$ is the Maurer-Cartan form of $\mathcal{G}$ and where $A = \phi^{-1} d'' \phi + \phi^{-1} A_0 \phi$. Then

$$D_1^2 = \phi^{-1} D_0^2 \phi = \phi^{-1} F_{A_0} \phi \quad , \quad D_0^2 = 0$$

$$D_1 - D_0 = \theta + A.$$

computed on M, then pulled back.

so

$$D_t^2 = (1-t) D_0^2 + t D_1^2 + (t^2 - t)(D_1 - D_0)^2$$

$$= t\, \phi^{-1} F_{A_0} \phi + (t^2 - t)\left(A^2 + [A, \theta] + \theta^2\right).$$

At this point I specialize to a point $g \in \mathcal{G}$ and ask what kind of form I get on the tangent space at g. Then

$$\phi^{-1} F_{A_0} \phi = g^{-1} F_{A_0} g = F_A = (d'' A + A^2).$$

so

$$D_t^2 = t\, d'' A + t^2 A^2 + (t^2 - t)\left([A, \theta] + \theta^2\right)$$

and I get the form

$$\frac{1}{n!} \int_0^1 dt \; \cancel{\mathrm{tr}}\left((\theta + A)\left(t\, d'' A + t^2 A^2 + (t^2 - t)\left([A, \theta] + \theta^2\right)\right)^n \right)$$

This is still very opaque. What formula should you write down?

$$\int_0^1 dt \; \mathrm{tr}\left((A + \theta)\, e^{t(d'' A + A^2) + (t^2 - t)(A + \theta)^2} \right)$$

Ultimately over $\mathcal{G} \times M$ you will define matrix forms Θ, A of types $(1,0)$ and $(0,1)$ such that

$$\phi^{-1} D_0 \phi = d + \Theta + A$$

Thus on $g \times M$, A is the form $g * A_0 = g^{-1} d_M g + g^{-1} A_0 g$

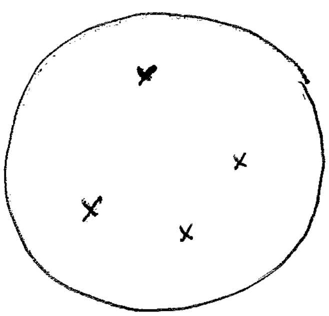
November 16, 1983

[redacted] Notes from the talk yesterday:

Luis' collaborator said they prove the anomaly formula

$$c_1(\bar{\mathcal{L}}) = \int_{M^{2n}} \left(ch(E)\hat{A}(M)\right)_{n+1}$$

geometrically as follows: It is enough to check the formula by integrating both sides over singular 2-spheres in $\mathcal{A}/\mathcal{G}$. [redacted] (Assume $\mathcal{G}$ connected). Hence one has a map of a disk into $\mathcal{A}$ whose boundary lies on a $\mathcal{G}$-orbit, [redacted] and we can assume the boundary consists of invertible operators and that the disk is transversal to the singular [redacted] hypersurface. Then $c_1(\bar{\mathcal{L}})$ is going to count the number of zeroes of the canonical section.



[redacted] Now they have a way of trivializing $\mathcal{L}$ over $\mathcal{A}$ based on a fixed $\not{\partial}_B^*$

$$\begin{bmatrix} 0 & \not{\partial}_B^* \\ \not{\partial}_A & 0 \end{bmatrix}$$

This gives them a map [redacted] of the boundary to $\mathbb{C}^*$ on S^1, and they can compare the phase differences with operator $\not{\partial}_A$, and conclude that $\int_{S^2} c_1(\bar{\mathcal{L}}) =$ winding number of this map.

Next they can somehow relate the winding number to the winding number of E over $S^2 \times M$

$$\mathcal{G} = \Omega^4 SU(2) \qquad \begin{aligned} \pi_0 \mathcal{G} &= \pi_4 S^3 = \mathbb{Z}/2 \\ \pi_1 \mathcal{G} &= \pi_5 S^3 = \mathbb{Z}/2 \end{aligned}$$

Witten's anomaly is connected with $\pi_4 S^3$. The following shows that only the torsion in $\pi_0 \mathcal{G}$ can contribute to $H^2(B\mathcal{G}, \mathbb{Z})$.

$$\begin{array}{ccccccc} H^1(B\mathcal{G}, \mathbb{Q}) & \longrightarrow & H^1(B\mathcal{G}, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^2(B\mathcal{G}, \mathbb{Z}) & \longrightarrow & H^2(B\mathcal{G}, \mathbb{Q}) \\ \| & & \| & & & & \\ \mathrm{Hom}(\pi_0\mathcal{G}, \mathbb{Q}) & \longrightarrow & \mathrm{Hom}(\pi_0\mathcal{G}, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{Ext}^1(\pi_0\mathcal{G}, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

Thus
$$0 \longrightarrow \mathrm{Ext}^1(\pi_0\mathcal{G}, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^2(B\mathcal{G}, \mathbb{Z}) \longrightarrow H^2(B\mathcal{G}, \mathbb{Q})$$
is exact. Thus there is a line bundle with structural group ± 1 over $B\mathcal{G}$, $\mathcal{G} = \Omega^4 SU(2)$, and it seems to be exactly what Witten looks at. In any case $\pi_5 S^3 = \mathbb{Z}/2$ doesn't contribute to line bundles over $B\mathcal{G}$.

Also $\mathcal{G} = \Omega^6 SU(3)$ has $\pi_0 = \mathbb{Z}/6$, $\pi_1 = 0$, so there might be a $\mathbb{Z}/6$ - anomaly in 6-dimension. I think Bott showed $\pi_{2n} SU(n) \cong \mathbb{Z}/n!$ so that over S^{2n+2} one could look for a $\mathbb{Z}/n!$ anomaly?

Concerning your lecture, you say that there is a map
$$H^{2k}(\mathcal{A}/\mathcal{G} \times M) \longrightarrow H^{2k-1}(\mathcal{G} \times M).$$
This isn't correct; the first group should be $H^{2k}(\mathcal{A}/\mathcal{G} \times M, M)$. In any case, the point is that $ch_{2k}(E)$ does transgress if done carefully.

Look at $\tilde{E}$ over $(\mathcal{A}/\mathcal{G}) \times M$, and pull it back to $\mathcal{G} \times M$. We have the commutative square

$$\begin{tikzcd} & \mathcal{A} \times M \arrow[dr] & \\ \mathcal{G} \times M \arrow[ur] \arrow[dr] & & (\mathcal{A}/\mathcal{G}) \times M \\ & pt \times M \arrow[ur] & \end{tikzcd}$$

and the map of $\mathcal{G} \longrightarrow \mathcal{A}$ gives a connection A_0 on E, [redacted] which agrees with the connection on $\tilde{E}$ over $pt \times M$. If I start with $\bar{D}$ on $\tilde{E}$ over $\mathcal{A}/\mathcal{G} \times M$ and pull back to [redacted] $\mathcal{A} \times M$, then I can join to the connection B pulled up to $\mathcal{A} \times E$:

$$ch_{\text{[redacted]}k}(\bar{D}) - ch_{\text{[redacted]}k}(B) = du$$

Now when you pull-back further to $\mathcal{G} \times M$ you get

$$ch_{\text{[redacted]}k}\left(\text{[redacted]}\,\hat{g}^{-1} D_{A_0} \hat{g}\right) - ch_{\text{[redacted]}k}(B) = du$$

If I take $B = A_0$, which was the old viewpt I get 0. However since I want to integrate over M I have $2k > \dim M$, and since A_0, B come from M the left-side is zero anyway.

I have decide to learn how to do connections up in the principal bundle. Let (P, M, G) be a principal G-bundle over M and let $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ be a connection form:

(i) $\iota_X \theta =$ [redacted] X

(ii) G-invariant: $\cancel{\mathrm{Ad}(g) R_g^* \theta}$

$$R_g^* \otimes \mathrm{Ad}(g)(\theta) = \theta \qquad \forall g \in G$$

$$\Rightarrow \quad \mathcal{L}_X \theta + [X, \theta] = 0 \qquad \forall X \in \mathfrak{g}$$

Then the curvature is

$$\Omega = d\theta + \theta^2 \qquad \theta^2 = \tfrac{1}{2}[\theta, \theta]$$

and (ii) $\Rightarrow$ Ω is horizontal: $\iota_X \Omega = 0$. In fact we have

$$\Omega \in \left[\Omega^2(P) \otimes \mathfrak{g}\right]_{basic} = \Omega^2(M, \underbrace{P \times^G \mathfrak{g}}_{\mathrm{Ad}(P)})$$

Similarly, a difference of two connections $\theta' - \theta$ is horizontal and invariant, so

$$\theta' - \theta \in \left[\Omega^1(P) \otimes \mathfrak{g}\right]_{basic} = \Omega^1(M, P \times^G \mathfrak{g})$$

So we have our usual transgression formula for the characteristic classes [redacted] computed via different connections:

$$\mathrm{tr}(e^{\Omega_1}) - \mathrm{tr}(e^{\Omega_0}) = d \int_0^1 dt \; \mathrm{tr}((\theta_1 - \theta_0) e^{\Omega_t})$$

where Ω_t is the curvature of $\theta_t = (1-t)\theta_0 + t\theta_1$ and so

$$\Omega_t = d\theta_t + \theta_t^2$$
$$= (1-t)d\theta_0 + t\,d\theta_1 + \underbrace{(1-t)^2}_{1-t+t^2-t}\theta_0^2 + (t-t^2)[\theta_0,\theta_1] + \underbrace{t^2}_{t+t^2-t}\theta_1^2$$
$$= (1-t)\Omega_1 + t\,\Omega_0 + (t^2-t)(\theta_1-\theta_0)^2$$

Next let's consider gauge transformations. In general given a bundle map

$$\begin{tikzcd} P' \arrow[r, "\varphi"] \arrow[d] & P \arrow[d] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

the pull-back $\varphi^*(\theta)$ of a connection form is still a connection form. Thus if $f = id$ and φ is an auto. of P, the effect of φ on connections is
$$\theta \longmapsto \varphi^*\theta.$$
(So $\mathcal{G}$ is a group acting on the space of connections $\mathcal{A}$ which is an affine space for $\Omega^1(M, Ad(P))$. Does this give an element of $H^1(\mathcal{G}, \Omega^1(M, Ad(P)))$?)

Now let's go back to the situation:

$$\begin{tikzcd} \mathcal{G}\times P \arrow[r] \arrow[d] & \mathcal{A}\times P \arrow[r] \arrow[d] & \mathcal{A}\times^{\mathcal{G}} P = \mathcal{Q} \arrow[d] \\ \mathcal{G}\times M \arrow[r] & \mathcal{A}\times M \arrow[r] & \mathcal{A}/\mathcal{G}\times M . \end{tikzcd}$$

Recall that on $\mathcal{A}\times P$ there is a canonical connection which I will call A. This is $\mathcal{G}$-invariant so there is a Higgs field φ, which when combined with a connection form $\overset{\eta}{\mathcal{O}}$ for $\mathcal{A} \to \mathcal{A}/\mathcal{G}$ gives the connection $A + \varphi \mathcal{O}^\eta$ which descends to $\mathcal{Q}$. When I pull back to $\mathcal{G} \times P$, [redacted] $\varphi \mathcal{O}^\eta$ restricts to the Maurer Cartan form Θ on $\mathcal{G}$, so we get the connection form $A + \Theta$ on $\mathcal{G} \times P$. This should turn out to be $m^* A_0$ where

$$m : \mathcal{G} \times P \longrightarrow P \qquad \text{is the mult.}$$

and where $A_0 \in \mathcal{A}$ is the point such that $\mathcal{G} \to \mathcal{A}$ is $g \longmapsto$ [redacted] $g^* A_0$. Thus we have

$$\Omega_{(A+\Theta)} = m^* \Omega_{A_0}$$

and so $\operatorname{tr}\left(\Omega_{(A+\Theta)}\right)^k = m^* \operatorname{tr}\left(\Omega_{A_0}\right)^k = 0$ once $2k > \dim M$. [redacted]

I think now that my confused calculations of the past weekend will be intelligible in this notation.

November 17, 1983

We have decided to try to do the calculations up in the principal bundle. So let's go over equivariant characteristic classes. We have a principal U-bundle P over M and an action of G on it. So a G-invariant connection on P is given by a connection form $\theta \in \Omega^1(P) \otimes \mathfrak{u}$ such that $g^*\theta = \theta$ for all $g \in G$. What is the Higgs field?

To answer this suppose V is a representation of U so that we can consider $E = P \times^U V$ over M. Then we have

$$\underset{\substack{\text{basic for}\\ \text{total } U \text{ action}}}{\left[\Omega(P) \otimes V\right]} \xleftarrow{\sim} \Omega(M, E)$$

$$d + \theta \qquad\qquad\qquad D$$

Recall that the Higgs field $\varphi_X \in \Omega^0(M, \operatorname{End} E)$ is defined on the right by the formula

$$\mathcal{L}_X = [\iota_X, D] + \varphi_X \qquad\qquad X \in \mathfrak{g}$$

On the left [redacted]

$$[\iota_X, d+\theta] = \mathcal{L}_X + \iota_X\theta$$

so we conclude that $\boxed{\varphi_X = -\iota_X\theta}$

(Digression: You have [redacted] the following notational problem concerning just connections $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ in a principal G-bundle. G acts on both factors of this tensor product so that when you write

$$\mathcal{L}_X \Theta + [X, \Theta] = 0$$

the $\mathcal{L}_X$ refers to only the action on $\Omega(P)$. This is OK, but when you come to $[\Omega(P) \otimes \mathfrak{g}]_{basic}$ you want to use the total action.)

So let's suppose given this G-invariant connection Θ in the [redacted] G-equivariant principal U-bundle P and let's consider equiv. characteristic classes. Introduce the Weil alg $W(\mathfrak{g})$. Now we have notational problems.

Let's use Θ, Ω in $W(\mathfrak{g}) \otimes \mathfrak{g}$ as usual and use the notation $A, F \in \Omega(P) \otimes \mathfrak{u}$. Then we modify the connection A to $A + \Theta\varphi$ where $\Theta \in W^1(\mathfrak{g}) \otimes \mathfrak{g}$ is the universal connection form and $\varphi \in \mathfrak{g}^* \otimes \Omega^0(P) \otimes \mathfrak{u}$ is the [redacted] Higgs field:

$$\varphi_X = -\iota_X A.$$

(Here $\Theta\varphi$ denotes the contraction of Θ and φ, so that $\Theta\varphi \in W^1(\mathfrak{g}) \otimes \Omega^0(P) \otimes \mathfrak{u}$.)

Thus $\iota_X (A + \Theta\varphi) = 0$ so that the Higgs map associated to $A + \Theta\varphi$ is 0.

Wait: $\quad A + \Theta\varphi \;=\; A - \theta^a \iota_a A$

is the canonical projection of A onto the horizontal forms.

Let's go over the problem again. We have a G equivariant principal $\mathcal{U}$-bundle P and an invariant connection on it. We are interested in the characteristic classes of P in equivariant cohomology. If we use the invariant connection to compute the char. classes then we get invariant forms on the base M. But there is a problem lifting a closed invariant form to a closed equivariant form. So one works directly with the connection.

$$\left[W(\mathfrak{g}) \otimes \Omega(P) \otimes \mathfrak{U}\right]_{horiz} \xrightarrow{\sim} S(\mathfrak{g}^*) \otimes \Omega(P) \otimes \mathfrak{U}$$

$$A - \theta^a i_a A \qquad\qquad \overset{\omega}{A}$$

$$d(A - \theta^a i_a A) + (A - \theta^a i_a A)^2 \longmapsto (d - \Omega^a i_a)A + A^2$$

$$= F - \Omega^a i_a A$$

Conclusion: Before I wrote

$$\left[W(\mathfrak{g}) \otimes \Omega(M, E)\right]_{basic} \xrightarrow{\sim} \left[S(\mathfrak{g}^*) \otimes \Omega(M, E)\right]^G$$

$$\bar{D} = D + \theta^a \varphi_a \longmapsto D - \Omega^a i_a$$

$$\bar{D}^2 \longmapsto D^2 + \Omega^a \varphi_a$$

(see p. 97)

Next I want to consider the G-map $G \to M$ given by a point of M. The U-bundle P over M pulls back to an equivariant G-bundle over G which is a bundle over G is trivial in a definite way. Hence the characteristic forms for P over M computed using the invariant connections must become coboundaries over G is a canonical way.

Summary of today's work: I want to consider an equivariant principal U-bundle P over the G-manifold M. If V is a representation of U, then $E = P \times^U V$ is an equivariant vector bundle over M for G. If we are given G-invariant connection in P, we get a G-invariant connection D in E. I have already described how to calculate the equivariant characteristic classes of E using D^2 and the Higgs field belonging to D.

I wanted today to do the same calculation using the principal bundle P and the form $A \in \Omega^1(P) \otimes \mathfrak{u}$ associated to the connection. This I have done: The Higgs field is $\varphi_X = -\iota_X A$ so the connection that descends is $\bar{A} = A - \theta^a \iota_a A$ or $A + \theta\varphi$. The equivariant curvature is

$$(d - \Omega^a \iota_a)A + A^2 = F + \Omega\varphi.$$

Then I looked at a different question related to transgression, namely, how the form $ch(e^{D^2})$

becomes a coboundary when pulled back via the map $G \to M$ determined by a point of M. The idea here is that the bundle becomes trivial so the cohomology classes should be trivial.

Let's look at this problem topologically:

$$\begin{tikzcd} & H^*(G) & H^*(M) \arrow[l] \\ H^*(pt) = & H^*_G(G) \arrow[u] & H^*_G(M) \arrow[l] \arrow[u] \end{tikzcd}$$

Replace by forms:

$$\begin{tikzcd} \Omega(G)^G & \Omega(M)^G \arrow[l] \\ \left[S(\mathfrak{g}^*) \otimes \Omega(G)\right]^G \arrow[u] & \left[S(\mathfrak{g}^*) \otimes \Omega(M)\right]^G \arrow[l] \arrow[u] \end{tikzcd}. \qquad ?$$

Let's instead look at an equivariant principal U-bundle over G, and suppose it is equipped with an invariant connection. To simplify suppose that $U = \mathrm{Aut}(V)$ so that we have G-vector bundle E over G equipped with an invariant connection D. Then $E \cong G \times V$ with G acting on itself, and this has an obvious invariant connection, so that we can write $D = d + A$ where $A \in \Omega^1(G) \otimes \mathrm{End}(V)$ is left-invariant, and hence

$$A \in \mathfrak{g}^* \otimes \mathrm{End}(V) = C^1(\mathfrak{g}, \mathrm{End}(V))$$

The curvature of D is

$$F = dA + A^2 \in \Lambda^2 \mathfrak{g}^* \otimes \mathrm{End}(V) = C^2(\mathfrak{g}, \mathrm{End}(V)).$$

(Note it is clear more or less that $X \mapsto i_X A \in \mathrm{End}\, V$ is a Lie homomorphism $\Longleftrightarrow F = 0$.)

The question is why the form $\mathrm{tr}(e^F)$ is a coboundary. One answer is to use the linear path between d and $d+A$; this leads to the Chern-Simons form

$$\int_0^1 dt\ \mathrm{tr}\left\{ A e^{t\,dA + t^2 A^2} \right\}.$$

An equivalent answer is to use the linear path from A to $\bar{A} = A + \theta\varphi$; in the present case $\theta \in \Omega^1(G) \otimes \mathfrak{g}$ is the Maurer-Cartan form and $\varphi_X = -\iota_X A$. Now $A \in \mathfrak{g}^* \otimes \mathrm{End}\, V$, so

$$A = \theta^a A_a \quad \text{is} \quad \varphi_a = -\iota_a A = -A_a$$

and then $\bar{A} = A + \theta^a \varphi_a = A + \theta^a(-A_a) = 0.$

Next let's go onto the case where $G = \mathcal{G}$ the gauge group. We have

$$\begin{array}{ccccc}
\mathcal{G} \times P & \longrightarrow & \mathcal{A} \times P & \longrightarrow & \mathcal{Q} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{G} \times M & \longrightarrow & \mathcal{A} \times M & \longrightarrow & \mathcal{A}/\mathcal{G} \times M
\end{array}$$

and we are looking at $ch_j(\mathcal{Q}) \in H^{2j}(\mathcal{A}/\mathcal{G} \times M)$ where $2j > \dim M$. Then if $ch_j(\mathcal{Q})$ is pulled up to $\mathcal{A} \times M$ it vanishes. On the other hand, if $\mathcal{Q}$ is pulled back to $\mathcal{G} \times M$

it becomes isomorphic to $G \times P$ another way.
Too confusing. Try another approach:

$$\begin{array}{ccc} G \times M & \longrightarrow & a \times M \\ \downarrow & & \downarrow \\ pt \times M & \longrightarrow & a/G \times M \end{array}$$

<u>November 18, 1983</u>

Let's return to our old viewpoint that we are primarily interested in $\mathcal{G}$-bundles. A $\mathcal{G}$-bundle $\mathcal{P}$ over Y is the same as a U-bundle $\tilde{P}$ over $Y \times M$ such that $\tilde{P}$ over each $\{y\} \times M$ is isom. to P:

$$B\mathcal{G} \sim \mathrm{Map}(M, BU)_{(P)}.$$

We construct char. classes for $\mathcal{G}$-bundles using the char. classes for U-bundles. Given $\mathcal{P} \longleftrightarrow \tilde{P}$ we need a connection on $\tilde{P}$ over $Y \times M$. This is the same as (i) horiz conn. $\longleftrightarrow$ connection in $\mathcal{P}$ over Y

(ii) vert. conn. $\longleftrightarrow$ $\mathcal{G}$-map $\mathcal{P} \longrightarrow \mathcal{A}$.

Picture:

$$\left(\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d, dotted] \\ Y \arrow[r, dotted] & \mathcal{A}/\mathcal{G} \end{tikzcd}\right) \times M \qquad \xleftarrow[U]{\text{fibre}} \qquad \begin{tikzcd} \mathcal{P} \times P \arrow[r] \arrow[d] & \mathcal{A} \times P \arrow[d] \\ \tilde{P} \arrow[r] & \mathcal{Q} \end{tikzcd} \qquad \Big\downarrow{\scriptstyle \text{fibre } \mathcal{G}}$$

When we calculate the char. classes of $\tilde{P}$, we use a connection form in $\tilde{P}$ which comes by $\mathcal{G}$-descent from a connection form on $\mathcal{P} \times P$.

Therefore given $\mathcal{P} \longleftrightarrow \tilde{P} = \mathcal{P} \times^{\mathcal{G}} P$ we are going to calculate with forms on $\mathcal{P} \times P$. The connection form on $\tilde{P}$ is going to lift back to a connection on $\mathcal{P} \times P$ which will have two pieces:

(i) vertical component. This comes from the map

$\mathcal{P} \longrightarrow \mathcal{A}$ and lies in

$$\Omega^{0,1}(\mathcal{P}\times P)\otimes \mathfrak{u} = \Omega^0(\mathcal{P}, \Omega^1(P)\otimes \mathfrak{u})$$

It is the map

$$\mathcal{P}\longrightarrow \mathcal{A} \subset \Omega^1(P)\otimes\mathfrak{u}.$$

(ii) horizontal component: This comes from the connection in $\mathcal{P}$ which is an element

$$\theta \in \Omega^1(\mathcal{P}, \tilde{\mathfrak{g}}).$$

Now $\tilde{\mathfrak{g}} = (\Omega^0(P)\otimes\mathfrak{u})_{bas} \subset \Omega^0(P)\otimes \mathfrak{u}$ so if I denote this inclusion by φ, I get

$$\varphi\theta \text{ or } \theta\varphi \in \Omega^1(\mathcal{P}, \Omega^0(P)\otimes\mathfrak{u}) = \Omega^{1,0}(\mathcal{P}\times P)\otimes \mathfrak{u}$$

The universal situation is the following: On $\mathcal{A}\times P$ is a canonical $\mathfrak{u}$-connection form

$$\hat{A} \in \Omega^{0,1}(\mathcal{A}\times P)\otimes\mathfrak{u} = \Omega^0(\mathcal{A}, \Omega^1(P)\otimes \mathfrak{u}))$$

which comes from the inclusion $\mathcal{A}\subset \Omega^1(P)\otimes \mathfrak{u}$.
In the Weil algebra $W(\tilde{\mathfrak{g}})$ is the canonical connection form θ. Then the $\mathfrak{u}$-connection which descends to the G-orbit space is

$$\hat{A} - \theta\iota\hat{A} \in \left(W(\tilde{\mathfrak{g}})\otimes \Omega(\mathcal{A}\times P)\otimes \mathfrak{u}\right)$$

If $v\in\tilde{\mathfrak{g}} = (\Omega^0(P)\otimes\mathfrak{u})_{bas} \subset \Omega^0(P)\otimes\mathfrak{u}$, then what is $\iota_v \hat{A}$?

November 19, 1983

I have to get the formulas for gauge transf. on the principal bundle straight. [redacted] Let P be a principal $\mathcal{U}$-bundle over M, let V be a representation of $\mathcal{U}$, let $E = P\times^{\mathcal{U}} V$ be the associated vector bundle on M. Then we have

$$\Omega(M,E) \xrightarrow{\sim} \big(\Omega(P)\otimes V\big)_{basic}$$

A connection A on P induces one in E:

$$D \longmapsto d+A$$

where $A\in\Omega^1(P)\otimes\mathfrak{u}$ is interpreted as an element of $\Omega^1(P)\otimes \mathrm{End}(V)$ via the representation $\mathfrak{u}\longrightarrow \mathrm{End}\,V$.

[redacted] Let $\mathcal{G}$ = group of gauge transformations, and $\tilde{\mathfrak{g}}$ its Lie algebra. Given $g\in\mathcal{G}$, [redacted] or $v\in\tilde{\mathfrak{g}}$ we have to explain how it acts on connections. If g is viewed as an auto of E, then

$$g:\quad D\longmapsto g^{-1}\cdot D\cdot g \;=\; g^{-1}[D,g]$$

defines a right action of $\mathcal{G}$ on the space of connections. Infinitesimally we have

$$v:\quad D\longmapsto [D,v]\;\in\Omega^1(M,\mathrm{End}\,E)$$

or in another notation

$$\delta_v(D) = [D,v]\in\Omega^1(M,\mathrm{End}\,E).$$

Here v is interpreted as a vector field on $\mathcal{A}$ and $\delta_v(D)$ is its values at D which lies in the tangent space to $\mathcal{A}$ at D; because $\mathcal{A}$ is an affine space this tangent space can be identified with $\Omega^1(M, \operatorname{End} A)$

Now I want to pass to the principal bundle picture. Here g is interpreted as an autom. of P commuting with the U-action. We have a commutative square

$$\begin{array}{ccc}
\Omega(M,E) & \hookrightarrow & \Omega(P)\otimes V \\
\big\downarrow g & & \big\downarrow (g^{-1})^*\otimes 1 \\
\Omega(M,E) & \hookrightarrow & \Omega(P)\otimes V
\end{array}$$

hence

$$g^{-1}\circ D\circ g \text{ [redacted]} \longmapsto g^*(d+A)(g^*)^{-1}$$
$$= d + g^*(A)$$

since d commutes with diffeomorphisms. So we see that viewed as forms on P, the action of gauge transformations on connections is

$$g: \quad A \longmapsto g^*A.$$

Next I want the infinitesimal version of this formula. This means that an infinitesimal gauge transf. v determines a vector field on the space of connections, and I want the formula for $\delta_v(A)$ the value of the vector field at A. We have

$$\delta_v(D) = [D,v] \longmapsto [d+A,v] = dv + [A,v]$$

hence

$$\delta_v(A) = dv + [A,v] \in \left(\Omega^1(P)\otimes \mathfrak{u}\right)_{basic}$$

Now I have explained the $\mathcal{G}$-action on $\mathcal{A}$ and the $\tilde{\mathfrak{g}}$-action on $\mathcal{a}$. But [redacted] $\mathcal{G}$ also acts on P and so there are vector fields on P associated to the $\mathcal{G}$ action. In particular we have operations ι_v and $\mathcal{L}_v$ on $\Omega(P)$, and we can ask what are $\iota_v(A)$ and $\mathcal{L}_v(A)$ for a connection form $A \in \Omega^1(P) \otimes \mathfrak{u}$.

<u>Prop</u>:
$$\iota_v(A) = v \qquad \in (\Omega^0(P)\otimes \mathfrak{u})_{bas}$$
$$\mathcal{L}_v(A) = \delta_v(A) = dv + [A,v] \in (\Omega^1(P)\otimes\mathfrak{u})_{bas}$$

<u>Proof</u>: The second formula is obvious because $\mathcal{L}_v(A)$ is just the tangent vector to $(e^{tv})^*A$ at $t=0$ which is also $\delta_v(A)$.

First formula: The vector field v on P points vertically, and we know that a connection form in the vertical direction gives the natural identification of the vertical tangent space with $\mathfrak{u}$. Try a trivialization of P. Then $v: M \longrightarrow \mathfrak{u}$ and $A = \theta + A_b$? Not clear.

In any case we get the following check:

$$\begin{aligned}\mathcal{L}_v(A) &= d\iota_v A + \iota_v dA \\ &= dv + \iota_v(-A^2) \\ &= dv + [A,v]\end{aligned}$$

since $dA + A^2$ is horizontal

The formula $\iota_v(A) = v$ is tautological, and it should come right out of the [redacted] identifications. $\mathcal{G}$ is those autos. of P commuting with the right U-action $\Longrightarrow$ $\tilde{\mathfrak{g}}$ = right U-invariant vector fields on P. So how do you identify an element v of $\tilde{\mathfrak{g}}$ with an element of

$(\Omega^0(P)\otimes \mathfrak{U})^{\mathfrak{U}}$? At a point p of P you have $\mathfrak{U} \xrightarrow{\sim} T_p(\text{Fibre})$ given by the right action. In other words there is an [redacted] $X(v)(p) \in \mathfrak{U}$ such that $v_p = X(v,p)_p$. Then $v \longmapsto X(v) \in (\Omega^0(P)\otimes\mathfrak{U})^{\mathfrak{U}}$ is the identification $\tilde{\mathfrak{g}} \simeq (\Omega^0(P)\otimes\mathfrak{U})^{\mathfrak{U}}$. Now we have $i_X A = X$ for $X \in \mathfrak{U}$, then

$$(i_v A)(p) = \left(i_{X(v,p)} A\right)(p) = X(v,p)_p = v_p$$

which shows $i_v A = v$.

[redacted] One point that should be elucidated using connection forms is the nature of the transgression map

$$W(\mathfrak{g})_{\text{basic}} \longrightarrow [redacted]\ \Lambda(\mathfrak{g})^G$$

[redacted] This lowers degree by one, but [redacted] algebraically it is a mysterious map

$$S^p(\mathfrak{g}^*)^G \longrightarrow \Lambda^{2p-1}(\mathfrak{g}^*)^G.$$

The nature of this map was explained by Chern-Simons. I'll do it for $\mathfrak{g} = \mathfrak{gl}_n$, [redacted] and the invariant polynomial $\operatorname{tr}(e^F)$, $F \in \mathfrak{g}$.

Think in terms of having a P with connection Θ, and curvature Ω. The bundle P when pulled back to itself becomes trivial canonically, so in addition to the pull-back of Θ, there is also a flat connection form describing this canonical trivialization. [redacted] Maybe it is simplest to replace P by $P\times^G V \simeq E$. Then the connection D on E becomes $d+\Theta$ on $P\times V$

which can be compared to the connection d [redacted] on $\Omega(P)\otimes V$ associated to the trivialization. We use the linear path $d+t\theta$ and get

[redacted] $$\operatorname{tr}(e^{\Omega}) - \operatorname{tr}(1) = d\int_0^1 dt\ \operatorname{tr}(\theta e^{\Omega_t})$$

where $\Omega_t = (d+t\theta)^2 = t\,d\theta + t^2\theta^2$ and $F = F_1$.

Also $\Omega_t = t\Omega + (t^2-t)\theta^2$.

So the Chern–Simons, or transgression, form is the element

$$\int_0^1 dt\ \operatorname{tr}\{\theta\, e^{t\Omega + (t^2-t)\theta^2}\}$$

which I can think of as being in the Weil algebra $W(\mathfrak{g})$.

Now restrict to the fibre and one gets the class $\int_0^1 dt\ \operatorname{tr}\{\theta\, e^{(t^2-t)\theta^2}\}$ in $\Lambda(\mathfrak{g})$. So under [redacted] the map $W(\mathfrak{g})_{basic} \longrightarrow \Lambda(\mathfrak{g})$, we have

$$\operatorname{tr}(e^{\Omega}) \longmapsto \int_0^1 dt\ \operatorname{tr}\{\theta\, e^{(t^2-t)\theta^2}\}$$

It will be useful to have the general formula.

Let $p \in S(\mathfrak{g}^*)^G$ be an invariant polynomial, and define $p'(X;Y)$ by

$$p(X+\varepsilon Y) = p(X) + \varepsilon p'(X;Y) + O(\varepsilon^2).$$

Then the transgression formula is

$$p(\Omega) - p(0) = d\int_0^1 dt\ p'(\Omega_t;\theta)$$

hence we get the map

$$(*) \qquad S(\mathfrak{g}^*)^G \longrightarrow \Lambda(\mathfrak{g}^*)$$

$$p(\Omega) \longmapsto \int_0^1 dt\; p'((t^2-t)\theta^2; \theta)$$

The map $(*)$ kills decomposables.

<u>November 20, 1983</u>

Consider a principal G-bundle $\mathcal{P}/Y$ and let $\tilde{P} = \mathcal{P}\times^{G} P$ be the corresponding $\mathcal{U}$-bundle over $Y\times M$. To compute the char. classes of $\tilde{P}$ we choose a connection in it, which is equiv. to a connection θ in $\mathcal{P}$ and a G-map $\mathcal{P}\longrightarrow \mathcal{A}$. The latter gives us over $\mathcal{P}\times P$ a $\mathcal{U}$-connection form $\hat{A} \in \Omega^{0}(\mathcal{P}, \Omega^1(P,\mathfrak{u})) = \Omega^{0,1}(\mathcal{P}\times P, \mathfrak{u})$. (The former is an element [redacted] $\theta \in \Omega^1(\mathcal{P}, \mathfrak{g}) \subset \Omega^1(\mathcal{P}, \Omega^0(P,\mathfrak{u}))$ $= \Omega^{1,0}(\mathcal{P}\times P, \mathfrak{u})$, but maybe I should leave it in the Weil algebra.)

Anyway what we are doing is to construct a connection in $\tilde{P}$ by descending from a connection in $\mathcal{P}\times P$. $\hat{A}$ is G-invariant and gets replaced by $\hat{A} - \theta\iota\hat{A}$ which descends.

Yesterday I computed how $v\in\tilde{\mathfrak{g}}$ acts on connection forms and found

$$\iota_v A = v \qquad \text{under the ident.} \quad \tilde{\mathfrak{g}} = \Omega^0(P,\mathfrak{u})_{basic}$$

$$\mathcal{L}_v A = dv + [A,v] \qquad \text{where we identify all}$$

the tangent spaces to $\mathcal{A}$ with $\Omega^1(P,\mathfrak{u})_{basic}$.

[redacted] On $\mathcal{P}\times P$ we also have a vector field associated to $v\in\tilde{\mathfrak{g}}$. We have

$$\iota_v \hat{A} = v \quad \text{under} \quad \tilde{\mathfrak{g}} = \Omega^0(P,\mathfrak{u})_{bas} \xrightarrow{pr_2^*} \Omega^0(\mathcal{P}\times P,\mathfrak{u})$$

$$\begin{aligned}\mathcal{L}_v \hat{A} &= d\,\iota_v\hat{A} + \iota_v(d\hat{A} + \hat{A}^2 - \hat{A}^2) \\ &= \underbrace{dv + [\hat{A},v]}_{(0,1)\ \text{comes from } P} + \underbrace{\iota_v F_{\hat{A}}}_{(1,0)}\end{aligned}$$

Let's defer the calculation of $i_v F_{\tilde{A}}$ and instead turn to the transgression. Let $p \in S(\mathfrak{u}^*)^U$ and $\gamma \in H_*(M)$. We are interested in the char. class for $\mathcal{G}$-bundles given by associating to $\mathcal{P}/Y$ the class

$$\int_\gamma p(\tilde{P}) \in H^*(Y)$$

(If $\deg p = k$, $\dim \gamma = d$, then this class on Y has degree $2k-d$ and we suppose that $2k > d$ since $B\mathcal{G}$ is connected.) We now want to compute the transgression of this class which lies in $H^*(\mathcal{G})$ of degree $2k-d-1$.

Recall the geometric description of the transgression. Consider the maps

$$H^*(Y) \longrightarrow H^*(\mathcal{P}) \longrightarrow H^*(\mathcal{G})$$

The bundle $\mathcal{P}/Y$ becomes canonically trivial when pulled back over itself. The bundle $\mathcal{P}$ also becomes trivial over $\mathcal{G}$ because $\mathcal{G}$ maps to a point in Y. So really I use the following square

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\
pt \arrow[r] & Y
\end{tikzcd}$$

and then you see the two reasons that the bundle $\mathcal{P}/Y$ becomes trivial over $\mathcal{G}$.

It might help to think in terms of an associated fibre bundle. Then we have

$$\begin{tikzcd}
& G \times V \arrow[r, "\sim"] \arrow[d] & P_y \times V \arrow[r, hook] \arrow[d] & P \times V \arrow[d] \\
V & G \times^G V \arrow[l, "m"', "\sim"] \arrow[r, "\sim"] & P_y \times^G V \arrow[r, hook] & P \times^G V
\end{tikzcd}$$

What this says is that when we pull $P \times^G V$ back to P we get $P \times V$ with right G action $(p, v)g = (pg, g^{-1}v)$, and hence when pulled back to G we get $G \times V$ with the same diagonal action. On the other hand we have the multiplication map $m : G \times V \twoheadrightarrow G \times^G V \cong V$ and hence an isomorphism

$$\begin{aligned} G \times V &\longrightarrow G \times V \\ (g, v) &\longmapsto (g, gv) \end{aligned}$$

where G acts on the ~~latter~~ just on the first factor.

Now take $V = G$ [redacted] as a G-torsor with the natural G-action, and then one sees that P pull back [redacted] over itself is $P \times G$, then restricting to the fibre we get $G \times G$ with action $(g_1, g_2)g = (g_1 g, g^{-1} g_2)$. We then have the isom

$$\begin{aligned} G \times G &\xrightarrow{\sim} G \times G \\ (g_1, g_2) &\longrightarrow (g_1, g_1 g_2) \end{aligned}$$

where G acts latter just on the first factor.

Next let's turn to the case of interest where we have a principal $\mathcal{G}$-bundle $\mathcal{P}/Y$ and an associated fibre space $\mathcal{P} \times^{\mathcal{G}} P = \tilde{P}$ which is a bundle of U-bundles over M, i.e. a U-bundle over $Y \times M$. The analogue of the diagram at the bottom of the preceding page is

$$\begin{tikzcd}
& \mathcal{G} \times P \arrow[r, "\simeq"] \arrow[d] & \mathcal{P}_y \times P \arrow[r, hook] \arrow[d] & \mathcal{P} \times P \arrow[d] \\
& \mathcal{G} \times^{\mathcal{G}} P \arrow[r, "\sim"] \arrow[d] & \mathcal{P}_y \times^{\mathcal{G}} P \arrow[r, hook] \arrow[d] & \mathcal{P} \times^{\mathcal{G}} P = \tilde{P} \arrow[d] \\
P \arrow[r, shift left, "m"] & M \arrow[l, shift left, "s"] & y \times M & Y \times M
\end{tikzcd}$$

Geometrically we have $\tilde{P}$ over $Y \times M$ which when pulled back to $P \times M$ is $\simeq pr_2^*(P)$ (this is the analogue of P [redacted] pulled back to itself being trivial). Hence $\tilde{P}$ pulled back [redacted] to $P \times M$ and then $G \times M$ is isomorphic to $G \times P$ with g acting diagonally $(g, p) g_1 = (g g_1, g_1^{-1} p)$. On the other hand pulling $\tilde{P}$ back to $y \times M = M$ gives P, and then pulling back to $G \times M$ gives $G \times P$ with G acting just on the first factor. The isomorphism between these is

$$\begin{array}{ccc} G \times P & \xrightarrow{\sim} & G \times P \\ (g, p) & \longmapsto & (g, gp) \\ \text{diag. action} & & \text{trivial action on } P \end{array}$$

I think that this explains the geometric side of transgression. Next we go on to the charac. classes. Let's write the basic squares

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, "\text{quotient by } G"'] & P \arrow[d] \\ y \arrow[r, hook] & Y \end{tikzcd} \qquad \begin{tikzcd} G \times M \arrow[r, hook] \arrow[d] & P \times M \arrow[d] \\ M \arrow[r, hook] & Y \times M \end{tikzcd} \qquad \begin{tikzcd} G \times P \arrow[r, hook] \arrow[d, "m"] & P \times P \arrow[d] \\ P \arrow[r, hook] & \tilde{P} \end{tikzcd}$$

The last square is the important one because the connection forms live on these spaces. The middle diagram is where the characteristic forms live.

So I will begin by describing what happens to the characteristic forms. Take the old viewpoint of connections D living on the spaces in the middle diagram

$p(\tilde{D}^2)$ represents $p(\tilde{P}) \in H^{2k}(Y \times M)$ where $\tilde{D}$ is a connection on $\tilde{D}$. When $\tilde{P}$ is lifted to $P \times M$ it becomes isom. to $pr_2^*(P)$ and $\int_\gamma p(pr_2^*(P)) = 0$ as $2k > d = \dim \gamma$. Thus $\int_\gamma p(\tilde{D}^2) = du$, where in practice u is found by choosing a connection B in P and using the linear path from D_B to $\tilde{D}$ up over $P \times M$. On the other hand if $\tilde{P}$ is restricted to $y \times M$ it becomes isom. to P, this isom. being the point of P giving the map $\mathcal{G} \to P$. Then $\tilde{D}$ restricts to a connection on P and $\int_\gamma p(\tilde{D}^2)$ restricts to zero for dimensional reasons.

So we conclude that the transgression of $\int_\gamma p(\tilde{P}) \in H^{2k-d}(Y)$ is to be found by using the linear path from D_B to $\tilde{D}$ over $P \times M$, and then restricting to $\mathcal{G} \times M$. We therefore have to describe the connections D_B and $\tilde{D}$ over $\mathcal{G} \times M$.

Next we recall that $\tilde{D}$ is constructed in the following way. We must give both a horizontal + vertical connection in $\tilde{P}$ over $Y \times M$. The vertical connection is the same as a $\mathcal{G}$-map $P \longrightarrow \mathcal{A}$ and gives us a connection form $\hat{A}$ on $P \times P$ of type $(0,1)$. It is the pull-back of the canonical connection on $\mathcal{A} \times P$ over $\mathcal{A} \times M$ and one has

$$D \longleftrightarrow d_P + d_{\mathcal{P}} + \hat{A}$$

$$\Omega(\mathcal{P} \times M, pr_2 E) \rightleftharpoons \Omega(\mathcal{P} \times P, V)_{U\text{-basic}}$$

This is the $\mathcal{G}$-invariant connection on $pr_2 E$ over $\mathcal{P} \times M$ that I ~~start~~ with. Then $\tilde{D}$ is the basic connection which I denoted ?

$$D + \theta\varphi \longleftrightarrow d_P + d_{\mathcal{P}} + \hat{A} - \theta i \hat{A}$$

where θ is a connection form in $\mathcal{P}$.

Repeat: We have the $\mathcal{G}$-invariant connection

$$\underset{\text{on}}{D} \longleftrightarrow d_{\mathcal{P}} + d_P + \hat{A}$$

$$\Omega(\mathcal{P}\times M, pr_2^* E) = \Omega(\mathcal{P}\times P, V)_{\mathfrak{u}\text{-basic}}$$

We modify it to

$$\bar{D} = D + \theta\varphi \longleftrightarrow d_{\mathcal{P}} + d_P + \overbrace{\hat{A} - \theta i \hat{A}}^{\bar{A}}$$

which then descends to $\tilde{D}$ on

$$\Omega(Y\times M, \tilde{E}) = \Omega(\tilde{P}, V)_{\mathfrak{u}\text{-basic}}$$

$$= \Omega(\mathcal{P}\times P, V)_{\mathcal{G},\, \mathfrak{u}\text{-basic}}$$

To compute $\check{u}$ we use a linear path between $\bar{D} = D + \theta\varphi$ and a pull-back connection on $pr_2^* E$:

$$D_B \longleftrightarrow d_{\mathcal{P}} + d_P + pr_2^* B$$

Then

$$\check{u} = \int_0^1 dt\; p'\big((1-t)F_B + tF_{\bar{A}}\,;\, \bar{A} - B\big)$$

will satisfy

$$p(F_{\bar{A}}) - p(F_B) = d\check{u}$$

and then

$$\int_\gamma p(F_{\bar{A}}) = d\int_\gamma \check{u}$$

Finally we must restrict $\check{u}$ to $\mathcal{G}\times P$, and this requires knowing $\bar{A}$ restricted to $\mathcal{G}\times P$. At this point it is useful to go around the other way and argue that $\tilde{A}$ down on $\Omega(\tilde{P}, \mathfrak{u})$ restricts to a connection $A_0 \in \Omega^1(P, \mathfrak{u})$:

$$\begin{tikzcd} \mathcal{G}\times P \arrow[r, hook] \arrow[d, "m"] & P\times P \ \bar{A} \arrow[d] \\ P \arrow[r, hook, "A_0"] & \tilde{P} \ \tilde{A} \end{tikzcd}$$

$\downarrow \mathcal{G}$ quotient

hence I am <u>calculating $m^*(A_0)$.</u>

Digression: I want to write a letter to Iz

Let A, B be two connections on P. Then Iz uses the process of multiplying by vol_B and integrating over the fibre to go from $\Omega(P)$ to $\Omega(P)_{basic}$. This operation is the same as the composition

$$\Omega(P) \cong \Omega(P)_{hor} \otimes \Lambda(\mathfrak{g}^*) \xrightarrow{1\otimes\varepsilon} \Omega(P)_{hor} \xrightarrow{\not{4}} \Omega(P)_{basic}$$

where the last map is averaging over the compact group

The Chern-Simons form for A is

$$\int_0^1 dt\ tr\left(A e^{tF_A + (t^2-t)A^2}\right)$$

The map $1\otimes\varepsilon$ is a ring homomorphism such that

$$F_A \longmapsto F_A \qquad \text{since } F_A \text{ is horizontal}$$

$$A \longmapsto A - B \qquad \text{since } A-B \text{ is horiz. } + B \mapsto 0.$$

Thus the image of the Chern-Simons form under $1\otimes\varepsilon$ is

$$\int_0^1 dt\ tr\left((A-B) e^{tF_A + (t^2-t)(A-B)^2}\right)$$

which is already basic, so is reproduced by $\not{4}$. On the other hand the good transgression class is

$$\int_0^1 dt\ tr\left((A-B) e^{(1-t)F_B + tF_A + (t^2-t)(A-B)^2}\right)$$

November 21, 1983

I want to go over the transgression computation again. (See p. 220). We have a square of principal G-bundles

$$\begin{tikzcd} \mathcal{G}\times P \arrow[r, hook] \arrow[d, "m"] & \mathcal{A}\times P \arrow[d] \\ P \arrow[r, phantom, "\subset"] & \mathcal{Q} \end{tikzcd} \qquad \text{over} \qquad \begin{tikzcd} \mathcal{G}\times M \arrow[r, hook] \arrow[d] & \mathcal{A}\times M \arrow[d] \\ M \arrow[r, phantom, "\subset"] & \mathcal{A}/\mathcal{G}\times M \end{tikzcd}$$

On $\mathcal{A}\times P$ we have the canonical connection $\hat{A}$ of type $(0,1)$. It is then modified to $\bar{A} = \hat{A} - \theta i \hat{A}$ so that it descends to a connection form $\bar{A}_{\mathcal{Q}}$ on $\mathcal{Q}$. We have $p \in S(\mathfrak{g}^*)^G$ of degree k with $2k > \dim M$ since we want to take $p(\mathcal{Q}) \in H^{2k}(\mathcal{A}/\mathcal{G}\times M)$ and ~~integrate~~ over M to get $\int_M p(\mathcal{Q}) \in H^{2k-d}(\mathcal{A}/\mathcal{G})$. To compute the transgression we take the connection $\bar{A}_{\mathcal{Q}}$ on $\mathcal{Q}$, whence $p(\mathcal{Q})$ is represented by $p(F_{\bar{A}_{\mathcal{Q}}})$, lift to $\mathcal{A}\times P$. One point is that $p(\mathcal{Q})$ lifted to $\mathcal{A}\times M$ is zero since $\mathcal{Q} \simeq pr_2^*(P)$