474 – 547 March 20, 1982 – April 8, 1982

Feynman's Inequality + Atiyah Bott convexity	474-478
Corrigan et al paper	486
Green's fn. over $\mathbb{C}/\Gamma$	518

March 20, 1982:

Recall Feynman's inequality: One has a perturbation $H = H_0 + V$, and one looks at the free energy change. Free energy at inv. temperature β is defined by

$$e^{-\beta F} = \underbrace{tr\left(e^{-\beta H}\right)}_{Z} \qquad \text{or} \qquad F = -\frac{1}{\beta}\log Z$$

[redacted] Feynman's inequality is:

$$F \leq F_0 + \langle V \rangle_0$$

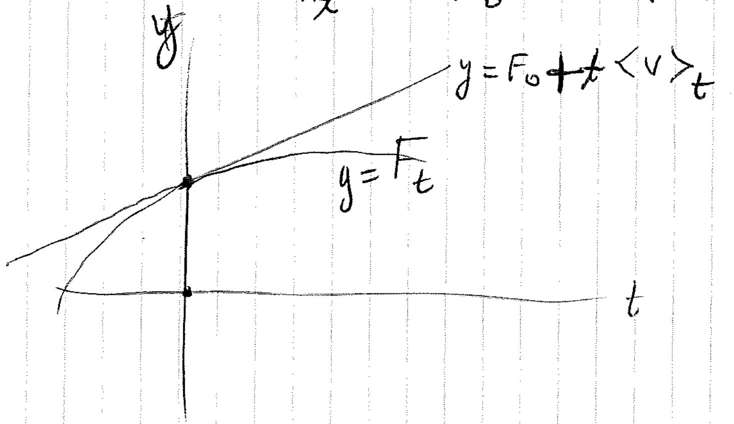
where $\langle V\rangle_0 = \dfrac{tr\left(e^{-\beta H_0}V\right)}{tr\left(e^{-\beta H_0}\right)}$ is the average value of V in the unperturbed situation. [redacted]

An equivalent formullation is that if we introduce F_t = free energy of $H_t = H_0 + tV$, then F_t is a concave function of t. This is because

$$\frac{d}{dt}F_t = -\frac{1}{\beta}\frac{1}{Z_t}\underbrace{\frac{d}{dt}tr\left(e^{-\beta H_t}\right)}_{tr\left(e^{-\beta H_t}(-\beta V)\right)} = \langle V\rangle_t$$

hence the Feynman inequality says [redacted]

$$F_t \leq F_0 + t\langle V\rangle_t$$



If we let $\beta \longrightarrow +\infty$, then $F \longrightarrow$ ground energy of H. so the Feynman inequality says the ground energy of $H_0 + tV$ is a concave function of [redacted] t.

Can I find a direct proof that on the set of self-adjoint matrices H, the function [redacted]

$$F = -\frac{1}{\beta} \log \operatorname{tr}(e^{-\beta H})$$

is concave for $\beta > 0$. This is equivalent to $\operatorname{tr}(e^{-\beta H})$ being logarithmically [redacted] convex, or more simply the function $H \longmapsto \operatorname{tr}(e^H)$ is log. convex.

There is a stronger [redacted] assertion which might be true, namely, that $\operatorname{tr}(e^H)$ is the Laplace transform of a positive measure on the space of s. a. matrices

$$\circledast \qquad \operatorname{tr}(e^H) = \int e^{(H,A)} d\mu(A). \qquad (H,A) = \operatorname{tr}(HA)$$

Why [redacted] is this stronger? [redacted]

$$\frac{\partial}{\partial B} \log(\operatorname{tr} e^H) = \int (B,A)\, e^{(H,A)} d\mu(A) \Big/ \int e^{(H,A)} d\mu(A)$$

This is ~~too~~ confusing. Take coordinates $H_1, \ldots$ on the space of s.a. matrices. Then

$$\frac{\partial}{\partial H_i} \log \int e^{\Sigma H_i A_i} d\mu(A) = \frac{\int A_i e^{\Sigma H_i A_i} d\mu}{\int e^{\Sigma H_i A_i} d\mu} = \langle A_i \rangle$$

$$\frac{\partial^2}{\partial H_i \partial H_j} \underline{\qquad\qquad\qquad} = \langle A_i A_j \rangle - \langle A_i \rangle \langle A_j \rangle$$

and we know this is positive-definite, since

$$\sum x_i x_j \left(\langle A_i A_j \rangle - \langle A_i \rangle \langle A_j \rangle \right) = \langle A_x^2 \rangle - \langle A_x \rangle^2 \geq 0.$$

Let's rewrite $\circledast$ as a Fourier transform:

$$\operatorname{tr}(e^{iH}) = \int e^{i(H,A)} \rho(A)\, dA$$

whence the inverse formula gives

$$\rho(A) = (2\pi)^{-h} \int e^{-i(H,A)} \operatorname{tr}(e^{iH})\, d^h H \qquad h = n^2.$$

The function $\mathrm{tr}(e^{iH})$ is constant on the orbits of $U(n)$, and hence following Weyl one can write it as an integral over diagonal matrices? ? NO because $e^{-i(H,A)}$ is not $U(n)$-invariant.

Clearly the distribution $\rho(A)$ is invariant under $U(n)$, so we can assume A is diagonal in trying to calculate $\rho(A)$.

Let's make a serious effort to show that $\mathrm{tr}(e^{H})$ is logarithmically convex. First check out the lowest eigenvalue, the point being that

$$\underset{\text{energy}}{\text{ground}}(H) = \lim_{\beta\to\infty} \frac{\log \mathrm{tr}(e^{-\beta H})}{-\beta}$$

should be concave as a function of H. This we should be able to see from perturbation theory.

Let's suppose H_0 has distinct eigenvalues E_n and consider a perturbation $H_0 + V = H$. Then

$$P = \frac{1}{2\pi i}\oint \frac{1}{\omega - H}\,d\omega$$

is the projection on the ground state of H, provided the contour is a small circle around E_0, and V is very small. Also

$$E\langle 0|P|0\rangle = \langle 0|HP|0\rangle = \langle 0|PH|0\rangle$$

$$= \frac{1}{2\pi i}\oint \langle 0|\frac{1}{\omega-H}\underbrace{H}_{H_0+V}|0\rangle\,d\omega$$

$$= E_0\langle 0|P|0\rangle + \frac{1}{2\pi i}\oint \underbrace{\langle 0|\frac{1}{\omega-H}V|0\rangle}_{\langle 0|\frac{1}{\omega-H_0}V + \frac{1}{\omega-H_0}V\frac{1}{\omega-H_0}V + \cdots|0\rangle}\,d\omega$$

$$\therefore\ (E-E_0)\langle 0|P|0\rangle = \underset{\omega=E_0}{\mathrm{res}}\left\{\frac{1}{\omega-E_0}\langle 0|V|0\rangle + \frac{1}{\omega-E_0}\sum_n \langle 0|V|n\rangle\frac{1}{\omega-E_n}\langle n|V|0\rangle\right\}$$

$$= \langle 0|V|0\rangle + \sum_{n\neq 0} \frac{|\langle 0|V|n\rangle|^2}{E_0 - E_n} + O(V^3).$$

Also

$$\langle 0|P|0\rangle = \frac{1}{2\pi i}\oint \langle 0| \frac{1}{\omega - H_0} + \frac{1}{\omega-H_0}V\frac{1}{\omega-H_0} + \cdots |0\rangle\, d\omega$$

$$= \operatorname*{res}_{\omega=E_0}\left\{ \frac{1}{\omega-E_0} + \frac{1}{(\omega-E_0)^2}\langle 0|V|0\rangle + \frac{1}{\omega-E_0}\langle 0|V|n\rangle \underbrace{\frac{1}{\omega-E_n}}_{\frac{1}{E_0-E_n} - \frac{\omega-E_0}{(E_0-E_n)^2}} \langle n|V|0\rangle \frac{1}{\omega-\omega_0}\right\}$$

$$= 1 - \sum_{n\neq 0}{}' \frac{|\langle 0|V|n\rangle|^2}{(E_0-E_n)^2}$$

So we get to 2nd order in V:

$$E - E_0 = \langle 0|V|0\rangle + \sum_{p\neq 0}\frac{|\langle 0|V|p\rangle|^2}{E_0 - E_p} + O(V^3)$$

When E_0 is the smallest eigenvalue, then the $E_0 - E_n$ are all < 0 for $n \neq 0$ and so one sees that E is concave in the perturbation V.

Moregenerally we have

$$E_n(V) - E_n(0) = \langle n|V|n\rangle + \sum_{p\neq n}\frac{|\langle p|V|n\rangle|^2}{E_n - E_p} + O(V^3)$$

Now put tV in for V and take derivatives at $t=0$.

$$Z_0 = \sum e^{E_n(tV)}\Big|_0 = \sum e^{E_n}$$

$$\dot Z_0 = \sum e^{E_n}\dot E_n = \sum e^{E_n}\langle n|V|n\rangle = \operatorname{tr}(e^H V)$$

$$\ddot Z_0 = \sum e^{E_n}(\dot E_n)^2 + \sum e^{E_n}\boxed{\ddot E_n} \quad 2\sum_{p\neq n}\frac{|\langle p|V|n\rangle|^2}{E_n - E_p}$$

$$(\log Z)^{\cdot\cdot}_0 = \left(\frac{\dot Z}{Z}\right)^{\cdot}_0 = \frac{\ddot Z_0}{Z_0} - \left(\frac{\dot Z_0}{Z_0}\right)^2$$

$$= \sum p_n\langle n|V|n\rangle^2 + 2\sum_n p_n \sum_{p\neq n}{}' \frac{|\langle p|V|n\rangle|^2}{E_n - E_p} - \left(\sum p_n\langle n|V|n\rangle\right)^2$$

always ≥ 0
by abelian
case $[H,V]=0$

So the [redacted] critical case is where $\langle n|V|n\rangle = 0$ in which case we want to look at

$$2\sum_{n}{}'\sum_{p\neq n} pr_n \frac{|\langle p|V|n\rangle|^2}{E_n - E_p}$$

$$= \sum_{n\neq p} |\langle p|V|n\rangle|^2 \cdot \frac{pr_n - pr_p}{E_n - E_p}$$

and now [redacted] one has $pr_n = \frac{e^{E_n}}{\sum' e^{E_n}} > pr_p$ when $E_n > E_p$. Thus $(\log Z)^{\cdot\cdot}_0 \geqslant 0$, [redacted] and so we have:

Prop: The function $H \longmapsto tr(e^H)$ on self-adjoint matrices is logarithmically convex.

Atiyah-Bott paper has a much more general statement. Namely take a function ϕ on $\mathbb{R}^n$ which is convex and symmetric (under Σ_n), and extend it to hermitian matrices, then you obtain a convex fn. on the space of hermitian matrices. The above is the case $\phi(\lambda_1, \ldots, \lambda_n) = \log\left(\sum e^{\lambda_i}\right)$.

Remark: Atiyah-Bott result can be proved by the same method as above. One needs the inequality

$$\left(\frac{\partial\phi}{\partial\lambda_n} - \frac{\partial\phi}{\partial\lambda_p}\right)/(\lambda_n - \lambda_p) \geqslant 0$$

which can be obtained by using convexity of ϕ and its symmetry:

$$\phi(\cdot, \lambda_p + t(\lambda_n - \lambda_p), \ldots, \lambda_n + t(\lambda_p - \lambda_n), \ldots)$$

has same value at $t = 0, 1$ so its derivative at $t = 0$ must be < 0. $\therefore (\lambda_n - \lambda_p)\left(\frac{\partial\phi}{\partial\lambda_p} - \frac{\partial\phi}{\partial\lambda_n}\right) \leqslant 0$

March 23, 1982:

Problem: I consider the clutching construction:

$$\mathcal{G} = \Gamma(\alpha, \underline{\mathrm{Aut}}(E)) \longrightarrow H^1(X, \underline{\mathrm{Aut}}(E)) = \mathrm{Vect}(X)$$

which gives me a family of vector bundles on X parameterized by $\mathcal{G}$, and hence a cohomology-det-line bundle $\mathcal{L}$ over $\mathcal{G}$. The question is whether the natural $\mathcal{G}$ action on itself can be lifted to a projective representation of $\mathcal{G}$ on $\mathcal{L}$. This is more or less equivalent to being able to find a projective repn. V of $\mathcal{G}$ in which $\mathcal{L}$ can be embedded.

So consider the rank 1 case, whence we have the map $\quad \Gamma(\alpha, \mathcal{O}^*) \longrightarrow H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$.

Now over $X \times \mathrm{Pic}(X)$ is the canonical Poincaré line bundle, which gives by determinant-of-cohomology a line bundle over $\mathrm{Pic}(X)$. Thus we find ourselves lead to the case of the canonical determinant line bundle over $\mathrm{Pic}(X)$.

(Interesting Point: The above map is surjective and so gives us an interesting description of $\mathrm{Pic}(X)$ as a quotient of the loop group. Hence the Jacobian $\mathrm{Pic}^{(0)}(X)$ will appear as a quotient of the Lie algebra $\Gamma(\alpha, \mathcal{O})$.)

Problem: Describe the cohomology determinant line bundle for the canonical family of line bundles over X parameterized by $\mathrm{Pic}(X)$. This means that over $\mathrm{Pic}(X)$ is a canonical line bundle $\mathcal{L}$ to be understand.

Suppose X is an elliptic curve. Then

$$X = \mathrm{Pic}^{(1)}(X)$$
$$P \longmapsto \mathcal{O}(P)$$

Prop: [redacted] Under the isom.

$$\begin{tikzcd} X \arrow[r, "\sim"] & \mathrm{Pic}^{(n)}(X) \\ P \arrow[r, mapsto] & \mathcal{O}(P+(n-1)\infty) \end{tikzcd}$$

the **cohomology** determinant line bundle [redacted] $\mathcal{L}$ corresponds to $\mathcal{O}((n-1)\infty)$.

Next we want to consider the translation action of $\mathrm{Pic}(X)$ on itself. Fix $L_0 \in \mathrm{Pic}(X)$, then we have the map

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r, "\mu_{L_0}"] & \mathrm{Pic}(X) \\ L \arrow[r, mapsto] & L_0 \otimes L \end{tikzcd}$$

and we can try to relate $\mathcal{L}$ with $\mu_{L_0}^*(\mathcal{L})$ i.e.

$$\lambda(R\Gamma(L_0\otimes L)) \qquad \lambda(R\Gamma(L)).$$

~~Now I can produce an [redacted] isomorphism between these by trivializing $\lambda(L_0;\mathcal{O}) = \lambda(R\Gamma(L_0)) \otimes \lambda(R\Gamma(\mathcal{O}))^*$.~~

~~Now I think it's clear how the translation action of $\mathrm{Pic}(X)$ **on** itself lifts to a projective action on $\mathcal{L}$.~~

~~[redacted] Define the covering group $\widetilde{\mathrm{Pic}}(X)$ as follows. An element is a line bundle L_0 together with an element of $\lambda(L_0;\mathcal{O})$. Now to [redacted] L_0 belongs a canonical isomorphism~~

~~$$\lambda(R\Gamma(L_0\otimes L)) \cong \lambda(R\Gamma(L)) \otimes \lambda(L_0;\mathcal{O})$$~~

In effect suppose one has an embedding $\mathcal{O} \subset L_0$ of codim 1. Then

$$0 \to L \to L_0\otimes L \to (L_0/\mathcal{O})\otimes L \to 0$$

hence

$$\lambda(R\Gamma(L_0\otimes L)) = \lambda(R\Gamma(L)) \otimes [(L_0/\mathcal{O})\otimes L]$$

Thus we see that $\mu_{L_0}^*(\mathcal{L})$ and [redacted] $\mathcal{L}$ are not going to be isomorphic.

and the canonical line bundle $\mathcal{L}$ is trivial over $\mathrm{Pic}^{(1)}(X)$ because we have $H^0(X, \mathcal{O}(P)) = \mathbb{C}$, $H^1(X, \mathcal{O}(P)) = 0$ independent of P.

Next pick a base point ∞ of X. Then

$$\circledast \qquad \begin{array}{ccc} X & \xrightarrow{\sim} & \mathrm{Pic}^{(0)}(X) \\ P & \longmapsto & \mathcal{O}(P-\infty) \end{array}$$

so that ∞ corresponds to the trivial line bundle. We know that for $L \in \mathrm{Pic}^{(0)}(X)$ not the trivial bundle, one has $H^0(X, L) = H^1(X, L) = 0$, and that $\mathcal{L}^{-1}$ has a canonical section vanishing at the trivial bundle. So we should know that under the isomorphism $\circledast$ we have

$$\mathcal{L}^{-1} \cong \mathcal{O}(\infty) \qquad \text{or} \qquad \mathcal{L} \cong \mathcal{O}(-\infty).$$

On the other hand

$$0 \longrightarrow \mathcal{O}(P-\infty) \longrightarrow \mathcal{O}(P) \longrightarrow \mathcal{O}(P) \otimes k(\infty) \longrightarrow 0$$

gives

$$\underbrace{\lambda(R\Gamma\, \mathcal{O}(P))}_{\substack{\text{trivial bdle} \\ \text{over } X}} = \underbrace{\lambda(R\Gamma\, \mathcal{O}(P-\infty))}_{\text{bdl. } \mathcal{O}(-\infty)} \otimes [\mathcal{O}(P) \otimes k(\infty)]$$

so we conclude that the line bundle $P \longmapsto \mathcal{O}(P) \otimes k(\infty)$, $P \in X$ is isomorphic to $\mathcal{O}(\infty)$. One can see this directly since $\mathcal{O}(P)$ has a canonical section vanishing at P.

Next consider deg 2.

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \mathrm{Pic}^{(2)}(X) \\ P & \longmapsto & \mathcal{O}(P+\infty) \end{array}$$

Use
$$0 \to \mathcal{O}(P) \longrightarrow \mathcal{O}(P+\infty) \longrightarrow \mathcal{O}(P) \otimes k(\infty) \longrightarrow 0$$

$$\underbrace{\lambda(R\Gamma(\mathcal{O}(P)))}_{\mathcal{O}} \otimes \underbrace{[\mathcal{O}(P) \otimes k(\infty)]}_{\mathcal{O}(\infty)} = \underbrace{\lambda(R\Gamma(\mathcal{O}(P+\infty)))}_{\mathcal{L}}$$

Now it's clear what happens in general:

Conclusion: I am considering the canonical family of line bundles over X parameterized by $Pic(X)$ and the associated cohomology-determinant line bundle $\mathcal{L}$ over $Pic(X)$. I have found that for the map $\mu_{L_0} : Pic(X) \longrightarrow Pic(X)$, $L \mapsto L_0 \otimes L$ one has $\mu_{L_0}^* \mathcal{L} \not\cong \mathcal{L}$ in general, so we are going to have to lift back to some group over $Pic(X)$ before we can expect an action on $\mathcal{L}$.

Actually we should first understand the line bundles over $Pic(X)$ of the form $L \longmapsto L \otimes k(Q)$ where $P \in X$. Over $Pic^{(1)}(X) = X$ we get the line bundle $P \longmapsto \mathcal{O}(P) \otimes k(Q)$, which has a section vanishing at Q obtained from the canon. section of $\mathcal{O}(P)$. Thus we get $\mathcal{O}(Q)$ over $Pic^{(1)}(X)$. Over $Pic^{(n)}(X) \cong X$ we get

$$P \longmapsto \mathcal{O}(P) \otimes \underbrace{\mathcal{O}((n-1)(\infty)) \otimes k(Q)}_{\cong\ k(Q)}$$

the same line bundle $\mathcal{O}(Q)$, however the isomorphism is not canonical

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Improvement in the Prop. on p. 480: We get an isomorphism $X \xrightarrow{\sim} Pic^{(n)}(X)$; $P \longmapsto \mathcal{O}(P) \otimes L_0$ for any L_0 of degree $n-1$. Then $\mathcal{L}$ pulls back to L_0 under this map.

In general for a Riemann surface we have line bundles over $Pic(X)$ given by

$$L \longmapsto \lambda(R\Gamma(L))$$

$$L \longmapsto L \otimes k(Q) \qquad Q \text{ a point of } X$$

in terms of which we can express the line bundle

$$L \longmapsto \lambda(R\Gamma(L_0 \otimes L))$$

for any line bundle L_0, more generally any coherent sheaf.

Questions: 1). What is known about the line bundle over $\mathrm{Pic}(X) \times \mathrm{Pic}(X)$ given by $L_1, L_2 \longmapsto \lambda(R\Gamma(L_1 \otimes L_2))$? Maybe it gives the self-duality of the Jacobian.

2) For any vector bundle E we associate the line bundle $L \longmapsto \lambda(R\Gamma(E \otimes L))$ on $\mathrm{Pic}(X)$. Is there a canonical isomorphism between this line bundle and the one obtained from $\lambda(E)$? Perhaps not, but a better question is whether this line bundle $\lambda(R\Gamma(E\otimes ?))$ depends up to canonical isomorphism on $\lambda(E)$ and the rank of E. A way to formulate this is maybe to consider flags in E. So if E is of rank 2 and we have a sequence

$$0 \longrightarrow L_1 \longrightarrow E \longrightarrow L_2 \longrightarrow 0$$

then we get an isomorphism

$$\lambda(R\Gamma(E\otimes L)) = \lambda(R\Gamma(L_1 \otimes L)) \otimes \lambda(R\Gamma(L_2 \otimes L))$$

Is this canonically isomorphic to $\lambda(R\Gamma(L_1 \otimes L_2 \otimes L)) \otimes \lambda(R\Gamma(L))$?

Perhaps the way to proceed is as follows: To each E one has associated a line bundle $\Phi(E)$ over $\mathrm{Pic}(X)$ compatible with isomorphism and exact sequences. So Φ must factor thru the universal Picard category associated to vector bundles. So you get a nice map from the Picard category with groups $K_1(X), K_0(X)$ to the Picard cat of line bundles on $\mathrm{Pic}(X)$, hence ~~an interesting~~ a map $K_1(X) \to \mathbb{C}^*$. This map probably isn't interesting

March 24, 1982

Fix an elliptic curve $M = \mathbb{C}/\Gamma$, and consider the family of line bundles of degree 0 over M described by the operators $\bar\partial_u - z : \mathcal{O} \longrightarrow \mathcal{O}$ with z running over $\mathbb{C}$. More generally over a Riemann surface I can consider the family of line bundles described by

$$\bar\partial - \bar\omega : \mathcal{O} \longrightarrow \Omega^{0,1}$$

where $\omega \in H^0(X, \Omega^1)$, so that $\bar\omega$ runs over $H^1(X, \mathcal{O})$. Associated to this family is a line bundle $\mathcal{L}$ over the parameter space. In the elliptic curve case we have a canonical map $\mathcal{L} \longrightarrow \mathcal{O}$ over $H^1(X,\mathcal{O}) = \mathbb{C}$, and I think we know that this map enables us to identify $\mathcal{L}^{-1}$ with the bundle $\mathcal{O}$(divisor with points in dual lattice).

Now consider the translation action of $H^1(X, \mathcal{O})$ on itself. The bundle $\mathcal{L}^{-1}$ is trivial, and a global non-vanishing section is furnished by the Weierstrass σ-fn which is unique up to a quadratic function of z.

$$\sigma(z) = z \prod_\mu{}' \left(1 - \frac{z}{\mu}\right) e^{\frac{z}{\mu} + \frac{z^2}{2\mu^2}}$$

The point is that if I wish to lift the translation action of $H^1(X, \mathcal{O})$ on itself to $\mathcal{L}$, this is completely equivalent to trivializing $\mathcal{L}$. Hence there is no canonical $H^1(X, \mathcal{O})$ action on $\mathcal{L}$, but perhaps there is an action by an extension group.

I am ultimately interested in the case where α is a curve on M and the family I get by clutching:

$$\Gamma(\alpha, \mathcal{O}^*) \longrightarrow H^1(M, \mathcal{O}^*).$$

If I restrict to degree 0, this map factors through

$H^1(M,\mathcal{O}) \xrightarrow{\exp} H^1(M,\mathcal{O}^*)^{(0)} = \mathrm{Pic}^{(0)}(M)$. Hence an action of $H^1(M,\mathcal{O})$ on $\mathcal{L}$ would produce one over $\Gamma(\alpha,\mathcal{O}^*)^{(0)}$.

But we know what to expect in the case where α is a small circle, hence we should check the central extensions.

$$\Gamma(\alpha,\mathcal{O}) = \text{holom. functions on } S^1$$

and the central extension is given by

$$f,g \longmapsto \frac{1}{2\pi i}\oint f\,dg.$$

This is non-degenerate off scalars; hence doesn't descend to a finite-diml. quotient.

March 25, 1982

Corrigan, Goddard, Osborn, + Templeton: Zeta fn. regularization and multi-instanton determinants. Nucl. Phys B159 (469-496)

θ pos. s.a. operator with eigenvalues $\lambda_1, \lambda_2, \dots$

$$\zeta_\theta(s) = \sum \lambda_n^{-s}$$

so that in finite dim.

$$\zeta_\theta(0) = \dim V$$

$$\zeta'_\theta(0) = -\ln \det \theta$$

Hence $$\zeta_{\theta/\mu}(s) = \mu^s \zeta_\theta(s) \qquad \zeta'_{\theta/\mu}(s) = \ln\mu \, \zeta_\theta(s) + \mu^s \zeta'_\theta(s)$$

$$\zeta'_{\theta/\mu}(0) = \ln\mu \, \zeta_\theta(0) + \zeta'_\theta(0)$$

$$\Rightarrow \quad \det(\mu^{-1}\theta) = \mu^{-\zeta_\theta(0)} \det \theta$$

hence one should think of $\zeta_\theta(0)$ as dim V.

Look at heat kernel $g(t) = e^{-\theta t}$ $\qquad \theta = -D^2$

$$g(x,y;t) \sim \frac{1}{16\pi^2 t^2} \exp\left\{-\frac{1}{4t}(x-y)^2\right\} \sum_{n=0}^{\infty} a_n(x,y)\, t^n \qquad t \downarrow 0$$

where the a_n can be evaluated iteratively by

$$(x-y)_\mu D_\mu a_0(x,y) = 0 \qquad a_0(x,x) = 1$$

$$n a_n(x,y) + (x-y)_\mu D_\mu a_n(x,y) = D^2 a_{n-1}(x,y) \qquad n \geq 1.$$

Ignoring infrared problems

$$\operatorname{Res}_{s=2} \zeta(s) = \frac{1}{16\pi^2} \int \operatorname{tr} a_0(x,x)\, d^4x$$

$$\operatorname{Res}_{s=1} \zeta(s) = \frac{1}{16\pi^2} \int \operatorname{tr} a_1(x,x)\, d^4x$$

$$\zeta(0) = \frac{1}{16\pi^2} \int \operatorname{tr} a_2(x,x)\, d^4x$$

Now
$$a_0(x,y) = P\exp\left\{-\int_y^x A_\mu \, dx^\mu\right\}$$
taken along the straight line from x to y.

~~Rep~~ Repeated diffn. yields
$$a_1(x,x) = \left[D^2 a_0(x,y)\right]_{x=y} = 0$$
$$a_2(x,x) = \left[\frac{1}{6} D^2 D^2 a_0(x,y)\right]_{x=y} = \frac{1}{12} F_{\mu\nu} F_{\mu\nu}$$
Thus the residue of $\zeta(s)$ at $s=2$ is infrared divergent whilst at $s=1$ it vanishes, and
$$\zeta(0) = \frac{1}{12.16\pi^2}\int d^4x \ \mathrm{tr}\,(F_{\mu\nu}F_{\mu\nu})$$
$$\text{⊛} \qquad = -\frac{1}{12}k$$
for a self-dual solution, where k = top. quantum no.

Now we want $\zeta'(0)$. Its easier to compute $\delta\zeta'(0)$, because the residue at $s=2$ disappears, hence $\delta\zeta(s)$ is regular for $Re(s) \geq 0$. Also if we stick to self-dual solutions $\delta\zeta(0) = 0$ by ⊛, and hence
$$\delta\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt \, t^s \ \mathrm{Tr}\left(e^{tD^2}\delta D^2\right)$$
$$\delta\zeta'(0) = \int_0^\infty dt \, t^s \, \mathrm{Tr}\left(e^{tD^2}\delta D^2\right)\Big|_{s=0}$$
$$= \left[s\int_0^\infty dt \, t^{s-1} \, \mathrm{Tr}\left(e^{tD^2}\frac{1}{-D^2}\delta D^2\right)\right]_{s=0}$$
Now to obtain a local expression for this.
$$\frac{1}{-D^2} = G \quad \text{is the Green's fn.}$$
$$D^2 G(x,y) = -\delta(x-y)$$
and it has the form
$$G(x,y) = \frac{1}{4\pi^2}\left\{\frac{\Phi(x,y)}{(x-y)^2} + \underbrace{R(x,y)}_{\text{non-singular along } x=y.}\right\}$$

Claim only the regular part $R(x,y)$ contributes to $\delta\zeta'(0)$. Admit this.

$$\delta D^2 = D_\mu \delta A_\mu + \delta A_\mu D_\mu$$

Define

$$\overrightarrow{D_\mu} R(x,y) = \left\{ \frac{\partial}{\partial x^\mu} + A_\mu(x) \right] R(x,y)$$

$$R(x,y)\overleftarrow{D_\mu} = -\frac{\partial}{\partial y^\mu} R(x,y) + R(x,y) A_\mu(y)$$

Then one gets

$$\delta \zeta'(0) = \int d^4x \; tr\left[\delta A_\mu(x) \, J^\mu(x) \right]$$

where

$$J_\mu(x) = \frac{1}{4\pi^2}\left[\overrightarrow{D_\mu} R(x,y) + R(y,x)\overleftarrow{D_\mu} \right]_{x=y}.$$

The $\mathbb{C}P^{n-1}$ model in flat $\mathbb{R}^2$ describes fields

$$z : \mathbb{R}^2 \longrightarrow \text{unit sphere in } \mathbb{C}^n$$

under gauge transformations of multiplying by a map $\mathbb{R}^2 \longrightarrow U(1)$. The action is

$$S = \| D z \|^2 = \int \overline{D_\mu z} \cdot D_\mu z \;\; d^2x$$

where

$$D_\mu z = \partial_\mu z - z(z \mid \partial_\mu z).$$

Given a field z approaching "a" classical vacuum as $|x| \longrightarrow \infty$

$$z(x) = h(x)\, v \qquad |h| = 1, \quad v \text{ constant}$$

it has attached a winding number

$$Q = \frac{i}{2\pi} \int_{|x|=\infty} dx_\mu \cdot h^{-1}(x)\, \partial_\mu h(x)$$

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Monopoles and Geodesics

Static Yang-Mills - Higgs monopoles are described by the Bogomolny equations:

Given a principal $SU(2)$ bundle with connection ∇ over $\mathbb{R}^3$ and a section Φ of the adjoint bundle, (called) the Higgs field. The Bogomolny equation is

$$\nabla\Phi = *F$$

Static YMH-monopoles can be described as the self-duality equations in Euclidean 4 space which are in addition time translation invariant.

Take a Riemannian 3 manifold geodesically convex, then the space of oriented geodesics has an almost complex structure, which is integrable when the traceless Ricci tensor vanishes. Real structure from reversing geodesics. Each point give a $\mathbb{P}^1$ in this surface, and any two $\mathbb{P}^1$'s intersect in 2 points.

Or. Geodesics in $\mathbb{R}^3$ are described by a point in $u \in S^2$ (the direction) and a vector $v \in u^\perp$, namely the point closest to zero. Hence oriented geodesics are the same as points in the tangent bundle to S^2.

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Consider $M = \mathbb{C}/\Gamma$ and the family of holomorphic structures on the trivial bundle; these are described by their $\bar\partial$-operators which are of the form

$$\partial \xrightarrow{\bar\partial - \omega} \Omega^{0,1} \qquad \omega \in \Gamma(\Omega^{0,1}).$$

Use the basis $d\bar u$ for $\Omega^{0,1}$ whence we are looking at the operator $\frac{\partial}{\partial \bar u} - f$ where $f \in \Gamma(\mathcal{O})$. Now I propose to compute the Singer torsion of

$$\Delta = \left(\frac{\partial}{\partial \bar u} - f\right)^* \left(\frac{\partial}{\partial \bar u} - f\right)$$

by calculating its variation with respect to f.

First review what happens when f is a constant, call it z. Then the eigenvalues of Δ are $|\mu - z|^2$ where $\mu \in \{\mu \in \mathbb{C} \mid \mu\bar\gamma - \bar\mu\gamma \in 2\pi i\mathbb{Z}\}$ is the dual lattice. So

$$\zeta_\Delta(s) = \sum \frac{1}{|\mu - z|^{2s}}.$$

~~Note the index is [illegible]~~ One analytically continues via the standard Kronecker method:

$$\pi^{-s}\Gamma(s)\,\zeta_\Delta(s) = \sum \frac{1}{|\mu - z|^{2s}} \int_0^\infty e^{-\pi t}\, t^s \frac{dt}{t}$$

$$= \int_0^\infty \underbrace{\left(\sum_\mu e^{-\pi|\mu - z|^2 t}\right)}_{\substack{\text{exp. decay at } t \to \infty \text{ assuming } z \notin \mu\text{-lattice} \\ \text{like } \frac{1}{t}(\text{const} + \text{exp decay at } t \searrow 0)}} t^s \frac{dt}{t}$$

so there is a simple pole at $s = 1$ and no other singularities. Thus $\zeta_\Delta(0) = 0$.

The residue at $s = 1$ is a volume of some sort, and is independent of z. Hence if we differentiate w.r.t. we should get something entire. Formalism:

$$\delta \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{tr}\left(e^{-t\Delta}\, \delta\Delta\right)(-t)\, t^s \frac{dt}{t}$$

$$= \frac{s}{\Gamma(s)} \int_0^\infty \mathrm{tr}\left(e^{-t\Delta} \frac{1}{(-\Delta)} \delta\Delta\right) t^{s-1}\, dt \qquad ?$$

Before I can make any sense out of this in general I have to really understand the asymptotic expansion of the heat kernel as $t \downarrow 0$. So I shouldn't be working with the eigenvalues but rather the actual kernel.

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Heat equation methods: Constructing the asymptotic form of the heat kernel at $t \downarrow 0$.

Example: Take the operator $-\Delta + q$. We want the kernel of $e^{t(\Delta - q)}$ knowing the heat kernel for $-\Delta$:

$$\langle x | e^{t\Delta} | y \rangle = \left(\frac{1}{\sqrt{4\pi t}}\right)^d e^{-\frac{(x-y)^2}{4t}} \qquad d = \text{dim}.$$

Then put

$$\langle x | e^{t(\Delta - q)} | y \rangle = \underbrace{\left(\frac{1}{\sqrt{4\pi t}}\right)^d e^{-\frac{(x-y)^2}{4t}}}_{\text{Call this } K_0(x,y,t)} A(x,y;t)$$

To simplify put $y = 0$ and forget it. Then

$$\partial_x K_0 = K_0 \cdot \left(-\frac{x}{2t}\right)$$

$$\delta(x)\delta(t) = (\partial_t - \partial_x^2 + q)(K_0 A)$$

$$= K_0\left((\partial_t - \partial_x^2 + q)A\right) - 2\partial_x K_0 \cdot \partial_x A$$

$$+ \cancel{(\partial_t - \partial_x^2)K_0} \cdot A$$

so A has to satisfy the equation

$$(\partial_t - \partial_x^2 + q)A + \frac{1}{t} x \cdot \partial_x A = 0$$

or

$$\left(\partial_t + \frac{1}{t} x \cdot \partial_x\right) A = (\partial_x^2 - q)A.$$

This can now be solved formally by

$$A(x,t) = \sum_{n \geq 0} a_n(x) t^n$$

with the condition that $A(0,0) = 1.$?

$$n a_n + x \cdot \partial_x a_n = (\partial_x^2 - q) a_{n-1}$$

This is still confused. Let's work on the line. We have $a_0(x,y) = 1$

$$a_1(x,y) + (x-y)\frac{\partial a_1}{\partial x}(x,y) = -q(x)$$

$$\frac{\partial}{\partial x}\Big[(x-y)\,a_1(x,y)\Big] = -q(x)$$

$$(x-y)\,a_1(x,y) = -\int_y^x q$$

since it must vanish when $x=y$

$$a_1(x,y) = -\frac{1}{(x-y)}\int_y^x q$$

which leads to $\boxed{a_1(x,x) = -q(x)}$. Next equation

$$2\,a_2(x,y) + (x-y)\frac{\partial}{\partial x}a_2(x,y) = \left(\partial_x^2 - q(x)\right)\Big[-\frac{1}{(x-y)}\int_y^x q\Big]$$

is too complicated. What is clear is that the nth equation can be put in the form (go back to $y=0$)

$$\left(x^n a_n\right)' = x^{n-1}\left(n a_n + x a_n'\right) = x^{n-1}\left(\partial^2 - q\right)a_{n-1}$$

and hence the integration constant is fixed since $x^n a_n$ has to vanish at $x=0$.

Do 2nd equation via power series:

$$2\,a_2 + x\,a_2' = \left(\partial_x^2 - q\right)a_1$$

suppose $\quad q = q_0 + \frac{q_1}{1!}x + \frac{q_2}{2!}x^2 + \cdots$, then

$$a_1 = -\frac{1}{x}\int_0^x q = -\Big[q_0 + \frac{1}{2!}q_1 x + \frac{1}{3!}q_2 x^2 + \cdots\Big]$$

so

$$2a_2(0) = q(0)^2 - \frac{1}{3}q''(0)$$

or

$$\boxed{a_2(x,x) = \frac{1}{2}q(x)^2 - \frac{1}{6}q''(x)}$$

Now it is clear to me that I can grind out the coefficients $a_n(x,y)$. What is not clear is why this asymptotic expansion for $A(x,y;t)$ is valid.

Let's take the symbol ~~[illegible]~~ and pseudo-differential operator viewpoint. Let $P(x,D)$ be a pseudo-diffl operator, use Hörmander notation $D = \frac{1}{i}\partial$. Then it has a symbol

$$p(x,\xi) = \sum_{r\leq m} p_r(x,\xi)$$

where p_r is homogeneous of degree r in ξ. Usually ~~[illegible]~~ $m-r \in \mathbb{N}$.

Now take a P whose leading symbol $p_m(x,\xi)$ is >0. Then on the symbol level it is possible to define P^{-s} whose order ~~[illegible]~~ is $-ms$ and leading symbol is $p_m(x,\xi)^{-s}$.

Take for example $P = -\partial^2 + q(x) = D^2 + q(x)$.

Then on the symbol level

$$P^{-s} = \left[\sum_0^{\infty}{}' a_n(x,s)\,\partial^{-n}\right](-\partial^2)^{-s}$$

where ~~[illegible]~~ the coefficients a_n are known for $-s \in \mathbb{N}$ to be given by polynomials which one extends to all complex numbers.

$$PP^{-s} = (-\partial^2 + q)\sum a_n(x,s)\,\partial^{-n}(-\partial^2)^{-s}$$

$$= \sum_n (+a_n'' \mp q a_n)\,\partial^{-n-2}(-\partial^2)^{-s+1} = \sum{}' a_n(x,s-1)\,\partial^{-n}(-\partial^2)^{-s+1}$$

$$(+2a_n')\,\partial^{-n-1}(-\partial^2)^{-s+1}$$

$$a_n\,\partial^{-n}(-\partial^2)^{-s+1}$$

Gives recursion relation

$$a_n(s-1) = a_n(s) + 2a_{n-1}'(s) + a_{n-2}''(s) - q\,a_{n-2}(s)$$

So start with $a_0 = 1$. $a_1(s-1) = a_1(s) = 0$.

$$a_2(s-1) = a_2(s) + 0 - q$$

$$a_2(s) = sq$$

$$a_3(s-1) = a_3(s) + 2sq'$$

$$a_3(s) = -s(s+1)\,q'$$

So we get

$$(*) \qquad (-\partial^2+q)^{-s} = \{1 \quad + sq\partial^{-2} - s(s+1)q'\partial^{-3} \cdots\}(\partial^2)^{-s}$$

Is there any connection between this symbol and things we are interested in such as the trace, i.e. the ζ fn., evaluated at interesting values of s?

Go back to the heat equation. What corresponds to ⊛ above is writing

$$e^{t(\partial^2-q)} = A(t)\, e^{t\partial^2}.$$

Thus

$$A(t) = e^{t(\partial^2-q)}e^{-t\partial^2}$$

$$\partial_t A(t) = e^{t(\partial^2-q)}(-q)\,e^{-t\partial^2}$$

$$= -e^{t(\partial^2-q)}\, q\, e^{-t(\partial^2-q)}\, A(t)$$

$$= -\sum_{n=0}^{\infty} \frac{t^n}{n!}\underbrace{[\partial^2-q\,[\partial^2-q\,[\cdots q]\cdots]}_{n \text{ times}}\, A(t)$$

$$= -\sum_{n=0}^{\infty} \frac{t^n}{n!}[\partial^2[\partial^2[\cdots\cdots q]\cdots]\, A(t)$$

where using the exponential series is formal. Hence A(t) has a formal series expansion in t whose coefficients are differential operators. To order t^2:

$$\partial_t A = -(q + t[\partial^2,q] + \cdot)\,A \qquad A_0 = 1$$

$$A_1 = -qA_0 = -q$$

$$2A_2 = -qA_1 - [\partial^2,q]A_0$$

$$= +q^2 - [\partial^2,q] \qquad A_2 = \frac{q^2-[\partial^2,q]}{2}$$

$$\boxed{A(t) = 1 - qt + \frac{q^2-[\partial^2,q]}{2}t^2 + \cdots}$$

But what does it means to express $e^{t(\partial^2-q)}$ in terms of $t^k e^{t\partial^2}$ as far as the resolvent is concerned?

So if $$e^{-tA} = \frac{1}{2\pi i}\oint \frac{1}{\lambda - A} e^{-\lambda t} d\lambda$$
is differentiated wrt A
$$(-t)^k e^{-tA} = \frac{1}{2\pi i}\oint \frac{(k-1)!}{(\lambda-A)^k} e^{-\lambda t} d\lambda$$
Thus expanding $e^{t(\partial^2 - q)}$ in terms of $t^k e^{t\partial^2}$ corresponds to expanding $\frac{1}{\lambda+\partial^2-q}$ in powers of $\frac{1}{\lambda+\partial^2}$

The idea is as follows. Let's work to first order in q. Then for $A = -\partial^2 + q$, $A_0 = -\partial^2$ we have
$$\frac{1}{\lambda - A} = \frac{1}{\lambda - A_0} + \frac{1}{\lambda - A_0} q \frac{1}{\lambda - A_0} + O(q^2)$$
Now $$\frac{1}{\lambda - A_0} q - q\frac{1}{\lambda - A_0} = \frac{1}{\lambda - A_0}[q, \lambda - A_0]\frac{1}{\lambda - A_0}$$
Better:
$$\frac{1}{A} B = B\frac{1}{A} + \frac{1}{A}[B,A]\frac{1}{A}$$
$$= B\frac{1}{A} + [B,A]\frac{1}{A^2} + \frac{1}{A}[[B,A]A]\frac{1}{A^2}$$
$$= B\frac{1}{A} + [B,A]\frac{1}{A^2} + [[BA]A]\frac{1}{A^3} + \cdots$$
So we get to first order in q:
$$\frac{1}{\lambda+\partial^2-q} = \frac{1}{\lambda+\partial^2} + q\frac{1}{(\lambda+\partial^2)^2} + [\partial^2, q]\frac{1}{(\lambda+\partial^2)^3} + [\partial^2[\partial^2 q]]\frac{1}{(\lambda+\partial^2)^4}$$
$$+ \cdots\cdots$$
Now the idea is that we want to get at the trace of powers $(-\partial^2+q)^k$ of a differential operator defined possibly by
$$tr(A^k) = \frac{1}{2\pi i}\oint tr\left(\frac{1}{\lambda - A}\right)\lambda^k dk$$

Terms [redacted] decaying as $|\lambda| \to \infty$ will not contribute to the contour integral. This whole business raises some questions: Things like the ζ function and heat kernel make sense even when the resolvent $\frac{1}{\lambda - A}$ doesn't have a trace. What goes wrong with inverting

$$\mathrm{tr}\left(e^{-tA}\right) = \frac{1}{2\pi i} \int \mathrm{tr}\left(\frac{1}{\lambda - A}\right) e^{-\lambda t} \, d\lambda$$

to define the trace of the resolvent:

$$\int_0^\infty \mathrm{tr}\left(e^{-tA}\right) e^{\lambda t} \, dt = -\mathrm{tr}\left(\frac{1}{\lambda - A}\right) . \qquad ?$$

The answer has to do with the behavior of $\mathrm{tr}(e^{-tA})$ as $t \downarrow 0$.

$$\frac{1}{A - B} = \sum_{n=1}^{\infty} p_n(A, B) \frac{1}{A^n}$$

$$1 = \sum_1^\infty (A - B) p_n \frac{1}{A^n} = \sum_1^\infty \left(p_n A + [A, p_n]\right) \frac{1}{A^n} - B p_n \frac{1}{A^n}$$

Recursion fmla: $\boxed{p_{n+1} = B p_n - [A, p_n] \qquad n \geq 1}$

Note that adding a scalar to A doesn't change the p_n.

$$\frac{1}{\lambda + \partial^2 - B} = \sum_1^\infty p_n(\partial^2, B) \frac{1}{(\lambda + \partial^2)^n}$$

$$\frac{1}{2\pi i} \oint \frac{1}{(\lambda + \partial^2)^n} e^{-\lambda t} \, d\lambda = e^{t\partial^2} \frac{(-t)^{n+1}}{(n+1)!}$$

$$\therefore \quad e^{t(\partial^2 - B)} = \sum_1^\infty{}' \, p_n(\partial^2, B) \frac{(-t)^{n+1}}{(n+1)!} e^{t\partial^2}$$

Unfortunately the $p_n(\partial^2, B)$ have to be applied to $\langle x | e^{t\partial^2} | y \rangle$ $= \left(\frac{1}{\sqrt{4\pi t}}\right)^d e^{-\frac{(x-y)^2}{4t}}$ and this brings down factors $\frac{1}{t}$

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Let's return to the problem of computing the variation of the ζ fn. determinant of D^*D where $D = \bar\partial + \omega$ is the $\bar\partial$ operator for a [redacted] holom. structure on the trivial line bundle over $M = \mathbb{C}/\Gamma$. [redacted] Use the coordinate z for $\mathbb{C}$, then we have the basis $d\bar z$ for $\Omega^{0,1}$ and so our operator is

$$D = \frac{\partial}{\partial \bar z} + \alpha \ : \ C^\infty(M) \longrightarrow C^\infty(M)$$

where $\alpha \in C^\infty(M)$. Now the adjoint D^* will depend on the volume on M and the metrics on $\mathcal{O}$ and $\Omega^{0,1}$. In this case ~~have volumes~~ we use constant volumes + metrics.

[Digression for general Riem. surf. M. The $\bar\partial$ operator on E goes

$$\bar\partial : E \longrightarrow E \otimes \Omega^{0,1}$$

and to define $\bar\partial^*$ we need metrics on the two bundles + volume element on M. ~~Now the Riem. metric on M gives a metric~~ Because M is a Riemann surface, the Riemann metric on M is equivalent to the volume element. Hence once the metrics on E and M are chosen, [redacted] the adjoint $\bar\partial^*$ is determined. The symbol of $\bar\partial^*\bar\partial$ depends only on the symbol of $\bar\partial$ which is $e \longmapsto e \otimes \xi''$, so it's pretty clear that the symbol of $\bar\partial^*\bar\partial$ is multiplication by $|\xi|^2$ up to a constant. Hence the symbol of $\bar\partial^*\bar\partial$ is a scalar operator, which may have some virtues when we try to compute the heat kernel.]

[redacted] $\underline{M = \mathbb{C}/\Gamma}$ $\quad D = \frac{\partial}{\partial \bar z} + \alpha \quad D^* = -\frac{\partial}{\partial z} + \bar\alpha$

Notice that when E has a metric we have a natural inner product on $\Gamma(E \otimes \Omega^{0,1})$. For if $\alpha, \beta \in \Gamma(E \otimes \Omega^{0,1})$ then their pointwise inner product using the inner product of E is a section of $\Gamma(\Omega^{1,1})$ which can be integrated. For example, if E = trivial line bundle, we have

$$\underline{\left(g\,d\bar z \,\middle|\, f\,d\bar z\right) = \int \bar g f \, dz\,d\bar z} . \qquad \text{factor of } i \ ?$$

Now I want to compare D^*D with $\mathcal{D}^*\mathcal{D}$, where D is the covariant derivative. Work locally using an orthonormal basis for E. We have then

$$D = d + A \ : \ E\otimes\Omega^{0,0} \longrightarrow E\otimes\left(\Omega^{1,0}\oplus\Omega^{0,1}\right)$$

$$= \left(\frac{\partial}{\partial z} - \alpha^*\right)dz + \left(\frac{\partial}{\partial \bar z}+\alpha\right)d\bar z = \mathcal{D}' + \mathcal{D}.$$

because A is a skew-hermitian matrix of 1-forms, hence of the form $-\alpha^* dz + \alpha\, d\bar z$ for some matrix of functions α.

Compute adjoints

$$\left(\beta\, d\bar z \,\middle|\, \mathcal{D} f\right) = \left(\beta\, d\bar z \,\middle|\, \left(\frac{\partial}{\partial \bar z}+\alpha\right) f\, d\bar z\right)$$

$$= \int \beta^* \left(\frac{\partial}{\partial \bar z}+\alpha\right) f \; dz\,d\bar z$$

$$= \int \left[\left(-\frac{\partial}{\partial z}+\alpha^*\right)\beta\right]^* \cdot f \, dz\,d\bar z$$

Now you need to know ~~how~~ $dz\,d\bar z$ compares with your volume. The standard thing for $\mathbb{C}$ is $dx\,dy = \frac{i}{2}\,dz\,d\bar z$. This is confusing. Let's work with

$$\mathcal{D} = \frac{\partial}{\partial \bar z} + \alpha \qquad \mathcal{D}^* = 2\left(-\frac{\partial}{\partial z} + \alpha^*\right)$$

$$\mathcal{D}' = \frac{\partial}{\partial z} - \alpha^* \qquad \mathcal{D}'^* = 2\left(-\frac{\partial}{\partial \bar z} - \alpha\right)$$

Then

$$\tfrac{1}{2}\mathcal{D}^*\mathcal{D} = \left(-\frac{\partial}{\partial z}+\alpha^*\right)\left(\frac{\partial}{\partial \bar z}+\alpha\right) = -\frac{\partial^2}{\partial z\partial \bar z} + \alpha^*\frac{\partial}{\partial \bar z} - \alpha\frac{\partial}{\partial z} + \alpha^*\alpha - \frac{\partial\alpha}{\partial z}$$

$$\tfrac{1}{2}\mathcal{D}'^*\mathcal{D}' = \left(-\frac{\partial}{\partial \bar z}-\alpha\right)\left(\frac{\partial}{\partial z}-\alpha^*\right) = -\frac{\partial^2}{\partial z\partial \bar z} - \alpha\frac{\partial}{\partial z} + \alpha^*\frac{\partial}{\partial \bar z} + \alpha\alpha^* + \frac{\partial\alpha^*}{\partial \bar z}$$

We see from these formulas that $D^*D = \mathcal{D}'^*\mathcal{D}' + \mathcal{D}^*\mathcal{D}$

is not twice $\mathcal{D}^*\mathcal{D}$, however only because of 0th order terms, so I can expect determinant calculations for D^*D to carry over.

So lets try to do the heat kernel calculation. Set

~~[illegible]~~

$$\langle z \,|\, e^{-t\mathcal{D}^*\mathcal{D}} \,|\, z_0 \rangle = \underbrace{\frac{1}{\pi t}\, e^{-\frac{|z-z_0|^2}{t}}}_{\phi} \underbrace{\left\{ a_0(z,z_0) + a_1(z,z_0)t + \cdots \right\}}_{A}$$

Then $\partial_t(\phi A) + \mathcal{D}^*\mathcal{D}(\phi A) = 0$. Things will be easier if I write

$$\mathcal{D} = \frac{\partial}{\partial \bar z} + \alpha \qquad \tilde{\mathcal{D}} = \frac{\partial}{\partial z} - \alpha^*$$

so that $\mathcal{D}^* = -\tilde{\mathcal{D}}$. Then let's put z_0 whence

$$\phi = \frac{1}{\pi t} e^{-|z|^2/t}$$

and

$$\phi^{-1}\mathcal{D}\phi = \mathcal{D} - \frac{z}{t} \qquad \phi^{-1}\tilde{\mathcal{D}}\phi = \tilde{\mathcal{D}} - \frac{\bar z}{t}$$

$$\phi^{-1}\partial_t\phi = \partial_t - \frac{1}{t} + \frac{|z|^2}{t^2}$$

Then our equation

$$\partial_t(\phi A) - \tilde{\mathcal{D}}\mathcal{D}(\phi A) = 0$$

becomes

$$\left[\partial_t - \frac{1}{t} + \frac{|z|^2}{t^2} \underbrace{- \left(\tilde{\mathcal{D}} - \frac{\bar z}{t}\right)\left(\mathcal{D} - \frac{z}{t}\right)}_{-\tilde{\mathcal{D}}\mathcal{D} + \frac{1}{t}(\bar z\mathcal{D} + z\tilde{\mathcal{D}}) - \frac{|z|^2}{t^2} + \frac{1}{t}}\right] A = 0$$

or

$$\left[\partial_t - \tilde{\mathcal{D}}\mathcal{D} + \frac{1}{t}(\bar z\mathcal{D} + z\tilde{\mathcal{D}})\right] A = 0$$

So if $A = a_0(z) + a_1(z)t$, then we get

$$\{\bar z\mathcal{D} + z\tilde{\mathcal{D}}\}\, a_0 = 0$$

$$(1 + \bar z\mathcal{D} + z\tilde{\mathcal{D}})\, a_1 = \tilde{\mathcal{D}}\mathcal{D}\, a_0$$

Ultimately I need the value of a_1 at $z=0$. This should be $\tilde{\mathcal{D}}\mathcal{D}a_0$ at $z=0$. A natural question is whether a_0 is smooth at $z=0$, because the operator $\bar{z}\mathcal{D}+z\tilde{\mathcal{D}}$ vanishes there. The equation for a_0 says that a_0 is flat with respect to the connection in the radial direction

$$\bar{z}\mathcal{D}+z\tilde{\mathcal{D}} = \underbrace{\bar{z}\frac{\partial}{\partial \bar{z}}+z\frac{\partial}{\partial z}}_{x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y} \,=\, r\frac{\partial}{\partial r}} + \bar{z}\alpha - z\alpha^*$$

So what I should look at is the diff. equation:

$$\sum_\mu x_\mu \frac{\partial}{\partial x_\mu} u = \sum_\mu x_\mu B_\mu u$$

where the B_μ are matrices. Try a power series solution $u = \sum a_\alpha x^\alpha$. Then

$$\sum_\alpha a_\alpha \underbrace{\sum_\mu x_\mu \frac{\partial}{\partial x_\mu} \text{[redacted]} x^\alpha}_{|\alpha|\, x^\alpha} = \sum_\mu x_\mu B_\mu(x) \cdot \sum a_\alpha x^\alpha$$

and there is no problem grinding out the homogeneous terms starting from $a_0 = 1$. Put $\sum_{|\alpha|=m} a_\alpha x^\alpha = a_m(x)$

$$a_1(x) = \sum x_\mu B_\mu(0)$$

$$2a_2(x) = \left(\sum x_\mu B_\mu(0)\right)^2 + \sum x_\mu x_\nu \frac{\partial B_\mu}{\partial x_\nu}(0)$$

etc. [redacted] It seems clear then that we get a smooth solution.

So let's put $a_0(z) = u(z)$ and solve

$$\left(\bar{z}\mathcal{D}+z\tilde{\mathcal{D}}\right)u = 0$$

to the second order in z.

$$\left(z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial \bar{z}}\right)u = \underbrace{\left(\alpha^* z - \alpha\bar{z}\right)}_{\alpha_0^* z - \alpha_0 \bar{z} + \left(\frac{\partial\alpha^*}{\partial z}\big|_0 z + \frac{\partial\alpha^*}{\partial\bar{z}}\big|_0 \bar{z}\right) z - \left(\frac{\partial\alpha}{\partial z}\big|_0 z + \frac{\partial\alpha}{\partial\bar{z}}\big|_0 \bar{z}\right)\bar{z}} u$$

So

$$u = 1 + \alpha_0^* z - \alpha_0 \bar{z} + \text{quad. terms.}$$

$$\left(z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial\bar{z}}\right)(\text{quad terms}) = \frac{\partial\alpha^*}{\partial z}\Big|_0 z^2 + \left(\frac{\partial\alpha^*}{\partial\bar{z}}\Big|_0 - \frac{\partial\alpha}{\partial z}\Big|_0\right) z\bar{z} + \left(-\frac{\partial\alpha}{\partial\bar{z}}\Big|_0\right)\bar{z}^2 + \left(\alpha_0^* z - \alpha_0\bar{z}\right)^2$$

So

$$u = 1 + \alpha_0^* z - \alpha_0\bar{z} + \frac{z^2}{2}\left(\frac{\partial\alpha^*}{\partial z}\Big|_0 + \alpha_0^{*2}\right) + \frac{z\bar{z}}{2}\left(\frac{\partial\alpha^*}{\partial\bar{z}} - \frac{\partial\alpha}{\partial z} - \alpha^*\alpha - \alpha\alpha^*\right)_0 + \frac{\bar{z}^2}{2}\left(-\frac{\partial\alpha}{\partial\bar{z}}\Big|_0 + \alpha_0^{*2}\right)$$

and what I want is

$$a_1(0) = (\tilde{D}Du)_0 = \left(\frac{\partial}{\partial z} - \alpha^*\right)\left(\frac{\partial}{\partial\bar{z}} + \alpha\right)u\Big|_0$$

$$= \left(\frac{\partial^2}{\partial z\partial\bar{z}} - \alpha^*\frac{\partial}{\partial\bar{z}} + \alpha\frac{\partial}{\partial z} - \alpha^*\alpha + \frac{\partial\alpha}{\partial z}\right)u_0$$

trace goes somewhere

$$= \left[-\alpha^*\alpha + \frac{\partial\alpha}{\partial z} + \alpha^*\alpha + \alpha\alpha^* + \frac{1}{2}\left(\frac{\partial\alpha^*}{\partial\bar{z}} - \frac{\partial\alpha}{\partial z} - \alpha^*\alpha - \alpha\alpha^*\right)\right]_0$$

$$= \frac{1}{2}\left[\frac{\partial\alpha^*}{\partial\bar{z}} + \frac{\partial\alpha}{\partial z} + \alpha\alpha^* - \alpha^*\alpha\right]_0$$

Contrast this with the heat operator for D^*D where one gets $a_1(x,x) = 0$ always.

Summary:

$$\boxed{a_1(z,z) = \frac{1}{2}\left(\frac{\partial\alpha^*}{\partial\bar{z}} + \frac{\partial\alpha}{\partial z} + \alpha\alpha^* - \alpha^*\alpha\right)(z)}$$

Next let's go over the formulas for the ζ-values.

$$\Gamma(s)\,\zeta_A(s) = \int_0^\infty tr(e^{-tA})\,t^s\,\frac{dt}{t}$$

$$\langle z|e^{t\tilde{\mathcal{D}}\mathcal{D}}|z\rangle \sim \frac{1}{\pi t}\left(1 + a_1(z,\bar z)\,t + \cdots\cdots\right)$$

gives

[redacted]

$$tr\left(e^{t\tilde{\mathcal{D}}\mathcal{D}}\right) \sim \frac{Vol(M)}{\pi t} + \frac{1}{\pi}\int_M tr\,a_1(z,z) + \cdots$$

Then
$$\operatorname*{Res}_{s=1}\ \zeta(s) = \frac{Vol(M)}{\pi}$$

$$\zeta(0) = \frac{1}{\pi}\int_M tr\,a_1(z,z)$$

From the explicit form of $a_1(z,z)$ we see that it's integral over $M = \mathbb{C}/\Gamma$ is 0. So it[redacted] seems we have proved

Theorem: $\zeta(0) = 0$ for the ζ function of $\tilde{\mathcal{D}}^*\mathcal{D}$, $\mathcal{D} = \bar\partial$ operator for a holomorphic structure on the trivial bundle [redacted] over an [redacted] elliptic curve. (Must assume $\operatorname{Ker}\mathcal{D} = 0$, see p. 506).

But we are after $\zeta'(0)$. Let us review formulas:

$$\zeta_A(s) = \frac{1}{\Gamma(s)}\int_0^\infty tr(e^{-tA})\,t^s\,\frac{dt}{t}$$

Consider a variation in A:

$$\delta\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty tr\left(e^{-tA}\,\delta A\right)\left(-t^s\right)dt$$

and integrating by parts:

$$\delta\zeta(s) = \frac{+1}{\Gamma(s)}\int_0^\infty tr\left(e^{-tA}\,\frac{1}{A}\,\delta A\right)\left(-s\,t^{s-1}\right)dt$$

or

$$\delta\zeta(s) = \frac{-s}{\Gamma(s)}\int_0^\infty tr\left(e^{-tA}\frac{1}{A}\delta A\right) t^s\frac{dt}{t}$$

This is equivalent to the formula of the Atiyah Pat. Singer paper

$$\delta\, tr(A^{-s}) = -s\, tr\left(A^{-s-1}\delta A\right)$$

(see March 10, p. 440). Now we are in the situation where $\zeta(0)$ is constant, in fact 0, hence

$$\frac{\delta\zeta(s)}{s} \longrightarrow \delta\zeta'(0) \qquad as \quad s\to 0.$$

which is consistent with the formula

$$\delta \log\det(A) = tr(A^{-1}\delta A).$$

when the latter exists.
In any case we have

$$-\frac{\delta\zeta(s)}{s} = +tr\left(A^{-s-1}\delta A\right) = \frac{+1}{\Gamma(s)}\int_0^\infty tr\left(e^{-tA}\frac{1}{A}\delta A\right)t^s\frac{dt}{t}$$

And so if we can find a an asymptotic expansion for $tr\left(e^{-tA}\frac{1}{A}\delta A\right)$ as $t\downarrow 0$, we will get a formula for $\delta\zeta'(0)$.

In dimension 1 the ζ function of $-\partial^2+q$, say, over S^1 has poles at $s=\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \cdots$ and zeroes at 0, $-1, -2, \cdots$ so its Singer determinant is always defined. I knew that $\delta\log\det(A) = tr(A^{-1}\delta A)$ was defined, but didn't know any way to specify the integration constant. Let's compute for

$$A = -\partial^2 + q^2 \qquad q \text{ real} \qquad \text{over } S^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

The eigenvalues are n^2+q^2 $\quad n\in\mathbb{Z}$.

$$\pi^{-s}\Gamma(s)\zeta_A(s) = \int_0^\infty \sum_{n\in\mathbb{Z}} e^{-\pi t(n^2+q^2)}\, t^s\frac{dt}{t} = \int_0^\infty \underbrace{\sum e^{-\pi t n^2}}_{\frac{1}{\sqrt{t}}\sum e^{-\pi n^2/t}} e^{-tq^2} t^s\frac{dt}{t}$$

A quick calculation using the formulas on p. 407 seems to yield the value

$$-\zeta_A'(0) = \log\left[4(\sinh \pi g)^2\right]$$

On the other hand a naive calculation gives

$$\frac{\text{"}\det(-\partial^2+g^2)\text{"}}{\det(-\partial^2)} = \text{"}\prod_{n\in\mathbb{Z}} \frac{n^2+g^2}{n^2}\text{"} = g^2 \prod_1^\infty \left(1+\frac{g^2}{n^2}\right)^2$$

$$= \left(\frac{\sinh \pi g}{\pi}\right)^2$$

So it's really not clear whether the actual [redacted] ζ constant [redacted] has any real meaning.

---

0 eigenvalues: For

$$\Gamma(s)\,\zeta_A(s) = \int_0^\infty tr\left(e^{-tA}\right) t^s \frac{dt}{t}$$

to be defined for $Re(s) >> 0$ we [redacted] need all eigenvalues of A to be > 0. When A has zero eigenvalues, then

$$\Gamma(s)\,\zeta_A(s) = \int_0^\infty tr\left(e^{-tA} - P_0\right) t^s \frac{dt}{t}$$

where P_0 = projection on the null-space $= \lim_{t\to+\infty} e^{-tA}$.
So

$$[redacted]\ \Gamma(s)\,\zeta_A(s) = \int_0^1 \left[tr(e^{-tA}) - d\right] t^s \frac{dt}{t} + \underbrace{\int_1^\infty \left[tr(e^{-tA}) - d\right] t^s \frac{dt}{t}}_{\text{entire in } s}$$

$d = \dim(\operatorname{Ker} A)$

$$= \underbrace{\int_0^1 tr(e^{-tA})\, t^s \frac{dt}{t}}_{\substack{\text{meromorphic in } s \\ \text{with singularities related to the} \\ \text{asymptotic expansion of } tr(e^{-tA}) \text{ as } t \downarrow 0.}} - \frac{d}{s}$$

Hence we see that the kernel of A contributes the value $-d$ to $\zeta(0)$ beyond what you get from local expressions.

So in the elliptic curve case and structures on the trivial bundle the local contribution to $\zeta(0)$ we know vanishes. Hence $\boxed{\zeta(0) = -\dim \operatorname{Ker} D.}$

In the case of $-\partial^2$ over S^1 we get

$$\zeta_A(s) = \sum_{n \neq 0} \frac{1}{n^{2s}} = 2\,\zeta_R(2s)$$

$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta_R(s) \sim \frac{1}{s-1} \quad \text{as } s \to 1$$

and is symmetric under $s \leftrightarrow 1-s$, so $\sim -\frac{1}{s}$ as $s \to 0$.

Hence

$$\frac{2}{s}\,\zeta_R(s) \sim \frac{-1}{s}, \quad \frac{2}{2s}\,\zeta_R(2s) \sim \frac{-1}{2s}$$

$$\zeta_{-\partial^2}(s) = 2\,\zeta_R(2s) \sim -1 \quad \text{as } s \to 0.$$

In fact recall that

$$\Gamma(s)\,\zeta_R(s) = \int_0^\infty \frac{1}{e^t-1}\, t^s \,\frac{dt}{t}$$

and

$$\frac{1}{e^t-1} = \frac{1}{t} - \frac{1}{2} + \sum_{n \text{ odd}} b_n t^n$$

where b_n is related to the Bernoulli nos. Hence $\zeta_R(s)$ vanishes at $-2, -4, -6$ and its values at $0, -1, -3, \dots$ are given by Bernoulli nos. e.g.

$$\zeta_R(0) = -\frac{1}{2} \qquad \zeta_R(-1) = \pm\frac{1}{12}$$

March 29, 1982

[redacted] Today I want to calculate the variation in [redacted] $\zeta_A'(0)$ where $A = \mathcal{D}^*\mathcal{D} = -\tilde{\mathcal{D}}\mathcal{D}$ over $M = \mathbb{C}/\Gamma$

$$\mathcal{D} = \frac{\partial}{\partial\bar{z}} + \alpha \qquad \tilde{\mathcal{D}} = \frac{\partial}{\partial z} - \alpha^*$$

and I assume [redacted] $\operatorname{Ker}\mathcal{D} = 0$, so that $\zeta_A(0) = 0$. The idea is to use

$$\delta\zeta(s) = -s\, tr(A^{-s-1}\delta A), \text{ so } -\delta\zeta'(0) = \lim_{s\to 0} tr(A^{-s-1}\delta A)$$

and

$$tr(A^{-s-1}\delta A) = \frac{1}{\Gamma(s)}\int_0^\infty tr(e^{-tA}A^{-1}\delta A)\, t^s\frac{dt}{t}$$

and to find an asymptotic expansion for $tr(e^{-tA}A^{-1}\delta A)$ as $t\downarrow 0$, then the value of $tr(A^{-s-1}\delta A)$ at $s=0$ will be given by the coeff. of t^0 in this asymptotic exp.

~~$tr\left(e^{-tA}\frac{1}{+\tilde{\mathcal{D}}\mathcal{D}}(\delta\tilde{\mathcal{D}}\mathcal{D} + \tilde{\mathcal{D}}\delta\mathcal{D})\right)$~~

~~$= \underbrace{tr\left(e^{-tA}\frac{1}{-\tilde{\mathcal{D}}\mathcal{D}}(+\delta\alpha^*)\mathcal{D}\right)}_{tr\left(\delta\alpha^*\mathcal{D}\frac{1}{\mathcal{D}^*\mathcal{D}}\right.} + tr\left(e^{-tA}\underbrace{\frac{1}{\tilde{\mathcal{D}}\mathcal{D}}\tilde{\mathcal{D}}}_{1/\mathcal{D}}\delta\alpha\right)$~~

$$tr\left(e^{-t\mathcal{D}^*\mathcal{D}}\frac{1}{\mathcal{D}^*\mathcal{D}}(\delta\mathcal{D}^*\mathcal{D} + \mathcal{D}^*\delta\mathcal{D})\right)$$

$$= \underbrace{tr\left(e^{-t\mathcal{D}^*\mathcal{D}}\frac{1}{\mathcal{D}^*\mathcal{D}}\delta\mathcal{D}^*\mathcal{D}\right)}_{tr\left(\delta\mathcal{D}^*\underbrace{\mathcal{D}\frac{1}{\mathcal{D}^*\mathcal{D}}}_{1/\mathcal{D}^*}e^{-t\mathcal{D}^*\mathcal{D}}\right)} + tr\left(e^{-t\mathcal{D}^*\mathcal{D}}\underbrace{\frac{1}{\mathcal{D}^*\mathcal{D}}\mathcal{D}^*}_{1/\mathcal{D}}\delta\mathcal{D}\right)$$

use here $tr(XY) = tr(YX)$

$X = \underset{\text{smooth}}{e^{-t\mathcal{D}^*\mathcal{D}}}\delta\mathcal{D}^*\mathcal{D} \qquad Y = \frac{1}{\mathcal{D}^*\mathcal{D}}$

$$= 2\,\mathrm{Re}\; tr\left(e^{-t\mathcal{D}^*\mathcal{D}}\mathcal{D}^{-1}\delta\mathcal{D}\right)$$

Therefore we are led to conjecture that we can define

$$\delta \log \det \mathcal{D} = \lim_{s\to 0} tr\left((\mathcal{D}^*\mathcal{D})^{-s} \mathcal{D}^{-1} \delta\mathcal{D} \right)$$

whence we will have

$$-\delta\, \zeta'_{\mathcal{D}^*\mathcal{D}}(0) = 2 \operatorname{Re}\, \delta \log \det \mathcal{D} .$$

Try to compute the asymptotic expansion of $e^{-t\mathcal{D}^*\mathcal{D}} \frac{1}{\mathcal{D}^*\mathcal{D}}$. First do with $\mathcal{D} = \frac{\partial}{\partial \bar{z}}$. Put

$$\langle z | e^{t\tilde{\mathcal{D}}\mathcal{D}} \frac{1}{\tilde{\mathcal{D}}\mathcal{D}} | 0 \rangle = \frac{1}{\pi} e^{-\frac{|z|^2}{t}} B(z,t).$$

Then

$$\partial_t \frac{1}{\pi} e^{-\frac{|z|^2}{t}} B = \frac{1}{\pi t} e^{-\frac{|z|^2}{t}}$$

or

$$\left(\partial_t - \frac{1}{t} + \frac{|z|^2}{t^2} \right) B = \frac{1}{t}$$

Look for a series solution in t. First term has to be $\frac{t}{|z|^2}$, [redacted] and this one checks is a complete soln. Thus

$$\langle z | e^{t\tilde{\mathcal{D}}\mathcal{D}} \frac{1}{\tilde{\mathcal{D}}\mathcal{D}} | 0 \rangle = \frac{t}{\pi |z|^2} e^{-\frac{|z|^2}{t}} .$$

This looks a little strange because you expect the kernel to be smooth for $t > 0$, but that's probably because of the 0 eigenvalue in $\mathcal{D}$. No, one has to be careful because the kernel of $\frac{1}{\tilde{\mathcal{D}}\mathcal{D}}$ is not smooth on the diagonal.

Example: On the line the kernel of $e^{t\partial^2} \frac{1}{-\partial^2}$ is

$$\langle x | e^{t\partial^2} \frac{1}{-\partial^2} | 0 \rangle = \int \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} |y|\, dy$$

because the Green's fn. for $-\partial^2$ is $|x|$. (Actually this can be altered by any linear function of x, so that I have made a choice [illegible] because of the 0 eigenvalue.)

The asymptotic expansions as $t \longrightarrow 0$ are

$$\langle x | e^{t\partial^2} \frac{1}{-\partial^2} | 0 \rangle \longrightarrow |x|$$ exponentially as $t \downarrow 0$ when $x \neq 0$.

But if $x = 0$ we have the value

$$\int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}} \underbrace{2|y|\,dy}_{dy^2} = \frac{1}{\sqrt{4\pi t}} \frac{\Gamma(1)}{1/4t} = \sqrt{\frac{4t}{\pi}}$$

So what this means is that the process of taking asymptotic expansions doesn't commute with restriction to the diagonal.

Try now $\tilde{\mathcal{D}}\mathcal{D}$, $\mathcal{D} = \frac{\partial}{\partial \bar{z}}$.

$$\langle z | e^{t\tilde{\mathcal{D}}\mathcal{D}} | 0 \rangle = \frac{1}{\pi t} e^{-\frac{|z|^2}{t}}$$

What would be a Green's function for $\mathcal{D} = \frac{\partial}{\partial \bar{z}}$. Thus we want $\frac{\partial}{\partial \bar{z}} g = \delta(z)$, hence g must be holom. outside of 0.

$$\frac{\partial}{\partial \bar{z}}(zg) = z\,\delta(z) = 0$$

Thus zg must be holomorphic, and so making a choice, the simplest thing is $g = \frac{c}{z}$, c = some const.

Can determine c in two ways

$$\frac{1}{c} = \iint \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right) dx\,dy = \iint \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z}\right) \frac{d\bar{z}\,dz}{2i} = \frac{1}{2i}\iint d\left(\frac{1}{z}\right) dz$$

$$= \frac{1}{2i} \iint d\left(\frac{1}{z}\,dz\right) = \frac{1}{2i} \oint \frac{1}{z}\,dz = \pi \qquad \therefore g = \frac{1}{\pi z}$$

Or one computes the Green's fn. for Δ.

$$\iint \Delta \log r \; dx\,dy = \oint \frac{\partial}{\partial r} \log r \underbrace{ds}_{r\,d\theta} = \int_0^{2\pi} d\theta = 2\pi$$

So $\frac{1}{2\pi}\log r$ is a Green's fn. for $\Delta = 4\frac{\partial^2}{\partial\bar z\partial z}$

$$\frac{\partial^2}{\partial\bar z\partial z}\ \underset{\parallel}{\frac{2}{\pi}\log r} = \delta(z)$$

$$\frac{\partial}{\partial\bar z}\ \frac{\partial}{\partial z}\ \frac{1}{\pi}\log(z\bar z) = \frac{\partial}{\partial\bar z}\left(\frac{1}{\pi z}\right)$$

Table:	$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$	has G. fn.	$\frac{1}{2\pi}\log r$
	$\frac{\partial}{\partial\bar z}$	has G. fn.	$\frac{1}{\pi z}$

[redacted] Both $\log r$ and $\frac{1}{\pi z}$ are locally L^1 functions and so define distributions

$$\langle z|\, e^{t\tilde{\mathcal D}\mathcal D}\frac{1}{\mathcal D}|0\rangle = \int\frac{1}{\pi t}e^{-\frac{|z-y|^2}{t}}\ \frac{1}{\pi y}\ d^2y$$

$$\longrightarrow \frac{1}{\pi z} \quad \text{Exponentially if } z\neq 0$$

On the other hand

$$\langle 0|e^{t\tilde{\mathcal D}\mathcal D}\frac{1}{\mathcal D}|0\rangle = 0$$

by symmetry.

So now we want to do the general case $\mathcal D = \frac{\partial}{\partial\bar z}+\alpha$. It is important to keep in mind that the operator $e^{t\tilde{\mathcal D}\mathcal D}\frac{1}{\mathcal D}$ has a smooth kernel which one can restrict to the diagonal, and then take asymptotic expansion. So

$$\langle 0|e^{t\tilde{\mathcal D}\mathcal D}\mathcal D^{-1}|0\rangle = \int\underbrace{\langle 0|e^{t\tilde{\mathcal D}\mathcal D}|z\rangle}_{\frac{1}{\pi t}e^{-\frac{|z|^2}{t}}\underset{\text{smooth}}{A(t,z)}}\underbrace{\langle z|\mathcal D^{-1}|0\rangle}_{\substack{\text{distribution}\\ \text{like }\frac{1}{\pi z}}}\ d^2z$$

It seems desirable to know more about $\langle z|D^{-1}|0\rangle$. This brings up the classical (Hilbert) problem of using a parametrix.

Two approaches to solving $(D-B)f=g$ when D^{-1} is known:

$$(D-B)^{-1}=\left[D(1-D^{-1}B)\right]^{-1}=(1-D^{-1}B)^{-1}D^{-1}$$

$$(D-B)^{-1}=\left[(1-BD^{-1})D\right]^{-1}=D^{-1}(1-BD^{-1})^{-1}$$

The first leads to an integral equation

$$(1-D^{-1}B)f=D^{-1}g$$

[redacted] and the second to

$$(1-BD^{-1})Df=g$$

The kernel when $D=\frac{\partial}{\partial\bar z}$ and $B=-\alpha$ are

$$D^{-1}B: \qquad -\frac{1}{\pi(z-z')}\alpha(z')$$

$$BD^{-1}: \qquad -\alpha(z)\frac{1}{\pi(z-z')}$$

At this point one should ask about the integral operator with the [redacted] Cauchy kernel $\frac{1}{z-z'}$.

$$K(z,z')=\frac{1}{z-z'}$$

[redacted] I know how to modify this so it works over an elliptic curve (Weierstrass ζ + term linear in $\bar z$ to make it doubly-periodic), so I can ignore "infrared problems" and see when K^n is of trace class.

The problem is now how to understand $\langle z|D^{-1}|0\rangle$ where $D=\frac{\partial}{\partial\bar z}+\alpha$. This is a distribution presumably locally L^1 beginning with $\frac{1}{\pi z}$ and certainly smooth away from

$z = 0$. The standard way to describe this distribution is as a pseudo-differential operator kernel.

$$(\mathcal{D}^{-1} f)(z) = (2\pi)^{-2} \iint d^2\xi \; \phi(z,\xi)\, e^{-i\xi(z)} \hat{f}(\xi)$$

where $\phi(z,\xi)$ has an asymptotic expansion in terms of homogeneous functions of ξ. Thus if $f = \delta(z)$, $\hat{f} = 1$ and

$$\langle z | \mathcal{D}^{-1} | 0 \rangle = (2\pi)^{-2} \iint d^2\xi \; \phi(z,\xi)\, e^{i\xi(z)}.$$

Actually a good question is how this standard ΨDO expression for $\mathcal{D}^{-1}$ compares with the ones you might be able to obtain from

$$\frac{1}{A-B} = \frac{1}{A} + \frac{1}{A} B \frac{1}{A} + \cdots$$

$$= \sum_{1}^{\infty} p_n(A,B) \frac{1}{A^n}$$

taking $A - B = \frac{\partial}{\partial \bar{z}} + \alpha$. Recall the recursion relation

$$p_{n+1} = B p_n - [A, p_n]$$

In the present situation $A = \frac{\partial}{\partial \bar{z}}$ so that if $p_n(A,B)$ is assumed to be a zero-th order operator, then inductively p_{n+1} will be zero-th order.

March 30, 1982:

Let's look a little bit at homogeneous distributions, and their Fourier transforms in $\mathbb{R}^n$. Use standard fmlas

$$f(x) = (2\pi)^{-n}\int d^n\xi \; e^{i\xi x}\hat{f}(\xi) \qquad \hat{f}(\xi) = \int d^nx \; e^{-i\xi x} f(x)$$

Suppose f homogeneous of degree ρ ; $f(tx) = t^\rho f(x)$. Then

$$\hat{f}(t\xi) = \int d^nx \; e^{-it\xi x} f(x) = t^{-n}\int d^nx \; e^{-i\xi x} f(t^{-1}x)$$
$$= t^{-n-\rho}\hat{f}(\xi).$$

so

$$\boxed{f \text{ homog. of deg } \rho \Longleftrightarrow \hat{f} \text{ homog. of deg } -n-\rho}$$

For example $\delta(x)$ is homogeneous of degree $-n$ because $\delta(x)d^nx$ is invariant, and this checks: δ homog deg $-n$ $\Rightarrow \hat{\delta} = 1$ homog. of deg $-n-(-n) = 0$.

A smooth function f on $\mathbb{R}^n - 0$ homog. of degree $\rho > -n$ is locally L^1, so defines a distribution. The transform $\hat{f}$ has degree $-n-\rho < -n-(-n) = 0$. Thus ~~homog. smooth~~ perhaps smooth homog. fns. on $\mathbb{R}^n - 0$ of any degree have a natural interpretation as homog. distributions. Let's do carefully for $\mathbb{R}^1$. NO.

In $\mathbb{R}^2$ consider the homogeneous fn. $\frac{1}{|\xi|^2}$ defined for $\xi \neq 0$ of degree -2. Extending this to a distribution over $\xi = 0$ is the same thing as makeing sense of the F.T.

$$\int \frac{d^2\xi}{(2\pi)^2} e^{i\xi x}\frac{1}{|\xi|^2}$$

which will then be a fundamental solution for $-\Delta$. Such a fundamental soln is $\frac{1}{2\pi}\log r$ + harmonic fn. There is no way you can make a homogeneous fundamental solution of degree 0.

Let's go back to our problem of constructing $\mathcal{D}^{-1}$ where $\mathcal{D} = \frac{\partial}{\partial \bar z} + \alpha$. Fundamental solutions for $\left(\frac{\partial}{\partial \bar z}\right)^n$ are as follows. Put $D = \frac{\partial}{\partial \bar z}$

$$\langle z | D^{-1} | 0 \rangle = \frac{1}{\pi z}$$

$$\frac{\partial}{\partial \bar z} \langle z | D^{-2} | 0 \rangle = \frac{1}{\pi z} \qquad \therefore \quad \langle z | D^{-2} | 0 \rangle = \frac{\bar z}{\pi z}$$

Similarly $\langle z | D^{-3} | 0 \rangle = \frac{\bar z^2}{2\pi z}$ $\qquad \langle z | D^{-n-1} | 0 \rangle = \frac{\bar z^n}{n!\,\pi z}$

Then we want to use

$$\frac{1}{A - B} = \sum_1^\infty p_n(A,B) \frac{1}{A^n}$$

where $p_{n+1} = B p_n - [A, p_n]$. In this situation $A = D = \frac{\partial}{\partial \bar z}$ and $B = -\alpha$. So

$$p_1 = 1$$
$$p_2 = (-\alpha)$$
$$p_3 = (-\alpha)^2 - [\bar\partial, -\alpha] = \alpha^2 + \bar\partial \alpha$$
$$p_4 = -\alpha(\alpha^2 + \bar\partial\alpha) - [\bar\partial, \alpha^2 + \bar\partial\alpha]$$
$$= -\alpha^3 - \alpha\bar\partial\alpha - \bar\partial(\alpha^2) - \bar\partial^2\alpha$$

$$\boxed{\frac{1}{\bar\partial + \alpha} = \frac{1}{\bar\partial} - \alpha\frac{1}{\bar\partial^2} + (\alpha^2 + \bar\partial\alpha)\frac{1}{\bar\partial^3} + \cdots}$$

$$\langle z | \frac{1}{\bar\partial + \alpha} | 0 \rangle = \frac{1}{\pi z} - \alpha(z)\frac{\bar z}{\pi z} + \left[\alpha(z)^2 + \bar\partial\alpha(z)\right]\frac{\bar z^2}{2\pi z} + \cdots\cdots$$

Because of the repeated derivatives this is probably only valid as an asymptotic expansion as $z \to 0$. However we can see from this that it is likely that

$$\langle z | \frac{1}{\bar\partial + \alpha} | 0 \rangle = \frac{1}{\pi z}(\text{smooth fn of } z).$$

But this is clear, because one can solve $(\bar\partial + \alpha) f = 0$, $f(0) = 1$ locally. Then $(\bar\partial + \alpha)\left(f \frac{1}{\pi z}\right) = f\left[f^{-1}(\bar\partial + \alpha) f\right]\left(\frac{1}{\pi z}\right)$

$= f \cdot \bar\partial\left(\frac{1}{\pi z}\right) = f\,\delta(z) = \delta(z).$ Thus.

Prop: $\langle z|\left(\frac{\partial}{\partial \bar z}+\alpha\right)^{-1}|0\rangle = \frac{1}{\pi z} f(z)$ where f is a solution of $(\bar\partial+\alpha)f = 0$, $f(0) = 1$. More generally $\langle z|\left(\frac{\partial}{\partial \bar z}+\alpha\right)^{-1}|y\rangle = \frac{1}{\pi(z-y)} f(z) f(y)^{-1}$

Next consider

$$\langle 0|e^{t\tilde{\mathcal{D}}\mathcal{D}}\mathcal{D}^{-1}|0\rangle = \int d^2z\, \underbrace{\langle 0|e^{t\tilde{\mathcal{D}}\mathcal{D}}|z\rangle}_{\frac{1}{\pi t}e^{-\frac{|z|^2}{t}}\left(A(t,z)\right)} \underbrace{\langle z|\mathcal{D}^{-1}|0\rangle}_{\frac{1}{\pi z}f(z)}$$

where $A(t,z) \sim a_0(0,z) + t\,a_1(0,z) + O(t^2)$ as $t \to 0$ is smooth in both t, z. We should have

$$A(t,z) = a_0(0,z) + t\,B(t,z) \qquad B \text{ smooth}$$

Now $\displaystyle\int d^2z\, \frac{1}{\pi \cancel{t}} e^{-\frac{|z|^2}{t}} \underbrace{\cancel{t}B(t,z)\,\frac{1}{\pi z}f(z)}_{\text{integrable}} \longrightarrow 0$

by dominated convergence. Hence

$$\langle 0|e^{t\tilde{\mathcal{D}}\mathcal{D}}\cdot\mathcal{D}^{-1}|0\rangle = \int d^2z\, \frac{1}{\pi t} e^{-\frac{|z|^2}{t}}\, a_0(0,z)\,\frac{1}{\pi z} f(z) + o(t)$$

Now $a_0(0,z)\,f(z)$ is smooth in z, say $= 1 + bz + c\bar z + \cdots$ and

$$\int d^2z\, \underbrace{\frac{1}{\pi t} e^{-\frac{|z|^2}{t}}}_{\text{approaches } \delta(z)} \frac{1}{\pi z}\left(1 + bz + c\bar z + O(|z|^2)\right) = \frac{1}{\pi} b = \frac{1}{\pi}\frac{\partial}{\partial z}\left(a_0(0,z)\,f(z)\right)\Big|_{z=0}$$

This will involve more precise idea of what $f(z)$ is. $f(z)$ can be locally altered by multiplying by a holomorphic function $= 1$ at 0.

Let's recall (p. 502)

$$a_0(z,y) = 1 + \alpha^*(y)(z-y) - \alpha(y)\overline{(z-y)} + O((z-y)^2)$$

so that $$\frac{\partial}{\partial y} a_0(z,y)\Big|_{y=z} = -\alpha^*(z)$$

$$\boxed{\frac{\partial}{\partial z} a_0(0,z)\Big|_{z=0} = -\alpha^*(0)}$$

We are after

$$\frac{\partial}{\partial z}\left(a_0(0,z)f(z)\right)\Big|_{z=0} = \underbrace{\frac{\partial}{\partial z} a_0(0,z)\Big|_{z=0}}_{-\alpha^*(0)} + \frac{\partial f}{\partial z}(0).$$

I don't understand the first term yet but the second has the following interpretation. Put

$$G(z,y) = \langle z | \frac{1}{\partial + \alpha} | y \rangle$$ ~~$= \frac{f(z,y)}{\pi(z-y)}$ locally~~

Recall that we are trying to define

$$\mathrm{tr}(\mathcal{D}^{-1}\delta\alpha)$$

and are having trouble because the kernel $G(z,y)$ for $\mathcal{D}^{-1}$ [illegible] blows up along the diagonal. What our regularization process does is to write

$$G(z,y) = \frac{f(z,y)}{\pi(z-y)} \quad \text{locally} \quad f \text{ smooth}, \ f(y,y) = 1.$$

and then give us the [illegible] function

$$y \longmapsto \frac{1}{\pi}\frac{\partial f}{\partial z}(z,y)\Big|_{z=y} = \frac{\partial}{\partial z}(z-y)G(z,y)\Big|_{z=y}$$

It looks simpler if $y=0$. We have $G(z) = \langle z|\mathcal{D}^{-1}|0\rangle \overset{\text{locally}}{=} \frac{f(z)}{\pi z}$ where f is smooth, $f(0)=1$. Then

$$\text{renormalized value for } G(0) = \frac{\partial}{\partial z} zG(z).$$

Is this invariantly defined? i.e. change z to another coordinate $zh(z)$, where h is analytic, $h(0)=1$.

Then
$$\frac{\partial}{\partial z}\, zh(z)G(z)\Big|_{z=0} = h(0)\frac{\partial}{\partial z}(zG(z))\Big|_0 + h'(0)\big[zG(z)\big]_{z=0}$$
$$= \frac{\partial}{\partial z}(zG(z))\Big|_{z=0} + h'(0)\,\frac{1}{\pi}$$

Therefore we see that the renormalized value of $G(0)$ is subtle. Also one should observe that your formal expression
$$G(z) \approx \frac{1}{\pi z}\left(1 - \alpha(z)\bar{z} + \frac{\alpha(z)^2 + \bar{\partial}\alpha(z)}{2}\,\bar{z}^2 + \cdots\right)$$
gives 0 for this renormalized $G(0)$.

You [redacted] should get curvature straight.
$$D = d + A = \frac{\partial}{\partial z}dz + \frac{\partial}{\partial \bar z}d\bar z + \big(-\alpha^* dz + \alpha\, d\bar z\big)$$
$$\uparrow$$
because A must be skew-herm.
$$= \underbrace{\left(\frac{\partial}{\partial z} - \alpha^*\right)}_{\tilde{\mathcal{D}}}dz + \underbrace{\left(\frac{\partial}{\partial \bar z} + \alpha\right)}_{\mathcal{D}}d\bar z$$

Hence
$$D^2 = \underbrace{\big[\tilde{\mathcal{D}}\mathcal{D} - \mathcal{D}\tilde{\mathcal{D}}\big]}_{\frac{\partial\alpha}{\partial z} + \frac{\partial\alpha^*}{\partial\bar z} - \alpha^*\alpha + \alpha\alpha^*}dz\,d\bar z$$

which checks with our value for a_1.

One of the problems with the $-\alpha^*(0)$ contribution is that we would like $tr\left((\bar\partial+\alpha)^{-1}\delta\alpha\right)$ to be holomorphic in α. [redacted] Hence I should check the previous calculations when α is constant and the rank is 1.

So I need the Green's fn. $G(z) = \langle z|\,\frac{1}{\bar\partial+\alpha}\,|0\rangle$ which I know locally around $z=0$ has the form
$$\frac{1}{\pi z}\,e^{-\alpha\bar z}\cdot(\text{holom. fn.})$$

We want to make a doubly-periodic function of this form, so the obvious candidate is

$$ⓧ \qquad \frac{1}{\pi\sigma(z)} e^{-\alpha\bar{z} + \beta z + \gamma z^2/2}$$

with β, γ adjusted so that it becomes doubly-periodic. Here

$$\sigma(z) = z \prod_{\gamma}{}' \left(1 - \frac{z}{\gamma}\right) e^{\frac{z}{\gamma} + \frac{z^2}{\gamma^2}}$$

is the Weierstrass σ-function, which puts simple poles at each of the lattice points. Now one knows that $-\frac{d^2}{dz^2}\log\sigma = \wp$ which is doubly-periodic, hence

$$\frac{\sigma(z+\omega)}{\sigma(z)} = e^{\eta(\omega)z + c(\omega)}$$

where $\eta(\omega), c(\omega)$ are additive in the period $\omega \in \Gamma$. [redacted]
I also know $\eta(\omega)$ is not of the form $s\omega$ with $s \in \mathbb{C}$.
If

$$f(z) = e^{-\alpha\bar{z} + \beta z + \gamma z^2/2}$$

then

$$\frac{f(z+\omega)}{f(z)} = e^{-\alpha\bar{\omega} + \beta\omega + \gamma\omega z + \gamma\omega^2/2}$$

so it's clear that ⓧ doesn't work. So try a ratio

$$\frac{\sigma(z+\omega+z_0)}{\sigma(z+\omega)} \bigg/ \frac{\sigma(z+z_0)}{\sigma(z)} = e^{\eta(\omega)(\cancel{z}+z_0) + \cancel{c(\omega)}} = e^{+\eta(\omega)z_0}$$

so if

$$f(z) = e^{-\alpha\bar{z} + \beta z}, \qquad \frac{f(z+\omega)}{f(z)} = e^{-\alpha\bar{\omega} + \beta\omega}$$

Thus we want to choose z_0, β such that

$$-\alpha\bar{\omega} + \beta\omega = -\eta(\omega)z_0$$

and it's possible to do this. The conditions $z_0 \notin \Gamma$ and $\alpha \notin \mu$ lattice are probably equivalent. So

$$G(z) = \boxed{\frac{\sigma(z+z_0)}{\pi\sigma(z)\sigma(+z_0)} e^{-\alpha\bar{z} + \beta z} = \langle z | \frac{1}{\bar{\partial} + \alpha} | 0 \rangle}$$

Now evaluate $\frac{\partial}{\partial z} zG(z)\Big|_{z=0}$. First $zG(z)\Big|_{z=0} = 1/\pi$

so

$$\frac{\partial}{\partial z} zG(z)\Big|_{z=0} = \frac{1}{\pi}\frac{\partial}{\partial z}\log zG(z)\Big|_{z=0}$$

(*) $$= \frac{1}{\pi}\Big[\frac{1}{z} - \zeta(z) + \zeta(z+z_0) + \beta\Big]_{z=0}$$

$$\boxed{\frac{\partial}{\partial z} zG(z)\Big|_{z=0} = \frac{1}{\pi}\big(\zeta(z_0)+\beta\big)}$$

where $\zeta(z) = \frac{\partial}{\partial z}\log\sigma(z) = \frac{1}{z} + \sum{}'\left(\frac{1}{z-\mu}+\frac{1}{\mu}+\frac{z}{\mu^2}\right)$

Now $\boxed{\eta(\omega) = l\omega + m\bar{\omega}}$ for constants l, m

hence $-\alpha\bar{\omega} + \beta\omega = -\eta(\omega)z_0 = -z_0 l\omega - z_0 m\bar{\omega}$

say $\alpha = z_0 m \qquad \beta = -z_0 l \qquad$ or

$$z_0 = \frac{\alpha}{m} \qquad \beta = -\frac{l}{m}\alpha$$

This means that our normalized $G(0)$ is analytic in α. Hence unless there is a mistake with the $-\alpha^*$ term, it seems the regularized value for $\mathrm{tr}(\mathcal{D}^{-1}\delta\alpha)$ is not analytic in α.

Check: Legendre reln. from p. 378 is

$$\boxed{2\pi i = \tau\eta(1) - \eta(\tau)}$$

$$= \tau(l+m) - (l\tau + m\bar{\tau}) = m\,2i\,\mathrm{Im}\,\tau$$

$$\therefore \boxed{m = \frac{\pi}{\mathrm{Im}\,\tau}}$$

Recall that

$$\mu\text{-lattice} = \underbrace{\frac{\pi}{\mathrm{Im}\,\tau}}\Gamma$$

$$\alpha = m\cdot z_0$$

so we check that $\alpha \in \mu\text{-lattice} \Longleftrightarrow z_0 \in \Gamma$.

What I seem to be getting is that if I use the analytic continuation or heat kernel method to define the expression $\sum_\mu' \frac{1}{\mu+\alpha}$ then the result is not analytic in α. Maybe I can see this directly. The analytic continuous method uses

$$\sum_\mu \frac{1}{\mu+\alpha}\frac{1}{|\mu+\alpha|^{2s}} = \frac{1}{\Gamma(s)}\int_0^\infty \sum_\mu' \frac{e^{-t|\mu+\alpha|^2}}{\mu+\alpha} \, t^s \frac{dt}{t}$$

which converges for $Re(s) > 1$, and then you let $s \to 0$ which gives a finite result. Note that the result is doubly-periodic in α as long as we stay away from $\alpha \in \mu$-lattice.

$$\zeta(z_0) = \zeta\left(\frac{\alpha}{m}\right) = \frac{m}{\alpha} + \sum_\gamma{}' \frac{1}{\frac{\alpha}{m}-\gamma} + \frac{1}{\gamma} + \frac{\alpha}{m\gamma^2}$$

$$= m\underbrace{\left\{ \frac{1}{\alpha} + \sum{}' \frac{1}{\alpha-\mu} + \frac{1}{\mu} + \frac{\alpha}{\mu^2} \right\}}_{\text{obvious analytic candidate for } \sum_\mu' \frac{1}{\mu+\alpha}}$$

Our answer for

$$tr\, (\mathcal{D}^*\mathcal{D})^{-s}\mathcal{D}^{-1}\Big|_{s=0} = \int_{\mathbb{C}/\Gamma} d^2z \; \frac{1}{\pi}\left(-\alpha^* + \zeta(z_0) + \beta\right) \qquad vol\ \mathbb{C}/\Gamma = Im\,\tau$$

$$= \frac{1}{m}\left(\zeta(z_0) + \beta - \alpha^*)\right)$$

$$= \left(\frac{1}{\alpha} + \sum{}' \frac{1}{\alpha-\mu} + \frac{1}{\mu} + \frac{\alpha}{\mu^2}\right) + \Big(\underset{-\frac{l}{m}\alpha}{\beta} - \alpha^*\Big)\frac{1}{m}$$

The two linear terms at the end are exactly what is needed to render the sum doubly-periodic. In fact:

$$\zeta(z+\gamma) - \zeta(z) = l\,\gamma + m\bar{\gamma}$$

$$\zeta\left(\frac{\alpha}{m}+\frac{\mu}{m}\right) - \zeta\left(\frac{\alpha}{m}\right) = \frac{l}{m}\mu + \bar{\mu}$$

March 31, 1982

Let's recall how to compute the connection and curvature of a hermitian holomorphic line bundle. Choose a non-vanishing holom. section s locally. Then $Ds = s\theta$ where θ is of type $(1,0)$, and also $d|s|^2 = (Ds|s) + (s|Ds)$ $= (\bar\theta + \theta)|s|^2$. Thus

$$d\log|s|^2 = \theta + \bar\theta \quad \text{so} \quad \theta = d'\log|s|^2$$

and the curvature is obtained from

$$D(Ds) = D(s\theta) = s(d\theta + \theta^2) = s\,d\theta$$

Thus
$$K(D) = d\theta = d''d'\log|s|^2$$

Standard example: Consider the trivial line bundle over $\mathbb{C}$ with the section 1 having norm^2 $= e^{-|z|^2}$. Thus

$$|s|^2 = e^{-|z|^2}$$

$$\theta = d'\log|s|^2 = -d'(\bar z z) = -\bar z\,dz$$

$$K = d\theta = -d\bar z\,dz = dz\,d\bar z = -2i\,dx\,dy$$

$$\therefore \frac{i}{2\pi}K = \frac{dx\,dy}{\pi} \quad \text{is the Chern form.}$$

Now I want to apply this to the cohomology-determinant line bundle. Suppose given a Riemann surface M and a C^∞ hermitian vector bundle E and consider $\mathcal{a}$ = space of all holom. structures on E. Let's assume we are in the index 0 case: $\deg E = (g-1)\operatorname{rank}(E)$, so for most $A \in \mathcal{a}$, the $\bar\partial$ operator: $\bar\partial_A : E \to E\otimes\Omega^{0,1}$ is an isomorphism. Use the notation $\mathcal{D} = \bar\partial_A$. In the case the dual-coh-det. line bundle $\mathcal{L}^*$ has a canonical section s which is non-vanishing at those $\mathcal{D}$ which are isoms. Let's call this open set $\mathcal{a}'$.

I propose to define a metric on $\mathcal{L}^*$ over $\mathcal{A}'$ by

$$|s|^2 \text{ at } \mathcal{D} = \text{"}\det(\mathcal{D}^*\mathcal{D})\text{"} = e^{-\zeta'(0)}$$

where $\zeta(s)$ is the ζ function of $\mathcal{D}^*\mathcal{D}$ which is a strictly positive self-adjoint operator. I know from general results that $\zeta(s)$ is analytic at $s=0$, so that this formula makes sense. As long as I stay on the open set $\mathcal{A}'$ so that $\mathcal{D}^*\mathcal{D}$ doesn't acquire 0 eigenvalues, $-\zeta'(0)$ will be a smooth fn. of $\mathcal{D}$.

Now I want to compute the curvature. When I compute $-\delta\zeta'(0)$ for an inf. variation $\delta\mathcal{D}$, I am computing the differential of the function $-\zeta'(0)$. I find

$$\delta \log|s|^2 = -\delta\zeta'(0) = \text{"tr"}(\mathcal{D}^{-1}\delta\mathcal{D}) + \overline{\text{"tr"}(\mathcal{D}^{-1}\delta\mathcal{D})}$$

where $\quad \text{"tr"}(\mathcal{D}^{-1}\delta\mathcal{D}) = \text{value of } \operatorname{tr}\left((\mathcal{D}^*\mathcal{D})^{-s}\mathcal{D}^{-1}\delta\mathcal{D}\right) \text{ at } s=0.$

Therefore, comparing with $d\log|s|^2 = \theta + \bar{\theta}$, we conclude that $d'\log|s|^2 = \theta$ at a point $\mathcal{D}$ of $\mathcal{A}'$ is the complex linear function on the tangent space

$$\delta\mathcal{D} \longmapsto \text{"tr"}(\mathcal{D}^{-1}\delta\mathcal{D})$$

We know this can be written

$$\text{"tr"}(\mathcal{D}^{-1}\delta\mathcal{D}) = \int_{z\in M} \operatorname{tr}\Big(\underbrace{\lim_{t\to 0}\langle z|e^{-t\mathcal{D}^*\mathcal{D}}\mathcal{D}^{-1}|z\rangle}_{J_{\mathcal{D}}(z)}\,\delta\mathcal{D}(z)\Big)$$

Hence the connection form θ for my line bundle can be identified with the quantity $J_{\mathcal{D}}$. The curvature is $d''\theta$, and hence measures how much θ deviates from being a holomorphic 1-form, i.e.

$$\Omega^{1,0} \xrightarrow{d''} \Omega^{1,1}$$

is the $\bar{\partial}$-complex for Ω^1.

Go back to yesterday's formulas for the case of $M = \mathbb{C}/\Gamma$. We found for $D = \frac{\partial}{\partial \bar z} + \alpha$

$$J_D(0) = -\frac{1}{\pi}\alpha^*(0) + \frac{\partial}{\partial z}\left(z\langle z|D^{-1}|0\rangle\right)\Big|_{z=}$$ [redacted] 0

and an analogous formula at any other point of M. The second term is obviously holomorphic in D, and so doesn't contribute to the curvature. So I will get the same curvature if I look at the 1-form

$$\delta\alpha \longmapsto -\frac{1}{\pi}\int_M d^2 z \; tr\left(\alpha^*(z)\,\delta\alpha(z)\right)$$

and so the curvature is

$$-\frac{1}{\pi}\int d^2 z \; tr\left((d\alpha)^*(z)\; d\alpha(z)\right).$$

Let's check this when α is constant. Then I get

$$-\frac{1}{\pi}\, vol(M)\;\; d\bar\alpha\, d\alpha = +\frac{\operatorname{Im}\tau}{\pi}\, d\alpha\, d\bar\alpha$$

Our next project will be to work this out for a general Riemann surface, but again in the index 0 case. This time [redacted] we don't have a global 1-form dz, [redacted] so the formulas for D^* will be more complicated. What I will really have to understand is how the operators $\partial, \bar\partial, \partial^*, \bar\partial^*$ look on a general Riemann surface. The important case perhaps is constant curvature, i.e. the UHP, or Riemann sphere.

Upper-Half-Plane formulas: $w = \frac{az+b}{cz+d}$, $ad-bc=1$, $dw = \frac{dz}{(cz+d)^2}$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ real, then $\operatorname{Im} w = \frac{\operatorname{Im} z}{|cz+d|^2}$ so $\frac{|dw|}{\operatorname{Im} w} = \frac{|dz|}{\operatorname{Im} z}$

and so on the UHP one gets an $SL_2(\mathbb{R})$-invariant metric

$$ds = \frac{|dz|}{y} \quad \text{or} \quad ds^2 = \frac{|dz|^2}{y^2} = \frac{dx^2+dy^2}{y^2}$$

Riemann sphere: If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_2$, then

$$1+|w|^2 = \frac{|cz+d|^2+|az+b|^2}{|cz+d|^2} = \frac{1+|z|^2}{|cz+d|^2}$$

so that $\frac{|dw|}{1+|w|^2} = \frac{|dz|}{1+|z|^2}$ and so on the Riemann sphere we get a SU_2-invariant metric

$$ds = \frac{|dz|}{1+|z|^2}.$$

(Notice that $a\bar{b} + c\bar{d} = 0 \Rightarrow c = -\frac{a\bar{b}}{\bar{d}}$ so $1 = ad - bc = ad + \frac{a|b|^2}{\bar{d}} = \frac{a}{\bar{d}}$ hence any matrix in SU_2 is of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{with} \quad |a|^2+|b|^2 = 1$$

which shows clearly the isom. $SU_2 = S^3$.)

Unit circle: Take $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1,1)$ so that $|a|^2-|b|^2=1$. Then one has that $ds = \frac{|dz|}{1-|z|^2}$ is invariant under $SU(1,1)$.

Next step is to take an open set in $\mathbb{C}$ on which a Riemann metric $ds^2 = g(dx^2+dy^2)$ is given. It's important at this point to get straight the fact that ds, dx, dy here are not differentials in the usual sense. The correct interpretation is to think of having a curve $z(t)$; then its speed is $\frac{ds}{dt} = \sqrt{g}\left|\frac{dz}{dt}\right|$. Or it's tangent vector

$$\frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y}$$

has this length. In other words

$$\left| a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right|^2 = g(a^2+b^2).$$

Now transfer this metric on T by duality to T^*. The linear fnl dx on T is represented by $\frac{1}{g}\frac{\partial}{\partial x}$ since

$$\left\langle dx, a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right\rangle = a = \left(\frac{1}{g}\frac{\partial}{\partial x}, a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right)$$

hence as a section of T^* we have

$$|dx|^2 = \left|\frac{1}{g}\frac{\partial}{\partial x}\right|^2 = \frac{1}{g}$$

and similarly for dy. Hence an orthonormal basis for T^* is $\sqrt{g}\,dx$, $\sqrt{g}\,dy$ and

$$|dz|^2 = |dx + i\,dy|^2 = \frac{2}{g} \qquad |dz| = \sqrt{\frac{2}{g}}$$

Since I like g for functions I will now change g into ρ.

Let's now compute d^* using the given metrics on T^* and the volume $\rho\,dxdy$.

$$\langle df \mid g\,dx\rangle = \int \overline{\frac{\partial f}{\partial x}}\, g \underbrace{|dx|^2}_{1/\rho} \rho\, dxdy$$

$$= -\int \bar{f} \frac{\partial g}{\partial x}\, dx\, dy$$

so that $d^*(g\,dx) = -\frac{1}{\rho}\frac{\partial g}{\partial x}$

similarly $\boxed{d^*(g\,dx + h\,dy) = -\frac{1}{\rho}\left(\frac{\partial g}{\partial x} + \frac{\partial h}{\partial y}\right).}$ Thus the Laplacean on fns. is

$$\Delta f = -d^* d f = \frac{1}{\rho}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f.$$

For example the Laplacean on the UHP is $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

Next compute the adjoint of d on one-forms

$$\langle d(f\,dx + g\,dy) \mid h\,dxdy\rangle = \int \overline{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}\, h \underbrace{|dxdy|^2}_{1/\rho^2} \rho\, dxdy$$

$$= \int \left(\frac{\partial \bar{g}}{\partial x} - \frac{\partial \bar{f}}{\partial y}\right) \frac{1}{\rho} h\, dx\, dy$$

$$= \int \bar{g}\left(-\frac{\partial}{\partial x}\frac{1}{\rho} h\right) dxdy \quad + \;\ldots\ldots$$

$$= -\int \bar{g}\left(\frac{\partial}{\partial x}\frac{1}{\rho} h\right) |dy|^2 \rho\, dxdy$$

$$= \left\langle g\,dy \;\middle|\; -\frac{\partial}{\partial x}\left(\frac{1}{\rho} h\right) dy\right\rangle$$

April 2, 1982

If T^* is an oriented Euclidean vector space, then one has the **Hodge** $*$ operator on ΛT^* defined by

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle \, \mathrm{vol}$$

where $\langle\,,\,\rangle$ is the natural inner product on ΛT^*, and vol is the element in $\Lambda^m T^*$, $m = \dim T^*$, of length 1 and having the orientation.

On a Riemann surface with metric $ds^2 = \rho\,(dx^2 + dy^2)$ then we have the orth. basis $1, \sqrt{\rho}\,dx, \sqrt{\rho}\,dy, \rho\,dx\,dy$ for ΛT^* and $\mathrm{vol} = \rho\,dx\,dy$. So

$$*1 = \rho\,dx\,dy, \quad * \sqrt{\rho}\,dx = +\sqrt{\rho}\,dy, \quad * \sqrt{\rho}\,dy = -\sqrt{\rho}\,dx$$

$$*(\rho\,dx\,dy) = 1.$$

Hence $\quad * dx\,dy = \frac{1}{\rho}, \quad *1 = \rho\,dx\,dy, \quad \begin{array}{l} * dx = dy \\ * dy = -dx \end{array}$

so that $\quad * = \begin{array}{ll} -i & \text{on } \Omega^{1,0} \\ +i & \text{on } \Omega^{0,1} \end{array} \qquad *(dz) = -i\,dz \quad$ etc.

and $$\boxed{*^2 = (-1)^p \quad \text{on } \Omega^p. \quad \leftarrow \text{generally true if } m = \dim(M) \text{ is even}}$$

Next compute d^* in terms of $d, *$:

$$\begin{aligned} (d\alpha, \beta) &= \int d\alpha \wedge * \beta, && \text{suppose } \deg \alpha = p-1 \\ &= (-1)^p \int \alpha \wedge d * \beta && \text{int. by parts} \\ &= -\int \alpha \wedge *(* d * \beta) \end{aligned}$$

Thus $$\boxed{d^* = - * d * \quad \text{when } \dim(M) \text{ is even}}$$

Lets compute the Laplacean $-\Delta = d^*d + dd^*$. On fns.

$$-\Delta f = * d * df = * d * (\partial_x f\,dx + \partial_y f\,dy) = * d(\partial_x f\,dy - \partial_y f\,dx)$$

$$= * (\partial_x^2 f + \partial_y^2 f)\,dx\,dy = \frac{1}{\rho}(\partial_x^2 + \partial_y^2) f$$

On 2 forms: $\quad -\Delta(f \rho\,dx\,dy) = d * d * (f \rho\,dx\,dy) = d * df = (\partial_x^2 f + \partial_y^2 f)\,dx\,dy$

$$= \frac{1}{\rho}\left(\partial_x^2 f + \partial_y^2 f\right)\rho\, dx\, dy$$

The Laplacean on 1-forms seems to be messy and I haven't been able to get a simple formula, possibly because I don't understand the curvature.

Let's go on to the $\bar\partial$ operator:

$$\bar\partial f = \frac{1}{2}(\partial_x + i\partial_y) f \cdot dx - i dy$$

$$|d\bar z|^2 = |dx|^2 + |dy|^2 = \frac{2}{\rho}$$

$$\|\bar\partial f\|^2 = \int |\partial_{\bar z} f|^2 \cdot |d\bar z|^2 \cdot \rho\, dx\, dy$$

$$= 2\int |\partial_{\bar z} f|^2\, dx\, dy$$

similarly $\|\partial f\|^2 = 2\int |\partial_z f|^2\, dx\, dy$, so

$$\|\partial f\|^2 + \|\bar\partial f\|^2 = 2\int \left\{ |\partial_z f|^2 + |\partial_{\bar z} f|^2 \right\} dx\, dy$$

$$\frac{1}{4}\cdot 2\left(|\partial_x f|^2 + |\partial_y f|^2\right) \quad '' \quad = \int \left(|\partial_x f|^2 + |\partial_y f|^2\right) dx\, dy$$

$$= \|df\|^2$$

which checks. To find the $\bar\partial$ Laplacean; integrate by parts

$$\|\bar\partial f\|^2 = 2\int \overline{\partial_{\bar z} f}\; \partial_{\bar z} f\; dx\, dy = -2\int \bar f\, \partial_z \partial_{\bar z} f\, dx\, dy$$

$$= -2\int \bar f \left(\frac{1}{\rho}\partial_z \partial_{\bar z} f\right) \rho\, dx\, dy$$

Hence

$$\boxed{-\bar\partial^* \bar\partial = +2\frac{1}{\rho}\partial^2_{z\bar z} = \frac{1}{2}\frac{1}{\rho}\left(\partial_x^2 + \partial_y^2\right)}$$

on functions

and so it seems that the effect of the non-flat metric will be to introduce the function $\frac{1}{\rho}$ into the calculations. New heat kernel

$$\langle z| \qquad |y\rangle = \qquad e^{-\rho(y)|z-y|^2/t} \cdots$$

April 2, 1982

Atiyah-Bott-Patodi (Inv. Math 19(1973)).

Seeley's method for obtaining the asymptotic expansion of the heat kernel. Take A to be a constant coefficient s.a. non-negative elliptic operator of order $2m$ over a torus T. Then

$T = (\mathbb{R}/2\pi\mathbb{Z})^n$
dual lattice is $\mathbb{Z}^n$.

$$tr\left(e^{-tA}\right) = \sum_{\xi\in\mathbb{Z}^n} e^{-tA(\xi)}$$

$$\sim \int e^{-tA(\xi)}\, d^n\xi$$

Write $\quad A(\xi) = A_m(\xi) + E(\xi) \quad , \qquad \deg E < 2m.$

$$\int e^{-tA_m(\xi)} \sum \frac{(-t)^\alpha}{\alpha!} E(\xi)^\alpha \; d^n\xi$$

do the integral over polar coordinates. This gives things like

$$t^\alpha \int_0^\infty e^{-tr^{2m}} r^k\, r^{n-1} dr = c t^\alpha \int_0^\infty e^{-tr}\, r^{(k+n)/2m} \frac{dr}{r}$$

$$= c\, t^{-(k+n)/2m + \alpha} \qquad\qquad k < 2m\alpha.$$

and so one gets an expansion

$$\langle x | e^{-tA} | x\rangle \sim \sum t^k \underset{\uparrow \text{ local measures}}{\mu_k(A)} \sim (2\pi)^{-n} \int e^{-tA(\xi)} d^n\xi.$$

Seeley's general formula

$$\sum t^k \mu_k(A) \sim (2\pi)^{-n} \lim_{\varepsilon\to 1} \int e^{-\lambda t} \left(A_\varepsilon - \lambda\varepsilon^{-2m}\right)^{-1} d\lambda\; d\xi$$

$$A_\varepsilon = e^{-ix\xi/\varepsilon} A\, e^{ix\xi/\varepsilon} = A(\xi) + \frac{1}{\varepsilon} A_1(\xi) + \cdots + \frac{1}{\varepsilon^{2m}} A_{2m}(\xi)$$

Invert $A_\varepsilon - \lambda\varepsilon^{-2m}$ formally, integrate λ over a contour containing $\mathbb{R}_{>0}$, then over ξ, then set $\varepsilon = 1$.

April 3, 1982

Over a Riemann surface M I consider a holom. vector bundle with hermitian inner product E, and let $\bar\partial : E \to E \otimes T^{01}$ be the assoc. $\bar\partial$ operator, and $\bar\partial^*$ the adjoint, a volume supposed given over M. I want to compute $\zeta_{\bar\partial^*\bar\partial}(0)$, and $\zeta_{\bar\partial\bar\partial^*}(0)$. Quite generally

$$\zeta_{\bar\partial^*\bar\partial}(0) = -\dim \operatorname{Ker}\bar\partial + \text{local integral obtained from the heat kernel of } \bar\partial^*\bar\partial$$

$$\zeta_{\bar\partial\bar\partial^*}(0) = -\dim \operatorname{Cok}\bar\partial + \text{local integral from } \bar\partial\bar\partial^*$$

and the two ζ fns. coincide because the positive eigenvalues of $\bar\partial^*\bar\partial$ and $\bar\partial\bar\partial^*$ are the same. Hence we will get a local integral formula for the index of $\bar\partial$, which should be the RR thm.

To calculate the local terms I can work in an open set with coordinate z and choose an orthonormal basis for E over this open set, whence I can write

$$\bar\partial f = \mathcal{D} f \, d\bar z \qquad \mathcal{D} = \partial_{\bar z} + \alpha$$

where α is a matrix of functions. Suppose the volume on M is $\rho\, dx dy$, whence $\sqrt{\rho}\, dx$, $\sqrt{\rho}\, dy$ form an orthonormal base for T^*, and

$$|d\bar z|^2 = |dx|^2 + |dy|^2 = \frac{2}{\rho}$$

Thus if I want to compute $\bar\partial^*$

$$\left(\bar\partial f \,\middle|\, g\, d\bar z\right) = \int (\partial_{\bar z} + \alpha) f^* g \overset{2/\rho}{\boxed{|d\bar z|^2}} \rho\, dx dy$$

$$= 2\int (\partial_{\bar z} + \alpha) f^* g \, dx\, dy = \int f^* \frac{2}{\rho}(-\partial_z + \alpha^*) g \,\rho\, dx dy$$

$$= \left(f \,\middle|\, \frac{2}{\rho}(-\partial_z + \alpha^*) g\right).$$

Thus

$$\boxed{-\bar\partial^*(g\, d\bar z) = \frac{2}{\rho}\tilde{\mathcal{D}} g \qquad \tilde{\mathcal{D}} = \partial_z - \alpha^*}$$

(The reason I like $\tilde{\mathcal{D}}$ is that the canonical connection $D : E \rightarrow E \otimes T^*$ on E is
$$D = d + (-\alpha^* dz + \alpha d\bar{z})$$
$$= \tilde{\mathcal{D}}\, dz + \mathcal{D}\, d\bar{z}\,.)$$

So now I want to compute the ~~[illegible]~~ constant term of the asymptotic expansion of the kernel of e^{-tA} restricted to the diagonal, where
$$A = \mathcal{D}^* \mathcal{D} = -\frac{2}{\rho} \tilde{\mathcal{D}} \mathcal{D}.$$

This is going to be like our calculation for the ~~[illegible]~~. elliptic curve case except that now $\frac{2}{\rho}$ is not constant.

We now must understand the analysis behind the asymptotic solution of the heat equation. I want to especially understand the example over $\mathbb{C}$
$$\frac{\partial u}{\partial t} = a\, \partial_z \partial_{\bar{z}}\, u$$
where $a > 0$ needn't be constant. Simpler might be $\partial_t u = a \partial_x^2 u$ over $\mathbb{R}$. There is the Feynman integral approach, and probably some sort of slick ΨDO approach, also your pedestrian resolvent techniques.

April 4, 1982

[redacted] To understand asymptotics of the heat equations.

Claim: The theory of the Laplace transform gives a relation between small t asymptotics for e^{-tA} and large λ asymptotics for the resolvent $\frac{1}{\lambda - A}$.

We have

$$\boxed{\begin{matrix}\langle\ \rangle \\ = \langle x|\ |x\rangle\end{matrix}} \qquad \underbrace{\langle e^{-tA}\rangle}_{f(t)} = \frac{1}{2\pi i}\int e^{-t\lambda}\left\langle\frac{1}{\lambda - A}\right\rangle d\lambda = \frac{1}{2\pi i}\int e^{t\lambda}\underbrace{\left\langle\frac{1}{\lambda + A}\right\rangle}_{g(\lambda)} d\lambda$$

so that

$$g(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt$$

If f has an asymptotic expansion as $t \downarrow 0$

$$f(t) \approx \sum_{[redacted]} a_m t^m$$

then g [redacted] has an asymptotic expansion as $Re(\lambda) \to +\infty$.

$$g(\lambda) \sim \sum_{[redacted]} a_m \frac{\Gamma(m+1)}{\lambda^{m+1}}$$

The converse is also true: suppose for example

$$g(\lambda) = \frac{a}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \qquad \text{as} \quad Re(\lambda) \to \infty$$

Then

$$f(t) = \underbrace{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\frac{a}{\lambda}\,d\lambda}_{\substack{\text{conditionally cvgt.} \\ \text{but OK if [redacted]} \\ \text{deformed to}}} + \underbrace{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t}\, O\left(\frac{1}{\lambda^2}\right) d\lambda}_{\substack{\text{cnvt as } t \to 0 \\ \text{and then it must} \\ \text{be zero because one} \\ \text{can push } c \to +\infty.}}$$

so that $f(t) \longrightarrow a$ as $t \downarrow 0$.

An interesting point whose significance I don't yet understand is that

$$g(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\,dt$$

doesn't make immediate sense unless $f(t)$ is integrable near $t=0$. So in the interesting cases like $e^{t\Delta}$ in 2-dims. we have problems defining g, i.e. the resolvent $\langle x|\frac{1}{\lambda-A}|x\rangle$ along the diagonal. Nevertheless Seeley is able to get at the heat kernel on the diagonal using the resolvent.

I want to understand this for the operator $A=-a\Delta$ where Δ is the Laplacean and a is a positive function. The first step is to construct a **parametrix** for the resolvent $\frac{1}{\lambda-A}$ of the form

$$(Bf)(x)=\int\frac{d^n\xi}{(2\pi)^n}\;B(x,\xi)\,e^{-i(x\xi)}\hat{f}(\xi)$$

where

$$B(x,\xi)=\frac{1}{\lambda-a(x)\xi^2}+\cdots\cdots$$

In other words I am approximating $\frac{1}{\lambda-A}$ by the operator B_0 with kernel

$$\langle x|B_0|y\rangle=\int\frac{d^n\xi}{(2\pi)^n}\;\frac{1}{\lambda-a(x)\xi^2}\,e^{i\xi(x-y)}$$

Notice for $\lambda\notin\mathbb{R}_{\geq 0}$ the integrand is defined for all ξ, so there is no problem integrating it against $\hat{f}(\xi)$, however for $n\geq 2$ one can't set $x=y$.

Seeley's method **at this** point is to do the contour integration over λ, before the ξ integration

$$\frac{1}{2\pi i}\int\frac{e^{-\lambda t}}{(\lambda-a(x)\xi^2)^{n+1}}d\lambda=\frac{1}{2\pi i}\int\frac{e^{-(\lambda+a(x)\xi^2)t}}{\lambda^{n+1}}\,d\lambda$$

$$=e^{-ta(x)\xi^2}\frac{(-t)^n}{n!}$$

So what should be true is that the resolvent is given by a $\tilde{B}(x,\xi)$ having an asymptotic expansion

$$B(x,\xi)\sim\frac{1}{\lambda-a(x)\xi^2}+\frac{b_1(x,\xi)}{(\lambda-a(x)\xi^2)^2}+\frac{b_2(x,\xi)}{(\lambda-a(x)\xi^2)^2}+\cdots$$

which leads some expansion for the heat kernel with terms

$$\frac{(-t)^n}{n!}\int \frac{d^n\xi}{(2\pi)^n}\; \underline{e^{-ta(x)\xi^2} b(x,\xi)}\; e^{i\xi(x-y)}$$

$$= \frac{(-t)^n}{n!}\frac{1}{(\sqrt{t})^n}\int \frac{d^n\xi}{(2\pi)^n}\; e^{-a(x)\xi^2}\, b(x, \xi/\sqrt{t})\, e^{i\xi\left(\frac{x-y}{\sqrt{t}}\right)}$$

All this leads to a basic Gaussian type formula with the exponent $e^{-\frac{(x-y)^2}{4a(x)t}}$

Problem: If I try to write down an asymptotic formula for $\langle x|e^{-tA}|y\rangle$ of the form

$$\frac{1}{t^{n/2}}\, e^{-\frac{(x-y)^2}{4a(x)t}}\left(a_0(x,y) + a_1(x,y)\,t + \cdots\right)$$

then by symmetry I expect an asymptotic formula of the form

$$\frac{1}{t^{n/2}}\, e^{-\frac{(x-y)^2}{4a(y)t}}\left(a_0(y,x) + a_1(y,x)\,t + \cdots\right)$$

Unfortunately it doesn't seem to be possible to express

$$e^{-\frac{(x-y)^2}{4a(y)t}} \quad \text{asymptotically as} \quad e^{-\frac{(x-y)^2}{4a(x)t}}\left(1 + b_1(x,y)\,t + \cdots\right)$$

but maybe you are missing something. In any case it appears that to expect

$$\langle x|e^{-tA}|y\rangle = \frac{1}{t^{n/2}}\, e^{-\frac{(x-y)^2}{4a(x)t}}\, A(t,x,y),$$

with **A** having an asymptotic expansion in t for $t \downarrow 0$ [redacted] as a C^∞ fn. of (x,y), is too much.

April 5, 1982

Notation: volume $= \rho\, dx\, dy$ so that $\sqrt{\rho}\, dx, \sqrt{\rho}\, dy$ is an orthonormal basis for T^*. $\mathcal{D} = \partial_{\bar z} + \alpha$, $\tilde{\mathcal{D}} = \partial_z - \alpha^*$ and $\mathcal{D}^* = -\frac{2}{\rho}\tilde{\mathcal{D}}$. Put $a = \frac{2}{\rho}$. We put

$$\langle z | e^{ta\tilde{\mathcal{D}}\mathcal{D}} | 0 \rangle = \frac{1}{t} e^{-\frac{u}{t}} A$$

and try to choose $u(z)$ so that A has an asymptotic exp. as $t \downarrow 0$. With $\phi = \frac{1}{t}e^{-u/t}$ we have

$$\phi^{-1}(\partial_t - a\tilde{\mathcal{D}}\mathcal{D})\phi = \partial_t - \frac{1}{t} + \frac{u}{t^2} - a\left(\tilde{\mathcal{D}} - \frac{1}{t}\partial_z u\right)\left(\mathcal{D} - \frac{1}{t}\partial_{\bar z} u\right)$$

$$= \frac{1}{t^2}\left(u - a\ \partial_z u\, \partial_{\bar z} u\right) + \frac{1}{t}\left(-1 + a\,\partial^2_{z\bar z}u \underset{\wedge}{+}{}^{a}\!\left(\partial_z u\cdot\mathcal{D} + \partial_{\bar z}u\,\tilde{\mathcal{D}}\right)\right)$$

$$+ \partial_t - a\tilde{\mathcal{D}}\mathcal{D}$$

We choose u so the coeff of t^2 vanishes

$$u = a\left|\frac{\partial u}{\partial z}\right|^2 = \frac{a}{4}\left[(\partial_x u)^2 + (\partial_y u)^2\right]$$

Now ∇u is the $\underset{\wedge}{}^{\text{tgt}}$ vector corresponding to $du = \frac{1}{\sqrt{\rho}}\partial_x u(\sqrt{\rho}\,dx) + \frac{1}{\sqrt{\rho}}\partial_y u \underset{\wedge}{}(\sqrt{\rho}\,dy)$

so
$\circledast$ $$\nabla u = \frac{1}{\sqrt{\rho}}\partial_x u\left(\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial x}\right) + \frac{1}{\sqrt{\rho}}\partial_y u\left(\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial y}\right)$$

hence $$|\nabla u|^2 = \frac{1}{\rho}\left[(\partial_x u)^2 + (\partial_y u)^2\right]$$ and so

$$u = \frac{1}{2}|\nabla u|^2 \quad \text{or} \quad |\nabla u^{1/2}|^2 = \left|\frac{1}{2}u^{-1/2}\nabla u\right|^2 = \frac{1}{4}\frac{1}{u}2u = \frac{1}{2}$$

or $|\nabla(\sqrt{2}u^{1/2})| = 1$. Thus $\sqrt{2}u^{1/2} = r(z) =$ distance of z from 0.

$$u(z) = \frac{1}{2}r(z)^2$$

(Check: If $\rho = 1$, then $a = 2$, $a\tilde{\mathcal{D}}\mathcal{D} \sim 2\,\partial^2_{z\bar z}$ so $u = \frac{1}{2}|z|^2$.)

Next I want to interpret $a(\partial_z u\cdot\mathcal{D} + \partial_{\bar z}u\,\tilde{\mathcal{D}})$. Recall that $D = \mathcal{D}\,d\bar z + \tilde{\mathcal{D}}\,dz$, hence the former operator is D contracted with respect to the vector field

$$a\left(\partial_z u\frac{\partial}{\partial \bar z} + \partial_{\bar z}u\frac{\partial}{\partial z}\right) = \overset{a/2=}{\frac{1}{\rho}}\left(\partial_x u\,\partial_x + \partial_y u\,\partial_y\right)$$

which we have seen above $\circledast$ is ∇u.

$$\therefore\ a\left(\partial_z u\,\mathcal{D} + \partial_{\bar z}u\,\tilde{\mathcal{D}}\right) = i(\nabla u)D$$

$\nabla u = r$ (unit vector in outward radial direction)

Now $A = A_0 + tA_1 + \cdots$ where

$$\begin{cases} \left(\left(-1 + a\partial^2_{z\bar{z}} u \right) + i(\nabla u) D \right) A_0 = 0 \\ \left(a\partial^2_{z\bar{z}} u + i(\nabla u) D \right) A_1 - a\tilde{\mathcal{D}}\mathcal{D} A_0 = 0 \end{cases}$$

Note that $-1 + a\,\partial^2_{z\bar{z}} u$ is a scalar, hence if we solve

$$i(\nabla u)\, d\log(g) = -1 + a\partial^2_{z\bar{z}} u$$

we will have

$$i(\nabla u) D (g A_0) = g\left(i(\nabla u)\cdot D + i(\nabla u) d\log g \right) A_0 = 0,$$

which means that gA_0 is flat in the radial directions.

I want to work everything out in the case of a flat line bundle, whence $\mathcal{D} = \partial_{\bar{z}}$ and $A_0 = \frac{\text{const}}{g}$.

Pseudo-differential operators.

Formula for composition:

$$\boxed{P(x,\xi) \circ Q(x,\xi) = \sum \frac{1}{\alpha!} \partial^\alpha_\xi P \circ D^\alpha_x Q}$$

where $D_x = \frac{1}{i}\partial_x$. Check for diffl. ops.

$$D^m(fu) = \sum_\alpha \binom{m}{\alpha} D^\alpha f\, D^{m-\alpha} u$$

$$= \sum_\alpha \frac{1}{\alpha!} D^\alpha f \left(\frac{m!}{(m-\alpha)!} D^{m-\alpha} \right) u$$

$$\therefore\ \xi^m \circ f = \sum_\alpha \frac{1}{\alpha!} D^\alpha f \frac{\partial^\alpha}{\partial\xi} \xi^m$$

Check formally

Take $n=1$

$$\langle x | P(x,D) Q(x,D) | z \rangle = \int \frac{d\xi\, dy}{(2\pi)} P(x,\xi) e^{i\xi(x-y)} \int \frac{d\eta}{2\pi} Q(y,\eta) e^{i\eta(y-z)}$$

$$= \int \frac{d\eta}{2\pi} \int \frac{d\xi\, dy}{2\pi} P(x,\xi) Q(y,\eta)\, e^{-i\xi x + i(\eta - \xi) y - i\eta z}$$

$$= \int \frac{d\eta}{2\pi} \int \frac{d\xi\, dy}{2\pi}\, P(x, \xi+\eta)\, Q(y,\eta)\, e^{i\eta(x-z) + i\xi(x-y)}$$

$$= \int \frac{d\eta}{2\pi} \int \frac{d\xi\, dy}{2\pi}\, \underbrace{P(x, \xi+\eta)}_{\sum \frac{1}{\alpha!} \partial_\eta^\alpha P(x,\eta)\, \xi^\alpha}\, Q(y+x,\eta)\, e^{-i\xi y}\, e^{i\eta(x-z)}$$

So you need
$$\int \frac{d\xi}{2\pi}\, \xi^\alpha \int dy\, e^{-i\xi y}\, Q(y+x,\eta) = D_y^\alpha\, Q(y+x,\eta)\Big|_{y=0}$$
whence you get
$$\int \frac{d\eta}{2\pi} \sum_\alpha \frac{1}{\alpha!} \partial_\eta^\alpha P(x,\eta)\, D_x^\alpha Q(x,\eta)\ e^{i\eta(x-z)}$$

So now let's try the Seeley method to compute e^{-tA} where $A = -a\partial_x^2$. We want the resolvent $(\lambda - A)^{-1}$ and so we look for a formal series $B = B_0 + B_1 + \cdots$ such that
$$(\lambda - a\xi^2) \circ (B_0 + B_1 + \cdots\cdots) = 1$$
So we start with $B_0 = \frac{1}{\lambda - a\xi^2}$. Then
$$(\lambda - a\xi^2) \circ \left(\frac{1}{\lambda - a\xi^2}\right) = 1 - 2a\xi \cdot D_x\left(\frac{1}{\lambda - a\xi^2}\right) - a\, D_x^2\left(\frac{1}{\lambda - a\xi^2}\right)$$
so we can take
$$B_1 = \frac{2a}{\lambda - a\xi^2}\, \xi \cdot D_x\left(\frac{1}{\lambda - a\xi^2}\right) + \frac{a}{\lambda - a\xi^2}\, D_x^2\left(\frac{1}{\lambda - a\xi^2}\right).$$
Now the problem with this is that as I differentiate I bring ξ-terms into the numerator. How can I estimate what will happen? A typical term in B_1 involves
$$\frac{\xi^3}{(\lambda - a\xi^2)^3} \quad \text{or} \quad \frac{\xi^4}{(\lambda - a\xi^2)^4} \quad \text{or} \quad \frac{\xi^2}{(\lambda - a\xi^2)^3}$$
and in B_2 we will apply $\frac{\xi D_x}{\lambda - a\xi^2}$ to get

[redacted] $\frac{\xi^6}{(\lambda - a\xi^2)^5}$ or $\frac{\xi^7}{(\lambda - a\xi^2)^6}$ $\frac{\xi^5}{(\lambda - a\xi^2)^5}$

or we apply $\frac{D_x^2}{\lambda - a\xi^2}$ to get

$\frac{\xi^7}{(\lambda - a\xi^2)^6}$ or $\frac{\xi^8}{(\lambda - a\xi^2)^7}$

Now we want to use

$$\frac{1}{2\pi i}\int e^{-\lambda t}\frac{1}{(\lambda - a\xi^2)^{n+1}}\,d\lambda = e^{-a\xi^2 t}\frac{(-t)^n}{n!}$$

so $\frac{1}{(\lambda - a\xi^2)^n}$ counts t^{n-1}. But a ξ^k in the [redacted] numerator $\int e^{-a\xi^2 t}\xi^k\,d\xi \sim \frac{t^{k/2}}{t^{n/2}}$ ← fixed.

So count t powers:

$\frac{\xi^3}{(\lambda - a\xi^2)^3} \longrightarrow t^{-3/2}t^2$ $\frac{\xi^4}{(\lambda - a\xi^2)^4} \longrightarrow t^{-2+3}$ $\begin{matrix}2\\3\end{matrix} \mapsto t^{-1+2}$

$\begin{matrix}6\\5\end{matrix} \longrightarrow t^{-3+4}$ $\begin{matrix}7\\6\end{matrix}$ $t^{-7/2+5}$

$\begin{matrix}8\\7\end{matrix} \longrightarrow t^{-4+6}$

so the things contributing to coefficient of t are the terms $\cancel{\frac{\xi^3}{(\lambda - a\xi^2)^3}}$ $\frac{\xi^4}{(\lambda - a\xi^2)^4}$ $\frac{\xi^2}{(\lambda - a\xi^2)^3}$ of B_1

and the term $\frac{\xi^6}{(\lambda - a\xi^2)^5}$ of B_2. ~~So~~ in principle it is clear we will get the desired [redacted] series. Plot $\frac{b\xi^k}{(\lambda - a\xi^2)^l}$ and then what the operators $\frac{1}{\lambda - a\xi^2}\xi D_x$, $\frac{1}{\lambda - a\xi^2}D_x^2$ can do [redacted] to k, l: $k, l \longmapsto (k+1, l+1), (k+2, l+2)$

$\longmapsto (k, l+1), (k+2, l+2), (k+4, l+3)$

April 6, 1982

Riemann geometry formulas

$ds^2 = g_{ab}\, dx^a dx^b$ means $\langle X_a | X_b \rangle = g_{ab}$ where $X_a = \frac{\partial}{\partial x^a}$. What vector field corresp. to the 1-form dx^a? Put $dx^a \longleftrightarrow g^{ac} X_c$. Then

$$\delta^a_b = \langle X_b | dx^a \rangle = \langle X_b | g^{ac} X_c \rangle = g^{ac} g_{bc}$$

so g^{ac} is the inverse matrix to g_{ab}.

Geodesic equations:

$$E(x(t)) = \int \frac{1}{2} g_{ab}\, \dot{x}^a \dot{x}^b \, dt$$

Euler equations are

$$\underbrace{\frac{d}{dt}\left(g_{cb}\dot{x}^b\right)} = \frac{1}{2} \frac{\partial}{\partial x^c} g_{ab}\, \dot{x}^a \dot{x}^b$$

$$g_{cb}\ddot{x}^b + \frac{\partial}{\partial x^a} g_{cb}\, \dot{x}^a \dot{x}^b$$

or

$$g_{cb}\ddot{x}^b = \left(\frac{1}{2}\frac{\partial}{\partial x^c} g_{ab} - \frac{\partial}{\partial x^a} g_{cb}\right)\dot{x}^a \dot{x}^b$$

$$= \underbrace{\left(\frac{1}{2}\frac{\partial}{\partial x^c} g_{ab} - \frac{1}{2}\frac{\partial}{\partial x^a} g_{cb} - \frac{1}{2}\frac{\partial}{\partial x^b} g_{ca}\right)}_{-\Gamma_{c,ab}} \dot{x}^a \dot{x}^b$$

or

$$\boxed{\ddot{x}^c = -\Gamma^c_{ab}\, \dot{x}^a \dot{x}^b}$$

~~Jacobi fields represent first order variations of a geodesic. $Y = y^a X_a$ is a Jacobi field along the geodesic $x(t)$ when~~

~~$$\ddot{y}^c = \Gamma^c_{ab}\, \dot{x}^a \dot{y}^b$$~~

Covariant differentiation: $\nabla_a(X_b) = \Gamma^c_{ab} X_c$ where The Γ^c_{ab} should turn out the same as above. Here $\nabla_a = \nabla_{X_a}$.

The Levi-Civita connection preserves metric:

$$\frac{\partial}{\partial x^c} g_{ab} = \langle \nabla_c X_a | X_b \rangle + \langle X_a | \nabla_c X_b \rangle$$

$$\frac{\partial}{\partial x^c} g_{ab} = \Gamma^{e}_{ca}\, g_{eb} + \Gamma^{e}_{cb}\, g_{ea} = \Gamma_{b,ca} + \Gamma_{a,cb}$$

and has 0 torsion: $\nabla_X(Y) - \nabla_Y(X) = [X,Y] \Rightarrow \Gamma_{c,ab} = \Gamma_{c,ba}$

$$\frac{\partial}{\partial x^c} g_{ab} = \cancel{\Gamma_{b,ca}} + \cancel{\Gamma_{a,bc}}$$

$$-\frac{\partial}{\partial x^a} g_{bc} = -\Gamma_{c,ab} + {}^{-}\cancel{\Gamma_{b,ca}}$$

$$-\frac{\partial}{\partial x^b} g_{ca} = -\cancel{\Gamma_{a,bc}} + {}^{-}\Gamma_{c,ab}$$

$$\frac{\partial}{\partial x^c} g_{ab} - \frac{\partial}{\partial x^a} g_{bc} - \frac{\partial}{\partial x^b} g_{ca} = -2\Gamma_{c,ab}$$

$$\boxed{\Gamma_{c,ab} = \frac{1}{2}\left(\frac{\partial g_{bc}}{\partial x_a} + \frac{\partial g_{ca}}{\partial x_b} - \frac{\partial g_{ab}}{\partial x_c}\right)}$$

Let's compute the Laplacean on functions. Need volume. We have $\langle dx^a | dx^b \rangle = g^{ab}$. Let h_{ab} be a positive square root of g_{ab}: $g_{ab} = h_{ab} h_{bc}$. Then

$$\langle h_{da} dx^a | h_{cb} dx^b \rangle = h_{da} g^{ab} h_{bc} = \delta_{dc}$$

so $h_{ab} dx^a$ is an orthonormal basis for T^*, hence

$$vol = h_{1b} dx^b \wedge \dots \wedge h_{nb} dx^b = \det(h)\, dx^1 \dots dx^n$$

$$\boxed{vol = \sqrt{\det(g)}\, dx^1 \dots dx^n}$$

Now $\langle df | h_a dx^a \rangle = \int \frac{\partial f}{\partial x^b} h^a g^{ab} \sqrt{\det g}\; d\overset{n}{x}$

$$= -\int f \frac{1}{\sqrt{\det}} \left(\frac{\partial}{\partial x^b} \sqrt{\det}\, g^{ab} h_a \right) \sqrt{\det g}\; d^n x$$

$$\therefore -d^*(h_a dx^a) = + \frac{1}{\sqrt{\det}} \left(\frac{\partial}{\partial x^b} \sqrt{\det}\, g^{ab} h_a \right)$$

$$\Delta f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^b} \sqrt{\det g}\; g^{ab} \frac{\partial}{\partial x^a} f$$

Calculation shows that

$$\Delta f = \left(g^{ab} \frac{\partial^2}{\partial x^a \partial x^b} - g^{ij} \Gamma^a_{ij} \frac{\partial}{\partial x^a} \right) f$$

April 7, 1982:

Goal: to understand the Patodi formulas in the case of a Riemann surface.

The key idea seems to be to write the Laplacean for the $\bar\partial$ complex $E \otimes \Omega^{0,*}$ of a hermitian holom. bundle in terms of the covariant differentiation operators on the bundle. Let's start with the trivial bundle.

I recall that for a Riemann surface with volume element $\rho dx dy$ one has $|dz|^2 = \frac{2}{\rho}$. Put $g = \frac{\rho}{2}$. Then

$$\text{vol} = g \, i dz d\bar z \qquad |dz|^2 = g^{-1}$$

[illegible] (In general a hermitian metric on the tangent bundle of a complex manifold is given in the form $g_{ab} dz^a d\bar z^b$, [illegible] i.e. $\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^b}\right) = g_{ab}$. So in dim. 1, $\left|\frac{\partial}{\partial z}\right|^2 = g$, hence $g = \frac{\rho}{2}$ in the old notation.) Presumably a Kähler manifold is one such that the two connections on the tangent bundle, one as a holom. bdle with metric, other from Levi-Civita, coinc

So $\Omega^{1,0}$ has the metric $|dz|^2 = g^{-1}$, hence

$$\nabla_{\bar z}(dz) = 0 \qquad \nabla_z (dz) = (\partial_z \log g^{-1}) dz,$$

I am writing

~~$D(dz)$ [illegible] ∂dz [illegible] $\theta = \partial \log g^{-1}$~~

$$\begin{cases} Ds = \theta s & \theta = \partial \log |s|^2 \\ \nabla_z = i(dz) D, & \nabla_{\bar z} = i(d\bar z) D. \end{cases}$$

Similarly

$$\nabla_z (d\bar z) = 0 \qquad \nabla_{\bar z}(d\bar z) = (\partial_{\bar z} \log g^{-1}) d\bar z$$

Now that we understand covariant differentiation in the bundles $\Omega^{p,q}$, let's compute the two Laplaceans for the complex $\bar\partial : \Omega^{0,0} \longrightarrow \Omega^{0,1}$.

$$\bar\partial f = (\partial_{\bar z} f)\, d\bar z$$

$$(\bar\partial f \mid \varphi\, d\bar z) = \int \overline{\partial_{\bar z} f}\, \varphi \underbrace{|d\bar z|^2 \, vol}_{\cancel{g^{-1} g}\, i dz d\bar z} = -\int \bar f\, g^{-1} \partial_z \varphi \;\; vol$$

Hence

$$\boxed{\bar\partial f = (\partial_{\bar z} f)\, dz \qquad -\bar\partial^*(\varphi\, d\bar z) = g^{-1}\partial_z \varphi}$$

and so the Laplacean on functions + (0,1)-forms is

$$-\bar\partial^* \bar\partial f = g^{-1} \partial^2_{z\bar z} f \qquad -\bar\partial\bar\partial^*(\varphi\, d\bar z) = \partial_{\bar z}(g^{-1}\partial_z \varphi)\, d\bar z$$

$$= g^{-1} \nabla_z \nabla_{\bar z} f$$

Let's write the Laplacean on (0,1)-forms in terms of the covariant diffn.

$$[redacted]\ \nabla_{\bar z} \nabla_z (\varphi\, d\bar z) = [redacted]\ \nabla_{\bar z}(\partial_z \varphi\ [redacted]\ d\bar z)$$

$$= [redacted]\ (\partial^2_{\bar z z} \varphi + \partial_{\bar z} \log(g^{-1})\, \partial_z \varphi)\, d\bar z$$

$$= g\, \partial_{\bar z}\, g^{-1} \partial_z \varphi\, d\bar z$$

or $\quad g^{-1}\nabla_{\bar z} \nabla_z (\varphi\, d\bar z) = \partial_{\bar z}(g^{-1}\partial_z \varphi)\, d\bar z$

Hence the Laplacean on (0,1) forms is $g^{-1}\nabla_{\bar z}\nabla_z$. But

$$[redacted] \quad \nabla_{\bar z}\nabla_{z} = \nabla_z \nabla_{\bar z} + [\nabla_{\bar z}, \nabla_z]$$

$$= \nabla_z \nabla_{\bar z} + \underbrace{[\partial_{\bar z} + \partial_{\bar z} \log g^{-1}, \partial_z]}_{-\partial_z \partial_{\bar z} \log g^{-1}}$$

So

$$\boxed{\begin{aligned} \Delta \text{ on } \Omega^{0,0} &= \frac{1}{g} \nabla_z \nabla_{\bar z} \\ \Delta \text{ on } \Omega^{0,1} &= \frac{1}{g} \nabla_z \nabla_{\bar z} + \frac{1}{g} \partial^2_{z\bar z} \log g \end{aligned}}$$

Next consider $\bar\partial : E \longrightarrow E\otimes\Omega^{0,1}$. Let $s_1,\dots,s_n$ be a local basis of holomorphic sections for E. Think $\mathbf{s}=(s_1,\dots,s_n)$ as a [redacted] row vector and a typical section of E as sf where f is a column vector of holom. functions. Let $h=s^\dagger s$ denote the matrix of inner products $\langle s_a|s_b\rangle$ for the given hermitian product on E. Hence

$$\langle sf_1|sf_2\rangle = f_1^\dagger s^\dagger s f_2$$

Now calculate the Laplacean:

$$\bar\partial(sf) = s\,\partial_{\bar z} f\, d\bar z$$

$$-(\bar\partial(sf)|\,s\varphi d\bar z) = \int -(\partial_{\bar z}f)^\dagger s^\dagger s\varphi\, \overbrace{|d\bar z|^2}^{i dz d\bar z} vol$$

$$= \int f^\dagger s^\dagger s\,(s^\dagger s)^{-1}\partial_z(s^\dagger s\,\varphi)\ \frac{1}{g}\ vol.$$

$$\therefore\quad -\bar\partial^*(s\varphi d\bar z) = s\cdot g^{-1}h^{-1}\partial_z h\varphi$$

$$-\bar\partial^*\bar\partial(sf) = -\bar\partial^*(s\,\partial_{\bar z}f\,d\bar z) = s\,g^{-1}h^{-1}\partial_z h\,\partial_{\bar z}f$$

I need the covariant differentiation operator on E.

$\nabla_{\bar z}(sf) = s\,\partial_{\bar z}f$ since s is holom. Put $\nabla_z s = s\alpha$

$$\partial_z(s^\dagger s) = \underbrace{\nabla_z s^\dagger}_{(\nabla_{\bar z}s)^\dagger = 0}\cdot s + s^\dagger\nabla_z s = s^\dagger\nabla_z s = s^\dagger s\alpha$$

so that
$$\nabla_z(sf) = s(\partial_z f + \alpha f) \qquad \alpha = (s^\dagger s)^{-1}\partial_z(s^\dagger s)$$
$$= s\,h^{-1}\partial_z h f \qquad\qquad = h^{-1}\partial_z h$$

Hence on E one has

$$\boxed{-\bar\partial^*\bar\partial = g^{-1}\nabla_z\nabla_{\bar z} \quad \text{on } E}$$

On $E\otimes\Omega^{0,1}$ we have

$$\nabla_{\bar z}(s\varphi d\bar z) = (s\,h^{-1}\partial_z(h\varphi))d\bar z$$ [redacted]

$$\nabla_{\bar z}(s\varphi\, d\bar z) = s\,\nabla_{\bar z}(\varphi\, d\bar z) = s\, g\,\partial_{\bar z} g^{-1}\varphi\; d\bar z$$

Laplacian on $E\otimes\Omega^{0,1}$ is

$$-\bar\partial\,\bar\partial^*(s\varphi\, d\bar z) = +\bar\partial(s g^{-1}h^{-1}\partial_z h\,\varphi)$$

$$= s\;\partial_{\bar z}(g^{-1}h^{-1}\partial_z h\,\varphi)\; d\bar z$$

$$\nabla_{\bar z}\nabla_z(s\varphi\, d\bar z) = \nabla_{\bar z}\, s(h^{-1}\partial_z h\varphi)\, d\bar z$$

$$= s\, g\,\partial_{\bar z} g^{-1}h^{-1}\partial_z h\,\varphi\; d\bar z$$

so we get

$$-\bar\partial\bar\partial^* = \frac{1}{g}\nabla_{\bar z}\nabla_z \qquad \text{on } E\otimes\Omega^{0,1}$$

$$[\nabla_{\bar z},\nabla_z] = [g\,\partial_{\bar z} g^{-1},\, h^{-1}\partial_z h] = [\partial_{\bar z} + \overbrace{\partial_{\bar z}\log g^{-1}}^{\text{scalar matrix}},\; \partial_z + h^{-1}\partial_z h]$$

$$= \partial_{\bar z}(h^{-1}\partial_z h) + \partial^2_{z\bar z}\log g$$

So

$$\boxed{\begin{aligned} &\Delta \text{ on } E = \frac{1}{g}\nabla_z\nabla_{\bar z} \\ &\Delta \text{ on } E\otimes\Omega^{0,1} = \frac{1}{g}\nabla_z\nabla_{\bar z} + \frac{1}{g}\left(\partial_{\bar z}(h^{-1}\partial_z h) + \partial^2_{z\bar z}\log g\right) \end{aligned}}$$

Finally notice in this complex situation

$$Ds = s\theta \qquad\qquad \theta = h^{-1}\partial h$$

$$D^2 s = s(d\theta + \theta\theta)$$

$$d\theta + \theta\theta = \bar\partial(h^{-1}\partial h) + \underbrace{\partial(h^{-1}\partial h)}_{\partial(h^{-1})\partial h = -\cancel{h^{-1}\partial h\, h^{-1}\partial h}} + \cancel{h^{-1}\partial h\, h^{-1}\partial h}$$

$$\therefore \qquad d\theta + \theta\theta = \bar\partial(h^{-1}\partial h)$$

April 8, 1982

Rapid review of WKB. Consider a Schroedinger equation for a wave function

$$\left[\frac{\hbar}{i}\partial_t + H\left(t,x,\frac{\hbar}{i}\partial_x\right)\right]\psi = 0$$

and put $\psi = e^{\frac{i}{\hbar}S}u$ where u is to have an asymptotic expansion $u_0 + \hbar u_1 + \cdots$ as $\hbar \downarrow 0$. Then

$$\circledast \quad e^{-\frac{i}{\hbar}S}\left[\frac{\hbar}{i}\partial_t + H\right]e^{\frac{i}{\hbar}S}u = \left[\frac{\hbar}{i}\partial_t + \partial_t S + H\left(t,x,\frac{\hbar}{i}\partial_x + \partial_x S\right)\right]u = 0,$$

and the first equation is formed by letting $\hbar \to 0$:

$$\left[\partial_t S + H(t,x,\partial_x S)\right]u_0 = 0 \qquad \text{Hamilton-Jacobi equation}$$

I am going to assume I am working with functions (not vector fns.) and that $H(t,x,p)$ is a quadratic function of p, say $H(t,x,p) = \ell\frac{p^2}{2} + \boxed{mp} + n$ where ℓ, m, n are fns. of t,x.

not self-adjoint

$$H\left(t,x,\frac{\hbar}{i}\partial_x + \partial_x S\right) = \frac{\ell}{2}\left(-\hbar^2\partial_x^2 + 2\frac{\hbar}{i}\partial_x S\,\partial_x + (\partial_x S)^2 + \frac{\hbar}{i}\partial_x^2 S\right)$$
$$+ m\left(\frac{\hbar}{i}\partial_x + \partial_x S\right) + n$$

Now we want the first order terms in $\hbar$ in $\circledast$, which gives

$$\left[\frac{1}{i}\partial_t + \ell\frac{1}{i}\partial_x S\,\partial_x + m\frac{1}{i}\partial_x + \frac{\ell}{2}\frac{1}{i}\partial_x^2 S\right]u_0 = 0$$

or

$$\left[\partial_t + (\ell\,\partial_x S + m)\partial_x\right]\log u_0 + \frac{\ell}{2}\partial_x^2 S = 0$$

Now the problem is to calculate the function u_0.

Solution for a quadratic Hamiltonian

$$H = \frac{\ell}{2}p^2 + \frac{m}{2}[xp + px] + \frac{n}{2}x^2 \qquad \left(\ell, m, n \text{ fns. of } t\right)$$

$$e^{-iS}\left(\frac{1}{i}\partial_t + H\right)e^{iS} = \frac{1}{i}\partial_t + \overset{\checkmark}{\partial_t S} + \frac{\ell}{2}\left(\frac{1}{i}\partial_x + \partial_x S\right)^2 + \frac{m}{2}x\left(\frac{1}{i}\partial_x + \overset{\checkmark}{\partial_x S}\right)$$
$$+ \frac{m}{2}\left(\frac{1}{i}\partial_x + \overset{\checkmark}{\partial_x S}\right)x + \frac{\overset{\checkmark}{n}}{2}x^2$$

$$= \text{[redacted]}\ \partial_t S + \frac{\ell}{2}(\partial_x S)^2 + mx(\partial_x S) + \frac{n}{2}x^2$$
$$+ \ell\, \partial_x S\, \frac{1}{i}\partial_x \ \text{[redacted]} + \frac{\ell}{2}(-\partial_x^2) + \frac{\ell}{2}\frac{1}{i}\partial_x^2 S + \frac{m}{2}\frac{1}{i} + \frac{1}{i}\partial_t$$

So choose S to satisfy the HJ equ.

$$\partial_t S + \frac{\ell}{2}(\partial_x S)^2 + mx\,\partial_x S + \frac{n}{2}x^2 = 0$$

Put $$S = \frac{a}{2}x^2 + bxx' + \frac{c}{2}x'^2$$

$$\partial_x S = ax + bx'$$

$$\frac{\dot{a}}{2}x^2 + \dot{b}\,xx' + \frac{\dot{c}}{2}x'^2 + \frac{\ell}{2}(ax+bx')^2 + mx(ax+bx') + \frac{n}{2}x^2 = 0$$

or
$$\begin{cases} \frac{\dot{a}}{2} + \frac{\ell}{2}a^2 + ma + \frac{n}{2} = 0 \\ \dot{b} + \ell ab + mb = 0 \\ \frac{\dot{c}}{2} + \frac{\ell}{2}b^2 = 0. \end{cases}$$

These can be solved for a, b, c and so one gets an S which is quadratic in x. But then one gets a solution of the Schroedinger equation of the form

$$e^{iS}\varphi(t)$$

where φ satisfies

$$\text{[redacted]}\ \left[\partial_t + \frac{1}{2}(\ell a + m)b\right]\varphi = 0$$

and hence $\varphi = \text{const}\,(b)^{1/2}$. This agrees with the fact that

$$\langle x|U|x'\rangle = c\, e^{i\left(\frac{a}{2}x^2 + bxx' + \frac{c}{2}x'^2\right)}$$

is a unitary kernel iff $|c| = \left(\frac{|b|}{2\pi}\right)^{1/2}$.

Go back to [redacted] a Riemann surface; $ds^2 = \rho(dx^2+dy$
means $\frac{1}{\sqrt{\rho}}\partial_x, \frac{1}{\sqrt{\rho}}\partial_y$ is an orthonormal base for T.
Hence $|\partial_z|^2 = \frac{1}{4}(|\partial_x|^2 + |\partial_y|^2) = \frac{1}{4}(\rho+\rho) = \frac{\rho}{2}$ which

is what I called g. Thus

$$g^{-1/2}\frac{\partial}{\partial z} \quad \text{is a unit vector.}$$

In general $\left\langle \frac{\partial}{\partial z^a}, \frac{\partial}{\partial z^b} \right\rangle = g_{a\bar{b}}$ or $ds^2 = \sum g_{a\bar{b}}\, dz^a d\bar{z}^b$ for a Kähler manifold.

The curvature form of the tangent bundle is

$$\bar{\partial}\partial \log g = -\partial^2_{z\bar{z}} \log g \quad dz\,d\bar{z}$$

because $\frac{\partial}{\partial z}$ is a holomorphic section with norm^2 g.

The [redacted] volume form is

$$g\, i\, dz\, d\bar{z} \;=\; \underbrace{2g}_{\rho}\, dx\, dy$$

Comparing the 2 gives you the curvature of the Riemann surface which is a function on the surface.

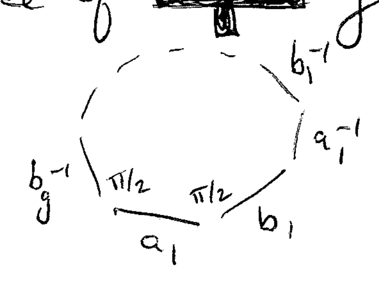
Actually the definition is normalized by the Gauss Bennet thm. You want for a geodesic triangle

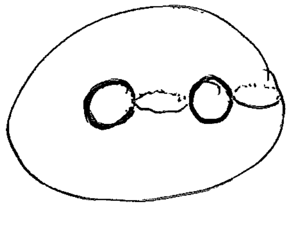
$$\alpha + \beta + \gamma - \pi = \int_{\Delta} R \cdot vol$$

or more generally for [redacted] a polygon with n sides

$$\sum \text{angles} - (n-2)\pi = \int_{\bigcirc} R \cdot vol$$

For a surface of [redacted] genus g, take its standard presentation





and you get

$$\int_M R \cdot vol = 4g \cdot \frac{\pi}{2} - (4g-2)\pi = (2-2g)\pi$$

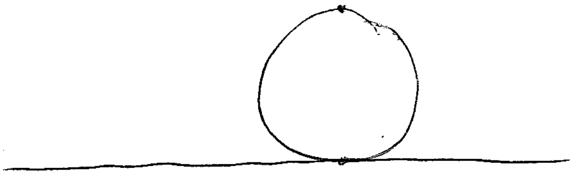
Hence we have

$$\boxed{R \cdot vol = \pi \frac{i}{2\pi} \bar{\partial}\partial \log g}$$

$$R\, g\, i\, dz\, d\bar{z} = -\frac{i}{2}\, \partial^2_{z\bar{z}} \log g \; dz\, d\bar{z}$$

and so $$R = -\frac{1}{2g}\, \partial^2_{z\bar{z}} \log g = -\frac{1}{\rho}\, \partial^2_{z\bar{z}} \log \rho$$

So for the Reimann sphere of radius $\frac{1}{2}$



$$ds^2 = \frac{|dz|^2}{(1+|z^2|)^2} \qquad \rho = \frac{1}{(1+|z|^2)^2}$$

[redacted] $\partial_{\bar{z}} \partial_z \log \rho = -2\, \partial_{\bar{z}} \frac{\bar{z}}{1+|z|^2}$ [redacted]

$= -2\, \frac{(1+|z|^2)}{}$ [redacted] $- \bar{z} z$ [redacted] $\frac{}{(1+|z|^2)^2}$ [redacted] $= \frac{-2}{(1+|z|^2)^2}$

and hence $R =$ [redacted] 2. In general for a sphere of radius a one has $\int R \cdot vol = R\, 4\pi a^2 = 2\pi$ and so $R = \frac{1}{2a^2}$ by these conventions.

$$\boxed{\text{Thus } g = \frac{1}{(1+\varepsilon|z|^2)^2} \text{ has } R = \varepsilon}$$