August 16, 1986

The problem would be to find a way to show $H^*(\mathbb{Z}\times BU)$ is naturally isomorphic to the fermion Fock space of $L^2(S^1)$ and in this way to explain why the Jacobi formula and the previously obtained decomposition of $H^*(BU)$ are related.

The first step will be to look at the case where the index is different from 0. Thus we look at $X = Gr_{n+d}(\overset{m}{V^0} \oplus \overset{n}{V^1})$ and we have correspondences

$$\begin{array}{ccc}
 & \{(K,I,W)\} & \longrightarrow \; Gr_{n+d}(V) \\
 & \Big\downarrow \text{dim } s(s+d) & \\
\{(K,I,\Gamma)\} & \longrightarrow \; Gr_{s+d}(\overset{m}{V^0}) \times \check{Gr}_s(\overset{n}{V^1}) & \\
\Big\downarrow & & \\
Gr_{n+d}(V) & &
\end{array}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \overset{n}{V^1} \arrow[dr] & \\
\overset{n-s}{I} \arrow[ur] \arrow[dr, "s+d"'] & & K \oplus \overset{n}{V^1} \\
 & \overset{n+d}{W} \arrow[ur, "s"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & \overset{m}{V^0} \arrow[dr] & \\
\overset{s+d}{K} \arrow[ur] \arrow[dr] & & \overset{m}{V^0} \oplus \overset{n-s}{I} \\
 & \overset{n+d}{\Gamma} \arrow[ur] &
\end{tikzcd}$$

The decomposition formula stably as $n, m \to \infty$ is

$$P.S.\ H^*(BU) = \sum_{s \geq 0} \frac{q^{s(s+d)}}{(1-q)\cdots(1-q^{s+d})(1-q)\cdots(1-q^s)} \qquad \text{say } d \geq 0.$$

From the t^d term in

$$\prod_{i>0} \left(1 + t q^{-\frac{1}{2}+i}\right)\left(1 + t^{-1} q^{-\frac{1}{2}+i}\right) = \frac{\sum q^{\frac{d^2}{2}} t^d}{\prod_{i>0}(1-q^i)}$$

we have

$$\frac{q^{d^2}}{\prod(1-q^i)} = \sum_{s \geq 0} \left(\sum_{i_1 < \cdots < i_{s+d}} q^{-\frac{1}{2}+i_1} \cdots q^{-\frac{1}{2}+i_{s+d}} \right) \left(\sum_{i_1 < \cdots < i_s} q^{-\frac{1}{2}+i_1} \cdots q^{-\frac{1}{2}+i_s} \right)$$

$$= \sum_{s \geq 0} \frac{q^{\frac{(s+d)^2}{2}}}{(1-q)\cdots(1-q^{s+d})} \frac{q^{\cancel{-\frac{s}{2}+\frac{s(s+1)}{2}}\ \frac{s^2}{2}}}{(1-q)\cdots(1-q^{s})}$$

So it checks as $\frac{(s+d)^2+s^2}{2} - d^2 = s(s+d)$

Let's now concentrate on the $s=0$ term. This corresponds to embedding $H^*(BU_d)$ as as direct factor of $H^*(BU)$. The relevant maps are

$$K \longmapsto K \oplus V^1$$

$$Gr_d(V^0) \longrightarrow Gr_{n+d}(V)$$

(note that if $\overset{0}{I} \begin{array}{c} \nearrow V^1 \searrow \\ \searrow \underset{W}{\overset{n+d}{}} \nearrow \end{array} \overset{d}{K} \oplus \overset{n}{V^1} \Rightarrow \begin{array}{l} W = K \oplus V^1 \\ I = 0 \end{array}$)

and the map the other way is the correspondence which roughly sends Γ to its kernel $\Gamma \cap V^0$ which generically has dim d.

This correspondence defines a natural injection

$$H^*(BU_d) \hookrightarrow H^*(BU)$$

which is a section of the obvious map the other way. A natural question is what this map does.

Let's work this out for $d = 1$. The map

$$\mathbb{P}V^0 \longrightarrow Gr_{n+1}(V)$$
$$K \longmapsto K \oplus V^1$$

induces on cohomology the map

$$H^*(BU_1) \longleftarrow H^*(BU)$$

which is the ring homomorphism sending $c_1(S) \to c_1(\mathcal{O}(-1))$ and $c_i(S) \to 0$ for $i \geq 2$, where S is the subbundle over $X = Gr_{n+1}(V)$.

The map going the other way is given by the correspondence

$$\begin{tikzcd} & \mathbb{P}V^0 \arrow[d, phantom, "\cap" rotate=90] \\ Y = \{(\overset{1}{K}, \overset{n+1}{\Gamma})\} \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathbb{P}V \\ X = Gr_{n+1}(V) & \end{tikzcd}$$

Note that $Y = \mathbb{P}(S)$ where S is the subbundle; Y is the subvariety of $\mathbb{P}V \times X$ where the canonical map

$$pr_1^* \mathcal{O}_{\mathbb{P}V}(-1) \subset \underset{m+n}{\tilde{V}} \longrightarrow \tilde{V} / pr_2^* \underset{n+1}{S}$$

vanishes. The coh. class of Y in $\mathbb{P}V \times X$ is

$$c_{m-1}\left(pr_2^* Q \otimes pr_1^* \mathcal{O}_{\mathbb{P}V}(1)\right) = \sum_{i=0}^{m-1} u^{m-1-i} c_i(Q)$$

where Q is the ~~quotient~~ bdle over X, $u = c_1(\mathcal{O}_{\mathbb{P}V}(+1))$.
To get $(p_2)_* p_1^*(u^i)$ we lift u^i to $\mathbb{P}V \times X$ multiply by the above Euler class, then integrate over $\mathbb{P}V$ which means find the coeff. of u^{m+n-1}.
If we want the map

$*$ $\quad H^*(\mathbb{P}V^0) \longrightarrow H^*(X)$

then we multiply by u^n as $\mathbb{P}V^0 \subset \mathbb{P}V$ has the class u^n. Thus the map $*$ we want sends u^j to

$$\left(\text{coeff of } u^{m+n-1} \text{ in } \sum u^{m-1-i} c_i(Q) \cdot u^n u^j\right) = c_j(Q).$$

Summarizing we have

<u>Prop</u>. The map $H^*(\mathbb{P}V^0) \longrightarrow H^*(Gr_{n+1}(V))$ associated to the correspondence $\Gamma \longmapsto \Gamma \cap V^0$ sends u^i, where $u = c_1(\mathcal{O}(1))$, to $c_i(\mathcal{Q})$.

Let's next check this is a section of the map in the other direction, which sends [redacted] K to $K \oplus V'$. If f is this map, then $f^*\mathcal{S} = \mathcal{O}(-1) \oplus \tilde{V}'$ and so $f^* c_i(\mathcal{S}) = \begin{cases} -u & i=1 \\ 0 & i>1 \end{cases}$

[redacted] Thus $f^* c_t(\mathcal{Q}) = f^* \frac{1}{c_t(\mathcal{S})} = \frac{1}{1-tu} = \sum t^i u^i$, showing that $f^* c_i(\mathcal{Q}) = u^i$ as expected.

August 17, 1986

Let's review the integration over the fibre map for the map

$$D_{12\cdots d}(V) \xrightarrow{\pi} Gr_d(\overset{N}{V})$$

We factor π into

$$\begin{tikzcd} D_{12\cdots d}(V) \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & Gr_d(V) \times (\mathbb{P}V)^d \arrow[d] \\ & Gr_d(V) \end{tikzcd}$$

Here I think of the flag manifold $D_{12\cdots d}(V)$ as the space of d tuples of orthogonal lines $(L_1, \dots, L_d)$ in V. The image of i is defined by the conditions

$L_i \perp L_j \qquad i<j$

$L_i \subset \Gamma \qquad$ i.e. where $pr_i^* \mathcal{O}(-1) \to \tilde{V} \to Q^{N-d}$ vanishes

so

$$i_* 1 = \pm \prod_{i<j}(u_i - u_j) \prod_i \left(u_i^{N-d} + \cdots + c_{N-d}(Q)\right)$$

In order to determine the sign we can use the flag that

$$\pi_*\left(u_1^{d-1} u_2^{d-2} \cdots u_d^0\right) = 1.$$

This comes the fact that π is an iterated projective bundles. So this means that the vander Mond determinant we want is to contain $u_1^0 u_2^1 \cdots u_d^{d-1}$,

$$\begin{vmatrix} u_1^0 & & & \\ & u_2^1 & & \\ & & \ddots & \\ & & & u_d^{d-1} \end{vmatrix} = \prod_{i>j}(u_i - u_j)$$

So the formula is

$$\pi_*(\alpha) = \text{coeff of } u_1^{N-1} \cdots u_d^{N-1} \text{ in}$$

$$\prod_{i>j} (u_i - u_j) \prod_i \left(u_i^{N-d} + \cdots + c_{N-d}(Q) \right) \alpha$$

Let's see what this gives when $\alpha = u_1^{a_1} \cdots u_d^{a_d}$.
The answer seems to be

$$* \quad \begin{vmatrix} c_{(a_1)+1-d} & c_{(a_2)+1-d} & & c_{(a_d)+1-d} \\ \vdots & & & \vdots \\ c_{(a_1)\text{[redacted]}1} & c_{(a_2)\text{[redacted]}1} & & c_{(a_d)\text{[redacted]}1} \\ c_{a_1}\text{[redacted]} & c_{a_2}\text{[redacted]} & & c_{a_d}\text{[redacted]} \end{vmatrix} \qquad c_i = c_i(Q)$$

For example if $a_i = d - i$, then we get

$$\begin{vmatrix} c_0 & 0 & \cdots & & 0 \\ c_1 & c_0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & & 0 \\ c_{d-1} & c_{d-2} & \cdots & & c_0 \end{vmatrix} = 1$$

Next let's turn to the problem of finding the map $H^*(BU_d) \longrightarrow H^*(BU)$, which is a section of the obvious map the other way, and which comes from the correspondence

$$\begin{tikzcd} \{(\overset{d}{K}, \overset{n+d}{\Gamma})\} \arrow[r] \arrow[d] & Gr_d(V^0) \\ Gr_{n+d}(V) & \end{tikzcd}$$

$$\begin{tikzcd} & \overset{m}{V^0} \arrow[dr, dash] & \\ \overset{d}{K} \arrow[ur, dash] \arrow[dr, dash] & & \overset{m+n}{V} \\ & \overset{n+d}{\Gamma} \arrow[ur, dash] & \end{tikzcd}$$

This correspondence is given by the subvariety of $Gr_{n+d}(V) \times Gr_d(V)$ described by the conditions

$$K \subset \Gamma \qquad \text{i.e. where } \ pr_2^*(\mathcal{S}_2) \hookrightarrow \tilde{V} \longrightarrow pr_1^*(\mathcal{Q}_1) \ \text{vanishes}$$

$$K \subset V^0 \qquad \text{i.e. where } \ pr_2^*(\mathcal{S}_2) \hookrightarrow \tilde{V} \longrightarrow \widetilde{V/V^0} \ \text{vanishes}$$

Here $\mathcal{S}_2^d$ is the subbundle over $Gr_d(V)$ and $\mathcal{Q}_1^{m-d}$ is the quotient bundle over $Gr_{n+d}(V^0)$. It follows that the class of the subvariety is

$$c_{d(m-d)}\left(pr_1^*(\mathcal{Q}_1) \otimes pr_2^*(\check{\mathcal{S}}_2)\right) \cdot c_{nd}\left(pr_2^*(\check{\mathcal{S}}_2) \otimes \widetilde{V/V^0}\right).$$

The map on cohomology consists of multiplying by this class and integrating over $Gr_d(V)$, which we can do by lifting to $D_{12\cdots d}(V)$ multiplying by $u_1^{d-1}\cdots u_d^0$ and then integrating over $D_{12\cdots d}(V)$; this last step can be done by multiplying by $\prod_{i>j}(u_i-u_j)$ and then finding the coefficient of $u_1^{N-1}\cdots u_d^{N-1}$. So the map we want sends $\alpha \in H^*(Gr_d(V))$ to

$$\text{coeff of } u_1^{N-1}\cdots u_d^{N-1} \text{ in } \prod_{i>j}(u_i-u_j)\ u_1^{d-1}\cdots u_d^0 \prod_i \left(u_i^{m-d}+\cdots+c_{m-d}(\mathcal{Q})\right) \prod_i u_i^n\ \alpha$$

$$= \text{coeff of } u_1^{m-1}\cdots u_d^{m-1} \text{ in } \prod_{i>j}(u_i-u_j) \prod_i \left(u_i^{m-d}+\cdots+c_{m-d}(\mathcal{Q})\right)\ u_1^{d-1}\cdots u_d^0\ \alpha$$

But this can be evaluated by writing the van der Monde determinant and gives the map

$$\alpha = u_1^{a_1} \cdots u_d^{a_2} \longmapsto \begin{vmatrix} c_{(a_1)} & c_{(a_2)-1} & \cdots & c_{(a_d)-d+1} \\ c_{(a_1)+1} & c_{a_2} & & \\ \vdots & & & \vdots \\ c_{(a_1)+d-1} & c_{(a_2)+d-2} & \cdots & c_{(a_d)} \end{vmatrix}$$

Notice that such a monomial α doesn't come from $Gr_d(V)$ unless the a_i are all equal. So the general formula for the map

$$H^*(BU_d) \longrightarrow H^*(BU)$$

still involves symmetrizing the above.

Observation. In the Jacobi setup for degree d we look at

$$\bigoplus_{s \geq 0} \Lambda^{d+s} H^+ \otimes \Lambda^s \overline{H^-}$$

and identify the s th piece somehow with a space of polys in $(d+s)+s$ variables. For example taking $s=0$ we see

$$P.S.\left(\Lambda^d H^+\right) = \sum_{0<\ell_1<\cdots<\ell_d} q^{-\frac{d}{2}} q^{\ell_1+\cdots+\ell_d} = \frac{q^{\frac{d^2}{2}}}{\prod_{i=1}^{d}(1-q^i)}$$

showing that additively $\Lambda^d H^+ \cong H^*(BU_d)$. A natural question is how to realize this isomorphism geometrically.

This is easily done using the integration over the fibre for the map

$$D_{12\cdots d}(V) \xrightarrow{\pi} Gr_d(V)$$

in the limit $\dim(V) \to \infty$. Say V is Hilbert space so that $D_{12\cdots d}(V)$ is d-tuples $(L_1, \ldots, L_d)$ of orthogonal lines and $\pi(L_1, \dots, L_d) = L_1 \oplus \cdots \oplus L_d$. We know the action of the symmetric group Σ_d acts on the orientation of the flag manifold via the sign character, so the map π_* induces a map

$$\pi_* : \Lambda^d H^*(BU_1) \longrightarrow H^*(BU_d)$$

lowering degree by $\frac{d(d-1)}{2}$. We know π_* is onto, and so the above calculation with P.S.'s shows it's an isomorphism. The explicit formula for this map is given on page 66, *.

Let $Gr(V) = \coprod_n Gr_n(V)$ and note that we have rational maps

$$\mathbb{P}V \times Gr_n(V) \dashrightarrow Gr_{n+1}(V) \qquad (L, K) \mapsto L + K$$

$$\check{\mathbb{P}}V \times Gr_n(V) \dashrightarrow Gr_{n-1}(V) \qquad (H, K) \mapsto H \cap K$$

which should induce maps on cohomology. It looks like $H^*(Gr(V))$ is a module over the Clifford algebra of $H^*(\mathbb{P}V) \oplus H^*(\check{\mathbb{P}}V)$. Note that the dimensions are correct:

$$\dim H^*(Gr_n(V)) = \binom{N}{n}$$

and

$$\sum_n \binom{N}{n} = 2^N \qquad N = \dim H^*(\mathbb{P}V) = \dim(V).$$

This means that $H^*(Gr_n(V)) \simeq \Lambda^n V$. This checks with the Morse theory. The fixed points of the maximal torus correspond to the fixed lines in $\Lambda^n V$ which are the coordinate axes.

August 18, 1986

Theorem: There is a canonical isomorphism

$$\Lambda^k H^*(\mathbb{P}V) \simeq H^*(Gr_k V)$$

lowering degree by $\frac{1}{2}k(k-1)$.

Proof: Let $D_{12\cdots k}(V)$ = space of flags $0 \subset W_1 \subset \cdots \subset W_k$ with $\dim W_i = i$, or equivalently the space of k-tuples $(L_1, \ldots, L_k)$ of orthogonal lines. We have maps

$$\begin{array}{ccc} D_{12\cdots k}(V) & \xhookrightarrow{i} & (\mathbb{P}V)^k \\ \downarrow \pi & & \\ Gr_k(V) & & \end{array}$$

which induce a map on cohomology

$$\pi_* \iota^* : H^*((\mathbb{P}V)^k) \longrightarrow H^*(Gr_k V)$$

lowering degree by $\frac{1}{2}k(k-1)$. This map is surjective because one knows this is true for both π_* and i^*.

Next look at the action of Σ_k on $D_{12\cdots k}(V)$ and $(\mathbb{P}V)^k$. One knows that the action of Σ_k on a flag manifold induces the sign action on the orientation. This means that $\pi_* \iota^*$ induces a map

$$\Lambda^k H^*(\mathbb{P}V) \longrightarrow H^*(Gr_k V)$$

(further argument required if char = 2). As this map is surjective and both sides have the same dimension $\binom{N}{k}$, it's an isomorphism.

This results suggests that we ought to be able to define interior + exterior multiplication operators on $H^*(Gr\, V)$ geometrically.

Here are the obvious candidates. We have maps

$$\begin{tikzcd} D_{n,n+1} \arrow[r] \arrow[d] & Gr_{n+1} \\ Gr_n & \end{tikzcd}$$

and a canonical line bundle $\mathcal{L}$ on $D_{n,n+1}$. (Note that $D_{n,n+1}$ is $\mathbb{P}(Q \text{ over } Gr_n)$ with $\mathcal{L} = \mathcal{O}(-1)$ and also $D_{n,n+1}$ is $\check{\mathbb{P}}(S \text{ over } Gr_{n+1})$ with $\mathcal{L} = \mathcal{O}(1)$.)

So using the class $c_1(\mathcal{L})^i$ over $D_{n,n+1}$ we can define maps $H^*(Gr_n) \rightleftarrows H^*(Gr_{n+1})$.

It seems likely that these operators satisfy the standard relations for ι_a, e_a on the exterior algebra. Sketch of a possible proof:

First show $\iota_a \iota_b + \iota_b \iota_a = 0$

$$\begin{tikzcd} D_{n-2,n} \arrow[rrr, bend left, "q"] \arrow[ddr, "p"'] & D_{n-2,n-1,n} \arrow[l, "\pi"'] \arrow[r] \arrow[d] & D_{n-1,n} \arrow[r] \arrow[d] & Gr_n \\ & D_{n-1,n-2} \arrow[r] \arrow[d] & Gr_{n-1} & \\ & Gr_{n-2} & & \end{tikzcd}$$

$u_2 = c_1(\mathcal{L} \text{ on } D_{n-1,n})$

$u_1 = c_1(\mathcal{L} \text{ on } D_{n-1,n-2})$

The operation $\iota_a \iota_b$ involves lifting horizontally multiplying by u_2^b, integrating vertically, then mult. by u_1^a and integ. vert. again. It's the same as

$$\alpha \longmapsto (p\pi)_*(u_1^a u_2^b (q\pi)^* \alpha) = p_*\{\pi_*(u_1^a u_2^b) \cdot q^* \alpha\}$$

But π is the projective space bundle of a 2 plane bundle, so one should have $\pi_*(u_1^a u_2^b) = -\pi_*(u_1^b u_2^a)$.

2nd show $\iota_a e_b + e_b \iota_a = \delta_{ab}$.

The composition $\iota_a e_b$ involves

$$\begin{tikzcd} Y = \{ \overset{n}{W} \subset \overset{n+1}{\Gamma} \supset \overset{n}{W'} \} \arrow[r] \arrow[d] & D_{n,n+1} \arrow[r] \arrow[d] & Gr_n \\ D_{n,n+1} \arrow[r] \arrow[d] & Gr_{n+1} & \\ Gr_n & & \end{tikzcd}$$

and $e_b \iota_a$ involves

$$\begin{tikzcd} Y' = \{ \overset{n}{W} \supset \overset{n-1}{K} \subset \overset{n}{W'} \} \arrow[r] \arrow[d] & D_{n-1,n} \arrow[r] \arrow[d] & Gr_n \\ D_{n-1,n} \arrow[r] \arrow[d] & Gr_{n-1} & \\ Gr_n & & \end{tikzcd}$$

Generically on Y, Y' one has $\Gamma = W + W'$, $K = W \cap W'$. So I get degree one maps to Y and Y' from

$$Z = \left\{ \begin{matrix} & \overset{n+1}{\Gamma} & \\ \overset{n}{W} & & \overset{n}{W'} \\ & K & \end{matrix} \right\} = D_{n-1,n,n+1} \times_{D_{n-1,n+1}} D_{n-1,n,n+1}$$

which is the fibre product with itself of the projective line bundle of the canonical two plane bundle over $D_{n-1,n+1}$.

On Z we have ~~two~~ cohomology classes

$$c_1(\Gamma/W)^a \, c_1(\Gamma/W')^b, \qquad c_1(W/K)^b \, c_1(W'/K)^a$$

which give rise to $\iota_a e_b$ and $e_b \iota_a$ respectively (ignoring $\pm$). On the complement of the diagonal of Z these classes coincide, so their difference comes from a class on the diagonal $D_{n-1,n,n+1}$. This class should be obtainable via a calculation with 2 plane bundles. In any case we see that the map $\iota_a e_b + e_b \iota_a$ will be represented by a correspondence supported in the diagonal, namely what happens to the class after being pushed forward under $D_{n-1,n,n+1} \longrightarrow Gr_n$. This involves integrating powers of the ~~classes~~ of the two lines bundles and I expect explicit calculation to give 0 or 1.

Further questions + problems.

How to handle the limit

$$\mathbb{Z} \times BU = \lim_{n\to\infty} Gr_{n+?}(\overset{n}{V}{}^0 \oplus \overset{n}{V}{}^1)?$$

Is there any nice result about the character classes on the components of index $\neq 0$ relative to the decomposition

$$H^*(\{d\} \times BU) = \bigoplus H^*(BU_{s+d} \times BU_s)?$$

August 19, 1986:

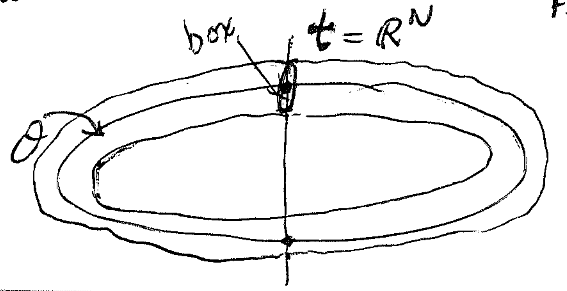
According to Berry's talk this spring Dyson (Missed Opportunities?) observed a link between the statistics of random hermitian matrices and the Riemann ζ. I would like to reconstruct this link.

For N fixed one considers the space of $N \times N$ hermitian matrices equipped with the Gaussian prob. measure with exponent $(\text{const}) \operatorname{tr}(X^* X)$. One is then interested in the ~~statistical~~ properties of the eigenvalues of the random variable X, for example the distribution of $\operatorname{tr}(X)$. One wants to take a large N limit; Berry uses the phrase G.U.E. = grand unitary ensemble. I interpret this as follows. The Gaussian measure on $\mathfrak{U}_N = \operatorname{Lie}(U_N)$ pushes forward to a measure on

$$U_N \backslash \mathfrak{U}_N \simeq \Sigma_N \backslash \mathbb{R}^N$$

Thus we get a prob measure on $\mathbb{R}^N$ invariant under Σ_N, and this is like having an N particle gas on $\mathbb{R}$. We put these together for all N according to the grand ensemble procedure.

Let's determine the symmetric measure on $\mathbb{R}^N$ obtained by pushing forward Lebesgue measure on $\mathfrak{U}_N$. Fix $(x_1, \ldots, x_N)$ with distinct coords and a small box of volume $\prod dx_i$ around this point. We want the volume of the set of hermitian matrices with eigenvalues in this box. We're talking about the volume of a tube around the U_N-orbit of the diagonal matrix with entries $x_1, \ldots, x_N$. Call this orbit $\mathcal{O}$. Picture



The volume we seek is $\mathrm{vol}(\mathcal{O})\prod dx_i$. Now we have $G/T \xrightarrow{\sim} \mathcal{O}$ and so we only have to compare the fixed volume of G/T $(G = \mathcal{U}_N, T = \text{diags})$ with that of $\mathcal{O}$. Look at the tangent space

$$\begin{aligned} \mathfrak{g}/\mathfrak{t} &\longrightarrow \mathfrak{t}^{\perp} \\ X &\longmapsto [X, \xi] \end{aligned}$$

$\xi =$ diag matrix entries x_i

Over $\mathbb{C}$ one has the basis X_α for $\mathfrak{g}/\mathfrak{t}$ and one has $[X_\alpha, \xi] = -\alpha(\xi) X_\alpha$. So each pos. root contributes a volume change $|\alpha(\xi)|^2$. So we get

$$\prod_{\alpha>0} |\alpha(\xi)|^2 \cdot \mathrm{vol}(G/T) = \mathrm{vol}(\mathcal{O})$$

Thus we can conclude that the following holds.

<u>Prop</u>: The Lebesgue measure on $\mathcal{U}_N$ pushes forward to the measure

$$\frac{\mathrm{vol}(G_N/T_N)}{N!} \prod_{i<j} (x_i - x_j)^2 \prod dx_i$$

on $\mathbb{R}^N$.

At the moment we don't know what $\mathrm{vol}(\mathcal{U}_N/T_N)$ is. Notice that we have

$$\int_{\mathbb{R}^N} \frac{\mathrm{vol}(\mathcal{U}_N/T_N)}{N!} e^{-\frac{\lambda}{2}\sum x_i^2} \prod_{i<j} (x_i - x_j)^2 \prod dx_i$$

$$= \int_{\mathcal{U}_N} e^{-\frac{\lambda}{2}\|X\|^2} d\mathrm{Lebes} = \int_{\mathbb{R}^{N^2}} e^{-\frac{\lambda}{2}|x|^2} d^{N^2}x = \left(\sqrt{\frac{2\pi}{\lambda}}\right)^{N^2}$$

so that $\mathrm{vol}(\mathcal{U}_N/T_N)$ is linked to $\int e^{-\frac{\lambda}{2}\sum x_i^2} \prod_{i<j} (x_i - x_j)^2 \cdot d^N x$.

Let's continue with the physics viewpoint. We are considering the N-particle gas on $\mathbb{R}$ described by the probability measure proportional to

$$* \qquad e^{-\frac{\lambda}{2}\sum x_j^2} \prod (x_i - x_j)^2 \, d^N x$$

The first questions to be asked concern the averages of "1-particle functions" $\sum_j f(x_j)$, and this means we are concerned with the density $\rho(x)$. Let $\rho_N(x_1, \ldots, x_N)\, dx^N$ be the prob. measure proportional to $*$. Then

$$\langle \sum_j f(x_j) \rangle = \int \sum_j f(x_j)\, \rho_N(x_1, \ldots, x_N)\, dx^N$$

$$= N \int f(x_1)\, dx_1 \int \rho_N(x_1, x_2, \ldots, x_N)\, dx_2 \ldots dx_N$$

$$= \int f(x)\, \rho_N(x)\, dx$$

$$\therefore \quad \rho_N(x) = N \int \rho_N(x_1, \ldots, x_N)\, dx_2 \ldots dx_N$$

$$= N e^{-\frac{\lambda}{2}x^2} \frac{\int \prod_2^N (x - x_j)^2 e^{-\frac{\lambda}{2}\sum_2^N x_j^2} \prod_{2 \le i < j} (x_i - x_j)^2 \, dx_2 \ldots dx_N}{\int e^{-\frac{\lambda}{2}\sum x_j^2} \prod_{i<j} (x_i - x_j)^2 \, dx_1 \ldots dx_N}$$

The problem is to choose λ as a function of N so that this function $\rho_N(x)$ converges as $N \to \infty$.

Note that $\rho_N(x) = e^{-\frac{\lambda}{2}x^2} \times (\text{poly in } x^2 \text{ of degree } N-1)$

August 20, 1986

Suppose we consider Grassmannian graph in the case of $U(n)$. In order to explain what I mean, I have to recall the analogy. In the usual case we have the graph embedding

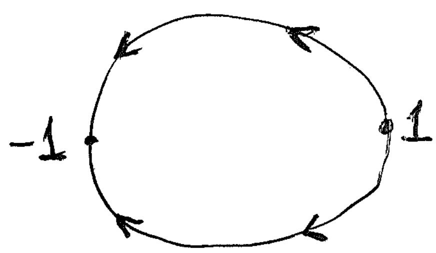
$$\mathrm{Hom}(V^0, V^1) \subset Gr_n(V^0 \oplus V^1)$$

and we view the latter as a compactification of the former. Then we do the rescaling transf. $\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$ on $V^0 \oplus V^1$.

In the odd case instead of the graph we use the Cayley transform map

$$\mathfrak{u}(V) \longrightarrow U(V)$$

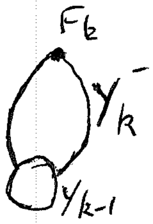
which associates to a skew-hermitian X the unitary $\frac{1+X}{1-X}$. We then have the rescaling transf. $X \mapsto tX$ and this ~~associates to~~ extends to an action of $\mathbb{R}_{>0}$ on $U(V)$. It really just moves the eigenvalues around the unit circle. We have an action of $\mathbb{R}_{>0}$ on the unit circle with $+1$ as expanding and -1 as contracting fixpt and this action is extended to $U(V)$ via the spectral thm.



The fixpoint set for this flow on $U(V)$ is

$$Gr(V) = \coprod_k \underbrace{Gr_k(V)}_{\text{call this } F_k}$$

Let's look at the Morse decomposition. Let Y_k^- be the set of points which under the backwards flow go into $\coprod_{s\le k} F_s$.



Now F_k consists of unitaries with k eigenvalues $= +1$ and the ~~rest~~ $= -1$. So Y_k^- consists of g with $\le k$ eigenvalues $\neq -1$, and $Y_k^- - Y_{k-1}^-$ " " " " k eigenvalues $\neq -1$. So $Y_k^- - Y_{k-1}^-$ fibres over F_k. A point of $Y_k^- - Y_{k-1}^-$ over a k plane W ~~consists of~~ is given by a unitary transformation on W with no eigenvalues $= -1$. In other words $Y_k^- - Y_{k-1}^-$ is diffeomorphic via the Cayley transform to the vector bundle of skew-adjoint endos. of the subbundle over $Gr_k(V)$. In particular the Morse index of F_k is k^2.

We can see immediately that the Morse theory is perfect because we know

$$H^*(U(V)) = \Lambda[e_1, e_3, \dots, e_{2N-1}]$$

$$H^*(Gr(V)) \simeq \Lambda H^*(\mathbb{P}V)$$

Note the isomorphism

$$\begin{aligned} H^*(Y_k^-, Y_{k-1}^-) &= H^*(Gr_k(V))[k^2] \\ &= \left(\Lambda^k H^*(\mathbb{P}V)\right)[-k(k-1)][k^2] \\ &= \Lambda^k H^*(\mathbb{P}V)[k] \\ &= \Lambda^k \{ H^*(\mathbb{P}V)[1] \} \end{aligned}$$

A natural question is whether there is a natural isom. of $H^*(\mathbb{P}V)[1]$ with Prim $H^*\{U(V)\}$. Also whether the filtration of $H^*\{U(V)\}$ defined by the Morse decomposition splits naturally in the same way as for the Grassmannian.

Let's try to construct the splitting using the upward "bowls" of the critical submanifolds. Recall

$$Y_k^{-} = \{ g \mid \leq k \text{ eigenvalues} \neq -1 \}$$

so this has the desingularization

$$\tilde{Y}_k^{-} = \{ (g, \overset{k}{W}) \mid g = -1 \text{ on } W^\perp \}.$$

We have

$$Y_\ell^{+} = \{ g \mid \geq \ell \text{ eigenvalues} = +1 \}$$

so this has the desingularization

$$\tilde{Y}_\ell^{+} = \{ (g, \overset{\ell}{K}) \mid g = 1 \text{ on } K \}$$

Now we wish to compute the correspondence

$$\begin{tikzcd}
\tilde{Y}_k^{-} \times_{U(N)} \tilde{Y}_\ell^{+} \arrow[r] \arrow[d] & \tilde{Y}_\ell^{+} \arrow[r] \arrow[d] & F_\ell \\
\tilde{Y}_k^{-} \arrow[r] \arrow[d] & U(N) & \\
F_k & &
\end{tikzcd}$$

The fibre product consists of $(g, \overset{k}{W}, \overset{\ell}{K})$ such that $g = 1$ on K and $g = -1$ on $W^\perp$. This implies that $\overset{\ell}{K} \subset \overset{k}{W}$, so if the fibre product is $\neq \phi$ we must have $\ell \leq k$. If $\ell = k$ the fibre product is just F_k. If $\ell < k$, the fibre product is a $U(k-\ell)$ bundle over the flag man of $(K^{\ell} \subset W^{k})$. If we check that the fibre square is transversal, then because the fibre product has image of the wrong dim in $F_k \times F_\ell$ it will follow the correspondence is zero.

So we compute dims.

$$\dim \tilde{Y}_\ell^+ = \dim Gr_\ell + \dim U(N-\ell)$$
$$= 2\ell(N-\ell) + (N-\ell)^2 = N^2 - \ell^2$$
$$\dim \tilde{Y}_k^- = \dim Gr_k + \dim U(k)$$
$$= 2k(N-k) + k^2 = 2kN - k^2$$
$$\dim\left(\tilde{Y}_k^- \times_{U(N)} \tilde{Y}_\ell^+\right) = \dim \{K^\ell \subset W^k\} + \dim U(k-\ell)$$
$$= 2\ell(N-\ell) + 2(k-\ell)(N-k) + (k-\ell)^2$$
$$= \cancel{2\ell N} - 2\ell^2 + 2\{kN - \cancel{\ell N} - k^2 + \cancel{\ell k}\} + k^2 - \cancel{2k\ell} + \ell^2$$
$$= 2kN - k^2 - \ell^2$$

$$\dim \tilde{Y}_k^- + \dim \tilde{Y}_\ell^+ - \dim U(N) = 2kN - k^2 - \ell^2$$

These two are equal so the correspondence is zero for $\ell < k$. Thus we get a canonical decomposition of $H^*(U(V))$ into $\bigoplus_k H^*(Gr_k(V))[k^2]$.

Next comes the question of whether the $k=1$ summand is the primitive part of $H^*(U(V))$.

The first point is that $\tilde{Y}_1^- = S^1 \times PV$ and the map on cohomology associated to the correspondence

$$(*)\qquad \begin{array}{ccc} S^1 \times \mathbb{P}V = \tilde{Y}_1^- & \longrightarrow & U(V) \\ \downarrow & & \\ \mathbb{P}V & & \end{array}$$

is surjective. On the other hand it kills decomposable elements in $H^*(U(V))$ and so the above gives an isomorphism

$$Q\{H^*(U(V))\} \xrightarrow{\sim} H^*(\mathbb{P}V)[1]$$

What's more we know the map associated to $(*)$ must kill the summands for $k \neq 1$. Thus the image of $H^*(\mathbb{P}V)[1]$ in $H^*(U(V))$ is a complement to the decomposable subspace. This makes it likely it is the primitive subspace.

August 21, 1986

Nice observation: The [redacted] desingularization

$$\tilde{Y}_k^- = \{(g, \overset{k}{W}) \mid g = -1 \text{ on } W^\perp\}$$

is isomorphic to the bundle of automorphisms of the subbundle over $Gr_k(V)$. So for $N = \dim V$ large it is universal for a k-plane bundle together with automorphism. This is the twisted version of $U(k)$ which is the universal space for an automorphism of the trivial k-plane bundle. In other words we see the twisted version for rank k sitting inside the trivial situation of rank N.

Let's check that the Morse decomposition is perfect for $\tilde{Y}_k^-$. The fixpts for the flow are pairs (g, W) where g has eigenvalues ± 1 on W. Thus the fixpts are

$$\coprod_{0 \leq s \leq k} D_{s,k}(V) = \text{full Grass bundle of } S \text{ over } Gr_k(V)$$

We know

$$H^*(\tilde{Y}_k^-) = H^*(Gr_k(V)) \otimes H^*(U_k)$$

because in the universal case the fibre is totally nonhomologous to zero. Since

$$\bigoplus_s H^*(D_{s,k}(V)) = H^*(Gr_k(V)) \otimes \underbrace{\bigoplus_s H^*(Gr_s(\mathbb{C}^k))}_{\cong H^*(U_k)}$$

we see the cohomology of $\tilde{Y}_s^-$ and of the fixpt set are of the same sizes. Thus the Morse theory will be perfect.

The hope is to be able to handle the link of the super connection classes and the Grassmannian graph in the general twisted case, possibly by gaining insight from the above fact that the twisted setup is embedded in the trivial setup.

Let's look at the next case where

$$\tilde{Y}_s^- = \left\{ (\overset{s}{K}, \overset{n-s}{I}, \overset{n}{W}) \;\middle|\; \overset{n-s}{I} \begin{array}{c} \nearrow \overset{n}{V^1} \searrow \\ \searrow \overset{n}{W} \nearrow \end{array} \overset{s}{K}\oplus\overset{n}{V^1} \right\}$$

$$= Gr_s(E^0 \oplus E^1)$$

where $E^0 \oplus E^1$ is the vector bundle over

$$F_s = Gr_s(V^0) \times \check{Gr}_s(V^1)$$

whose fibre at (K, I) is $K \oplus V^1/I$. In $\tilde{Y}_s^-$ we have the open set where $W \cap V^1 = I$, i.e. where $W/I \subset K \oplus V^1/I$ is the graph of a map from K to V^1/I. This is the open set

$$\mathrm{Hom}(E^0, E^1) \subset Gr_s(E^0 \oplus E^1)$$

On $Gr_s(E^0 \oplus E^1)$ we are looking for a cohomology class which restricts to $ch(E^0) - ch(E^1)$ over $\mathrm{Hom}(E^0, E^1)$ and which is supported in the singular set: This is where $S \cap E^0 \neq 0$. On the complement of the singular set we have $S \xrightarrow{\sim} E^1$ and $Q \xleftarrow{\sim} E^0$. So the obvious candidate for our class is

$$ch(S) - ch(E^1) = ch(E^0) - ch(Q)$$

So we consider this class on $Gr_s(E^0 \oplus E^1)$, which dies on the complement of the singular set. The first question is whether it comes from a unique class in the cohomology with supports in the singular set.

But we have this backwards flow on $Gr_s(E^0 \oplus E^1)$ under which the singular set is Y_1^+, so the fact that the Morse decomposition is perfect should tell that the cohomology with supports in the singular set is isomorphic to the cohomology vanishing on the complement of the singular set.

At this point it seems desirable to determine whether the superconnection forms,

$$tr_s\left\{e^{u(L^2+dL)}\right\} \qquad L=\begin{pmatrix}0 & -T^*\\ T & 0\end{pmatrix} \quad T: V^0\to V^1$$

which are defined on $Hom(V^0,V^1)$, extend to $Gr_n(V^0\oplus V^1)$. In order to treat this, I propose to use the fmla

$$\frac{1}{u}\, tr_s\left\{e^{u(L^2+dL)}\right\}_{(2k)} = \frac{1}{2\pi i}\int e^{\lambda u}\frac{1}{2k}\, tr_s\left(\frac{1}{\lambda-L^2}dL\right)^{2k} d\lambda$$

and the fact that $tr_s\left(\frac{1}{\lambda-L^2}dL\right)^{2k}$ for $\lambda>0$ is essentially the ~~...~~ kth character form on the Grassmannian relative to a certain metric on V.

The first problem is to see if ~~...~~ the form $tr_s\left(\frac{1}{\lambda-L^2}dL\right)^{2k}$ is globally defined on the Grassmannian for $\lambda\notin\mathbb{R}_{\le 0}$.

Let's first look at the odd case. If $X\in \mathfrak{u}_n$, then its Cayley transform is

$$g=\frac{a+X}{a-X}=(a+X)(a-X)^{-1}$$

where a is fixed >0. Then

$$g^{-1}dg = (a-X)(a+X)^{-1}\left\{dX\frac{1}{a-X}+(a+X)\frac{1}{a-X}dX\frac{1}{a-X}\right\}$$

$$=(a-X)\frac{1}{a+X}\left[1+(a+X)\frac{1}{a-X}\right]dX\,\frac{1}{a-X}$$

$$=2a\,(a-X)\left(\frac{1}{a^2-X^2}dX\right)\frac{1}{a-X}$$

so

$$\operatorname{tr}(g^{-1}dg)^{2k+1} = (2a)^{2k+1} \operatorname{tr}\left(\frac{1}{a^2-X^2}\,dX\right)^{2k+1}$$

Let's now use $a=1$ to embed $\mathfrak{U}_n \subset U_n$.
The above forms on the right are essentially the transform of the forms

$$2^{2k+1} \operatorname{tr}\left(\frac{1}{1-X^2}\,dX\right)^{2k+1}$$

under the rescaling map $X \mapsto \frac{X}{a}$ which we know extends from $\mathfrak{U}_n$ to U_n. Thus we see that the forms

$$\operatorname{tr}\left(\frac{1}{\lambda-X^2}\,dX\right)^{2k+1}$$

☆

are globally defined on U_n for any $\lambda > 0$.

But now we can check directly that ☆ is globally defined on U_n starting from

$$g = \frac{1+X}{1-X} \qquad X = \frac{g-1}{g+1} = 1 - \frac{2}{g+1}$$

$$dX = 2\,\frac{1}{g+1}\,dg\,\frac{1}{g+1}$$

$$\operatorname{tr}\left(\frac{1}{\lambda-X^2}\,dX\right)^{2k+1} = 2^{2k+1} \operatorname{tr}\left(\frac{1}{g+1}\,\frac{1}{\lambda-\left(\frac{g-1}{g+1}\right)^2}\,\frac{1}{g+1}\,dg\right)^{2k+1}$$

$$= 2^{2k+1} \operatorname{tr}\left(\frac{1}{\lambda(g+1)^2-(g-1)^2}\,dg\right)^{2k+1}$$

Now this is smooth in $g \in U_n$, unless

$$\lambda(\zeta+1)^2 = (\zeta-1)^2 \qquad \text{for some } |\zeta| = 1.$$

$$\Longrightarrow \lambda = \left(\frac{\zeta-1}{\zeta+1}\right)^2 \leq 0$$

August 22, 1986

Let's go back to $Gr_n(V^0 \oplus \overset{n}{V^1})$, the critical submanifold $F_s = Gr_s(V^0) \times \check{Gr}_s(V^1)$, and the resolution

$$\tilde{Y}_s^- = \left\{ (\overset{s}{K}, \overset{n-s}{I}, \overset{n}{W}) \;\middle|\; \overset{n-s}{I} \begin{smallmatrix} & \overset{n}{V^1} & \\ \nearrow & & \searrow \\ \searrow & & \nearrow \\ & \overset{n}{W} & \end{smallmatrix} \overset{s}{K} \oplus \overset{n}{V^1} \right\}$$

of the descending "bowl" from F_s (I could also say the resolution of the closure of the expanding submanifold through F_s).

We have pointed out that

$$\tilde{Y}_s^- = Gr_s(E^0 \oplus E^1)$$

where E^0, E^1 are the bundles over $Gr_s(V^0) \times \check{Gr}_s(V^1)$ with fibres $K, V^1/I$ resp. at (K, I). Thus

$$E^0 = pr_1^*(\overset{s}{\mathcal{S}}) \qquad E^1 = pr_2^*(\mathcal{Q}^s).$$

So [redacted] as $\dim V^0, V^1 \longrightarrow \infty$, $\tilde{Y}_s^-$ is a universal bundle for describing a pair of rank s vector bundles and a correspondence (of degree 0) between them, where by such a corresp. we mean a [redacted] rank s subbundle $\Gamma \subset E^0 \oplus E^1$.

The singular set for such a correspondence is where $\Gamma \cap E^0 \neq 0$. The cohomology class

$$* \qquad ch(\Gamma) - ch(E^1) = ch(E^0) - ch(E/\Gamma)$$

vanishes off the singular set, and so it can be [redacted] refined to a class with [redacted] supports in the singular set.

Returning to $\tilde{Y}_s^-$ we have $\Gamma = W/I \subset K \oplus V^1/I$ at (K, I, W). So denoting by script letters the vector [redacted] bundles we have

$$\Gamma = \mathcal{W}/\mathcal{I} \subset \underbrace{\mathcal{K}}_{E^0} \oplus \underbrace{V^1/\mathcal{I}}_{E^1}$$

$$ch(\mathbb{T}) - ch(E^1) = ch(\mathcal{W}/\mathcal{J}) - ch(\tilde{V}^1/\mathcal{J})$$
$$= ch(\mathcal{W}) - ch(\tilde{V}^1)$$

In other words the character class on $Gr_s(\overset{s}{E}{}^0 \oplus \overset{s}{E}{}^1)$ we are interested in is the restriction of the same class over $Gr_n(V)$.

How to describe the above. One should think of the W in (K, I, W) as being a correspondence from V^0 to V^1. If $W \cap V^1 = 0$, then it is the graph of a map from V^0 to V^1 and conversely. So $\tilde{Y}_s^- = \{(K, I, W)\}$ is ~~the~~ the space of correspondences W with given domain K and indeterminacy I, at least. (The open set where $I = W \cap V^1$ and $K = V^0 \cap (W + V^1)$ is the space of correspondences with domain of codim $n-s$ and indeterminacy of dim $n-s$.)

Somehow by allowing maps "with infinities" we reach a twisted situation.

Now we know already that the basic character class $ch(\mathcal{W}) - ch(V^1)$ on $Gr_n(V)$ comes from a class on $\mathbb{P}(V^0) \times \check{\mathbb{P}}(V^1)$ via the correspondence

$$\begin{tikzcd} \tilde{Y}^+ \arrow[r] \arrow[d] & \mathbb{P}(V^0) \times \check{\mathbb{P}}(V^1) \\ Gr_n(V) & \end{tikzcd}$$

I decided before going on to check the result that the form $ch(\mathcal{W}) - ch(V^1)$ really does come via the above correspondence. To do this it will suffices to show that for any $s \geq 2$, the correspondence

$$\begin{tikzcd} \tilde{Y}_s^- \arrow[r] \arrow[d, "\pi"] & Gr_n(V) \\ Gr_s(V^0) \times \check{Gr}_s(V') & \end{tikzcd}$$

V'

$I \quad W \quad K \oplus V'$

kills $ch(\mathcal{W}) - ch(V')$. Now we have seen

$$\tilde{Y}_s^- = Gr_s(\underset{\substack{\parallel \\ pr_1^*(S)}}{E^0} \oplus \underset{\substack{\parallel \\ pr_2^*(Q)}}{E^1}) \quad \text{over} \quad \underbrace{Gr_s(V^0) \times \check{Gr}_s(V')}_{F_s}$$

and that the class $ch(\mathcal{W}) - ch(V')$ pulls back to

$$ch(\mathbb{T}) - ch(E')$$

$\mathbb{T}$ denoting the subbundle over $Gr_s(E)$. Now E' comes from F_s = the base of π, so $\pi_* ch(E') = (\pi_* 1) \cdot ch(E') = 0$ provided $s > 0$. We are therefore reduced to proving the

<u>Lemma</u>: Let $\pi : Gr_s(E) \longrightarrow X$ be the Grassmannian bundle of s planes in the rank n bundle E/X. Then $\pi_* ch(S) = 0$ provided $1 < s < n-1$.

Proof. We can suppose E is the canon. subbundle over $X = Gr_n(V)$, whence $Gr_s(E) = D_{s,n}(V)$.

$$\begin{tikzcd} D_{1,2,\ldots,s,n} \arrow[r] \arrow[d] \arrow[dd, bend right, "f"'] & D_{1,2,\ldots,s} \arrow[d] \\ D_{s,n} \arrow[r] \arrow[d, "\pi"] & Gr_s \\ Gr_n & \end{tikzcd}$$

Now $ch(S)$ on $D_{s,n}$ or Gr_s pulls up to $\sum_1^s e^{-u_i}$ on $D_{1,2,\ldots,s,n}$ or $D_{1,2,\ldots,s}$. So we want to compute

$$f_* \left\{ u_1^{s-1} \cdots u_s^0 \sum_1^s e^{-u_i} \right\}$$

To do this we embed $D_{1,2,\cdot s,n} \subset Gr_n \times (\mathbb{P}V)^s$ and integrate over $(\mathbb{P}V)^s$. The cohomology class of this submanifold is

$$\underbrace{\prod_{i>j}(u_i-u_j)}_{\substack{\text{condition of}\\ \text{orthogonality}\\ \text{for } L_1,\dots,L_n}} \cdot \prod_i \underbrace{\left\{u_i^{N-n}+\cdots+c_{N-n}(Q)\right\}}_{\substack{\text{condition that}\\ L_i \subset E \quad \text{or that } L_i \to V/E\\ \text{be zero.}}}$$

So we need the coefficient of $u_1^{N-1} \cdots u_s^{N-1}$ in

$$\prod_{i>j}(u_i-u_j) \prod_i \left\{u_i^{N-n}+\cdots+c_{N-n}(Q)\right\} u_1^{s-1}\cdots u_s^0 \sum e^{-u_i}$$

Recall that the coefficient of $u_1^{N-1}\cdots u_s^{N-1}$ in

$$\prod_{i>j}(u_i-u_j) \prod_i \left\{u_i^{N-n}+\cdots+c_{N-n}(Q)\right\} u_1^{a_1}\cdots u_s^{a_s}$$

$$\begin{vmatrix} u_1^{a_1} & u_2^{a_2} & & u_s^{a_s} \\ u_1^{a_1+1} & u_2^{a_2+1} & & \\ & & & \\ u_1^{a_1+s-1} & u_2^{a_2+s-1} & & u_s^{a_s+s-1} \end{vmatrix}$$

$N-1 \Rightarrow a_1-n-j = -1$

$u_1^{a_1} \; u_1^{N-n-j} \; c_j$

$\Rightarrow j = a_1+1-n$

is

$$\begin{vmatrix} c_{a_1+1-n} & c_{a_2+1-n} & \cdots & c_{a_s+1-n} \\ c_{a_1+2-n} & & & \\ & & & \\ c_{a_1+s-n} & c_{a_2+s-n} & & a_{a_s+s-n} \end{vmatrix}$$

Take the case $a_i = s-i+b_i$

$$\begin{vmatrix} c_{b_1+s-n} & c_{b_2+s-n-1} & \cdots & c_{b_s+s-n-s-1} \\ c_{b_1+s-n+1} & c_{b_2+s-n} & & \\ \vdots & & & \\ c_{b_1+s-n+s-1} & & & c_{b_s+s-n} \end{vmatrix}$$

Now when we integrate $ch_k(8)$ we want to add up the results when one b_i is k and the rest are zero. We are assuming $1 < s < n-1$, so the above determinant is obtained from

$$\begin{vmatrix} c_{s-n} & c_{s-n-1} & \\ c_{s-n+1} & c_{s-n} & \ddots \\ & c_{s-n+1} & \ddots \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{vmatrix}$$

1's in first non-zero diag

by replacing one column. By the minor expansion relative to this column we get zero.

If $s = n-1$, then we have one non-zero minor for the last column, and we get the sign $(-1)^{n-1}$. Then $ch_k(8)$ integrates to

$$(-1)^{n-1} c_{k-n-1}(2)$$

If $s=1$, $ch_k(8)$ integrates to

$$c_{k+1-n}(2)$$

Now where are we? I think we now can see in the general twisted case that the class $ch($[illegible]$) - ch(E')$ on $Gr_A(E^0 \oplus E')$ is in the $s=1$ summand. I ought to be able to give two proofs, the first using that $ch_k(\mathcal{S})$ comes from a class on $\mathbb{P}V^0 \times \check{\mathbb{P}}V^1$ via $\tilde{Y}_1^+$, the second being to show that for $s \neq 1$ the class $ch_k(\Gamma) - ch_k(E^1)$ is killed if one maps it to $H^*(Gr_s(E^0) \times \check{Gr}_s(E^1))$ via the twisted $\tilde{Y}_s^-$.

Let's work out the first proof:

$$\begin{tikzcd} \tilde{Y}_s^- \times_X \tilde{Y}_1^+ \arrow[r] \arrow[d] & \tilde{Y}_1^+ = \{(L,H,\Gamma)\} \arrow[r] \arrow[d] & \mathbb{P}V^0 \times \check{\mathbb{P}}V^1 \\ \tilde{Y}_s^- = \{(K,I,W)\} \arrow[r] & Gr_n(V) = X & \end{tikzcd}$$

The fibre product consists of $(\overset{s}{K}, \overset{1}{L}, \overset{n-s}{I}, \overset{n-1}{H}, \overset{n}{W})$, with K, L in V^0 and I, H in V^1,

such that

$$\begin{tikzcd}[column sep=small] & V^1 \arrow[dr, dash] & \\ I \arrow[ur, dash] \arrow[dr, dash] & & K \oplus V^1 \\ & W \arrow[ur, dash] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & V^0 \arrow[dr, dash] & \\ L \arrow[ur, dash] \arrow[dr, dash] & & V^0 \oplus H \\ & W \arrow[ur, dash] & \end{tikzcd}$$

It then follows that $L \subset K$, $I \subset H$ and

$$L \oplus I \subset W \subset K \oplus H$$

But from the viewpoint of $E^0 = K$, $E^1 = \check{V}/I$ we have $L \in \mathbb{P}(E^0)$, $H/I \in \check{\mathbb{P}}(E^1)$, and $\Gamma = W/I \in Gr_s(E^0 \oplus E^1)$ such that

$$\begin{tikzcd}[column sep=small] & K \arrow[dr, dash] & \\ L \arrow[ur, dash] \arrow[dr, dash] & & K \oplus H/I \\ & W/I \arrow[ur, dash] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & E^0 \arrow[dr, dash] & \\ L \arrow[ur, dash] \arrow[dr, dash] & & E^0 \oplus \mathcal{H} \\ & \Gamma \arrow[ur, dash] & \end{tikzcd}$$

Thus the fibre product sits over $\mathbb{P}(E^0) \times \check{\mathbb{P}}(E^1)$ and it is the twisted version of $\tilde{Y}_1^+$.

Note that we have an obvious map

$$\mathbb{P}(E^0) \times \check{\mathbb{P}}(E^1) \longrightarrow \mathbb{P}(V^0) \times \check{\mathbb{P}}(V^1)$$

so that the class we found on the latter which gives rise to the character over $Gr_n(V)$ will induce a similar case in the twisted case. It might be interesting to do the calculation, although I remember this was very tricky.

August 25, 1986

Recall that if g is an automorphism of a vector bundle equipped with inner product E and if D is a connection on E preserving the inner product, then we can define a closed odd degree form

$$1) \qquad \int_0^1 dt \; tr_E \left(e^{(D+t g^{-1}[D,g])^2} g^{-1}[D,g] \right).$$

This is the Chern-Simons difference form associated to the linear path of connections

$$(1-t)D + t\, g^{-1} D g \;=\; D + t\, g^{-1}[D,g]$$

On the other hand suppose $E \xrightarrow{i} \tilde{V}$ is an isometric embedding into a trivial bundle and $D = i^* . d . i$ is the induced connection on E; according to a theorem of some Indians any D can be obtained this way. We can extend g to a unitary auto $\tilde{g}$ by letting $\tilde{g} =$ constant on $(iE)^\perp$. Then $\tilde{g}$ is a map from the underlying manifold X to $U(V)$ and we can pull back the odd character forms. This gives the form

$$2) \qquad \int_0^1 dt \; tr_V \left(e^{(d + t \tilde{g}^{-1} d\tilde{g})^2} \tilde{g}^{-1} d\tilde{g} \right)$$

A natural question is whether the forms 1), 2) are the same. This appears unlikely although it seems likely that a) the forms are cohomologous and b) the form 2) depends only on g and $D = i^* d\, i$ but not on the embedding i.

<u>August 31, 1986</u>

Let's recall that

$$\tilde{Y}_s^- = \left\{ (\overset{s}{K}, \overset{n-s}{I}, \overset{n}{W}) \;\middle|\; \begin{tikzcd}[column sep=small, row sep=small] & \check{V}' \arrow[dr] & \\ \overset{n-s}{\mathbb{I}} \arrow[ur] \arrow[dr] & & \overset{s}{K} \oplus \overset{n}{\check{V}}' \\ & \overset{n}{W} \arrow[ur] & \end{tikzcd} \right\}$$

is $Gr_s(\mathcal{K} \oplus \tilde{V}'/\mathcal{I})$ over $Gr_s(V^0) \times \check{Gr}_s(V')$, and that if $\mathbb{\Gamma} = \mathcal{W}/\mathcal{I}$ is the canonical subbundle, we are interested in the class

$$ch(\mathbb{\Gamma}) - ch(\tilde{V}'/\mathcal{I}) = ch(\mathcal{W}) - ch(\tilde{V}')$$

We have three ways to represent this class:

i) character forms of the subbundle $\mathcal{W}$ on $Gr_n(V)$

ii) " " associated to canonical connections on $\mathbb{\Gamma}$ and $\tilde{V}'/\mathcal{I}$

iii) superconnection forms associated to the connections on $\mathcal{K}$ and $\tilde{V}'/\mathcal{I}$ and the maps $\mathcal{K} \Longrightarrow \tilde{V}'/\mathcal{I}$ associated to $\mathbb{\Gamma}$.

It seems likely that these are all different and that the exact relation is complicated. One way to try to understand the situation might be to use the flow φ_t and to identify the limits as $t \to \infty$.

So a natural question or problem is to investigate $\lim \varphi_t^*(\gamma_k)$ where γ_k is say the kth character form. We have seen the limit is a current which is the image of a smooth form on $\tilde{Y}_1^+$. <u>Question</u>: Is it possible to refine this fact to some sort of asymptotic behavior in analogy to the way deMoivre-Laplace refines the Law of Large Nos?

Let us consider the ungraded case.

$$g = \frac{1+X}{1-X}, \quad \theta = g^{-1}dg = 2\,\frac{1}{1+X}\,dX\,\frac{1}{1-X}$$

$$X = \frac{g-1}{g+1} \qquad dX = 2\,\frac{1}{g+1}\,dg\,\frac{1}{g+1}$$

$$\varphi_t^*(\theta) = 2t\,\frac{1}{1+tX}\,dX\,\frac{1}{1-tX}$$

$$= 2t\,\frac{1}{1+t\frac{g-1}{g+1}}\,2\,\frac{1}{g+1}\,dg\,\frac{1}{g+1}\,\frac{1}{1-t\frac{g-1}{g+1}}$$

$$= 4t\,\frac{1}{(g+1)+t(g-1)}\,dg\,\frac{1}{(g+1)-t(g-1)}$$

$$\varphi_t^*(\operatorname{tr}\theta^{2k+1}) = \operatorname{tr}\left(\frac{4t}{(g+1)^2-t^2(g-1)^2}\,dg\right)^{2k+1}$$

This form approaches 0 as $t \to +\infty$ near any point g where g does not have the eigenvalue $+1$; this is the complement of $\overline{Y}_1^+$. Recall that

$$\tilde{Y}_s^+ = \{(\overset{s}{K}, g) \mid g = 1 \text{ on } K\} = \text{auto bdle of } Q \text{ over } Gr_s(V)$$

$$\tilde{Y}_s^- = \{(\overset{s}{W}, g) \mid g = -1 \text{ on } W^\perp\} = \text{auto bundle of } S \text{ over } Gr_s(V)$$

Note that we have a map

$$\tilde{Y}_s^- \times_{F_s} \tilde{Y}_s^+ \longrightarrow U(V)$$

which assembles an auto of K and $K^\perp$ into an auto of V. This map is a branched covering of degree $\binom{N}{s}$, since given g with N distinct eigenvalues one has to choose s of them to specify the subspace $\overset{s}{K}$.

Let's next work out the limit of the graph of φ_t in $U(V) \times U(V)$. If (g, g') is the limit of $(g_n, \varphi_{t_n}(g_n))$, then it's clear, because φ_t moves the eigenvalues from $+1$ to -1, that where $g \neq +1$ must be contained in where $g' = -1$:

$$\operatorname{Ker}(g-1)^{\perp} \subset \operatorname{Ker}(g'+1)$$

Similarly where $g' \neq -1$ must be contained in where $g = 1$:

$$\operatorname{Ker}(g'+1)^{\perp} \subset \operatorname{Ker}(g-1)$$

These two conditions are the same, and I'll suppose that this condition specifies the limit of the graph of φ_t as $t \to +\infty$. Note that

$$\tilde{Y}_s^- \times_{F_s} \tilde{Y}_s^+ = \left\{ (g', g, \overset{s}{K}) \;\middle|\; \begin{array}{l} g' = -1 \text{ on } W^\perp \\ g = +1 \text{ on } W \end{array} \right\}$$

$$\Downarrow$$

$$\begin{array}{l} W^\perp \subset \operatorname{Ker}(g'+1) \\ W \subset \operatorname{Ker}(g-1) \end{array}$$

$$\Downarrow$$

$$\operatorname{Ker}(g-1)^\perp \subset W^\perp \subset \operatorname{Ker}(g'+1)$$

Thus

$$\lim_{t \to +\infty} \Gamma_{\varphi_t} = \coprod_s \operatorname{Im}\left\{ \tilde{Y}_s^- \times_{F_s} \tilde{Y}_s^+ \longrightarrow U(V) \times U(V) \right\}$$

September 2, 1986

Look at superconnection forms associated to a graded v.b. $E = E^0 \oplus E^1$ with connection $D = D^0 \oplus D^1$ and $L = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$.

$$\int_0^\infty \left\{ tr_s\, e^{u(L^2+[D,L]+D^2)} - tr_s\, e^{uL^2} \right\} e^{-\lambda u} \frac{du}{u}$$

$$= \sum_{k=1}^\infty \frac{1}{k}\, tr_s \left(\frac{1}{\lambda - L^2}([D,L]+D^2) \right)^k$$

Let us look at the component of degree 2

$$* \quad \boxed{\frac{1}{2}\, tr_s \left(\frac{1}{\lambda - L^2}[D,L]\right)^2 + tr_s\left(\frac{1}{\lambda - L^2} D^2\right)}$$

and let's review what we once worked out ~~about~~ this form relative to the Grassmannian graph construction.

Let $\Gamma = \operatorname{Im}\begin{pmatrix} 1 \\ T \end{pmatrix} \subset E^0 \oplus E^1$

$\Gamma^\perp = \operatorname{Im}\begin{pmatrix} -T^* \\ 1 \end{pmatrix} \subset E^0 \oplus E^1$.

We want the connection induced on $\Gamma, \Gamma^\perp$ from $D = D^0 \oplus D^1$. Let's use the isomorphism (not unitary)

$$E^0 \oplus E^1 \xrightarrow{\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}} E^0 \oplus E^1$$

to identify E^0, E^1 with $\Gamma, \Gamma^\perp$. Then the connection on $\Gamma \oplus \Gamma^\perp$ which is the sum of the induced connections on $\Gamma, \Gamma^\perp$ becomes under this isomorphism

diag part of $\frac{1}{1+L} D(1+L)$

$$= \text{diag part of } \underbrace{D + \frac{1}{1+L}[D,L]}_{D + \frac{1}{1-L^2}(1-L)[D,L]}$$

$$= D - \frac{1}{1-L^2} L[D,L]$$

In the above we are treating L as an endo of E and $[D,L] = DL - LD$. In other words we are working with the usual algebra of $\mathrm{End}(E)$-valued forms.

Let's next compute the curvature of this connection on $E^0 \oplus E^1$ which we recall is isomorphic to the sum of the connections on Π and $\Pi^\perp$.

$$\left(D - \frac{1}{1-L^2}L[D,L]\right)^2 = D^2 - \left[D, \frac{1}{1-L^2}L[D,L]\right] + \left(\frac{1}{1-L^2}L[D,L]\right)^2$$

$$= D^2 - \frac{1}{1-L^2}\left([D,L]L + \cancel{L[D,L]}\right)\frac{1}{1-L^2}L[D,L] + \left(\cancel{\frac{1}{1-L^2}L[D,L]}\right)^2$$

$$- \frac{1}{1-L^2}[D,L]^2 - \frac{1}{1-L^2}L[D,[D,L]]$$

$$= \left(D^2 - \frac{1}{1-L^2}L[D^2,L]\right) - \left(\frac{1}{1-L^2}[D,L]\right)^2$$

So the class ~~[illegible]~~ $\frac{1}{2}\left(ch_1(\Pi) - ch_1(\Pi^\perp)\right)$ is represented by

$$-\frac{1}{2}\,tr_s\left(\frac{1}{1-L^2}[D,L]\right)^2 + \frac{1}{2}\,tr_s\left(D^2 - \frac{1}{1-L^2}LD^2L + \frac{1}{1-L^2}L^2D^2\right)$$

as $tr_s(XL) = tr(\varepsilon XL) = tr(L\varepsilon X) = -tr(\varepsilon LX)$
$= -tr_s(LX)$, we see this is

$$-\frac{1}{2}\operatorname{tr}_s\left(\frac{1}{1-L^2}[D,L]\right)^2 + \frac{1}{2}\operatorname{tr}_s\left(D^2 + 2\frac{1}{1-L^2}L^2D^2\right)$$

$$= -\frac{1}{2}\operatorname{tr}_s\left(\frac{1}{1-L^2}[D,L]\right)^2 + \operatorname{tr}_s\left(\frac{1}{1-L^2}D^2\right) - \frac{1}{2}\operatorname{tr}_s(D^2)$$

Now the form $*$ on p. 97 is computed in the superconnection formalism, so $\frac{1}{1-L^2}[D,L]$ which is a 1 form with values in odd endos when squared in

$$\Omega^\cdot(M)\,\hat{\otimes}\,\Omega^0(\operatorname{End}E)$$

is minus the square in

$$\Omega^\cdot(M)\otimes\Omega^0(\operatorname{End}E)$$

Thus working in the latter algebra we see the form $*$ is

$$-\frac{1}{2}\operatorname{tr}_s\left(\frac{1}{\lambda-L^2}[D,L]\right)^2 + \operatorname{tr}_s\left(\frac{1}{\lambda-L^2}D^2\right)$$

Therefore we have

$$\text{super conn } ch_1 = \underbrace{\text{Grass graph form}}_{\text{rep } \frac{1}{2}(ch_1(\Gamma)-ch_1(\Gamma^\perp))} + \underbrace{\frac{1}{2}\operatorname{tr}_s(D^2)}_{\text{rep } \frac{1}{2}\{ch_1(E^0)-ch_1(E^1)\}}$$

so we conclude

$$\boxed{\begin{array}{l}\text{superconn } ch_1 \text{ represents } \frac{1}{2}\{ch_1(\Gamma)-ch_1(E^1) + ch_1(E^0)-ch_1(\Gamma^\perp)\} \\ \quad = ch_1(\Gamma)-ch_1(E^1)\end{array}}$$

Now recall from our earlier work the problem encountered with linking Grass graph forms + superconn. ones, namely that the former didn't appear to be expressible in terms of D^2, $[D,L]$, L^2. Today I noticed another nice feature of the superconnection forms, namely they seem to transform nicely under rescaling $L \mapsto tL$.

To check this let us write

$$\sum \frac{1}{k} \operatorname{tr}\left(\frac{1}{\lambda - L^2}([D,L]+D^2)\right)^k = \sum_{a \geq 0} \omega_\lambda^a$$

where ω_λ^a is a form of degree a. Think of these forms as living on $Gr_s(E^0 \oplus E^1)$, except that I haven't checked this, so at the moment all I know is that they lives on $\mathrm{Hom}(E^0 \oplus E^1)$. Notice that ω_λ^a is a sum

$$\omega_\lambda^a = \sum_{b+2c=a} \frac{1}{b+c} \operatorname{tr}\Big(\text{sum of all } \tbinom{b+c}{c} \text{ monomials in } X_\lambda = \frac{1}{\lambda - L^2}[D,L],\ Y_\lambda = \frac{1}{\lambda - L^2} D^2 \text{ of bidegree } b, c\Big)$$

Now apply the transformation $\varphi_t : L \longrightarrow tL$. As

$$\varphi_t^*\left(\frac{1}{\lambda - L^2}[D,L]\right) = \frac{1}{\lambda - t^2L^2}\, t\,[D,L] = \frac{1}{t}\,\frac{1}{t^{-2}\lambda - L}[D,L]$$

$$\varphi_t^*\left(\frac{1}{\lambda - L^2} D^2\right) = \frac{1}{t^2}\,\frac{1}{t^{-2}\lambda - L^2} D^2$$

we see that

$$\boxed{\varphi_t^* \omega_\lambda^a = \frac{1}{t^a}\, \omega_{t^{-2}\lambda}^a}$$

<u>September 3, 1986</u>

I want to show that the superconnection forms are globally defined on the grassmannian bundle even in the twisted case.

In more detail let E^0, E^1 be vector bundles/M with inner product and let D^i be a connection on E^i compatible with the inner product. Over M we have bundles

$$\mathrm{Hom}(E^0,E^1) \subset Gr_s(E^0\oplus E^1) \qquad s = \mathrm{rank}\, E^0$$

and if we lift E^0, E^1 up to $\mathrm{Hom}(E^0,E^1)$ we have a tautological $L = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$, $T : \tilde{E}^0 \to \tilde{E}^1$ (resp. a tautological subbundle $\Gamma \subset \widetilde{E^0\oplus E^1}$ in the case of the Grassmannian bundle). Then over $\mathrm{Hom}(E^0,E^1)$ we have the superconnection forms

$$tr_s\, e^{u(L^2+[D,L]+D^2)} \qquad Re(u)>0$$

and what we intend to prove is that these forms extend smoothly to $Gr_s(E^0\oplus E^1)$. [redacted] We use the transforms

$$\int_0^\infty \left\{ tr_s\left(e^{u(L^2+[D,L]+D^2)}\right) - tr_s\left(e^{uL^2}\right)\right\} e^{-\lambda u}\, \frac{du}{u}$$

$$(*) \quad = \sum_{k=1}^{\infty} \frac{1}{k}\, tr_s\left(\frac{1}{\lambda - L^2}\left([D,L]+D^2\right)\right)^k$$

Now all this is calculated in the algebra

$$* \qquad \left(\Omega(M)\hat{\otimes}\Gamma(\mathrm{End}\,E)\right)^{ev} = \Omega^{ev}(M)\otimes\Gamma(\mathrm{End}^{ev}E) \oplus \Omega^{odd}(M)\otimes\Gamma(\mathrm{End}^{odd}E)$$

and it will be nicer to calculate in the usual algebra of endomorphism-valued forms.

$$\Omega(M) \otimes \Gamma(\mathrm{End}\, E)$$

Let us denote the product in the former algebra with a $*$. Then we have relative to the decomposition

$$(a+b)*(a'+b') = aa' + ab' + ba' - bb'$$

So if ψ is the auto of $\left(\Omega(M) \otimes \Gamma(\mathrm{End}\, E)\right)^{ev}$ given by

$$\psi(a+b) = a + ib \qquad \begin{matrix} a \in \Omega^{ev}(M) \otimes \Gamma(\mathrm{End}^{ev} E) \\ b \in \Omega^{odd}(M) \otimes \Gamma(\mathrm{End}^{odd} E) \end{matrix}$$

we have

$$\psi\left((a+b)*(a'+b')\right) = aa' + i(ab' + ba') - bb'$$

$$= (a+ib)(a'+ib') = \psi(a+b)\,\psi(a'+b')$$

Thus to compute the product in the superconnection sense all I have to do is put in i's, i.e. apply ψ, then use the usual product and then remove the i's. This means that the form $\circledast$ above is the form

$$\sum_{k=1}^{\infty} \frac{1}{k}\, tr_s\left(\frac{1}{\lambda - L^2}\left(i[D,L] + D^2\right)\right)^k \qquad \text{computed in the algebra } \Omega(M) \otimes \Gamma(\mathrm{End}\, E)$$

since the supertrace kills odd endos of E.

Next let's recall that the embedding

$$\mathrm{Hom}(E^0, E^1) \subset Gr(E^0 \oplus E^1)$$

is given by the Cayley transform

$$g = \frac{1+L}{1-L}$$

in the following sense. Since $\varepsilon L \varepsilon^{-1} = -L$, one has $\varepsilon g \varepsilon^{-1} = g^{-1}$ so $(g\varepsilon)^2 = 1$; this involution defines a point of the Grassmannian. Formulas:

$$L = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \qquad g = (1+L)^2 \frac{1}{1-L^2}$$

$$g\varepsilon = \begin{pmatrix} \frac{1-T^*T}{1+T^*T} & +2T^*/1+TT^* \\ 2T/1+T^*T & -\frac{1-TT^*}{1+TT^*} \end{pmatrix} \qquad \begin{aligned} g\varepsilon &= +1 \text{ on } \mathrm{Im}\begin{pmatrix}1\\1\end{pmatrix} \\ &= -1 \text{ on } \mathrm{Im}\begin{pmatrix}-T^*\\1\end{pmatrix}\end{aligned}$$

Thus we propose to parametrize the Grassmannian using unitaries g such that $\varepsilon g \varepsilon^{-1} = g^{-1}$.

$$g = \frac{1+L}{1-L} \qquad L = \frac{g-1}{g+1}$$

$$[D,L] = \frac{1}{g+1}[D,g] - \frac{1}{g+1}[D,g]\frac{1}{g+1}\, g-1$$

$$= \frac{1}{g+1}[D,g]\left(1 - \frac{g-1}{g+1}\right) = 2\frac{1}{g+1}[D,g]\frac{1}{g+1}$$

$$\frac{1}{\lambda - L^2} = \frac{1}{\sqrt{\lambda}-L}\;\frac{1}{\sqrt{\lambda}+L}$$

$$= \frac{1}{\sqrt{\lambda} - \frac{g-1}{g+1}}\;\frac{1}{\sqrt{\lambda}+\frac{g-1}{g+1}}$$

$$= \frac{g+1}{\sqrt{\lambda}(g+1)-(g-1)}\cdot\frac{g+1}{\sqrt{\lambda}(g+1)+(g-1)}$$

Note that
$$\frac{1}{\sqrt{\lambda}\pm L} = \frac{g+1}{\sqrt{\lambda}\,(g+1)-(g-1)} \qquad \text{for } \sqrt{\lambda}\notin i\mathbb{R} \text{ or } \lambda \notin \mathbb{R}_{\leq 0}$$

is smooth as a function ~~on~~ the unitary group. Also the form

$$\frac{1}{\sqrt{\lambda}+L}[D,L]\frac{1}{\sqrt{\lambda'}-L} = 2\frac{1}{\sqrt{\lambda}(g+1)+(g-1)}[D,g]\frac{1}{\sqrt{\lambda}(g+1)-(g-1)}$$

is smooth. Since

$$tr_s\left(\frac{1}{\lambda-L^2}\left(i[D,L]+D^2\right)\right)^k$$

$$\cancel{= tr_s\left(\frac{1}{\sqrt{\lambda}+L}\left(i[D,L]+D^2\right)\frac{1}{\sqrt{\lambda}-L}\right)^k}$$

$$= tr\Bigg(\underbrace{\varepsilon\frac{1}{\sqrt{\lambda}-L}}_{\frac{1}{\sqrt{\lambda}+L}\varepsilon}\ \frac{1}{\sqrt{\lambda}+L}\left(i[D,L]+D^2\right)\frac{1}{\sqrt{\lambda}-L}\,\frac{1}{\sqrt{\lambda}+L}\left(\qquad\right)\cdots$$

$$\cdots\frac{1}{\sqrt{\lambda}-L}\,\frac{1}{\sqrt{\lambda}+L}\left(i[D,L]+L^2\right)\Bigg)$$

$$= tr\left(\varepsilon\left(\frac{1}{\sqrt{\lambda}+L}\left(i[D,L]+D\right)^2\frac{1}{\sqrt{\lambda}-L}\right)^k\ \frac{\sqrt{\lambda}-L}{\sqrt{\lambda}+L}\right)$$

is smooth as a function on the unitary grp.

Another proof is based on the fact that when we expand $\left(\frac{1}{\lambda-L^2}\left(i[D,L]+D^2\right)\right)^k$ and use the cyclic property of the trace the bad term $[D,L]$ always appears ~~[illegible]~~ in a grouping

$$\frac{1}{\lambda-L^2}[D,L]\frac{1}{\lambda-L^2}[D,L]\frac{1}{\lambda-L^2}\cdots\frac{1}{\lambda-L^2}[D,L]\frac{1}{\lambda-L^2}$$

which as we have seen is a smooth form on the unitary group which is analytic for $\lambda\notin\mathbb{R}_{\leq 0}$.

So now just use the inverse Laplace transform to see that the original super connection forms are defined on the Grassmannian bundle.

September 5, 1986

Let's try to compute what happens to the form
$$\operatorname{tr}\left(\frac{\sqrt{\lambda}}{\lambda - X^2}\,dX\right)^{2k}$$
as $\lambda \to 0$ near a skew-hermitian X_0 of corank 1. Note that
$$\frac{\sqrt{\lambda}}{\lambda + x^2} \longrightarrow \pi\,\delta(x) \qquad \text{as } \lambda \to 0,$$
but that the product of δ functions is tricky.

~~[illegible]~~

For X near X_0 there is one eigenvalue of X close to zero and the rest are bounded away from zero. So we can use the map
$$\{(L, x, Y) \mid L \in \mathbb{P}(V),\ x \in \mathbb{R},\ Y \text{ skew on } L^\perp\}$$
$$\downarrow$$
$$\mathfrak{u}(N)$$
which assigns to (L, x, Y) the endo. which is ix on L and Y on $L^\perp$. This map is a diffeomorphism near X_0 in a suitable sense.

It seems to be simpler to go ~~to~~ a larger manifold in which we give a basis for L and $L^\perp$. Thus we replace $\mathbb{P}(V)$ by $U(V)$ and we consider triples (g, x, Y), where $g \in U(V)$, $x \in \mathbb{R}$ and $Y \in \mathfrak{u}(N-1)$. The map sends this triple to
$$X = g\begin{pmatrix} ix & 0 \\ 0 & Y \end{pmatrix} g^*.$$

Then

$$dX = g\begin{pmatrix} i\,dx & 0 \\ 0 & dY \end{pmatrix} g^* + dg\begin{pmatrix} ix & 0 \\ 0 & \alpha \end{pmatrix} g^* + g\begin{pmatrix} ix & 0 \\ 0 & Y \end{pmatrix} \underbrace{dg^*}_{-g^* dg\, g^*}$$

$$= g\left\{ \begin{pmatrix} i\,dx & 0 \\ 0 & dY \end{pmatrix} + \left[g^* dg, \begin{pmatrix} ix & 0 \\ 0 & Y \end{pmatrix} \right] \right\} g^*$$

and so

$$\operatorname{tr}\left(\frac{\sqrt{\lambda}}{\lambda - X^2}\, dX \right)^{2k+1} = \operatorname{tr}\left[\begin{pmatrix} \frac{\sqrt{\lambda}}{\lambda + x^2} & 0 \\ 0 & \frac{\sqrt{\lambda}}{\lambda - Y^2} \end{pmatrix} \left\{ \begin{pmatrix} i\,dx & 0 \\ 0 & dY \end{pmatrix} + \left[g^* dg, \begin{pmatrix} ix & 0 \\ 0 & Y \end{pmatrix} \right] \right\} \right]^{2k+1}$$

Let us write

$$g^* dg = \begin{pmatrix} c & -\omega^* \\ \omega & \alpha \end{pmatrix}$$

Then

$$\left[g^* dg, \begin{pmatrix} ix & 0 \\ 0 & Y \end{pmatrix} \right] = \begin{pmatrix} c\,ix & -\omega^* Y \\ \omega\, ix & \alpha Y \end{pmatrix} - \begin{pmatrix} ixc & -ix\omega^* \\ Y\omega & Y\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \omega^*(-Y + ix) \\ (-Y + ix)\omega & [\alpha, Y] \end{pmatrix}$$

$$\begin{pmatrix} \frac{\sqrt{\lambda}}{\lambda + x^2} & 0 \\ 0 & \frac{\sqrt{\lambda}}{\lambda - Y^2} \end{pmatrix} \left\{ \begin{matrix} i\,dx & \omega^*(-Y + ix) \\ (-Y + ix)\omega & dY + [\alpha, Y] \end{matrix} \right\}$$

Thus the form we want to understand is

$$\operatorname{tr}\begin{pmatrix} \frac{\sqrt{\lambda}}{\lambda + x^2}\, i\,dx & \frac{\sqrt{\lambda}}{\lambda + x^2}\, \omega^*(-Y + ix) \\ \frac{\sqrt{\lambda}}{\lambda - Y^2}(-Y + ix)\omega & \frac{\sqrt{\lambda}}{\lambda - Y^2}\left(dY + [\alpha, Y] \right) \end{pmatrix}^{2k+1}$$

in the limit as $\lambda \to 0$. Let us now rescale and put $x = \sqrt{\lambda}\, y$ whence we have

$$\mathrm{tr}\begin{pmatrix} \frac{i\,dy}{1+y^2} & \frac{1}{\sqrt{\lambda}}\frac{1}{1+y^2}\,\omega^*(-Y+i\sqrt{\lambda}\,y) \\ \frac{\sqrt{\lambda}}{\lambda-Y^2}(-Y+i\sqrt{\lambda}y)\omega & \frac{\sqrt{\lambda}}{\lambda-Y^2}(dY+[\alpha,Y]) \end{pmatrix}^{2k+1}$$

$\Vert$ conjugation by $\begin{pmatrix}\sqrt{\lambda} & 0\\ 0 & 1\end{pmatrix}$

$$\mathrm{tr}\begin{pmatrix} \frac{i\,dy}{1+y^2} & \frac{1}{1+y^2}\,\omega^*(-Y+i\sqrt{\lambda}y) \\ \frac{1}{\lambda-Y^2}(-Y+i\sqrt{\lambda}y)\omega & \frac{\sqrt{\lambda}}{\lambda-Y^2}(dY+[\alpha,Y] \end{pmatrix}^{2k+1}$$

$\downarrow$ as $\lambda \to 0$ assuming Y^{-1} exists

$$\mathrm{tr}\begin{pmatrix} \frac{i\,dy}{1+y^2} & -\frac{\omega^* Y}{1+y^2} \\ \frac{1}{Y}\omega & 0 \end{pmatrix}^{2k+1} = \mathrm{tr}\,(a+b)^{2k+1}$$

where $a = \begin{pmatrix} \frac{i\,dy}{1+y^2} & 0 \\ 0 & 0\end{pmatrix}$ $\quad b = \begin{pmatrix} 0 & -\frac{\omega^* Y}{1+y^2} \\ \frac{1}{Y}\omega & 0\end{pmatrix}$ are matrices

of 1-forms and $a^2 = 0$ $\quad b^2 = \begin{pmatrix} -\frac{\omega^*\omega}{1+y^2} & 0 \\ 0 & -\frac{1}{1+y^2}\frac{1}{Y}\omega\omega^*\frac{1}{Y}\end{pmatrix}$.

so

$$\mathrm{tr}(a+b)^{2k+1} = \cancel{\mathrm{tr}(b^{2k+1})} + \mathrm{tr}(ab^{2k}) + \mathrm{tr}(bab^{2k-1}) + \cdots$$

$$= (2k+1)\,\mathrm{tr}(ab^{2k}) = (2k+1)\frac{i\,dy}{1+y^2}\left(\frac{(-\omega^*\omega)}{1+y^2}\right)^k$$

$$\boxed{= (2k+1)\frac{i\,dy}{(1+y^2)^{k+1}}(-\omega^*\omega)^k}$$

Recall that

$$\int_{-\infty}^{\infty}\frac{dy}{(1+y^2)^s} = \frac{1}{\Gamma(s)}\int_{-\infty}^{\infty}\left(\int_0^\infty e^{-t(1+y^2)}\,t^s\,\frac{dt}{t}\right)dy$$

$$= \frac{1}{\Gamma(s)}\int_0^\infty e^{-t}\,\frac{\sqrt{\pi}}{\sqrt{t}}\,t^s\,\frac{dt}{t} = \frac{\sqrt{\pi}\;\Gamma(s-\frac{1}{2})}{\Gamma(s)}$$

so $\displaystyle\int_0^\infty \frac{dy}{(1+y^2)^{k+1}} = \frac{\sqrt{\pi}\,\Gamma(k+\frac{1}{2})}{k!} = \frac{\sqrt{\pi}\,(k-\frac{1}{2})\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{k!}$

$$= \frac{\pi\,(2k-1)!!}{2^k\,k!}$$

On the other hand the Chern character form of degree $2k+1$ is

$$\int_0^1 tr\left(e^{(t^2-t)\theta^2}\theta\right)_{(2k+1)} dt = \frac{1}{k!}tr\left(\theta^{2k+1}\right)\underbrace{\int_0^1 (t^2-t)^k dt}_{(-1)^k\beta(k+1,k+1) = (-1)^k\frac{k!k!}{(2k+1)!}}$$

$$= (-1)^k\frac{k!}{(2k+1)!}\,tr\left(\theta^{2k+1}\right)$$

Thus the character form in the limit at least on the set of X of corank 1 is

$$(-1)^k\frac{k!}{(2k+1)!}(2k+1)\,\frac{i\,dy}{(1+y^2)^{2k+1}}\,(-\omega^*\omega)^k\,2^{2k+1}$$

4 comes from $g^{-1}dg$ "=" $\frac{2}{1+X^2}dX$

Now $-\omega^*\omega$ is probably the curvature of $\mathcal{O}(1)$ over $\mathbb{P}V$ and the $\frac{dy}{(1+y^2)^{2k+1}}$ is the constant $\frac{\pi\,(2k-1)!!}{2^k k!}$ times the Thom class in the normal bundle. So what we have is the numerical factor

$$(-1)^k\frac{k!}{\cancel{(2k+1)!}\;2^k k!}\cancel{(2k+1)}\,i\pi\,\frac{\cancel{(2k-1)!!}}{2^k k!}\,2^{2k+1} = \pm(2i\pi)\frac{1}{k!}$$

and we conclude

Prop: The limit of the character form of degree $2k+1$ on $U(N)$ is the current obtained from the character form of degree $2k$ on $\mathbb{P}V$ via the correspondence

$$\begin{array}{ccc} \tilde{Y}_1^+ & \longrightarrow & \mathbb{P}V \\ \downarrow & & \\ U(N) & & \end{array}$$

plus possible pieces coming from the Y_s^+, $s>1$. (these extra terms should be zero).

September 7, 1986

I want to try to do the above for higher rank, that is, to understand the limit of $\varphi_t(\omega)$ as $t \to \infty$, where ω is a character form, near the submanifold $Y_s^+ - Y_{s+1}^+$ consisting of g having the eigenvalue 1 of multiplicity s.

Recall
$$\tilde{Y}_s^+ = \{(g, \overset{s}{W}) \mid g = 1 \text{ on } W\} = \mathrm{Aut}(\mathcal{Q})$$
$$\tilde{Y}_s^- = \{(g, \overset{s}{W}) \mid g = -1 \text{ on } W^\perp\} = \mathrm{Aut}(\mathcal{S})$$

and that we have a covering (ramified)

1) $$\tilde{Y}_s^+ \times_{F_s} \tilde{Y}_s^- \longrightarrow U(N)$$

of degree $\binom{N}{s}$. This enables us to identify the normal bundle of $Y_s^+ - Y_{s+1}^+$ with the pullback of the normal bundle of F_s in $\tilde{Y}_s^-$ which is simply the skew-adjoint endom. bundle of $\mathcal{S}$.

~~In the following I want to work with skew-adjoint endos instead of unitaries. Thus $g \in U(N)$ will be replaced by $X \in \mathfrak{u}(N)$ the relation being the Cayley transform~~

~~$\mathfrak{u}$~~

~~$$g = \frac{1+X}{1-X}$$~~

~~and the char. form is const. $\mathrm{tr}\left(\frac{1}{1-X^2}dX\right)^{odd}$~~

What I want to do is to pull-back the form $\varphi_t^*(\omega)$ via the map 1) then restrict to a neighborhood of Y_s^+ and rescale in the normal direction with the hope of getting a limit as $t \to \infty$.

Let's use the Cayley transform to work with skew adjoint matrices instead of unitaries. ~~Instead~~ We

~~describe $\tilde{Y}_s^- \times \tilde{Y}_s^+$ in terms of triples (g, g', g'') where~~ need a convenient description of $\tilde{Y}_s^- \times_{F_s} \tilde{Y}_s^+$ which consists of triples consisting of a subspace $\overset{s}{W}$ and autos of W and $W^\perp$. We use the frame bundle idea and consider triples $(g, g', g'') \in \mathcal{U}(N) \times \mathcal{U}(s) \times \mathcal{U}(N-s)$. g gives isom of $\mathbb{C}^s$ with W and $\mathbb{C}^{N-s}$ with $W^\perp$ and g', g'' give the corresponding autos of W and $W^\perp$. In other words

$$\tilde{Y}_s^- \times_{F_s} \tilde{Y}_s^+ = \mathcal{U}(N) \times^{\mathcal{U}(s) \times \mathcal{U}(N-s)} \big(\mathcal{U}(s) \times \mathcal{U}(N-s)\big)$$

where $\mathcal{U}(s) \times \mathcal{U}(N-s)$ acts on itself by conjugation. The map 1) is given by

$$(g, g', g'') \longmapsto g \begin{pmatrix} g' & 0 \\ 0 & g'' \end{pmatrix} g^{-1}$$

Let's now use the Cayley transform to replace g', g'' by skew-adjoint matrices. We are then looking at the bundle

$$\mathcal{U}(N) \times^{\mathcal{U}(s) \times \mathcal{U}(N-s)} \big(\mathfrak{u}(s) \times \mathfrak{u}(N-s)\big)$$

over G_s of skew endos of $(1, 2)$ and the map

$$(g, X', X'') \longmapsto g \left(\begin{array}{c|c} X' & 0 \\ \hline 0 & X'' \end{array}\right) g^{-1}$$

to $\mathfrak{u}(N)$.

We propose to pull back $\varphi_t^*(\omega)$, where

$$\omega = \operatorname{tr}\left(\frac{1}{1-X^2}\, dX\right)^{2k+1}$$

and then rescale suitably in the normal direction to $\tilde{Y}_s^+$ with the idea of getting some limit as $t \to \infty$.

First note that $\varphi_t : X \longrightarrow tX$ so that

$$\varphi_t^*(\omega) = \operatorname{tr}\left(\frac{t}{1-t^2X^2}dX\right)^{2k+1} = \operatorname{tr}\left(\frac{t^{-1}}{t^{-2}-X^2}dX\right)^{2k+1}$$

$$= \operatorname{tr}\left(\frac{\sqrt{\lambda}}{\lambda - X^2}dX\right)^{2k+1} \qquad \lambda = t^{-2}$$

Life will be simpler if we change to

$$\varphi_{t^{-1}}^*(\omega) = \operatorname{tr}\left(\frac{t}{t^2-X^2}dX\right)^{2k+1} \qquad \text{and let } t \to 0$$

We have the map

$$(g, X', X'') \longmapsto g\begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix} g^{-1} = X$$

so

$$dX = g\begin{pmatrix} dX' & 0 \\ 0 & dX'' \end{pmatrix} g^{-1} + dg\begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix} g^{-1}$$

$$+ g\begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix}(-g^{-1}dg\, g^{-1})$$

$$= g\left\{\begin{pmatrix} dX' & 0 \\ 0 & dX'' \end{pmatrix} + \left[g^{-1}dg, \begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix}\right]\right\} g^{-1}$$

Let us write

$$g^{-1}dg = \begin{pmatrix} A' & -B^* \\ B & A'' \end{pmatrix}$$

so that

$$\begin{pmatrix} dX' & 0 \\ 0 & dX'' \end{pmatrix} + \left[g^{-1}dg, \begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix}\right] = \begin{pmatrix} dX' + [A', X'] & -B^*X'' + X'B^* \\ BX' - X''B & dX'' + [A'', X''] \end{pmatrix}.$$

In the case of $s = 1$ we rescaled by letting $X' \longrightarrow tX'$. But notice that

$$g\begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix} g^{-1} \longrightarrow g\begin{pmatrix} tX' & 0 \\ 0 & X'' \end{pmatrix} g^{-1} \xrightarrow{\varphi_{t^{-1}}} g\begin{pmatrix} X' & 0 \\ 0 & t^{-1}X'' \end{pmatrix} g^{-1}$$

so that the effect of the rescaling is to pull back

ω to $\tilde{Y}_S^- \underset{F_S}{\times} \tilde{Y}_S^+$ and then apply the transformation $\begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix} \longrightarrow \begin{pmatrix} X' & 0 \\ 0 & \frac{1}{t}X'' \end{pmatrix}$

Thus we are asking whether

$$\mathrm{tr}\left\{\begin{pmatrix} \frac{1}{1-X'^2} & 0 \\ 0 & \frac{1}{1-t^{-2}X''^2} \end{pmatrix}\begin{pmatrix} dX'+[A',X'] & -\frac{1}{t}B^*X''+X'B^* \\ BX'-\frac{1}{t}X''B & \frac{1}{t}(dX''+[A'',X'']) \end{pmatrix}\right\}^{2k+1}$$

has a limit as $t\to 0$. This

$$\mathrm{tr}\left\{\begin{pmatrix} \frac{1}{1-X'^2} & 0 \\ 0 & \frac{t^{1}}{t^2-(X'')^2} \end{pmatrix}\begin{pmatrix} dX'+[A',X'] & -\frac{1}{t}B^*X''+X'B^* \\ tBX'-X''B & dX''+[A'',X''] \end{pmatrix}\right\}^{2k+1}$$

$\|$ conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}$

$$\mathrm{tr}\left\{\begin{pmatrix} \frac{1}{1-X'^2} & 0 \\ 0 & \frac{1}{t^2-X''^2} \end{pmatrix}\begin{pmatrix} dX'+[A',X'] & -B^*X''+tX'B^* \\ tBX'-X''B & t(dX''+[A'',X'']) \end{pmatrix}\right\}^{2k+1}$$

$\downarrow$ as $t\to 0$ assuming $(X'')^{-1}$ ∃

$$\mathrm{tr}\left\{\begin{pmatrix} \frac{1}{1-X'^2} & 0 \\ 0 & -\frac{1}{X''^2} \end{pmatrix}\begin{pmatrix} dX'+[A',X'] & -B^*X'' \\ -X''B & 0 \end{pmatrix}\right\}^{2k+1}$$

$\|$

$$\mathrm{tr}\left\{\begin{pmatrix} \frac{1}{1-X'^2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} dX'+[A',X'] & -B^*X'' \\ \frac{1}{X''}B & 0 \end{pmatrix}\right\}^{2k+1}$$

$\|$ — conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & X'' \end{pmatrix}$

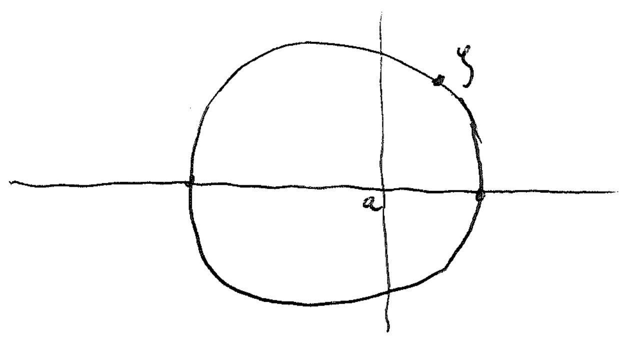
$$\mathrm{tr}\left\{\begin{pmatrix} \frac{1}{1-X'^2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} dX'+[A',X'] & -B^* \\ B & 0 \end{pmatrix}\right\}^{2k+1}$$

September 8, 1986

[redacted] It is necessary to improve our understanding of yesterday's calculations. We want to study the behavior of the character form $\varphi_{t^{-1}}^*(\omega)$, $\omega = \operatorname{tr}\left(\frac{1}{1-X^2}dX\right)^{2k+1}$ as $t \to 0$.

We know it converges to a distribution (current) supported on $Y_1^+ = \{g \in U(V) \mid$ [redacted] $1 \in sp(g)\}$. We would like to understand [redacted] the nature of the limiting process in the nbd of the stratum $Y_s^+ - Y_{s+1}^+ = X_s$ $= \{g \mid 1$ is an s fold eigenvalue of $g\}$.

A natural nbd of the stratum X_s is the open set U_s of g for which $\exists$ $a \in (-1,1)$ such that g has exactly s eigenvalues ξ with $a < \operatorname{Re}(\xi) \le 1$



This nbd is stable under $\varphi_{t^{-1}}$ and it fibres over X_s the fibre over g being a norm ball in the vector [redacted] space of skew-adjoint endomorphisms in the $+1$ eigenspace of g.

As $t \to 0$, the form $\varphi_{t^{-1}}^*(\omega)$ blows up along Y_1^+ and goes to zero outside. I proposed to rescale in the normal direction to X_s in the following sense. [redacted] We first apply φ_t to the s eigenvalues near 1. Then $\varphi_{t^{-1}}$ moves all the eigenvalues to -1. The composition sends the outer $N-s$ eigenvalues to -1 and keeps the inner s eigenvalues first. It's clear this has a perfectly good limit: We get the natural projection of U_s onto $\widetilde{Y}_s$.

We conclude that under this normal rescaling the limit of $\varphi_{t^{-1}}^*(\omega)$ is just the pull back of ω via the composition

$$U_s \longrightarrow \tilde{Y}_s^- \longrightarrow U(V)$$

Thus the form obtained yesterday (bottom p 112) should be the pull back under the map $Y_s^- \to U(V)$ ~~[illegible]~~ of the character form.

Let's compute this directly. Y_s^- is the auto bundle of S over $G_s(V)$ so it is universal for having a vector bundle E with isometric embedding $E \xrightarrow{i} \tilde{V}$ and with unitary auto g. We extend g to $\tilde{g}$ on $\tilde{V}$ defined to be -1 on $E^\perp$. Let $j : E^\perp \to \tilde{V}$ be the embedding. Let us compute $\tilde{g}^{-1}[d,\tilde{g}]$.

$$\begin{aligned} i^*[d,\tilde{g}]i &= i^* d\tilde{g} i - i^*\tilde{g} di \\ &= i^* di g - g i^* di = [D,g] \end{aligned}$$

where $D = i^* d i$ is the induced connection on E.

$$\begin{aligned} j^*[d,\tilde{g}]i &= j^* d\tilde{g} i - j^*\tilde{g} di \\ &= j^* di\, g - (-j^*) di = j^* di\,(g+1) \\ i^*[d,\tilde{g}]j &= i^* d\tilde{g} j - i^*\tilde{g} dj = i^* d(-j) - g i^* dj \\ &= -(g+1)\, i^* dj \\ j^*[d,\tilde{g}]j &= j^* d\tilde{g} j - j^*\tilde{g} dj \\ &= j^* d(-j) - (-j^*) dj = 0 \end{aligned}$$

So relative to the decomposition $\tilde{V} = E \oplus \tilde{E}$ we have

$$\tilde{g}^{-1} d\tilde{g} = \begin{pmatrix} g^{-1} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} [D,g] & -(g+1)\, i^* dj \\ j^* di\,(g+1) & 0 \end{pmatrix}$$

Suppose $\quad g = \dfrac{1+X}{1-X} \qquad g+1 = \dfrac{2}{1-X}$

$$[D,g] = 2\,\frac{1}{1-X}[D,X]\frac{1}{1-X}$$

$$\therefore\ \tilde{g}^{-1}d\tilde{g} = 2\begin{pmatrix} g^{-1} & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} \frac{1}{1-X}[D,X]\frac{1}{1-X} & -\frac{1}{1-X}i^*dj \\ j^*di\,\frac{1}{1-X} & 0\end{pmatrix}$$

$$= 2\begin{pmatrix} \frac{1}{1+X}[D,X]\frac{1}{1-X} & -\frac{1}{1+X}i^*dj \\ -j^*di\,\frac{1}{1-X} & 0\end{pmatrix}$$

$$= 2\begin{pmatrix} \frac{1}{1+X}[D,X] & +\frac{1}{1+X}i^*dj \\ -j^*di & 0\end{pmatrix}\begin{pmatrix} \frac{1}{1-X} & 0 \\ 0 & -1\end{pmatrix}$$

is conjugate to

$$2\begin{pmatrix} \frac{1}{1-X^2}[D,X] & +\frac{1}{1-X^2}i^*dj \\ +j^*di & 0\end{pmatrix}$$

$$\|$$

$$2\begin{pmatrix} \frac{1}{1-X^2} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} [D,X] & +i^*dj \\ +j^*di & 0\end{pmatrix}$$

Note that
$$d - \begin{pmatrix} i^*di & 0 \\ 0 & j^*dj\end{pmatrix} = \begin{pmatrix} 0 & i^*dj \\ j^*di & 0\end{pmatrix}$$

is skew-hermitian being the difference of two unitary connections. So setting

$$B = j^*di \quad \text{we have} \quad i^*dj = -B^*$$

and
$$B^*B = -i^*dj\,j^*di = -i^*d(1-ii^*)di = (i^*di)^2$$
is the curvature of the subbundle E.

So far we have checked yesterday's calculation. But we have learned that the curvature D^2 is not only the square of the covariant derivative but that it's also expressible as B^*B in the larger bundle.

Also we see that

$$2^{2k+1}\,\mathrm{tr}\left(\begin{pmatrix}\frac{1}{1-X^2} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}[D,X] & -B^*\\ B & 0\end{pmatrix}\right)^{2k+1}$$

is a function of $\frac{1}{1-X^2}$, $[D,X]$, and $B^*B = D^2$. The question is whether it is the [redacted] same as, or at least closely related to, the superconnection character form of degree $2k+1$.

Let's tackle this as follows:

$$\sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{tr}\Bigg(\underbrace{\begin{pmatrix}\frac{1}{1-X^2} & 0\\ 0 & 1\end{pmatrix}}_{\frac{1}{H_0}}\underbrace{\begin{pmatrix}[D,X] & -B^*\\ B & 0\end{pmatrix}}_{V}\Bigg)^{k}$$

$$= -\,\mathrm{tr}\log\left(1-\frac{1}{H_0}V\right) = -\log\det\left(1-\frac{1}{H_0}V\right)$$

$$= -\log\big(\det(H_0-V) - \det(H_0)\big)$$

But
$$H_0 - V = \begin{pmatrix}1-X^2 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}[D,X] & -B^*\\ B & 0\end{pmatrix}$$

$$= \begin{pmatrix}1-X^2-[D,X] & B^*\\ -B & 1\end{pmatrix}$$

so
$$\det(H_0-V) = \det\left\{\begin{pmatrix}1-X^2-[D,X] & B^*\\ -B & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ B & 1\end{pmatrix}\right\}$$

$$= \det \begin{pmatrix} 1 - X^2 - [D,X] + B^*B & B^* \\ 0 & 1 \end{pmatrix}$$

$$= \det \left(1 - X^2 - [D,X] + D^2\right)$$

$$-\log\left(\det(H_0 - V) - \det(H_0)\right) \simeq -\operatorname{tr}\log\left(1 - \frac{1}{1-X^2}\left([D,X] - D^2\right)\right)$$

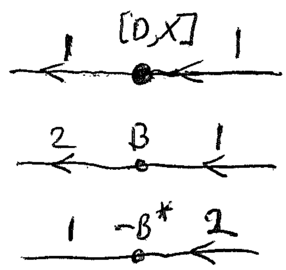
$$= \sum \frac{1}{k} \operatorname{tr}\left(\frac{1}{1-X^2}\left([D,X] - D^2\right)\right)^k$$

This manipulation shows that heuristically one is on the right track. There is a sign problem [redacted] – we get $[D,X] - D^2$ instead of $[D,X] + D^2$. This might be due to the superconnection algebra. Also it's not clear to what extent we can manipulate determinants of matrices of differential forms.

Let's do some low degree calculations. Let's evaluate

$$\operatorname{tr}\Bigg(\underbrace{\begin{pmatrix} \frac{1}{1-X^2} & 0 \\ 0 & 1 \end{pmatrix}}_{G_0} \underbrace{\begin{pmatrix} [D,X] & -B^* \\ B & 0 \end{pmatrix}}_{V}\Bigg)^k$$

using Feynman diagrams. Here our particle has two ~~states~~ with propagators $\frac{1}{1-X^2}$ and 1 and the interaction allows the following transitions

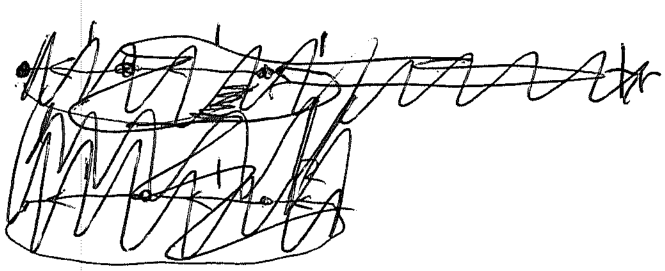


To simplify put $C = -B^*$.

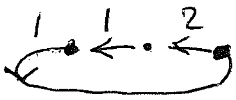
$B = j^* di$

$C = -i^* dj$

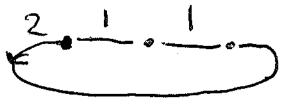
The diagrams for $k = 3$ are



1 1 1  $tr\left(\frac{1}{1-X^2}[D,X]\right)^3$

1 1 2  $tr\left(\frac{1}{1-X^2}[D,X]\frac{1}{1-X^2}CB\right)$

1 2 1 $tr\left(\frac{1}{1-X^2}CB\frac{1}{1-X^2}[D,X]\right)$

2 1 1  $tr\left(B\frac{1}{1-X^2}[D,X]\frac{1}{1-X^2}C\right)$

Notice that when we are dealing with odd degree forms, really I mean a product of an odd number of matrix one-forms then we have

$$tr(\omega_1\cdots\omega_{2k+1}) = tr(\omega_{a+1}\cdots\omega_{2k+1}\,\omega_1\cdots\omega_a).$$

Thus

$$\frac{1}{3}tr\left(\begin{pmatrix}\frac{1}{1-X^2} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}[D,X] & C\\ B & 0\end{pmatrix}\right)^3 = \frac{1}{3}tr\left(\frac{1}{1-X^2}[D,X]\right)^3 + tr\left(\frac{1}{1-X^2}[D,X]\frac{1}{1-X^2}CB\right)$$

Next do $k=5$.

1 1 1 1 1 — $tr\left(\frac{1}{1-X^2}[D,X]\right)^5$

1 1 1 1 2 — $tr\left(\left(\frac{1}{1-X^2}[D,X]\right)^3\frac{1}{1-X^2}CB\right)$

1 1 1 2 1

1 1 2 1 1 — $tr\left(\left(\frac{1}{1-X^2}[D,X]\right)^2\frac{1}{1-X^2}CB\frac{1}{1-X^2}[D,X]\right)$

1 2 1 1 1

2 1 1 1 1

2 1 2 1 1 — $tr\left(B\frac{1}{1-X^2}CB\frac{1}{1-X^2}[D,X]\frac{1}{1-X^2}C\right)$

2 1 1 2 1

1 2 1 2 1 — $tr\left(\frac{1}{1-X^2}CB\frac{1}{1-X^2}CB\frac{1}{1-X^2}[D,X]\right)$

1 2 1 1 2

1 1 2 1 2

Thus

$$\frac{1}{5}tr\left(\begin{pmatrix}\frac{1}{1-X^2} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}[D,X] & C\\ B & 0\end{pmatrix}\right)^5 = \frac{1}{5}tr\left(\frac{1}{1-X^2}[D,X]\right)^5 + tr\left(\left(\frac{1}{1-X^2}[D,X]\right)^3\frac{1}{1-X^2}CB\right) + tr\left(\frac{1}{1-X^2}[D,X]\left(\frac{1}{1-X^2}CB\right)^2\right)$$

On the other hand consider

$$\sum \frac{1}{k} \operatorname{tr}\left(\frac{1}{1-X^2}([D,X]+D^2)\right)^k$$

Then the degree 3 term is

$$\frac{1}{3}\operatorname{tr}\left(\frac{1}{1-X^2}[D,X]\right)^3 + \frac{1}{2}\operatorname{tr}\left(\frac{1}{1-X^2}[D,X]\frac{1}{1-X^2}D^2 + \frac{1}{1-X^2}D^2\frac{1}{1-X^2}[D,X]\right)$$

and the degree 5 term is

$$\frac{1}{5}\operatorname{tr}\left(\frac{1}{1-X^2}[D,X]\right)^5$$
$$+\frac{1}{4}\operatorname{tr}\left(\left(\frac{1}{1-X^2}[D,X]\right)^3\frac{1}{1-X^2}D^2 + \left(\frac{1}{1-X^2}[D,X]\right)^2\frac{1}{1-X^2}D^2\frac{1}{1-X^2}[D,X]\right.$$
$$\left. + \frac{1}{1-X^2}[D,X]\frac{1}{1-X^2}D^2\left(\frac{1}{1-X^2}[D,X]\right)^2 + \frac{1}{1-X^2}D^2\left(\frac{1}{1-X^2}[D,X]\right)^3\right)$$
$$+\frac{1}{3}\operatorname{tr}\left(\frac{1}{1-X^2}[D,X]\left(\frac{1}{1-X^2}D^2\right)^2 + \frac{1}{1-X^2}D^2\frac{1}{1-X^2}[D,X]\frac{1}{1-X^2}D^2 + \qquad\right)$$

Thus it's clear we have the identity

$$\left\{\sum \frac{1}{k}\operatorname{tr}\left(\frac{1}{1-X^2}([D,X]-D^2)\right)^k\right\}_{(2k+1)} = \frac{1}{2k+1}\operatorname{tr}\left(\begin{pmatrix}\frac{1}{1-X^2} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}[D,X] & C\\ B & 0\end{pmatrix}\right)^{2k+1}$$

since $CB = -B^*B = -D^2$.

September 9, 1986

Here's a simple proof that if $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$, then $F = \underbrace{\left(\frac{1+X}{1-X}\right)}_{g}\varepsilon$ is the involution which is $+1$ on the graph $\mathrm{Im}\begin{pmatrix} 1 \\ T \end{pmatrix}$ and -1 on $\mathrm{Im}\begin{pmatrix} -T^* \\ 1 \end{pmatrix}$.

Clearly $\varepsilon g \varepsilon^{-1} = \frac{1-X}{1+X} = g^{-1}$ so $F = g\varepsilon$ is an involution.

$$(1-X)\begin{pmatrix} 1 \\ T \end{pmatrix} = \begin{pmatrix} 1 & +T^* \\ -T & 1 \end{pmatrix}\begin{pmatrix} 1 \\ T \end{pmatrix} = \begin{pmatrix} 1-T^*T \\ 0 \end{pmatrix}$$

$$(1-X)\begin{pmatrix} -T^* \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & T^* \\ -T & 1 \end{pmatrix}\begin{pmatrix} -T^* \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1+TT^* \end{pmatrix}$$

Thus

$$\underbrace{\varepsilon(1-X)}_{(1+X)\varepsilon}\begin{pmatrix} 1 \\ T \end{pmatrix} = (1-X)\begin{pmatrix} 1 \\ T \end{pmatrix} \implies g\varepsilon\begin{pmatrix} 1 \\ T \end{pmatrix} = \begin{pmatrix} 1 \\ T \end{pmatrix}$$

and similarly $g\varepsilon\begin{pmatrix} -T^* \\ 1 \end{pmatrix} = -\begin{pmatrix} -T^* \\ 1 \end{pmatrix}$.

Computation of the curvature: Yesterday we learned that if $E \oplus E^\perp \xrightarrow[\sim]{i+j} \tilde{V}$, then the curvature of $D = i^* d i$ is given by $D^2 = B^* B$ where $B = j^* d i$. The principle seems to be that it's easier to multiply the off-diagonal blocks:

$$d - \begin{pmatrix} i^*di & 0 \\ 0 & j^*dj \end{pmatrix} = \begin{pmatrix} 0 & \overbrace{i^*dj}^{=-B^*} \\ \underbrace{j^*di}_{=B} & 0 \end{pmatrix}$$

to get the curvature than to compute D^2. Here are two examples:

Suppose $E = \mathrm{Im}\begin{pmatrix} 1 \\ T \end{pmatrix}$ $\qquad i = \begin{pmatrix} 1 \\ T \end{pmatrix}(1+T^*T)^{-1/2}$

$$j = \begin{pmatrix} -T^* \\ 1 \end{pmatrix}(1+TT^*)^{-1/2}, \qquad B = j^* d i = (1+TT^*)^{-1/2}(-T \ \ 1)\begin{pmatrix} 0 \\ dT \end{pmatrix}(\quad)^{-1/2}$$

$$B = (1+TT^*)^{-1/2}\, dT\, (1+T^*T)^{-1/2}$$

and so the curvature is

$$D^2 = B^*B = (1+T^*T)^{-1/2}\, dT^* \,(1+TT^*)^{-1} dT\,(1+T^*T)^{-1/2}$$

The other example is

$$d - \big(ede + (1-e)d(1-e)\big) = (1-e)de + ed(1-e)$$

The curvature is

$$-e.d.(1-e).(1-e).d.e. = -e[d,1-e][d,e] = e[d,e]^2$$

What I propose to do now is to take the character form on $Gr_n(V)$

$$\frac{1}{2^{2k+1}}\frac{1}{k!}\, tr\big(F(dF)^{2k}\big)$$

and to pull it back to $\tilde{Y}_s$ and hopefully obtain the superconnection character form natural living on $\tilde{Y}_s$ which is a Grassmannian bundle $Gr_s(E^0 \oplus E^1)$.

If $F = g\varepsilon$, then

$$tr\big(F(dF)^{2k}\big) = tr\Big(g\varepsilon \underbrace{(dg.\varepsilon.dg.\varepsilon)\cdots(dg.\varepsilon.dg.\varepsilon)}_{k \text{ times}}\Big)$$

$$= tr\big(g(\varepsilon dg\, \varepsilon dg)^k \varepsilon\big) = tr\big(\varepsilon g\,(d(g^{-1})dg)^k\big)$$

$$= (-1)^k\, tr\big(\varepsilon g\,(g^{-1}dg)^{2k}\big)$$

Next putting $g = \frac{1+X}{1-X}$ $\qquad g^{-1}dg = 2\frac{1}{1+X}dX\frac{1}{1-X}$

the character form becomes

$$\frac{1}{2}\frac{1}{k!}(-1)^k\, tr\Big(\varepsilon\, \frac{1+X}{1-X}\Big(\frac{1}{1+X}dX\frac{1}{1-X}\Big)^{2k}\Big)$$

Now take $1+X$ thru ε and around

$$= \frac{1}{2}\frac{1}{k!}(-1)^k\, tr\Big(\varepsilon\Big(\frac{1}{1-X^2}dX\Big)^{2k}\Big)$$

which checks.

Now we want to take the character form of degree $2k$ on $Gr_n(V)$ and pull back to $\tilde{Y}_s^-$. Recall $\tilde{Y}_s^-$ consists of $(\overset{s}{K}, \overset{n-s}{I}, \overset{n}{W})$,

$$\begin{tikzcd}[arrows=dash] & \overset{n}{\tilde{V}^1} \arrow[dr, "s"] & \\ \overset{n-s}{I} \arrow[ur] \arrow[dr] & & K \oplus \overset{n}{\tilde{V}^1} \\ & \overset{n}{W} \arrow[ur] & \end{tikzcd}$$

so it is the Grassmannian bundle $Gr_s(E^0 \oplus E^1)$ where $E^0 = K$, $E^1 = \tilde{V}^1/\mathcal{I} = \mathcal{I}^\perp$ over $Gr_s(V^0) \times Gr_{n-s}(V^1)$

I want to find the $\tilde{g} = F\varepsilon$ reversed by ε corresponding to W. On I one has $F = 1$, $\varepsilon = -1$ so $\tilde{g} = -1$, and on $K^\perp = V^0 \ominus K$ one has $F = -1$, $\varepsilon = 1$ so $\tilde{g} = -1$. Now $W \ominus I \subset K \oplus I^\perp$ corresponds to a unitary g on $K \oplus I^\perp = E^0 \oplus E^1$ reversed by ε. Thus it is clear that $\tilde{g} = g$ on E and -1 on $E^\perp = K^\perp \oplus I$, and so we conclude that the map

$$\tilde{Y}_s^- = Gr_s(E^0 \oplus E^1) \longrightarrow Gr_n(V)$$

is just the map $g \longmapsto \tilde{g}$ described above but restricted to g reversed by ε. Then we can use the formula (bottom p. 114)

$$\tilde{g}^{-1} d\tilde{g} = \begin{pmatrix} g^{-1} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} [D, g] & -(g+1) i^* dj \\ j^* di (g+1) & 0 \end{pmatrix}$$

$$= 2 \begin{pmatrix} \frac{1-X}{1+X} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-X} [D, X] \frac{1}{1-X} & \frac{1}{1-X} (-i^* dj) \\ j^* di \frac{1}{1-X} & 0 \end{pmatrix}$$

$$= 2 \begin{pmatrix} \frac{1}{1+X} [D, X] & \frac{1}{1+X} \overbrace{(-i^* dj)}^{-C} \\ \underbrace{-j^* di}_{-B} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{1-X} & 0 \\ 0 & 1 \end{pmatrix}$$

The character form is

$$\frac{(-1)^k}{2^{2k+1}\cdot k!}\operatorname{tr}\left(\varepsilon\tilde g(\tilde g^{-1}d\tilde g)^{2k}\right)$$

$$=\frac{(-1)^k}{2\cdot k!}\operatorname{tr}\left[\varepsilon\begin{pmatrix}\frac{1+X}{1-X} & 0\\ 0 & -1\end{pmatrix}\left\{\begin{pmatrix}1-X & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{1-X^2}[D,X] & \frac{1}{1-X^2}(-C)\\ -B & 0\end{pmatrix}\begin{pmatrix}\frac{1}{1-X} & 0\\ 0 & 1\end{pmatrix}\right\}^{2k}\right]$$

$$\boxed{=\frac{(-1)^k}{2\cdot k!}\operatorname{tr}\left[\varepsilon\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{1-X^2}[D,X] & \frac{1}{1-X^2}C\\ B & 0\end{pmatrix}^{2k}\right]}$$

(Recall that $\varepsilon=\begin{pmatrix}\varepsilon' & 0\\ 0 & \varepsilon''\end{pmatrix}$ $\quad$ $\varepsilon'=\varepsilon$ on E, $\varepsilon''=\varepsilon$ on $E^\perp$).

As a check on this calculation I could use the other method:

$$X=g\begin{pmatrix}X' & 0\\ 0 & X''\end{pmatrix}g^{-1}$$

where $g\in U(V^0)\times U(V^1)$ is used to trivialize E^0 and E^1. Then $\quad g^{-1}dg=\begin{pmatrix}A' & C\\ B & A''\end{pmatrix}$

$$dX=g\begin{pmatrix}dX'+[A',X'] & CX''-X'C\\ BX'-X''B & dX''+[A'',X'']\end{pmatrix}g^{-1}$$

and the character form is

$$\frac{(-1)^k}{2\cdot k!}\operatorname{tr}_\varepsilon\left\{\begin{pmatrix}\frac{1}{1-X'^2} & 0\\ 0 & \frac{1}{1-X''^2}\end{pmatrix}\begin{pmatrix}dX'+[A',X'] & CX''-X'C\\ BX'-X''B & dX''+[A'',X'']\end{pmatrix}\right\}^{2k}$$

Now rescale $X',X''\longrightarrow X',\frac{1}{t}X''$ with $t\to 0$.

$$\frac{(-1)^k}{2\cdot k!}\operatorname{tr}_\varepsilon\left\{\begin{pmatrix}\frac{1}{1-X'^2} & 0\\ 0 & \cancel{\frac{1}{1-\frac{1}{t^2}X''^2}}\end{pmatrix}\begin{pmatrix}[D,X'] & t\left(\frac{1}{t}CX''-X'C\right)\\ \cancel{t}\left(BX'-\frac{1}{t}X''B\right) & t^2\frac{1}{t}[D'',X'']\end{pmatrix}\right\}^{2k}$$

$$\frac{1}{t^2-X''^2}\qquad t$$

As $t\to 0$ this becomes

$$\frac{(-1)^k}{2\ k!}\ \mathrm{tr}_\varepsilon \left\{ \underbrace{\begin{pmatrix} \frac{1}{1-X'^2} & 0 \\ 0 & -\frac{1}{X''^2} \end{pmatrix} \begin{pmatrix} [D',X'] & CX'' \\ -X''B & 0 \end{pmatrix}} \right\}^{2k}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{X''} \end{pmatrix} \begin{pmatrix} \frac{1}{1-X'^2}[D',X'] & \frac{1}{1-X'^2}C \\ B & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & X'' \end{pmatrix}$$

Now when you bring the $\begin{pmatrix} 1 & 0 \\ 0 & X'' \end{pmatrix}$ around thru the ε its changes sign yieldig

$$\frac{(-1)^k}{2\ k!}\ \mathrm{tr}_\varepsilon \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{1-X'^2}[D',X'] & \frac{1}{1-X'^2}C \\ B & 0 \end{pmatrix}^{2k} \right\}$$

Now that I have the character form I want to see that it coincides with the superconn. form, which recall is the degree $2k$ part of

$$\sum \frac{1}{m}\ \mathrm{tr}_s \left(\frac{1}{1-X^2}\left(i[D,X] + D^2\right) \right)^m$$

up to a constant. Let's denote this by ω_{2k}.

We have

$$\omega_{2k} = \sum_{a+2b=2k} \frac{1}{a+b}\ \omega_{a,b}$$

where $\omega_{a,b}$ is a sum of tr_s of monomials with a copies of $iA = i\frac{1}{1-X^2}[D,X]$ and b copies of $-B = \frac{1}{1-X^2}D^2$. a is necessarily even $a = 2(k-b)$ and so we get the sign $i^a = (-1)^{k-b}$ times the same mononimal constructed from $A = \frac{1}{1-X^2}[D,X]$ and $-B = \frac{1}{1-X^2}D^2$. This is $(-1)^k$ times the same mononmial constructed from $A = \frac{1}{1-X^2}[D,X]$ and $B = \frac{1}{1-X^2}(-D^2)$. Thus up to the constant $(-1)^k$ we are after the degree $2k$ part of

$$\sum \frac{1}{m}\ tr_s \left(\frac{1}{1-X^2}([D,X]-D^2)\right)^m$$

Let's now start with

$$* \qquad \frac{1}{2k}\ tr_\varepsilon \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-X^2}[D,X] & \frac{1}{1-X^2}C \\ B & 0 \end{pmatrix} \right\}^{2k}$$

where B, C are matrix 1-forms such that $CB = -D^2$. Note that B, C commute with ε. Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}$ where J is invertible and $\varepsilon J \varepsilon^{-1} = -J$. ($J$ exists provided $(E^0)^\perp$ and $(E^1)^\perp$ have the same rank, which is the case when $\dim K^\perp = m-s = \dim I = n-s$, i.e. $\dim V^0 = \dim V^1$. At any rate choosing α is a device to simplify.) Then

$$\varepsilon\alpha\varepsilon\alpha^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and so the form $*$ can be written

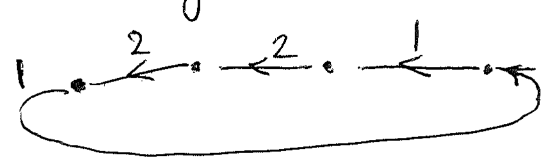
$$\frac{1}{2k}\ tr\{\varepsilon\alpha^{-1}(\qquad)^{2k}\alpha\}$$

$$= \frac{1}{2k}\ tr_s \begin{pmatrix} \frac{1}{1-X^2}[D,X] & \frac{1}{1-X^2}CJ \\ J^{-1}B & 0 \end{pmatrix}^{2k}$$

so what has been achieved is to make all the matrix 1-forms $P = \frac{1}{1-X^2}[D,X]$, $Q = \frac{1}{1-X^2}CJ$, $R = J^{-1}B$ anti-commute with ε.

When the $2k$ fold product is expanded one gets a sum over diagrams labelled by sequences of 1, 2's

1 2 2 1



arrows rep. states and vertices are the matrix elts

and then one takes the corresp. product $a_{12}a_{22}a_{21}a_{11}$

The important point for the following is that in such a product where the entries a_{ij} are matrix 1-forms anti-commuting with ε, the tr_s is cyclically symmetric.

~~Now~~ $\cancel{\sum_{k=1}^{\infty} \frac{1}{k} tr_s \begin{pmatrix} P & Q \\ R & 0 \end{pmatrix}^k = -tr_s \log\left(1 - \begin{pmatrix} P & Q \\ R & 0 \end{pmatrix}\right)}$

Let's look at the even degree part of

$$(1) \qquad \sum_{m=1}^{\infty} \frac{1}{m} tr_s \Big(\underbrace{\frac{1}{1-X^2}([D,X] - D^2)}_{P+QR} \Big)^m$$

Again because $\frac{1}{1-X^2}[D,X]$ is a matrix 1-form anti-commuting with ε, and QR is a matrix 2 form commuting with ε, it follows that in the expansion of ~~the~~ even degree part we have the cyclic symmetry of the trace.

Let's put things together. When

$$(2) \qquad \sum_{k=1}^{\infty} \frac{1}{2k} tr_s \begin{pmatrix} \frac{1}{1-X^2}[D,X] & \frac{1}{1-X^2}CJ \\ J^{-1}B & 0 \end{pmatrix}^{2k}$$

is expanded because the matrix elt $a_{22} = 0$, the result is a sum of the supertraces of monomials made out of $a_{11} = \frac{1}{1-X^2}[D,X]$ and $a_{12}a_{21} = \frac{1}{1-X^2}CJJ^{-1}B$ $= \frac{1}{1-X^2}(-D^2)$. Moreover cyclic symmetry prevails just as if we were dealing with ordinary matrices and the usual trace.

The same holds for the even part of (1), so both expressions (1), (2) are sums of terms involving tr_s of monomials in a_{11} and $a_{12}a_{21}$, but with certain rational coefficients. So to check the coeffs. are equal its enough to work with usual matrices and the usual traces and prove

$$\sum_1^\infty \frac{1}{m} \operatorname{tr}(P+QR)^m \overset{?}{=} \sum_1^\infty \frac{1}{m} \operatorname{tr}\begin{pmatrix} P & Q \\ R & 0 \end{pmatrix}^m$$

$$\begin{array}{ccc} \Vert & & \Vert \\ -\operatorname{tr}\log(1-P-QR) & & -\operatorname{tr}\log\left(1-\begin{pmatrix} P & Q \\ R & 0 \end{pmatrix}\right) \\ \Vert & & \Vert \\ -\log\det(1-P-QR) & & -\log\det\begin{pmatrix} 1-P & -Q \\ -R & 1 \end{pmatrix} \end{array}$$

But these are equal because

$$\begin{pmatrix} 1-P & -Q \\ -R & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ R & 1 \end{pmatrix} = \begin{pmatrix} 1-P-QR & -Q \\ 0 & 1 \end{pmatrix}$$

I think the above constitutes a proof although there ought to be a simpler one.

<u>September 12, 1986</u>

Let's consider a block matrix

$$\begin{array}{cc} \overset{m}{\longleftrightarrow} & \overset{n}{\longleftrightarrow} \\ \end{array}$$
$$\begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

which is invertible and assume D is invertible.

$$\underbrace{\begin{pmatrix} 1 & -CD^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & C \\ B & D \end{pmatrix}}$$

$$\underbrace{\begin{pmatrix} (A-CD^{-1}B)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}\begin{pmatrix} A-CD^{-1}B & 0 \\ B & D \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 0 \\ -D^{-1}B & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ D^{-1}B & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -D^{-1}B & 1 \end{pmatrix}\begin{pmatrix} (A-CD^{-1}B)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix}\begin{pmatrix} 1 & -CD^{-1} \\ 0 & 1 \end{pmatrix}$$

<u>Application:</u> Suppose that A, B, C, D depend on a parameter t in such a way that

$$\begin{aligned} A(t) &= a + o(t) \\ B(t) &= \tfrac{1}{t}(b + o(t)) \\ C(t) &= \tfrac{1}{t}(c + o(t)) \qquad \text{as } t \to 0 \\ D(t) &= \tfrac{1}{t^2}(d + o(t)) \end{aligned}$$

Then $A - CD^{-1}B = A - (tC)(t^2D)^{-1}tB \longrightarrow a - cd^{-1}b$ provided $d^{-1}\ \exists$. I will assume this and that $a - cd^{-1}b$ is invertible. Then I see from the above formula that

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix}^{-1} \longrightarrow \begin{pmatrix} (a-cd^{-1}b)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

Next I want to apply this to the superconnection situation.

Let us consider two vector bundles with compatible inner products + connections E^0, E' and let $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$ on $E^0 \oplus E'$. Then we have the superconnection character form

$$\mathrm{tr}_s\, e^{u(X^2 + [D,X] + D^2)} \qquad \mathrm{Re}(u) > 0.$$

Now suppose there is a super subbundle $F = F^0 \oplus F' \subset E$ which is stable under X, so that

$$X = \begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix} \quad \text{relative to the decomposition } E = F \oplus F^\perp.$$

Suppose also that X'' is invertible, and put

$$X_t = \begin{pmatrix} X' & 0 \\ 0 & t^{-1}X'' \end{pmatrix}$$

Let's now compute the limit of the superconnection character form with X replaced by X_t as $t \to 0$.

Let i be the inclusion $F \hookrightarrow E$ and j the inclusion $F^\perp \subset E$. We want to compute the block description of $[D,X] \in \Omega(M, \mathrm{End}\,E)$. Recall that D, X can be interpreted as operators on $\Omega(M,E)$ and that one has

$$[D,X] = DX + XD$$

Thus

$$\begin{aligned} j^*[D,X]i &= j^*DXi + j^*XDi \\ &= j^*DiX' + X''j^*Di \end{aligned}$$

Thus we have the following block decomp. for

$$X_t^2 + [D, X_t] + D^2 = \begin{pmatrix} X'^2 + [D', X'] + i^*D^2 i & \frac{1}{t} i^*Dj X'' + X' i^*Dj \\ (j^*Di)X' + \frac{1}{t} X''(j^*Di) & \frac{1}{t^2}X''^2 + \frac{1}{t}[D'', X''] + j^*D^2 j \end{pmatrix}$$

where $D' = i^*Di$, $D'' = j^*Dj$ are the connections induced by D on F and $F^\perp$.

Now by our previous calculation

$$\frac{1}{\lambda - \begin{pmatrix} A & C \\ B & D \end{pmatrix}} = \frac{1}{\begin{pmatrix} \lambda - A & -C \\ -B & \lambda - D \end{pmatrix}} \longrightarrow \begin{pmatrix} \frac{1}{\lambda - a + c\frac{1}{d}b} & 0 \\ 0 & 0 \end{pmatrix}$$

provided $A \to a$, $tC \to c$, $tB \to b$, $t^2 D \to d$ and λ is not in the spectrum of $a - c\frac{1}{d}b$. Check signs:

$$(\lambda - a) - (-c)\frac{1}{-d}(-b) = \lambda - a + c\frac{1}{d}b$$

Thus

$$\frac{1}{\lambda - (X_t^2 + [D, X_t] + D^2)} \longrightarrow \begin{pmatrix} \frac{1}{\lambda - Q} & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$Q = X'^2 + [D', X'] + \underbrace{i^*D^2 i - (i^*Dj X'')(X''^2)^{-1}(X'' j^*Di)}_{}$$

$$\begin{aligned} & i^*D^2 i - i^*Djj^*Di \\ &= i^*D(1 - jj^*)Di = i^*Di\, i^*Di \\ &= D'^2. \end{aligned}$$

$$\therefore\ Q = X'^2 + [D', X'] + D'^2$$

is the superconnection curvature of the bundle F with its induced connection and automorphism.

September 11, 1986

Let's look at the odd case, where according to my paper one works in the superalgebra

$$\Omega(M, \operatorname{End} E) \hat{\otimes} C_1$$

with the supertrace

$$tr_\sigma(a + b\sigma) = tr_E(b) \qquad a, b \in \Omega(M, \operatorname{End} E).$$

Given D on E and X a skew adjoint endo, the curvature of the superconnection $D + X\sigma$ is

$$(D + X\sigma)^2 = D^2 + [D, X]\sigma + X^2$$

and the supercon. character form is

$$tr_\sigma\left(e^{u(D+X\sigma)^2}\right) = tr_\sigma\left(e^{u(X^2+[D,X]\sigma+D^2)}\right)$$

Let's check the formula

$$\boxed{\begin{aligned}&\int_0^\infty tr_\sigma\left(e^{u(X^2+[D,X]\sigma+D^2)} - e^{uX^2}\right) e^{-\lambda u}\,\frac{du}{u} \\ &= \sum_{m=1}^{\infty} \frac{1}{m}\, tr_\sigma\left\{\frac{1}{\lambda - X^2}([D,X]\sigma + D^2)\right\}^m.\end{aligned}}$$

We first note that both sides vanish as $\lambda \to +\infty$. Then we differentiate with respect to λ. One has

$$-\partial_\lambda \frac{1}{\lambda - X^2} = \frac{1}{(\lambda - X^2)^2}$$

Notice that $K = [D,X]\sigma + D^2$ is even in $\Omega(M, \operatorname{End} E)\hat{\otimes} C_1$, hence can be moved around in the supertrace. Thus

$$-\partial_\lambda\, tr_\sigma\left(\frac{1}{\lambda - X^2}K\right)^m = tr_\sigma\left(\frac{1}{(\lambda-X^2)^2}K\left(\frac{1}{\lambda-X^2}K\right)^{m-1} + \frac{1}{\lambda-X^2}K\frac{1}{(\lambda-X^2)^2}K\cdots\right)$$

$$= m\, tr_\sigma\Big(\frac{1}{\lambda - X^2}\underbrace{\left(K\frac{1}{\lambda-X^2}\right)\cdots\left(K\frac{1}{\lambda-X^2}\right)}_{m \text{ factors}}\Big)$$

Thus

$$-\partial_\lambda \sum_{m=1}^{\infty} \tfrac{1}{m} tr_\sigma \left(\frac{1}{\lambda - X^2} K\right)^m = \sum_{m=1}^{\infty} tr_\sigma \left(\frac{1}{\lambda - X^2} \left(K \frac{1}{\lambda - X^2}\right)^m \right)$$

$$= tr_\sigma \left(\frac{1}{\lambda - X^2 - K} - \frac{1}{\lambda - X^2} \right)$$

and the rest is clear

Next I need to evaluate

$$tr_\sigma \, (a\sigma + b)^m \qquad a \in \Omega^1(M, \mathrm{End}\, E), \; b \in \Omega^2(\dots)$$

Since the tr_σ [redacted] sees only odd numbers of σ it follows this form is of odd degree.

$$tr_\sigma (a\sigma + b)^m_{(2k+1)} = tr_\sigma \sum \underbrace{\cdots a\sigma \cdots a\sigma \cdots a\sigma \cdots}$$

monomial involving j factors $a\sigma$ and $m-j$ factors b

where $j + 2(m-j) = 2k+1$. Now σ commutes with b and anti-commutes with a so if we move all σ's to the right we get the [redacted] sign

$$(-1)^{(j-1)+(j-2)+\cdots+1} = (-1)^{j(j-1)/2} = (-1)^{k+m-j}$$

The $m-j$ could be made up by changing b to $-b$. So we have

$$tr_\sigma \, (a\sigma + b)^m_{(2k+1)} = (-1)^k \, tr\, (a-b)^m_{(2k+1)}.$$

Now apply this with $a = \frac{1}{\lambda - X^2}[D, X]$, $b = \frac{1}{\lambda - X^2} D^2$ and we obtain

$$\int_0^\infty tr_\sigma\left\{e^{u(X^2+[D,X]\sigma+D^2)}\right\}_{(2k+1)} e^{-\lambda u}\frac{du}{u}$$

$$= (-1)^k \sum_{m=1}^{\infty} \frac{1}{m}\, tr\left(\frac{1}{\lambda - X^2}([D,X] - D^2)\right)^m_{(2k+1)}$$

$$= (-1)^k \int_0^\infty tr\left(e^{u(X^2+[D,X]-D^2)}\right)_{(2k+1)} e^{-\lambda u}\frac{du}{u}$$

Simpler

$$\boxed{tr_\sigma\left\{e^{u(X^2+[D,X]\sigma+D^2)}\right\}_{(2k+1)} = (-1)^k\, tr\left(e^{u(X^2+[D,X]-D^2)}\right)_{(2k+1)}}$$

Now I can fill in the details of the proof that when $E \overset{i}{\hookrightarrow} \tilde{V}$, $D = i^* d i$, $g \in Aut(E)$ is extended to $\tilde{g} \in Aut(\tilde{V})$ by setting $\tilde{g} = -1$ on $E^\perp$, then the superconnection forms associated to $\tilde{g}, d$ [redacted] coincide with the superconnection forms associated to g, D.

It just occurred to me that I can carry out the construction entirely within the superconnection formalism. Let $i : E' \hookrightarrow E$ be an isometric embedding, let $D' = i^* D i$ be the induced connection on E' from the connection D on E, and let X be a skew-adjoint endom of E preserving E' such that X is non singular on $E \ominus E' = E''$. In fact we should start with X' on E' and extend to E, say by using a $\neq 0$ scalar on E''. Then relative to the decomposition $E = E' \oplus E''$ we have

$$X^2 + [D,X]\sigma + D^2 = \begin{pmatrix} X'^2 + [D',X']\sigma + i^*D^2 i & (CX'' - X'C)\sigma + i^*D^2 j \\ (BX' - X''B)\sigma + j^*D^2 i & X''^2 + [D'',X'']\sigma + j^*D^2 j \end{pmatrix}$$

So now with $X'' = ti$, $t \to \infty$ we have by previous understanding

$$\frac{1}{\lambda - (X^2 + [D,X]\sigma + D^2)} \longrightarrow \begin{pmatrix} \frac{1}{\lambda - (X'^2 + [D',X']\sigma + i^*Di - CX''\sigma \frac{1}{X''^2}(-X''B)\sigma)} & 0 \\ 0 & 0 \end{pmatrix}$$

But

$$-CX''\sigma \frac{1}{X''^2}(-X''B)\sigma = -CB$$

because [redacted] B are odd [redacted] and so anti-commutes with σ. Thus we get

$$i^*Di - (i^*Dj)(j^*Di) = (i^*Di)^2 = (D')^2 .$$

September 12, 1986

Review stuff on Γ. The starting point was to find a continuous version of the identity

$$1) \qquad -\log \det(1-F) = \sum_1^\infty \frac{1}{m} \operatorname{tr}(F^m)$$

or more simply

$$2) \qquad -\log(1-z) = \sum_1^\infty \frac{1}{m} z^m .$$

Let $z = e^{-\varepsilon s}$ with $\varepsilon \downarrow 0$. $\quad 1-z \sim \varepsilon s$.

$$\sum_1^\infty \frac{1}{m\varepsilon} e^{-(m\varepsilon)s} \cdot \varepsilon \sim \int_0^\infty e^{-ts} \frac{dt}{t}$$

$$-\log(1-e^{-\varepsilon s}) \sim -\log(\varepsilon s) = -\log(s) + \log(\tfrac{1}{\varepsilon})$$

Thus we find the formal identities

$$-\log s = \int_0^\infty e^{-ts} \frac{dt}{t} \qquad \text{renormalized}$$

$$-\log \det A = \int_0^\infty \operatorname{tr}(e^{-tA}) \frac{dt}{t} \qquad \text{"} .$$

as the continuous versions of 1), 2).

Other versions are

$$-\log \det(s+A) = \int_0^\infty e^{-st} \operatorname{tr}(e^{-tA}) \frac{dt}{t} \qquad \text{ren.}$$

showing that the Laplace transform of $\operatorname{tr}(e^{-tA})\frac{1}{t}$ gives the log of the characteristic polynomial in some sense.

One would like to apply this to $\Gamma(s)$ because it occurs in the Reimann zeta theory. Review the formulas

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t} = s^s \int_0^\infty e^{-s(t-\log t)} \frac{dt}{t}$$

$$= s^s \int_{-\infty}^\infty e^{-s(e^u - u)} du \sim s^s e^{-s} \int_{-\infty}^\infty e^{-s\frac{u^2}{2}} du$$

$$\sim s^s e^{-s} \sqrt{\frac{2\pi}{s}} \left(1 + O(\tfrac{1}{s})\right)$$

$$\frac{\Gamma(s+n)}{\Gamma(n)} = \frac{\int_0^\infty e^{-t}\, t^{n+s}\frac{dt}{t}}{\int_0^\infty e^{-t} t^n \frac{dt}{t}} = \frac{n^{n+s}\int_0^\infty e^{-nt} t^n t^s \frac{dt}{t}}{n^n \underbrace{\int_0^\infty e^{-nt} t^n \frac{dt}{t}}_{\text{measure peaks near } t \doteq 1}}$$

$$\therefore \quad \frac{\Gamma(s+n)}{\Gamma(n)} \sim n^s .$$

Then can derive infinite product expansion

$$\frac{1}{\Gamma(s)} = \frac{s(s+1)\cdots(s+n-1)}{\Gamma(s+n)} = s\prod_1^{n-1}\left(1+\frac{s}{m}\right) \frac{\Gamma(n)}{\Gamma(s+n)}$$

$$= s\prod_1^{n-1}\left(1+\frac{s}{m}\right)e^{-\frac{s}{m}} \quad e^{s\underbrace{\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}-\log n\right)}_{\downarrow\ \gamma>0}} \quad \underbrace{\frac{\Gamma(n)n^s}{\Gamma(s+n)}}_{\downarrow\ 1}$$

$$= s\prod_1^{\infty}\left(1+\frac{s}{m}\right)e^{-\frac{s}{m}} \quad e^{\gamma s}$$

Next I want to try to play around with making $\frac{1}{\Gamma(s)}$ appear as $\det(s+A)$ where $A =$ harmonic oscillator operator with eigenvalues $n \geq 0$.

$$\log\frac{1}{\Gamma(s)} = \log s + \sum_1^\infty \log\left(1+\frac{s}{n}\right) - \frac{s}{n} \quad + \gamma s$$

$$(*) \qquad \frac{d}{ds}\log\frac{1}{\Gamma(s)} = \frac{1}{s} + \sum_1^\infty\left(\frac{1}{s+n}-\frac{1}{n}\right) + \gamma$$

$$-\frac{d^2}{ds^2}\log\frac{1}{\Gamma(s)} = \sum_0^\infty \frac{1}{(s+n)^2} = \mathcal{L}\left\{ \operatorname{tr}(e^{-tA})\, t\right\}$$

$$= \int_0^\infty e^{-st}\frac{1}{1-e^{-t}}\, t\, dt$$

We want to integrate this. We have

$$(**) \qquad +\frac{d}{ds}\log\frac{1}{\Gamma(s)} = \int_0^\infty e^{-st}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)dt \ - \log(s) + c_1$$

as both sides have same derivative. Similarly

$$\log \Gamma(s) = \int_0^\infty e^{-st}\left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\right)\frac{dt}{t} + s\log s - s - \frac{1}{2}\log s + c_1 s + c_2$$

From Stirling one sees that $c_1 = 0$, $c_2 = \log\sqrt{2\pi}$. The integral has an asymptotic expansion in terms of negative powers of s coming from the Taylor series of $\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}$, so we get the complete asymptotic formula for $\log\Gamma(s)$ in terms of Bernoulli nos.

Notice also that because $c_1 = 0$ we get by setting $s = 1$ in (*) $\binom{*}{*}$ that

$$\gamma = \int_0^\infty e^{-t}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right) dt$$

This review being completed I want to examine the questions raised by Graeme yesterday namely the links between

asymptotic expansion of $tr(e^{-tA})$ as $t \to 0$

residues of $\zeta_A(s) = tr(A^{-s})$

asymptotics of $tr\left(\frac{1}{\lambda - A}\right)$ as $\lambda \to \infty$.

According to him if one considers $A = -\partial_x^2 + u$, then these three things lead to the conserved functionals of u for the KdV flow. (Maybe one looks not at the actual $\zeta_A(s)$ but the formal ψDO A^{-s}.)

Now one connection is easy:

$$tr(A^{-s}) = \frac{1}{\Gamma(s)}\int_0^\infty tr(e^{-tA})\, t^s \frac{dt}{t}$$

This tells us that the coefficients of the asymptotics of $tr(e^{-tA})$ will appear as residues or values of $tr(A^{-s})$ depending on the zeros of $\frac{1}{\Gamma(s)}$ at $s = 0, -1, -2, \ldots$.

Next one has

$$\operatorname{tr} \frac{1}{s+A} = \int_0^\infty e^{-st} \operatorname{tr}(e^{-tA})\, dt$$

except that the Laplace transform need not be defined because of negative powers of t in the asymptotic expansion of $\operatorname{tr}(e^{-tA})$ as $t \to 0$.

So instead we consider

$$\Gamma(k) \operatorname{tr} \frac{1}{(s+A)^k} = \int_0^\infty e^{-st} \left\{ \operatorname{tr}(e^{-tA})\, t^{k-1} \right\} dt$$

The asymptotics of this as $s \to \infty$ in the RHP depend on the behavior at $t=0$. Thus the coefficients of powers of t in $\operatorname{tr}(e^{-tA})$ will turn into coefficients of negative powers of s.

For example

$$\sum_0^\infty \frac{1}{(s+n)^2} = \mathcal{L}\left\{ \underbrace{\frac{1}{1-e^{-t}}}_{\frac{1}{t} + \frac{1}{2} + \sum_{\substack{k \text{ odd} \\ \geq 1}} b_k t^k} t \right\}$$

$$\sim \mathcal{L}\left\{ 1 + \tfrac{1}{2} t + \sum b_k t^{k+1} \right\}$$

$$\sim \frac{1}{s} + \frac{1}{2}\frac{1}{s^2} + \sum_{\substack{k \geq 1 \\ k \text{ odd}}} b_k \frac{\Gamma(k+2)}{s^{k+2}}$$

Next let's try to understand the Euler-Maclaurin summation formula. It's based on the formal identity

$$\sum_0^\infty e^{-nD} = \frac{1}{1-e^{-D}} = \frac{1}{D} + \frac{1}{2} + \sum_{\substack{k \text{ odd} \\ \geq 1}} b_k D^k$$

In order to avoid confusion I'll work with finite sum:

$$\underbrace{\frac{e^{n\omega D}-1}{e^{\omega D}-1}} f(x) = \sum_{j=0}^{n-1} f(x+j\omega)$$

$$\frac{D}{e^{\omega D}-1} \cdot \frac{e^{n\omega D}-1}{D}$$

Now $\displaystyle \frac{1}{e^t-1} + 1 = \frac{1+e^t-1}{e^t-1} = \frac{e^t}{e^t-1} = \frac{1}{1-e^{-t}}$

$$= \frac{1}{t} + \frac{1}{2} + \sum b_k t^k$$

$$\therefore \quad \frac{1}{e^t-1} = \frac{1}{t} - \frac{1}{2} + \sum b_k t^k$$

$$\frac{e^{n\omega D}-1}{D} f(x) = \int_0^{n\omega} f(x+t)\,dt$$

Thus

$$\sum_{j=0}^{n-1} f(x+j\omega) = \left\{\frac{1}{\omega} - \frac{1}{2}D + \sum b_k \omega^k D^{k+1}\right\} \int_0^{n\omega} f(x+t)\,dt$$

or

$$\boxed{\begin{aligned}\sum_{j=0}^{n-1} f(x+j\omega) = \frac{1}{\omega}\int_0^{n\omega} f(x+t)\,dt &- \frac{1}{2}\big(f(x+n\omega) - f(x)\big) \\ &+ \sum b_k \omega^k \left(f^{(k)}(x+n\omega) - f^{(k)}(x)\right)\end{aligned}}$$

This formula should hold at least for polynomials

Apply it to $f(x) = \frac{1}{x^2}$ $\quad \omega = 1$

$$\sum_0^{n-1} \frac{1}{(x+j)^2} = \int_0^n \frac{dt}{(x+t)^2} - \frac{1}{2}\left(\frac{1}{(x+n)^2} - \frac{1}{x^2}\right)$$

$$\left[-\frac{1}{x+t}\right]_{t=0}^{t=n} \qquad + \sum_{k\ \text{odd}} b_k (-1)^k (k+1)! \left(\frac{1}{(x+n)^{k+2}} - \frac{1}{x^{k+2}}\right)$$

or letting $n \to \infty$

$$\sum_0^\infty \frac{1}{(x+n)^2} = \frac{1}{x} + \frac{1}{2x^2} + \sum_{k \text{ odd}} b_k \frac{(k+1)!}{x^{k+2}}$$

I guess this means that the Euler Maclaurin series is an asymptotic one at best.

I get the feeling that the summation formula can be replaced by working with the function $f(x)$ given as a transform, Laplace or Fourier, where the manipulations with D are done algebraically on the transform side. To be more specific, the expansion

$$\frac{1}{1-e^{-t}} = \frac{1}{t} + \frac{1}{2} + \cdots$$

is only valid for $0 < |t| < 2\pi$.