September 14, 1981

$$\tilde{\mathfrak{g}} = \underbrace{z^{-1}\mathfrak{g}[z^{-1}] \oplus (Y)}_{\mathfrak{n}^*} \oplus (H) \oplus \underbrace{(X) \oplus z\mathfrak{g}(z)}_{\mathfrak{n}}$$

So let us take a character (1 dim'l rep. L_λ of $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ $L_\lambda = (e_\lambda)$ where $He_\lambda = \lambda e_\lambda$, $X_i e_\lambda = 0$) and try to understand the module

$$U(\tilde{\mathfrak{g}}) \otimes_{U(\mathfrak{b})} L_\lambda$$

restricted to our torus. Since

$$\tilde{\mathfrak{g}} = \mathfrak{n}^* \oplus \mathfrak{b}$$

we have
$$U(\tilde{\mathfrak{g}}) = U(\mathfrak{n}^*) \otimes U(\mathfrak{b}) \quad \text{as left } \mathfrak{b}\text{-modules}$$
so
$$U(\tilde{\mathfrak{g}}) \otimes_{U(\mathfrak{b})} L_\lambda \cong \underbrace{U(\mathfrak{n}^*)}_{S(\mathfrak{n}^*)} \otimes_{\mathbb{C}} L_\lambda$$

and it should be very easy to write down the PS for this representation of $S^1 \times T$.

So we begin by getting the P.S. of $\mathfrak{n}^*$.

$$\mathfrak{n}^* = (Y) + z^{-1}\mathfrak{g}[z^{-1}]$$

has basis

	D-value	H-value		
$z^n X$	$n \geq 1$	2	$u^2 t^n$	$n \geq 1$
$z^n Y$	$n \geq 0$	-2	$u^{-2} t^n$	$n \geq 0$
$z^n H$	$n \geq 1$	0	t^n	$n \geq 1$

Use variable t to register D-degree
u to —— H-value

So the P.S. of $\mathfrak{n}^*$ is simply

$$\sum_{n\geq 1} u^2 t^n + \sum_{n \geq 1} t^n + \sum_{n \geq 0} u^{-2} t^n$$

and when we take $S(\mathfrak{n}^*)$ a character described by a monomial μ gives rise to a factor $\frac{1}{1-\mu s}$ where s is the variable giving the degree in the symmetric algebra. Thus I get

$$P.S.\ S(\mathfrak{n}^*) = \prod_{n\geq 1} \frac{1}{1-u^2t^n s} \prod_{n\geq 1} \frac{1}{1-t^n s} \prod_{n\geq 0} \frac{1}{1-u^{-2}t^n s}$$

It's now time to review the Jacobi identity. This is derived using difference equations of the form

$|q|<1$.
$$c_1 f(x) + c_2 f(qx) + c_3 f(q^2x) = x\left(c_3 f(x) + c_4 f(qx) + c_5 f(q^2x)\right)$$

which lead to power series, as does the hypergeometric DE. Actually one looks only at some first order cases:

$$f(x) = (1+ax)f(qx)$$

Iteration leads to

$$f(x) = (1+ax)(1+aqx)(1+aq^2x)\cdots(1+aq^{n-1}x)f(q^nx)$$

If f is a power series in x, then $f(q^nx) \to f(0)$ and so you get

$$f(x) = \text{const} \prod_{n\geq 0}(1+aq^nx)$$

On the other hand if

$$f(x) = \sum a_n x^n$$

then
$$a_n x^n = a_n q^n x^n + ax\, a_{n-1} q^{n-1} x^{n-1}$$

rec. fmla
$$a_n = \frac{aq^{n-1}}{1-q^n} a_{n-1}$$

so that
$$f(x) = a_0 \sum_{n\geq 0} \frac{q^{\frac{n(n-1)}{2}}}{\prod_1^n (1-q^i)} a^n x^n$$

which gives the identity

$$\boxed{\prod_{n\geq 0}(1+q^n x) = \sum_{n\geq 0}' \frac{q^{n(n-1)/2}}{\prod_1^n (1-q^i)} x^n}$$

Similarly look at

$$\theta(x) = ax\,\theta(qx) \qquad f(x) = \sum a_n x^n$$

~~gives~~

$$a_n x^n = ax\, a_{n-1} q^{n-1} x^{n-1}$$

or

$$a_n = a q^{n-1} a_{n-1} \implies a_n = q^{\frac{n(n-1)}{2}} a^n a_0$$

So you get a unique Laurent series solution up to a constant

$$\theta(x) = \sum_{n\in\mathbb{Z}}' q^{\frac{n(n-1)}{2}} a^n x^n$$

Finally look at (take $a=1$)

$$f(x) = (1+x) f(qx)$$
$$\theta(x) = \quad x\,\theta(qx)$$

$$\implies \frac{f}{\theta}(x) = (1+x^{-1})\frac{f}{\theta}(qx)$$

$$\frac{f}{\theta}\left(\frac{x}{q}\right) = \left(1+\frac{q}{x}\right)\frac{f}{\theta}(x) \qquad \frac{\theta}{f}(x) = \left(1+\frac{q}{x}\right)\frac{\theta}{f}\left(\frac{x}{q}\right)$$

so by iterating, we get an obvious power series solution in x^{-1}:

$$\frac{\theta}{f}(x) = \prod_{n\geq 1}(1+q^n x^{-1}) = \sum_{n\geq 0} \frac{q^{n(n-1)/2}}{\prod_1^n (1-q^i)} q^n x^{-n}$$

Therefore (and one can see this directly) if you multiply this with the old f you get

$$\prod_{n\geq 1}(1+q^n x^{-1}) \prod_{n\geq 0}(1+q^n x) = \text{const}\cdot\theta(x)$$

To determine the const (which depends on q but not x) let $x\to\infty$. Then one can apply dominant term to

$$\prod_{n\geq 0}(1+q^n x) = \sum \frac{q^{\frac{n(n-1)}{2}}}{\prod_1^n (1-q^i)} x^n$$

and it's clear that one can prove this series is asymptotic to the series.

$$\frac{1}{\prod_1^\infty (1-q^i)} \underbrace{\sum q^{n(n-1)/2} x^n}_{\Theta(x)}.$$

Hence we get Jacobi's identity

$$\boxed{\prod_{n\geq 1}(1+q^n x^{-1}) \prod_{n\geq 0}(1+q^n x) \prod_{i=1}^\infty (1-q^i) = \sum_{n\in\mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n}$$

which tells us that $\Theta(x)$ has simple zeroes at $x = -q^n$, $n \in \mathbb{Z}$.

Go back to

$$\text{PS of } S(\mathfrak{n}^*) = \prod_{n\geq 1} \frac{1}{1-u^2t^n} \prod_{n\geq 0} \frac{1}{1-u^{-2}t^n} \prod_{n\geq 1} \frac{1}{1-t^n}$$

$$z^n X \qquad z^n Y \qquad z^n H$$

If we use the Jacobi identity we get

$$\text{PS of } S(\mathfrak{n}^*) = \frac{1}{\sum_{n\in\mathbb{Z}} t^{\frac{n(n-1)}{2}} (-u^{-2})^n}$$

This denominator can be written in an interesting way.

$$\sum_{n\in\mathbb{Z}} t^{\frac{n(n-1)}{2}}(-u^{-2})^n = \sum_{n=1}^\infty t^{\frac{n(n-1)}{2}} \underbrace{\left[(-u^{-2})^n + (-u^{-2})^{-n}\right]}_{(-1)^{n+1}(u^{2n-1}-u^{-2n+1})u^{-1}}$$

$\frac{u^{2n-1}-u^{-2n+1}}{u-u^{-1}}$ is the character of the irred. repn of $\mathfrak{g}$ with highest weight $2n-1$.

and maybe the $\frac{n(n-1)}{2}$ has something to do with the Casimir operator.

On sl_2 one has the invariant inner product [redacted] $tr(AB) = \langle A, B\rangle$. One has the basis X, Y, H and the dual basis is $Y, X, \frac{1}{2}H$ so an[redacted] invariant operator is clearly

$$YX + XY + \tfrac{1}{2}H^2 = 2YX + H + \tfrac{1}{2}H^2$$

On the irreducible repn. with highest weight n this operator has the [redacted]value $n + \frac{1}{2}n^2 = \frac{n(n+2)}{2}$

Now

$$|\lambda+\rho|^2 - |\rho|^2 = (n+1)^2 - 1^2 = n^2 + 2n$$

This isn't very clear. Formula

$$\boxed{\tfrac{1}{2}\left(|\lambda+\rho|^2 - |\rho|^2\right) = \text{eigenvalue of } YX+XY+\tfrac{1}{2}H^2 \text{ on irreducible repn. with highest weight } \lambda.}$$

For the Killing form $tr(\operatorname{ad}A \operatorname{ad}B)$ on sl_2 one has $tr((\operatorname{ad}H)^2) = 4+4 = 8$, so Casimir should contain the term $\frac{1}{8}H^2$. If so the eigenvalue of Casimir in the representation of heighest weight $2n-2$ is

$$\frac{1}{8}(2n-2)(2n) = \frac{n(n-1)}{2}$$

September 17, 1981

Kac-Moody Lie algebras: This begins with Serre's existence proof for the simple Lie algebras. Take $\mathfrak{g} = sl_n$ also called A_{n-1}. One has the root space decomp.

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}^{\alpha}$$

where $\alpha(h) = (h)_i - (h)_j$, $1 \le i \ne j \le n$. $\mathfrak{g}$ is generated by the simple root vectors

$$e_i = \begin{array}{c} \\ \end{array}\begin{array}{c} i+1 \\ \downarrow \end{array} \quad e_i = \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix} \leftarrow i \qquad f_i = e_i^* \qquad 1 \le i \le n-1$$

and by

$$h_i = [e_i, f_i] = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} \begin{array}{l} \leftarrow i \\ \leftarrow i+1 \end{array}$$

These satisfy the following relations.

$$[h_i, e_j] = \alpha_{ij} e_j \qquad \text{where } \alpha_{ij} = \begin{cases} 2 & i = j \\ -2 & |i-j| = 1 \end{cases}$$

$$[h_i, f_j] = -\alpha_{ij} f_j \qquad \text{is the Cartan matrix}$$

$$[e_i, f_j] = \delta_{ij} h_i$$

Finally there is the Serre relation:

$$\begin{cases} (\operatorname{ad} f_i)^{-\alpha_{ij}+1} f_j = 0 & i \ne j \\ (\operatorname{ad} e_i)^{-\alpha_{ij}+1} e_j = 0 & i \ne j \end{cases}$$

In the general theory α_{ij} needn't be symmetric but in the important examples it is.

The Kac-Moody idea is to consider the Lie alg. defined by the above relations for any Cartan matrix (generalized).

Example: Take the loop algebra for sl_2 i.e.

$$\tilde{\mathfrak{g}} = sl_2[z, z^{-1}] = \underset{(z^{-1}\mathfrak{g}[z^{-1}] + (Y))}{\tilde{n}^-} \oplus \underset{(H)}{\mathfrak{h}} \oplus \underset{((X) + z\mathfrak{g}[z])}{\tilde{n}^+}$$

In this algebra we have the generators

$$e_1 = X \qquad f_1 = Y \qquad h_1 = [X, Y] = H$$
$$e_2 = zY \qquad f_2 = z^{-1}X \qquad h_2 = [zY, z^{-1}X] = -H$$

and the relations

$$\left[h_1, \begin{pmatrix} e_1 \\ e_2 \\ f_1 \\ f_2 \end{pmatrix}\right] = \begin{pmatrix} 2e_1 \\ -2e_2 \\ -2f_1 \\ 2f_2 \end{pmatrix} \qquad \left[h_2, \begin{pmatrix} e_1 \\ e_2 \\ f_1 \\ f_2 \end{pmatrix}\right] = \begin{pmatrix} -2e_1 \\ 2e_2 \\ 2f_1 \\ -2f_2 \end{pmatrix}$$

which gives the Cartan matrix

$$(\alpha_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

However in the Kac-Moody algebra with these relations one has $h_1 \neq -h_2$ and hence it seems that the KM algebra is a central extension of $\tilde{\mathfrak{g}}$.

The Serre relations are

$$(\operatorname{ad} f_i)^3 f_j = 0 \qquad i \neq j$$
$$(\operatorname{ad} e_i)^3 e_j = 0 \qquad i \neq j$$

Now
$$(\operatorname{ad} e_i)^2 h_j = (\operatorname{ad} e_i)^1 \underbrace{[e_i, h_j]}_{-\alpha_{ji} e_i} = 0$$

and
$$(\operatorname{ad} e_i)^3 f_j = (\operatorname{ad} e_i)^2 \underbrace{[e_i, f_j]}_{\delta_{ij} h_i} = 0$$

Unfortunately $(\operatorname{ad} e_i)^3$ is not a derivation, so it's not immediately clear that $(\operatorname{ad} e_i)^3 = 0$ in the KM algebra.

A first problem is to determine exactly what the root spaces in the KM algebra are.

Digression: Look at the Lie algebra of vector fields on S^1.
$$z = e^{2\pi i t} \qquad \frac{dz}{z} = 2\pi i \, dt$$

$$\frac{1}{2\pi i} \frac{d}{dt} = z \frac{d}{dz}$$

So we will describe a vector field on S^1 by a smooth function $f(z)$ via the formula

$$f(z)\, z\frac{d}{dz} = \frac{f(z)}{2\pi i}\frac{d}{dt}$$

Then the vector field is real $\Leftrightarrow$ $\overline{f(z)} = -f(z)$. Also

$$\left[f\, z\frac{d}{dz},\, g\, z\frac{d}{dz}\right] = \left(f\, z\frac{dg}{dz} - g\, z\frac{df}{dz}\right) z\frac{d}{dz}$$

So the Lie algebra has the generators z^n $n \in \mathbb{Z}$ with

$$[z^m, z^n] = (n-m)\, z^{m+n}$$

and the real vector fields are described by skew-hermitian element under the involution

$$(z^m)^* = z^{-m}.$$

$$(c z^m)^* = \bar{c}\, z^{-m}.$$

Call this Lie algebra $\mathfrak{g}$. Then we have

$$\mathfrak{g} = \underset{(z^{-n},\, n\geq 1)}{\mathfrak{n}^*} \oplus \underset{(z^0)}{\mathfrak{h}} \oplus \underset{(z^n,\, n\geq 1)}{\mathfrak{n}}$$

and $\mathfrak{n}$ has the generators z^1, z^2 because

$$[z^1, z^2] = (2-1)z^3 = z^3$$

$$[z^1, z^3] = (3-1)z^4 \qquad \text{etc.}$$

Unfortunately it won't work to put $e_1 = z$, $e_2 = z^2$ because then $f_1 = z^{-1}$, $f_2 = z^{-2}$ is reasonable, and then $[f_1, e_2] = [z^{-1}, z^2] = 3z \neq 0$. So this string algebra doesn't seem to be a KM type algebra.

Central extensions of Lie algebras. Suppose we want a central extension

$$0 \to V \to \mathfrak{g}_1 \to \mathfrak{g} \to 0$$

where V is a vector space. Choose a splitting $\mathfrak{g}_1 \cong \mathfrak{g} + V$ and then the bracket will be given by

$$[x+v, y+w] = [x,y] + f(x,y)$$

where $f: \mathfrak{g} \otimes \mathfrak{g} \longrightarrow V$ satisfies some conditions: 1) f must be skew-symmetric. 2) For the Jacobi identity

$$[x+v, [y+w, z+u]] = [x+v, [y,z] + f(y,z)]$$
$$= [x,[y,z]] + f(x,[y,z])$$

so we must have

$$\boxed{\begin{aligned} &f(x,y) = -f(y,x) \\ &f(x,[y,z]) + f(y,[z,x]) + f(z,[x,y]) = 0 \end{aligned}}$$

If we choose another splitting: $s(x) = x + h(x)$, then we have the new cocycle

$$[s(x), s(y)] - s[x,y] = [x+h(x), y+h(y)] - [x,y] - h([x,y])$$
$$= f(x,y) - h([x,y])$$

So a cocycle f is a coboundary when it is of the form

$$f(x,y) = h([x,y]) \quad \text{where} \quad h: \mathfrak{g} \longrightarrow V \text{ is linear.}$$

Anyway the string algebra perhaps has a more or less canonical central extension. The cocycle is ~~perhaps~~

$$\langle f, g \rangle = \int f\, dg = i \int f g' \, d\theta$$

where $g' = z \dfrac{dg}{dz} = e^{i\theta} \dfrac{dg}{i e^{i\theta} d\theta} = \dfrac{1}{i} \dfrac{dg}{d\theta}$. Then

$$\langle f, [g,h] \rangle = i \int f(gh' - hg')\, d\theta$$

and when cyclically permuted & added we get

$$\begin{aligned} fgh' + ghf' + hfg' &= (fgh)' \\ -(fhg' + hgf' + gfh') &\quad -(fhg)' \end{aligned}$$

which integrates to give 0.

Calculation: Let $\mathfrak{g}$ be $sl_2[z, z^{-1}]$ and denote by $\tilde{\mathfrak{g}}$ the central extension given by the Kac-Moody algebra. Then the map $\tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is an isom on all weight spaces except that on the 0 weight space we have $(h_1, h_2) \longmapsto (H)$ with kernel $h_1 + h_2$. So we choose a section s by mapping H into h_1. Then we can compute a cocycle

$$\Phi(\xi, \eta) = [s\xi, s\eta] - s[\xi, \eta]$$

This will be zero unless $[\xi, \eta] \in \mathfrak{h}$. So let's do some computations.

$$\Phi(X, Y) = [e_1, f_1] - s[X, Y]$$
$$= h_1 - s(H) = 0$$

$$\Phi(zY, z^{-1}X) = [e_2, f_2] - s(-H)$$
$$= 1(h_2 + h_1)$$

Let's identify $h_2 + h_2 \longleftrightarrow 1$. Next

$$\Phi(zH, z^{-1}H) = \quad ?$$

$$zH = [X, zY] \qquad s(zH) = [e_1, e_2]$$
$$z^{-1}H = [z^{-1}X, Y] \qquad s(z^{-1}H) = [f_2, f_1]$$

$$\Phi(zH, z^{-1}H) = [[e_1, e_2], [f_2, f_1]]$$
$$= [e_1 \underbrace{[e_2[f_2, f_1]]}_{[h_2, f_1] + \cancel{[f_2, 0]}}] - [e_2 \underbrace{[e_1[f_2, f_1]]}_{[e_2[f_2, h_1]] = -2h_2}]$$
$$2f_1 \quad = 2[e_1, f_1] = 2(h_1 + h_2)$$

Thus we have

$$\Phi(X,Y)=0$$

$$\Phi(zY,z^{-1}X)=1$$

$$\Phi(zH,z^{-1}H)=2$$

$$\Phi(zX,z^{-1}Y)=\frac{1}{2}\Phi([zH,X],z^{-1}Y)$$

$$=\frac{1}{2}\left(-\Phi([X,z^{-1}Y],zH)-\Phi([z^{-1}Y,zH],X)\right)$$

$$=\frac{1}{2}\left(-\Phi(z^{-1}H,zH)-\Phi(2Y,X)\right)$$

$$=\frac{1}{2}\left(\Phi(zH,z^{-1}H)+2\Phi(X,Y)\right)$$

$$=1$$

$$\Phi(z^2Y,z^{-2}X)=-\frac{1}{2}\Phi([zH,zY],z^{-2}X)$$

$$=+\frac{1}{2}\Big[\Phi(\underbrace{[zY,z^{-2}X]}_{-z^{-1}H},zH)+\Phi(\underbrace{[z^{-2}X,zH]}_{-2z^{-1}X},zY)\Big]$$

$$=\frac{1}{2}\big[\,2\qquad+2\,\big]$$

$$=2$$

Somehow this is too hard. Another possibility is to use the standard SL_2 inner product

$$\Phi(f(z),g(z))=\frac{1}{2\pi i}\int \mathrm{tr}(f(z)\,dg(z))$$

e.g. $$\Phi(z^2Y,z^{-2}X)=\frac{1}{2\pi i}\int z^2\,d(z^{-2})=\frac{1}{2\pi i}\int z^2(-2)\frac{dz}{z^3}=-2$$

$$\Phi(zH,z^{-1}H)=\frac{1}{2\pi i}\,2\int z\,dz^{-1}=-2$$

Hence it is OKAY up to sign.

September 18, 1981

Let $\mathfrak{g}$ be a Lie algebra with $H_1(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] = 0$. Then we know it has a universal central extension

$$0 \longrightarrow H_2(\mathfrak{g}) \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

I want to classify all homogeneous symplectic manifolds for the simply-connected Lie group G with Lie algebra $\mathfrak{g}$.

Suppose first that $\mathfrak{g} = \tilde{\mathfrak{g}}$, and let M be a symplectic manifold on which $\mathfrak{g}$ acts. This means we have a Lie homomorphism Θ:

$$\begin{tikzcd} & C^\infty(M) \arrow[d] \\ \mathfrak{g} \arrow[ur, dotted] \arrow[r, "\Theta"] & \mathrm{HamVectf}(M) \end{tikzcd}$$

$\leftarrow$ central extension of Lie algebras

[redacted] Now because $H_2(\mathfrak{g}) = 0$ the homomorphism Θ lifts to $C^\infty(M)$, and [redacted] because $H_1(\mathfrak{g}) = 0$, the lift is unique. Thus we get a canonical map

$$\mathfrak{g} \longrightarrow C^\infty(M) \qquad x \longmapsto H_x$$

which is a Lie homomorphism for the Poisson bracket. Hence we get a map

$$M \longrightarrow \mathfrak{g}^\vee$$

compatible with the action of $\mathfrak{g}$. If M is homogeneous then this map should be a covering of an orbit, so it coincides with the orbit when the orbit is 1-connected. If this is the case it follows that the adjoint group [redacted] of G acts on M.

So next consider the situation where $\tilde{\mathfrak{g}} \neq \mathfrak{g}$. Then [redacted] the orbits of $\tilde{\mathfrak{g}}$ on $\tilde{\mathfrak{g}}^\vee$ are the different $\tilde{\mathfrak{g}}$ homogeneous symplectic manifolds up to coverings. But because $\tilde{\mathfrak{g}}$ acting on $\tilde{\mathfrak{g}}$ is really an action of

$\tilde{\mathfrak{g}}/\mathfrak{z}$ on $\tilde{\mathfrak{g}}$, we see that the center acts trivially on the homogeneous symplectic manifolds. So we see that up to coverings homogeneous symplectic manifolds for $\mathfrak{g}$ are simply orbits in $\tilde{\mathfrak{g}}^\vee$.

Now we have

$$0 \longrightarrow \mathfrak{g}^\vee \longrightarrow \tilde{\mathfrak{g}}^\vee \longrightarrow H^2(\mathfrak{g}) \longrightarrow 0$$

so that each symplectic manifold determines a definite central extension of $\mathfrak{g}$ by $\mathbb{R}$. This kernel $\mathbb{R}$ will be important perhaps for the quantization.

[redacted] In the Kirillov-Kostant picture one has a [redacted] line bundle (at least infinitesimally) over the orbit $\mathcal{O}$ defined as follows. Take $\lambda \in \mathcal{O}$. Then if $\mathfrak{g}_\lambda = \{x \mid (\operatorname{ad} x)^t \lambda = 0\}$ we have $\lambda([x,y]) = 0$ for all $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}$. In particular $\lambda : \mathfrak{g}_\lambda \to \mathbb{R}$ is a character.

Thus in the case of a $\tilde{\mathfrak{g}}$ orbit on $\tilde{\mathfrak{g}}^\vee$ we will have $\tilde{\mathfrak{g}}_\lambda \supset H_2(\mathfrak{g})$ and so $\lambda : \tilde{\mathfrak{g}}_\lambda \longrightarrow \mathbb{R}$ will be constant on $H_2(\mathfrak{g})$. What this means is that if we look at the symplectic manifold together with the line bundle, [redacted] a definite central extension of $\mathfrak{g}$ by $\mathbb{R}$ acts, and the kernel $\mathbb{R}$ acts as a definite scalar on the line bundle.

So there seems to be a problem. [redacted] The orbit $\mathcal{O}$ in $(\tilde{\mathfrak{g}})^\vee$ is actually a $\mathfrak{g}$-orbit, so that we have

$$\tilde{G}/\tilde{H} \simeq G/H.$$

So any line bundle [redacted] over $\tilde{G}/\tilde{H}$ will also be a line bundle over G/H.

So look at the following. Take $\lambda \in (\tilde{\mathfrak{g}})^\vee$ and look at its orbit $\mathcal{O}$ under $\tilde{\mathfrak{g}}$. This is not going to be an orbit of $\mathfrak{g}$ on $\mathfrak{g}^\vee$. However the orbit should be some kind of homogeneous space for $\mathfrak{g}$, and so you can ask for the stabilizer of a point λ. It should be the case that $\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}_\lambda = \mathfrak{g}/\mathfrak{g}_\lambda$. Thus the question is what kind of $\mathfrak{g}_\lambda$ occurs in the case of loop groups.

Problem: $\mathcal{K} =$ alg maps $S^1 \longrightarrow K$ seems to have a central extension with kernel $\mathbb{C}^*$, possibly with kernel $= S^1$. For example if $K = SU_n$, then

$$\mathcal{K} \subset SL_n(F) \qquad F = \mathbb{C}[[z]][z^{-1}]$$

and there is a central extension $\quad n \geq 3$

$$\begin{tikzcd} 1 \arrow[r] & K_2 F \arrow[r] \arrow[d, "\text{tame symbol}"] & St_n(F) \arrow[r] & SL_n(F) \arrow[r] & 1 \\ & \mathbb{C}^* & & & \end{tikzcd}$$

Recall how Matsumoto constructs a central extension of $SL_n(F) = G$. One uses the Bruhat decomp.

$$G = UNU \qquad W = N/H$$

where N is the group of monomial matrices and U the unipotent radical of the Borel. Then the central extension is constructed in the form

$$\tilde{G} = U\tilde{N}U$$

where $1 \to K_2F \longrightarrow \tilde{N} \longrightarrow N \longrightarrow 1$ is an appropriate central extension.

So what we would like to have first of all is a central extension

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1$$

and it seems to me that there is an obvious candidate given by the Heisenberg algebra. L

September 20, 1981

My goal is now to understand the canonical central extension of $SL_n(\mathbb{C}[z,z^{-1}])$ [redacted] given by the tame symbol

$$K_2([redacted]\ \mathbb{C}[z,z^{-1}]) \longrightarrow K_2(F) \xrightarrow[\text{symbol}]{\text{tame}} \mathbb{C}^*$$

where $F = \mathbb{C}[[z]][z^{-1}]$. What I would like to do is to construct explicitly a representation of the central extension [redacted] in which $\mathbb{C}^*$ acts in the standard way. Question: What is this central extension when restricted to diagonal matrices? This question should be answerable from the theory of the Steinberg group.

Another [redacted] idea is to first understand Graeme's ideas for the string algebra. One start with $G = \mathrm{Diff}(S^1)$, and $\mathrm{Lie}(G) =$ smooth vector fields on S^1. Also we have $T = S^1$ as rotations inside of G. Then we have that $\mathrm{Lie}(G)_c = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{n}^+$ is spanned by $z^n z\frac{d}{dz}$ $n \geq 0$ etc. [redacted] Somehow one of the points is that this splitting of $\mathrm{Lie}(G)_c$ is invariant under the adjoint action modulo compact operators, and this defines a mapping of G into the group of symplectic matrices congruent to I mod compacts and this last group has a canonical Heisenberg repn. Not very clear.

One thing worth remembering is that there is a good class of vector fields in $\mathrm{Lie}(G)$, namely the non-vanishing vector fields. This [redacted] class is stable under the adjoint action, and the orbits are described by a non-zero real number, namely the rotation number.

[redacted] The string algebra works on the space $S(\mathfrak{n}^-)$ ion a way I really ought to understand. This is because of the isomorphism

$$U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes \underbrace{U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)}_{U(\mathfrak{h} \oplus \mathfrak{n}^+)}$$

so that

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}^+)} L_\lambda \cong U(\mathfrak{n}^-) \otimes_{\mathbb{C}} L_\lambda \cong S(\mathfrak{n}^-) \otimes_{\mathbb{C}} L_\lambda$$

where one has used the Poincaré-Birkhoff-Witt thm. [redacted]

[redacted]

Let's go back to the holomorphic function repn. of the canonical commutation relations.

$$\left.\begin{aligned} \|f\|^2 &= \int |f(z)|^2 e^{-|z|^2} \frac{dxdy}{\pi} \\ a &= \frac{d}{dz} \qquad a^* = z \end{aligned}\right\} \text{ in one dimension}$$

However there is a way to extend this to d dimensions and even ∞-dimensions. An orthonormal basis in d dimensions is given by monomials $\frac{z^n}{\sqrt{n!}} = \frac{z_1^{n_1} \cdots z_d^{n_d}}{\sqrt{n_1!} \cdots \sqrt{n_d!}}$.

In infinitely many dimensions, one looks at the Hilbert space with the orthonormal basis consisting of all $\frac{z^n}{\sqrt{n!}}$ where $n = (n_1, n_2, \cdots)$ has only finitely many $n_j \neq 0$. On this Hilbert space one has the $2d$-diml subspace V of operators of the form

$$\alpha a + \beta \bar{a} \qquad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \qquad \bar{a} = \begin{pmatrix} a_1^* \\ \vdots \\ a_n^* \end{pmatrix}$$

α, β are row vectors. On this space one has an involution

$$(\alpha a + \beta \bar{a})^* = \bar{\beta} a + \bar{\alpha} a^*$$

hence a real structure. Also one has the skew-symm. form

$$[x, y] = xy - yx$$

which is non-degenerate. Thus the space V is the complexification of a real symplectic vector space.

Next suppose given a transformation of V:

$$a' = Aa + B\bar{a}$$

A, B are d dim'l matrices

compatible with the conjugation:

$$\overline{a'} = \bar{B}a + \bar{A}\bar{a}$$

and the bracket:

$$[a', (a')^t] = [Aa + B\bar{a},\ a^tA^t + \bar{a}^tB^t]$$

$$= A\underbrace{[a, a^t]}_{0}A^t + A[a, \bar{a}^t]B^t$$

$$+ B[\bar{a}, a^t]A^t + B\underbrace{[\bar{a}, \bar{a}^t]}_{0}B^t$$

$$\boxed{0 = AB^t - BA^t} \qquad \therefore AB^t \text{ symmetric}$$

(Here $a^t = (a_1, \ldots, a_d)$ and I am writing the commutation relations in the form $[a, \bar{a}^t] = \left[\begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix}, (a_1^* \ldots a_d^*)\right] = [a_i, a_j^*] = \delta_{ij}$ etc.) Similarly

$$[a', \overline{a'}^t] = A\bar{A}^t - B\bar{B}^t$$

so we want

$$\boxed{I = AA^* - BB^*}$$

<u>Check</u>: Look at these conditions infinitesimally:

$A = I + \varepsilon\mathring{A}$, $B = \varepsilon\mathring{B}$ with $\varepsilon^2 = 0$. Then we get

$\mathring{B} = \mathring{B}^t$ gives $2\,\frac{d(d+1)}{2}$ poss for $\mathring{B}$

$\mathring{A} + \mathring{A}^* = 0$ gives d^2 poss. for $\mathring{A}$

total $2d^2 + d$

which is the dimension of the symplectic group.

Notice that

$$\begin{pmatrix} A & B \\ \bar B & \bar A\end{pmatrix}\begin{pmatrix} A^* & -B^t \\ -B^* & A^t\end{pmatrix} = \begin{pmatrix} AA^*-BB^* & -AB^t+BA^t \\ \bar B\bar A^t-\bar A\bar B^t & -\bar B\bar B^t+\bar A\bar A^t\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}$$

If we also have

$$\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix} = \begin{pmatrix} A^* & -B^t \\ -B^* & A^t\end{pmatrix}\begin{pmatrix} A & B \\ \bar B & \bar A\end{pmatrix} = \begin{pmatrix} A^*A-B^t\bar B & A^*B-B^t\bar A \\ -B^*A+A^t\bar B & -B^*B+A^t\bar A\end{pmatrix}$$

i.e.

$$A^*A-(\bar B)^*(\bar B)=I$$

$$A^t\bar B = (\bar B)^t A \quad \Longrightarrow \quad A^t\bar B \text{ symmetric}$$

then we have
$$\begin{pmatrix} A & B \\ \bar B & \bar A\end{pmatrix}^{-1} = \begin{pmatrix} A^* & -B^t \\ -B^* & A^t\end{pmatrix}.$$

This would be automatic in finite dimensions from the conditions on the previous page.

The next project is take the transformation of commutation relations:

$$\begin{pmatrix} a' \\ \bar a'\end{pmatrix} = \begin{pmatrix} A & B \\ \bar B & \bar A\end{pmatrix}\begin{pmatrix} a \\ \bar a\end{pmatrix}$$

AB^t symmetric

$AA^*-BB^*=I$

and to implement this by a unitary operator S on the Hilbertspace. Thus I want

$$\begin{pmatrix} a' \\ \bar a'\end{pmatrix} = S\begin{pmatrix} a \\ \bar a\end{pmatrix}S^{-1}$$

by which I mean simply $\boxed{a_i' = Sa_iS^{-1}}$ for each i.

Recall that the exponential functions $e^{\lambda z}$ form a very nice generating set for the Hilbert space.

$$\|e^{\lambda z}\|^2 = \int e^{\overbrace{+\lambda z+\bar\lambda\bar z - z\bar z}^{-|z-\bar\lambda|^2+|\lambda|^2}}\prod \frac{dx_i\,dy_i}{\pi}$$

$$= e^{+|\lambda|^2}$$

using translation invariance

It's clear in ∞ dims we want $|\lambda|^2 < \infty$.
The exponential functions are the eigenfunctions for the operators a_i. Thus

$$(a_i - \lambda_i)\psi = 0 \implies \psi = \text{const.}\, e^{\lambda z}$$

So

$$(a_i' - \lambda_i) S\psi = (S a_i S^{-1} - \lambda_i) S\psi$$
$$= S(a_i - \lambda_i)\psi = 0.$$

and we see that $\psi' = S e^{\lambda z}$ is an eigenfunction for a'.

$$a' = Aa + B\bar{a}$$

$$\left(A\frac{d}{dz} + Bz\right)\psi' = \lambda\psi'$$

$$\left(\frac{d}{dz} + A^{-1}Bz - A^{-1}\lambda\right)\psi' = 0$$

so we get
$$\psi' = \text{const}\; e^{-\frac{1}{2}z^t A^{-1}Bz + z^t A^{-1}\lambda}$$

and for this to work we must have that $A^{-1}B$ is symmetric, among other things. This follows from

$$AB^t = BA^t \implies B^t(A^t)^{-1} = A^{-1}B$$

Let's generalize this calculation as follows. Put e_λ for the function $e^{\lambda z}$. Recall

$$\langle e^{\lambda z} | f \rangle = \left\langle \sum_\alpha \frac{\lambda^\alpha z^\alpha}{\alpha!} \,\Big|\, \sum_\alpha c_\alpha z^\alpha \right\rangle$$
$$= \sum_\alpha \frac{\bar\lambda^\alpha}{\alpha!} c_\alpha \alpha! = \sum c_\alpha \bar\lambda^\alpha = f(\bar\lambda).$$

or that

$$f(\bar\lambda) = \int e^{\bar\lambda \bar z} f(z) e^{-|z|^2} dL$$

$$= \int e^{\bar\lambda a} f(\bar a) e^{-|a|^2} dL =$$

hence

$$\boxed{|f\rangle = \int |e_u\rangle e^{-|u|^2} dL \,\langle e_u | f\rangle}$$

which shows how to reconstruct $|f\rangle$ from the family $|e_u\rangle$.

[redacted] I want to compute $S|e_\lambda\rangle$, and it will be enough to compute the matrix element $\langle e_\mu | S | e_\lambda\rangle$.

[redacted] We've already seen that

$$\langle e_\mu | S | e_\lambda \rangle = (Se_\lambda)(\bar\mu) = \underset{\text{on } \lambda}{\text{const dep}}\; e^{-\frac{1}{2}\bar\mu^t A^{-1} B \bar\mu + \bar\mu^t A^{-1}\lambda}$$

because we knew $(a-\lambda)e_\lambda = 0$.

Go over this

$$\lambda \langle e_{\bar\mu} | S | e_\lambda\rangle = \langle e_{\bar\mu} | S | \overbrace{\lambda e_\lambda}^{a e_\lambda} \rangle$$

$$= \langle e_{\bar\mu} | \underbrace{S a S^{-1}}_{a' = Aa + B\bar a} S | e_\lambda\rangle$$

$$= A \underbrace{\langle e_{\bar\mu} |}\, a S | e_\lambda \rangle + B \underbrace{\langle e_{\bar\mu} |}\, \bar a S | e_\lambda\rangle$$

$$\langle z e_{\bar\mu} | = \frac{d}{d\mu} \langle e_\mu | \qquad \langle a e_{\bar\mu} | = \mu \langle e_{\bar\mu} |$$

$$= \left(A \frac{d}{d\mu} + B\mu \right) \langle e_{\bar\mu} | S | e_\lambda \rangle$$

which was the D.E. obtained before. Now

$$\frac{d}{d\lambda} \langle e_{\bar\mu} | S | e_\lambda \rangle = \langle e_{\bar\mu} | S \bar a | e_\lambda\rangle$$

$$= \langle e_{\bar\mu} | (\bar B a + \bar A \bar a) S | e_\lambda \rangle$$

$$= \left(\bar B \frac{d}{d\mu} + \bar A \mu \right) \langle e_{\bar\mu} | S | e_\lambda \rangle$$

$$= \left(\bar B (-A^{-1} B\mu + A^{-1}\lambda) + \bar A \mu \right) \langle e_{\bar\mu} | S | e_\lambda \rangle$$

So $\log\langle e_{\bar\mu}|S|e_\lambda\rangle = \frac{1}{2}\lambda^t \bar{B}A^{-1}\lambda + \lambda^t(-\bar{B}A^{-1}B+\bar{A})\mu + \text{const}$

($A^t\bar{B} = \bar{B}^tA \implies \bar{B}A^{-1} = (A^t)^{-1}\bar{B}^t = (\bar{B}A^{-1})^t$ is symmetric

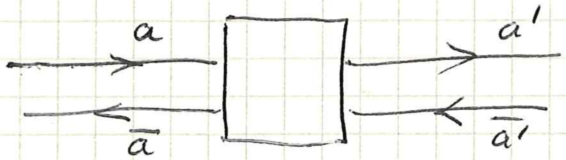
Also $(-\bar{B}A^{-1}B+\bar{A})^t = -(A^{-1}B)B^* + A^* = A^{-1}(-BB^*+AA^*) = A^{-1}$

Thus we get the formula

$$\boxed{\langle e_{\bar\mu}|S|e_\lambda\rangle = \text{const}\; e^{-\frac{1}{2}\mu^t(A^{-1}B)\mu + \mu^tA^{-1}\lambda + \frac{1}{2}\lambda^t\bar{B}A^{-1}\lambda}}$$

which I can use to write an integral formula for the operator S.

Next I should review the scattering coefficients



$$\begin{pmatrix} a' \\ \bar{a}' \end{pmatrix} = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}\begin{pmatrix} a \\ \bar{a} \end{pmatrix}$$

I want to solve for $\begin{matrix} a \\ \bar{a}' \end{matrix}$ in terms of $\begin{matrix} a' \\ \bar{a} \end{matrix}$

$$a' = Aa + B\bar{a} \qquad a = A^{-1}a' - A^{-1}B\bar{a}$$

$$\begin{aligned} \bar{a}' &= \bar{B}(A^{-1}a' - A^{-1}B\bar{a}) + \bar{A}\bar{a} \\ &= \bar{B}A^{-1}a' + \underbrace{(A+\bar{B}A^{-1}B)}_{(A^{-1})^t}\bar{a} \end{aligned}$$

so

$$\boxed{\begin{pmatrix} a \\ \bar{a}' \end{pmatrix} = \begin{pmatrix} -A^{-1}B & A^{-1} \\ (A^{-1})^t & \bar{B}A^{-1} \end{pmatrix}\begin{pmatrix} \bar{a} \\ a' \end{pmatrix}}$$

this is a symmetric unitary matrix, the scattering matrix.

So now I should be in a good position to understand the (Shale?) theorem on implementing all symp.

transformations. Unitary transformations on the z-variable can be implemented easily. These correspond to matrices of the form $\begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}$ with U unitary.

It seems that in finite dimensions the symplectic group has the unitary group as maximal compact subgroup. Recall that inf-symplectics are

$$\begin{pmatrix} \dot{A} & \dot{B} \\ \bar{\dot{B}} & \bar{\dot{A}} \end{pmatrix} \qquad \dot{B} = \dot{B}^t \quad , \quad \dot{A} + \dot{A}^* = 0$$

so if we divide out by inf unitaries, i.e. skew-hermitian matrices we get

$$\mathfrak{p} = \text{set of } \begin{pmatrix} 0 & C \\ \bar{C} & 0 \end{pmatrix} \qquad C = C^t$$

and so the symmetric space should consist of

$$\exp\begin{pmatrix} 0 & C \\ \bar{C} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & C \\ \bar{C} & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} C\bar{C} & \\ & \bar{C}C \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & C\bar{C}C \\ \bar{C}C\bar{C} & 0 \end{pmatrix}.$$

Thus

$$A = 1 + \frac{1}{2!} C\bar{C} + \frac{1}{4!}(C\bar{C})^2 + \dots = \cosh\sqrt{C\bar{C}}$$

$$B = C + \frac{1}{3!} C\bar{C}C + \dots = C\,\frac{\sinh\sqrt{\bar{C}C}}{\sqrt{\bar{C}C}} = \frac{\sinh\sqrt{C\bar{C}}}{\sqrt{C\bar{C}}}\,C$$

and we have $C\bar{C} = CC^* \geq 0$, so $A = A^* \geq I$.

Notice also that $B = B^t$ is symmetric. It's also clear that we have

$$1 + BB^* = 1 + \sinh^2\sqrt{C\bar{C}} = \cosh^2(\sqrt{C\bar{C}}) = A^2$$

and

$$AB^t = AB = \cosh\sqrt{C\bar{C}}\,\frac{\sinh\sqrt{C\bar{C}}}{\sqrt{C\bar{C}}}\,C$$

is symmetric.

Thus we would like to show any $\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$ with $B = B^t$ and $A = A^* > 0$ is uniquely in the form $\exp\begin{pmatrix} 0 & C \\ \bar{C} & 0 \end{pmatrix}$ with C symmetric. We can conjugate by unitaries

$$\begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}\begin{pmatrix} U^* & \\ & U^t \end{pmatrix} = \begin{pmatrix} UAU^* & UBU^t \\ \bar{U}\bar{B}U^* & \bar{U}\bar{A}U^t \end{pmatrix}$$

without changing that $B = B^t$ and $A = A^* > 0$. Thus we can assume that A is diagonal with real positive entries. Then because $A = A^t$ and $AB^t = AB$ is symmetric we have

$$AB = (AB)^t = B^t A^t = BA$$

so that B commutes with A. ∎

Actually suppose only that $A > 0$, but not that B is symmetric. Then transform by a unitary so that A is real and positive diagonal. [redacted] From the scattering matrix we get that

$$A^{-1}B \quad \text{and} \quad \bar{B}A^{-1} \quad \text{are symmetric}$$

$$\therefore \quad BA^{-1} \quad \text{is symmetric}$$

Thus
$$a_i^{-1} b_{ij} = a_j^{-1} b_{ji} \qquad b_{ij} a_j^{-1} = b_{ji} a_i^{-1}$$

$$\Rightarrow \quad \text{[redacted]} \quad a_i^{-2} b_{ij} = a_j^{-1} b_{ji} a_i^{-1} = a_j^{-2} b_{ij}$$

$$\Rightarrow \quad b_{ij} = 0 \quad \text{if} \quad a_i \neq a_j$$ remember these are > 0.

So we see that A and B commute. Also we see that B is symmetric. So in general one can conclude that $A > 0 \Rightarrow B$ symmetric

Yesterday we associated to a symplectic transformation

$$\begin{pmatrix} a' \\ \bar{a}' \end{pmatrix} = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \begin{pmatrix} a \\ \bar{a} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a \\ \bar{a}' \end{pmatrix} = \begin{pmatrix} -A^{-1}B & A^{-1} \\ A^{-1} & \bar{B}A^{-1} \end{pmatrix} \begin{pmatrix} \bar{a} \\ a' \end{pmatrix}$$

a unitary operator S with $a_i' = S a_i S^{-1}$ and found

$$\langle e_{\bar{\mu}} | S | e_\lambda \rangle = \text{const}\; e^{+\frac{1}{2}\mu^t(-A^{-1}B)\mu + \mu^t A^{-1}\lambda + \frac{1}{2}\lambda^t(\bar{B}A^{-1})\lambda}$$

A better way to say this is that if S is a unitary operator such that conjugation by S preserves the operator space spanned by the a_i, a_i^*, then the matrix elements $\langle e_{\bar{\mu}} | S | e_\lambda \rangle$ has the above form.

It's likely that the above formulas do not associate to a product of operators TS the corresponding product of matrices. So suppose that T has the matrix

$$\begin{pmatrix} C & D \\ \bar{D} & \bar{C} \end{pmatrix} \qquad \text{i.e.}$$

$$T a T^{-1} = Ca + D\bar{a}$$

Then to the product TS we have

$$\begin{aligned} TSaS^{-1}T^{-1} &= T(Aa + B\bar{a})T^{-1} \\ &= A(Ca + D\bar{a}) + B(\bar{D}a + \bar{C}\bar{a}) \\ &= (AC + B\bar{D})a + (AD + B\bar{C})\bar{a} \end{aligned}$$

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \begin{pmatrix} C & D \\ \bar{D} & \bar{C} \end{pmatrix} = \begin{pmatrix} AC + B\bar{D} & AD + B\bar{C} \\ & \end{pmatrix} \quad \text{etc.}$$

Therefore if we associate to S the matrix with

$$SaS^{-1} = Aa + B\bar{a}$$

then composition of operators corresponds to reverse mult.

of matrices. This makes sense because $a_1, \dots a_n, \bar{a}_1, \dots, \bar{a}_n$ is a basis for V, so the matrix of the transformation $b \longmapsto SbS^{-1}$ should be written using the row vector $(a^t \ \bar{a}^t)$:

$$S(a^t \ \bar{a}^t)S^{-1} = a^t A^t + \bar{a}^t B^t \quad \bar{a}^t \bar{A}^t + a^t \bar{B}^t$$

$$= (a^t \ \bar{a}^t)\begin{pmatrix} A^t & \bar{B}^t \\ B^t & \bar{A}^t \end{pmatrix}$$

Now I want to get at the symmetric space which I can identify as the orbit under the group of these unitary operators S of the line gener. by ground state e_0. Thus if to S belongs $\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$ we find $S|e_0\rangle$ is

$$(S|e_0\rangle)(z) = \text{const } e^{\frac{1}{2} z^t(-A^{-1}B)z}$$

So what we have is a symmetric complex matrix $A^{-1}B = B^t(A^t)^{-1}$ associated to S. To the product TS belongs the symmetric matrix

$$(AD + B\bar{C})^t((AC + B\bar{D})^t)^{-1}$$

$$= (\bar{C}^t B^t + D^t A^t)(C^t A^t + \bar{D}^t B^t)^{-1}$$

$$= (\bar{C}^t B^t (A^t)^{-1} + D^t) A^t \left[(C^t + \bar{D}^t B^t (A^t)^{-1}) A^t \right]^{-1}$$

$$= \left\{ \bar{C}^t \left[B^t (A^t)^{-1} \right] + D^t \right\} \left\{ \bar{D}^t (B^t (A^t)^{-1}) + C^t \right\}^{-1}$$

Thus $T \longleftrightarrow \begin{pmatrix} C & D \\ \bar{D} & \bar{C} \end{pmatrix}^t = \begin{pmatrix} C^t & \bar{D}^t \\ D^t & \bar{C}^t \end{pmatrix}$ acts on $Z = B^t(A^t)^{-1}$

by $\quad T * Z = (\bar{C}^t Z + D^t)(\bar{D}^t Z + C^t)^{-1}$

So now it is clear that you want to revise all previous formulas so that to the operator T belongs the matrix

$$\begin{pmatrix} \bar{C}^t & D^t \\ \bar{D}^t & C^t \end{pmatrix}$$

Projects: Understand Shale thms. and to compute the cocycle describing this extension.

We have seen that to $\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$ belongs the symmetric matrix $A^{-1}B$ which is independent of multiplying on the left by unitaries.

$$\begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}\begin{pmatrix} V & 0 \\ 0 & \bar{V} \end{pmatrix} = \begin{pmatrix} UAV & UB\bar{V} \\ & \end{pmatrix}$$

$$(UAV)^{-1}(UB\bar{V}) = V^{-1}A^{-1}B\bar{V} = V^*(A^{-1}B)\bar{V}$$
$$= (\bar{V})^t(A^{-1}B)\bar{V}$$

Thus I want to know the orbits for symmetric complex matrices under the action $U^t(A^{-1}B)U$ by unitary matrices. This amounts to describing bilinear quadratic forms on a finite dimensional Hilbert space. Call such a form Q on V. Look at Q on the unit sphere of V and choose v_1 so that $\operatorname{Re} Q(v_1,v_1)$ is maximum on the unit sphere. If $\langle v_2 | v_1 \rangle = 0$, then $v_1 + \varepsilon v_2$ is tangent to the unit sphere so

$$\operatorname{Re} Q(v_1+\varepsilon v_2, v_1+\varepsilon v_2) = \operatorname{Re} Q(v_1,v_1) + 2\operatorname{Re}(\varepsilon Q(v_1,v_2))$$

has to vanish to first order in ε. Thus we must have $Q(v_1,v_2) = 0$. So now continue and you get an orth. basis $v_1,\dots,v_n$ for V such that $Q(v_i,v_j) = \delta_{ij} r_j$ with $r_1 \geq r_2 \geq \dots \geq r_n \geq 0$.

Here's how to lift Z to an $\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$: We want $Z = A^{-1}B$, so

$$ZZ^* = A^{-1}BB^*(A^{-1})^* = A^{-1}(AA^*-I)(A^*)^{-1}$$
$$= I - A^{-1}(A^{-1})^*$$

so if $A > 0$ we have that $\quad A^{-1} = \sqrt{1-ZZ^*}$

and then
$$B = AZ = (1-ZZ^*)^{-1/2}Z.$$

B will be given by a series with terms $ZZ^* \cdots Z^*Z$

and hence will be symmetric. The rest is clear.

Especially if Z is diagonal, then so will A and B. So if $Z = (r_i \delta_{ij})$ with $r_i \geq 0$ we have

$$A = \frac{1}{\sqrt{1-r_i^2}}\,\delta_{ij}$$
$$B = \frac{r_i}{\sqrt{1-r_i^2}}\,\delta_{ij}$$

Now to see when this sort of transformation comes from a unitary operator. The first thing to compute is the norm of

$$e^{-\frac{1}{2} z^t A^{-1} B z} = \prod e^{-\frac{1}{2} r_i z_i^2}$$

$$\left\| e^{-\frac{1}{2} r z^2} \right\|^2 = \sum_n \left\| \frac{\left(-\frac{1}{2} r z^2\right)^n}{n!} \right\|^2 = \sum_n \frac{1}{(n!)^2} \left(\frac{|r|}{2}\right)^{2n} (2n)!$$

$$\frac{(2n)!}{n!\,n!\,2^{2n}} = \frac{(2n-1)\cdots 3 \cdot 1}{n!\,2^n} = \frac{1}{n!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)(-1)^n$$

$$\therefore \quad \boxed{\left\| e^{-\frac{1}{2} r z^2} \right\|^2 = \left(1-|r|^2\right)^{-1/2} \quad \text{in one-dim}}$$

Conclude that

$$\left\| e^{-\frac{1}{2} z^t (A^{-1} B) z} \right\|^2 = \det\left(1 - Z Z^*\right)^{-1/2}$$

and so therefore the condition of unitary implementability is that $Z = A^{-1}B$ is Hilbert-Schmidt.

Next we want to compute the cocycle of the central extension of the symplectic group given by the unitary operators preserving V. So I need a section and therefore to a matrix [illegible] $\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$ I will associate the operator S given by

$$\langle e_z | S | e_\lambda \rangle = e^{-\frac{1}{2}\bar{z}^t A^{-1} B \bar{z} + \bar{z}^t A^{-1}\lambda + \frac{1}{2}\lambda^t \bar{B} A^{-1}\lambda}$$

This is not unitary but differs from it by a scalar, so we will get a cocycle with values in $\mathbb{C}^*$.

Now to compute the composition of these operators I will use the completeness formula

$$f(z) = \langle e_{\bar{z}} | f \rangle = \int e^{z\bar{u}}\, e^{-|u|^2} \underbrace{f(u)}_{\langle e_{\bar{u}} | f \rangle}$$

or

$$id = \int |e_z\rangle\, e^{-|z|^2} \langle e_z|$$

Thus if T is the operator belonging to $\begin{pmatrix} C & D \\ \bar{D} & \bar{C} \end{pmatrix}$ we we get

$$\langle e_\mu | TS | e_\lambda \rangle = \int \langle e_\mu | T | e_z \rangle\, e^{-|z|^2} \langle e_z | S | e_\lambda \rangle$$

$$= \int e^{-\frac{1}{2}\bar{\mu}^t C^{-1} D \bar{\mu} + \bar{\mu}^t C^{-1} z + \frac{1}{2} z^t \bar{D} C^{-1} z - |z|^2 - \frac{1}{2}\bar{z}^t A^{-1} B \bar{z} + \bar{z}^t A^{-1}\lambda + \frac{1}{2}\lambda^t \bar{B} A^{-1}\lambda}$$

This is a Gaussian integral which consists of an exponential factor which you get by evaluating at the critical point, and a determinantal factor. The latter gives the cocycle. Thus we want to compute

$$(*) \qquad \int e^{\frac{1}{2} z^t \beta z - |z|^2 + \frac{1}{2}\bar{z}^t \alpha \bar{z}} \qquad \begin{aligned} \alpha &= -A^{-1}B \\ \beta &= \bar{D} C^{-1} \end{aligned}$$

with respect to the volume $\prod \frac{dx\,dy}{\pi}$. In general for a Gaussian integral

$$\int e^{-x^t A x} \prod \frac{dx}{\sqrt{\pi}} = (\det A)^{-1/2}$$

when A is a symmetric matrix with positive definite real part. The quadratic form on $\mathbb{C}^n = \mathbb{R}^{2n}$ involved in $(*)$ has the matrix when complexified

$$\frac{1}{2}\begin{pmatrix} -\beta & 1 \\ 1 & \alpha \end{pmatrix}$$

relative to the basis with coords $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$,
hence its determinant should be some constant (universal) times $\det(1 - \alpha\beta) = \det(1 - \beta\alpha)$ $\quad (\det A = \det A^t)$

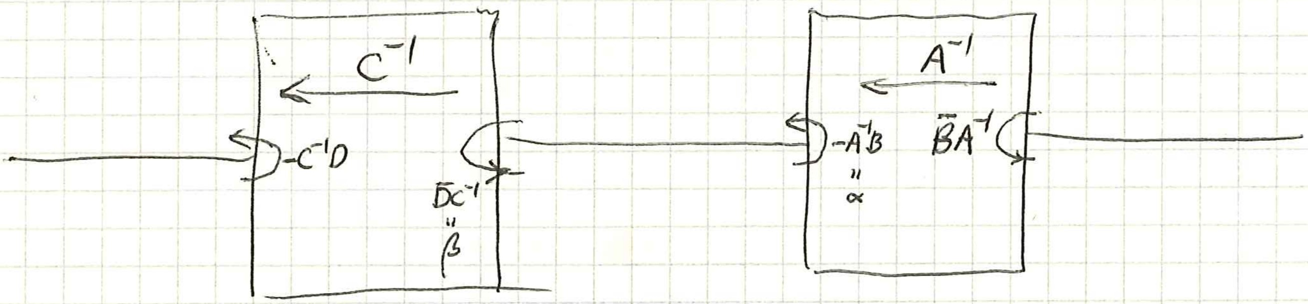
(Note $\begin{pmatrix} -\beta & 1 \\ 1 & -\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-\alpha\beta & \alpha \end{pmatrix}$ etc.)

Thus we have the formula

$$\boxed{\int e^{\frac{1}{2}z^t\beta z - |z|^2 + \frac{1}{2}\bar{z}^t\alpha\bar{z}} = \det(1 - \alpha\beta)^{-1/2}}$$

where here α, β are symmetric and satisfy $\alpha\alpha^* < 1$ $\beta^*\beta < 1$ so that the determinant is definable by a powerseries.

It seems desirable to work in terms of the transmission and reflection coefficients



The transmission coefficient for the two connected together is

$$C^{-1}(1 + \alpha\beta + \alpha\beta\alpha\beta + \cdots)A^{-1} = C^{-1}\frac{1}{1-\alpha\beta}A^{-1}$$

Check:

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}\begin{pmatrix} C & D \\ \bar{D} & \bar{C} \end{pmatrix} = \begin{pmatrix} AC+B\bar{D} & AD+B\bar{C} \\ & \end{pmatrix}$$

so the new transmission coefficient is

$$\frac{1}{AC+B\bar{D}} = \left[A(1 + A^{-1}B\bar{D}C^{-1})C\right]^{-1}$$
$$= C^{-1}(1-\alpha\beta)^{-1}A^{-1}.$$

The cocycle now is as follows. Let

$$g_1 \longleftrightarrow \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \qquad g_2 \longleftrightarrow \begin{pmatrix} C & D \\ \bar{D} & \bar{C} \end{pmatrix}$$

be the symplectic transformations belonging to the given matrices. Then we know

$$g_2 g_1 \longleftrightarrow \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}\begin{pmatrix} C & D \\ \bar{D} & \bar{C} \end{pmatrix} = \begin{pmatrix} AC+B\bar{D} & AD+B\bar{C} \\ \cdot & \cdot \end{pmatrix}$$

and the cocycle is given by

$$f(g_2,g_1)\, S_{g_2 g_1} = S_{g_2} S_{g_1}$$

Here S_g is the lift to the unitary gp. Thus we have the formula

$$f(g_2,g_1) = \det\left(1 + A^{-1}B\bar{D}C^{-1}\right)^{-1/2}$$

Note that

$$\begin{aligned} f(g_2,g_1)^2 &= \det\left(A^{-1}(AC+B\bar{D})C^{-1}\right) \\ &= h(g_1)\, h(g_2 g_1)^{-1}\, h(g_2) \end{aligned}$$

where

$$h : \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} = (\det A)^{-1}$$

Thus we see that f^2 is a coboundary, and this seems true in ∞ dimensions.

September 22, 1981

So let

$$g_1 \longleftrightarrow \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \qquad g_2 \longleftrightarrow \begin{pmatrix} C & D \\ \bar{D} & \bar{C} \end{pmatrix} \qquad g_3 \longleftrightarrow \begin{pmatrix} E & F \\ \bar{F} & \bar{E} \end{pmatrix}$$

$$g_2 g_1 \longleftrightarrow \begin{pmatrix} AC + B\bar{D} & AD + B\bar{C} \\ \bar{B}C + \bar{A}\bar{D} & \bar{B}D + \bar{A}\bar{C} \end{pmatrix} \qquad g_3 g_2 \longleftrightarrow \begin{pmatrix} CE + D\bar{F} & CF + D\bar{E} \\ \cdot & \cdot \end{pmatrix}$$

$$g_3 g_2 g_1 \longleftrightarrow \begin{pmatrix} (AC+B\bar{D})E + (AD+B\bar{C})\bar{F} & \cdot \\ \cdot & \cdot \end{pmatrix}$$

Now we saw that the cocycle was

$$f(g_2, g_1) = \det\left(A^{-1}(AC+B\bar{D})C^{-1}\right)^{-1/2}$$
$$= \det\left(1 + A^{-1}B\bar{D}C^{-1}\right)^{-1/2}$$

where the square root is calculated by a power series using the fact that $\|A^{-1}B\bar{D}C^{-1}\| < 1$. The question is how to see this is a cocycle, and so we need to see why

$$f(g_3, g_2g_1)f(g_2,g_1) = \det\left((AC+B\bar{D})^{-1}[(AC+B\bar{D})E + (AD+B\bar{C})\bar{F}]E^{-1}\right)^{-1/2}$$
$$\cdot \det\left(A^{-1}(AC+B\bar{D})C^{-1}\right)^{-1/2}$$

and

$$f(g_3g_2, g_1)\, f(g_3,g_2) = \det\left(A^{-1}[A(CE+D\bar{F}) + B(\bar{C}\bar{F}+\bar{D}\bar{E})](CE+D\bar{F})^{-1}\right)^{-1/2}$$
$$\cdot \det\left(C^{-1}(CE+D\bar{F})E^{-1}\right)^{-1/2}$$

are equal. Formally there is no problem; that is, if the square root weren't there then one uses that the determinant is a homomorphism.

In the case of $SU(1,1)$ one can argue that

$$\log(1+\sigma) + \log(1+\tau) = \log[(1+\sigma)(1+\tau)]$$

for $|\sigma|, |\tau| < 1$. Consequently

$$\tilde{f}(g_2, g_1) = \log\left(1 + A^{-1}B\bar{D}C^{-1}\right)$$

is a cocycle with complex values. One [redacted] ought to be able to shove this into $\mathbb{Z}$. The method is as follows. We have

$$\exp \tilde{f} = \delta h \qquad h(g_1) = A^{-1}$$

so we lift h to $\tilde{h}$ and then $\tilde{f}$ is cohomologous to $\tilde{f} - \delta\tilde{h}$. This means we work with

$$\tilde{f}'(g_2, g_1) = \log(1 + A^{-1}B\bar{D}C^{-1}) + \log(A) - \log(AC + B\bar{D}) + \log(C)$$

where the second log involves making a section of $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$.

[redacted] Recall the fibration $U^{n-1} \to U^n \to S^{2n-1}$. This shows that $\pi_1 U_n = \pi_1 U_1 = \mathbb{Z}$. Since the symplectic group Sp_{2n} has U_n as maximal compact subgroup we have

$$\pi_1 Sp_{2n} = \pi_1 U_n = \mathbb{Z}$$

and hence Sp_{2n} has a central extension with kernel $\mathbb{Z}$. This central extension which = the universal covering of Sp_{2n} has to pull back to the universal covering of U_n, which is obtained by pull-back

$$\begin{array}{ccc} \tilde{U}_n & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow {\scriptstyle \exp 2\pi i} \\ U_n & \xrightarrow{\det} & S^1 \end{array}$$

i.e. an element of $\tilde{U}_n$ is given by a unitary matrix A together with a choice for $\log(\det A)$.

We see from the above that the double covering of Sp_{2n} can be viewed as consisting of operator kernels of the form

$$\frac{1}{(\det A)^{1/2}} \, e^{-\frac{1}{2} z^t \bar{A} B z + \bar{z}^t A^{-1} \lambda + \frac{1}{2} \lambda^t \bar{B} A^{-1} \lambda}$$

where a choice for $(\det A)^{1/2}$ has been made. Thus it is reasonable to expect that the universal covering of the symplectic group will be given by a symplectic transf. $\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$ together with a choice for $\log(\det A)$. Thus if we denote by $\hat{\log}$ a fixed choice for a section of $\exp : \mathbb{C} \to \mathbb{C}^*$ we get the specific cocycle

$$f'(g_2, g_1) = \log \det(1 + A^{-1}B\bar{D}C^{-1}) + \hat{\log}\det(A)$$
$$- \hat{\log}\det(AC + B\bar{D}) + \hat{\log}\det(C)$$

To prove this one would have to understand the proof that the two products in the middle of page 81 are equal, and then carry the argument over.

What makes it work maybe is:

<u>Lemma</u>: Define log to have the argument in $(-\pi, \pi)$ off the negative real axis. Thus

$$\log(1+z) = z - \frac{z^2}{2} + \cdots\cdots \qquad \text{for } |z| < 1.$$

Then if z_1, z_2 are two points with $|z_1|, |z_2| < 1$ we have

$$\log(1+z_1) + \log(1+z_2) = \log((1+z_1)(1+z_2)).$$

(Clearly $\mathrm{Re}(1+z_1) > 0$, $\underline{\mathrm{Re}(1+z_2) > 0}$ is sufficient)

NO One applies this to

$$1 + z_1 = \det(A^{-1}(AC + B\bar{D})C^{-1}) = \det(1 + A^{-1}B\bar{D}C^{-1})$$
$$1 + z_2 = \det((AC + B\bar{D})[\qquad\qquad]E^{-1})$$

as on page 81. Unfortunately this doesn't work since one could start off 1 dimension with $1 + A^{-1}B\bar{C}\bar{D}^{-1}$ and then repeat it n-times to get any number for $\det(1 + A^{-1}B\bar{D}C^{-1})$.

The good argument that

$$\tilde{f}(g_2,g_1) = \log\det(1+A^{-1}B\bar{D}C)$$

defined by the power series using that $\|A^{-1}B\bar{D}C\|^n \to 0$ is a cocycle uses analyticity. Fix A, C, E and replace B, D, F by zB, zD, zF. Then both sides of the cocycle equation are analytic for $|z| \leq 1$ and agree for small z.

<u>September 24, 1981</u>

Let's next try to get at the universal, or at least metaplectic, covering of $SL_2(\mathbb{R})$ using the real repn of $p = \frac{1}{i}\frac{d}{dx}$, $q = x$ on $L_2(\mathbb{R})$. Let's consider the operator given by a Gaussian kernel:

$$* \quad f \longmapsto \int dy\, e^{i\alpha\frac{x^2}{2} + i\beta xy + i\gamma\frac{y^2}{2}} f(y) \qquad \alpha, \beta, \gamma \in \mathbb{R}$$

This is the composite of three operators the first being $e^{i\gamma\frac{q^2}{2}}$ the last $e^{-i\alpha\frac{q^2}{2}}$. Now compute the matrices on the space of operators spanned by p and q.

$$\cancel{e^{i\gamma\frac{q^2}{2}} \begin{pmatrix} p \\ q \end{pmatrix} e^{-i\gamma\frac{q^2}{2}} = \begin{pmatrix} p-\gamma q \\ p-\gamma q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}}$$

~~since~~ $\cancel{i[\frac{q^2}{2}, p] = -iq[p,q] = -q}$. ~~It would be better to write this~~

$$\cancel{e^{i\gamma\frac{q^2}{2}} (p \;\; q) e^{-i\gamma\frac{q^2}{2}} = (p \;\; p-\gamma q) = (p, q)\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}}$$

$$e^{i\gamma\frac{q^2}{2}} \begin{pmatrix} p \\ q \end{pmatrix} e^{-i\gamma\frac{q^2}{2}} = \begin{pmatrix} p-\gamma q \\ q \end{pmatrix} = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

since infinitesimally $\quad [i\gamma\frac{q^2}{2}, p] = i\gamma q\,[q, p] = -\gamma q$

Next next

$$\int dy\, e^{ixy} \underbrace{\begin{pmatrix} p \\ \; \end{pmatrix} \int e^{-iyz} f(z)\frac{dz}{2\pi}}$$

$$\int \qquad \int -z\, e^{-iyz} f(z)\, \frac{dz}{2\pi} \quad = \quad (-qf)(x)$$

$$\int dy\, e^{ixy}\, \underset{\substack{\shortparallel \\ y}}{q} \int e^{-iyz} f(z)\frac{dz}{2\pi} \quad = \quad (pf)(x)$$

Thus to e^{ixy} belong $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$

and to $f(x) \longmapsto \sqrt{\beta} f(\beta x)$ belongs

$$\left(p\, f(\tfrac{1}{\beta}x) \right)(\beta x) = \frac{1}{\beta} p f(x)$$

so we get the matrix $\begin{pmatrix} \frac{1}{\beta} & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$

Thus the integral operator $(*)$ gives the matrix

$$\begin{pmatrix} 1 & -\gamma \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\beta} & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\beta} & -\frac{\alpha}{\beta} \\ & \beta \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\gamma \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & -\beta \\ \frac{1}{\beta} & -\frac{\alpha}{\beta} \end{pmatrix} = \begin{pmatrix} -\frac{\gamma}{\beta} & -\beta + \frac{\gamma\alpha}{\beta} \\ \frac{1}{\beta} & -\frac{\alpha}{\beta} \end{pmatrix}$$

which is a typical element of $SL_2(\mathbb{R})$ having lower left entry $\neq 0$. Recall the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}\begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}$$

<u>September 25, 1981</u>

At the moment I am trying to understand the metaplectic extension of $SL_2(\mathbb{R})$ using the repn. of the commutation relations on $L_2(\mathbb{R})$. Integral operator has the kernel

$$(*)\qquad (\text{const})\, e^{i\alpha\frac{x^2}{2} + i\beta xy + i\gamma\frac{y^2}{2}}\frac{dy}{\sqrt{2\pi}}$$

and we first compute the matrix of the operator on the space $\mathbb{R}p + \mathbb{R}q$.

$$e^{i\gamma\frac{q^2}{2}}(p\ \ q)\,e^{-i\gamma\frac{q^2}{2}} = (p-\gamma q\ \ q) = (p, q)\begin{pmatrix}1 & 0\\ -\gamma & 1\end{pmatrix}$$

because $\left[i\gamma\frac{q^2}{2}, p\right] = i\gamma q\underbrace{[q,p]}_{i} = -\gamma q$

Next consider $(Ff)(x) = \int e^{-ixy} f(y)\frac{dy}{\cancel{\sqrt{2\pi}}}$

$$(pFf)(x) = \int e^{-ixy}\, y f(y)\frac{dy}{\sqrt{2\pi}} = (Fqf)(x)$$

$$\therefore\quad F(p\ \ q)F^{-1} = (-q\ \ +p) = (p\ \ q)\begin{pmatrix}0 & +1\\ -1 & 0\end{pmatrix}$$

$$(Tf)(x) = f(\beta x)$$

$$(qTf)(x) = x(Tf(x)) = xf(\beta x) = \frac{1}{\beta}T(qf)(x)$$

$$\therefore\quad T(p\ \ q)T^{-1} = \left(\tfrac{1}{\beta}p\ \ \beta q\right) = (p\ \ q)\begin{pmatrix}\frac{1}{\beta} & 0\\ 0 & \beta\end{pmatrix}$$

and so to the integral operator belongs the matrix

$$\begin{pmatrix}1 & 0\\ -\alpha & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\beta} & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ -\gamma & 1\end{pmatrix} = \begin{pmatrix}-\frac{\beta}{\gamma} & \frac{1}{\beta}\\ \frac{\alpha\gamma}{\beta}-\beta & -\frac{\alpha}{\gamma}\end{pmatrix}$$

Next we [redacted] need the composition of [redacted] two operators in the form $(*)$.

$$(**)\qquad \int \frac{dy}{\sqrt{2\pi}}\, e^{i\alpha \frac{x^2}{2} + i\beta xy + i\gamma \frac{y^2}{2} + i\delta \frac{y^2}{2} + i\varepsilon yz + i\lambda \frac{z^2}{2}}$$

This is a Gaussian integral which can be evaluated by locating the critical point of

$$\Phi = \beta xy + (\gamma+\delta)\frac{y^2}{2} + \varepsilon yz$$

$$\beta x + (\gamma+\delta) y_c + \varepsilon z = 0 \implies y_c = \frac{-(\beta x + \varepsilon z)}{\gamma+\delta}$$

$$\Phi(y_c) = -(\gamma+\delta)\frac{y_c^2}{2} = \frac{-(\beta x+\varepsilon z)^2}{2(\gamma+\delta)}$$

and so the coefficient of xz in the composition is

$$\frac{-\beta\varepsilon}{(\gamma+\delta)}$$

Also when we do the Gaussian integral we get the determinantal factor

$$\int \frac{dy}{2\pi}\, e^{i(\gamma+\delta)\frac{y^2}{2}} = \frac{1}{\sqrt{-i(\gamma+\delta)}}$$

where the square root is computed in the right half plane. (The point is that the number $-i(\gamma+\delta) \in i\mathbb{R}$ is thought of as being approached from the RHP and if >0 we want the >0 square root)

So we see that the composition $(**)$ is the operator with kernel

$$\frac{1}{\sqrt{-i(\gamma+\delta)}}\, e^{i\left(\alpha - \frac{\beta^2}{\gamma+\delta}\right)\frac{x^2}{2} + i\left(\frac{-\beta\varepsilon}{\gamma+\delta}\right)xy + i\left(\lambda - \frac{\varepsilon^2}{\gamma+\delta}\right)\frac{y^2}{2}}$$

Now we see what the constant factor to put in $(*)$ should be in order to get the double covering. Namely, if we put $\sqrt{i\beta}$, then when we compute the composition we get the factors

$$\sqrt{i\beta}\ \frac{1}{\sqrt{-i(\gamma+\delta)}}\ \sqrt{i\varepsilon} = \sqrt{i\left(\frac{-\beta\varepsilon}{\gamma+\delta}\right)}$$

To compute the cocycle we have to select a value for $\sqrt{i\beta}$ and so let us take the RHP square root. Then the cocycle will be -1 when the three factors on the left above, each of which individually lies in the RHP, have a product outside the RHP.

Now let's rewrite this using the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\frac{\gamma}{\beta} & \frac{1}{\beta} \\ \frac{\alpha\gamma}{\beta}-\beta & -\frac{\alpha}{\beta} \end{pmatrix}$$

We are computing the composition

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g_1}\underbrace{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}_{g_2} = \begin{pmatrix} -\gamma/\beta & 1/\beta \\ \alpha\gamma/\beta-\beta & -\alpha/\beta \end{pmatrix}\begin{pmatrix} -\lambda/\varepsilon & 1/\varepsilon \\ \frac{\lambda\delta}{\varepsilon}-\varepsilon & -\delta/\varepsilon \end{pmatrix}$$

$$= \begin{pmatrix} & \frac{-(\gamma+\delta)}{\beta\varepsilon} \\ & \end{pmatrix} = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}_{g_1 g_2}$$

and so we are getting a cocycle by taking the value of

$$\sqrt{\frac{i}{b}\ \frac{i}{b'}\ \frac{b''}{i}} = \sqrt{-i(\gamma+\delta)}$$

which lies in the RHP

Let's try next to understand the central extension of $SL_2(F)$ defined by a symbol on F. The idea is that we use the Bruhat decomposition of $G = SL_2(F)$

$$G = \underset{\overset{\shortparallel}{HU}}{B} \amalg \underset{\overset{\shortparallel}{UwHU}}{BwB}$$

and then $\tilde{G} = \tilde{H}U \amalg Uw\tilde{H}U$ where $\tilde{H}$ is the central extension of H. I have to see exactly what I need to make $\tilde{G}$ into a group.

The first thing we need to know is how to [redacted] construct $\tilde{H}$. [redacted] Is it a split extension of H by the kernel in a canonical way? [redacted]

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} 1 & y \\ x & 1+xy \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} = \begin{pmatrix} 1+yz & y \\ x+z+xyz & 1+xy \end{pmatrix}$$

So if we take $y = -x^{-1}$, $z = x$ we get

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 0 & -x^{-1} \\ x & 0 \end{pmatrix}$$

However we also have

$$\begin{pmatrix} 1 & -x^{-1} \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & -x^{-1} \\ & 1 \end{pmatrix} = \begin{pmatrix} 0 & -x^{-1} \\ x & 0 \end{pmatrix},$$

so that in $\tilde{G}$, these two products might give different elements. So how could we [redacted] understand what is happening? Let's calculate in the metaplectic gp.

$$e^{-i\alpha\frac{q^2}{2}} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}$$

$$e^{i\beta\frac{p^2}{2}} \longmapsto \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}$$

Note that if F is the Fourier transform

$$(Ff)(x) = \int \frac{dy}{\sqrt{2\pi}} e^{ixy} f(y)$$

Then $F \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and we have

$$F e^{i\beta\frac{p^2}{2}} F^{-1} = e^{-i\beta\frac{q^2}{2}}$$

$$\left(e^{i\beta\frac{p^2}{2}}\, e^{+i\alpha\frac{q^2}{2}} f\right)(x) = \int \frac{dp}{\sqrt{2\pi}} e^{-ipx} e^{i\beta\frac{p^2}{2}} \int \frac{dq}{\sqrt{2\pi}} e^{-ipq} e^{i\alpha\frac{q^2}{2}} f(q)$$

$$= \int \frac{dq}{\sqrt{2\pi}} \underbrace{\left[\int \frac{dp}{\sqrt{2\pi}} e^{i\left[-px+\beta\frac{p^2}{2}+pq\right]} \right]} e^{i\alpha\frac{q^2}{2}} f(q)$$

$$\frac{1}{\sqrt{-i\beta}} e^{-i\frac{(x-q)^2}{2\beta}} e^{i\alpha\frac{q^2}{2}} f(q)$$

Take $\alpha = +\frac{1}{\beta}$ and then you get

$$\left(e^{i\frac{1}{\beta}\frac{q^2}{2}}\, e^{i\beta\frac{p^2}{2}}\, e^{i\frac{1}{\beta}\frac{q^2}{2}} f\right)(x) = \frac{1}{\sqrt{-i\beta}} \int \frac{dq}{\sqrt{2\pi}} e^{i\frac{1}{\beta}xq} f(q)$$

$$e^{i\frac{1}{\beta}\frac{q^2}{2}}\, e^{i\beta\frac{p^2}{2}}\, e^{i\frac{1}{\beta}\frac{q^2}{2}} = \frac{1}{\sqrt{-i\beta}} T_{1/\beta} F$$

Now conjugate with F to get

$$e^{i\frac{1}{\beta}\frac{p^2}{2}}\, e^{i\beta\frac{q^2}{2}}\, e^{i\frac{1}{\beta}\frac{p^2}{2}} = \frac{1}{\sqrt{-i\beta}} F T_{1/\beta}$$

Now

$$(F T_{1/\beta})(f)(x) = \int \frac{dy}{\sqrt{2\pi}} e^{ixy} f\left(\tfrac{1}{\beta}y\right)$$

$$= |\beta| \int \frac{dy}{\sqrt{2\pi}} e^{i\beta xy} f(y) = \left(|\beta| T_\beta F f\right)(x)$$

$$\therefore \quad F T_{1/\beta} = |\beta| T_\beta F$$

and so

$$e^{i\beta\frac{p^2}{2}}\, e^{i\frac{1}{\beta}\frac{q^2}{2}}\, e^{i\beta\frac{p^2}{2}} = \frac{1}{\sqrt{-i\beta^{-1}}} \overbrace{F T_\beta}^{\left|\frac{1}{\beta}\right| T_{1/\beta} F} = \frac{1}{\sqrt{-i\beta^{-1}}} \frac{1}{|\beta|} T_{1/\beta} F$$

So the question of whether the two lifts are the same is whether one has

$$\sqrt{-i\beta} = \sqrt{-i\beta^{-1}}\,|\beta|.$$

~~[illegible]~~

However the good way to do these computations is to calculate how these operators work on Gaussian functions $e^{ia\frac{x^2}{2}}$, $\operatorname{Im} a > 0$.

$$\boxed{e^{-it\frac{p^2}{2}}\, e^{ia\frac{x^2}{2}} = \frac{1}{\sqrt{1+at}}\, e^{i\left(\frac{a}{ta+1}\right)\frac{x^2}{2}}}$$

From this viewpoint we have

$$e^{it\frac{p^2}{2}} \longmapsto \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$$

$$e^{i\alpha\frac{q^2}{2}} \longmapsto \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$$

Calculate

$$e^{i\beta^{-1}\frac{q^2}{2}}\, e^{i\beta\frac{p^2}{2}}\, \underbrace{e^{i\beta^{-1}\frac{q^2}{2}}\, e^{ia\frac{x^2}{2}}}_{e^{i(a+\beta^{-1})\frac{x^2}{2}}}$$

$$\underbrace{\qquad\qquad}_{\frac{1}{\sqrt{1-\beta(a+\beta^{-1})}}\, e^{i\,\underbrace{\frac{a+\beta^{-1}}{-\beta(a+\beta^{-1})+1}}_{-\frac{1}{\beta}-\frac{1}{\beta^2 a}}\,\frac{x^2}{2}}} \qquad = \frac{1}{\sqrt{-\beta a}}\, e^{i\left(-\frac{1}{\beta^2 a}\right)\frac{x^2}{2}}$$

square root in RHP

$$e^{i\beta\frac{p^2}{2}}\, e^{i\frac{1}{\beta}\frac{q^2}{2}}\, \underbrace{e^{i\beta\frac{p^2}{2}}\, e^{ia\frac{x^2}{2}}}_{\frac{1}{\sqrt{1-\beta a}}\, e^{i\frac{a}{1-\beta a}\frac{x^2}{2}}} \qquad \frac{a}{1-\beta a} + \frac{1}{\beta} = \frac{1}{\beta(1-\beta a)}$$

$$\underbrace{\qquad\qquad}_{\frac{1}{\sqrt{1-\beta\frac{1}{\beta(1-\beta a)}}}\, \frac{1}{\sqrt{1-\beta a}}\, e^{i\,\underbrace{\frac{1}{-\beta+\beta(1-\beta a)}}_{\frac{1}{-\beta^2 a}}\,\frac{x^2}{2}}}$$

So the question is whether

$$\frac{1}{\sqrt{\frac{-\beta a}{1-\beta a}}}\;\frac{1}{\sqrt{1-\beta a}} = \frac{1}{\sqrt{-\beta a}}$$

when the square roots are taken in the RHP. This should be independent of the value of a in the UHP, so let a approach $-\frac{1}{\beta}$ and then you get

$$\frac{1}{\sqrt{\frac{1}{1+1}}}\cdot\frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{1}}$$

and so it works.

I found out that the products

$$e^{i\frac{1}{\beta}\frac{q^2}{2}}\, e^{i\beta\frac{p^2}{2}}\, e^{i\frac{1}{\beta}\frac{q^2}{2}}, \quad e^{i\beta\frac{p^2}{2}}\, e^{i\frac{1}{\beta}\frac{q^2}{2}}\, e^{i\beta\frac{p^2}{2}}$$

in the metaplectic gp. are equal and ~~are~~ gives the operator

$$\frac{1}{\sqrt{-i\beta^{-1}}}\, F T_\beta = \frac{1}{\sqrt{-i\beta}}\, T_{1/\beta} F \qquad (T_\beta f)(x) = f(\beta x)$$

Call this operator $\varphi(\beta)$. It's defined for $\beta \in \mathbb{R}^\circ$ and belongs to the matrix:

$$\begin{pmatrix} 1 & 0 \\ -\beta^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\beta^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \beta^{-1} & \\ & \beta \end{pmatrix}$$

Thus we we can lift the matrix $\begin{pmatrix} \beta^{-1} & \\ & \beta \end{pmatrix}$ into the metaplectic group by the formula

$$h(\beta) \overset{\text{defn.}}{=} \varphi(1)^{-1}\varphi(\beta) = \sqrt{-i}\,\frac{1}{\sqrt{-i\beta^{-1}}}\, T_\beta$$

$$= |\beta|^{1/2} T_\beta \begin{cases} 1 & \text{if } \beta > 0 \\ -i & \text{if } \beta < 0 \end{cases}$$

The interesting thing is that this is not a homomorphism and so it defines a cocycle on the diagonal subgroup. Clearly we have

$$h(\beta_1\beta_2) = h(\beta_1)h(\beta_2) \qquad \text{if at least one of } \beta_1, \beta_2 > 0$$

$$= -h(\beta_1)h(\beta_2) \qquad \text{if both } \beta_1, \beta_2 < 0.$$

and so the cocycle is the standard symbol $\mathbb{R}^\circ \times \mathbb{R}^\circ \to \mathbb{Z}/2$.

September 27, 1981

Simple calculations and questions:

F is a field. According to the Bruhat decomposition $SL_2(F)$ is generated by matrices $\begin{pmatrix}1&a\\&1\end{pmatrix}\begin{pmatrix}1&\\a&1\end{pmatrix}$ for $a\in F$ and $\begin{pmatrix}a^{-1}&\\&a\end{pmatrix}$ for $a\in F^\cdot$. Also one has

$$\varphi(a)=\begin{pmatrix}1&a\\&1\end{pmatrix}\begin{pmatrix}1&\\-a^{-1}&1\end{pmatrix}\begin{pmatrix}1&a\\&1\end{pmatrix}=\begin{pmatrix}0&a\\-a^{-1}&0\end{pmatrix}$$

$$\varphi(a)\varphi(1)^{-1}=\begin{pmatrix}a&\\&a^{-1}\end{pmatrix}$$

so that one sees in general $SL_2(F)$ is generated by the elementary matrices $\begin{pmatrix}1&*\\&1\end{pmatrix}$ and $\begin{pmatrix}1&\\ *&1\end{pmatrix}$. On the other hand

$$\left(\begin{pmatrix}a&\\&a^{-1}\end{pmatrix},\begin{pmatrix}1&b\\&1\end{pmatrix}\right)=\begin{pmatrix}1&(a^2-1)b\\&1\end{pmatrix}$$

so that if $\exists\ a\in F^\cdot$ with $a^2\neq 1$, then [redacted] all elem. matrices are commutators, and so $SL_2(F)$ is perfect. $\therefore$

$SL_2(F)$ is perfect except for $F=\mathbb{F}_2,\mathbb{F}_3$.

For $\mathbb{F}_2$ $SL_2=GL_2$ has $(4-1)(4-2)=6$ elements and the projective line has 3 elements so $SL_2(\mathbb{F}_2)=\Sigma_3$

For $\mathbb{F}_3$, GL_3 has $(9-1)(9-3)=48$ elements, so PGL_3 has 24 elements and acts triply transitively on $\mathbb{P}(\mathbb{F}_3)$ which has 4 elements. Thus $PGL_2=\Sigma_4$ and GL_2 is a double covering of Σ_4. Now I know $H^*(GL_2(\mathbb{F}_3),\mathbb{Z}/2)=\mathbb{Z}/2[\underset{2}{c_1},\underset{4}{c_2},\underset{1}{e_1},\underset{3}{e_2}]$ and $H^*(SL_2(\mathbb{F}_3),\mathbb{Z}/2)=\mathbb{Z}/2[\underset{4}{c_2},\underset{3}{e_2}]$, so that $SL_2(\mathbb{F}_3)$ has no [redacted] central extensions by 2-gps. $SL_2(\mathbb{F}_3)=(\mathbb{Z}/3)\ltimes Q_8$

September 28, 1981

We want generators and relations for $SL_2(F)$. Put

$$x(a) = \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \qquad y(a) = \begin{pmatrix} 1 & \\ a & 1 \end{pmatrix} \qquad a \in F^{\circ}$$

These generate. Here are some formulas

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}}_{\begin{pmatrix} 1 & -c^{-1} \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & -c^{-1} \\ & 1 \end{pmatrix}} \begin{pmatrix} 1 & c^{-1}d \\ & 1 \end{pmatrix}$$

for $c \neq 0$

$$= \begin{pmatrix} 1 & (a-1)c^{-1} \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & c^{-1}(d-1) \\ & 1 \end{pmatrix}$$

Thus we can parameterize the fat cell BwB of the Bruhat decomposition as

$$x(\alpha)\, y(\beta)\, x(\gamma) \qquad \alpha, \gamma \in F,\ \beta \in F^{\circ}$$

Next we want to be able to compute products in this parameterization. Thus we need to write $y(\lambda)x(\mu)$ in this form.

$$\begin{pmatrix} 1 & \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} 1 & \mu \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ \lambda & \lambda\mu+1 \end{pmatrix}$$

No. We need to compute $y(\lambda)\,x(\mu)\,y(\nu)$ in the x, y, x form.

$$\begin{pmatrix} 1 & \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} 1 & \mu \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ \nu & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mu \\ \lambda & \lambda\mu+1 \end{pmatrix}\begin{pmatrix} 1 & \\ \nu & 1 \end{pmatrix} = \begin{pmatrix} 1+\mu\nu & \mu \\ \lambda+\lambda\mu\nu+\nu & \lambda\mu+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \mu\nu(\lambda+\lambda\mu\nu+\nu)^{-1} \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda+\lambda\mu\nu+\nu & 1 \end{pmatrix}\begin{pmatrix} 1 & (\lambda+\lambda\mu\nu+\nu)^{-1}\lambda\mu \\ & 1 \end{pmatrix}$$

September 30, 1981

F = field, $X = \mathbb{P}_1(F)$, $G = PGL_2(F)$. Then G acts on the semi-simplicial set whose p-simplices are sequences $x_0, \ldots, x_p$ in X.

$$\cdots\ X\times X\times X \ \substack{\rightarrow\\\rightarrow\\\rightarrow}\ X\times X \rightrightarrows X$$

The chains on this s-set form a resolution of $\mathbb{Z}$

$$\substack{\rightarrow\\\rightarrow\\\rightarrow}\ \underset{C_1}{\mathbb{Z}[X\times X]} \rightrightarrows \underset{C_0}{\mathbb{Z}[X]} \longrightarrow \mathbb{Z} \longrightarrow 0$$

and according to the normalization theorem we can replace this s. abelian gp by its normalized chain complex. This is the quotient by the image of the generacies.

$$N_q(C.) = C_q \Big/ \sum_{i=0}^{q-1} s_i C_{q-1}$$

$$\cong \mathbb{Z}\big[(\underbrace{X\times\cdots\times X}_{q+1})_{reg}\big]$$

where $(X\times\cdots\times X)_{reg}$ means one has distinct sequences $x_0, \ldots, x_q$. Thus for example

$$N_2(C.) = \mathbb{Z}[X\times X]\big/ s_0\mathbb{Z}[X] = \mathbb{Z}[X\times X]\big/\mathbb{Z}[\Delta X]$$

$$= \mathbb{Z}[(X\times X)_{reg}].$$

Notice that this normalized complex is not the same thing as the chains on the simplex with vertices X, for which in dimension 1 one has a basis (x_0, x_1) with $x_0 \neq x_1$ and the relations $(x_0, x_1) = -(x_1, x_0)$.

So thus we have an acyclic complex

$$\longrightarrow \mathbb{Z}[X^2_{reg}] \longrightarrow \mathbb{Z}[X] \longrightarrow \mathbb{Z} \longrightarrow 0$$

which gives rise to a spec. sequence in homology

$$E^1_{pq} = H_q(G, \mathbb{Z}[X^{p+1}_{reg}]) \Longrightarrow H_\cdot(G)$$

Now for $G = PGL_2(F)$, one knows G acts simply-transitively on X^3_{reg}. Thus the spectral sequence becomes

$$\begin{array}{cccccccccccc}
 & & & & & & H_2(T) & \xrightarrow{i-iw} & H_2(B) & \longrightarrow & H_2(G) \\
 & & & & & & H_1(T) & \xrightarrow{i-iw} & H_1(B) & \longrightarrow & H_1(G) \\
\mathbb{Z}[(F^\bullet-1)^2_{reg}] & & \mathbb{Z}[F^\bullet-1] & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\sim} & H_0(T) & \xrightarrow{0} & H_0(B) & \xrightarrow{\sim} & H_0(G)
\end{array}$$

Here $T =$ stabilizer of $(\infty, 0) \simeq F^\bullet$
$B =$ " " $(\infty) \cong F^\bullet \times F$

and we have the inclusion $i : T \to B$ and the embedding $T \xrightarrow{w=-1} T \xrightarrow{i} B$.

Except for $\mathbb{F}_2$ one has $H_1(B) = H_1(T) = F^\bullet$ because

$$\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & (a-1)x \\ & 1 \end{pmatrix} \quad \text{and } \exists\ a \in F^\bullet - 1.$$

Hence one gets

$$H_1(G) = F^\bullet/(F^\bullet)^2$$

and also one has

$$\mathrm{Ker}\left\{ H_1(T) \xrightarrow{i-wi} H_1(B) \right\} = \{\pm 1\}$$

I know for F of characteristic 0 at least that $H_2(B) = H_2(T)$. Also in general for any abelian group T one has

$$H_2(T) = \Lambda^2 T$$

Thus in general $i - iw : H_2(T) \to H_2(B)$ is zero. So E_2 is as follows in general. $(F \neq \mathbb{F}_2)$ char $\neq 2$.

$$
\begin{array}{cccc|c}
0 & 0 & H_2(T) & H_2(B) & \dashrightarrow H_2(G) \\
0 & 0 & \{\pm 1\} & F^\circ/F^{\circ 2} & \xrightarrow{\sim} H_1(G) \\
\mathbb{Z}[F^\circ - 1]/\mathrm{Ind}_1 & 0 & 0 & \mathbb{Z} & \xrightarrow{\sim} H_0(G)
\end{array}
$$

and thus we get an exact sequence

$$\mathbb{Z}[(F^\circ-1)^2_{reg}] \xrightarrow{d_1} \mathbb{Z}[(F^\circ-1)] \xrightarrow{d_3} H_2(B) \longrightarrow H_2(G) \to \{\pm 1\} \to 0$$

[redacted] Unfortunately what we want is [redacted] information about the group $PSL_2(F)$ which fits into sequences

$$\cdot \longrightarrow PSL_2 \longrightarrow \underbrace{PGL_2}_{G} \longrightarrow F^\circ/F^{\circ 2} \longrightarrow \cdot$$

$$\cdot \longrightarrow \{\pm 1\} \longrightarrow SL_2 \longrightarrow PSL_2 \longrightarrow \cdot$$

From first sequence we get

$$H_3(F^\circ/F^{\circ 2}) \to H_2(PSL_2)_{F^\circ/F^{\circ 2}} \xrightarrow{[redacted]} H_2(G) \longrightarrow H_2(F^\circ/F^{\circ 2}) \longrightarrow 0$$
$$H_2(F^\circ/F^{\circ 2}) = \Lambda^2(F^\circ/F^{\circ 2})$$

an do the second in general

$$\cdot \longrightarrow A \longrightarrow E \longrightarrow Q \longrightarrow \cdot \qquad \begin{array}{l} H_1(Q)=0 \\ H_1(E)=0 \end{array}$$

$$E^2 = H_*(Q, H_*(A)) \Rightarrow H_*(E).$$

$$H_2(E) \longrightarrow H_2(Q) \xrightarrow{d_2} H_0(Q, H_1A) \longrightarrow H_1(E) \longrightarrow H_1(Q) \to 0$$
$$H_0(Q, H_1 A) = A, \quad H_1(E) = 0, \quad H_1(Q) = 0$$

$$
\begin{array}{|ccc}
(\Lambda^2 A)^Q & 0 & H_2(Q, \Lambda^2 A) \\
[redacted]A & 0 & H_2(Q,A) = H_2(Q) \otimes A \\
H_0(Q) & H_1(Q) & H_2(Q)
\end{array}
$$

should be onto

seems to give

$$0 \longrightarrow H_2(E) \longrightarrow H_2(Q) \longrightarrow H_2(A) \longrightarrow 0 .$$

as it should.

So now it's clear that knowing $H_2(G) = H_2(PGL_2)$ leaves a lot to be desired. However let's make some educated guesses. We have

$$H_2(B) = H_2(T) = \Lambda^2(F^\circ)$$

Interesting point: We established about a map $\quad H_2(G) \longrightarrow \{\pm 1\}$, (but this should be checked) By universal coefficients there has to be a central extension of $G = PGL_2$ by ± 1.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1(H_1(G), \mathbb{Z}/2) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^2(G, \mathbb{Z}/2) \arrow[r] & \mathrm{Hom}(H_2(G), \mathbb{Z}_2) \arrow[r] & 0 \\ & \mathrm{Hom}(F^\circ/F^{\circ 2}, \mathbb{Z}/2) & & & \end{tikzcd}$$

But we have

$$\circ \longrightarrow F^\circ \longrightarrow GL_2 \longrightarrow PGL_2 \longrightarrow \circ$$

$$\circ \longrightarrow F^\circ/F^{\circ 2} \longrightarrow GL_2/F^{\circ 2} \longrightarrow PGL_2 \longrightarrow \circ$$

so we can see the subgroup $\mathrm{Hom}(F^\circ/F^{\circ 2}, \mathbb{Z}/2)$ sitting inside $H^2(G, \mathbb{Z}/2)$. Actually we would like to have

$$\begin{tikzcd} H_2(SL_2) \arrow[d] & & & \\ H_2(PSL_2) \arrow[r] \arrow[d] & H_2(G) \arrow[r] & \Lambda^2(F^\circ/F^{\circ 2}) \arrow[r] & \circ \\ \mathbb{Z}/2 & & & \end{tikzcd}$$

And

$$H_2(B) = \Lambda^2(F^\circ) \quad \text{mapping to } \Lambda^2(F^\circ/F^{\circ 2})$$

in the obvious way with kernel $2\Lambda^2 F^\circ$

Here's a good way to see the homomorphism $H_2(G) \to \{\pm 1\}$. Pass to a larger field $\bar{F}$ with $\bar{F}^{\cdot}/\bar{F}^{\cdot 2} = 0$. Then $PSL_2 = PGL_2$. So we have

$$\begin{array}{ccccccc}
0 \to & H_2(SL_2(F)) & \to & H_2(PSL_2(F)) & \to & H_2(PGL_2(F)) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \to & H_2(SL_2(\bar{F})) & \to & H_2(PSL_2(\bar{F})) & = & H_2(PGL_2(\bar{F})) \\
& & & \downarrow & & \\
& & & \{\pm 1\} & &
\end{array}$$

Thus we have a canonical exact sequence

$$H_2(SL_2(F)) \longrightarrow H_2(PGL_2(F)) \longrightarrow \{\pm 1\} \times \Lambda^2(F^{\cdot}/F^{\cdot 2}) \to 0$$

Next if $F^{\cdot} = F^{\cdot 2}$ we can let $SL_2(F)$ act on our complex and we get a spectral sequence

E_2

$$\begin{array}{cccc}
 & \overset{0}{\cancel{H_2(\pm 1)}} & H_2(T') & \xrightarrow{0} H_2(B') \\
\mathbb{Z}/2[F^{\cdot}-1] & H_1(\pm 1) \longrightarrow & H_1(T') & \xrightarrow{2} H_1(B') \\
\mathbb{Z}[F^{\cdot}-1] \xrightarrow{0} & \mathbb{Z} \xrightarrow{1} & \mathbb{Z} & \xrightarrow{0} \mathbb{Z}
\end{array}$$

which gives an exact sequence

$$\mathbb{Z}[F^{\cdot}-1] \to H_2(B') \longrightarrow H_2(G') \longrightarrow 0$$

where B' is the Borel in $SL_2 = G'$. Map to same for G:

$$\mathbb{Z}[F^{\cdot}-1] \longrightarrow H_2(B) \longrightarrow H_2(G) \longrightarrow \{\pm 1\} \to 0$$

OKAY because $H_2(B') = \Lambda^2 F^{\cdot}$, $H_2(B) = \Lambda F^{\cdot}$ and $F^{\cdot} \to F^{\cdot}$ is squaring.

In general I looked at the situation

$$\mathbb{Z}[F^\cdot - 1] \longrightarrow H_2(B) \longrightarrow H_2(G) \longrightarrow \{\pm 1\}$$ [redacted] $$\to 0$$

$$\begin{array}{c} H_2(B) \\ \shortparallel \\ \Lambda^2 F^\cdot \end{array}$$

and $\quad H_2(SL_2(F)) \longrightarrow H_2(G) \longrightarrow \{\pm 1\} \times \Lambda^2(F^\cdot / F^{\cdot 2}) \to 0$

and I conclude that we [redacted] can't expect $H_2(B')$ to map surjectively on $H_2(SL_2)$ because the map $H_2(B') \longrightarrow H_2(B)$ is $\Lambda^2 F^\cdot \xrightarrow{4} \Lambda^2 F^\cdot$.

Example: $SL_2(\mathbb{R})$. Work topologically. Then we know that $H_2(BSL_2(\mathbb{R})) = \mathbb{Z}$ and $H_2(BPGL_2(\mathbb{R})) = H_2(B(O_2/\pm 1))$ $=$ [redacted] $\mathbb{Z}/2\mathbb{Z}$. Thus $\mathbb{R}^\cdot/\mathbb{R}^{\cdot 2}$ [redacted] acts non-trivially on $H_2(PSL_2(\mathbb{R}))$. Also $H_2(B) = H_2(\pm 1) =$ [redacted] 0 can't generate $H_2(SL_2)$.

Tomorrow use new approach. Let's start with the fact that the symbol measures the failure of the lifting of T to be a homomorphism. Formulas

$$w_\alpha(t) = x_\alpha(t)\, x_{-\alpha}(-t^{-1})\, x_\alpha(t)$$

$$h_\alpha(t) = w_\alpha(t)\, w_\alpha(1)^{-1}$$

Symbol $\quad \{u, v\} = h_\alpha(uv)\, h_\alpha(u)^{-1} h_\alpha(v)^{-1}$

Now use this to understand the tame symbol.

October 1, 1981

Let's take the case of $SL_2(F)$ where F is a Laurent series field, say $F = \mathbb{C}[[z]][z^{-1}]$. From Steinberg, Matsumoto, etc. one has a central extension associated to any symbol, in particular the tame symbol:

$$(*) \qquad \{u,v\} = (-1)^{\mathrm{ord}(u)\,\mathrm{ord}(v)} \frac{u^{\mathrm{ord}\, v}}{v^{\mathrm{ord}\, u}} \quad \text{evaluated at } z=0.$$

How to understand this formula. One has the central extension

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \widetilde{SL_2(F)} \longrightarrow SL_2(F) \longrightarrow 0$$

and one can restrict to the diagonal subgp $H = \left\{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\right\}$ to get a central extension

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 0$$

Because H is abelian one has for any abel. gp A

$$0 \to \underset{\text{abel ext.}}{\mathrm{Ext}^1(H,A)} \longrightarrow H^2(H,A) \xrightarrow{\text{comm. pairing}} \mathrm{Hom}(\Lambda^2 H, A) \longrightarrow 0$$

The first vanishes for $A = \mathbb{C}^*$ since it's injective.

But the symbol is not the commutator pairing. One has a lifting of H into $\tilde{H}$ given by

$$h_{12}(u) = w_{12}(u)\, w_{12}(1)^{-1}$$

where

$$w_{12}(u) = x_{12}(u)\, x_{21}(-u^{-1})\, x_{12}(u) \longmapsto \begin{pmatrix} & u \\ -u^{-1} & \end{pmatrix}$$

and the symbol is the cocycle defined by this lifting

$$\{u,v\} = h_{12}(uv)\, h_{12}(u)^{-1} h_{12}(v)^{-1}$$

Where $(*)$ comes from. It is bilinear skew-symmetric (not alternating) on the group $F^*/1+\mathfrak{m} \cong \mathbb{Z} \times \mathbb{C}^* = \{z^n \lambda\}$

and one has

$$\{z,z\} = -1 \qquad \{\lambda, z\} = \lambda \qquad \{\lambda,\mu\} = 1$$

Thus

$$\{z^m\mu, z^n\nu\} = \{z,z\}^{mn}\{z,\nu\}^m\{\mu,z\}^n\{\mu,\nu\}$$

$$= (-1)^{mn}\frac{\mu^n}{\nu^m} = (-1)^{mn}\frac{(z^m\mu)^n}{(z^n\nu)^m}(0)$$

How does this central extension $\tilde{H}$ of H compare with the canonical Heisenberg extension? Answer: The iso. class of $\tilde{H}$ is determined by the commutator pairing which is obtained from a cocycle by anti-symmetrization. Thus if $h : H \to \tilde{H}$ is a section

$$h(uv) = \{u,v\}\, h(u)\, h(v)$$
$$\overset{\shortparallel}{h}(vu) = \{v,u\}\, h(v)\, h(u)$$

so

$$[h(u), h(v)] = \{v,u\}\{u,v\}^{-1}$$

$$= \{v,u\}^2 \qquad \text{for the tame symbol.}$$

$$= \left(\frac{v^{\operatorname{ord} u}}{u^{\operatorname{ord} v}}\right)^2$$

Now the Heisenberg group is the extension defined by the bilinear [redacted] cocycle $(\mathbb{Z}\times\mathbb{C}^\cdot)^2 \longrightarrow \mathbb{C}^\cdot$

$$(z^m\lambda,\ z^n\mu) \longmapsto \lambda^n$$

[redacted]

Check this out. The Heisenberg group acts on $\mathbb{C}[z,z^{-1}]$ with $(z^m\lambda)\, f(z) = z^m f(\lambda z)$. Then

$$(z^m\lambda)[(z^n\mu)f] = (z^m\lambda)(z^n f(\mu z)) = z^m(\lambda z)^n f(\lambda\mu z)$$
$$[z^{m+n}\lambda\mu f] = z^{m+n} f(\lambda\mu z)$$

so indeed the cocycle is λ^n. Then the commutator pairing is

$$(\underbrace{z^m\lambda}_{u}, \underbrace{z^n\mu}_{v}) \longmapsto \frac{\mu^m}{\lambda^n} = \frac{v^{\operatorname{ord} u}}{u^{\operatorname{ord} v}}$$

So therefore we conclude that we don't get the Heisenberg extension. First notice that the group of extensions (topological) should be

$$\operatorname{Hom}\big(\underbrace{\Lambda^2(\mathbb{Z}\times \mathbb{C}^\cdot)}_{\underset{\overset{\parallel}{0}}{\Lambda^2\mathbb{Z}} \,\oplus\, \mathbb{Z}\otimes\mathbb{C}^\cdot \,\oplus\, \Lambda^2\mathbb{C}^\cdot}, \mathbb{C}^\cdot\big)$$

$\Lambda^2\mathbb{C}^\cdot$ should be zero because the μ_n are dense and $\Lambda^2\mu_n = 0$. $\mathbb{C}^\cdot \sim S^1$

$$= \operatorname{Hom}(\mathbb{C}^\cdot, \mathbb{C}^\cdot) = \mathbb{Z}.$$

Thus the extensions are infinite-cyclic generated by the Heisenberg extension.

Perhaps one can make more sense of this by looking at SL_n. The group H is effectively $\pi_1(T)\times T = \operatorname{Hom}(S^1, T)\times T$. The Heisenberg group would be formed from $\operatorname{Hom}(T, S^1)\times T$. Hence what sort of duality exists between

$$\operatorname{Hom}(S^1, T) \quad \text{and} \quad \operatorname{Hom}(T, S^1)$$

in the case of SL_n? They are canonically dual abelian gp, because one has composition

$$\operatorname{Hom}(S^1, T) \times \operatorname{Hom}(T, S^1) \longrightarrow \operatorname{Hom}(S^1, S^1) = \mathbb{Z}$$

But I need an isomorphism between these, or maybe a natural map

$$\operatorname{Hom}(S^1, T) \longrightarrow \operatorname{Hom}(T, S^1)$$

so that I can pull-back Heisenberg.

Idea: $Hom(S^1, T)$ is generated by the [redacted] root vectors H_α. Thus for SU_n one has

$$T = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \qquad \prod a_i = 1$$

is generated by the 1-parameter subgroups

$$a \longmapsto \begin{pmatrix} a & \\ & a^{-1} \\ & & \ddots \end{pmatrix} \qquad i,j \text{ th position}.$$

[redacted] Now associate to H_α the root α:

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \longmapsto a_i a_j^{-1}$$

Put another way $Hom(S^1, T)$ is generated by the root vectors H_α and $Hom(T, S^1)$ is the lattice of weights which includes as a subgroup the lattice [redacted] generated by roots. Thus find the embedding

$$[redacted] \quad \mathrm{Ker}\{\mathbb{Z}^n \xrightarrow{\Sigma} \mathbb{Z}\} \longrightarrow \mathrm{Coker}\{\mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}^n\}$$

E.g. for $n=2$ have

$$\{(m, -m)\} \qquad \mathbb{Z} \qquad (m_1, m_2) \mapsto m_1 - m_2$$

so you have index 2. In general

$$\mathrm{Coker}\{\mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}^n\} \cong \mathbb{Z}^{n-1}$$

$$m_1, \ldots, m_n \longmapsto (m_1 - m_2, m_2 - m_3, \ldots, m_{n-1} - m_n)$$

$\mathrm{Ker}\{\mathbb{Z}^n \to \mathbb{Z}\}$ has basis

$$\begin{matrix} 1, -1 & & \\ & 1, -1 & \\ & & \ddots \\ & & & 1, -1 \end{matrix}$$

One should be able to compute index as the abs. value of

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & -1 & & \\ & 1 & -1 & \\ & & & \\ & & 1 & -1 \end{vmatrix} = \begin{vmatrix} n & 0 & \cdots & 0 \\ 1 & -1 & & \\ & & \ddots & \\ & & 1 & -1 \end{vmatrix} = \pm n$$

which also agrees with n = order of center of SU_n.

So, it seems clear now that the central extension of the diagonal part of $SL_n(F)$ defined by the tame symbol should be obtained from the Heisenberg extension of $\mathrm{Hom}(T, \mathbb{C}^\cdot) \times T$ by pulling back [redacted] via the canonical map $\mathrm{Hom}(\mathbb{G}_m, T) \longrightarrow \mathrm{Hom}(T, \mathbb{G}_m)$.

Let's go back to the loop algebra.

$$\mathfrak{g}_0[z, z^{-1}] = \underbrace{z^{-1}\mathfrak{g}_0[z^{-1}] + (Y)}_{\mathfrak{n}^*} + (H) + \underbrace{(X) + z\mathfrak{g}_0[z]}_{\mathfrak{n}}$$

Then we have the Kac-Moody covering algebra

$$e_1 \longmapsto X \qquad f_1 \longmapsto Y \qquad h_1 = [e_1, f_1] \longmapsto H$$
$$e_2 \longmapsto zY \qquad f_2 \longmapsto z^{-1}X \qquad h_2 = [e_2, f_2] \longmapsto -H$$

$$[h_i, e_j] = \alpha_{ij} e_j \qquad (\alpha_{ij}) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$
$$[h_i, f_j] = -\alpha_{ij} f_j$$

Then $h_1 + h_2 \mapsto 0$ generates the center. Next we take a representation with a highest weight vector v_λ:

$$e_i v_\lambda = 0$$
$$h_i v_\lambda = \lambda_i v_\lambda$$

and we want the f_i to be nilpotent, so that λ_1, λ_2 must be integers ≥ 0.

What I want ~~to~~ do now is to understand exactly what Kac says about this $\mathfrak{g}[z, z^{-1}]$ module. Supposedly he has an exact description of its Poincare series, and maybe I can see what sort of repn. it is over the central extension of the diagonal part of $SL_2(F)$.

October 2, 1981

Let $\bar{\mathfrak{g}}$ be the KM Lie algebra which covers $sl_2[z, z^{-1}]$. Then $\bar{h} = (h_1) + (h_2)$ and the center is $(h_1 + h_2)$. We want to understand the Weyl group and how it acts on the weights of a highest weight module for $\bar{\mathfrak{g}}$.

A key idea is to build in the degree, that is to work with the semi-direct product $\bar{\mathfrak{g}}^e = (D) \oplus \bar{\mathfrak{g}}$ where D acts as the derivation of $\bar{\mathfrak{g}}$ giving the z-degree. Thus

$$[D, \begin{matrix} e_1 \\ e_2 \end{matrix}] = \begin{matrix} 0 \\ e_2 \end{matrix} \qquad \text{since} \quad e_2 \mapsto zY \quad \text{has degree } 1$$

$$[D, \begin{matrix} f_1 \\ f_2 \end{matrix}] = \begin{matrix} 0 \\ -f_2 \end{matrix}$$

Then one works with $\bar{h}^e = (D) + (h_1) + (h_2)$ and the roots are now

$$\alpha_1 = (0, 2, -2)$$
$$\alpha_2 = (1, -2, 2)$$

relative to the basis D, h_1, h_2 of $\underline{h}^e$.

Let's now take a linear function $\Lambda : \underline{h}^e \to \mathbb{C}$ and consider the induced module

$$\tilde{V}(\Lambda) = U(\mathfrak{g}^e) \underset{U(h^e + n)}{\otimes} (v_\Lambda)$$

where v_Λ is killed by n = Lie subalg gen. by e_1, e_2 and

$$h v_\Lambda = \Lambda(h) v_\Lambda \qquad h \in \underline{h}^e.$$

This module has a smallest quotient $V(\Lambda)$ which is irreducible.

The good case is when a module with highest weight vector v_Λ satisfies $f_1^n v_\Lambda = f_2^n v_\Lambda = 0$ for n large, in which case Kac calls it quasi-simple. Necessarily $\Lambda(h_i)$ are integers ≥ 0; (look at the sl_2 subalg (f_i, h_i, e_i) acting on v_Λ. $V(\Lambda)$ has this property when $\Lambda(h_i) \in \mathbb{Z}_{\geq 0}$.

because otherwise one would get other highest weight vectors. It turns out that quasi-simple $\Rightarrow$ one gets $V(\Lambda)$.

For a quasi-simple module M_Λ with highest weight vector v_Λ one shows the generators e_i, f_i are locally nilpotent, and hence one can make sense of Weyl gp. elements

$$w_i = \exp(e_i)\exp(-f_i)\exp(e_i).$$

(Proof of local nilpotence is based on the formula

$$y^n X = (L_y)^n X = \text{[redacted]} \ (R_y + \mathrm{ad}\, Y)^n X$$

$$= \sum_{p=0}^{n} \binom{n}{p} \left[(\mathrm{ad}\, Y)^p X\right] y^{n-p}.$$

Hence if v is killed by $(f$ [redacted] $_i)^m$, [redacted] then $f_j v$ will be killed by a higher power

$$(f_i)^n (f_j v) = \sum_{p=0}^{n} \binom{n}{p} \big[\underbrace{(\mathrm{ad}\, f_i)^p f_j}_{0 \text{ for } p \geq -a_{ij}+1 \text{ defining relation}}\big] f_i^{n-p} v.)$$

Therefore the weights of a quasi-simple M_Λ are stable under the Weyl group transformations on $(\underline{h}^e)^\vee$ where

$$w_i(h) = h - \alpha_i(h) h_i \qquad \text{on } \underline{h}^e$$

Let's calculate in our example. Work in $(\underline{h}^e)^\vee$ so that

$$\alpha_1 = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \qquad \begin{matrix} h_1 = \text{[redacted]} \ (0\ 1\ 0) \\ h_2 = \phantom{\text{[redacted]}} \ (0\ 0\ 1) \end{matrix}$$

and then

$$w_i = \text{[redacted]} \ \mathrm{id} - \alpha_i \otimes h_i$$

$$w_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ +2 \\ -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

In the modules the center $h_1 + h_2$ acts as a scalar and $h_1 + h_2$ is killed by α_1, α_2 so the value of all the $\lambda \in (\underline{h}^e)^\vee$ in a Weyl gp orbit on $h_1 + h_2$ is constant. Suppose then we consider the [redacted] plane of λ with $\lambda(h_1 + h_2) = \varepsilon$. Write

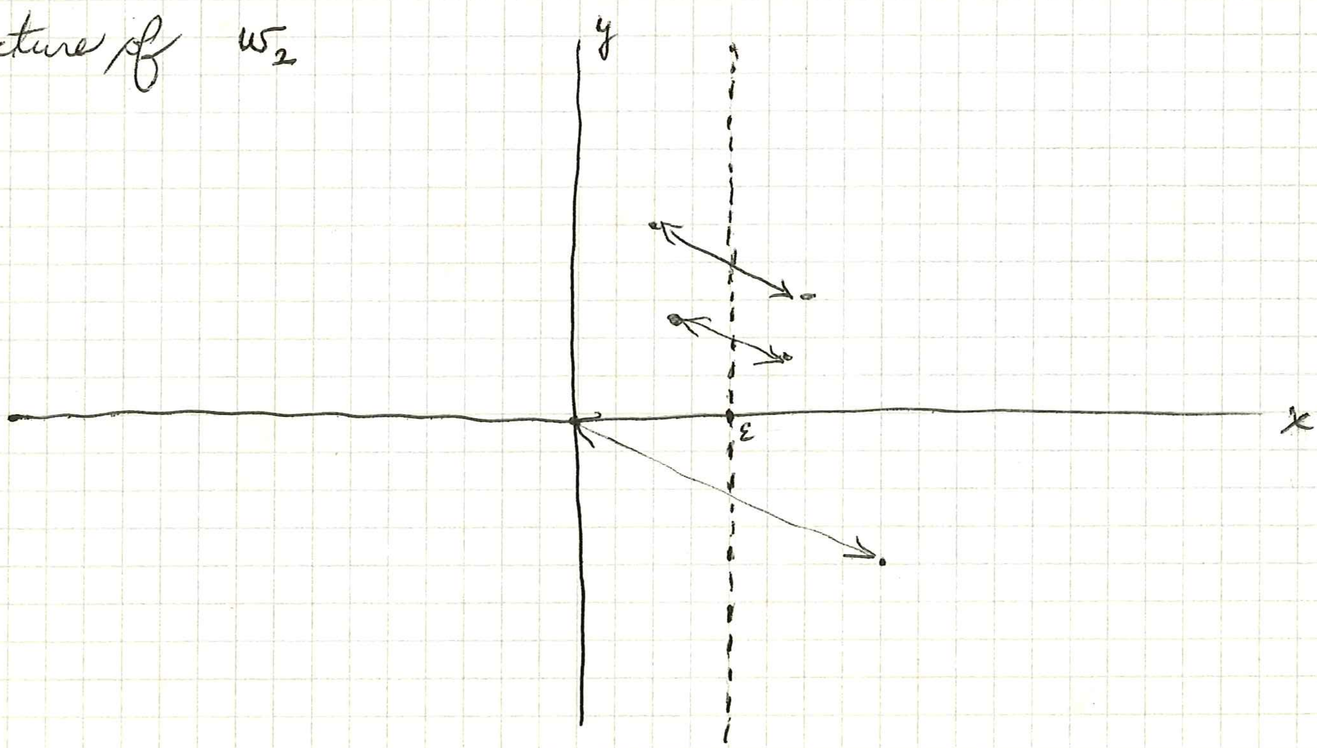
$$\lambda = \begin{pmatrix} y \\ x \\ -x+\varepsilon \end{pmatrix}$$

Then

$$w_1 \lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} y \\ x \\ -x+\varepsilon \end{pmatrix} = \begin{pmatrix} y \\ -x \\ x+\varepsilon \end{pmatrix} \qquad \text{fixes } x=0$$

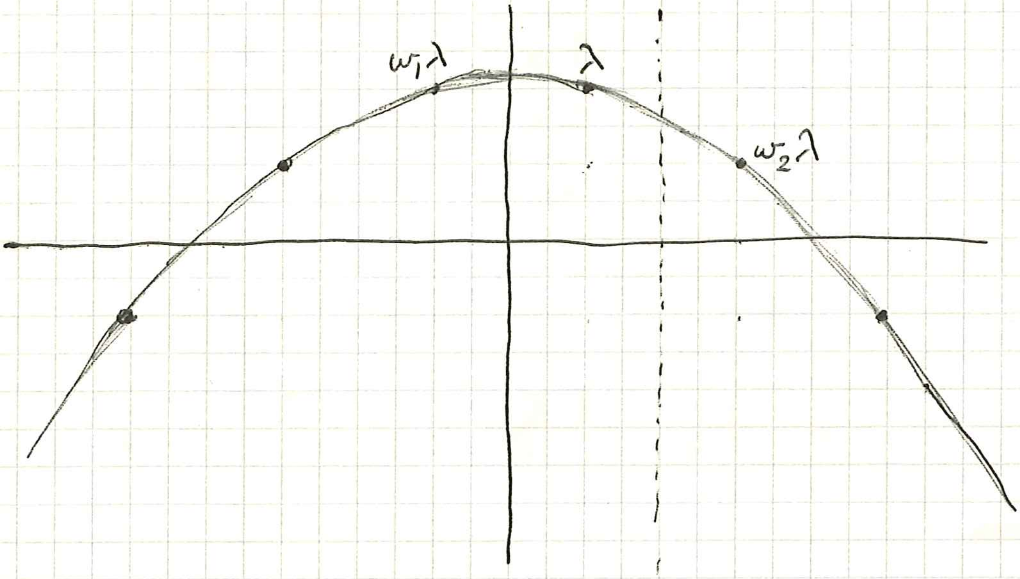
$$w_2 \lambda = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y \\ x \\ -x+\varepsilon \end{pmatrix} = \begin{pmatrix} y+x-\varepsilon \\ -x+2\varepsilon \\ x-\varepsilon \end{pmatrix} \qquad \text{fixes } x=\varepsilon$$

Picture of w_2



Look at the orbit under the Weyl group of a point [redacted] $\lambda =$ [redacted] $\begin{pmatrix} y \\ x \\ -x+\varepsilon \end{pmatrix}$ with $\lambda(h_1) = x \geq 0$

$\lambda(h_2) = \varepsilon - x \geq 0$



So the orbit seems to lie on a parabola

$$y + \frac{x^2}{4\varepsilon} = \text{constant}.$$

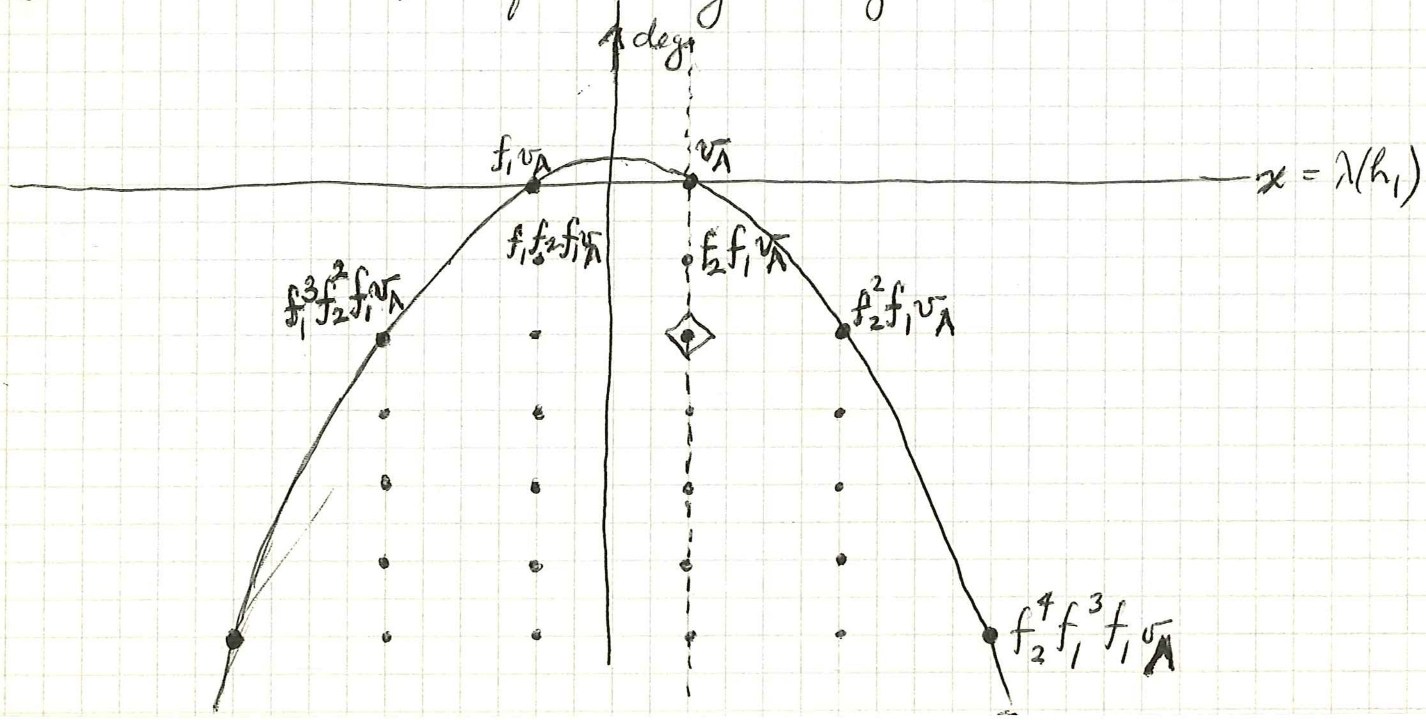
Check

$$(y+x-\varepsilon) + \frac{1}{4\varepsilon}(-x+2\varepsilon)^2 = y+\cancel{x}-\cancel{\varepsilon}+\frac{1}{4\varepsilon}(x^2-\cancel{4\varepsilon x}+\cancel{4\varepsilon^2})$$

$$= y + \frac{x^2}{4\varepsilon}.$$

Lets take an example: $\varepsilon = 1$, $\Lambda(h_1) = 1$, $\Lambda(h_2) = 0$.

Then we have the following weight vectors.



The multiplicity question arises at $\diamond$ where one has 2 possibilities:

$$f_2 f_1 f_2 f_1 v_\lambda \ , \quad f_1 f_2^2 f_1 v_\lambda$$

October 4, 1981

To understand the Kac-Weyl character formula for the loop algebras. Recall the algebra looks like

$$\mathfrak{g}^e = \underbrace{\bigoplus_{\alpha>0} \mathfrak{g}_{-\alpha}}_{\mathfrak{n}^*} \oplus \underline{\mathfrak{h}}^e \oplus \underbrace{\bigoplus_{\alpha>0} \mathfrak{g}_\alpha}_{\mathfrak{n}}$$

and the basic idea is to build up the module $V(\Lambda)$ out of the induced modules

$$\tilde{V}(\Lambda) = U(\mathfrak{g}^e) \otimes_{U(\mathfrak{h}^e \oplus \mathfrak{n})} (v_\Lambda)$$

$$\underset{\substack{\text{as } \mathfrak{h}^e \\ \text{module}}}{\cong} S(\mathfrak{n}^*) \otimes (v_\Lambda) \ .$$

Thus
$$ch(\tilde{V}(\Lambda)) = ch(S(\mathfrak{n}^*)) . e^\Lambda ,$$

and
$$ch(S(\mathfrak{n}^*)) = \frac{1}{\prod_{\alpha>0}(1-e^{-\alpha})}$$

One also puts
$$ch(S(\mathfrak{n}^*)) = \sum_\lambda K(\lambda) e^\lambda$$

where
$$K(\lambda) = \text{number of families } (n_\alpha) \quad n_\alpha \in \mathbb{Z}_{\geq 0}, \ \alpha \text{ root}$$
$$\text{such that } \lambda = -\sum n_\alpha \alpha$$

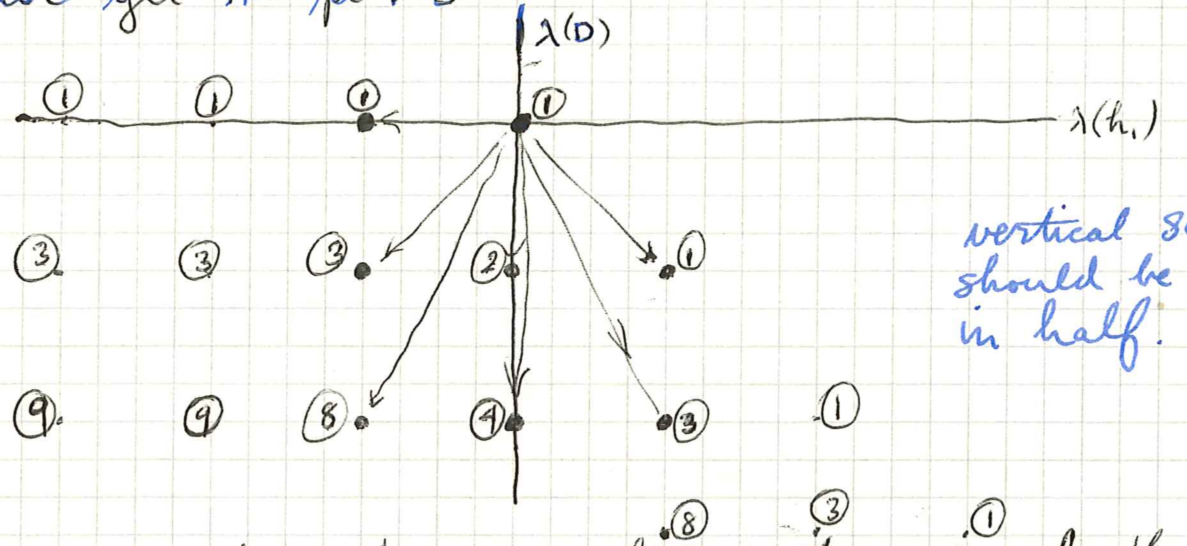
is the so-called Kostant function.

Example: ΩSU_2. Here
$$\mathfrak{g}^e = z^{-1}\mathfrak{g}_0[z^{-1}] + (Y) + \mathfrak{h}^e + (X) + z\mathfrak{g}_0[z]$$

and $h^e = (h_1) + (h_2) + (D)$. For any $\lambda : h^e \to \mathbb{C}$, we plot the point $(\lambda(h_1), \lambda(D))$ in the plane. For roots $\lambda(h_1 + h_2) = 0$.

Root vectors	roots [redacted]	[redacted] $\lambda(h_1)$	$\lambda(D)$
$Y \leftarrow f_1$	$-\alpha_1$	-2	0
$z^{-1}X \leftarrow f_2$	$-\alpha_2$	2	-1
$z^{-1}H$		0	-1
$z^{-1}Y$		-2	-1

So we get the points



vertical scale should be squashed in half.

Each arrow represents a negative root $-\alpha$. In the circle goes the multiplicity of the corresponding weight of $S(\mathfrak{m}^*)$, i.e. the Kostant fn.

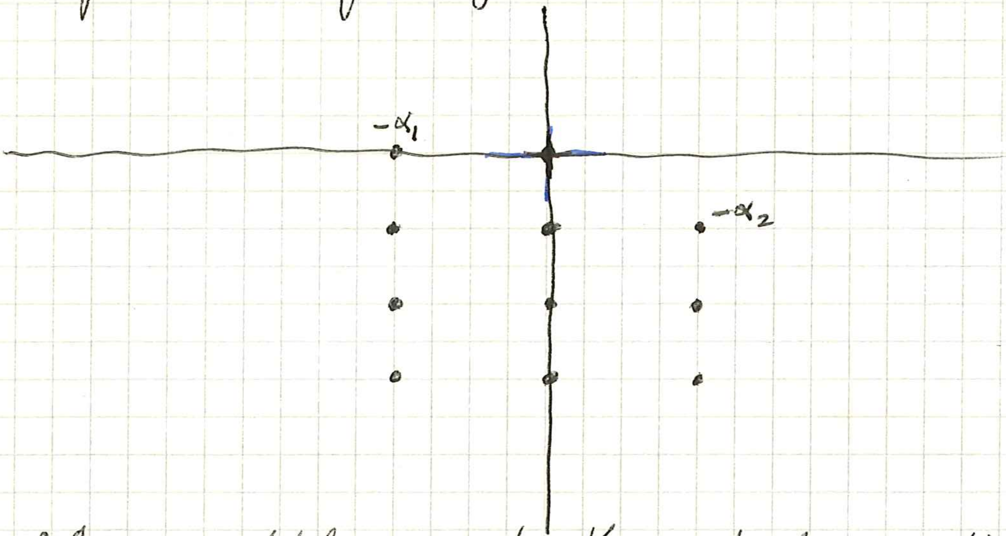
Now the Weyl group acts on the **roots** of the Lie algebra. [redacted] Here I am looking at those linear functions λ on h^e which vanish on $h_1 + h_2$ and hence they are really functions on $\bar{h}^e = (H) + (D)$. Now I have computed that if such λ are parameterized by $(x, y) = (\lambda(H), \lambda(D))$, then

$$w_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} \qquad w_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y + x \end{pmatrix}$$

(formulas on page 110 with $\varepsilon = \lambda(h_1 + h_2) = 0$.)

Thus w_2 fixes $x = 0$ and preserves the lines $y + \frac{1}{2}x =$ [redacted] constant. So because the vertical scale above is wrong, it looks like the roots in $S(\mathfrak{m}^*)$ are **invariant** under W.

Good picture of negative roots



Thus if we apply w_1 to the set of negative roots, they get permuted except for $-\alpha_1$ which goes to α_1 and nothing takes its place.

So now we need the quantity ρ, classically $= \frac{1}{2}\sum_{\alpha>0} \alpha$ which then is such that

$$e^{\rho} \prod_{\alpha>0} (1-e^{-\alpha}) = \prod_{\alpha>0} (e^{\alpha/2}-e^{-\alpha/2})$$

is anti-invariant under the Weyl group. Put

$$L = \prod_{\alpha>0} (1-e^{-\alpha})$$

Then we have

$$w_i L = L \, \frac{1-e^{\alpha_i}}{1-e^{-\alpha_i}} = (-1)e^{\alpha_i} L$$

and so we want that

$$w_i(\rho) = \rho - \alpha_i$$

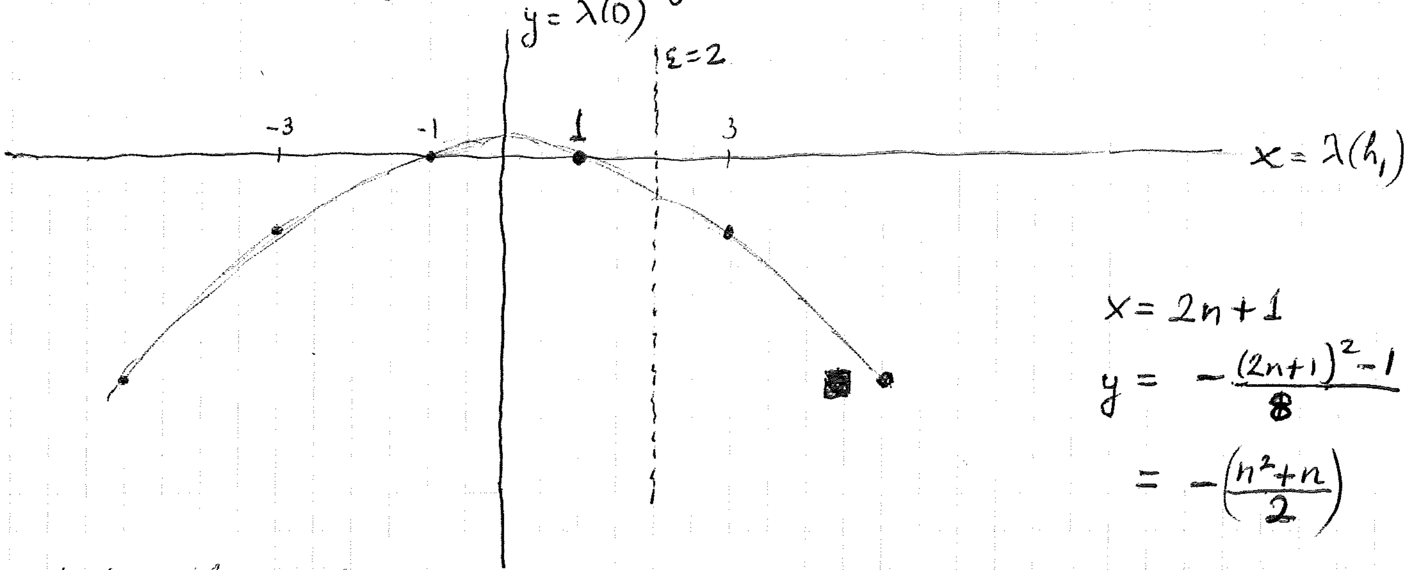
Suppose in the notation of page 110, $\rho = \begin{pmatrix} y \\ x \\ -x+\varepsilon \end{pmatrix}$

$$w_1(\rho) = \begin{pmatrix} y \\ -x \\ x+\varepsilon \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} y \\ x \\ -x+\varepsilon \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \Rightarrow x = +1$$

$$w_2(\rho) = \begin{pmatrix} y+x-\varepsilon \\ -x+2\varepsilon \\ x-\varepsilon \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} y \\ x \\ -x+\varepsilon \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \Rightarrow x-\varepsilon = -1$$

$\therefore$ Take $y = 0$ $x = 1$ $\varepsilon = 2$

Plot the Weyl orbit of this



$$x = 2n+1$$
$$y = -\frac{(2n+1)^2 - 1}{8} = -\left(\frac{n^2+n}{2}\right)$$

and take the alternating sum.

$$\sum_{w \in W} (-1)^w e^{w(\rho)}$$

Now this should be equal to $e^\rho L$ via the Jacobi formula. So we need some conventions: if $\lambda = (x, y)$ put

$$e^{\lambda} = u^{-x} q^{-y} \qquad \begin{matrix} e^{-\alpha_1} = u^2 \\ e^{-\alpha_2} = u^{-2} q \end{matrix}$$

where u and q are variables, [redacted] and we think of $|q| < 1$, so as to get convergent series as $y \to -\infty$. Then

$$L = \prod_{\alpha > 0} (1 - e^{-\alpha}) = \prod_{n \geq 0} (1 - q^n u^2) \prod_{n=1}^{\infty} (1 - q^n) \prod_{n=1}^{\infty} (1 - q^n u^{-2})$$

[redacted]

A typical $w(\rho)$ has coords. $\overset{x}{(2n+1}, \overset{y}{-\tfrac{1}{8}[(2n+1)^2 - 1]}, \overset{\varepsilon}{2})$

ρ " " $(1, 0, 2)$

Thus $w(\rho) - \rho$ " " $(2n, -\tfrac{1}{8}[(2n+1)^2 - 1], 0)$

$$-\tfrac{1}{8}[(2n+1)^2-1] = -\frac{n^2+n}{2}$$

So you get

$$\sum_{w \in W} (-1)^w e^{w(\rho) - \rho} = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2+n}{2}} u^{-2n} = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2+n}{2}} u^{2n}$$

which is the Jacobi formula.

Consider sl_n next. So

$$\mathfrak{g} = \underbrace{sl_n}_{\mathfrak{g}_0}[z,z^{-1}] = z^{-1}\mathfrak{g}_0[z^{-1}] + (X_{-\alpha})_{\alpha>0} + \mathfrak{h} + \underbrace{(X_\alpha)_{\alpha>0} + z\mathfrak{g}_0[z]}_{\mathfrak{n}}$$

and $\mathfrak{n}$ has the generators

$$e_i = X_{i,i+1} \qquad i = 1, 2, \dots, n-1$$
$$= zX_{d,1} \qquad i = n.$$

In the covering KM algebra, one has $\underline{h}$ with basis $h_1, \dots, h_n$ where

$$h_i \longmapsto H_i = \begin{pmatrix} \ddots & & \\ & 1 & \\ & & -1 \\ & & & \ddots \end{pmatrix} \begin{matrix} \leftarrow (i)\text{ row} \\ \leftarrow (i+1)\text{ "} \end{matrix} \qquad H_n = \begin{pmatrix} -1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 & \\ & & & & 1 \end{pmatrix}$$

so that $h_1 + \dots + h_n \longmapsto H_1 + \dots + H_n = 0$.

The simple roots $\alpha_i : \underline{h}^e = (D) + \underline{h} \longrightarrow \mathbb{C}$ are then given by $[h, e_i] = \alpha_i(h)e_i$. A little calculation gives the Cartan matrix

$$\big(\alpha_i(h_j), \alpha_i(D)\big) = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Notice that the Cartan matrix for sl_n is $\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & \\ & & & -1 \\ & & -1 & 2 \end{pmatrix}$.

Next I need the Weyl group.