July 5, 1981 1981 ~ 1982

The immediate project is to review the structure of rings like $\bigoplus_n H_*(BG_n)$ for the symmetric groups, then finite general linear groups, then maybe understand Green's work. Let's first go over Atiyah's description of the representations of the symmetric groups.

Representations of Σ_n lead to operations in complex K-theory as follows. Given a bundle E over X, then $E^{\otimes n}$ is a Σ_n-bundle over X. Since we are over the complex nos. we have

$$E^{\otimes n} \xleftarrow{\sim} \bigoplus_\alpha W_\alpha \otimes \operatorname{Hom}_{\Sigma_n}(W_\alpha, E^{\otimes n})$$

where W_α run over representatives for the irred. reps. of Σ_n. More generally, this decomposition holds for a G-bundle over X when G (a compact gp.) acts trivially on X. Then

$$K_G(X) = R(G) \otimes K(X).$$

In the case of the n-th tensor power, we get then a map

$$\begin{aligned} \mathrm{Vect}(X) &\longrightarrow K_{\Sigma_n}(X) = R(\Sigma_n)\otimes K(X) \\ E &\longmapsto cl(E^{\otimes n}) \end{aligned}$$

Since $R(\Sigma_n)$ is a free $\mathbb{Z}$-module with basis $cl(W_\alpha)$ we have

$$R(\Sigma_n)\otimes K(X) = \operatorname{Hom}_{\mathbb{Z}}\left(R(\Sigma_n)^\vee, K(X)\right)$$

and so for each element of $R(\Sigma_n)^\vee$, i.e. way of assigning integers to irred. reps. of Σ_n, we get an operation from $\mathrm{Vect}(X)$ to $K(X)$.

Let's put this together for all n. If the Σ_n-bundle $E^{\otimes n}$ is restricted to the subgp $\Sigma_p \times \Sigma_{n-p}$, then it is the product of the Σ_p-bundle $E^{\otimes p}$ and the Σ_{n-p} bundle $E^{\otimes(n-p)}$. This says

$$\begin{tikzcd}
R(\Sigma_p)^\vee \otimes R(\Sigma_{n-p}^\vee) \arrow[rr, "\text{belongs to } E^{\otimes p}\otimes E^{\otimes(n-p)}"] \arrow[d, equal] & & K(X)\otimes K(X) \arrow[dd] \\
R(\Sigma_p \times \Sigma_{n-p})^\vee \arrow[d] & & \\
R(\Sigma_n)^\vee \arrow[rr, "\text{belongs to } E^{\otimes n}", "\text{restricted to } \Sigma_p\times\Sigma_{n-p}"'] & & K(X)
\end{tikzcd}$$

commutes, or in other words that the map

$$\bigoplus_n R(\Sigma_n)^\vee \longrightarrow K(X),$$

associated to $E^{\otimes n}$ for all n, is a ring homomorphism, provided we equip the former with the product

$$R(\Sigma_p)^\vee \otimes R(\Sigma_q)^\vee = R(\Sigma_p \times \Sigma_q)^\vee \xrightarrow[\text{of restriction}]{\text{transpose}} R(\Sigma_{p+q})^\vee$$

We see that for each E over X we get a point of the ring $\bigoplus_{n\geq 0} R(\Sigma_n)^\vee$ with values in $K(X)$. Now we can add bundles and multiply them, so it is natural to ask if the points of $\bigoplus_{n\geq 0} R(\Sigma_n)^\vee$ form a ring, i.e. if this is an affine ring scheme. Clearly as representations of Σ_n we have

$$(E\oplus F)^{\otimes n} = \bigoplus_{p=0}^{n} \operatorname{Ind}_{\left(\substack{\Sigma_p\times\Sigma_{n-p} \\ \to \Sigma_n}\right)} E^{\otimes p}\otimes F^{\otimes(n-p)}$$

$$(E\otimes F)^{\otimes n} = E^{\otimes n}\otimes F^{\otimes n}$$

Thus if we have ring homomorphisms $\alpha, \beta : \bigoplus R(\Sigma_n)^\vee \longrightarrow R$ we define their sum and product by

$$(\alpha+\beta)_n = \sum_{p=0}^{n} \operatorname{Ind}_{\Sigma_p\times\Sigma_{n-p}\to\Sigma_n}(\alpha_p\otimes\beta_{n-p})$$

$$(\alpha \cdot \beta)_n = \alpha_n \cdot \beta_n$$

What does the first formula mean? One has $\alpha_p : R(\Sigma_p)^\vee \longrightarrow R$, $\beta_{n-p} : R(\Sigma_{n-p})^\vee \longrightarrow R$, so one has a map

$$R(\Sigma_p \times \Sigma_{n-p})^\vee = R(\Sigma_p)^\vee \otimes R(\Sigma_{n-p})^\vee \longrightarrow R$$

which I have denoted $\alpha_p \otimes \beta_{n-p}$. Then one has induction or transfer

$$R(\Sigma_p \times \Sigma_{n-p}) \longrightarrow R(\Sigma_n)$$

or

$$R(\Sigma_p \times \Sigma_{n-p})^\vee \longleftarrow R(\Sigma_n)^\vee .$$

Composition yields the map $R(\Sigma_n)^\vee \longrightarrow R$ I have denoted $\mathrm{Ind}_{\Sigma_p \times \Sigma_{n-p} \to \Sigma_n} (\alpha_p \otimes \beta_{n-p})$.

We should check that

$$\mathrm{Res}_{\Sigma_q \times \Sigma_{n-q} \to \Sigma_n} (\alpha+\beta)_n = (\alpha+\beta)_q \cdot (\alpha+\beta)_{n-q}$$

Use Mackey formula on

$$\mathrm{Res}_{\Sigma_q \times \Sigma_{n-q} \to \Sigma_n} \mathrm{Ind}_{\Sigma_p \times \Sigma_{n-p} \to \Sigma_n} (\alpha_p \otimes \beta_{n-p}).$$

This is a sum over double cosets:

$$\underbrace{\Sigma_q \times \Sigma_{n-q}}_{\text{stabilizer of } \{1,\dots,q\}} \backslash \underbrace{\Sigma_n / \Sigma_p \times \Sigma_{n-p}}_{p\text{-subsets of } \{1,\dots,n\}}$$

[redacted]

There is one double coset for each a with $0 \le a \le \min\{p,q\}$.
a is the size of the intersection of the p-subset with $\{1,\dots,q\}$.
A [redacted] nice coset representative is

$$\{1,\dots,a\} \cup \{q+1,\dots,q+p-a\}$$

We must have [redacted] $p-a \le n-q$ or $a \ge p+q-n$ also.

Maybe a better description is to have

$$\begin{cases} a+b=p & a,b\geq 0 \\ a\leq q \quad b\leq n-q \end{cases}$$

whence the coset representative is the p-set

$$\{1,\dots,a\}\cup\{q+1,\dots,q+b\}.$$

The stabilizer of this p-set [redacted] for the action of $\Sigma_q\times\Sigma_{n-q}$ is clearly

$$(\Sigma_a\times\Sigma_{q-a})\times(\Sigma_b\times\Sigma_{n-q-b})$$

and the twisted embedding into $\Sigma_p\times\Sigma_{n-p}$ is certainly given by interchanging Σ_{q-a}, Σ_b. Thus the Mackey formula should give

$$\mathrm{Res}_{\Sigma_q\times\Sigma_{n-q}\to\Sigma_n}\ \mathrm{Ind}_{\Sigma_p\times\Sigma_{n-p}\to\Sigma_n}(\alpha_p\otimes\beta_{n-p})$$

$$=\sum_{\substack{a+b=p\\ 0\leq a\leq q,\ 0\leq b\leq n-q}}\mathrm{Ind}_{(\Sigma_a\times\Sigma_{q-a})\times(\Sigma_b\times\Sigma_{q-b})\to\Sigma_q\times\Sigma_{n-q}}\Big[\underbrace{\mathrm{Res}_{\Sigma_a\times\Sigma_{q-a}\times\Sigma_b\times\Sigma_{n-q-b}\to\Sigma_p\times\Sigma_{n-p}}(\alpha_p\otimes\beta_{n-p})}_{(\alpha_a\otimes\beta_{q-a})\otimes(\alpha_b\otimes\beta_{q-b})}\Big]$$

$$=\sum_{\substack{a+b=p\\ 0\leq a\leq q,\ 0\leq b\leq n-q}}\mathrm{Ind}_{\binom{\Sigma_a\times\Sigma_{q-a}}{\to\Sigma_q}}(\alpha_a\otimes\beta_{q-a})\otimes\mathrm{Ind}_{\binom{\Sigma_b\times\Sigma_{n-q-b}}{\to\Sigma_q}}(\alpha_b\otimes\beta_{n-q-b})$$

Now sum over $p=0,\dots,n$ which means that you sum over all a,b, $0\leq a\leq q$, $0\leq b\leq n-q$ and you get

$$\mathrm{Res}_{\Sigma_q\times\Sigma_{n-q}\to\Sigma_n}(\alpha+\beta)_n=(\alpha+\beta)_q\otimes(\alpha+\beta)_{n-q}$$

as desired.

Let's abstract a bit the sort of thing that's going on here. We are considering collections $\alpha = (\alpha_n)$ where α_n is a kind of cohomology class of Σ_n. We want α to satisfy

$$\mathrm{Res}_{\Sigma_p \times \Sigma_{n-p} \longrightarrow \Sigma_n}(\alpha_n) = \alpha_p \otimes \alpha_{n-p}$$

Then we find that on the set of such $\alpha = (\alpha_n)$ we can define an addition by

$$(\alpha+\beta)_n = \sum_{p=0}^{n} \mathrm{Ind}_{\Sigma_p \times \Sigma_{n-p} \to \Sigma_n}(\alpha_p \otimes \beta_{n-p})$$

Here $\otimes$ denotes the external cup product. Similarly we can define a multiplication

$$(\alpha \cdot \beta)_n = \alpha_n \cdot \beta_n = \mathrm{Res}_{\Sigma_n \xrightarrow{\Delta} \Sigma_n \times \Sigma_n}(\alpha_n \otimes \beta_n)$$

(Check:
$$\begin{aligned}
\mathrm{Res}_{\Sigma_p \times \Sigma_{n-p} \to \Sigma_n}(\alpha_n \cdot \beta_n) &= \mathrm{Res}(\alpha_n) \cdot \mathrm{Res}(\beta_n) \\
&= (\alpha_p \otimes \alpha_{n-p}) \cdot (\beta_p \otimes \beta_{n-p}) \\
&= (\alpha_p \cdot \beta_p) \otimes (\alpha_{n-p} \cdot \beta_{n-p})
\end{aligned}$$

(recall $\alpha \otimes \beta = pr_1^*(\alpha) \cdot pr_2^*(\beta)$.)).

Finally let me check the distributive law

$$\begin{aligned}
((\alpha+\beta)\cdot\gamma)_n &= (\alpha+\beta)_n \cdot \gamma_n \\
&= \sum_{p=0}^{n} \underbrace{\underbrace{\mathrm{Ind}_{\Sigma_p \times \Sigma_{n-p} \to \Sigma_n}(\alpha_p \otimes \beta_{n-p}) \cdot \gamma_n}_{\mathrm{Ind}\left((\alpha_p \otimes \beta_{n-p}) \cdot \underbrace{\mathrm{Res}\,\gamma_n}_{\gamma_p \otimes \gamma_{n-p}}\right)}}_{\mathrm{Ind}(\alpha_p\gamma_p \otimes \beta_{n-p}\gamma_{n-p})} \\
&= (\alpha\cdot\gamma + \beta\cdot\gamma)_n
\end{aligned}$$

So it seems that whenever we have a suitable multiplicative type cohomology functor, we can compute this kind of ring out of its values over the family of symmetric groups.

The example I have been interested in is

$$R(G) \otimes_{\mathbb{Z}} R = \mathrm{Hom}_{\mathbb{Z}}(R(G)^\vee, R). \qquad R \text{ any coeff. ring}$$

In this case we can make $\bigoplus R(\Sigma_n)^\vee$ into a ring by the maps

$$R(\Sigma_n)^\vee \otimes R(\Sigma_m)^\vee \cong R(\Sigma_n \times \Sigma_m)^\vee \xrightarrow[\text{of restriction}]{\text{transpose}} R(\Sigma_{n+m})^\vee$$

some kind of canonical isomorphism which has to be assumed in general. The fact that we have a product of a class on Σ_n and a class on Σ_m to get a class on $\Sigma_n \times \Sigma_m$ suggests that one has a map

$$R(\Sigma_n \times \Sigma_m)^\vee \longrightarrow R(\Sigma_n)^\vee \otimes R(\Sigma_m)^\vee$$

the wrong way.

Let's return to the computation of the ring $\bigoplus_n R(\Sigma_n)^\vee$. One knows that $R(\Sigma_n)$ is a free abelian group of rank = the number of conjugacy classes in Σ_n. Conjugacy classes in Σ_n are the same as partitions of n, which are the number of monomials in the variables $\sigma_1, \sigma_2, \ldots, \sigma_N$ N large of degree n where σ_i has degree i. Thus to the monomial $\sigma_1^{r_1} \cdots \sigma_N^{r_N}$ belongs the partition

$$\underbrace{1 \cdots 1}_{r_1} \; \underbrace{2 \cdots 2}_{r_2} \; \cdots .$$

Next one takes a vector space V of dimension N ~~on~~ which a group G acts and considers the ring homomorphism

$$\bigoplus R(\Sigma_n)^\vee \longrightarrow R(G)$$

obtained from the tensor powers $V^{\otimes n}$. Suppose $V = \mathbb{C}^n$ and

G = the torus [redacted] of diagonal matrices in [redacted] U_n, so that $R(G) = \mathbb{Z}[G^\vee]$, $G^\vee$ = free abelian group with basis the characters [redacted] $t_i = pr_i : G \to S^1$.

Now $V = L_1 \oplus \cdots \oplus L_N$ where $L_i = \mathbb{C}$ with the G-action given by the character t_i. Thus

$$V^{\otimes n} = \bigoplus \Sigma_n \cdot \left(L_1^{\otimes s_1} \otimes L_2^{\otimes s_2} \otimes \cdots \otimes L_N^{\otimes s_N}\right)$$

where the sum is taken over all [redacted] $s_1, \dots, s_N \geq 0$ with $\sum s_i = n$. More exactly

$$V^{\otimes n} = \bigoplus_{\substack{s_1, \dots, s_N \geq 0 \\ \sum s_i = n}} \mathbb{C}[\Sigma_n / \Sigma_{s_1} \times \cdots \times \Sigma_{s_N}] \otimes \left(L_1^{s_1} \otimes \cdots \otimes L_N^{s_N}\right)$$

[redacted] gives the decomposition of $V^{\otimes n}$ with respect to

$$R(\Sigma_n \times G) = R(\Sigma_n) \otimes R(G).$$

Now the representation $\mathbb{C}[\Sigma_n / \Sigma_{s_1} \times \cdots \times \Sigma_{s_N}] = \mathrm{Ind}_{\Sigma_{s_1} \times \cdots \times \Sigma_{s_N} \to \Sigma_N}(1)$ depends only on the (unordered) partition determined by $s_1, \dots, s_N$, so the above can be rewritten

$$[V^{\otimes n}] = \bigoplus_{\substack{r_1 \geq \cdots \geq r_N \\ \sum r_i = N}} \mathrm{Ind}_{\Sigma_{r_1} \times \cdots \times \Sigma_{r_N} \to \Sigma_n}(1) \cdot \text{symmetrization of the monomial } t_1^{r_1} \cdots t_N^{r_N}$$

It follows that the map

$$\bigoplus_n R(\Sigma_n)^\vee \longrightarrow R(G) = \text{Laurent polys. in } t_1, \dots, t_N$$

has its image in the symmetric polys. ring

$$\mathbb{Z}[t_1, \dots, t_N]^{\Sigma_N} = \mathbb{Z}[\sigma_1, \dots, \sigma_N]$$

where $\sigma_1, \dots, \sigma_N$ are the elementary symmetric fns. Next let $\lambda_n \in R(\Sigma_N)^\vee$ pick out the sign representation, so that λ_n when applied to $V^{\otimes n}$ gives $\Lambda^n(V)$. Clearly the character of $\Lambda^n(V)$ is σ_n. Thus the map

$$\bigoplus R(\Sigma_n)^\vee \longrightarrow \mathbb{Z}[\sigma_1,\dots,\sigma_N]$$

is onto, and by comparing ranks one sees that it is an isomorphism in degrees $\leq N$. It also follows that the representations $\mathrm{Ind}_{\Sigma_{\lambda_1}\times\cdots\times\Sigma_{\lambda_N}\to\Sigma_n}(1)$ generate $R(\Sigma_n)$.

So we see that $\bigoplus R(\Sigma_n)^\vee \cong \mathbb{Z}[\lambda_1,\lambda_2,\dots]$ is a polynomial ring.

July 7, 1981.

We have a ring homomorphism

$$K(X) \longrightarrow \mathrm{Hom}_{\text{rings}}\left(\bigoplus R(\Sigma_n)^\vee, K(X)\right)$$

where the ring structure on the latter is due to the ring scheme structure of $\bigoplus R(\Sigma_n)^\vee$. In particular it doesn't depend on any λ-structure of $K(X)$.

We denote this functor from rings to rings by

$$W(R) = \mathrm{Hom}_{\text{rings}}\left(\bigoplus R(\Sigma_n)^\vee, R\right).$$

Using the fact that $\bigoplus R(\Sigma_n)^\vee \xleftarrow{\sim} \mathbb{Z}[\lambda_1,\lambda_2,\dots]$ we see that

$$W(R) \cong \text{set of power series } 1+a_1t+a_2t^2+\cdots \text{ with } a_i\in R.$$

and that the map above is

$$K(X) \longrightarrow W(K(X))$$

$$x \longmapsto \lambda_t(x) = \sum \lambda^n(x)t^n$$

So I can conclude that the ring $\bigoplus R(\Sigma_n)^\vee$ represents the Witt ring functor. For example, we have $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$ which forces the addition in $W(R)$ to be multiplication of power series.

I would like to try to do something [redacted] for the finite general linear groups similar to the above stuff for the symmetric groups. The idea is to consider instead of $\Sigma_p \times \Sigma_{n-p} \subset \Sigma_n$, the parabolic subgp $G_{p,n-p} \subset G_n$ which is the stabilizer of the subspace $\mathbb{C}^p \subset \mathbb{C}^n$ (first p coordinates). Then the double cosets

$$G_{q,n-q} \backslash G_n / G_{p,n-p}$$

are again described by integers a, b with $a+b=p$, $0 \le a \le q$, $0 \le b \le n-q$. Better: $G_n/G_{p,n-p}$ can be identified with the set of p-subspaces of $\mathbb{C}^n$, and the double cosets above are orbits for the stabilizer of $\mathbb{C}^q$. A convenient representative for this orbit is the subspace

$$(\mathbb{C}^a \oplus O) \oplus (\mathbb{C}^b \oplus 0) \subset \mathbb{C}^q \oplus \mathbb{C}^{n-q}$$

The stabilizer is the subgroup fixing the subspaces

$$\begin{array}{ccc} \mathbb{C}^a & \subset & \mathbb{C}^a \oplus O \oplus \mathbb{C}^b \\ \cap & & \cap \\ \mathbb{C}^q & \subset & \mathbb{C}^q \oplus \mathbb{C}^b \end{array}$$

It is the intersection of the stabilizer of the flag

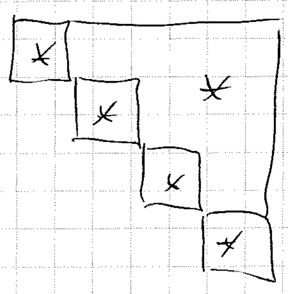
$$0 \subset \mathbb{C}^a \subset \mathbb{C}^q \subset \mathbb{C}^{q+b} \subset \mathbb{C}^n$$

a

$q-a$

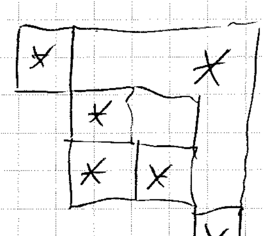
b

$n-q-b$

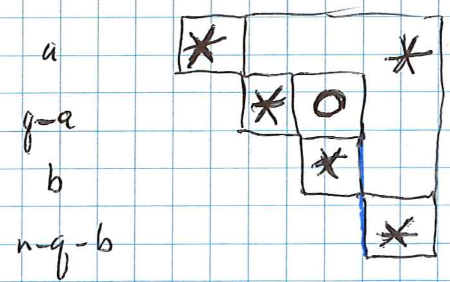


and that of the flag

$$O \subset \mathbb{C}^a \subset \mathbb{C}^a + O + \mathbb{C}^b \subset \mathbb{C}^{q+b}$$



so the stabilizer of interest is



So now we want to consider a family $\alpha = (\alpha_n)$ where $\alpha_n \in H(G_n)$. I am going to think of mod l cohomology of $GL(n, \mathbb{F}_q)$, $l \neq p$. Then I know that $G_{p,q} \to G_p \times G_q$ induces isos. on cohomology. As before I want my family of classes $\alpha = (\alpha_n)$ to satisfy

$$\mathrm{Res}_{G_p \times G_{n-p} \longrightarrow G_n} (\alpha_n) = \alpha_p \otimes \alpha_{n-p}$$

This [redacted] implies that if we have any flag [redacted] say [redacted] ξ: [redacted] $\mathbb{C}^{a_1} \subset \mathbb{C}^{a_1+a_2} \subset \cdots \subset \mathbb{C}^{a_1+\cdots+a_k} = \mathbb{C}^n$

then the restriction of α to the parabolic grp G_ξ stabilizing the flag is the product of the classes

$$\mathrm{Res}_{G_\xi \longrightarrow G_{a_i}} (\alpha_{a_i}) \qquad i = 1, \ldots, k,$$

and maybe this latter condition is a good one in general.

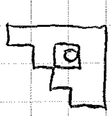
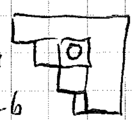
So now let us consider two such families $\alpha = (\alpha_n)$, and $\beta = (\beta_n)$. Let's define

$$(\alpha + \beta)_n = \sum_{p=0}^{n} \mathrm{Ind}_{G_{p,n-p} \longrightarrow G_n} (\alpha_p \otimes \beta_{n-p})$$

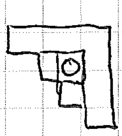
and see if this works. [redacted] Here $\alpha_p \otimes \beta_{n-p}$ denotes the product of α_p pulled back via $G_{p,n-p} \longrightarrow G_p$ and β_{n-p} pulled back similarly. Now we compute

$$\mathrm{Res}_{G_{q,n-q} \to G_n} \text{[redacted]} \; \mathrm{Ind}_{G_{p,n-p} \to G_n} (\alpha_p \otimes \beta_{n-p})$$

by the Mackey formula. We get the sum over $a+b=p$
$0\le a\le q$, $0\le b\le n-q$ of

Ind  $\hookrightarrow G_{q,n-q}$ Res $\begin{smallmatrix} a \\ q-a \\ b \\ n-q-b \end{smallmatrix}$  $\longrightarrow G_{p,n-p}$ $(\alpha_p\otimes\beta_{n-p})$

this embedding interchanges
to $q-a$ and b parts.

$=$ Ind  $\hookrightarrow G_{q,n-q}$ $\left(\alpha_a\otimes\beta_{q-a}\otimes\alpha_b\otimes\beta_{n-q-b}\right)$

What is it that you want to get?

$$\mathrm{Res}_{G_{q,n-q}\to G_q\times G_{n-q}}\left(\mathrm{Ind}_{G_{a,q-a}\to G_q}\left(\alpha_a\otimes\beta_{q-a}\right)\otimes\mathrm{Ind}_{G_{b,n-q-b}\to G_{n-q}}\left(\alpha_b\otimes\beta_{n-q-b}\right)\right)$$

Now in a situation:

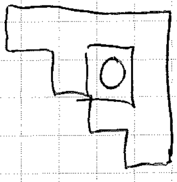
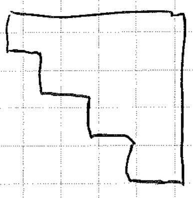
$$\begin{array}{ccc} H' & \longrightarrow & H \\ \downarrow & & \downarrow \\ G' & \longrightarrow & G \end{array} \qquad \text{cartesian}$$

we have $\mathrm{Res}_{G'\hookrightarrow G}\,\mathrm{Ind}_{H\to G} = \mathrm{Ind}_{H'\hookrightarrow G'}\,\mathrm{Res}_{H'\to H}$

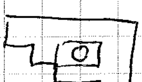
because $G'\backslash G/H = pt$. Thus the desired quantity is
the same as

$$\mathrm{Ind}_{G_{a,q-a,b,n-q-b}\longrightarrow G_n}\left(\alpha_a\otimes\beta_{q-1}\otimes\alpha_b\otimes\beta_{n-q-b}\right)$$

and we see that we are off because

 $\not\subset$ 

$G_{a,q-a,b,n-q-b}$

How far off: Put $K' =$  $\quad K = G_{a,\,q-a,\,b,\,n-q-b}$.

Then

$$\mathrm{Ind}_{K'\to G}\left(\mathrm{Res}_{K'\to K}(u)\right) = \mathrm{Ind}_{K\to G}\,\underbrace{\mathrm{Ind}_{K'\to K}\,\mathrm{Res}_{K'\to K}}_{\substack{\text{multiplication by } [K:K'] \\ \text{in cohomology}}}(u)$$

Thus if we are working in cohomology we are off by the factor $[K:K'] = (\operatorname{card} \mathbb{F}_q)^{(q-a)b}$. If we have cohomology mod ℓ where $\ell \mid q-1$, then $[K:K'] = 1$ and it all seems to work.

Let's digress to go over Green's paper on the representations of $G_n = GL_n(\mathbb{F}_q)$. He forms $\bigoplus_n R(G_n)$ and defines a multiplication as follows. Given $\alpha \in R(G_m)$, $\beta \in R(G_n)$ then

$$\alpha\cdot\beta = \mathrm{Ind}_{G_{m,n}\to G_{m+n}}\,\mathrm{Res}_{G_{m,n}\to G_m\times G_n}(\alpha\otimes\beta)$$

Associativity follows from

$$(\alpha\cdot\beta)\cdot\gamma = \mathrm{Ind}_{G_{m,n,p}\to G_{m+n+p}}\,\mathrm{Res}_{G_{m,n,p}\to G_m\times G_n\times G_p}(\alpha\otimes\beta\otimes\gamma)$$

To see this compute

$$(\alpha\cdot\beta)\cdot\gamma = \mathrm{Ind}_{G_{m+n,p}\to G_{m+n+p}}\,\mathrm{Res}_{G_{m+n,p}\to G_{m+n}\times G_p}\left((\alpha\cdot\beta)\otimes\gamma\right)$$

Use diagram:

$$\begin{tikzcd}
G_{m,n,p} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & G_{m+n,p} \arrow[r] \arrow[d] & G_{m+n+p} \\
G_{m,n}\times G_p \arrow[r, hook] \arrow[d] & G_{m+n}\times G_p & (\alpha\cdot\beta)\gamma \\
G_m\times G_n\times G_p & \alpha\otimes\beta\otimes\gamma & (\alpha\cdot\beta)\otimes\gamma
\end{tikzcd}$$

[redacted] Next we want to get at the size of $R(G_n)$. $R(G_n)$ is a free abelian group of rank equal to the number of conjugacy classes of G_n. Such conjugacy classes are known from the theory of the Jordan normal form. Thus let V be an n-dimensional vector space over $\mathbb{F}_q$ equipped with an automorphism θ. Choosing a basis for V gives an element of $GL_n(\mathbb{F}_q)$ corresponding to θ, and in this way one gets an [redacted] 1-1 correspondence between isom. classes of (V, θ) and conjugacy classes of G_n.

One knows each (V, θ) is a direct sum of cyclic pairs. Precisely, we are looking at finite length modules over $\mathbb{F}_q[T, T^{-1}]$ which is a P.I.D., so any module is a direct sum of cyclic modules which are of the form $\mathbb{F}_q[T, T^{-1}]/(f)^{k}$ [redacted] where f is an irreducible + monic polynomial [redacted] with non-zero constant term.

It follows that the Poincaré series of $\oplus R(G_n)$ is the same as that of a polynomial ring having one generator of degree $k.\deg(f)$ for each pair (f, k), consisting of an irreducible polynomial f and an integer $k > 0$. It would be nice if the ring $\oplus R(G_n)$ were a tensor product over such pairs (f, k). Also one would like the part belonging to an irreducible poly. f to be essentially independent of f in the same way that Jordan blocks with the same [redacted] diagonal blocks are given by just the standard Jordan nilpotent matrices.

By way of comparison let's first consider the modular character theory, denote it [redacted] $\bar{R}(G_n)$. Here the rank is given by the conjugacy classes of p-regular elements, so one finds a Poincaré series for $\oplus \bar{R}(G_n)$, the same as for a poly ring with one generator for each irreducible polynomial f with non-zero constant term.

July 8, 1981

Recall that for a finite group $R(G)$ is a free abelian group with basis the irreducible isomorphism classes. One has the character map

$$R(G) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \mathrm{Map}_{\mathrm{central}}(G, \mathbb{C})$$

which is an isomorphism since the characters of the irred. repns. form a basis for the latter.

How does one compute the character of an induced repn? Given H acting on W where $H \subset G$, the induced repn is

$$V = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} W = \bigoplus_{xH} xW$$

An element $g \in G$ permutes the factors of this decomposition around, hence the trace of g is zero unless g fixes a point xH of G/H, i.e. $g \in xHx^{-1}$ or $x^{-1}gx \in H$. Thus

$$\mathrm{tr}(g \text{ on } V) = \sum_{xH}' \begin{cases} \mathrm{tr}(x^{-1}gx \text{ on } W) & \text{if } x^{-1}gx \in H \\ 0 & \phantom{\text{if } x^{-1}gx} \notin H. \end{cases}$$

In general given a central fn. f on H, one has

$$\mathrm{Ind}_{H\to G}(\chi) = \sum_{xH}' (\chi \text{ extended by } 0 \text{ to } G)^x.$$

Hence if χ is supported on a conjugacy class of H, then $\mathrm{Ind}_{H\to G}(\chi)$ is supported on the image conjugacy class of G.

Let's try to use this to understand Green's multiplication on $\bigoplus R(G_n)$. Start off with two distinct irred. polys. f_1, f_2 over $\mathbb{F}_q$ of degrees a_1, a_2 respectively. Let χ_{f_1} etc. be the characteristic function of the conjugacy class in G_{a_1} consisting of all matrices with characteristic poly

f_1. To compute

$$\chi_{f_1} \cdot \chi_{f_2} = \mathrm{Ind}_{G_{a_1,a_2} \to G_{a_1+a_2}} \mathrm{Res}_{G_{a_1,a_2} \to G_{a_1} \times G_{a_2}} \chi_{f_1} \otimes \chi_{f_2}$$

Obviously $\chi_{f_1} \otimes \chi_{f_2}$ is the char. function of the product of the conjugacy classes belonging to f_1, f_2 in $G_{a_1} \times G_{a_2}$. When we pull this back to G_{a_1,a_2} we [redacted] get the char. fn. of the set of $\theta \in G_{a_1,a_2}$ such that the image of θ in G_{a_i} has char. poly f_i. Because f_1, f_2 are distinct we know that the θ invariant subspace $\mathbb{C}^{a_1} \subset \mathbb{C}^{a_1+a_2}$ has an invariant complement. Since these complements are conjugate in G_{a_1,a_2}, it follows that all these θ form a single conjugacy class in G_{a_1,a_2}, namely all autos with char. poly f_1 on $\mathbb{C}^{a_1}$ and char. poly f_2 on the quotient $\mathbb{C}^{a_1+a_2}/\mathbb{C}^{a_1}$.

Thus $\mathrm{Res}_{G_{a_1,a_2} \to G_{a_1} \times G_{a_2}} \chi_{f_1} \otimes \chi_{f_2}$ is the char. function of a single conjugacy class, and so when induced to $G_{a_1+a_2}$ it is supported in the image conjugacy class. I want to compute the multiplicity. The image conjugacy class consists of all autos θ having an invariant decomposition $\mathbb{C}^{a_1+a_2} = V_1 \oplus V_2$ with θ having char. poly f_i on V_i. Given such a θ we want the number of cosets xG_{a_1,a_2} such that $x^{-1}\theta x$ belonging to the conjugacy class in G_{a_1,a_2} described in the preceding paragraph. This coset xG_{a_1,a_2} is the same as a subspace of dim a_1 invariant under θ having the char. poly f_1 on this subspace, so it is unique.

It seems that the above argument only uses the fact that f_1, f_2 are relatively prime, so that one has the so-called primary decomposition. Thus given χ_i = char. fn of a conj. class in G_{a_i}, if the char. polys. of χ_1 and χ_2 are rel. prime then $\chi_1 \cdot \chi_2$ = char. fn. of [redacted] conjugacy class of the direct sion in $G_{a_1+a_2}$. This tells us that $\bigoplus R(G_n)\otimes\mathbb{C}$ is a tensor product over primary pieces, a primary piece consisting of all conjugacy classes with characteristic polynomial of the form f^k, $k>0$, f irreducible polynomial.

Summary: On $\bigoplus R(G_n)$ one has the Green product, and we've been analyzing $\bigoplus R(G_n)\otimes\mathbb{C}$ by using conjugacy classes. So far we've found that corresponding to the primary decomposition part of the Jordan normal form is a decomposition of this ring as a tensor product of primary pieces, each primary piece described by an irreducible poly over $\mathbb{F}_q$ with non-zero constant term. All the primary pieces appear to have the same structure, but over some finite extension field of $\mathbb{F}_q$. Now we need:

1) Structure of the primary part belonging to the irreducible polynomial $f = X-1$. Unipotent conjugacy classes

2) Representation theory refinements of the above calculations with conjugacy classes.

Let's now begin to understand the structure of the unipotent conjugacy classes and "unipotent representations". We have

$G_0 = 1$

$G_1 = \mathbb{F}_q^*$ $\qquad (1)$

$G_2 = GL_2(\mathbb{F}_q)$ $\qquad \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

$G_3 = GL_3(\mathbb{F}_q)$ $\qquad \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$

The number of unipotent classes in GL_n is the number of partitions of n. Also if we consider the induced representations

$\mathrm{Ind}_{G_{a_1,\dots,a_r} \longrightarrow G_n}(1)$ $a_1+\dots+a_r = n$

it should be true that this induced repn. depends only on the (unordered) partition $(a_1,\dots,a_r)$ of n. In this way we might obtain a basis for the unipotent representations of G_n.

So let's consider the support of these induced repns. Consider G_2 first, and begin with conjugacy classes. Take the identity in G_1 and multiply by itself: Then you pull the char. function of the identity in $G_1 \times G_1$ to $G_{1,1}$ and you get the set of matrices $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ $x \in \mathbb{F}_q$, which is the union of two conjugacy classes in $G_{1,1}$. As there is only one line invariant under $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ for $x \neq 0$ this conjugacy class should remain of multiplicity 1 when pushed into G_2. However the $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ leaves all lines invariant so the multiplicity should be $q+1$ in G_2.

<u>July 9, 1981</u>

So we try to understand the representations of G_2. First we have the Mackey formula

$$\langle \mathrm{Ind}_{H\to G}(W), \mathrm{Ind}_{K\to G}(V)\rangle$$
$$= \sum_{HxK} \langle \mathrm{Res}_{H\cap xKx^{-1}\to H}(W), \mathrm{Res}_{H\cap xKx^{-1}\to K}(V)\rangle$$

Lets $H = K = G_{1,1}$ = the Borel subgroup B and W, V should be characters. There are two double cosets the trivial on $BeB = B$ and the non-trivial one BsB which has $B\cap sBs^{-1} = T$. The above formula becomes

$$\langle \mathrm{Ind}_{B\to G}(W), \mathrm{Ind}_{B\to G}(V)\rangle = \langle W, V\rangle_B + \langle W, V^s\rangle_T$$

Here $T = \mathbb{F}_q^* \times \mathbb{F}_q^*$ and V is given by a pair (χ_1, χ_2) of characters of $\mathbb{F}_q^*$. $V^s = (\chi_2, \chi_1)$. The above fmla shows that if $\chi_1 \neq \chi_2$ so that $V \not\cong V^s$, then $\mathrm{Ind}_{B\to G}(V)$ is irreducible of dim. $q+1$. If $\chi_1 = \chi_2$, then the induced repn. splits into two irreducible. One of them is a character

$$G_2 \xrightarrow{\det} G_1 \xrightarrow{\chi} \mathbb{C}^*$$

because the restriction to T is (χ, χ). (Recall by Frobenius reciprocity $\langle \mathrm{Ind}_{B\to G}(V), W\rangle_G = \langle V, \mathrm{Res}_{B\to G}(W)\rangle_B$. Thus from each of the character (χ,χ) of T one obtains two irred repns. of dim q and dim. 1 respectively.

Now let's use the formula $|G| = \sum_{\chi \text{ irred.}} d_\chi^2$. So far we have

$$\underbrace{\frac{(q-1)^2-(q-1)}{2}}_{\substack{\text{no. of } (\chi_1,\chi_2) \\ \text{with } \chi_1\neq\chi_2 \\ \text{modulo action of } s}} (q+1)^2 + (q-1)(q^2+1^2)$$

$$= \frac{q-1}{2}\left\{(q-2)\underbrace{(q+1)^2}_{q^2+2q+1} + 2q^2 + 2\right\}$$

$$= \frac{q-1}{2}\left\{ q^3 \cancel{-2q^2} \cancel{+2q^2} + q - 4q \cancel{-2} + 2q^2 \cancel{+2} \right\}$$

$$= \frac{q(q-1)}{2}\left\{ q^2 + 2q - 3 \right\}$$

Taking this away from $|G_2| = (q^2-1)(q^2-q) = q(q-1)(q^2-1)$

gives $$\frac{q(q-1)}{2}\Big[\underbrace{2q^2-2 - q^2-2q+3}_{q^2-2q+1}\Big] = \frac{q(q-1)}{2}(q-1)^2$$

It should be true that the number of discrete series representations of G_2 is the same as the number of conjugacy described by irreducible polys. of degree 2. The number of these polys is

$$\left|\left(\mathbb{F}_{q^2}^* - \mathbb{F}_q^*\right)/\text{galois}\right| = \frac{q^2-q}{2}$$

Thus each discrete series representation should have dimension $q-1$.

At the moment I am more interested in the induction process than the discrete series repns. So a good problem is to understand why the Green multiplication is commutative. Thus given a rep. V of $G_a \times G_b$ why are the repns.

$$\mathrm{Ind}_{G_{a,b} \to G_{a+b}} \mathrm{Res}_{G_{a,b} \to G_a \times G_b}(V)$$

$$\mathrm{Ind}_{G_{b,a} \to G_{a+b}} \mathrm{Res}_{G_{b,a} \to G_a \times G_b}(V)$$

isomorphic? Recall

$$\mathrm{Hom}_G\left(\mathrm{Ind}_{H\to G} W, \mathrm{Ind}_{K\to G} V\right)$$

$$= \bigoplus_{H x K} \mathrm{Hom}_H\left(W, \mathrm{Ind}_{H\cap xKx^{-1}\to H} \mathrm{Res}_{H\cap xKx^{-1}\to K}(V)\right)$$

$$= \bigoplus_{H \times K} \mathrm{Hom}_{H \cap xKx^{-1}}\left(\mathrm{Res}_{H\cap xKx^{-1} \to H} W,\ \mathrm{Res}_{H\cap xKx^{-1}\to K} V\right)$$

For the last isomorphism one uses the coincidence of induction and coinduction for finite groups:

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \xrightarrow{\sim} \mathrm{Map}_H(G, M)$$

(The latter can be viewed as sections of $G \times^H M \longrightarrow G/H$; the isomorphism comes from the fact there is an evident system of imprimitivity on this space of sections.)

Now we want to take $K = G_{a,b}$, $G = G_{a+b}$, $H = G_{b,a}$. Then $G/K = G_{a+b}/G_{a,b}$ can be identified with the set of a-planes in $\mathbb{C}^{a+b}$, whereas $G_{b,a}$ is the stabilizer of the standard b-plane $\mathbb{C}^b \oplus 0 \subset \mathbb{C}^{a+b}$. The set of a-planes complementary to $\mathbb{C}^b$ form an orbit under $G_{b,a}$, hence one has a canonical (more or less) double coset. The group $H \cap xKx^{-1}$ is just $G_a \times G_b$ with the two obvious embeddings into $G_{b,a}$ and $G_{b,a}$.

Think of $\mathrm{Ind}_{H\to G}(W) = \mathrm{Map}_H(G, W) = \{ f : G \to W \mid f(hg) = hf(g)\}$

Let $K(x,y)$ be a function on $G \times G$ with values in $\mathrm{Hom}(W,V)$ such that

$$K(kx, hy) = k\,K(x,y)\,h^{-1}$$

and

$$K(xg, yg) = K(x,y)$$

Then for $f \in \mathrm{Map}_H(G,W)$ we can form

$$(Kf)(x) = \int_{H\backslash G} K(x,y) f(y)$$

and $$(Kf)(kx) = \int K(kx,y) f(y) = k \int K(x,y) f(y) = k\cdot (Kf)(x).$$

so $Kf \in \mathrm{Map}_K(G, V)$. Furthermore

$$(g\cdot Kf)(x) = (Kf)(xg) = \int K(xg, y) f(y) = \int \underbrace{K(xg, yg)}_{K(x,y)} f(yg)$$

$= K(gf)(x).$

Thus $f \longmapsto Kf$ is a G-map from $\mathrm{Ind}_{H \to G}(W)$ to $\mathrm{Ind}_{K \to G}(V)$. Furthermore, if we write $K(xy^{-1}) = K(x,y)$, then K is a function of G with values in $\mathrm{Hom}(W,V)$ satisfying

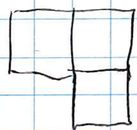
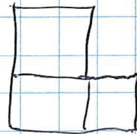
$$K(kxh^{-1}) = k\,K(x)\,h^{-1}.$$

It's clear that this allows K to vary over double cosets KxH, so that by the Mackey formula any G-map between induced representations is given by

$$f \longmapsto \int_{H\backslash G} K(xy^{-1})f(y)$$

by such a kernel <u>function</u> K.

<u>July 10, 1981</u>. I wanted to show that for a repn V of $G_a \times G_b$, the two repns of G_{a+b} induced from V on the subgroups

a  and  are isomorphic.
b

To simplify, let us assume that V is the trivial 1-dimensional representation, whence the induced repn. from the subgroup H is the set of $s : G \longrightarrow \mathbb{C}$ such that $s(hg) = s(g)$, with

$$(gs)(x) = s(xg)$$

To map this to the induced repn. from K, suppose given $f : G \longrightarrow \mathbb{C}$ such that $f(kxh) = f(x)$. Then

$$s \longmapsto (fs)(x) = \int_{y \in H\backslash G} f(xy^{-1})\,s(y)$$

is a G-map. Note that such a map amounts to a function on $K\backslash G/H$.

Here seems to be a good way to think. The induced representation from H is the set of sections over G/H. So one has the diagram

$$\begin{tikzcd} G/K \times G/H \arrow[r, "pr_2"] \arrow[d, "pr_1"] & G/H \\ G/K & \end{tikzcd}$$

and f denotes a G-invariant function on $G/K \times G/H$, for example the characteristic function of the orbit $\{(gxK, gH)\}$. Therefore we [redacted] can do computations by working with correspondences invariant [redacted] under G from G/H to G/K.

In particular the endomorphisms of $\mathrm{Ind}_{H \to G}(1)$ are given by elements of $\mathbb{Z}[H \backslash G/H]$, which is called a Hecke algebra.

[redacted] Let's compute the effect of the double coset KxH on the coset yH. The result should be a subset of G/K. Precisely the double coset HxK belongs to the G-orbit of (xK, H) in $G/K \times G/H$. So we have

$$\begin{tikzcd} yH/H\cap xKx^{-1} \arrow[r, "\sim"] & \{yh(xK,H)\} \arrow[r] \arrow[d, phantom, "\cap"] & yH \arrow[d, phantom, "\cap"] \\ G/H\cap xKx^{-1} \arrow[r, "\sim"] & \{g(xK,H)\} \arrow[r] \arrow[d] & G/H \\ & G/K & \end{tikzcd}$$

and so what we get is the family of [redacted] cosets

$$\{yhxK \mid h \in H/H\cap xKx^{-1}\}.$$

As a check note that $yh_1xK = yh_2xK \iff yh_1x \in yh_2xK \iff h_2^{-1}h_1 \in xKx^{-1}$. So we conclude the following: The effect of the double coset HxK on yH is the subset

$$yH \cdot HxK = yHxK \quad \text{of } G/K.$$

Next we compute the composition of the map associated to HxK and the one associated to KyL. Start with the coset zH. This goes to the family of cosets

$$zHxK = \{zhxK \mid h \in H/H\cap xKx^{-1}\}$$

A typical one of these, say $zhxK$ goes under KyL to the family

$$zhxKyL = \{zhxkyL \mid k \in K/K\cap yLy^{-1}\}.$$

So the result is the set of cosets

$$zhxkyL \qquad h \in H/H\cap xKx^{-1} \quad k \in K/K\cap yLy^{-1}.$$

except these have to be treated with attention to multiplicity. So we find the map belonging to double cosets of the form

$$HxkyL$$

except for multiplicity. It's clear we don't want to distinguish such double cosets for k's which differ by right multiplication by something in $K\cap yLy^{-1}$. Similarly we don't want to distinguish if k changes by left multiplying by an element of $K\cap x^{-1}Hx$. Hence I guess that the composition is given by

$$\boxed{HxK \cdot KyL = \sum_{k\in K\cap x^{-1}Hx\backslash K/K\cap yLy^{-1}}{}' HxkyL}$$

not quite correct see next p.

where this sum involves multiplicities.

Let's try to get at this formula differently. The double cosets HxK, KyL represent the correspondences

$$\begin{tikzcd}[column sep=small] & G/K\cap x^{-1}Hx \arrow[dl, "x^{-1}"'] \arrow[dr] & & G/K\cap yLy^{-1} \arrow[dl] \arrow[dr, "y"] & \\ G/H & & G/K & & G/L \end{tikzcd}$$

To compose we form the fibre product

(*) $\quad G/K_1 \times_{G/K} G/K_2 \qquad\qquad K_1 = K \cap x^{-1}Hx,\ K_2 = K \cap yLy^{-1}$

Break this into orbits. $G(xK_1, yK_2) \longleftrightarrow K_2 y^{-1} x K_1$ for $G/K_1 \times G/K_2$. For this to be in the fibre product over G/K we want $xK = yK$ or $y^{-1}x \in K$. Thus the G-orbits of (*) are

$$G(kK_1, K_2) \xleftarrow{\sim} G/K_2 \cap kK_1k^{-1}$$

for each double coset $K_2 k K_1$. Thus the composition of the correspondences is given by the sum over double cosets $K_2 k K_1$ of the correspondence

$$\begin{tikzcd} & G/K_2 \cap kK_1k^{-1} \arrow[dl, ".x^{-1}k"'] \arrow[dr, ".y"] & \\ G/H & & G/L \end{tikzcd} \qquad K_2 \cap kK_1k^{-1} = K \cap yLy^{-1} \cap kx^{-1}Hxk^{-1}$$

or better

$$\begin{tikzcd} & G/xKx^{-1} \cap H \cap xk^{-1}yLy^{-1}kx^{-1} \arrow[d] & \\ & G/H \cap xk^{-1}yLy^{-1}kx^{-1} \arrow[dl] \arrow[dr, "xk^{-1}y"] & \\ G/H & & G/L \end{tikzcd}$$

So in fact I see there is a multiplicity which has to be added to the formula on the preceding page.

$$\boxed{HxK \cdot KyL = \sum_{(K \cap x^{-1}Hx)k(K \cap yLy^{-1})} [x^{-1}Hx \cap yLy^{-1} : K \cap x^{-1}Hx \cap yLy^{-1}]\ HxkyL}$$

A simpler version of this formula seems to be to take the image with multiplicity of the map

$$HxK \times^K KyL \longrightarrow G$$

~~In effect $HxK \cong (K \cap x^{-1}Hx)\backslash K \qquad K/K \cap yLy^{-1} \cong KyL$~~

July 11, 1981

Problem: To get the Hecke algebra straight.

Think of $\mathrm{Ind}_{H\to G}(M)$ as the space of functions $s : G \to M$ with $s(hx) = h\, s(x)$ and $(gs)(x) = s(xg)$:

$$\mathrm{Ind}_{H\to G}(M) = \mathrm{Map}_H(G, M)$$

(This is the function viewpoint.)

Take $M = 1$. If $f : K\backslash G/H \to \mathbb{C}$, then f gives rise to a G-homomorphism

$$\begin{array}{ccc} \mathrm{Ind}_{H\to G}(1) & \longrightarrow & \mathrm{Ind}_{K\to G}(1) \\ s & \longmapsto & fs \end{array} \qquad (fs)(x) = \int_{y\in H\backslash G} f(xy^{-1})\, s(y)$$

and according to Mackey's formula all G-homomorphisms are in 1-1 correspondence with such functions f.

It seems to be useful to think of f as a function on G left-invariant under K, right-invariant under H. For example, f could be the characteristic function of a double coset KxH.

Let $f' : L\backslash G/K \to \mathbb{C}$. Then

$$(f'f\, s)(x) = \int_{y\in K\backslash G} f'(xy^{-1}) \int_{z\in H\backslash G} f(yz^{-1})\, s(z)$$

$$= \int_{z\in H\backslash G} \left[\int_{y\in K\backslash G} f'(xy^{-1}) f(yz^{-1}) \right] s(z)$$

and it should be possible to see the quantity in brackets as being in the form $(f' \underset{K}{*} f)(xz^{-1})$ for some $f' \underset{K}{*} f : L\backslash G/H \to \mathbb{C}$. This is clear since the bracketed quantity is invariant under $(x,z) \mapsto (xg, zg)$. So we have

$$(f' *_K f)(x) = \int_{y\in K\backslash G} f'(xy^{-1})\, f(y)$$

and the good way to interpret this is as follows: One has the multiplication map

$$\begin{array}{ccc} G \times^K G & \xrightarrow{\mu} & G \\ (g', g) & \longmapsto & g'g \end{array}$$

and the function $f' \otimes f : (g', g) \longmapsto f'(g') f(g)$ on $G \times^K G$.
Then

$$(\mu_*(f' \otimes f))(x) = \int\limits_{\substack{g'g = x \\ (g',g) \in G \times^K G}} f'(g') f(g) = \int\limits_{y \in K|G} f'(xy^{-1}) f(y)$$

where we use the fact that

$$\begin{array}{ccc} G \times^K G & \xrightarrow{\sim} & G \times K \backslash G \\ (g', g) & \longmapsto & (g'g, Kg) \end{array}$$

The above is very clean. The composition of the homomorphism induced by $f : K\backslash G/H \longrightarrow \mathbb{C}$ followed by the homom. induced by $f' : L\backslash G/K \longrightarrow \mathbb{C}$ is induced by the convolution $f' *_K f$ which is the image of $f' \otimes f$ on $G \times^K G$ under $\mu : G \times^K G \longrightarrow G$.

Consequently if I take the characteristic functions of two double cosets KaH, LbK the composition is given by the direct image of $\mathbf{1}$ under multiplication

$$LbK \times^K KaH \longrightarrow LbKaH \subset G.$$

Now I should apply this to the Hecke algebra which is the double coset algebra $\mathbb{C}[B\backslash G/B]$, where B is a Borel in G. This will be the algebra of endos. of $\mathrm{Ind}_{B \to G}(1)$. The point is that the theory of Tits systems says that the double cosets are in 1-1 correspondence with elements of the Weyl group. So the idea is to try to establish an isomorphism of $\mathbb{C}[B\backslash G/B]$ with the group ring $\mathbb{C}[W]$.

One has to be a bit careful. Take the case of G_2, whence there are 2 double cosets B, BsB. I need to compute the image of

$$BsB \times^B BsB \longrightarrow G$$

with multiplicities.

$B = H \ltimes N$, $\quad BsB \xleftarrow{\sim} N \times sB$. $\qquad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

is the stabilizer of $\infty \in \mathbb{P}^1$. $\qquad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

So

$$\underbrace{BsB}_{(N\times\{s\}\times B)} \times^B \underbrace{BsB}_{(N\times\{s\}\times B)} \cong N \times sNs \times B$$

We want the image of this under multiplication to G counted appropriately. So let's compute the map

$$N \times sNs \longrightarrow G/B = \mathbb{P}^1$$

$$\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ b & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & a \\ & 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & \\ b & 1 \end{pmatrix}(\infty)}_{}$$

$$\begin{pmatrix} 1 & a \\ & 1 \end{pmatrix} \frac{1}{b} = a + \frac{1}{b}$$

If $a + \frac{1}{b} = \infty$, then $b = 0$ and a is arbitrary so there are q possibilities for $\infty \leftrightarrow B$

If $a + \frac{1}{b} = c \neq \infty$, then $b \neq 0$ and $a = c - \frac{1}{b}$ so there are $q-1$ possibilities for c. Thus we get

$$\boxed{BsB \cdot BsB = (q-1)BsB + qB}$$

in the Hecke algebra. Check numbers

$$\text{card } BsB = \text{card}(N \times B) = q(q-1)^2 q$$

$$\text{card } BsB \times^B BsB = \text{card}(N \times sNs \times B) = q^2(q-1)^2 q$$

so is

$$q^2 \text{ card } B = (q-1)q \text{ card } B + q \text{ card } B \quad \checkmark.$$

So if we put $x = BsB$, the Hecke algebra is

$\mathbb{C} \oplus \mathbb{C}x$ where $x^2 = (q-1)x + q$. This polynomial has roots $x = -1$, $x = q$. So the integral Hecke algebra is not isomorphic to the group ring $\mathbb{Z}[W]$, because one is lousy at primes dividing $q+1$, and the other is lousy at 2.

However it would be nice to get an isomorphism of $\mathbb{C}[B\backslash G/B]$ with $\mathbb{C}[W]$ in general, for then corresponding to irreducible repns. of Σ_n would [redacted] be irreducible pieces of $\mathrm{Ind}_{B\to G}(\mathbb{1})$.

Notice that if l divides $q-1$. Then one has the relation

$$x^2 = (q-1)x + q$$
$$\equiv 1 \qquad \mathrm{mod}\,(q-1).$$

so it might happen that we get the group ring. This same thing occurred with <u>cohomology</u> - nice behavior if $q \equiv 1$.

So let's try to work out the Hecke algebra for general n. One knows there is an increasing filtration on the Hecke algebra given by the canonical length on the Weyl group. So it should be true that if

$$w = s_{i_1} \cdots s_{i_l} \qquad \text{with } l(w) = l$$

then

$$Bs_{i_1}B \times^B Bs_{i_2}B \times^B \cdots\cdots \times^B Bs_{i_l}B \xrightarrow{\sim} BwB.$$

This shows that the double cosets Bs_iB generate the Hecke algebra. For GL_n one has $s_1, \ldots, s_{n-1}$ and if $|i-j| \geq 2$, then

$s_is_j = s_js_i$:

e.g.

$$s_1 = \begin{pmatrix} \boxed{\begin{matrix} & 1 \\ 1 & \end{matrix}} & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad s_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \boxed{\begin{matrix} & 1 \\ 1 & \end{matrix}} \end{pmatrix} \qquad \text{in } GL_4$$

Hence we know the double cosets Bs_iB, Bs_jB commute for $|i-j| \geq 2$.

What I am doing is to use the Coxeter system relations for Σ_n. It's generated by s_i $i = 1, \dots, n-1$ with relations $s_i^2 = 1$, $(s_i s_{i+1})^3 = 1$, $(s_i s_j)^2 = 1$ for $|i-j| \geq 2$; (this last says $s_i s_j = s_j s_i$). It remains to work out the analogue of $(s_i s_{i+1})^3 = 1$ in the Hecke algebra. This means we effectively have to compute in GL_3.

Look at Σ_3. It has the elements $e, s_1, s_2, s_1 s_2, s_2 s_1$ and the element $s_1 s_2 s_1 = s_2 s_1 s_2$ and these are the only reduced [redacted] words in the s_i. So if put $x_i = Bs_iB$, then the Hecke algebra has the bases $1, x_1, x_2, x_1x_2, x_2x_1$ and $x_1x_2x_1 = x_2x_1x_2$. I guess I can now compute anything I need [redacted] using the relations $x_i^2 = (q-1)x_i + q$; established already. For example

$$x_1x_2x_1x_2 = x_1x_1x_2x_1 = (q-1)x_1x_2x_1 + q\,x_2x_1.$$

For example, modulo $q-1$ we see that we have the group ring of Σ_3.

Now the obvious thing to try is to see if there is a simple linear change of variables. Put

$$\sigma = \frac{2x + 1 - q}{1+q}$$

Then

$$\sigma^2 = \frac{4x^2 + 4(1-q)x + (1-q)^2}{(1+q)^2}$$

$$= \frac{4[(q-1)x + q] + 4(1-q)x + 1 - 2q + q^2}{(1+q)^2}$$

$$= 1$$

So what I'd like is for $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Say $\sigma_i = a x_i + b$.

Then
$$\sigma_1\sigma_2\sigma_1 = (ax_1+b)(ax_2+b)(ax_1+b)$$
$$= a^3 x_1x_2x_1 + a^2b(x_1x_2 + x_1^2 + x_2x_1)$$
$$+ b^2a(x_1 + x_2 + x_1) + b^3$$

One wants this not to change if 1, 2 are interchanged.

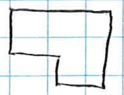
This is true only if
$$a^2b(x_1^2) + b^2a(x_1) = ab\left[\overbrace{\frac{2}{1+q}}^{a}((q-1)x_1+q) + \overbrace{\frac{1-q}{1+q}}^{b}x_1\right]$$
is a constant independent of x_1. Doesn't work.

What was the reason behind wanting the Hecke algebra to be the group ring. The Hecke alg. is the ring of endos. of the induced representation $\mathrm{Ind}_{B\to G}(1)$. It is therefore a direct product of matrix rings one for each irreducible constituent of the induced repn, the size of the matrix ring being the multiplicity. So if the Hecke alg. is isomorphic to $\mathbb{C}[W]$, then we get a 1-1 correspondence between irreducibles of W and irreducible constituents of $\mathrm{Ind}_{B\to G}(1)$, the dimension of the W repns. [redacted] being the multiplicity of the corresponding component.

In the case of GL_3, the Hecke algebra over $\mathbb{C}$ has dimension 6 and is non-abelian, hence the only possibility is $6 = 2^2+1+1$, and the Hecke alg. is isomorphic to the group ring of Σ_3.

July 13, 1981

I think we can see now why Green's multiplication is commutative. Take the parabolics

$P = \begin{matrix} a \\ b \end{matrix}$  $\qquad Q =$ 

The double cosets PQ, QP give rise to homomorphisms

$$\mathrm{Ind}_{P\to G}(1) \underset{QP}{\overset{PQ}{\rightleftarrows}} \mathrm{Ind}_{Q\to G}(1)$$

and we want to show these are isomorphisms, however they are not inverses of each other. Let's compute

$$PQ \times^Q QP.$$

Now $\quad PQ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \times Q \qquad$ where $* =$ all $a\times b$ matrices

$\qquad QP = \begin{pmatrix} 1 & 0 \\ *' & 1 \end{pmatrix} \times P \qquad$ where $*' =$ all $b\times a$ matrices

so

$$PQ \times^Q QP = \Big\{ \underbrace{\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}}_{\begin{pmatrix} 1+AB & A \\ B & 1 \end{pmatrix}} \times P \Big\}$$

The image of this in G is a union of double cosets for P with certain multiplicities. What are the double cosets $P \times P$? They describe the orbits of P ($=$ stabilizer of $\mathbb{C}^a \subset \mathbb{C}^{a+b}$) on the set of a-diml subspaces. The only invariant of a W^a is the dimension of $\mathbb{C}^a \cap W^a$, or better, $\dim (W^a/\mathbb{C}^a \cap W^a)$. The subspace W^a belonging to the coset

$$\begin{pmatrix} 1+AB & A \\ B & 1 \end{pmatrix} \times P$$

is spanned by the column vectors of $\begin{pmatrix} 1+AB \\ B \end{pmatrix}$
and $W^a/\mathbb{C}^a \cap W^a \subset \mathbb{C}^b$ is spanned by the colums of B. In principle I should be able to compute the multiplicities, but it looks ugly.

Let's proceed. The double cosets are described by the rank of B. The first case is where $B=0$, whence we get the double coset $PP = P$. The multiplicity is the number of A which is q^{ab}. Another case is where $a=b$ and B is non-singular. In this case the subspace is complementary to $\mathbb{C}^a$, so B can be chosen arbitrary and A is uniquely determined; hence the multiplicity is $|GL_a| = q^{a(a-1)/2}(q-1)(q^2-1)\cdots(q^a-1)$.

It should happen that for $B \neq 0$ the multiplicity is divisible by $q-1$. Assuming this we get that the composite $[PQ]\cdot[QP] \equiv 1 \mod q-1$ in the Hecke algebra $\mathbb{Z}[P|G/P]$. Now this alg. is a finitely generated free module over $\mathbb{Z}$, hence any element has a norm defined by determinant, and an element is invertible in $\mathbb{Q}[P|G/P]$ when its norm is $\neq 0$. Hence the element $[PQ][QP]$ has to be invertible over $\mathbb{Q}$ because its norm is 1 modulo $(q-1)$.

All this however is too hard.

<u>July 14, 1981</u>

Idea of discrete series repns: Consider the decomposable subspace of $R(G_n)$ for Green's multiplication, i.e. the image of

$$\bigoplus_{\substack{a+b=n \\ a,b>0}} R(G_a)\otimes R(G_b) \longrightarrow R(G_n)$$

It turns out that the orthogonal complement of this decomposable subspace has a nice description, namely, it is spanned by certain irreducible representations, the so-called discrete series representations. Let's work this out.

Put $G=G_n$. The decomposable subspace of $R(G)$ is spanned by representations of the form $\mathrm{Ind}_{P\to G}(W)$ where [redacted] P is a proper parabolic subgroup of G and W is an irreducible representation of P/P^u. Let V be an irreducible representation of G. By reciprocity

$$\begin{aligned}\langle \mathrm{Ind}_{P\to G}(W), V\rangle_G &= \langle W, \mathrm{Res}_{P\to G}(V)\rangle_P \\ &= \langle W, (\mathrm{Res}_{P\to G}(V))^{P^u}\rangle_{P/P^u}\end{aligned}$$

Therefore V will be orthogonal to the decomposable subspace of $R(G)$ $\iff$ $(\mathrm{Res}_{P\to G}(V))^{P^u}=0$ for all proper parabolics P.

Let's be a little more careful. Given a central function f on G, one can obtain a central function on P/P^u by restricting f to P and then integrating over P^u.

$$f \longmapsto \int_{y\in P^u} f(py)$$

and this is how one goes from χ_V to the character of $[\mathrm{Res}_{P\to G}(V)]^{P^u}$. The orthogonal complement to the decomposable space consists of all $\alpha\in R(G)$ whose character functions give zero under this map for all proper P.

The goal will be to show this orthogonal complement is spanned by irreducibles. At the same time one wants to establish a stronger result as follows: One gets generators for $\bigoplus R(G_n)$ out of the discrete series repns. on each of the G_a. One wants to prove that one gets a commutative polynomial ring with these generators. In other words if one takes an ordered partition

$$n = a_1 + \cdots + a_k$$

and discrete series reps. W_i of G_{a_i}, then the representation obtained by taking the Green product of the W_i, $i = 1, \ldots, k$, should be independent of the order, and all these products should be a basis for $R(G_n)$.

July 15, 1981

It seems necessary to work on the level of representations. So let M be a G-module. If M is contained in an induced module from a repn. W of P/P^u, then we have an epim.

$$\mathbb{C}[G] \otimes_{\mathbb{C}[P]} W \simeq \mathrm{Ind}_{P \to G}(W) \twoheadrightarrow M$$

so M is generated by the image of W, which is pointwise fixed under P^u. Thus $M = \mathbb{C}[G](M^{P^u})$. Conversely if M is generated as a G-module by M^{P^u}, then because $P^u \triangleleft P$, M^{P^u} is a repn. of P and so we get an epim.

$$\mathbb{C}[G] \otimes_{\mathbb{C}[P]} (M^{P^u}) \twoheadrightarrow M$$

Consequently for any repn. M of G, there is a largest submodule contained in a module induced from P/P^u, namely $\mathbb{C}[G](M^{P^u})$. A module is a direct sum of discrete series representations exactly when $M^{P^u} = 0$ for all proper parabolics P.

[redacted] Notice that the submodule $\mathbb{C}[G]M^{P^u}$ depends only on the conjugacy class of P. $\therefore$ Only finitely many.

So the next step is to let P vary and to consider the inclusions among the submodules $\mathbb{C}[G]M^{P^u}$. The smallest of these submodules occurs when $P = B$ a Borel, because when $P \subset Q$ we have $P^u \supset Q^u$. (Re[redacted]view:
P corresponds to a flag: it is the stabilizer of a flag and $P^u =$ those autos inducing the identity on gr wrt. the flag. So $P \subset Q$ ~~means~~ the flag of Q is part of the flag of P so that $Q^u \subset P^u$.)

Let's consider then the first case of interest, namely, where M is generated by M^{B^u}, where B is a Borel. M^{B^u} is a [redacted] repn. of B/B^u which is a torus, whose irred. reps. are characters. [redacted] So we get a class of representations, namely, those which are contained in the induced repn. from a character of B. One should understand the decomposition into irreducibles of these induced representations in particular when two characters of B give isomorphic representations.

So take $\chi : B \to \mathbb{C}^*$ and consider

$$\mathrm{Ind}_{B\to G}(\chi) = \{ f : G \to \mathbb{C} \mid f(bx) = \chi(b) f(x) \}$$

Obviously we want to know the [redacted] module over B/B^u

$$\left(\mathrm{Ind}_{B\to G}(\chi)\right)^{B^u}$$

because this will tell us about maps between these induced representations. Now

$$\left(\mathrm{Ind}_{B\to G}(\chi)\right)^{B^u} = \left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} f(bx) = \chi(b) f(x) \\ f(xb) = f(x) \end{array} \right\}$$

Actually it seems to be more sensible to fix another character χ'

of B and then look at

$$\operatorname{Hom}_B\left(\chi', \operatorname{Ind}_{B\to G}(\chi)\right) = \left\{ f \;\middle|\; \begin{array}{l} f(bx) = \chi(b) f(x) \\ f(xb) = f(x)\chi'(b) \end{array} \right\}$$

Such an f is determined on representatives for the double cosets BxB, hence by elements w of the Weyl group such that w transforms χ into χ' in some sense. Hence one can see that $\operatorname{Ind}_{B\to G}(\chi)$ is irreducible exactly when the character χ is not fixed under non-identity elements of the Weyl group.

Nice formula:

$$\langle \operatorname{Ind}_{B\to G}(\chi'), \operatorname{Ind}_{B\to G}(\chi) \rangle = \text{no. of } w \text{ in Weyl gp with } \chi' = \chi^w.$$

Next let us remove from a G-module M the part generated by M^{B^u} and let's consider a parabolic P just above B (i.e. of form $B \cup Bs_iB$) such that $M^{P^u} \neq 0$. Say in fact that P is a parabolic such that $M^{P^u} \neq 0$, but that for all parabolic $Q < P$ one has $M^{Q^u} = 0$. Then clearly M^{P^u} is a discrete series representation of P/P^u, because we know the parabolics of P/P^u are of the form Q/P^u, where $Q \subset P$, e.g.

$$P = \begin{pmatrix} * & * \\ & * \end{pmatrix} \qquad Q = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$$

then

$$P^u = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \qquad Q^u = \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$$

July 16, 1981

What I want to do is to take a representation M of G and then to filter it by the pieces

$$\mathbb{C}[G]\cdot M^{P^u}$$

as P ranges over the proper parabolics of G. I want to identify the "gr" of this filtration with induced repns. from discrete series repns.

So take take case where M is generated by M^{P^u} and remove the submodules generated by M^{Q^u} for all $Q<P$. Then $M^{Q^u}=0$ for all $Q<P$ and so M^{P^u} is a discrete series repn. of P/P^u, because the parabolics of P/P^u are of the form Q/P^u for Q a parabolic in P.

Now the real question we have to worry about concerns whether $\mathrm{Ind}_{P\to G}(V)$ with V a d.s. repn. of P/P^u determines the subgroup P. For example, I certainly need to know that

$$\left(\mathrm{Ind}_{P\to G}(V)\right)^{Q^u}=0$$

for $Q<P$.

First one should understand the conjugacy classes of parabolics. For $G=GL_n$ one gets all non-empty subsets of $\{1,\dots,n-1\}$.

Note first that

$$\left(\mathrm{Ind}_{P\to G}(V)\right)^{Q^u}\neq 0$$

$\iff$ there is a non-zero map

$$\mathrm{Ind}_{Q\to G}(W)\longrightarrow \mathrm{Ind}_{P\to G}(V)$$

for some repn. W of Q/Q^u. Such a map is given by a function

$$f: G\longrightarrow \mathrm{Hom}(W,V)$$

satisfying

$$f(pxq) = p \cdot f(x) \cdot q \qquad p \in P,\ q \in Q.$$

This condition says that f is separately determined on each double coset PxQ as we already know. The condition over this double coset is that

$$f(x) \in \mathrm{Hom}_{P \cap xQx^{-1}}(W, V)$$

For example, if $x = e$, then for $g \in P \cap Q$ we must have

$$\begin{array}{ccccc} f(ge) & = & f(g) & = & f(eg) \\ \shortparallel & & & & \shortparallel \\ g\,f(e) & & & & f(e)g \end{array}$$

so $f(e) : W \longrightarrow V$ is a $P \cap Q$-homomorphism.

Now suppose that $f(e) \neq 0$. Because W is fixed under Q^u, it follows that the image of $f(e)$ is a subspace of V fixed under $P \cap Q^u$, as well as P^u. So if V is a discrete series repn. of P/P^u it follows that $\mathrm{Im}\,(P \cap Q^u \longrightarrow P/P^u)$ can't contain a unipotent radical of a parabolic of P/P^u.

Our goal is to establish disjointness of the repns. $\mathrm{Ind}_{P \to G}(V)$, where V is in the discrete series of P/P^u. Thus I want to see that the image of $P \cap Q^u$ in P/P^u contains a unipotent radical for lots of Q.

P, Q belong to two filtrations, and $P \cap Q$ is the stabilizer of the distributive lattice of subspaces built out of these two filtrations. Think of P as fixed. If Q is enlarged then Q^u shrinks, so the first thing is to consider the case where Q is the stabilizer of a single subspace.

Let's consider carefully the case where Q stabilizes the subspace A, P stabilizes the subspace B in the vector space L.

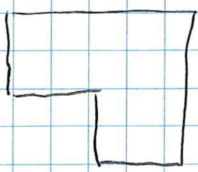
So we have the picture

$$L = \mathbb{C}^a \oplus \mathbb{C}^b \oplus \mathbb{C}^c \oplus \mathbb{C}^d$$

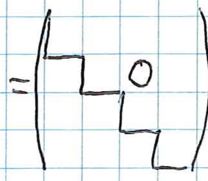
P stabilizes $\mathbb{C}^a \oplus \mathbb{C}^b$

Q stabilizes $\mathbb{C}^a \oplus 0 \oplus \mathbb{C}^c$

So

$P =$ 

$$Q = \begin{pmatrix} & & & \\ 0 & & 0 & \\ * & & & \\ 0 & & 0 & \end{pmatrix}$$

$P \cap Q =$ 

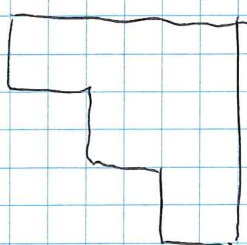
$$Q^u = \begin{pmatrix} 1 & * & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & * & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P \cap Q^u = \begin{pmatrix} 1 & * & 0 & * \\ & 1 & 0 & 0 \\ & & 1 & * \\ & & & 1 \end{pmatrix}$$

In this example, one sees that if either $ab \neq 0$, or $cd \neq 0$ that $\mathrm{Im}(P \cap Q^u \longrightarrow P/P^u)$ contains a non-trivial unipotent radical. What happens if $ab = 0$ and $cd = 0$? Note $a+b > 0$, $c+d > 0$ $a+c > 0$, $b+d > 0$.

If $a = 0$, then $c \neq 0 \Rightarrow d = 0 \Rightarrow L = \underbrace{\mathbb{C}^b}_{\text{st. by } P} \oplus \underbrace{\mathbb{C}^c}_{\text{st. by } Q}$

If $b = 0$, then $a \neq 0, d \neq 0 \Rightarrow c = 0. \Rightarrow P = Q$

So we see the following: that $\mathrm{Im}(P \cap Q^u \to P/P^u)$ contains a unipotent radical unless the two subspaces coincide or are complementary. Next let's try a 3 step filtration for P

$P =$ 

$$Q = \begin{matrix} * & * & * & * & * & * \\ 0 & * & 0 & * & 0 & * \\ * & * & * & * & * & * \\ 0 & * & 0 & * & 0 & * \\ * & * & * & * & * & * \\ 0 & * & 0 & * & 0 & * \end{matrix} \qquad Q^u = \begin{matrix} 1 & * & 0 & * & 0 & * \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & * & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & * & 0 & * & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$P \cap Q^u = \begin{matrix} 1 & * & 0 & * & 0 & * \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & * & 0 & * \\ & & & 1 & 0 & 0 \\ & & & & 1 & * \\ & & & & & 1 \end{matrix}$$

Therefore it seems to be true that

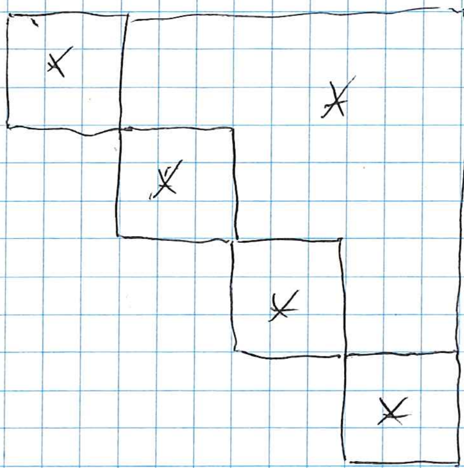
$$P \cap Q / P^u \cap Q \hookrightarrow P/P^u$$

is a parabolic of P/P^u and that its unipotent radical is

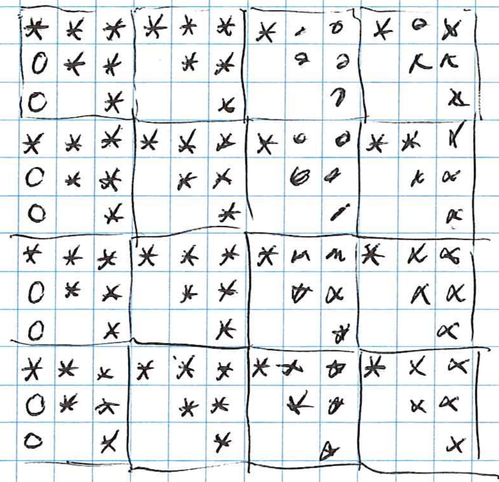
$$P \cap Q^u / P^u \cap Q^u \hookrightarrow P \cap Q / P^u \cap Q$$

I can see this from the viewpoint of block matrices as follows.

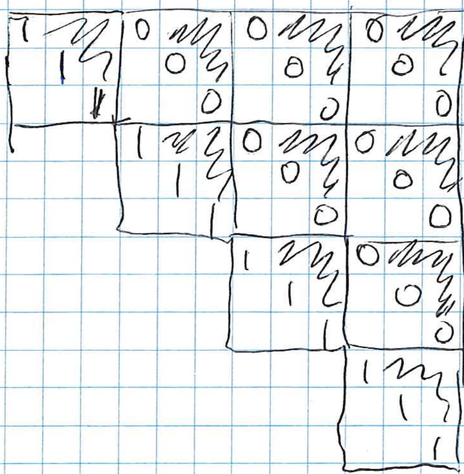
P :



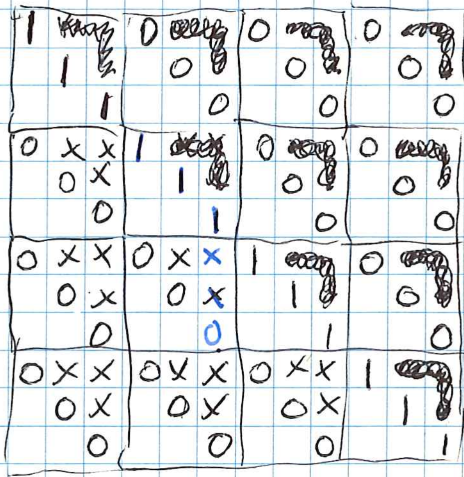
Q ;



$P \cap Q^u$:



Q^u :



So let's assume this assertion above to be true and see what the consequences are. When is it true that the unipotent radical of $P\cap Q/P^u \cap Q \subset P/P^u$ is trivial? It must be the case that $P \cap Q / P^u \cap Q = P/P^u$. In terms of flags, each quotient for the P-filtration must appear in exactly one quotient for the Q-filtration

For example if $P \subset Q$, then one has $P \cap Q^u = Q^u \subset P^u$ has image 1 in P/P^u.

Let's summarize where we presently are. I have two parabolics P, Q and discrete series repns. V, W of P/P^u, Q/Q^u respectively. I am computing

$$\mathrm{Hom}_G\left(\mathrm{Ind}_{Q\to G}(W), \mathrm{Ind}_{P\to G}(V)\right) = \left\{ f: G \to \mathrm{Hom}(W, V) \,\middle|\, \begin{array}{l} f(pxq) = \\ pf(x)q \end{array} \right\}$$

and I know that it is a sum over double cosets [redacted] PxQ of

$$\mathrm{Hom}_{P\cap xQx^{-1}}(W, V)$$

(Check $f(x)q = f(xq) = f((xqx^{-1})x) = (xqx^{-1})f(x)$ if

$$q \in Q \cap x^{-1}Px$$

so better $\quad pf(x) = f(px) = f(x(x^{-1}px)) = f(x)(x^{-1}px)$

if $\quad p \in P \cap xQx^{-1}$.)

Now [redacted] fix a double coset PxQ, and let us suppose that

$$\mathrm{Hom}_{P\cap xQx^{-1}}(W, V) \neq 0$$

say, there is a non-zero φ here. Now W is fixed under Q^u, so $\varphi(W)$ is a non-zero subspace of V fixed under

$$P \cap (xQ^u x^{-1}) / P^u \cap (xQ^u x^{-1}) \hookrightarrow P/P^u$$

But because of what we've seen above this subgroup is usually a non-trivial unipotent radical. Thus since V is a discrete series rep, we must have that this unipotent radical is trivial, which means that the [redacted] quotients of the flag of P appear inside the quotients of the flag of xQx^{-1}.

But now if we consider the orthogonal complement of the Kernel of φ, this subspace injects into V, hence is fixed under $P^u \cap xQx^{-1}$. So because W is a disc. series repn. of Q/Q^u, we conclude that the quotients of the flag of xQx^{-1} appear in the quotients of the flag of P.

To simplify notation let's replace xQx^{-1} by Q. We then have that

$$P\cap Q / P^u \cap Q \simeq P/P^u$$

because $P \cap Q / P^u \cap Q$ is a parabolic with 0 unipotent radical. Thus

$$P^u \cap Q = \text{unip. radical } P \cap Q$$

and so a Levi factor for P can be found in $P \cap Q$. Similarly one has

$$P \cap Q^u = \text{unip. radical of } P \cap Q$$

so that when a non-zero $\varphi \in \mathrm{Hom}_{P \cap Q}(W, V)$ exists we see that $P \cap Q$ contains a Levi subgroup which is also a Levi subgp. for P and for Q.

July 19, 1981

We have gone over the fact that any repn. M of $G_n = GL_n(\mathbb{C})$ has a canonical decomposition into pieces indexed by partitions α of n. Such a partition corresponds to an equivalence class of associated parabolics, and in the case of G_n one can think of the partition α: $n = n_1 + \cdots + n_r$, $n_1 \geq n_2 \geq \cdots \geq n_r$ as determining the reductive subgroup $G_\alpha = G_{n_1} \times \cdots \times G_{n_r}$. The piece of M belonging to α consists of all irred. subrepns. of M which occur in induced representations from an discrete series repn. of G_α.

Thus we have

$$R(G_n) = \bigoplus_{\alpha \vdash n} \underbrace{R(G_n)_\alpha}$$

spanned by irreducibles occurring in repns. of the form $\mathrm{Ind}_{P \to G}(V)$ where $P/P^u = G_\alpha$ and V is a discrete series repn of G_α.

Example: G_2. Here $G_1 \times G_1 = \mathbb{F}_q^* \times \mathbb{F}_q^*$ has irred. repns given by characters $\chi_1 \boxtimes \chi_2$. We know that if these are induced up to G_2 we get irred reps for $\chi_1 \neq \chi_2$, but if $\chi_1 = \chi_2 = \chi$, then the character

$$G_2 \xrightarrow{\det} G_1 \xrightarrow{\chi} \mathbb{C}^*$$

splits off. Thus we get a total of repns.

$$\chi_1 \neq \chi_2 \qquad \frac{(q-1)^2 - (q-1)}{2} \text{ repn. of dim } q+1$$

$$\chi_1 = \chi_2 \qquad (q-1) \text{ repns. of dim } 1, \text{ and } q$$

leaving over

$$q(q-1)(q^2-1) - \frac{(q-1)^2-(q-1)}{2}(q+1)^2 - (q-1)(1+q^2)$$

$$= (q-1)\left[q^3 - q - \frac{(q-2)}{2}(q^2+2q+1) - 1 - q^2\right] = (q-1)\left[\frac{q^3}{2} - q^2 + \frac{1}{2}\right]$$

$$\frac{1}{2}\left[q^3 + \cancel{2q^2} + q - \cancel{2q^2} - 4q - \cancel{2}\right]$$

$$= \frac{q(q-1)}{2}(q-1)^2 \qquad \frac{q(q-1)}{2} \text{ discrete series repn. of dim } q-1.$$

So therefore in the Green algebra $\bigoplus R(G_n)$, in $R(G_2)$ one has what sort of products? $R(G_1)$ has the basis of $\chi : \mathbb{F}_q^* \to \mathbb{C}^*$ and there are $q-1$ of these characters. Then $R(G_2)$ [redacted] 2 [redacted] has the basis consisting of the $\frac{q(q-1)}{2}$ discrete series representations. $R(G_2)_{1,1}$ contains the monomials

$\chi_1 \chi_2$ which are basis elts for $\chi_1 \chi_2$

but if $\chi_1 = \chi_2 = \chi$, then

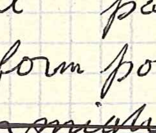
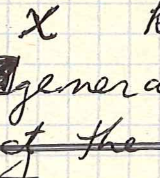
$$\chi^2 = \chi \circ \det + \text{basis element}.$$

So the good conjecture is that for each discrete series [redacted] representation χ and integer $k \geq 1$ there is a irred. repn. $\chi_{(k)}$ of degree $= k \cdot \deg \chi$, which is the "dominant" part of $\chi \cdots \chi$ k-times. Then these elements χ_k form polynomial [redacted] generators for the Green algebra. ~~One might even expect the monomials in these generators give all the irreducible~~

Notice that even in the case of G_2, the discrete series representations don't tell very much, because one does not get the new representations that seem to belong to non-semi-simple conjugacy classes. These one has to get by decomposing the induced representations.

Back to the commutativity of the Green product.

Fix

$P =$  $\quad Q =$ 

so that $\qquad P \cap Q = P/P^u = Q/Q^u$. For any repn. V of this group $P \cap Q$, call it $\bar{G}$, we get a canonical map

$$\mathrm{Ind}_{Q \to G}(V) \longrightarrow \mathrm{Ind}_{P \to G}(V)$$

defined by the function

$$F : G \longrightarrow \mathrm{Hom}(V, V)$$

specified by the conditions:

$$\begin{cases} F = 0 & \text{on double cosets } PxQ \text{ outside of } PQ. \\ F(pq) = \bar{p}\bar{q} & \text{on } V. \end{cases}$$

(This makes sense since $\quad pq = p'q' \implies (p')^{-1}p = q'q^{-1} \in P \cap Q$

$\implies$ [redacted] $\overline{(p')^{-1}p} = \overline{q'q^{-1}}$ on V

$$(\bar{p}')^{-1}\bar{p} = \bar{q}'\bar{q}^{-1} \implies \bar{p}\bar{q} = \bar{p}'\bar{q}'.)$$

Question: Is this canonical map an isomorphism for all representations V?

Since it's a natural transformation on the category of $\bar{G}$-modules, once it's an iso. for V it is also an isomorphism for any direct summand of V. So it would [redacted] be enough to check for $V =$ reg. rep. Probably by a suitable induction it would be enough to do it for a discrete series repn.

Try $V =$ regular repn. $= \mathrm{Maps}(\bar{G}, \mathbb{C})$. Then it should be clear that

$$\mathrm{Ind}_{Q \to G}(V) = \mathrm{Ind}_{Q^u \to G}(1) = \mathrm{Maps}(Q^u \backslash G, \mathbb{C}).$$

$$\mathrm{Ind}^-_{P \to G}(V) = \mathrm{Maps}(P^u \backslash G, \mathbb{C})$$

and it should follows that the canonical map in question is given by

$$(*) \qquad (Ff)(x) = \int_{Q^u \backslash G} F(x\,[\text{redacted}]\,y^{-1})\, f(y)$$

where $F = 1$ on the double coset $P^u Q^u$ and 0 outside. In any case this map transforms functions on $Q^u \backslash G$ to functions on $P^u \backslash G$ since it is left P^u, right Q^u invariant. Also it commutes with G-action

$$(F(gf))(x) = \int F(xy^{-1})(gf)(y) \;[\text{redacted}] = \int_{y \in Q^u \backslash G} F(xy^{-1}) f(yg)$$

$$= \int_{y \in Q^u \backslash G} F(xgy^{-1}) f(y) = (Ff)(xg) = [g(Ff)](x)$$

Finally it commutes with the right $\bar{G}$-action. Thus if $h \in \bar{G} = P \cap Q$ one has

$$(h^* f)(y) = f(hy)$$

and

$$\begin{aligned}[F(h*f)](x) &= \int F(xy^{-1}) f(hy) \\ &= \int F(xy^{-1}h) f(y) \\ &= \int F(hxy^{-1}) f(y) \\ &= (h * Ff)(x).\end{aligned}$$

So since the regular $\bar{G}$ repn. generates all the representations, this map induce a natural transf for all $\bar{G}$-modules.

One can ask at least if the map ⊛ is injective. First compute the composition with the similar map going backwards. This means we need to determine the direct image of the map

$$\underbrace{P^u \cancel{Q^u} \overset{Q^u}{\times} Q^u P^u}_{P^u \times Q^u \times P^u} \longrightarrow G$$

$$\begin{pmatrix} 1 & A \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} = \begin{pmatrix} 1+AB & A \\ B & 1 \end{pmatrix}$$

doesn't help.

So ask whether ⊛ is injective.

$$0 = \int_{y \in Q^u \backslash G} F(xy^{-1}) f(y) = \int_{\substack{y \in Q^u \backslash G \\ xy^{-1} \in P^u Q^u}} f(y) \qquad xy^{-1} \in P^u Q^u \iff y \in Q^u P^u x$$

$$= \int_{y^{-1} \in x P^u Q^u / Q^u} f(y^{-1})$$

So if you have a function on G/Q^u which adds up to zero on all subsets of the form xP^uQ^u/Q^u as x runs over G you want to be able to conclude that $f = 0$. ?

July 22, 1981

[redacted] It would seem to be useful to go over Deligne's proof of the $\lambda^i\lambda^j$ identities. As a preliminary we want to understand the Grothendieck group of algebraic functors F from P_A to P_A where A is a ring. Here algebraic means that F is really defined from P_B to P_B for all [redacted] A-algebras B so as to be compatible with base change.

The simplest case to understand first is when A is a field of characteristic 0. Let's write K instead of A, and consider a functor $F: \mathrm{Vect}(K) \longrightarrow \mathrm{Vect}(K)$ which is not necessarily additive. I seem to recall that Dold-Puppe have an [redacted] approach to decomposing F as a sum of functors which are homogeneous.

Given F we see that $F(0) \to F(V) \to F(0)$ splits off [redacted] the constant functor with value $F(0)$. So let's assume that $F(0) = 0$. Then the composition

$$F(V)\oplus F(W) \longrightarrow F(V\oplus W) \longrightarrow F(V)\oplus F(W)$$

is the identity: $\qquad F(pr_j)\,F(in_i) = F(pr_j\, in_i) = \delta_{ji}$

Dold-Puppe define the cross-effect of F to be the part left over:

$$F_2(V,W) = F(V\oplus W) \Big/ F(V)\oplus F(W)$$

so for example if $F = S^2$ or Λ^2, then $F_2(V,W)$ is [redacted] bi-additive.

Further examples: Suppose F is additive. Then we can define a map of abelian groups

$$V = [redacted]\ \mathrm{Hom}(K,V) \longrightarrow \mathrm{Hom}(F(K),F(V))$$

In fact $F(K)$ has two K-module structures. Thus we get

$$F(K)\otimes_{\mathbb{Z}} V \longrightarrow F(V)$$

and it should in fact be a map

$$F(K)\otimes_K V \longrightarrow F(V)$$

where we use the inside K-module structure on $F(K)$.
Since $V \simeq K^n$ this map is an isomorphism.

Therefore an additive functor $Vect(K) \to Vect(K)$ over a field is simply given by a $K \otimes_{\mathbb{Z}} K$ module which is a finite-diml vector space over K for the left multiplication.

Call F [redacted] quadratic when $F_2(V, W)$ is biadditive. Note that $F_2(K, K)$ then has two right K-multiplications which commute, and a single left multiplication. [redacted] Furthermore we have

$$F_2(K, K) \otimes_{(K \otimes K)} (V \otimes W) \xrightarrow{\sim} F_2(V, W)$$

Next, how do we reconstruct F from F_2? First of all we have to assume F has no [redacted] 0th and 1st order pieces.

Simpler approach is to [redacted] try to get the complete picture as follows. Given V we can multiply by K^* on V and hence $F(V)$ is a representation of the group K^* over the field K. If F is homogeneous of degree d, we expect that for the integer $n \geq 1$ one can prove that

$$F(n) = n^d.$$

[redacted] and hence $F(\frac{n}{m}) = (\frac{n}{m})^d$. [redacted]

Let's suppose that

$$F = F_{(0)} \oplus F_{(1)} \oplus F_{(2)} \oplus \cdots\cdots$$

is a decomposition of F into homogeneous functors. Then we [redacted] would like that

$$F_{(d)}(V) = \cancel{\boxed{W_d \otimes_{\Sigma_d} V^{\otimes d}}} \left(W_d \otimes_{(K^{\otimes d})} (V^{\otimes d}) \right)^{\Sigma_d}$$

where W_d is a left K, right $K^{\otimes d}$ module compatible with Σ_d action. In practice, one assumes F has finite degree, or else one has to arrange that the W_d are like the exterior powers Λ^d, so that F is always finite-dimensional.

So if $K = \mathbb{Q}$, one conjectures that the homogeneous degree d functors from $Vect(\mathbb{Q})$ to $Vect(\mathbb{Q})$ are in 1-1 correspondence with representations of Σ_d over $\mathbb{Q}$.

In any case we get a model for the kind of functors we are interested in, namely, functors of the form

$$V \longmapsto W \underset{\Sigma_d}{\otimes} V^{\otimes d}$$

where W is a representation of Σ_d. We determined these [redacted] the Grothendieck group ~~of~~ these and found a polyn. ring generated by the exterior power functors.

Let's try next to get some idea of algebraic functors So let's start with an additive functor $F: \mathrm{Vect}(K) \to \mathrm{Vect}(K)$. We know this is [redacted] in the form

$$F(V) = F(K) \otimes_K V$$

where $F(K)$ is a $K \otimes_{\mathbb{Q}} K$-module finite diml. over $K \otimes 1$. Which of these functors are algebraic. In particular for such an F, we know that the map

$$K^* \longrightarrow \mathrm{Aut}_K(F(K))$$

or better $K \longrightarrow \mathrm{End}_K(F(K))$ must be algebraic. Thus $F(K)$ has to be an algebraic representation of the multiplicative group. So it must decompose into characters

$$F(K) = \bigoplus_{n \geq 0} F(K_n)$$

where on $F(K_n)$ we have

$$x \cdot k = k^n x$$

[redacted] Something is strange, for if $K = \mathbb{Q}$ the two structures on $F(K)$ coincide. NO, only $n = $ [redacted] 1 occurs, because on $F(K)$ right multiplication is additive.

For the same reason, $F_d(K, \cdots, K)$ must be simply a K-module if the degree d functor F_d is algebraic. So its clear that our algebraic functors are given by representations of symmetric groups.

Now it seems to me that because repns. of GL_n are classically determined by [redacted] representations of the symmetric groups, perhaps there is a way to [redacted] extend suitable repns. of GL_n to these functors F. It might be a combinatorial method that would work in the non-algebraic case.