January 13, 1981

In 2 dimensional Euclidean space $\mathbb{R}^2$ with volume $dx\,dy$ I have seen that a "pure" gauge field $A_x dx + A_y dy$ satisfies the field equations [redacted] when $F = \partial_x A_y - \partial_y A_x + [A_x, A_y]$ is fixed under parallel translation. This means that A can be gauge transformed to a field where F is constant and hence the action won't be finite unless $F = 0$, whence A can be gauge transformed to zero. Therefore in $\mathbb{R}^2$, unlike $\mathbb{R}^4$, there is no interesting pure gauge theory.

To get something interesting one has to add a "matter" field which will give sources for the gauge field. Coleman considers the "abelian Higgs model" which consists of a $U(1)$ gauge field A and a complex scalar field φ. One can think of A as the EM field and φ as a charged meson field. The meson mass[redacted] perhaps arises via the Higgs mechanism.

Work in the Euclidean setup with coords x, y on $\mathbb{R}^2$. Then $A = A_x dx + A_y dy$ is a purely imaginary 1-form. It's curvature is

$$dA = \underbrace{(\partial_x A_y - \partial_y A_x)}_{F}\, dx\, dy$$

and the part of the action due to A is proportional to

$$\int |F|^2 \qquad\qquad \int = \int dx dy$$

If A is varied by δA, then

$$\delta \int |F|^2 = \int \overline{\partial_x \delta A_y - \partial_y \delta A_x} \cdot F \quad + \text{c.c.}$$

$$= \int \delta \bar{A}_y (-\partial_x F) + \delta \bar{A}_x (\partial_y F) \quad + \text{c.c.}$$

$$= 2 \int \delta A_y (\partial_x F) - \delta A_x (\partial_y F)$$

The part of the action connecting A, ψ is

$$\int |D\psi|^2 = \int \sum_\mu |(\partial_\mu + A_\mu)\psi|^2$$

It's variation wrt A is

$$\delta \int |D\psi|^2 = \int \overline{\delta A_\mu \psi} \cdot D_\mu \psi + \text{c.c.}$$

$$= \int \delta A_\mu \left(-\bar{\psi} D_\mu \psi + \psi \overline{D_\mu \psi} \right)$$

Its variation wrt ψ is

$$\int \overline{D_\mu \delta\psi}\, D_\mu \psi + \text{c.c.} \qquad\qquad D_\mu = \partial_\mu - A_\mu$$

$$= \int \overline{\delta\psi} \left(-D_\mu D_\mu \psi \right) + \text{c.c.}$$

If we combine: $\frac{1}{2}\int |F|^2 + \int |D\psi|^2$ then the vanishing of the variation w.r.t. A leads to the equations

$$\boxed{\left(\partial_y F, -\partial_x F \right) = \bar{\psi} D_\mu \psi - \psi \overline{D_\mu \psi}}$$

This is a linear equation for A given ψ. The right side is somehow the current due to the field ψ, and it has two parts

$$\bar{\psi}\partial_\mu \psi - \psi \partial_\mu \bar{\psi} + 2A_\mu |\psi|^2$$

The total action is

$$\frac{1}{2}\int |F|^2 + \int |D\psi|^2 + \int \lambda (|\psi|^2 - a^2)^2$$

and setting the variation wrt $\psi = 0$ leads to

$$\left(\delta \int g (|\psi|^2 - a^2)^2 = \int \lambda\, 2(|\psi|^2 - a^2)\, \overline{\delta\psi}\, \psi + \text{c.c.} \right)$$

the field equation

$$\boxed{D_\mu D_\mu \psi = 2\lambda\left(|\psi|^2 - a^2\right)\psi}$$

Now I would like to find solutions of these field equations. First of all we should mention gauge transformations:

$$A, \psi \longmapsto A + g\,dg^{-1},\ g\psi$$

where $g : \mathbb{R}^2 \to S^1$. This obviously doesn't change the action.

According to Coleman one first looks for finite action solutions. It's necessary that $|\psi| \to a$ as $r \to \infty$, assuming $\lambda > 0$. However the phase in the limit is arbitrary. Suppose

$$\lim_{r\to\infty} \psi(r,\theta) = g(\theta)\,a \qquad g : S^1 \to S^1 \qquad |g| = 1.$$

Then $\quad D\psi = (d + A)\psi = (dg + Ag)a$, hence

$$A \sim g\,dg^{-1} \qquad \text{as } r \to \infty.$$

(Coleman argues that $|dg + Ag|$ behaves like an integral power of r as $r \to \infty$, in any practical case. If $dg + gA \sim O(\frac{1}{r})$, then $\int |\ \ |^2 \sim \int \frac{r\,dr\,d\theta}{r^2}$ which diverges logarithmically. "Hence" it must be $O(\frac{1}{r^2})$:

$$A = g\,dg^{-1} + O\left(\tfrac{1}{r^2}\right)$$

and hence $\quad dA = O(\frac{1}{r^3})$ will give rise to $\int |dA|^2 < \infty$.

Next he takes a winding number say 1 and the simplest possible g with this winding number i.e. $g(\theta) = e^{i\theta}$ and then looks for a solution with this angular dependence and a radial dependence to be found:

$$\psi(r,\theta) = f(r) e^{i\theta a}$$
$$A = -i\rho(r)\, d\theta$$

In the following we suppose $a = 1$.

$$iA = \rho\, d\theta = (\rho\,\partial_x\theta)\,dx + (\rho\,\partial_y\theta)\,dy$$

$$D_x = \partial_x - i\rho\,\partial_x\theta \qquad D_y = \partial_y - i\rho\,\partial_y\theta$$

$$d\theta = d\tan^{-1}\left(\frac{y}{x}\right) = \frac{-\frac{y}{x^2}dx + \frac{1}{x}dy}{1+\frac{y^2}{x^2}} = \frac{-y\,dx + x\,dy}{r^2}$$

$$\partial_x\theta = -\frac{y}{r^2} \qquad \partial_y\theta = \frac{x}{r^2}$$

$$D_x^2 = \partial_x^2 - 2i\rho(\partial_x\theta)\,\partial_x - \rho^2(\partial_x\theta)^2 - i\,\partial_x(\rho\,\partial_x\theta)$$
$$D_y^2 = \partial_y^2 - 2i\rho(\partial_y\theta)\,\partial_y - \rho^2(\partial_y\theta)^2 - i\,\partial_y(\rho\,\partial_y\theta)$$

$$\left(D_x^2 + D_y^2\right) f(r) = \Delta f - \frac{\rho^2}{r^2} f \qquad \text{since } \nabla f \cdot \nabla\theta = 0,\ \Delta\theta = 0$$

But I want to apply this to $f e^{i\theta}$

and
$$e^{-i\theta} D_\mu e^{i\theta} = \partial_\mu + A_\mu + i\,\partial_\mu\theta$$
$$= \partial_\mu - i\rho\,\partial_\mu\theta + i\,\partial_\mu\theta = \partial_\mu + i(1-\rho)\,\partial_\mu\theta$$

which has the effect of changing ρ to $\rho - 1$, hence

$$e^{-i\theta}\left(D_x^2 + D_y^2\right) e^{i\theta} f = \Delta f - \frac{(1-\rho)^2}{r^2} f$$

Next $\quad \bar{f}\left(\partial_\mu + i(1-\rho)\partial_\mu\theta\right)f - f\overline{\left(\partial_\mu + i(1-\rho)\partial_\mu\theta\right)f}$

$$= \underbrace{\left(\bar{f}\partial_\mu f - f\partial_\mu \bar{f}\right)}_{0 \text{ if } f \text{ is real}} + 2i(1-\rho)\partial_\mu\theta\,|f|^2$$

~~Thus the equation~~ $\cancel{D_\mu D_\mu \psi = \bar{\psi}(D_\mu\psi) - \psi\overline{(D_\mu\psi)}}$

~~becomes~~

$$\cancel{\Delta f - \frac{(1-\rho)^2}{r^2} f = 2i(1-\rho)\partial_\mu\theta}$$

Thus the equation $\quad D_\mu D_\mu \psi = 2\lambda(|\psi|^2 - 1)\psi \quad$ becomes

$$\boxed{\Delta f - \frac{(1-\rho)^2}{r^2} f = 2\lambda(|f|^2 - 1) f}$$

where $\quad \Delta f(r) = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} f$

$$iA = \rho\, d\theta$$

$$i\,dA = d\rho\, d\theta = \frac{1}{r}\frac{\partial\rho}{\partial r}\underbrace{r\,dr\,d\theta}_{dx\,dy}$$

$$\therefore iF = \frac{1}{r}\frac{\partial\rho}{\partial r}$$

Assuming f is real we thus get that the equation

$$(-\partial_y, \partial_x)F = \bar{\psi} D_\mu\psi - \psi\overline{D_\mu\psi} \qquad \left(-\frac{y}{r^2}, \frac{x}{r^2}\right)$$

becomes

$$(-\partial_y, \partial_x)\frac{1}{i}\underbrace{\frac{1}{r}\frac{\partial\rho}{\partial r}}_{h(r)} = 2i(1-\rho)f^2\left(\partial_x\theta, \partial_y\theta\right)$$

$$(-\partial_y, \partial_x)\,h(r) = h'(r)\left(-\frac{\partial r}{\partial y}, \frac{\partial r}{\partial x}\right)$$

$$= h'(r)\left(-\frac{y}{r}, \frac{x}{r}\right).$$

So things are consistent and we get

$$\boxed{r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\rho}{\partial r}\right) = -2(1-\rho)f^2}$$

January 18, 1981

Look at inverse scattering, and review work of '78.

Discrete case: We are given a unitary operator U on a Hilbert space $\mathcal{H}$ and "incoming" and "outgoing" representations

$$L^2(S^1, \tfrac{d\theta}{2\pi}) \xleftarrow{\text{in}} \mathcal{H} \xrightarrow{\text{out}} L^2(S^1, \tfrac{d\theta}{2\pi})$$

These satisfy $\quad \text{in}\, U = z\, \text{in} \quad$ and $\quad \text{in}\,\text{in}^* = id$, so that in is orthogonal projection onto a closed invariant subspace generated by $e_{in} = \text{in}^*(1)$. We have

$$\langle e_{in} | U^n e_{in} \rangle = \delta_{n0}$$

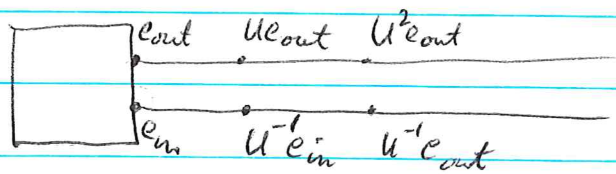
$$\text{in}(h) = \sum z^n \langle z^n | \text{in}(h) \rangle$$

$$= \sum z^n \langle \underbrace{\text{in}^*(z^n)}_{U^n e_{in}} | h \rangle = \sum z^n \langle U^n e_{in} | h \rangle.$$

Similarly for e_{out}. One puts

$$R = \text{out}(e_{in}) = \sum z^n \langle U^n e_{out} | e_{in} \rangle$$

The typical picture is that of a port connected to a line:



In this case R is a power series in z^{-1}, so it is analytic outside S^1.

Define

$$F_0\mathcal{H} = (\text{out}, \text{in})^{-1}(H^- \times H^+)$$

so that in the above picture it is the part of $\mathcal{H}$ supported in the port. This space is part of an increasing filtration

$$F_n\mathcal{H} = (\text{out}, \text{in})^{-1}(z^n H^- \times H^+)$$

whose union is $in^{-1}(H^+)$ and contains e_{in}. Put

$$\tilde{\alpha}_n = \text{projection of } e_{in} \text{ on } F_n\mathcal{H}. \qquad \alpha_n = \frac{\tilde{\alpha}_n}{\|\tilde{\alpha}_n\|}$$

Then we have the picture

$$U^{-n}\tilde{\beta}_n = \text{projection of } e_{out} \text{ on } U^{-n}F_n\mathcal{H}$$

(*)
$$\begin{tikzcd}[column sep=small]
 & & F_{n+1}\mathcal{H} \arrow[dl, dash, "\beta_{n+1}"'] & \\
 & F_n\mathcal{H} \arrow[dl, dash, "\beta_n"'] \arrow[dr, dash, "\alpha_n"] & & \\
F_{n-1}\mathcal{H} \arrow[dr, dash, "\alpha_{n-1}"] & & UF_{n-1}\mathcal{H} \arrow[dl, dash, "U\beta_{n-1}"] & \\
 & UF_{n-2}\mathcal{H} & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & U^{-1}F_1 \arrow[dl, dash, "U^{-1}\beta_1"'] \arrow[dr, dash] & & F_1 \arrow[dl, dash] \\
 & & F_0 \arrow[dl, dash, "\beta_0"'] \arrow[dr, dash] & \\
 & & & UF_{-1}
\end{tikzcd}$$

and
$$\alpha_{n-1} = \frac{\alpha_n - \beta_n\langle\beta_n|\alpha_n\rangle}{\sqrt{1-|\langle\beta_n|\alpha_n\rangle|^2}} \qquad U\beta_{n-1} = \frac{\beta_n - \alpha_n\langle\alpha_n|\beta_n\rangle}{\sqrt{1-|\langle\alpha_n|\beta_n\rangle|^2}}$$

$$\begin{pmatrix}1 & 0\\ 0 & U\end{pmatrix}\begin{pmatrix}\alpha_{n-1}\\ \beta_{n-1}\end{pmatrix} = \frac{1}{\sqrt{1-|h_n|^2}}\begin{pmatrix}1 & -h_n\\ -\bar{h}_n & 1\end{pmatrix}\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} \qquad h_n = \langle\beta_n|\alpha_n\rangle$$

or
$$\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} = \frac{1}{\sqrt{1-|h_n|^2}}\begin{pmatrix}1 & h_n\\ \bar{h}_n & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & U\end{pmatrix}\begin{pmatrix}\alpha_{n-1}\\ \beta_{n-1}\end{pmatrix}$$

Worth mentioning is the fact that if we [redacted] assume (out, in) injective, then $\mathcal{H}$ is spanned by $f(U)e_{in} + g(U)e_{out}$ with norm

$$\|fe_{in} + ge_{out}\|^2 = \|f\|^2 + \|g\|^2 + \langle g|Rf\rangle + \langle Rf|g\rangle$$
$$= \|Rf + g\|^2 + \langle f|(1-|R|^2)f\rangle$$

Thus $\|fe_{in} + ge_{out}\|^2 \geq$ ~~[illegible]~~ $\varepsilon\|f\|^2$ and similarly for g, which shows that provided $|R| \leq 1-\varepsilon$

$$(\text{out}, \text{in}) : \mathcal{H} \longrightarrow (L^2)^2$$

is a topological isomorphism. Hence each of the squares above as in $(*)$ are transversal, also

$$\bigcup F_n \mathcal{H} \quad \text{dense in} \quad in^{-1}(H^+)$$

which shows that $\alpha_n \longrightarrow e_{in}$ as $n \longrightarrow +\infty$.

Next look at what happens as $n \longrightarrow -\infty$.

$F_{-\infty}\mathcal{H} = (out, in)^{-1}(0 \times H_+) \subset Ker\,(out)$ which is spanned by elements $f(e_{in} - Re_{out})$ with norm

$$\| f(e_{in} - Re_{out})\|^2 = \langle f | (1-|R|^2) f \rangle$$

We have

$$\begin{tikzcd}
 & F_{\infty}\mathcal{H} \arrow[dl, dash] \arrow[dr, dash, "e_{in}"] & \\
F_{-\infty}\mathcal{H} \arrow[dr, dash, "\alpha_{-\infty}"'] & & UF_{\infty}\mathcal{H} \arrow[dl, dash] \\
 & UF_{-\infty}\mathcal{H} &
\end{tikzcd}$$

We obtain $\tilde{\alpha}_{-\infty}$ by projecting onto $F_{-\infty}\mathcal{H}$ (strictly, one removes a linear combination of β_n. We have

$$\|\tilde{\alpha}_{-\infty}\|^2 = \prod (1-|h_n|^2)$$

$$\boxed{\cancel{e_{in} = \tilde{\alpha}_{-\infty} + \beta \qquad \beta \in F_{\infty} \ominus F_{-\infty}}}$$

$$\boxed{\cancel{\text{Put} \quad \beta' = \beta - e_{in}\langle e_{in} | \beta \rangle \quad \text{proj}}}$$

Since $\|\tilde{\alpha}_{n-1}\| = \sqrt{1-|h_n|^2}\,\|\tilde{\alpha}_n\|$. ~~If~~ $\cancel{e_{in} = \tilde{\alpha}_{-\infty} + \beta}$ ~~where β is the appropriate linear combination of β_n, then also~~

$$\cancel{\tilde{\alpha}_{-\infty} = e_{in} - \beta \frac{1}{\|\beta\|^2}\langle \beta | e_{in} \rangle \implies \frac{1}{\|\beta\|^2}\langle \beta | e_{in} \rangle = 1}$$

$$\cancel{in(\tilde{\alpha}_{-\infty}) = 1 - in(\beta)\frac{1}{\|\beta\|^2}\langle \beta | e_{in} \rangle}$$

since $\quad \|\tilde{\alpha}_{n-1}\| = \sqrt{(1-|h_n|^2)}\ \|\tilde{\alpha}_n\|.$

If $\quad e_{in} = \tilde{\alpha}_{-\infty} + \beta \quad$ where β is the appropriate linear combination of the β_n, then

$$\langle e_{in} | \tilde{\alpha}_{-\infty} \rangle = \|\tilde{\alpha}_{-\infty}\|^2$$

$$\| \qquad\qquad \Longrightarrow \quad in(\alpha_{-\infty})(0) = \|\tilde{\alpha}_{-\infty}\|$$

$$in(\tilde{\alpha}_{-\infty})(0)$$

Next put $\quad \alpha_{-\infty} = h(e_{in} - Re_{out})$; this is possible since $out(\alpha_{-\infty}) = 0$. Then because $\alpha_{-\infty}$ is a unit vector in $F_\infty\mathcal{H} \ominus UF_\infty\mathcal{H}$ we have

$$\delta_{n0} = \langle \alpha_{-\infty} | U^n \alpha_{-\infty} \rangle = \langle h | z^n h(1-|R|^2) \rangle$$

so $\quad |h|^2(1-|R|^2) = 1.$ Thus if we put

$$T = in(\alpha_{-\infty}) = h(1-|R|^2) \in H^+$$

we have $\quad \bar{h}T = 1 \quad$ so $\quad h = \frac{1}{\bar{T}}$. Thus

$$|R|^2 + |T|^2 = 1$$

$$\alpha_{-\infty} = \frac{e_{in} - Re_{out}}{\bar{T}}$$

$$T(0) = \|\tilde{\alpha}_{-\infty}\| = \prod (1-|h_n|^2)^{1/2}$$

Also one knows T has no zeros for $|z|<1$, hence $\log|T|$ is a harmonic function in the disk with bdry values $\frac{1}{2}\log(1-|R|^2)$. So

$$\log T(0) = \int \frac{1}{2}\log(1-|R|^2)\frac{d\theta}{2\pi}$$

hence

$$\boxed{\prod (1-|h_n|^2) = \exp \int \log(1-|R|^2)\frac{d\theta}{2\pi}}$$

Next consider the continuous case. Suppose given $R(k)$ of modulus $\leq 1-\varepsilon$ and form a Hilbert space $\mathcal{H}$ of $f e_{in} + g e_{out}$ where $f, g \in L^2(\mathbb{R}, \frac{dk}{2\pi})$ with

$$\| f e_{in} + g e_{out} \|^2 = \|f\|^2 + \|g\|^2 + \langle g | Rf \rangle + \langle Rf | g \rangle$$

On $\mathcal{H}$ we have a 1-parameter unitary group $U(t) =$ mult. by e^{-ikt}. We use physics conventions, hence must change roles of $H^{\pm}$. In this situation H^+ is spanned by e^{ikx} with $x>0$, hence is analogous to span of z^{-n} since z "$=$" e^{-ik}.

Filtration is

$$F_x \mathcal{H} = (out, in)^{-1}\left(e^{-ikx} H^+ \times H^-\right).$$

This increases with closed union $in^{-1}(H^-)$. We define (heuristically) α_x as the projection of e_{in} onto $F_x\mathcal{H}$. Thus

$$\alpha_x = e_{in} + f_x e_{in} - \bar{g}_x e^{-ikx} e_{out}$$

where $f_x, g_x \in H^+$. Note that

$$(F_x\mathcal{H})^{\perp} = e^{-ikx} H^- e_{out} + H^+ e_{in}$$

so α_x differs from e_{in} by an element of $F_x\mathcal{H}^{\perp}$. For α_x to belong to $F_x\mathcal{H}$ means formally

$$out(\alpha_x) = R(1+f_x) - \bar{g}_x e^{-ikx} \in e^{-ikx} H^+$$

$$in(\alpha_x) = 1 + f_x - \bar{g}_x e^{-ikx} \bar{R} \in H^-$$

and we can make these precise as follows

$$\begin{cases} P_- R_x (1+f_x) = \bar{g}_x \\ f_x = P_+ \bar{R}_x \bar{g}_x \end{cases} \qquad R_x = R e^{ikx}$$

where $P_{\pm}$ are the projectors onto $H^{\pm}$ respectively.
We assume that $R \to \infty$ as $|k| \to \infty$ sufficiently fast. Solve

$$f_x = P_+ \overline{R}_x P_- R_x (1 + f_x)$$

$$\boxed{\begin{aligned} 1 + f_x &= \frac{1}{1 - \Gamma_x} |1\rangle \\ \overline{g}_x &= P_- R_x \frac{1}{1 - \Gamma_x} |1\rangle \end{aligned}} \qquad \Gamma_x = P_+ \overline{R}_x P_- R_x$$

Assuming that $|R|$ is bounded below 1, the operator Γ_x on L^2 has norm < 1 and so the Neumann series for $(1 - \Gamma_x)^{-1}$ converges.

Next we want to compute derivatives wrt x.

$$\Gamma_x = P_+ \overline{R} e^{-ikx} P_- e^{ikx} R$$

Take $f \in L^2(\mathbb{R}, \frac{dk}{2\pi})$, $\quad f = \int dy \, e^{iky} \underbrace{\langle e^{iky} | f \rangle}_{\hat{f}(y)}$

$$P_- e^{ikx} f(k) = \int_{-\infty}^{0} dy \, e^{iky} \hat{f}(y - x)$$

$$\boxed{\left(e^{-ikx} P_- e^{ikx} f\right)(k) = \int_{-\infty}^{-x} dy \, e^{iky} \hat{f}(y)}$$

$$\frac{d}{dx}\left(e^{-ikx} P_- e^{ikx} f\right) = -e^{-ikx} \hat{f}(-x) = -|e^{-ikx}\rangle\langle e^{-ikx} | f\rangle$$

Thus $\quad \dfrac{d}{dx} \Gamma_x = - P_+ \overline{R}_x |1\rangle\langle 1| R_x$

$$P_- R_x f = e^{ikx} \int_{-\infty}^{-x} dy \, e^{iky} \langle e^{iky} | R f \rangle$$

so $$\frac{d}{dx} P_- R_x \varphi = ik\, P_- R_x \varphi - |1\rangle\langle 1|R_x \varphi\rangle$$

Thus
$$\frac{d}{dx} f_x = \frac{d}{dx} \frac{1}{1-\Gamma_x} |1\rangle$$

$$= \frac{1}{1-\Gamma_x} \frac{d\Gamma_x}{dx} \frac{1}{1-\Gamma_x} |1\rangle$$

$$= \frac{1}{1-\Gamma_x} \left(- P_+ \bar{R}_x |1\rangle\langle 1 | R_x \right) \frac{1}{1-\Gamma_x} |1\rangle$$

Notice that
$$\bar{g}_x = \left(P_- R_x + P_- R_x P_+ \bar{R}_x P_- R_x + \cdots \right) |1\rangle$$

$$= P_- R_x \frac{1}{1 - \underbrace{P_+ \bar{R}_x P_- R_x}_{\Gamma_x}} |1\rangle \qquad \text{or}$$

$$= \frac{1}{1 - \underbrace{P_- R_x P_+ \bar{R}_x}_{\bar{\Gamma}_x}} P_- R_x |1\rangle$$

Thus
$$g_x = \frac{1}{1-\Gamma_x} P_+ \bar{R}_x |1\rangle$$

and we get
$$\boxed{\frac{df_x}{dx} = - g_x \langle 1 | R_x \frac{1}{1-\Gamma_x} | 1\rangle}$$

Next
$$\frac{d\bar{g}_x}{dx} = \frac{d}{dx}\left(P_- R_x \frac{1}{1-\Gamma_x} |1\rangle \right)$$

$$= ik\left(P_- R_x \text{———} \right) - |1\rangle\langle 1 | R_x \frac{1}{1-\Gamma_x} |1\rangle$$

$$+ \underbrace{P_- R_x}_{-\bar{f}_x} \left(-g_x \langle 1 | R_x \frac{1}{1-\Gamma_x} |1\rangle \right)$$

$$\boxed{\frac{d\bar{g}_x}{dx} = ik\,\bar{g}_x - (1+\bar{f}_x)\langle 1 | R_x \frac{1}{1-\Gamma_x} | 1 \rangle}$$

Let's put $\quad h(x) = \langle 1 | R_x \frac{1}{1-\Gamma_x} | 1 \rangle$. Then

$$\boxed{\begin{aligned} \frac{df_x}{dx} &= -\,g_x\, h(x) \\ \frac{dg_x}{dx} &= -ik\,g_x - (1+f_x)\overline{h(x)}. \\ h(x) &= \langle 1 | R_x \frac{1}{1-\Gamma_x} | 1 \rangle \end{aligned}}$$

Let's now consider a wave equation

$$\partial_t^2 u = (\partial_x^2 - q(x))u$$

on the line with $q \in C_0^\infty$ to simplify. Temporarily think of $\mathcal{H}$ as consisting of all solutions which for each x [redacted] are [redacted] rapidly decreasing C^∞ functions of t. Then we can replace u by its FT wrt t

$$u(x,t) = \int \frac{dk}{2\pi}\, \hat{u}(x,k)\, e^{-ikt}$$

Then

$$\left[-\partial_x^2 + q(x)\right]\hat{u} = k^2 \hat{u}$$

so $\hat{u}$ is a section of a 2-diml vector bundle over the k-line; it is smooth and rapidly decreasing as a fn. of k for x fixed. For $x \to \infty$ we have

$$\hat{u}(x,k) = A(k)e^{-ikx} + B(k)e^{ikx}$$

and hence each element of $\mathcal{H}$ determines two functions of k, except that there are slight problems defining A, B at $k=0$, which I would like to avoid facing as long as possible. I will assume $-\partial_x^2 + g(x)$ has no bound states, so that we don't lose solutions of the wave equation by requiring there are rapidly decreasing in t for fixed x. Also I will try to use in $\mathcal{H}$ only solutions given by a pair $A(k)$, $B(k)$ of C^∞ rapidly-decreasing functions.

If $\quad \hat{u}(x,k) = A(k)e^{-ikx} + B(k)e^{ikx} \qquad x >> 0$

then $\quad u(x,t) = \hat{A}(x+t) + \hat{B}(t-x) \qquad x >> 0$

and as $t \to +\infty$, the first term decays leaving only $\hat{B}(t-x)$ for $x >> 0$. Thus we have a natural in and out maps

$$\mathrm{out}(\hat{u}) = B(k)$$
$$\mathrm{in}(\hat{u}) = A(k).$$

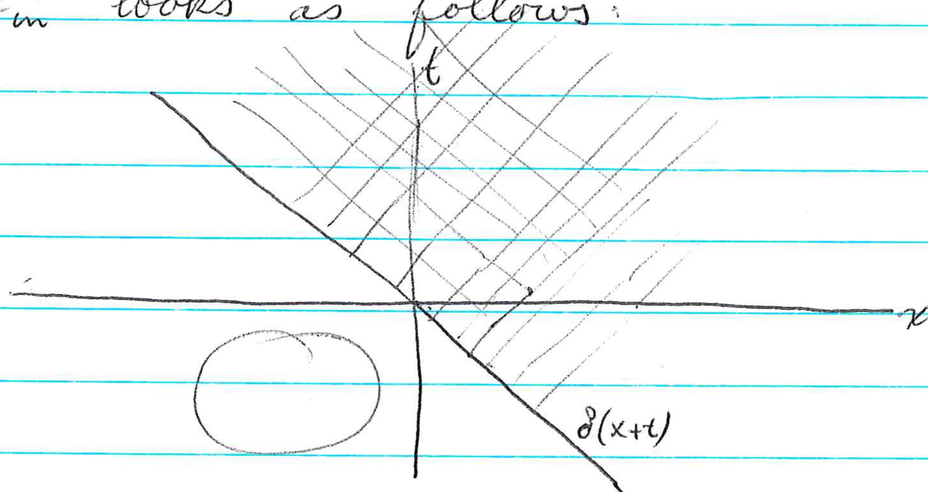
We can define e_{in} to be the element of $\mathcal{H}$ which for $t << 0$ is entirely an incoming δ-function from the right:

$$T(k)e^{-ikx} \xleftrightarrow{\;e_{in}\;} e^{-ikx} + R(k)e^{ikx}$$

and $$\overline{T(k)}\,e^{ikx} \xleftrightarrow{\;e_{out}\;} e^{ikx} + \overline{R(k)}\,e^{-ikx}$$

similarly.

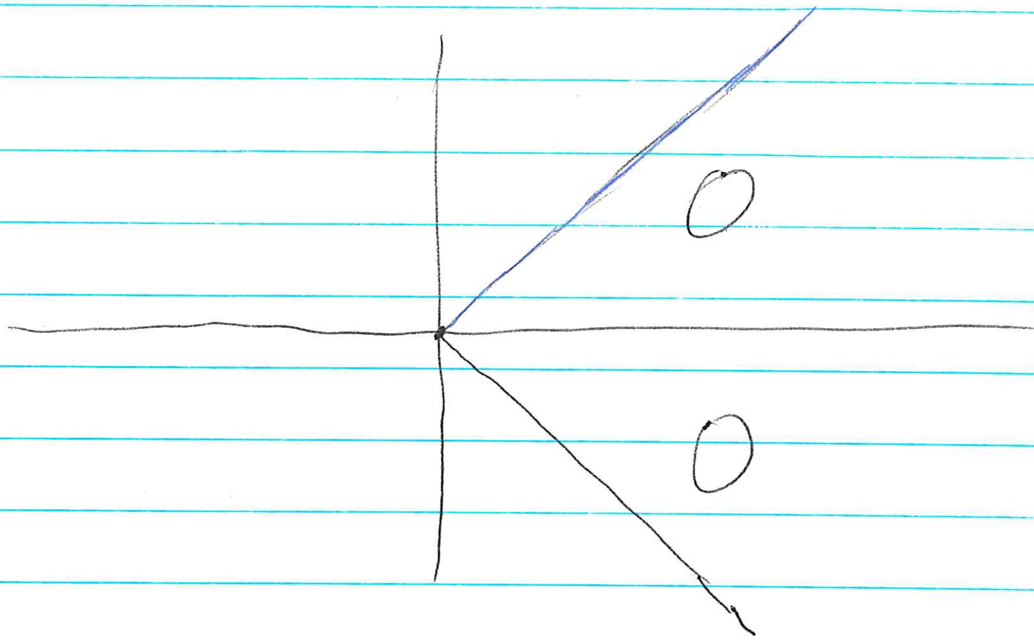
We can draw pictures of elements of $\mathcal{H}$ in the (x,t) plane. Thus e_{in} looks as follows:



Next consider $F_0 \mathcal{H} = (\text{out}, \text{in})^{-1}(H^+ \times H^-)$.
This consists of solutions with

$$\text{out}(\hat{u}) = B(k) = \int_0^\infty e^{iky} \hat{B}(y)\, dy$$

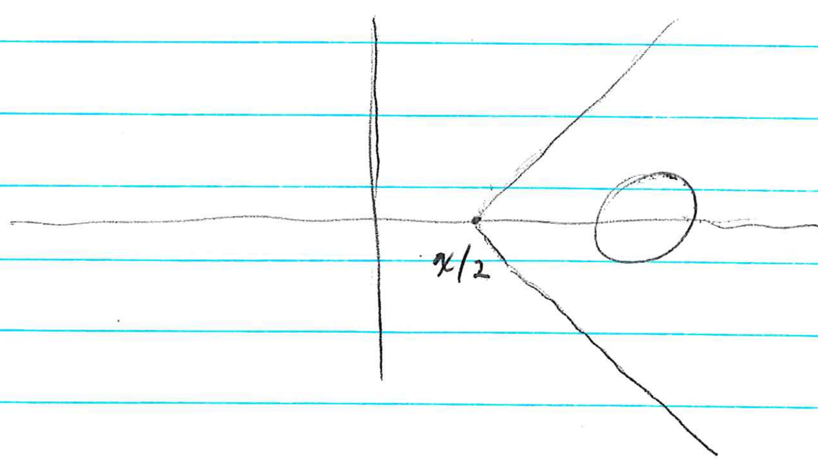
$$\int \frac{dk}{2\pi} B(k) e^{+ik(x-t)} = \hat{B}(\cancel{t} - x) = 0 \qquad t<x$$



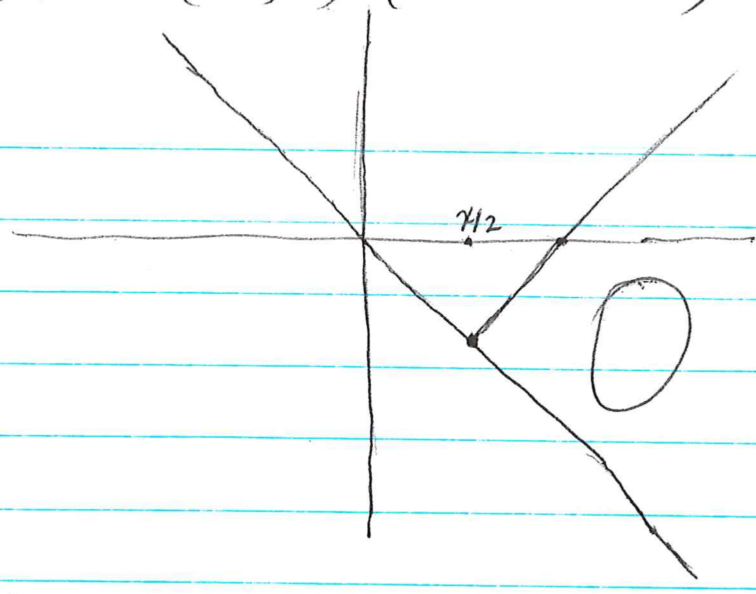
Hence $F_0 \mathcal{H}$ consists of solutions 0 for $|t| < x$, i.e. whose Cauchy data at $t=0$ has support $(-\infty, 0]$.

$$e^{+ik\frac{x}{2}} F_x \mathcal{H} = (\text{out}, \text{in})^{-1}\left(e^{-ik\frac{x}{2}} H^+ \times e^{+ik\frac{x}{2}} H^-\right)$$

contains solutions which look like



and $F_x \mathcal{H} = (\text{out}, \text{in})^{-1}\left(e^{+ikx} H^+ \times H^-\right)$ looks like



Notice that $\bigcap F_x \mathcal{H} = F_{-\infty}\mathcal{H}$ consists of solutions supported in $x+t \leq 0$, in particular the solution

$$e^{-ikx} + \tilde{R} e^{ikx} \longleftrightarrow T e^{-ikx}$$

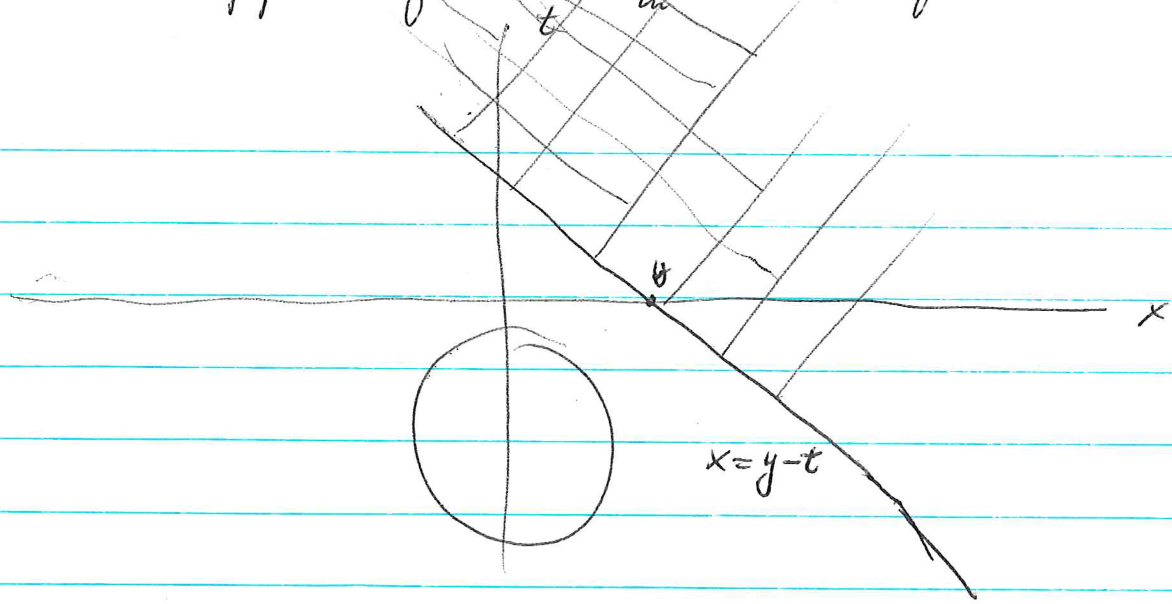
Question: We've seen how $e^{ik\frac{x}{2}} F_x \mathcal{H}$ is an increasing filtration of $\mathcal{H}$. Are the elements $e^{ik\frac{x}{2}} \alpha_x$ and $e^{+ik x/2}\beta_x$ orthogonal to $e^{ik\frac{y}{2}} F_y \mathcal{H}$ for $y < x$? And does these elements correspond to solutions with δ-function Cauchy data at $x/2$?

Consider $\alpha_0 = (1+f_0)e_{in} - \bar{g}_0 e_{out}$. Now f_0 is a linear combination of e^{iky} $y>0$ so $(1+f_0)e_{in}$ is a linear combination of

$$\left(e^{iky} e_{in}\right)(x,t) = \int \frac{dk}{2\pi} e^{ik(y-t)} \underbrace{e_{in}(x,k)}_{e^{-ikx} + \cdots}$$

$$= \delta(x-(y-t)) + \cdots$$

Thus the support of $e^{iky}e_{in}$ is as follows

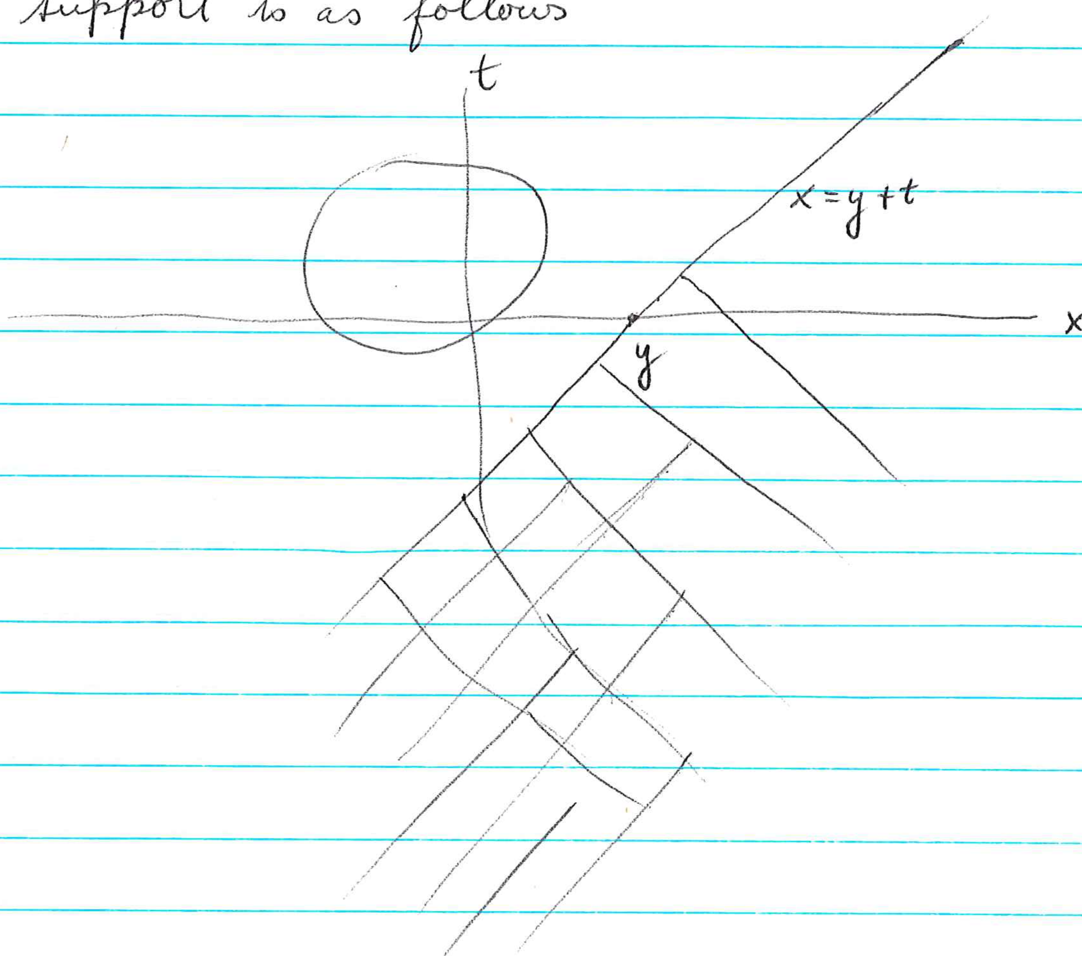


Similarly $\bar{g}_0 e_{out}$ is a linear combination of $e^{-iky}e_{out}$

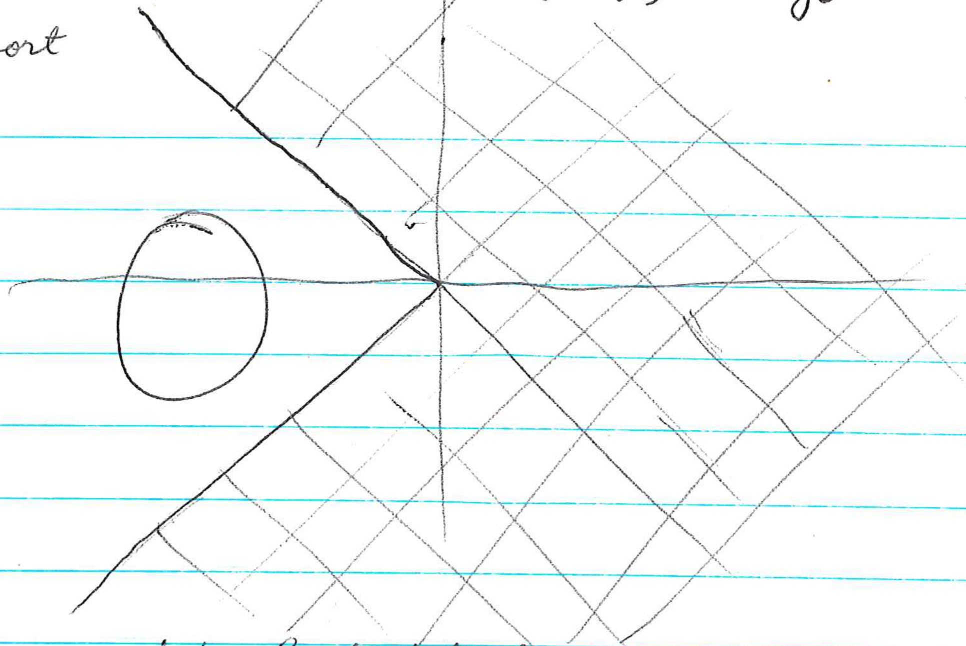
$$\left(e^{-iky}e_{out}\right)(x,t) = \int \frac{dk}{2\pi}\, e^{-ik(y+t)}\, \underbrace{e_{out}(x,k)}_{e^{+ikx}\,+\,\cdots}$$

$$= \delta(x-(y+t)) + \cdots$$

whose support is as follows



The conclusion is that $(1+f_0)e_{in} - \bar{g}_0 e_{out}$ has support



So therefore the $t=0$ Cauchy data of α_0 has support exactly at $x=0$.

January 21, 1981

Recall we are considering solutions of the wave eqn.

$$\partial_t^2 u = (\partial_x^2 - q) u$$

In general given a solution of $\partial_t^2 u = -Lu$ we can form its Laplace transform

$$\mathcal{L}(u) = \int_0^\infty dt\, e^{-st} u(t)$$

and

$$\mathcal{L}(u'') = \int_0^\infty dt \left[(e^{-st}u' + se^{-st}u)' + s^2 e^{-st} u \right]$$

$$-L\mathcal{L}(u) = -u'(0) - su(0) + s^2 \mathcal{L}(u)$$

so

$$\mathcal{L}(u) = \frac{u'(0) + su(0)}{s^2 + L}$$

similarly $\displaystyle \mathcal{L}_-(u) = \int_{-\infty}^0 dt\, e^{-st} u(t)$

will be given by

$$\mathcal{L}_-(u) = -\frac{u'(0) + su(0)}{s^2 + L}$$

Here $(s^2+L)^{-1}$ is computed in a right half-plane $Re(s) > \sigma_0$ for $\mathcal{L}_+(u) = \mathcal{L}(u)$ and a left half plane for $\mathcal{L}_-$. In good cases we can analytically continue to a strip around the imaginary s-axis, whence the [redacted] sum

$$\hat{u} = \mathcal{L}_+(u) + \mathcal{L}_-(u)$$

is an analytic function along the imaginary axis contained in the kernel of $s^2 + L$.

When L is hermitian so that the resolvent $(s^2+L)^{-1}$ is defined off $i\mathbb{R}$, a solution is determined by its Fourier transform $\hat{u}$. Think of $\hat{u}$ as like a Laurent series which then gets split into parts analytic in the inner and outer disks.

When L has discrete spectrum, the functions $\mathcal{L}_+(u)$ and $\mathcal{L}_-(u)$ are the same except for sign. The Fourier transform involves appropriate residue contributions at the eigenvalues.

Given the reflection coefficient $R(k)$ for $(-\partial_x^2 + q)\psi = k^2\psi$ with $q \in C_0^\infty$ say, we can define

$$\alpha_x = (1+f_x)\, e_{in} - \bar{g}_x e^{-ikx} e_{out}$$

So that
$$\begin{cases} out(\alpha_x) = (1+f_x)R - \bar{g}_x e^{-ikx} & \in e^{-ikx} H^+ \\ in(\alpha_x) = 1 + f_x - \bar{g}_x e^{-ikx} \bar{R} & \in 1 + H^- \end{cases}$$

for suitable $f_x, g_x \in H^+$. Arguments given above suggest that as a solution of the wave equation, α_x has Cauchy data on $t=0$ supported at x. However this can't be correct because α_x depends only on $R(k)$, hence the above α_x belongs to a q without bound states.

Let's consider an example $q = -h\delta(x)$. Then

$$(\partial_x^2 + q)\psi = k^2\psi$$

can be integrated in a small interval $[-\varepsilon, \varepsilon]$ to give

$$-(\partial_x \psi)_{0-}^{0+} - h\psi(0) = 0$$

or
$$[\partial_x \psi]_{0-}^{0+} = -h\psi(0)$$

Thus if
$$e^{-ikx} \xleftrightarrow{\ \psi\ } Ae^{-ikx} + Be^{ikx}$$

gives
$$1 = A + B \qquad\qquad A - B = 1 + \frac{h}{ik}$$
$$-ik(A-B-1) = -h \qquad\qquad A + B = 1$$
$$A = 1 + \frac{h}{2ik} \qquad B = -\frac{h}{2ik}$$

Notice if $h>0$, then $A=0 \implies k = \frac{h}{2}i \in UHP$
so we have the bound state $e^{-\frac{h}{2}|x|}$.

Next we want e_{in} which I thought should be

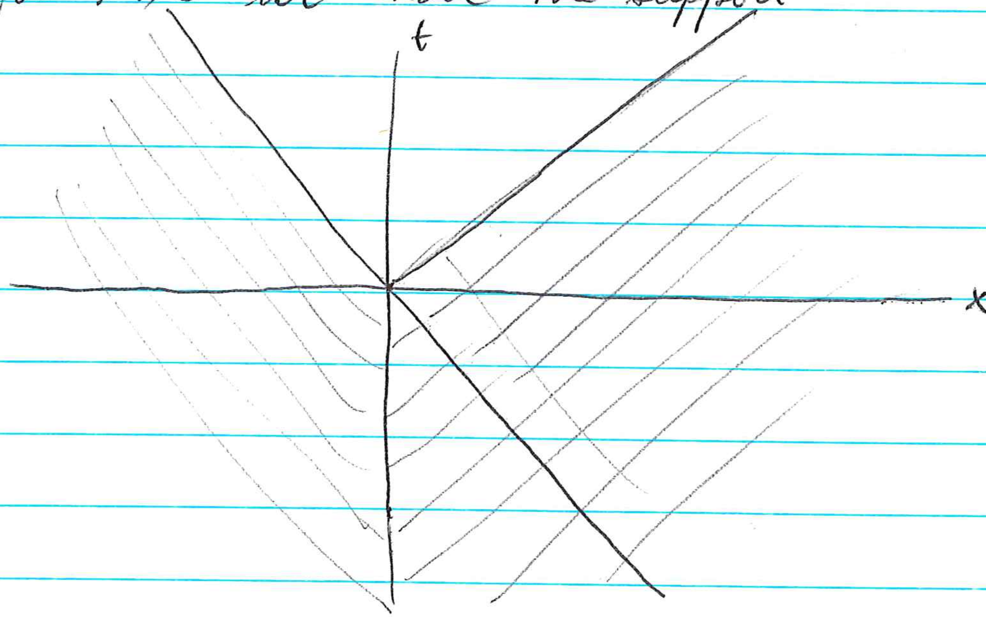
$$\underbrace{\frac{2ik}{2ik+h}}_{T(k)} e^{-ikx} \longleftrightarrow e^{-ikx} + \underbrace{\frac{-h}{2ik+h}}_{R(k)} e^{ikx}$$

But let's look at the corresponding solution of the wave equation:

$$\int \frac{dk}{2\pi} \frac{h}{2ik+h} e^{ik(x-t)} = \theta(x-t) \frac{2\pi i h e^{-\frac{h}{2}(x-t)}}{2\pi \, 2i}$$

$$= \theta(x-t) \frac{h}{2} e^{-\frac{h}{2}(x-t)}$$

Thus for $x \geq 0$ we have the support



and similarly on the other side.
Thus the solution is

$$\frac{2ik}{2ik+h} = 1 - \frac{h}{2ik+h}$$

$$\delta(x+t) - \theta(x-t)\frac{h}{2} e^{\frac{h}{2}(x+t)} \longleftrightarrow \delta(x+t) - \theta(x-t)\frac{h}{2} e^{-\frac{h}{2}(x-t)}$$

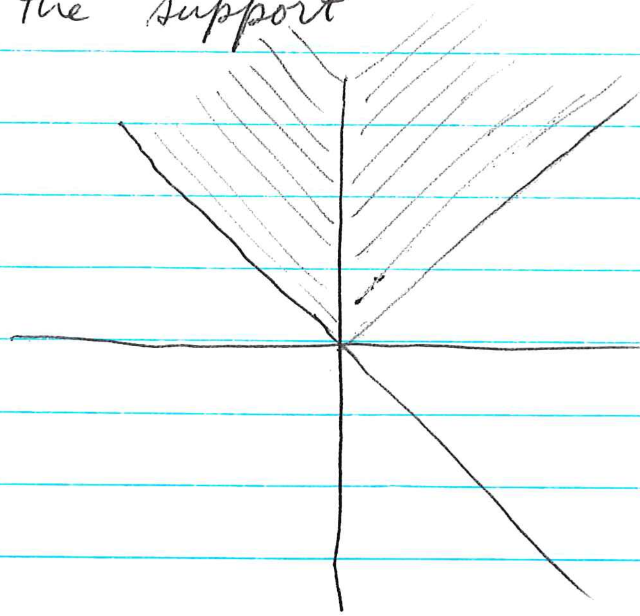
Note that if we add to this the bound state solution

$$\frac{h}{2} e^{-\frac{h}{2}|x|} e^{+\frac{h}{2}t}$$

we get

$$\delta(x+t) + \theta(x+t)\,\frac{h}{2}\,e^{\frac{h}{2}(x+t)} \longleftrightarrow \delta(x+t) + \theta(t-x)\,\frac{h}{2}\,e^{-\frac{h}{2}(x-t)}$$

which has the support



I expect for e_{in}.

January 23, 1981

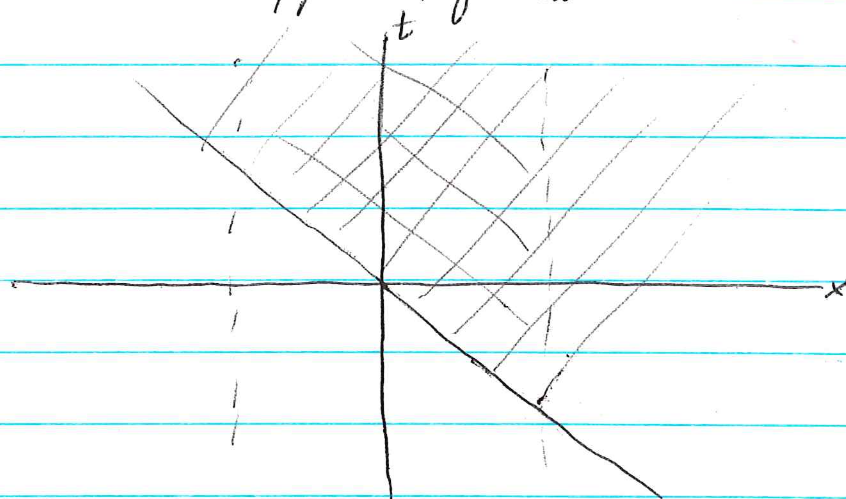
Let us consider a wave equation

$$\partial_t^2 u = (\partial_x^2 - q) u$$

with $q \in C_0^\infty(R)$. I want to identify the solution $e_{in}(x,t)$ which is [redacted] equal to an incoming δ-function disturbance for $t \ll 0$. Thus for $t \ll 0$ we have

$$e_{in}(x,t) = \delta(x+t).$$

[redacted] The support of e_{in} should look like



and hence the Fourier-Laplace transform of e_{in}

$$e_{in}(x,k) = \int dt\, e^{ikt} e_{in}(x,t)$$

should be defined and analytic for $\mathrm{Im}(k) \gg 0$. Moreover it should be a solution of

$$(k^2 + \partial_x^2)\psi = q(x)\psi$$

My candidate for $e_{in}(x,k)$ is as follows. Start with

$$e^{-ikx} \xleftrightarrow{\phi(x,k)} A(k) e^{-ikx} + B(k) e^{ikx}$$

and then form

$$\frac{1}{A(k)} e^{-ikx} \xleftrightarrow{\frac{\phi}{A}} e^{-ikx} + \frac{B(k)}{A(k)} e^{ikx}$$

We know that $A(k)$ is analytic in the UHP with zeroes belonging to the bound state and that (roughly)

$$A(k) = 1 + \frac{1}{2ik}\int q(x)\,dx + O\left(\frac{1}{k^2}\right) \quad \text{as } k \to \infty.$$

Moreover $\phi(x,k)$ is analytic in the UHP because e^{-ikx} is the small solution there. Thus for $\mathrm{Im}(k) >$ bound state region, the solution $\frac{1}{A(k)}\phi(x,k)$ is analytic.

January 24, 1981 (David is 17)

I want to work out the details of scattering on the line from the wave equation viewpoint. Consider

$$L = -\partial_x^2 + q(x)$$

where $q \in C_0^\infty(\mathbb{R})$ and suppose there are no bound states. Then Fourier transform allows us to pass between solutions of the wave equation

$$\partial_t^2 u = (\partial_x - q)\, u = -Lu$$

and functions of x, k for k real satisfying

$$(-\partial_x^2 + q)\,\psi = k^2\psi$$

We have for $x \gg 0$

$$\psi = A(k)e^{-ikx} + B(k)e^{ikx}$$

and we can define

$$\text{in}(\psi) = A(k)$$
$$\text{out}(\psi) = B(k)$$

Actually we have

$$A'e^{ikx} + B'e^{-ikx} \xleftrightarrow{\ \psi\ } Ae^{-ikx} + Be^{ikx}$$

and one knows that

$$|A|^2 + |A'|^2 = |B|^2 + |B'|^2,$$

for example

$$\begin{cases} Te^{-ikx} \Longleftrightarrow e^{-ikx} + Re^{ikx} \\ |T|^2 + |R|^2 = 1. \end{cases}$$

We shall define a norm on the solutions to the wave equation by putting

$$\|\psi\|^2 = \int \frac{dk}{2\pi}\left(|A|^2 + |A'|^2\right)$$

This is not the energy norm

$$E(u) = \frac{1}{2}\int\left(|\dot{u}|^2 + \bar{u}Lu\right)dx \qquad \text{any fixed } t$$

which can be evaluated by letting $t \to -\infty$ to get

$$E(u) = \int \frac{dk}{2\pi}\, k^2\left(|A|^2 + |A'|^2\right)$$

Thus we have

$$E(u) = \|k\psi\|^2$$

Next define the solutions e_{in}, e_{out} by

$$T(k)e^{-ikx} \xleftrightarrow{e_{in}(x,k)} e^{-ikx} + R(k)e^{ikx}$$

$$e_{out}(x,k) = \overline{e_{in}(x,k)} = e_{in}(x,-k)$$

Hence $e_{out}(x,t) = e_{in}(x,-t)$.

January 27, 1981

I want to understand how to compute the terms of the asymptotic series for $G_k(x,x)$ as $k \to i\infty$ in terms of the potential. Here I am considering the operator

$$L = -\partial_x^2 + u(x) = D^2 + u \qquad D = \frac{1}{i}\partial_x$$

Proceeding formally put

$$A = e^{-t(D^2+u)}\, e^{tD^2}$$

Then

$$\partial_t A = -(D^2+u)A + AD^2$$

hence if

$$A = \sum \frac{t^\ell}{\ell!} a_\ell$$

we get the recursion formula

$$\boxed{a_{\ell+1} = -[D^2, a_\ell] - u a_\ell}$$

Starting with $A\big|_{t=0} = a_0 = 1$ we find

$$a_1 = -u$$

$$a_2 = -[D^2, -u] - u(-u) = 2(-i)u'D + (-i)^2 u'' + u^2$$

$$= 2(Du)D + (D^2 u) + u^2$$

It's clear from the recursion formula that a_ℓ is a differential operator of order $\le \ell - 1$.

$$a_\ell = \left(-\,\mathrm{ad}(D^2) - u\right)^\ell 1$$

Heat kernel: $K(t,x,x') = \langle x | e^{-tL} | x' \rangle$

$$= \sum_\alpha e^{-t\lambda_\alpha} \phi_\alpha(x) \overline{\phi_\alpha(x')}$$

supposedly has some kind of asymptotic expansion as $t \downarrow 0$.

My first idea is that for each t, K_t which is a smooth function on $\mathbb{R}^2$, has an asymptotic expansion in the space of distributions on $\mathbb{R}^2$. However this seems to be the formal expansion

$$e^{-tL} = I - tL + \frac{t^2}{2!}L^2 \cdots$$

(the terms are distributions supported on the diagonal)

So instead what seems to be the good result is that one takes the restriction to the diagonal first: $K_t(x,x)$ and then asks for an asymptotic expansion.

In the same way we can [redacted] form the resolvent kernel

$$\langle x | \frac{1}{s-L} | x' \rangle$$

and ask about an asymptotic expansion in s as $s \to -\infty$. If we work in distributions on $\mathbb{R}^2$ we seem to get just

$$\frac{1}{s-L} = \frac{1}{s} + \frac{L}{s^2} + \frac{L^2}{s^3} + \cdots$$

however the interesting case arises when we set $x = x'$ and then ask for an asymptotic expansion.

Here is a possible method for computing the expansions of $\langle x | e^{-tL} | x \rangle$ or $\langle x | \frac{1}{s-L} | x \rangle$. Begin with

$$e^{-tL} = e^{-t(D^2+u)} = [redacted] \underbrace{A(t,x,D)}_{\sim \sum \frac{t^\ell}{\ell!} a_\ell(x,D)} e^{-tD^2}$$

where $\qquad a_\ell = (-\operatorname{ad} D^2 - u)^\ell (1)$. $\qquad$ Then

$$\langle x | e^{-tL} | x' \rangle = \int \frac{dk}{2\pi} \langle x | e^{-tL} | k \rangle \langle k | x' \rangle$$

$$= \int \frac{dk}{2\pi} e^{ikx} A(t,x,k) e^{-tk^2} e^{-ikx'}$$

Thus
$$\langle x | e^{-tL} | x \rangle = \int \frac{dk}{2\pi} A(t,x,k) e^{-tk^2}$$
$$\sim \sum \frac{t^\ell}{\ell!} \int \frac{dk}{2\pi} a_\ell(x,k) e^{-tk^2}$$

Since $a_\ell(x,k)$ is a poly. in k, one can do these Gaussian integrals and get the required asymptotic expansion. Unfortunately this introduces $\frac{1}{t}$ factors.

Different procedure: Put
$$(D^2+u)^n = \Big(\sum_{k\geq 0} a_k^n D^{-k}\Big) D^{2n}$$

Then we can derive recursion relations
$$(D^2+u)^{n+1} = \sum_k (D^2+u)\, a_k^n D^{-k+2n}$$
$$a_k^n D^{-k+2(n+1)} + 2(Da_k^n) D^{-k-1+2(n+1)}$$
$$+ (D^2+u)\, a_k^n D^{-k-2+2(n+1)}$$
$$= \sum_k \left(a_k^n + 2(Da_{k-1}^n) + (D^2+u) a_{k-2}^n\right) D^{-k+2(n+1)}$$

So
$$\boxed{a_k^{n+1} = a_k^n + 2\,Da_{k-1}^n + (D^2+u)\, a_{k-2}^n}$$

This can be used to grind out a_k^n recursively starting from $a_0^n = 1$, $a_k^0 = 0 \quad k>0$. Thus

$$a_0^n = 1$$
$$a_1^n = 0 \qquad\qquad a_2^{n+1} - a_2^n = u$$
$$a_2^n = nu \qquad\qquad a_3^{n+1} - a_3^n = 2D(nu)$$
$$a_3^n = n(n-1)(Du)$$

In general for $k \geq 1$ a_k^n is a polynomial of degree $\leq k-1$ in n

and hence can be expanded in terms of the basic binomial polynomials

$$\phi_\ell(n) = \frac{n(n-1)\cdots(n-\ell+1)}{\ell!} \qquad \Delta\phi_\ell = \phi_{\ell-1}$$

Hence we get

$$(D^2+u)^n = \sum_{0\le\ell<k} a_{k\ell}(x)\,\phi_\ell(n)\,D^{-k+2n}$$

Now we can use this to define the symbol of a pseudo-differential operator: $(D^2+u)^{-s}$. One has

$$\Gamma(s)L^{-s} = \int_0^\infty dt\; e^{-tL}\,t^{s-1}$$

$$\Gamma(s)(D^2+u)^{-s} = \sum_{0\le\ell<k} a_{k\ell}(x)\,\underbrace{\Gamma(s)\,\phi_\ell(-s)}_{\Gamma(s)\frac{(-1)^\ell}{\ell!}s(s+1)\cdots(s+\ell-1) \;=\; \frac{(-1)^\ell}{\ell!}\Gamma(s+\ell)}\,D^{-k-2s}$$

$$D^{-k+2\ell-2(s+\ell)}$$

So

$$e^{-t(D^2+u)} = \sum_{0\le\ell<k} a_{k\ell}(x)\frac{(-1)^\ell}{\ell!}D^{2\ell-k}\,t^\ell\,e^{-tD^2}$$

$$e^{-t(D^2+u)} = \sum_{0\le\ell<k} a_{k\ell}(x)\frac{(-1)^\ell}{\ell!}t^\ell\,D^{2\ell-k}e^{-tD^2}$$

This is a formal expression which gives a new way of computing the operators $a_\ell(x,D)$, but it's not really new.

I need the Gaussian moments

$$\sum\frac{u^m}{m!}\int dx\,x^m e^{-tx^2} = \int dx\,e^{-\frac{2t x^2}{2}+ux} = \sqrt{\frac{2\pi}{2t}}\,e^{\frac{u^2}{4t}}$$

$$= \sqrt{\frac{\pi}{t}}\sum\frac{u^{2n}}{(4t)^n n!}$$

$$\therefore \int\frac{d\xi}{2\pi}\,\xi^{2\ell-k}e^{-t\xi^2} = \frac{1}{\sqrt{4\pi t}}\,\frac{(2\ell-k)!}{(4t)^{\ell-k/2}(\ell-k/2)!} \qquad k \text{ even}$$

Hence it seems we get the asymptotic expansion

$$\langle x | e^{-t(D^2+u)} | x \rangle \sim \sum_{\substack{0 \le \ell < k \\ k \text{ even}}}{}' \; a_{k\ell}(x) \frac{(-1)^\ell}{\ell!} t^{k/2} \frac{1}{\sqrt{4\pi t}} \; \frac{(2\ell - k)!}{4^{\ell - k/2} (\ell - k/2)!}$$

here $\frac{k}{2} \le \ell < k$.

It's clear this derivation can be streamlined quite a bit.

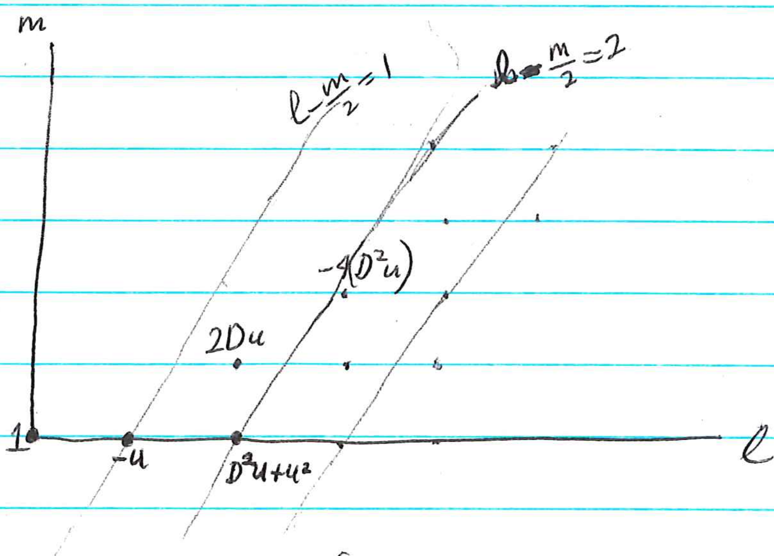
[redacted] $\qquad A = e^{-t(D^2+u)} e^{tD^2}$

$$\partial_t A = -(D^2+u)A + AD^2 = -[D^2, A] - uA$$

If
$$A \sim \sum_{0 \le m < \ell} \frac{t^\ell}{\ell!} a_{\ell m}(x) D^m$$ [redacted]

then
$$a_{\ell+1,m} = -2(Da_{\ell,m-1}) - (D^2+u)a_{\ell,m}$$

e.g.
$$a_0 = 1 \qquad\qquad a_{\ell+1} = -2(Da_\ell)D - (D^2+u)a_\ell$$
$$a_1 = -u$$
$$a_2 = \text{[redacted]} + 2(Du)D + (D^2+u)u$$
$$a_3 = -4(D^2u)D^2 + \cdots$$



$$\langle 0 | D^m e^{-tD^2} | 0 \rangle = \int \frac{dk}{2\pi} k^m e^{-tk^2} = \partial_x^m \int \frac{dk}{2\pi} e^{kx-tk^2} \Big|_{x=0}$$

$$= \partial_x^m \frac{1}{\sqrt{4\pi t}} e^{\frac{x^2}{4t}} \Big|_{x=0} = \frac{m!\ \text{[redacted]}}{\sqrt{4\pi t}\ (\text{[redacted]}\, m/2)!} \left(\frac{1}{4t}\right)^{m/2}$$

$$= \begin{cases} \dfrac{1}{\sqrt{4\pi t}} \dfrac{m!}{(m/2)!\,2^m} t^{-m/2} & m \text{ even} \\ 0 & m \text{ odd.} \end{cases}$$

Thus $\langle x | e^{-t(D^2+u)} | x \rangle \sim \sum_{\substack{0 \le m < l \\ m \text{ even}}} \frac{t^l}{l!} a_{lm}(x) \frac{1}{\sqrt{4\pi t}} \frac{m!}{(m/2)!\, 2^m} t^{-m/2}$

We want to collect according to powers of t, i.e. we want $l - \frac{m}{2} = k$ or $m = 2l - 2k$, which gives us

$$\langle x | e^{-t(D^2+u)} | x \rangle \sim \frac{1}{\sqrt{4\pi t}} \Big(1 + t(-u) + \frac{t^2}{2!}(D^2u + u^2)$$

$$+ \frac{t^2}{3!\,3}(-\cancel{4}D^2u) \frac{\cancel{2!}}{1!\,\cancel{2}^2}$$

$$= \frac{1}{\sqrt{4\pi t}} \Big(1 - tu + \frac{t^2}{2!}\big(u^2 + \tfrac{1}{3}D^2u\big) + \cdots\cdots \Big)$$

Now we have

$$\int_0^\infty dt\, e^{-tL} e^{-st} = \frac{1}{s+L}$$

$$\int_0^\infty dt\, e^{-st} \frac{t^{k-\frac{1}{2}}}{\sqrt{\pi}} = \frac{\Gamma(k+\frac{1}{2})}{s^{k+\frac{1}{2}}\, \Gamma(\frac{1}{2})} = \frac{\frac{1}{2}\,\frac{3}{2} \cdots \frac{2k-1}{2}}{s^{k+1/2}}$$

so that

$$\langle x | \frac{1}{s+D^2+u} | x \rangle \sim \left(\frac{1}{2\sqrt{s}} - \frac{u}{4 s^{3/2}} + \frac{1}{2\cdot 2!}\, \frac{1}{2}\, \frac{3}{2}\, \frac{(u^2 + \frac{1}{3}D^2u)}{s^{5/2}} + \cdots\cdots \right)$$