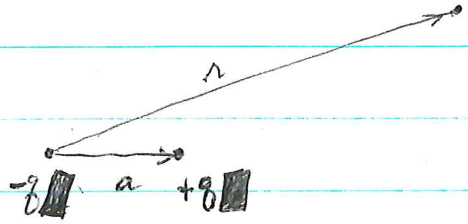
April 11, 1980

sound waves in fermi gas 746
plasma frequency
TF approx. for electron gas 732

Dielectrics: Begin by computing the field of a dipole.



$$\varphi = \frac{q[\text{redacted}]}{|\vec{r}-\vec{a}|} - \frac{q[\text{redacted}]}{|\vec{r}|}$$

$$\varphi = \frac{q}{[\text{redacted}]}\left[\left(r^2+a^2-2ar\cos\theta\right)^{-1/2} - \frac{1}{r}\right]$$

$$\cong \frac{q}{[\text{redacted}]\,r}\left[1 - \frac{1}{2}\left(-\frac{2ar\cos\theta}{r^2}\right) - 1\right] = \frac{qa\cos\theta}{r^2} = \frac{\vec{p}\cdot\vec{r}}{r^3}$$

where $\vec{p}$ = dipole moment = $q\vec{a}$. The above calculation becomes exact in the limit as $\vec{a}\to 0$, $q\vec{a}\to\vec{p}$.

Suppose we compute the electric field due to a lot of dipoles. Let $\vec{P}$ be the dipole moment density, so that $\vec{P}\,d^3x$ is the total dipole moment in the volume d^3x. Then

$$\varphi(r) = \int \frac{\vec{P}(r')\cdot(r-r')}{|r-r'|^3}\, d^3r'$$

Now

$$= \int \vec{P}(r')\cdot\nabla_{r'}\left(\frac{+1}{|r-r'|}\right) d^3r'$$

$$= \int -(\nabla\cdot\vec{P})(r')\,\frac{1}{|r-r'|}\, d^3r'$$

integrating by parts using $\nabla\cdot(u\vec{P}) = \nabla u\cdot P + u\nabla\cdot P$

$$+ \int_{\substack{\text{boundary}\\\text{surface}}} \frac{+(\vec{P}\cdot\hat{n})(r')\,dS}{|r-r'|}$$

This is the field of a surface charge density $+P\cdot\hat{n}$ on the bounding surface, and a charge density $-\nabla\cdot P$ inside.

Let's now consider a dielectric medium which polarizes.

when an electric field is around. Let ρ_{ext} be the external charges present, and ρ_{int} the charge density due to the dielectric. We saw

$$\rho_{int} = -\nabla \cdot \vec{P}$$

hence Poisson's equation becomes

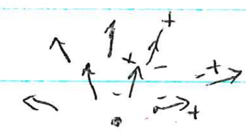
$$\nabla \cdot \vec{E} = 4\pi \rho_{ext} - 4\pi \nabla \cdot \vec{P}$$

or

$$\nabla \cdot \underbrace{(\vec{E} + 4\pi \vec{P})}_{} = 4\pi \rho_{ext}$$

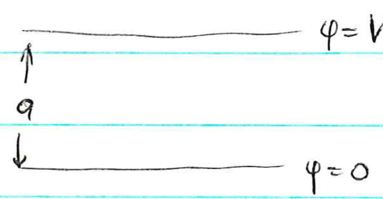
This defines $\vec{D}$

Picture of P



Clearly $\mathrm{div}(P) > 0$ and there is an effective neg. charge at center

Example: Consider a parallel plate condenser with voltage difference V. Then



$$\varphi = \begin{cases} V & z > a \\ \frac{V}{a} z & 0 < z < a \\ 0 & 0 < z \end{cases}$$

$$\vec{E} = -\nabla \varphi = \begin{cases} 0 & z > a \\ -\frac{V}{a} \hat{k} & 0 < z < a \\ 0 & z < 0 \end{cases}$$

so that one has a charge density

$$4\pi \rho = \nabla \cdot E = -\frac{V}{a} \delta(z) + \frac{V}{a} \delta(z-a)$$

supported on the plates. Now if between the plates is a dielectric with $\vec{P} = h\vec{E}$, then $\vec{D} = E + 4\pi P = \vec{E}(1 + 4\pi h)$ inside the plates so

$$\vec{D} = \begin{cases} 0 & a < z \\ -\frac{V}{a}(1 + 4\pi h)\hat{k} & 0 < z < a \\ 0 & z < 0 \end{cases}$$

and so

$$4\pi \rho_{ext} = \nabla \cdot \vec{D} = +\frac{V}{a}(1 + 4\pi h)\left(-\delta(z) + \delta(z-a)\right).$$

Thus the capacitance increases by the factor $1 + 4\pi h$.

Program: Consider a fermion gas of particles with charge e, mass m, moving in a static field (electric) with the potential φ. Better: I start with an external potential φ_{ext} such as the field $\frac{Z}{r}$ for a nucleus of charge Z. Then I want to understand the way the gas behaves. I make the approximation of assuming that the particles in the gas contribute to form a total potential φ which then governs their motions independently. More precisely given φ we solve the 1-particle Hamiltonian

$$(H_0 + e\varphi)\psi = E\psi$$

and then if ψ_n is an orth. basis of eigenfns with eigenvalues ε_n, we get the particle density of the gas.

$$n(x) = \sum_{\varepsilon_n < \varepsilon_F} |\psi_n(x)|^2 .$$

This defines n as a function of φ. Conversely we have Poisson's equation

$$-\Delta\varphi = -\Delta\varphi_{ext} + 4\pi e\, n(x)$$

The Thomas-Fermi approach is to use a semi-classical approximation to compute $n(x)$.

I want to compute the linear response of an electron gas to an external charge distribution ρ_{ext}. The basic approximation is to assume the electrons move independently but subject to an average potential φ. Given φ we solve the Schroedinger equation with

$$H = \frac{p^2}{2m} + e\varphi$$

to find the eigenfunctions: $H\psi_a = \varepsilon_a \psi_a$. Then we get the particle density of the electron gas

$$n(x) = \sum_a{}' |\psi_a(x)|^2 f(\varepsilon_a)$$

where f is the Fermi function

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\varepsilon_F)}+1} \overset{\text{as } \beta\to\infty}{=} \begin{cases} 1 & \varepsilon < \varepsilon_F \\ 0 & \varepsilon > \varepsilon_F \end{cases}$$

(Question: Can one do Thomas-Fermi at non-zero-temperature?)

Thus n is a function of φ.

On the other hand φ must satisfy the Poisson eqn

$$-\Delta\varphi = 4\pi\rho_{ext} + 4\pi e n$$

Notice that this isn't quite correct because if we take $\rho_{ext}=0$, then the situation is translation-invariant, so φ should be a constant, hence $n=0$. We have forgotten to include the uniform positive charge background which is needed so that $\varphi = 0$ when $\rho_{ext}=0$. The Poisson eqn. is then

$$-\Delta\varphi = 4\pi\rho_{ext} + 4\pi e(n - \langle n\rangle)$$

where $\langle n\rangle$ = average particle density $= N/\text{vol}$.

Let's now introduce the Thomas-Fermi approximation.

for computing $n(x)$:

$$(2\pi\hbar)^3 n(x) d^3x = \frac{4}{3}\pi \left[2m(\varepsilon_F - e\varphi(x))\right]^{3/2} d^3x$$

$$n(x) = \frac{1}{(2\pi\hbar)^3} \frac{4}{3}\pi \left[2m(\varepsilon_F - e\varphi(x))\right]^{3/2}$$

We want to consider a small change $\delta\rho_{ext}$ from 0 and compute the induced changes $\delta\varphi$, δn.

$$\delta n(x) = \underbrace{\frac{1}{(2\pi\hbar)^3} \frac{4\pi}{3} \frac{3}{2} \left[2m(\varepsilon_F \cancel{- e\varphi})\right]^{1/2} (-2me)}\delta\varphi(x)$$

$$= \gamma\, \delta\varphi(x)$$

where γ is a (positive) constant. Then

$$-\Delta\delta\varphi = 4\pi\,\delta\rho_{ext} + 4\pi e\,\delta n$$

$$\left[-\Delta + (-4\pi e\gamma)\right]\delta\varphi = 4\pi\,\delta\rho_{ext}$$

This is the Helmholz equation (modified) with $-4\pi e\gamma > 0$. Put $\mu = \sqrt{-4\pi e\gamma}$. The fundamental solution is

$$\frac{e^{-\mu r}}{4\pi r}$$

hence if $\delta\rho_{ext}$ is a charge q located at the origin we find

$$\delta\varphi = \frac{q}{r} e^{-\mu r}$$

which is the Coulomb potential $\frac{q}{r}$ "screened" with the factor $e^{-\mu r}$.

$$\mu = \left(-4\pi e \frac{1}{(2\pi\hbar)^3} 2\pi\, p_F (-2me)\right)^{1/2}$$

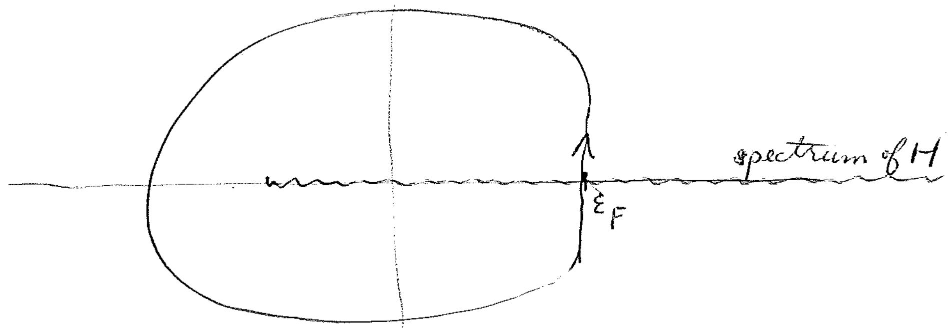
$$= \left(\frac{2}{\pi\hbar^3} m e^2 p_F\right)^{1/2}$$

2 becomes 4 if spins are counted.

Next let's avoid the TF approximation. We have

$$n(x) = \langle x | P^- | x \rangle = \frac{1}{2\pi i} \oint \langle x | \frac{1}{W-H} | x \rangle \, dW$$

with the contour:



Here $H_0 = \frac{p^2}{2m}$ is to be perturbed by $e\delta\varphi$. One has to first order

$$\delta \frac{1}{W-H} = \frac{1}{W-H_0} e\delta\varphi \frac{1}{W-H_0}$$

so

$$\delta n(x) = \frac{1}{2\pi i} \int \langle x | \frac{1}{W-H_0} e\delta\varphi \frac{1}{W-H_0} | x \rangle \, dW$$

Recall H_0 has the orthonormal basis of eigenfunctions

$$\langle x | k \rangle = u_k(x) = \frac{1}{\sqrt{vol}} e^{ikx} \quad , \quad H_0 | k \rangle = \frac{k^2}{2m} | k \rangle .$$

so

$$\delta n(x) = \frac{1}{2\pi i} \int \sum_{k,k'} \frac{\langle x|k\rangle \langle k | e\delta\varphi | k' \rangle \langle k' | x \rangle}{(W-\varepsilon_k)(W-\varepsilon_{k'})} \, dW$$

$$\langle k | \delta\varphi | k' \rangle = \frac{1}{vol} \int e^{-ikx} \delta\varphi(x) e^{ik'x} dx = \frac{1}{vol} \widehat{(\delta\varphi)}(k-k')$$

$$\langle x|k\rangle\langle k'|x\rangle = \frac{1}{vol} e^{+ikx - ik'x}$$

I need $\quad f(x) = \frac{1}{vol} \sum_k e^{ikx} \hat{f}(k) \qquad \hat{f}(k) = \int e^{-ikx} f(x) dx$

$$\therefore \quad \delta n(x) = \frac{1}{(vol)^2} \sum_{k,k'} e^{i(k-k')x} \widehat{(\delta\varphi)}(k-k') \frac{1}{2\pi i} \int \frac{dW}{(W-\varepsilon_k)(W-\varepsilon_{k'})}$$

It will be useful later, when we want to solve the Poisson equation, to work with $\hat{\delta n}(q)$ defined by

$$\delta n(x) = \frac{1}{vol}\sum_k e^{iqx}\,\hat{\delta n}(q)$$

Then we get

$$\hat{\delta n}(q) = e\,\hat{\delta\varphi}(q)\,\frac{1}{vol}\sum_{k-k'=q}\frac{1}{2\pi i}\int\frac{dW}{(W-\varepsilon_k)(W-\varepsilon_{k'})}$$

Now

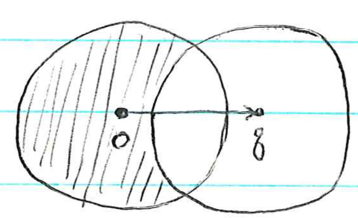
$$\frac{1}{2\pi i}\oint\frac{dW}{(W-a)(W-b)} = \begin{cases} 0 & a, b \text{ outside contour} \\ \dfrac{1}{a-b} & a \text{ inside } b \text{ outside} \\ \dfrac{1}{b-a} & b \text{ inside } a \text{ outside} \\ \dfrac{1}{a-b}+\dfrac{1}{b-a}=0 & \text{if both are inside} \end{cases}$$

$$\frac{1}{vol}\sum_{k-k'=q}\frac{1}{2\pi i}\oint\frac{dW}{(W-\varepsilon_k)(W-\varepsilon_{k'})} = \frac{1}{vol}\left\{\sum_{\substack{|k'|<k_F\\|k'+q|>k_F}}\frac{1}{\varepsilon_{k'}-\varepsilon_{k'+q}}+\sum_{\substack{|k|<k_F\\|k-q|>k_F}}\frac{1}{\varepsilon_k-\varepsilon_{k-q}}\right\}$$

Change k into $-k'$ in the 2nd term and it becomes the first. Hence we get the above

$$= \frac{2}{vol}\sum_{\substack{|k|<k_F\\|k+q|>k_F}}\frac{2m}{k^2-(k+q)^2} = \frac{-4m}{vol}\sum_{\substack{|k|<k_F\\|k+q|>k_F}}\frac{1}{2\underline{k}\cdot q+q^2}$$

$$\longrightarrow -4m\,\frac{1}{(2\pi)^3}\int_{\substack{|k|<k_F\\|k+q|>k_F}}\frac{d^3k}{2\underline{k}\cdot q+q^2}$$



$$= -4m\,\frac{1}{(2\pi)^3}\,k_F\int_{\substack{|k|<1\\|k+\frac{q}{k_F}|>1}}\frac{d^3k}{2\underline{k}\cdot\frac{q}{k_F}+\left(\frac{q}{k_F}\right)^2}$$

$u\left(\frac{q}{2k_F}\right)\pi$

So

$$\delta n(q) = e\,\delta\varphi(q)\,\frac{-4mk_F}{(2\pi)^3}\,u\!\left(\frac{q}{2k_F}\right)\pi$$

$$= \delta\varphi(q)\,\underbrace{\frac{-4\pi m e k_F}{(2\pi)^3}}_{\gamma \text{ of before}}\,u\!\left(\frac{q}{2k_F}\right)$$

The function u can be evaluated (see Doniach-Sondheimer: p149 for

$$u(x) = \frac{1}{2}\left\{1 + \frac{1}{2x}(1-x^2)\log\left|\frac{1+x}{1-x}\right|\right\} \longrightarrow 1 \quad \text{as } x \to 0$$

so that the above reduces to the TF formula on 728 as $q \to 0$.

Thus we [redacted] have an exact formula for $\delta n(q)$ in terms of $\delta\varphi(q)$. Now solve the Poisson equation

$$-\Delta\delta\varphi = 4\pi\delta\rho_{ext} + 4\pi e\,\delta n \text{[redacted]}$$

to get

$$q^2\,\delta\varphi(q) = 4\pi\delta\rho_{ext}(q) + 4\pi e\,\overbrace{\left(\frac{-4\pi m e k_F}{(2\pi)^3}\right)}^{\gamma}\,u\!\left(\frac{q}{2k_F}\right)\delta\varphi(q)$$

or

$$\delta\varphi(q) = \frac{4\pi}{q^2 + (-4\pi e\gamma)\,u\!\left(\frac{q}{2k_F}\right)}\,\delta\rho_{ext}(q)$$

$$= \frac{4\pi}{q^2 + \mu^2\,u\!\left(\frac{q}{2k_F}\right)}\,\delta\rho_{ext}(q)$$

Now I can connect this up with the dielectric response. It is not true, so it seems, that there is a dielectric constant $\varepsilon > 1$ such that the electric field E is weakened by the factor $\frac{1}{\varepsilon}$ from what it would be in a vacuum. This is the meaning of the equations

$$\vec{D} = \varepsilon \vec{E} \qquad \nabla \cdot D = 4\pi \rho_{ext}$$

What actually happens is that the electrons are free to move around, and so the relation of D and E depends on the wave number.

Notice that D is computed from ρ_{ext} as if one were is a vacuum: $D = -\nabla \cdot \varphi_{ext}$ where $-\Delta \varphi_{ext} = 4\pi \rho_{ext}$. Hence when $\rho_{ext} = e^{iqx} \rho_{ext}(q)$, then

$$\vec{D}(q) = -i\vec{q}\,\varphi_{ext}(q) \qquad \varphi_{ext}(q) = \frac{4\pi}{q^2}\rho_{ext}(q)$$

From

$$-\Delta\varphi = \nabla \cdot E = 4\pi\rho_{ext} + 4\pi e(n - \langle n \rangle) \qquad \text{(we drop } \delta$$

we get

$$q^2 \varphi(q) = 4\pi \rho_{ext}(q) + 4\pi e \gamma\, u\left(\frac{q}{2k_F}\right)\varphi(q)$$

$$\varphi(q) = \frac{4\pi}{q^2 + \mu^2 u\left(\frac{q}{2k_F}\right)} \rho_{ext}(q).$$

Now $\vec{E}(q) = -i\vec{q}\,\varphi(q)$ and the dielectric function is defined by

$$\vec{D}(q) = \varepsilon(q)\vec{E}(q) \qquad \text{or}$$

$$\varphi(q) = \frac{1}{\varepsilon(q)}\varphi_{ext}(q)$$

actual potential reduced by $\frac{1}{\varepsilon(q)}$ from external potential

Thus our calculations give

$$\varepsilon(q) = \frac{\varphi_{ext}(q)}{\varphi(q)} = \frac{\frac{4\pi}{q^2}\rho_{ext}(q)}{\frac{4\pi}{q^2+\mu^2 u}\rho_{ext}(q)}$$

$$\boxed{\varepsilon(q) = 1 + \frac{\mu^2}{q^2} u\left(\frac{q}{2k_F}\right)}$$

Interesting point is that the above answer agrees with the calculation of the static dielectric function $\varepsilon(q)$ in the so-called random phase approximation.

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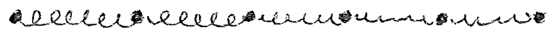
Speed of sound: First consider longitudinal vibrations of a bar. We use the mechanical analogy



Let $u(x)$ denote the displacement of the particle at rest position x, let m = mass of particle, a = spacing of particles at rest, k = spring constant. Then

$$m\ddot{u}(x) = k\big(u(x+a)-u(x)\big) - k\big(u(x)-u(x-a)\big)$$

Next we let $a \to 0$ such that $\frac{m}{a} \to \rho$ and $ka \to \tau$.

$$\frac{m}{a}\ddot{u}(x) = ak\left(\frac{u(x+a)-2u(x)+u(x-a)}{a^2}\right)$$

$$\downarrow$$

$$\rho\ddot{u} = \tau\,\partial_x^2 u$$

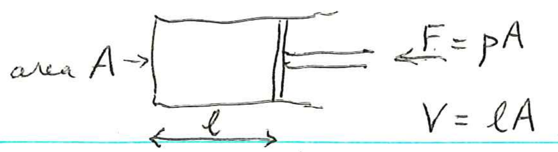
which gives waves $e^{i(kx-\omega t)}$ with

$$\rho\omega^2 = \tau k^2 \qquad \text{or speed} = \frac{\omega}{k} = \sqrt{\frac{\tau}{\rho}}$$

Notice that if one has a length made of N springs

then stretching this a distance x causes each spring to stretch $\frac{x}{N}$, causing a force $k\frac{x}{N} = \frac{ka}{Na}x$. Thus the spring constant of a length $l = Na$ of the bar is $\frac{\tau}{l}$. τ is some sort of elastic modulus.

Let's next consider a tube filled with gas having at rest a pressure p_0 and density ρ_0. We think of the gas as analogous to a bar which can vibrate longitudinally. We have to determine the elastic modulus τ. So consider a cylinder:



For adiabatic changes which is the sort that occur in sound waves we have

$$pV^\gamma = \text{const}$$

so for small changes about equilibrium we have

$$\frac{dp}{p_0} + \gamma \frac{dV}{V_0} = 0 \quad \text{or} \quad dF = dp \cdot A = -\frac{p_0 \gamma}{V_0} A\, dV$$

$$= -\frac{p_0 \gamma}{A \ell_0} A^2 d\ell = -\left(A \frac{\gamma p_0}{\ell_0}\right) d\ell$$

Hence the spring constant is

$$A \frac{\gamma p_0}{\ell_0}$$

so the elastic modulus τ is $A \gamma p_0$. (Really τ should be defined as the spring constant per unit area, since one wants to use it with a linear density $A \rho_0$). So the sound speed is

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}}$$

I am working toward understanding plasma oscillations in an electron gas. These are different in some way from sound waves, so it seems, because the frequency wave number relation $\omega = \omega(q)$ does not tend to zero as $q \to 0$. For sound waves $\omega(q) = cq$.

Question: Does a Fermi gas have pressure at 0 temperature, i.e. does it exert a pressure on the walls of the box?

Consider $V = L^3$ with N particles inside. The energy levels are $\varepsilon_k = \frac{\hbar^2 k^2}{2m}$ where $\underline{k} \in \frac{2\pi}{L} \mathbb{Z}^3$. Write $\underline{k} = \frac{2\pi}{L} \underline{\lambda}$ with $\underline{\lambda} \in \mathbb{Z}^3$. Then the Fermi sea will be given by $|\underline{\lambda}| < \lambda_F$ so

$$N = \sum_{|\underline{\lambda}| < \lambda_F} 1$$

$$U = \sum_{|\underline{\lambda}| < \lambda_F} \frac{\hbar^2}{2m} \left(\frac{2\pi}{L}\right)^2 \lambda^2 = \frac{1}{L^2} \frac{\hbar^2 (2\pi)^2}{2m} \sum_{|\underline{\lambda}| < \lambda_F} \lambda^2$$

so we get the relation

$$U = V^{-2/3} \cdot \text{const.}$$

The pressure should be given by

$$p = -\frac{\partial}{\partial V} U = \frac{2}{3} V^{-5/3} \cdot \text{const.}$$

since at $T=0$, U should = free energy. Another way to see this is to use

$$U = \sum{}' E_n \underbrace{\frac{e^{-\beta E_n}}{Z}}_{p_n} \qquad E_n(V) = \text{const} \cdot V^{-2/3}$$

$$p = \sum{}' - \frac{\partial E_n}{\partial V} p_n$$

and at $T=0$ one has $\begin{cases} p_n = 1 & E_n < \varepsilon_F \\ = 0 & E_n > \varepsilon_F . \end{cases}$

Notice that $pV^{5/3} = \text{constant}$ is the same as for adiabatic changes of an ideal gas.

Let's see if the pressure makes sense as $V \to \infty$, $N \to \infty$ with $\frac{N}{V} \to \rho$. We have

$$N = \sum_{\lambda < \lambda_F} 1 \qquad \lambda_F = \frac{L}{2\pi} k_F$$

$$\frac{N}{V} = \frac{1}{L^3} \sum_{\lambda < \lambda_F} 1 = \frac{1}{L^3} \sum_{|\underline{k}| < k_F} 1 \sim \int_{k < k_F} \frac{d^3k}{(2\pi)^3} = \frac{4}{3}\pi \left(k_F/2\pi\right)^3$$

$$U = \frac{1}{L^2} \overbrace{\frac{\hbar^2 (2\pi)^2}{2m}}^{C} \sum_{|\lambda| < \lambda_F} \lambda^2 = V^{-2/3} C$$

$$p = \frac{2}{3} L^{-5} \frac{\hbar^2 (2\pi)^2}{2m} \sum_{|\lambda| < \lambda_F}{}' \lambda^2 = \frac{2}{3} V^{-5/3} C \qquad \text{see above}$$

$$p = \frac{2}{3} L^{-5} \frac{\hbar^2 (2\pi)^2}{2m} \sum_{k<k_F} \left(\frac{L}{2\pi} k\right)^2 = \frac{2}{3} \frac{\hbar^2}{2m} \frac{1}{L^3} \sum_{k<k_F} k^2$$

$$\rightarrow \frac{2}{3} \frac{\hbar^2}{2m} \int_{k<k_F} \frac{d^3k}{(2\pi)^3} k^2 = \frac{2}{3} \frac{\hbar^2}{2m} \frac{4\pi}{(2\pi)^3} \frac{k_F^5}{5}$$

so we see indeed that the pressure has an infinite volume limit.

It should be possible to derive the same result via the grand canonical ensemble. Recall

$$Z_{gr} = \sum_N z^N Z_N = \sum_n \underbrace{z^{N(n)} e^{-\beta E_n}}_{p_n Z_{gr}}$$

is regarded as a function of z, β, V. One has

$$U = \sum E_n p_n = -\frac{\partial}{\partial \beta} \log Z_{gr}$$

$$N = \sum N_n p_n = z \frac{\partial}{\partial z} \log Z_{gr}$$

$$p = \sum -\frac{\partial E_n}{\partial V} p_n = +\frac{1}{\beta} \frac{\partial}{\partial V} \log Z_{gr}$$

As $V \to \infty$ one expects $\frac{1}{V} \log Z_{gr}$ to converge to $\Omega = \lim \frac{\log Z_{gr}}{V}$, so that $\frac{N}{V} \to \rho$, $\frac{U}{V} \to$ energy density, also $\log Z_{gr} \sim V \, \Omega$ and so we get

$$p = \frac{1}{\beta} \lim \left(\frac{\log Z_{gr}}{V} \right).$$

Consider now a Fermi gas

$$Z_{gr} = \prod_{k \in \frac{2\pi}{L} \mathbb{Z}^3} \left(1 + z e^{-\beta \varepsilon_k}\right) \qquad \varepsilon_k = \frac{\hbar^2 k^2}{2m}$$

$$\frac{1}{V} \log Z_{gr} = \frac{1}{V} \sum_k \log\left(1 + z e^{-\beta \varepsilon_k}\right) \longrightarrow \int \frac{d^3k}{(2\pi)^3} \log\left(1 + z e^{-\beta \varepsilon_k}\right)$$

$$\therefore \quad \Omega = \frac{4\pi}{(2\pi)^3} \int_0^\infty k^2 dk \, \log\left(1 + z e^{-\beta \varepsilon_k}\right)$$

Now suppose β is very large, and $z = e^{\beta \varepsilon_F}$. Then for ε_k slightly bigger that ε_F, we have $1 + z e^{-\beta \varepsilon_k} \sim 1$ and for ε_k sligtly less that ε_F, we have $1 + z e^{-\beta \varepsilon_k} \sim e^{\beta(\varepsilon_F - \varepsilon_k)}$.
Thus

$$\Omega \sim \frac{4\pi}{(2\pi)^3} \int_0^{k_F} k^2 dk \; \beta(\varepsilon_F - \varepsilon_k)$$

and so

$$p = \frac{\Omega}{\beta} \sim \frac{4\pi}{(2\pi)^3} \int_0^{k_F} \frac{\hbar^2}{2m} (k_F^2 - k^2) k^2 dk$$

$$= \frac{4\pi}{(2\pi)^3} \frac{\hbar^2}{2m} k_F^5 \left(\frac{1}{3} - \frac{1}{5}\right)$$

which is the same as on the preceding page. One also has

$$\rho = z \frac{\partial}{\partial z} \Omega = \frac{4\pi}{(2\pi)^3} \int_0^\infty k^2 dk \; \frac{z e^{-\beta \varepsilon_k}}{1 + z e^{-\beta \varepsilon_k}}$$

$$\longrightarrow \frac{4\pi}{(2\pi)^3} \int_0^{k_F} k^2 dk = \frac{4\pi}{(2\pi)^3} \frac{k_F^3}{3} \qquad \text{as } \beta \to \infty$$

$$\text{energy density} = -\frac{\partial}{\partial \beta} \Omega = \frac{4\pi}{(2\pi)^3} \int_0^\infty k^2 dk \; \frac{\varepsilon_k z e^{-\beta \varepsilon_k}}{1 + z e^{-\beta \varepsilon_k}}$$

$$\longrightarrow \frac{4\pi}{(2\pi)^3} \int_0^{k_F} k^2 \varepsilon_k \, dk = \frac{4\pi}{(2\pi)^3} \frac{\hbar^2}{2m} \frac{k_F^5}{5}$$

Paradox: As $\beta \to \infty$, one has $\frac{\Omega}{\beta} \to \frac{2}{3}\alpha$ where $\alpha =$ hence one expects

$$\Omega \sim \frac{2}{3} \alpha \beta \qquad \text{whence} \qquad \frac{\partial \Omega}{\partial \beta} \sim \frac{2}{3} \alpha$$

However $\frac{\partial \Omega}{\partial \beta} \longrightarrow -\alpha$. You've forgot $z = e^{\beta \varepsilon_F}$ depends on β.

Consider fermi gas in a box: $V = L^3$. The states with 1-particle are $u_k = \frac{1}{\sqrt{V}} e^{ikx}$, $k \in \frac{2\pi}{L}\mathbb{Z}^3$, $\varepsilon_k = \frac{1}{2m}k^2$.

$$Z_{gr} = \prod_k \left(1 + e^{\beta\mu - \beta\varepsilon_k}\right)$$

I want to write this so that its dependence on volume is clear. So put $\underline{k} = \frac{2\pi}{L}\underline{\lambda}$, $\underline{\lambda} \in \mathbb{Z}^3$. Then

$$Z_{gr} = \prod_{\lambda \in \mathbb{Z}^3} \left(1 + e^{\beta\mu - \beta\varepsilon(\lambda, V)}\right)$$

$$\varepsilon(\lambda, V) = \frac{1}{2m}\left(\frac{2\pi}{L}\right)^2 \lambda^2 = \frac{(2\pi)^2\lambda^2}{2m} V^{-2/3}$$

$$\boxed{\frac{\partial\varepsilon(\lambda, V)}{\partial V} = -\frac{2}{3}\frac{1}{V}\varepsilon(\lambda, V)}$$

The pressure is

$$p = \sum -\frac{\partial\varepsilon_n}{\partial V} p_n \qquad Z_{gr} = \sum_n \underbrace{\left(e^{\beta\mu}\right)^{N_n} e^{-\beta\varepsilon_n}}_{p_n Z_{gr}}$$

$$= \frac{1}{Z_{gr}} \sum -\frac{\partial\varepsilon_n}{\partial V} e^{\beta\mu N_n - \beta\varepsilon_n}$$

$$\boxed{p = \frac{1}{\beta}\frac{\partial}{\partial V}\log Z_{gr}}$$

$$= \frac{1}{\beta}\frac{\partial}{\partial V}\sum_\lambda \log\left(1 + e^{\beta\mu - \beta\varepsilon(\lambda, V)}\right)$$

$$= \frac{1}{\cancel{\beta}} \sum_\lambda \frac{e^{\beta\mu - \beta\varepsilon(\lambda, V)}}{1 + e^{\beta\mu - \beta\varepsilon(\lambda, V)}} \left(-\cancel{\beta}\frac{\partial\varepsilon}{\partial V}(\lambda, V)\right)$$

$$= \frac{2}{3}\frac{1}{V}\sum_\lambda \frac{e^{\beta\mu - \beta\varepsilon(\lambda, V)}}{1 + e^{\beta\mu - \beta\varepsilon(\lambda, V)}} \varepsilon(\lambda, V)$$

$$= \frac{2}{3}\frac{1}{V}\sum_{k \in \frac{2\pi}{L}\mathbb{Z}^3} \frac{e^{\beta\mu - \beta\varepsilon_k}}{1 + e^{\beta\mu - \beta\varepsilon_k}} \varepsilon_k \longrightarrow \frac{2}{3}\int \frac{dk^3}{(2\pi)^3} \frac{e^{\beta\mu - \beta\varepsilon_k}}{1 + e^{\beta\mu - \beta\varepsilon_k}} \varepsilon_k$$

On the other hand

$$\frac{\log Z_{gr}}{V} = \frac{1}{V}\sum_k \log\left(1+e^{\beta\mu-\beta\varepsilon_k}\right)$$

$$\longrightarrow \int \frac{d^3k}{(2\pi)^3} \log\left(1+e^{\beta\mu-\beta\varepsilon_k}\right)$$

In the preceding β, μ have been held fixed and only V has been allowed to vary. So we end up with the following paradox:

$$\log Z_{gr} \sim V \cdot C_1 \qquad C_1 = \int \frac{d^3k}{(2\pi)^3} \log\left(1+e^{\beta\mu-\beta\varepsilon_k}\right)$$

$$\frac{\partial}{\partial V}\log Z_{gr} \sim C_2 \qquad C_2 = \frac{2}{3}\int \frac{d^3k}{(2\pi)^3} \frac{e^{\beta\mu-\beta\varepsilon_k}}{1+e^{\beta\mu-\beta\varepsilon_k}}\,\beta\varepsilon_k$$

Maybe $C_1 = C_2$ by integration by parts. YES.

The good formula is

$$p = \lim \frac{\log Z_{gr}}{\beta V} = \frac{2}{3}\int \frac{d^3k}{(2\pi)^3}\, \varepsilon_k \frac{e^{\beta\mu-\beta\varepsilon_k}}{1+e^{\beta\mu-\beta\varepsilon_k}}$$

because the integral is the average kinetic energy density. It's useful to think of $-\mu$ as the potential energy inside the box and ε_k as the kinetic energy of a particle of momentum k.

It seems to be a good idea to use $z = e^{\beta\mu}$ instead of μ, i.e. you adjust μ as β varies so that $\beta\mu$ is constant. Then

$$-\frac{\partial}{\partial\beta}\log Z_{gr} = \sum \varepsilon_n p_n = U \qquad \text{internal energy}$$

and

$$\| \quad \sum_k \frac{z e^{-\beta\varepsilon_k}}{1+z e^{-\beta\varepsilon_k}}\,\varepsilon_k$$

Another way to write this is

$$\star \quad U = \langle \hat{H} \rangle = \sum_k \varepsilon_k \underbrace{\langle a_k^* a_k \rangle}_{\frac{z e^{-\beta\varepsilon_k}}{1+z e^{-\beta\varepsilon_k}}}$$

In the limit as $V \to \infty$ we get

$$\lim \frac{U}{V} = \int \frac{d^3k}{(2\pi)^3} \, \varepsilon_k \, \frac{z e^{-\beta\varepsilon_k}}{1+z e^{-\beta\varepsilon_k}}$$

Let's denote this by $\bar{\varepsilon}$. Then we have for Fermi gas

$$\boxed{\begin{array}{lll} & pV = \frac{2}{3} U & (V \text{ large}) \\ \text{or} & p = \frac{2}{3}\bar{\varepsilon} & \bar{\varepsilon} = \int \frac{d^3k}{(2\pi)^3} \, \varepsilon_k \, \frac{z e^{-\beta\varepsilon_k}}{1+z e^{-\beta\varepsilon_k}} \end{array}}$$

Interesting point: The formula $\star$ above suggests that

$$\hat{H} = \int \frac{d^3k}{(2\pi)^3} \, \varepsilon_k \, a_k^* a_k \qquad \hat{N} = \int \frac{d^3k}{(2\pi)^3} \, a_k^* a_k$$

are energy-density and particle-density operators in some sense. Or maybe it's the average $\langle \ \rangle$ which should be viewed as giving a density.

April 18, 1980

Question: Are there sound waves in a fermi gas at 0 temperature?

Take a wave function Ψ describing the gas at $t=0$. Then $\Psi(t) = e^{-i\hat{H}t}\Psi$ describes the gas at time t and

$$\rho(xt) = \langle \Psi(t) | \psi(x)^* \psi(x) | \Psi(t) \rangle = \langle \Psi | \psi^*(xt)\, \psi(xt) | \Psi \rangle$$

gives the density at position x and time t. Here

$$\psi(xt) = \sum \frac{1}{\sqrt{V}} e^{ikx - i\varepsilon_k t} a_k$$

$$\psi^*(xt) = \sum \frac{1}{\sqrt{V}} e^{-ikx + i\varepsilon_k t} a_k^*$$

and so

$$\rho(xt) = \frac{1}{V} \sum_{k,k'} e^{i(k-k')x - i(\varepsilon_k - \varepsilon_{k'})t} \langle \Psi | a_{k'}^* a_k | \Psi \rangle$$

Notice that $\langle \Psi | a_{k'}^* a_k | \Psi \rangle$ is a positive semi-definite matrix on the 1-particle Hilbert space $\mathcal{H}$. For example if Ψ is the ground state in $\Lambda^N \mathcal{H}$, then

$$\langle \Psi | a_{k'}^* a_k | \Psi \rangle = \delta_{k'k} \begin{cases} 1 & \text{if } \varepsilon_k < \varepsilon_F \\ 0 & \text{if } \varepsilon_k > \varepsilon_F \end{cases}$$

The question is whether one can approximately interpret $\rho(xt)$ as a wave motion, better a solution of a scalar wave equation $\partial_t^2 \rho = c^2 \Delta \rho$. Hence we would like to see $\rho(xt)$ as a superposition of exponentials of the form

$$e^{iqx - ic|q|t}$$

Let's try to take Ψ to be a small perturbation of the ground state, e.g.

$$\Psi = \Phi + \varepsilon \Psi_1$$

whence

$$\langle \Psi | a^*_{k'} a_k | \Psi \rangle = \langle \Phi | a^*_{k'} a_k | \Phi \rangle + \varepsilon \left\{ \langle \Psi_1 | a^*_{k'} a_k | \Phi \rangle + \langle \Phi | a^*_{k'} a_k | \Psi_1 \rangle \right\} + O(\varepsilon^2)$$

Let's take $\Psi_1 = a^*_p a_q \Phi$ where $\varepsilon_q < \varepsilon_F$, $\varepsilon_p > \varepsilon_F$. Thus Ψ_1 represents the state with a hole of momentum q and a particle of momentum p. We have

$$\langle \Phi | a^*_{k'} a_k a^*_p a_q | \Phi \rangle = \begin{cases} 1 & \text{if } k = p \text{ and } k' = q \\ 0 & \text{otherwise} \end{cases}$$

In effect $$\langle \Phi | a^*_{k'} a_k = \big(\underbrace{a^*_k a_{k'} | \Phi \rangle}_{} \big)^*$$

0 unless $\varepsilon_{k'} < \varepsilon_F$, $\varepsilon_k > \varepsilon_F$
in which case we have a hole of mom. k'
part ______ k

~~Thus~~

$$\cancel{\delta \langle a^*_{k'} a_k \rangle = 2\varepsilon \begin{cases} 1 & \text{if } k = p \text{ and } k' = q \\ 0 & \text{otherwise} \end{cases}}$$

~~and so~~

$$\cancel{\delta\rho(xt) = \frac{2\varepsilon}{V} e^{i(p-q)x - i(\varepsilon_p - \varepsilon_q)t}}$$

Also

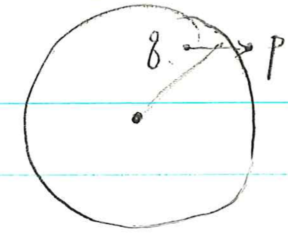
$$\langle a^*_p a_q \Phi | a^*_{k'} a_k | \Phi \rangle = \langle \Phi | a^*_q a_p a^*_{k'} a_k | \Phi \rangle$$

$$= \begin{cases} 1 & \text{if } p = k' \text{ and } q = k \\ 0 & \text{otherwise} \end{cases}$$

Thus the change in density due to the change Φ to $\Phi + \varepsilon a^*_p a_q \Phi$ is

$$\delta\rho(xt) = \frac{\varepsilon}{V} \left(e^{i(p-q)x - i(\varepsilon_p - \varepsilon_q)t} + e^{i(q-p)x - i(\varepsilon_q - \varepsilon_p)t} \right)$$

Here p is outside, and q is inside the Fermi sphere. If we require the wave-length of these density waves to be very long, i.e. $p-q$ is small, then maybe it is possible to relate $\varepsilon_p - \varepsilon_q$ to $|p-q|$.



$$\varepsilon_p - \varepsilon_q = \frac{1}{2m}(p^2 - q^2) = \frac{1}{m}\frac{\vec{p}+\vec{q}}{2}\cdot(\vec{p}-\vec{q})$$

Now $\frac{|\vec{p}+\vec{q}|}{2} \sim k_F$, so if we require p, q to point in the same direction, we get

$$\varepsilon_p - \varepsilon_q = \frac{k_F}{m}|p-q|$$

and so the sound speed is $\frac{k_F}{m}$. However I don't see how to eliminate the pairs where [redacted] $\vec{p}-\vec{q}$ points in a different direction from $\vec{p}$. [redacted] Probably what's happening is that we have a stationary phase situation in the expression

$$\rho(xt) = \frac{1}{V}\sum_{k',k}{}' e^{i(k-k')x - i(\varepsilon_k - \varepsilon_{k'})t}\langle \Psi | a^*_{k'} a_k | \Psi \rangle$$

$$f = (k-k')x - (\varepsilon_k - \varepsilon_{k'})t$$

$$\nabla_k f = x - \frac{k}{m}t = 0$$

$$\nabla_{k'} f = x - \frac{k'}{m}t = 0 \qquad \Rightarrow \qquad k = k' \qquad ?$$

This doesn't seem to work. However if we [redacted] use

$$\Psi = \Phi + \varepsilon \sum_{p,q}{}' f(p,q)\, a^*_p a_q \Phi$$

then

$$\delta\rho(xt) = \sum_{p,q} \frac{\varepsilon f(p,q)}{V}\ \text{[redacted]}\ e^{i[(p-q)x - \text{[redacted]}(\varepsilon_p - \varepsilon_q)t]} + c.c.$$

Maybe then stationarity forces $\vec{p}-\vec{q}$ in the direction of p.

April 20, 1980

Another calculation of the sound speed in a fermi gas at 0 [redacted] temperature can be done by using $c = \sqrt{\frac{\gamma p_0}{\rho_0}}$ together with the gas laws. These are obtained as follows. [redacted] The partition function is

$$Z_{gr} = \prod_{k \in \frac{2\pi}{L}\mathbb{Z}^3} (1 + z e^{-\beta \varepsilon_k}) \qquad \varepsilon_k = \frac{k^2}{2m}, \quad V = L^3$$

$$\log Z_{gr} = \sum_k \log(1 + z e^{-\beta \varepsilon_k})$$

Recall $\varepsilon_k(V) = V^{-2/3} \varepsilon_k(1)$

$$p = \frac{1}{\beta} \frac{\partial}{\partial V} \log Z_{gr} = \sum_k \frac{z e^{-\beta \varepsilon_k}}{1 + z e^{-\beta \varepsilon_k}} \underbrace{-\frac{\partial \varepsilon_k}{\partial V}}_{\frac{2}{3}\frac{\varepsilon_k}{V}}$$

or

$$p [redacted] \sim \frac{2}{3} \int \frac{d^3k}{(2\pi)^3} \frac{z e^{-\beta \varepsilon_k}}{1 + z e^{-\beta \varepsilon_k}} \varepsilon_k = \frac{2}{3} \frac{U}{V} \qquad U = \text{internal energy.}$$

[redacted] also

$$N = z \frac{\partial}{\partial z} \log Z_{gr} = \sum_k \frac{z e^{-\beta \varepsilon_k}}{1 + z e^{-\beta \varepsilon_k}}$$

$$n = \frac{N}{V} \longrightarrow \int \frac{d^3k}{(2\pi)^3} \frac{z e^{-\beta \varepsilon_k}}{1 + z e^{-\beta \varepsilon_k}}$$

As $\beta \to \infty$ we have with $z = e^{\beta \varepsilon_F}$

$$n = \int_{k < k_F} \frac{d^3k}{(2\pi)^3} = \frac{4\pi}{(2\pi)^3} \frac{k_F^3}{3} \qquad \text{particle density}$$

$$p = [redacted] \frac{2}{3} \int_{k < k_F} \varepsilon_k \frac{d^3k}{(2\pi)^3} = \frac{2}{3} \frac{4\pi}{(2\pi)^3} \frac{1}{2m} \frac{k_F^5}{5} \qquad \text{pressure}$$

hence the adiabatic relation is that

$$p = n^{5/3} \text{ const} \qquad \text{or} \qquad p V^{5/3} = \text{const}.$$

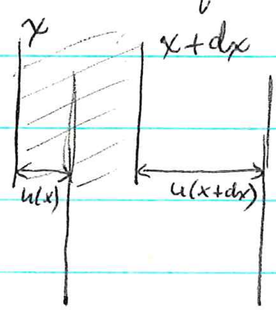
Thus $\gamma = 5/3$. Also

$$\frac{\gamma p_0}{\rho_0} = \frac{\frac{5}{3}\cancel{\gamma}\;\frac{2}{\cancel{3}}\;\frac{1}{\cancel{2}m}\;\frac{k_F^5}{5}}{m\cdot\frac{k_F^3}{\cancel{3}}} = \frac{k_F^2}{3m^2}$$

which gives a sound speed of $\boxed{\dfrac{k_F}{\sqrt{3}\,m}}$.

Actually from yesterday's work I know that the density fluctuations of long wave-length come with many frequencies, so that maybe there isn't ~~any~~ well-defined sound speed. On the other hand it might be possible to single out among the density waves those which are longitudinal.

Plasma frequency – from Feynman lectures. Look at 1-dim'l motion in the x-direction, and let $u(x)$ denote the displacement of the gas which at rest would be a position x.



The electron density of the slab between x and $x+dx$ when displaced is found as follows. Let A denote the area of the tube. Then $n_0 A dx$ is the number of electrons in the slab. The new volume is $\big(x+dx+u(x+dx) - (x+u(x))\big)A = (1+u'(x))A dx$. The new electron density is

$$n = \frac{n_0 \cancel{Adx}}{(1+u')\cancel{Adx}} = n_0(1-u')$$

assuming the displacement is small. We calculate the electric field using

$$\Delta\cdot E = 4\pi\rho = 4\pi(en - en_0)$$

fixed positive background

(One assumes the positive ions in a plasma don't move much)

Thus $$\frac{dE}{dx} = -4\pi e n_0 \frac{du}{dx} \quad \text{or} \quad E = -4\pi e n_0 u + C$$

where $C = 0$ because $E = 0$ when $u = 0$. Then Newton's law gives

$$m\,\partial_t^2 u(x) = eE(x) = -4\pi e^2 n_0 u(x)$$

or $$\left(\partial_t^2 + \frac{4\pi e^2 n_0}{m}\right) u(x) = 0$$

which means that $u(x)$ oscillates with frequency

$$\omega_p = \sqrt{\frac{4\pi e^2 n_0}{m}}$$

Note that these are oscillations, i.e. each $u(x)$ moves independently. They are not waves where the motion of $u(x)$ is propagated to the electrons nearby. A better way to say this perhaps is that the frequency as a function of wave number is constant

$$\omega(k) = \omega_p$$

For sound waves we have

$$\omega(k) = c|k|$$

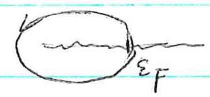
April 21, 1980

The problem is to understand the "effective potential" method for describing the interacting electron gas. We did this in the static case (see p. 729); let's review it.

We begin with a fermi gas of independent particles described by $H_0 = \frac{p^2}{2m}$. Then we take a small electric potential $\varphi(x)$ and compute the change $\delta n(x)$ in particle density. We have

$$n(x) = \langle \Phi | \psi^*(x)\psi(x) | \Phi \rangle = \langle x | P^- | x \rangle$$

$$= \frac{1}{2\pi i} \int \langle x | \frac{1}{W-H} | x \rangle \, dW$$



hence
$$\delta n(x) = \frac{1}{2\pi i} \oint \langle x | \frac{1}{W-H_0} \underbrace{\delta H}_{e\varphi(x)} \frac{1}{W-H_0} | x \rangle \, dW \qquad \text{etc.}$$

Let's now consider a small time-dependent electric potential $e\varphi(xt)$ and compute the linear response $\delta n(xt)$.

$$n(xt) = \langle \Phi(t) | \psi^*(x)\psi(x) | \Phi(t) \rangle$$

where $\Phi(t)$ satisfies Schroedinger equation for $H = H_0 + e\phi$ on Fock space, and $\Phi(t) \sim e^{-iH_0 t}\Phi$ as $t \to -\infty$. Thus

$$\Phi(t) = \underbrace{e^{-iH_0 t}\Phi}_{\Phi(t)} + \underbrace{\int_{-\infty}^{t} dt_1 \, e^{-iH_0(t-t_1)} \frac{1}{i} e\phi(t_1) e^{-iH_0 t_1} \Phi}_{\delta\Phi(t)}$$

to first order in e. Hence

$$\delta n(xt) = \langle \Phi(t) | \psi^*(x)\psi(x) | \delta\Phi(t) \rangle + \text{c.c.}$$

$$= \langle \Phi | e^{iH_0 t} \psi^*(x)\psi(x) \int_{-\infty}^{t} dt_1 \, e^{-iH_0(t-t_1)} \frac{1}{i} e\phi(t_1) e^{-iH_0 t_1} | \Phi \rangle$$

$$+ \text{c.c.}$$

The operator $\phi(t_1)$ on Fock space is

$$\int dx_1\ \phi(x_1t_1)\ \psi^*(x_1)\,\psi(x_1)$$

and so we obtain

$$\delta n(xt) = \int_{-\infty}^{t} dt_1 \int dx_1\ \frac{1}{i}\langle \Phi |\, \psi^*(xt)\psi(xt)\,\psi^*(x_1t_1)\,\psi(x_1t_1)\,| \Phi\rangle\, e\varphi(x_1t_1)$$
$$+\ \text{c.c.}$$

or finally

$$\delta n(xt) = \int dt_1 dx_1\ K(xt, x_1t_1)\ e\varphi(x_1t_1)$$

where
$$K(xt, x_1t_1) = \frac{1}{i}\langle \Phi |\, [\hat{n}(xt), \hat{n}(x_1t_1)]\,| \Phi\rangle\ \theta(t-t_1)$$

and
$$\hat{n}(xt) = \psi^*(xt)\,\psi(xt)$$

Now we want to compute this retarded Green's function K. Let's first use brute calculation:

$$\hat{n}(xt) = \sum_{k_1, k_2}{}'\ \overline{u_{k_1}(x)}\ u_{k_2}(x)\ e^{+i(\varepsilon_{k_1}-\varepsilon_{k_2})t}\ a^*_{k_1}a_{k_2}$$

hence

$$K(xt, x_1t_1) = \frac{1}{i}\sum_{k_1k_2k_3k_4} \overline{u_{k_1}(x)}\, u_{k_2}(x)\ \overline{u_{k_3}(x_1)}\, u_{k_4}(x_1)\ e^{i(\varepsilon_{k_1}-\varepsilon_{k_2})t + i(\varepsilon_{k_3}-\varepsilon_{k_4})t_1}$$
$$\langle \Phi |\, [a^*_{k_1}a_{k_2}, a^*_{k_3}a_{k_4}]\,| \Phi\rangle\ \theta(t-t_1)$$

Now

$$[a^*_{k_1}a_{k_2}, a^*_{k_3}a_{k_4}] = [a^*_{k_1}a_{k_2}, a^*_{k_3}]\,a_{k_4} + a^*_{k_3}\,[a^*_{k_1}a_{k_2}, a_{k_4}]$$
$$= a^*_{k_1}\,\delta_{k_2k_3}\,a_{k_4} - a^*_{k_3}\,\delta_{k_1k_4}\,a_{k_2}$$

$$\langle [a^*_{k_1}a_{k_2}, a^*_{k_3}a_{k_4}]\rangle = \delta_{k_1k_4}\delta_{k_2k_3}\begin{cases} 1 & \text{if } \varepsilon_{k_4}<0 \text{ and } \varepsilon_{k_2}>0 \\ -1 & \text{if } \varepsilon_{k_4}>0 \text{ and } \varepsilon_{k_2}<0 \\ 0 & \text{otherwise.} \end{cases}$$

$$K(xt,x_1t_1) = \frac{1}{i}\sum_{k_2,k_4} \overline{u_{k_4}(x)}\, u_{k_4}(x_1)\, u_{k_2}(x)\, \overline{u_{k_2}(x_1)}\; e^{-i(\varepsilon_{k_2}-\varepsilon_{k_4})(t-t_1)}\,\theta(t-t_1)$$

$$\begin{cases} 1 & \varepsilon_{k_4}<0 \quad \varepsilon_{k_2}>0 \\ -1 & \varepsilon_{k_4}>0 \quad \varepsilon_{k_2}<0 \\ 0 & \text{otherwise} \end{cases}$$

This is $n_{k_4}-n_{k_2}$

Let's now put it $\quad u_k(x) = \dfrac{e^{ikx}}{\sqrt{V}}$

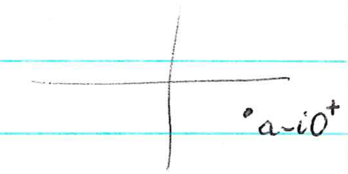
$$K(xt,x_1t_1) = \frac{1}{i}\sum_{k_2,k_4}\frac{1}{V^2} e^{-ik_4x+ik_4x_1+ik_2x-ik_2x_1-i(\varepsilon_{k_2}-\varepsilon_{k_4})(t-t_1)}\,\theta(t-t_1) \times \begin{cases} 1 \\ -1 \\ 0 \end{cases}$$

Put $q = k_2-k_4,\ k = k_4$

$$K(xt,x_1t_1) = \sum_{q,k}\frac{1}{V^2} e^{iq(x-x_1)-i(\varepsilon_{k+q}-\varepsilon_k)(t-t_1)}\,\theta(t-t_1) \times \frac{1}{i}\begin{cases} 1 & \varepsilon_{k+q}>0,\ \varepsilon_k<0 \\ -1 & \varepsilon_{k+q}<0,\ \varepsilon_k>0 \\ 0 & \text{otherwise} \end{cases}$$

Next use

$$\frac{1}{i}e^{-iat}\theta(t) = \int\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega-a+i0^+}$$



so

$$K(xt,00) = \frac{1}{V}\sum_q e^{iqx}\int\frac{d\omega}{2\pi}e^{-i\omega t}\frac{1}{V}\sum_k \frac{1}{\omega-(\varepsilon_{k+q}-\varepsilon_k)+i0^+}\underbrace{\begin{cases} 1 & \varepsilon_{k+q}>0,\ \varepsilon_k<0 \\ -1 & <0 \quad >0 \\ 0 & \text{otherwise} \end{cases}}_{n_k-n_{k+q}}$$