On the Lehmann representation. Let A, B be two hermitian operator, H_0 a Hamiltonian, say H_0 has discrete spectrum: $H_0|n\rangle = E_n|n\rangle$ with $|0\rangle$ the ground state. Then

$$\langle A(t) \cdot B\rangle = \langle 0|e^{iH_0t}Ae^{-iH_0t}B|0\rangle$$

$$= \sum_n \langle 0|A|n\rangle\langle n|B|0\rangle e^{-i(E_n-E_0)t}$$

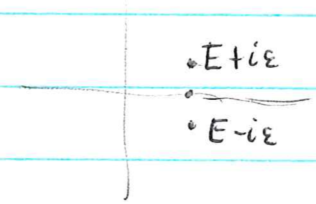
$$\langle B\cdot A(t)\rangle = \sum_n \langle 0|B|n\rangle\langle n|A|0\rangle e^{i(E_n-E_0)t}$$

Now we want to compare the two "Green's functions"

$$G^R(t) = \frac{1}{i}\theta(t)\langle[A(t),B]\rangle \qquad \text{retarded}$$

$$G^T(t) = \frac{1}{i}\{\theta(t)\langle A(t)B\rangle + \theta(-t)\langle B\,A(t)\rangle\} \qquad \text{time-ordered}$$

We have for E real, $\varepsilon = 0^+$



$$\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega - E + i\varepsilon}\frac{d\omega}{2\pi} = \begin{cases} 0 & t<0 \\ ie^{-iEt} & t>0 \end{cases}$$

$$\int \frac{e^{-i\omega t}}{\omega - E - i\varepsilon}\frac{d\omega}{2\pi} = \begin{cases} 0 & t>0 \\ -ie^{-iEt} & t<0 \end{cases}$$

hence

$$G^R(t) = \int\left\{\sum_n \frac{A_{on}B_{no}}{\omega-(E_n-E_0)+i\varepsilon} - \frac{B_{on}A_{no}}{\omega+(E_n-E_0)+i\varepsilon}\right\}e^{-i\omega t}\frac{d\omega}{2\pi}$$

$$G^T(t) = \int\left\{\sum_n \frac{A_{on}B_{no}}{\omega-(E_n-E_0)+i\varepsilon} - \frac{B_{on}A_{no}}{\omega+(E_n-E_0)-i\varepsilon}\right\}e^{-i\omega t}\frac{d\omega}{2\pi}$$

Because A, B are hermitian $B_{on} = \overline{B_{no}}$, $A_{on} = \overline{A_{no}}$. But this isn't enough, so also assume $B = A$ so that $B_{on}A_{no}$ is real. Then

$$\operatorname{Re} G^T(\omega) = \operatorname{Re} G^R(\omega) \qquad \operatorname{Im} G^T(\omega) = \left[\operatorname{Im} G^R(\omega)\right]\frac{\omega}{|\omega|}$$

because $\quad \mathrm{Im}\, \frac{1}{\omega - E + i\varepsilon} = \frac{-i\varepsilon}{(\omega-E)^2+\varepsilon^2}$

$$= -i\pi\, \delta(\omega - E)$$

is concentrated near $\omega = E$.

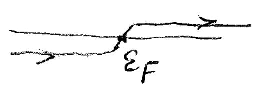
Maybe the good way to think about G^R, G^T is as follows. There is <u>one</u> analytic function of ω off the real axis

$$\sum_n \frac{A_{on} B_{no}}{\omega - (E_n - E_o)} - \frac{B_{on} A_{no}}{\omega + (E_n - E_o)}$$

which can be continued to the real axis from either side to get the Fourier transforms $G^R(\omega)$, $G^T(\omega)$.

Basic conventions

$$G(t) = \int \frac{e^{-i\omega t}}{\omega - H_o} \frac{d\omega}{2\pi} = \begin{cases} \frac{1}{i} e^{-iH_o t} P_+ & t > 0 \\ +i\, e^{-iH_o t} P_- & t < 0 \end{cases}$$



$$= \frac{1}{i} \langle T[\psi(t)\, \psi^*] \rangle$$

Let's consider the response function for the independent electron gas:

$$R(xt, x_1 t_1) = \frac{1}{i} \langle [\psi^*(xt)\psi(xt), \psi^*(x_1 t_1)\psi(x_1 t_1)] \rangle \theta(t - t_1)$$

Put in $\quad \psi(xt) = \sum u_\alpha(x) e^{-i\varepsilon_\alpha t} a_\alpha \quad$ etc. and you get

$$R(xt, x_1 t_1) = \frac{1}{i} \sum \overline{u_\alpha(x)}\, u_\beta(x)\, \overline{u_\gamma(x_1)}\, u_\delta(x_1)\, e^{i(\varepsilon_\alpha - \varepsilon_\beta)t + i(\varepsilon_\gamma - \varepsilon_\delta)t_1} \theta(t - t_1) \langle [a_\alpha^* a_\beta, a_\gamma^* a_\delta] \rangle .$$

$$\langle [a_\alpha^* a_\beta, a_\gamma^* a_\delta] \rangle = \langle a_\alpha^* [a_\beta, a_\gamma^* a_\delta] + [a_\alpha^*, a_\gamma^* a_\delta] a_\beta \rangle$$

$$= \langle a_\alpha^* \{a_\beta, a_\gamma^*\} a_\delta - a_\gamma^* \{a_\alpha^* a_\delta\} a_\beta \rangle$$

$$= \langle \delta_{\beta\gamma}\, a_\alpha^* a_\delta - \delta_{\alpha\delta}\, a_\gamma^* a_\beta \rangle$$

$$= \delta_{\alpha\delta}\delta_{\beta\gamma} \begin{cases} 1 & \varepsilon_\alpha < \varepsilon_F \quad \varepsilon_\beta > \varepsilon_F \\ -1 & \varepsilon_\alpha > \varepsilon_F \quad \varepsilon_\beta < \varepsilon_F \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad R(xt, x_1 t_1) = \overbrace{\Big\{}^{\theta(t-t_1)} \frac{1}{i} \sum_{\varepsilon_\alpha < \varepsilon_F} \overline{u_\alpha(x)}\, u_\alpha(x_1)\, e^{i\varepsilon_\alpha (t-t_1)} \sum_{\varepsilon_\beta > \varepsilon_F} u_\beta(x) \overline{u_\beta(x_1)}\, e^{i\varepsilon_\beta (t_1 - t)}$$

$$- \frac{1}{i} \sum_{\varepsilon_\alpha > \varepsilon_F} \overline{u_\alpha(x)}\, u_\alpha(x_1)\, e^{i\varepsilon_\alpha (t-t_1)} \sum_{\varepsilon_\beta < \varepsilon_F} u_\beta(x) \overline{u_\beta(x_1)}\, e^{i\varepsilon_\beta (t_1 - t)} \Big\}$$

Since for $t > t_1$

$$G(xt, x_1 t_1) = \frac{1}{i} \langle \psi(xt) \psi^*(x_1 t_1) \rangle = \frac{1}{i} \sum_{\varepsilon_\beta > \varepsilon_F} u_\beta(x) \overline{u_\beta(x_1)}\, e^{-i\varepsilon_\beta (t-t_1)}$$

$$G(x_1 t_1, xt) = i \langle \psi^*(xt) \psi(x_1 t_1) \rangle = i \sum_{\varepsilon_\alpha < \varepsilon_F} \overline{u_\alpha(x)}\, u_\alpha(x_1)\, e^{i\varepsilon_\alpha (t-t_1)}$$

we get

$$R(xt, x_1 t_1) = \frac{1}{i} \theta(t-t_1) \left\{ G(x_1 t_1, xt)\, G(xt, x_1 t_1) - \overline{G(x_1 t_1, xt) G(xt, x_1 t_1)} \right\}$$

If we look at the frequency version of R:

$$R(x, x', t) = \int \left\{ \sum_{\substack{\varepsilon_\alpha < \varepsilon_F \\ \varepsilon_\beta > \varepsilon_F}} \frac{\overline{u_\alpha(x)}\, u_\alpha(x_1)\, u_\beta(x) \overline{u_\beta(x_1)}}{\omega - (\varepsilon_\beta - \varepsilon_\alpha) + i\varepsilon} - \sum_{\substack{\varepsilon_\alpha < \varepsilon_F \\ \varepsilon_\beta > \varepsilon_F}} \frac{\overline{u_\beta(x)}\, u_\beta(x_1)\, u_\alpha(x) \overline{u_\alpha(x_1)}}{\omega + (\varepsilon_\beta - \varepsilon_\alpha) + i\varepsilon} \right\} e^{-i\omega t} \frac{d\omega}{2\pi}$$

It appears possible to interpret $R(x, x', \omega)$ as a sum of terms represented by a particle β, hole α created at x_1

propagating to x.

Here's another way of looking at this:

$$\frac{1}{i}\langle[\rho(xt),\rho(x_1)]\rangle = \frac{1}{i}\left[\langle\rho(xt)\rho(x_1)\rangle - \langle\rho(x_1)\rho(xt)\rangle\right]$$

$$\rho(x_1)|0\rangle = \sum \overline{u_\beta(x_1)}\, u_\alpha(x_1)\ \underbrace{a_\beta^* a_\alpha|0\rangle}$$

0 unless $\varepsilon_\alpha < \varepsilon_F$
and either $\beta=\alpha$ or $\varepsilon_\beta > \varepsilon_F$.

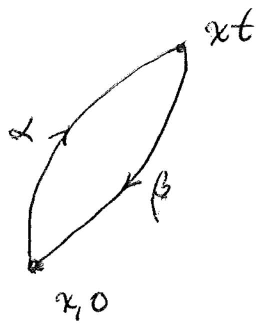
$$\rho(x_1)|0\rangle = \underbrace{\sum_{\varepsilon_\alpha<\varepsilon_F} |u_\alpha(x_1)|^2 \cdot |0\rangle}_{\langle 0|\rho(x_1)|0\rangle} + \sum_{\substack{\varepsilon_\alpha<\varepsilon_F\\ \varepsilon_\beta>\varepsilon_F}} \overline{u_\beta(x_1)}\, u_\alpha(x_1)\ \underbrace{a_\beta^* a_\alpha|0\rangle}_{\substack{\text{hole } \alpha\\ \text{part. } \beta}}$$

Hence

$$\langle\rho(xt)\rho(x_1)\rangle = \langle\rho(xt)\rangle\langle\rho(x_1)\rangle + \sum_{\substack{\varepsilon_\alpha<\varepsilon_F\\ \varepsilon_\beta>\varepsilon_F}} u_\beta(xt)\,\overline{u_\beta(x_1)}\;\overline{u_\alpha(xt)}\,u_\alpha(x_1)$$

$$= \langle\rho(xt)\rangle\langle\rho(x_1)\rangle + \sum_{\substack{\alpha \text{ hole}\\ \beta \text{ part}}} \overline{u_\alpha(x)}\,u_\beta(x)\;u_\alpha(x_1)\,\overline{u_\beta(x_1)}\;e^{-i(\varepsilon_\beta-\varepsilon_\alpha)t}$$

This seems to be made of diagrams



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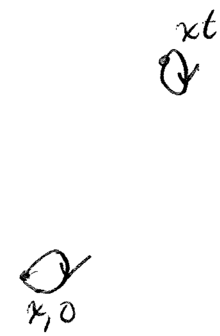


March 21, 1980

Review scattering (see Feb. 24, 1980) for potential $V(r)$:

$$(-\Delta+V)\psi = k^2\psi \qquad \frac{\hbar^2k^2}{2m}=E$$

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)\,P_l(\cos\theta)$$

$$j_l(r) \sim \frac{\sin(r-l\pi/2)}{r} \quad \text{as } r\to\infty$$

$$\sim \frac{r^l}{(2l+1)!!} \quad \text{as } r\to 0$$

$$e^{ikr\cos\theta} = \sum(2l+1)\underbrace{i^l j_l(kr)}\,P_l(\cos\theta)$$

$$\sim i^l\frac{\sin(kr-l\pi/2)}{kr} = \frac{e^{ikr}-(-1)^l e^{-ikr}}{2ikr}$$

$$\psi^+(k,r,\theta) = \sum(2l+1)\frac{1}{kr}\underbrace{\psi_l^+(k,r)}\,P_l(\cos\theta)$$

$$\sim \frac{S_l(k)e^{ikr}-(-1)^l e^{-ikr}}{2i}$$

$$\psi_l^+(k,r) \sim \text{Const.}\sin(kr-l\pi/2+\delta_l) \quad \text{as } r\to\infty \quad \Rightarrow e^{2i\delta_l}=S_l$$

$$\psi_l^+(k,r) - e^{ikr\cos\theta} \sim \underbrace{\left(\sum(2l+1)\frac{S_l-1}{2ik}P_l(\cos\theta)\right)}_{A(k,\theta)}\frac{e^{ikr}}{r}$$

The incoming wave function $\varphi = e^{ikz}$ has the particle current density

$$\vec{j} = \frac{\hbar}{2mi}\left(\varphi^*\nabla\varphi - \nabla\varphi^*\,\varphi\right) = \frac{\hbar}{2mi}(2i\vec{k}) = \frac{\hbar\vec{k}}{m} \quad \left(=\begin{smallmatrix}\text{classical}\\ \text{velocity}\end{smallmatrix}\right)$$

$$\vec{k}=k(0,0,1)$$

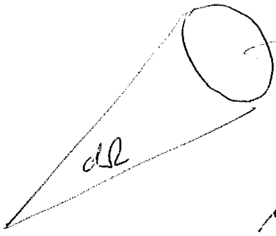
The scattered wave $\sim A(k,\theta)\dfrac{e^{ikr}}{r}$ has the current density

$$\vec{j}_{scat} = \frac{\hbar}{2mi}\left(\frac{\bar{A}e^{-ikr}}{r}\nabla\left(\frac{Ae^{ikr}}{r}\right) - c.c.\right)$$

Recall $\quad \nabla = \hat{u}_r \frac{\partial}{\partial r} + \hat{u}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{u}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$

and so taking θ derivative leads to $O(\frac{1}{r^3})$. Thus

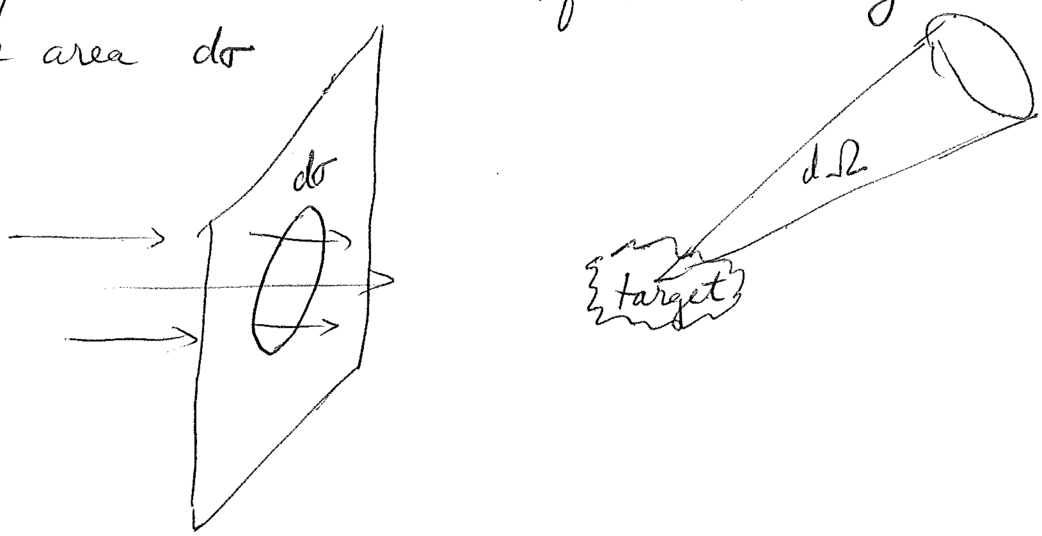
$$\vec{j}_{scat} = \frac{\hbar k}{m} |A|^2 \frac{1}{r^2} \hat{u}_r + O\left(\frac{1}{r^3}\right)$$



area $r^2 d\Omega$. The quantity

$$|\vec{j}_{scat}| r^2 d\Omega = \frac{\hbar k}{m} |A|^2 d\Omega$$

represents something like the amount scattered into the solid angle $d\Omega$, whereas $\cancel{|\vec{j}| = \frac{\hbar k}{m}}$ $|\vec{j}| d\sigma = \frac{\hbar k}{m} d\sigma$ represents the amount of the incoming beam passing thru the area $d\sigma$



Classically the trajectories ending up in $d\Omega$ cut out an area $d\sigma$, so one has a well-defined differential cross-section $\frac{d\sigma}{d\Omega}$. Quantum-mechanically you set the amounts equal:

$$\frac{\hbar k}{m} d\sigma = \frac{\hbar k}{m} |A|^2 d\Omega$$

or

$$\boxed{\frac{d\sigma}{d\Omega} = |A|^2}$$

The total cross-section is

$$\sigma(k) = \int |A|^2 d\Omega = 2\pi \int_0^\pi |A(k,\theta)|^2 \sin\theta \, d\theta = 2\pi \int_0^\pi \left| \sum (2l+1) \frac{S_l - 1}{2ik} P_l(\cos\theta) \right|^2 \times \sin\theta \, d\theta$$

$$\sigma(k) = \frac{4\pi}{k^2} \sum (2l+1) \underbrace{\left|\frac{S_l - 1}{2i}\right|^2}_{\sin^2 \delta_l}$$

Next project is the mystery of the T-matrix. If one has a complex value for the energy W one has the T-operator

$$T(W) = V + V \frac{1}{W-H_0} V + \cdots\cdots = V\left(1 - \frac{1}{W-H_0} V\right)^{-1}$$

The T matrix is defined by

$$T_{ba} = \langle \varphi_b | T(E_a + i\varepsilon) | \varphi_a \rangle \qquad \varepsilon \to 0^+$$

The mystery consists in why one concentrates on this gadget, especially why one considers b such that $E_b \neq E_a$ even though only "on the energy shell" has physical significance. The operator $T(W)$ is essentially equivalent to the resolvant $\frac{1}{W-H}$.

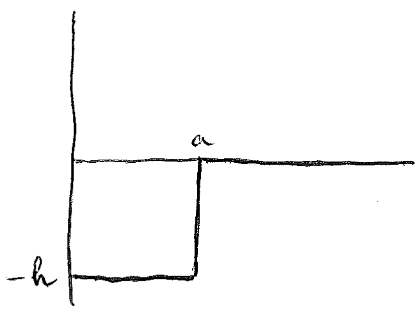
March 23, 1980

Consider scattering on the line by a potential $V(x) = V(|x|)$. The rotation group is $\mathbb{Z}/2\mathbb{Z}$, so $L^2(\mathbb{R})$ decomposes into two pieces – the odd and even functions. S-waves are given by even eigenfunctions, so that we [redacted] want solutions of

$$\left(-\frac{d^2}{dx^2} + V\right)\psi = k^2\psi \qquad \text{on } [0,\infty) \text{ satisfying}$$

$$\psi'(0) = 0$$

The free eigenfunction is $\cos(kx) = \dfrac{e^{-ikx}+e^{ikx}}{2}$. Consider now a potential well:



$$\psi(x,k) = \cos \overbrace{\sqrt{k^2+h}}^{K}\, x \qquad x<a$$

$$= Ae^{-ikx} + Be^{ikx} \qquad x>a$$

Equality of values + deriv. at $x=a$ gives

$$\cos Ka = Ae^{-ika} + Be^{ika}$$

$$-K\sin Ka = -ikAe^{-ika} + ikBe^{ika}$$

$$\begin{pmatrix} Ae^{-ika} \\ Be^{ika} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -ik & ik \end{pmatrix}^{-1} \begin{pmatrix} \cos Ka \\ -K\sin Ka \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2ik} \\ \frac{1}{2} & \frac{1}{2ik} \end{pmatrix} \begin{pmatrix} \cos Ka \\ -K\sin Ka \end{pmatrix}$$

$$S(k) = \frac{B}{A} = e^{2i\delta} = \frac{\left(\frac{1}{2}\cos Ka + \frac{i}{2k} K\sin Ka\right)e^{-ika}}{\left(\frac{1}{2}\cos Ka - \frac{i}{2k} K\sin Ka\right)e^{ika}}$$

$$\delta(k) = \arctan\left(\frac{K}{k}\tan Ka\right) - ka$$

Check: $h \to 0$, $K \to k$ and $\delta \to 0$.

this formula isn't too illuminating. A better one is

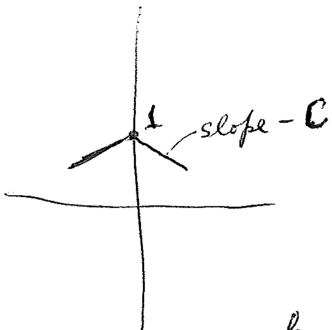
$$e^{i\delta(k)} = \frac{\cos Ka + i\frac{K}{k}\sin Ka}{\left|\cos Ka + i\frac{K}{k}\sin Ka\right|}\, e^{-ika} \qquad K = \sqrt{k^2+h}$$

Let $a \to 0$, $h \to \infty$ so that $ha \to C$ and $V \to -2C\delta(x)$. Then $Ka = \sqrt{k^2+h}\,a \sim \sqrt{a}$, $K\sin Ka \sim K^2 a = (k^2+h)a \to C$.

So $\quad e^{i\delta(k)} = \dfrac{1 + i\frac{C}{k}}{\left|1 + i\frac{C}{k}\right|}$ in the limit

As a check notice that $(k^2+\Delta)\psi = -2C\delta(x)\psi = -2C\psi(0)\delta(x)$ is satisfied by $\psi = \dfrac{\cos(kx+\delta)}{\cos\delta}$ provided

$$\psi'(0) = -k\frac{\sin\delta}{\cos\delta} = -C$$



which gives the equation $\boxed{\tan\delta = \dfrac{C}{k}}$

Here $A(k) = \frac{1}{2}\left(1 - i\frac{C}{k}\right)$ has a zero at $k = iC$ so that there is a bound state when $C > 0$.

Return to Weinberg: Weinberg speaks of a Regge pole as being a "real" particle. The ultimate goal is to understand what he means. He considers a perturbation situation $H = H_0 + V$ at a fixed energy W, and wants to compute the resolvant of H using the Born series:

$$\frac{1}{W-H} = \frac{1}{W-H_0} + \frac{1}{W-H_0}V\frac{1}{W-H_0} + \cdots\cdots$$

$$= \left(1 - \frac{1}{W-H_0}V\right)^{-1}\frac{1}{W-H_0}$$

This series converges provided that the eigenvalues of $\frac{1}{W-H_0}V$, which in good cases is a compact operator, are all inside the unit circle. He thinks of the energy W as being real and as it increases the eigenvalues of $\frac{1}{W-H_0}V$ move around. As they cross the unit circle, he speaks of there being a resonance.

Now he must have a picture in his mind other than simple potential scattering in order to worry about zeroes of $\det\left(1-\frac{1}{W-H_0}\lambda V\right)$ for λ complex.

To be more precise, consider S-wave scattering by a central potential:

$$(-\Delta + V)\psi = k^2\psi \qquad \text{on } 0 \le r < \infty$$

$$\psi(0) = 0.$$

Let $\varphi(k,r)$ be the solution normalized in same way, i.e. $\varphi'(0) = 1$, and let

$$\varphi(k,r) \sim A(k)e^{-ikr} + A(-k)e^{ikr} \qquad \text{as } r \to \infty$$

Then $(k^2-H)^{-1}$ has the kernel

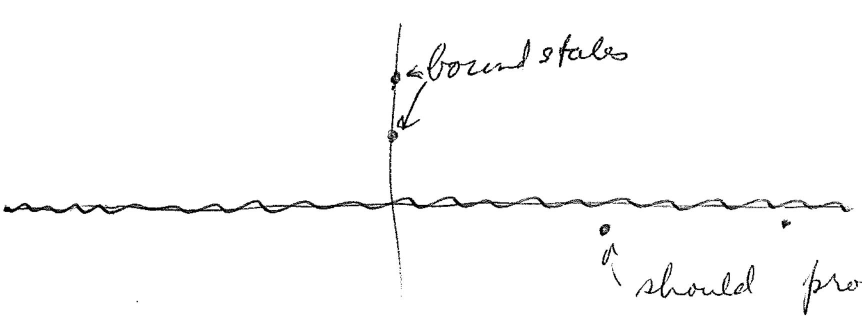
$$G(r,r') = \frac{\varphi(r_<)\,\psi(r_>)}{W(\varphi,\psi)}$$

where $\psi \sim e^{ikr}$ as $r \to \infty$; think of $\operatorname{Im}(k) > 0$. Also

$$W(\varphi,\psi) = W\left(A(k)e^{-ikr} + A(-k)e^{ikr}, e^{ikr}\right) = 2ik\,A(k)$$

φ, ψ are nice and analytic in k at least in the UHP, so that the singularities of G in k are due to $A(k)$. In particular analytically continuing $A(k)$ across the real axis, if we have a zero close to the real axis, then the Green's function peaks near this zero.

k^2 plane



k plane

$$\varphi^0(k,r) = \frac{\sin kr}{k} = \frac{e^{ikr} - e^{-ikr}}{2ik}$$

$$S(k) = -\frac{A(-k)}{A(k)} = \frac{+2ikA(-k)}{-2ikA(k)} = \frac{W(-k)}{W(k)}$$

where $W(k) = -W(\varphi, \psi)$ is the Jost function. Finally, what is the Fredholm determinant:

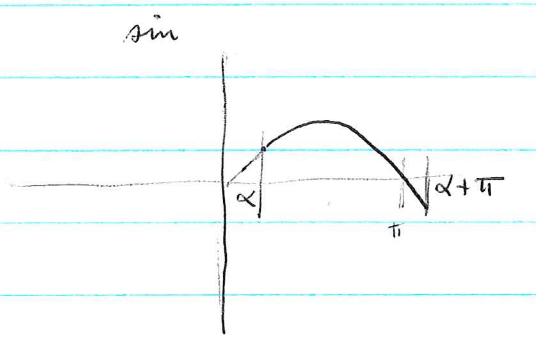
$$\det\left(1 - \frac{1}{k^2+\Delta}V\right) = \frac{\text{“}\det(k^2 - H)\text{”}}{\text{“}\det(k^2 - H_0)\text{”}} = \frac{W(\varphi,\psi)}{W(\varphi^0,\psi^0)} = \frac{2ikA(k)}{W\left(\frac{e^{ikr}-e^{-ikr}}{2ik}, e^{ikr}\right)}$$

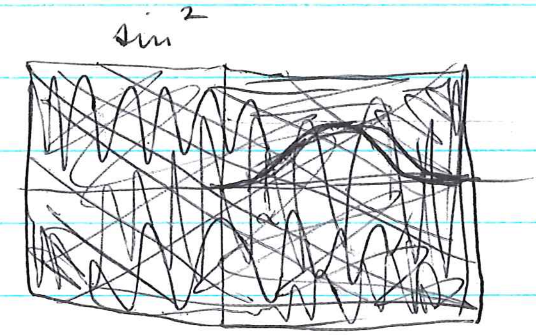
$$= \frac{2ikA(k)}{\frac{-1}{2ik}2ik} = -2ikA(k)$$

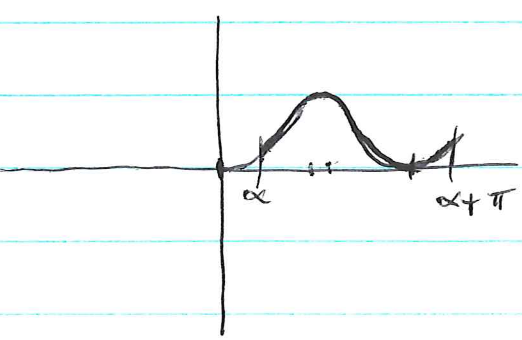
Thus the Jost function is the Fredholm determinant. Now notice that a zero of the Jost function just under the real axis produces a resonance in the scattering cross-section for S-waves:

$$\sigma_0 = \frac{4\pi}{k^2} \underbrace{\sin^2\delta_0(k)}_{\left|\frac{S_k - 1}{2i}\right|^2}$$

because $\delta_0(k) = -$ phase of Jost fn; $\delta_0(k)$ jumps by π as we pass thru the resonance.







But the point is that a zero of $\det(1 - \frac{1}{k^2+\Delta}\lambda V)$ with λ complex seems to have very little if anything to do with the Jost function $W(k)$.

So Weinberg must have a different picture, possibly dealing with field theory. Q: Are his quasi-particles related to the quasi-particles of the many body problem?

Return to the independent fermion gas. One is given a 1-particle Hamiltonian H_0 on $\mathcal{H}$ and a Fermi energy ε_F. Then one can split $\mathcal{H}$ into $\mathcal{H}^- \oplus \mathcal{H}^+$ and form Fock space with its vacuum state corresponding to the subspace $\mathcal{H}^-$. We next consider a perturbation $H = H_0 + V$ and ask about change in particle density, ground energy due to the perturbations. All of these can be computed from the resolvants $\frac{1}{W-H}$, $\frac{1}{W-H_0}$ and the fermi energy.

Density: If u_α is an orthonormal basis for $\mathcal{H}$ one has

$$\psi(x) = \sum u_\alpha(x)\, a_\alpha$$

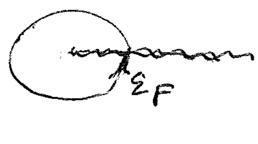
$$\psi^*(x) = \sum \overline{u_\alpha(x)}\, a_\alpha^*$$

$$\rho(x) = \langle \psi^*(x)\psi(x) \rangle = \sum u_\alpha(x)\overline{u_\beta(x)} \langle a_\alpha^* a_\beta \rangle$$

$$= \sum_{\varepsilon_\alpha < \varepsilon_F} |u_\alpha(x)|^2 \quad = \text{diagonal part of the kernel for } P^-.$$

Total number of particles is $\int \rho(x)\,dx = \operatorname{tr} P^-$. This is usually infinite, but using

$$P^- = \frac{1}{2\pi i} \oint \frac{1}{W-H}\, dW$$



one can ask about the change in [redacted] particles due to the perturbation:

$$\operatorname{tr}(P^- - P_0^-) = \frac{1}{2\pi i}\int \underbrace{\operatorname{tr}\left(\frac{1}{W-H} - \frac{1}{W-H_0}\right)}_{\frac{d}{dW}\operatorname{tr}\log\frac{W-H}{W-H_0} = \frac{d}{dW}\log\det\left(1-\frac{1}{W-H_0}V\right)} dW$$

so that

$$\Delta N = \operatorname{tr}(P^- - P_0^-) = \frac{1}{2\pi i}\int \frac{d}{dW}\log\det\left(1 - \frac{1}{W-H_0}V\right) dW$$

Similarly the change in ground energy is

$$\Delta E_0 = \operatorname{tr}(P^-H - P_0^-H_0) = \frac{1}{2\pi i}\int \operatorname{tr}\left(\frac{1}{W-H} - \frac{1}{W-H_0}\right) W\,dW$$

$$= \frac{1}{2\pi i}\int \frac{d}{dW}\log\det\left(1 - \frac{1}{W-H_0}V\right)\cdot W\,dW.$$

These integrals depend only on the jump of the resolvant across the real axis

March 27, 1980

X-ray singularity, single impurity problem, Kondo effect are all closely related. One starts with an independent fermion gas described by $H_0 = \sum \varepsilon_p a_p^* a_p$. An impurity introduces a perturbing [redacted] potential to the 1-particle Hamiltonian

$$V = \sum a_p^* V_{pp'} a_{p'}$$

In the X-ray problem one has in addition to the fermions in the gas a "core" electron which can be removed by an X-ray to form a core hole. One has a fermion Fock space with operators a_p, a_p^* for the gas, and d, d^* for the core electron. The Hamiltonian is

$$H = \sum \varepsilon_p a_p^* a_p + \varepsilon_{core}\, d^* d + \sum a_p^* V_{pp'} a_p \, d d^*$$

In the ground state $|0\rangle$ the core hole is filled so $d^*|0\rangle = 0$, and $H|0\rangle =$ [redacted] $\varepsilon_F + \varepsilon_{core}$. The Fock space is the direct sum of the gas Fock space $\mathcal{F}_0$ [redacted] with the hole filled and the gas Fock space with hole open: $d\mathcal{F}_0$.

$$\text{On } \mathcal{F}_0 \qquad H = H_0 + \varepsilon_{core}$$
$$\text{On } d\mathcal{F}_0 \qquad H = H_0 + V$$

In both these problems the interaction potential is taken to be a δ-function type at the origin, the idea being that there's some sort of screening. In the impurity case a potential $V(x)$ added to the 1-particle H_0 gives 2nd quantized operator

$$\int \psi^*(x)\, V(x)\, \psi(x)\, dx$$

Work in a box with normalized wave fns. $u_p(x) = \frac{1}{\sqrt{V}} e^{-ip\cdot x}$

whence $\qquad \psi(x) = \sum u_p(x)\, a_p = \frac{1}{\sqrt{V}} \sum_p e^{ipx} a_p$

and
$$\int \psi^*(x)\, V(x)\, \psi(x)\, dx = \sum_{p,p'} a_p^* V_{pp'} a_{p'}$$

$$V_{pp'} = \langle u_p | V | u_{p'} \rangle = \frac{1}{Vol} \int e^{-i(p-p')x} V(x)\, dx$$

Now one takes $\quad V(x) = C\,\delta(x) \quad$ and finds

$$V_{pp'} = \frac{C}{Vol}$$

whence
$$\sum_{p,p'} a_p^* V_{pp'} a_{p'} = \frac{C}{Vol} \sum_p a_p^* \sum_p a_p$$

$$= a_\gamma^* \, a_{\bar{\gamma}}$$

where $a_\gamma^* =$ ext. mult. by $|\gamma\rangle$ and $a_{\bar{\gamma}} =$ int. mult. by $\langle \bar{\gamma}|$. Such a V is a rank 1 operator, so it leads to a separable integral equation for the Green's function.

March 26, 1980

X-ray problem: We have a [redacted] gas of independent fermi particles described by a free Hamiltonian H_0 and we subject it to a potential V. The problem is to compute the amplitude $\langle 0|e^{-iHT}|0\rangle$, where $|0\rangle$ is the unperturbed ground state of the gas [redacted] Here I suppose a Fermi energy ε_F has been given so as to define $|0\rangle$; it might be simpler to remove a constant μ from the free 1-particle Hamiltonian, so as to make $\varepsilon_F = 0$. Then $H_0|0\rangle = 0$, and $|0\rangle$ is the ground state of H_0 on the [redacted] whole Fock space. Now the amplitude

$$\langle 0|e^{iH_0T}e^{-iHT}|0\rangle$$

can be computed by the Dyson-Wick formalism. Or I can use what I know about computing $\langle 0|S|0\rangle$ from Schwinger's paper. The result is roughly

$$\langle 0|e^{iH_0T}e^{-iHT}|0\rangle = \det(1+G_0\tilde{V})$$

Let's work in imaginary time. $\tilde{V}(t) = \begin{cases} V & t\in[0,T] \\ 0 & t\notin[0,T]\end{cases}$

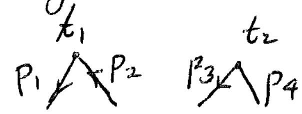
Then

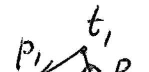
$$\langle 0|e^{iH_0T}e^{-iHT}|0\rangle = 1 - \int_{-\infty}^{\infty}dt_1\langle \tilde{V}_I(t_1)\rangle + \frac{1}{2!}\iint dt_1dt_2\langle T[\tilde{V}_I(t_1)\tilde{V}_I(t_2)]\rangle$$

Now $V = \sum a_p^* V_{pp'}a_{p'}$, [redacted], and

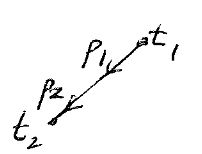
$$\langle T[\tilde{V}_I(t_1)\tilde{V}_I(t_2)]\rangle = \sum_{p_1,p_2,p_3,p_4}\tilde{V}_{p_1p_2}(t_1)\tilde{V}_{p_3p_4}(t_2)\langle T[a_{p_1}^*(t_1)a_{p_2}(t_1)a_{p_3}^*(t_2)a_{p_4}(t_2)]\rangle$$

leads to diagrams with vertices



with vertex contribution $-\tilde{V}_{p_1p_2}(t)$ for 

and

edge contribution 

$$G_0(p_2 t_2, p_1 t_1) = \langle T[a_{p_2}(t_2)\, a^*_{p_1}(t_1)] \rangle$$

and with -1 contribution for each fermion loop.

The connected diagram computations are loops

$$-\frac{1}{n}(-1)^n \int tr\left(G_0(t_1, t_2)\tilde{V}(t_2) \dots G_0(t_n, t_1)\,\tilde{V}(t_1)\right)$$

In the X-ray problem V is a rank 1 operator:

$V = |\gamma\rangle\langle\bar{\gamma}|$ and $H_0 = \sum_p |p\rangle \varepsilon_p \langle p|$ where $\varepsilon_p = \frac{p^2}{2m} - \mu$.

I'm thinking of periodic wave functions in a box:

$$|p\rangle = \frac{1}{\sqrt{Vol}}\, e^{ipx}$$

In the limit as the Vol goes to ∞ one gets

$$\sum_p \frac{1}{Vol} \longrightarrow \int \left(\frac{dp}{2\pi}\right)^d$$

$$H_0 = \int \left(\frac{dp}{2\pi}\right)^d |p\rangle\, \varepsilon_p \langle p| \qquad\qquad |p\rangle = e^{ipx}$$

We know G_0 is diagonal in the $|p\rangle$ basis:

$$G_0(p_1 t_1, p_2 t_2) = \delta_{p_1 p_2}\, G_0(p_1, t_1 - t_2)$$

$$G_0(p, t) = \langle T[a_p(t)\, a^*_p(0)] \rangle = e^{-\varepsilon_p t} \begin{cases} \langle a_p a^*_p \rangle & t > 0 \\ -\langle a^*_p a_p \rangle & t < 0 \end{cases}$$

$$= \begin{cases} e^{-\varepsilon_p t}\, \theta(t) & \varepsilon_p > 0 \\ -e^{-\varepsilon_p t}\, \theta(-t) & \varepsilon_p < 0 \end{cases}$$

This becomes even simpler if we change time to frequency.

$$G_0(p,\omega) = \int e^{i\omega t} G_0(p,t)\,dt \qquad = -\int_{-\infty}^{0} e^{i\omega t - \varepsilon_p t}\,dt \quad \text{if } \varepsilon_p < 0$$

$$= \int_0^\infty e^{i\omega t - \varepsilon_p t}\,dt = -\frac{1}{i\omega - \varepsilon_p} \quad //$$

$$\varepsilon_p > 0$$

[redacted] We want to compute $\det(1 + G_0\tilde{V})$. It seems simpler to work with $\tilde{V}G_0$ because $\det(1+G_0\tilde{V}) = \det(1+\tilde{V}G_0)$, and because $\tilde{V}$ has a small image, namely functions $f(t)\gamma_p$ with $f(t)$ supported in $[0,T]$. On these functions $\tilde{V}G_0$ has the effect

$$f(t)\gamma_p \xmapsto{G_0} \cancel{\quad} \int_0^T G_0(p, t-t')f(t')\gamma_p\,dt'$$

$$\xmapsto{\tilde{V}} \sum_{p'} \gamma_p \text{[redacted]} \bar{\gamma}_{p'} \int_0^T G_0(p', t-t')f(t')\gamma_{p'}\,dt'$$

which is the same as

$$f(t) \longmapsto \int_0^T \sum_{p'} \text{[redacted]}\bar{\gamma}_{p'} G_0(p', t-t')\gamma_{p'} \text{[redacted]}\, f(t')\,dt'$$

Thus we are interested in the kernel on functions on $[0,T]$ given by

$$K(t,t') = \sum_p G_0(p, t-t')\,\bar{\gamma}_p \gamma_p$$

(Lousy notation: $\langle\bar{\gamma}|$ is a row vector with components $\bar{\gamma}_p$).

The interesting case is when $\gamma_p = 1$ for all p, whence

$$K(t) = \sum G_0(p,t) = \sum_{\varepsilon_p<0} -e^{-\varepsilon_p t}\theta(-t) + \sum_{\varepsilon_p>0} e^{-\varepsilon_p t}\theta(t)$$

Now $\varepsilon_p = \frac{p^2}{2m} - \mu$ so that the sum over $\varepsilon_p < 0$ is finite, and is an integral over a bound interval in the continuum limit.

So the singularities of the kernel $K(t)$ come from the 2nd term, and they will be essentially the same for the kernel when $\mu = 0$:

$$K(t) = \sum e^{-\frac{p^2}{2m}t}\,\theta(t)$$

In the continuum limit this becomes

$$K(t) = \theta(t)\int\left(\frac{dp}{2\pi}\right)^d e^{-\frac{p^2}{2}\frac{t}{m}} = \theta(t)\left(\frac{1}{\sqrt{2\pi t/m}}\right)^d$$

This singularity is integrable for $d<2$, but presents problems for $d \geq 2$.

<u>Question:</u> What does one know about $\theta(t)t^\alpha$ as a distribution?

Let's change notation a bit and consider on the real line the function $(x_+)^{\lambda-1}$ where $x_+ = \theta(x)x$. For $\mathrm{Re}(\lambda) > 0$, this is integrable so, it defines a distribution. We would like to analytically continue it as a distribution. Look at the Fourier transform

$$\int e^{i\omega x}(x_+)^{\lambda-1}\,dx = \int_0^\infty e^{i\omega x}\,x^\lambda\frac{dx}{x} = \frac{\Gamma(\lambda)}{(-i\omega)^\lambda}$$

This has problems at $\lambda = 0, -1, -2, \ldots\ldots$ because of the Γ-function.

March 28, 1980

The distribution $(x_+)^{s-1}$ on $\mathbb{R}$.

If $f(x) \in \mathcal{S}(\mathbb{R})$, then

$$\int f(x)(x_+)^{s-1}\,dx = \int_0^\infty f(x)\, x^s \frac{dx}{x}$$

is an analytic function for $\mathrm{Re}(s) > 0$. Moreover if we integrate by parts we find for $\mathrm{Re}(s) > 0$, that

$$\int_0^\infty f(x) x^s \frac{dx}{x} = \frac{1}{s(s+1)\cdots(s+n)} \int_0^\infty f(x) \left(\frac{d}{dx}\right)^{n+1} (x^{s+n})\, dx$$

$$= \frac{(-1)^{n+1}}{s(s+1)\cdots(s+n)} \int_0^\infty f^{(n+1)}(x)\, x^{s+n}\, dx$$

so it is a meromorphic function of s with simple poles at $s=0,-1,-2,\ldots\ldots$. The residue at $s=-n$ is

$$\operatorname*{Res}_{s=-n} \int_0^\infty f(x) x^s \frac{dx}{x} = \frac{(-1)^{n+1}}{(-n)(-n+1)\cdots(-n+n-1)} \int_0^\infty f^{(n+1)}(x)\,dx = \frac{-1}{n!}\Big[f^{(n)}(x)\Big]_0^\infty = \frac{f^{(n)}(0)}{n!}$$

This analytic continuation process does not define $(x_+)^{-n-1}$, however this can be defined non-canonically as follows. Take $n=0$. Note that

$$L(f) = \int_0^\infty f(x)\, x^{-1}\, dx$$

is well-defined on the subspace of $f \in \mathcal{S}$ with $f(0)=0$. One can extend L in many ways to $\mathcal{S}$, i.e.

$$\tilde{L}(f) = L(f - f(0)\chi) + f(0)\,\tilde{L}(\chi)$$

where $\chi \in \mathcal{S}$ satisfies $\chi(0)=1$, and $\tilde{L}(\chi)$ is arbitrarily assigned. Any two extensions differ by a multiple of the homom. $f \mapsto f(0)$.

A standard choice for χ is e^{-ax} for some $a \in \mathbb{R}_{>0}$. One would like to take $\chi = 1$ and define $\tilde{L}(1) = 0$, but $1 \notin \mathcal{S}$. So the next best thing is to suppose there is a good extension $\tilde{L}$ such that $\tilde{L}(e^{-ax}) \to 0$ as $a \downarrow 0$. But then

$$\tilde{L}(e^{-ax}) - \tilde{L}(e^{-bx}) = L(e^{-ax} - e^{-bx}) = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$$

$$= \int_a^b dt \int_0^\infty -e^{-tx}\,dx = \int_a^b dt\left(-\frac{1}{t}\right) = -\log(b/a)$$

shows you have problems.

<u>Remark</u>: The business of defining $(x_+)^{-n-1}$ seems to be related to renormalization. Is there an analogue of the renormalization group? Is there some topological significance buried in these negative singularities?

Another approach is based on the Fourier transform

$$\int_0^\infty f(x)\, x^s \frac{dx}{x} = \int \frac{dk}{2\pi} \hat{f}(k) \underbrace{\int_0^\infty e^{ikx} x^s \frac{dx}{x}}_{\frac{\Gamma(s)}{(-ik)^s}}$$

$$= \Gamma(s) \int \frac{dk}{2\pi} \hat{f}(k)\,(-ik)^{-s}$$

This last integral is OK at $k = \infty$ as $f \in \mathcal{S}$ and at $k = 0$ it will be OK for $\mathrm{Re}(s) < 1$. So this gives the analytic continuation as well as the residues from the known properties of $\Gamma(s)$.

Renormalization gp:- Intro to, by Ma, Rev. Mod. Phys. 45 (1973), 589–614

The renormalization group is really an action of the semi-group $\mathbb{R}_{\geq 1}$ (under multiplication) on a space of probability distributions μ. μ is of the form

$$e^{-\mathcal{H}(\phi)}\,\mathcal{D}\phi$$

where $\mathcal{H}$ is a "Hamiltonian" depending on certain parameters.

Here's an example. Suppose given a d-diml lattice of volume L^d and at each point x of the lattice one has a vector $\underline{\phi}(x) = (\phi_1(x), \dots, \phi_n(x))$ which I can think of as the spin at x. A typical Hamiltonian is

$$\mathcal{H} = \int \left(a(\underline{\nabla}\phi)^2 + c\, U(\underline{\phi}^2) \right) d^d x$$

where the integral means sum over lattice points, and $\underline{\nabla}$ is to be interpreted as a suitable difference operator. Ultimately one works on the Fourier transform level so that $\phi(x)$ is replaced by its Fourier transform:

$$\phi(x) = \frac{1}{\sqrt{\mathrm{vol}}} \sum e^{ikx}\, \phi_k .$$

We have a summation because of the finite volume. Because we have a lattice k ranges over the first Brillouin zone Λ belonging to the lattice.

The Λ is important, but we can probably let $L \to \infty$ without trouble. Now the renormalization gp. idea consists in integrating out the ϕ_k for $k \in \Lambda - s\Lambda$, and then rescaling in a suitable way. Thus if $\mu = e^{-\mathcal{H}}\mathcal{D}\phi$,

$$R_s \mu = e^{-R_s(\mathcal{H})}\mathcal{D}\phi \quad \text{where}$$

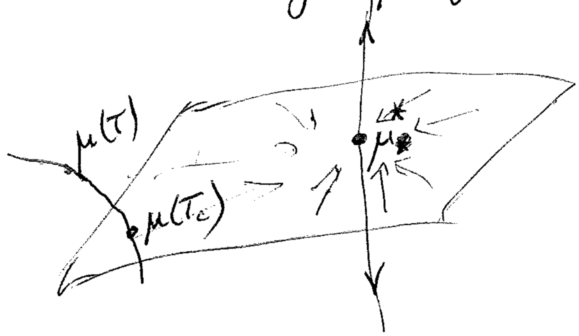
$$e^{-R_s(\mathcal{H})}\mathcal{D}\phi = \text{rescaling of } \left(\int e^{-\mathcal{H}} \prod_{k\in\Lambda - s\Lambda} d\phi_k\right)\prod_{k\in s\Lambda} d\phi_k$$

The point is that near the critical point one is interested in correlations amonst the ϕ_k with k very small.

Kadanoff idea: To understand Ising model near critical point let's assume the important configurations have all spins in a block in the same direction. (One divides the lattice into ~~blocks~~ blocks first.) This gives a new Ising model. The assumption says that important quantities don't change under this transformation, and as a consequence one gets scaling laws.

Wilson improves this "absurd" idea, by calculating the correct Hamiltonian for the block lattice, and somehow he can see the coarsening as integrating out the ϕ_k for $k\in\Lambda - s\Lambda$. Thus Wilson gets a transformation R_s on Hamiltonians.

One assumes $\{R_s\}$ has a fixpt μ^*, and maybe one can find it in examples. An important case is where locally around μ^* one has an attracted hypersurface



Now the Hamiltonian one starts with gives an 1-parameter family of measures $\mu(T)$ depending on T. At T_c this family crosses the attracting hypersurface. For large s, R_s moves $\mu(T_c)$ (and hence $\mu(T)$ for T near T_c) near μ^*. Since the basic physical quantities of interest are unchanged by R_s, we see that $\mu(T)$ is like the linearization of μ^* for T near T_c. This gives the desired scaling laws.

The above picture is approximate, and has to be worked out via examples.

March 29, 1980

Let's return to the independent fermion gas being perturbed by an external potential. We are given a Hilbert space $\mathcal{H}$ with a free Hamiltonian H_0 and a perturbed Hamiltonian $H_0 + V$; this describes the 1-particle situation. To describe the gas we must be given a fermi energy ε_F, supposed not to belong to the discrete spectrum of H_0. Then we can decompose $\mathcal{H}$ into $\mathcal{H}_0^- \oplus \mathcal{H}_0^+$, where $H_0 < \varepsilon_F$ and $H_0 > \varepsilon_F$ respectively. Associated to this decomposition is a Fock space obtained as follows. Choose an orthonormal basis u_p for $\mathcal{H}_0$ so that u_p spans $\mathcal{H}_0^-$ for $p < \eta$ and u_p spans $\mathcal{H}_0^+$ for $p > \eta$. Then

$$(1) \qquad \mathcal{F}_0 = \left(\Lambda \mathcal{H}_0^- \otimes \Lambda \mathcal{H}_0^+\right)^\wedge \quad \text{Hilbert space completion of}$$

with
$$a_p = \begin{cases} \text{int. mult by } \langle u_p| \text{ on } \Lambda\mathcal{H}_0^+ & \text{if } p > \eta \\ \text{ext } \quad '' \quad '' \quad |u_p\rangle \text{ on } \Lambda\mathcal{H}_0^- & \text{if } p < \eta \end{cases}$$

$$a_p^* = \begin{cases} \text{ext mult by } |u_p\rangle \text{ on } \Lambda\mathcal{H}_0^+ & \text{if } p > \eta \\ \text{int } \quad '' \quad '' \quad \langle u_p| \text{ on } \Lambda\mathcal{H}_0^- & \text{if } p < \eta \end{cases}$$

$$|0\rangle = 1 \otimes 1 \quad \text{in } \mathcal{F}.$$

This description of $\mathcal{F}_0$ corresponds to the picture of a many fermion system where the ground state has all the states in $\mathcal{H}_0^-$ filled.

It should be possible to describe $\mathcal{F}_0$ as the representation of the fermion CCRs belonging to $\mathcal{H}$ having a state $|0\rangle$ killed by the a_p for $p > \eta$, a_p^* for $p < \eta$.

$\mathcal{F}_0$ together with the operators a_p, a_p^* ~~constitutes~~ the kinematics of the fermion gas. To obtain dynamics we extend H_0 to an operator $\hat{H}_0$ on $\mathcal{F}_0$ in the obvious (*) way

(*) see below

using (1) and the fact that H_0 operates on both $\mathcal{H}_0^+$, $\mathcal{H}_0^-$. Then $\hat{H}_0 |0\rangle = 0$.

Notice that the formula

$$\hat{H}_0 = \sum' a_p^* \langle p|H_0|p'\rangle a_{p'} \quad ?$$

isn't correct, because

$$\langle 0| \sum a_p^* \langle p|H_0|p'\rangle a_{p'} |0\rangle = \sum_{p<\eta}' \langle p|H_0|p\rangle$$

$$= tr\left(H_0 \text{ on } \mathcal{H}_0^-\right).$$

To see things more easily, suppose $H_0 u_p = \varepsilon_p u_p$. Then

$$\sum' \varepsilon_p a_p^* a_p |0\rangle = \Big(\sum_{p<\eta}' \varepsilon_p\Big) |0\rangle$$

so the correct formula is

$$\hat{H}_0 = \sum_{p>\eta}' \varepsilon_p a_p^* a_p - \sum_{p<\eta} \varepsilon_p a_p a_p^*$$

or better

$$\hat{H}_0 = :\sum \varepsilon_p a_p^* a_p:$$

where $: \;:$ denotes normal ordering of some sort.

(*) We want $\hat{H}_0 = \sum \varepsilon_p a_p^* a_p$ + some constant to be true, hence for $p<\eta$

$$\hat{H}_0 a_p |0\rangle = \underbrace{\left[\sum \varepsilon_p a_p^* a_p, a_p\right]}_{-\varepsilon_p a_p} |0\rangle + a_p \underbrace{\hat{H}_0 |0\rangle}_{0}$$

$$= -\varepsilon_p a_p |0\rangle$$

Thus in defining $\hat{H}_0$ on $\mathcal{F}_0$ one defines it to be the derivation on $\Lambda(\mathcal{H}_0^- \oplus \mathcal{H}_0^+)$ which is $-H_0$ on $\mathcal{H}_0^-$ and H_0 on $\mathcal{H}_0^+$.

Also all these formulas are probably stupid unless $\varepsilon_F = 0$. (This is confusing - normal ordering removes $\sum_{p<\eta} \varepsilon_p$

from $\sum \varepsilon_p a_p^* a_p$. But replacing H_0 by $H_0 - \varepsilon_F$ removes $\varepsilon_F \sum a_p^* a_p$ from $\sum \varepsilon_p a_p^* a_p$. The good thing seems to always ~~start~~ with $\varepsilon_F = 0$, then $|0\rangle$ is the ground ~~state~~ for $\hat{H}_0$.)

Let's summarize the above. We begin with a Hamiltonian H_0 on $\mathcal{H}$ describing the 1-particle system. Assuming 0 not in the discrete spectrum of H_0, we split $\mathcal{H}$ into $\mathcal{H}_0^- \oplus \mathcal{H}_0^+$ according to where $H_0 < 0$ and $H_0 > 0$. Then

$$\mathcal{F}_0 = \text{Hilbert space completion of } \Lambda \mathcal{H}_0^- \otimes \Lambda \mathcal{H}_0^+$$

and

$$\hat{H}_0 = :\sum a_p^* \langle p | H_0 | p' \rangle a_{p'}: \quad \text{is the derivation}$$

equal to H_0 on $\mathcal{H}_0^+$ and $-H_0$ on $\mathcal{H}_0^-$.

The above describes the free fermion gas. Now we suppose given a perturbation $H = H_0 + V$ of the 1-particle system and try to understand its effect on the gas.

The first thing we can do is to decompose $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$ according to $H < 0$ or $H > 0$, assuming 0 not in the discrete spectrum of H. Then we can form the Fock space $\mathcal{F} = \Lambda \mathcal{H}^- \otimes \Lambda \mathcal{H}^+$ completed, which gives a description of the perturbed gas. But this doesn't allow one to see the ground energy shift

$$\operatorname{tr}(HP^- - H_0 P_0^-)$$

or the shift in the particle number

$$\operatorname{tr}(P^- - P_0^-)$$

due to the perturbation.

What I would really like to do is to define $\hat{H}$ on $\mathcal{F}_0$, show that it has a ground state Ψ of energy equal to the ground energy shift. [redacted] Furthermore the annihilation operators from $\mathcal{H}^+$ and the creation operators from $\mathcal{H}^-$ should kill Ψ, so that one gets an isomorphism of $\mathcal{F}$ with $\mathcal{F}_0$. It would be nice if [redacted] I could find a simple example which was non-trivial.

March 30, 1980

The goal is to understand the effect on an independent fermion gas of a perturbation $H = H_0 + V$ of the 1-particle Hamiltonian. The unperturbed gas is [redacted] described by a Fock space $\mathcal{F}_0$ with a ground state $|0\rangle$ determined by the splitting of the 1-particle Hilbert space $\mathcal{H}$ into the pieces where $H_0 < 0$ and > 0. [redacted] H_0 is extended to an operator $\hat{H}_0$ on $\mathcal{F}_0$ with additive constant normalization [redacted] such that $\hat{H}_0|0\rangle =$ [redacted] 0.

Now the good case occurs when we can find a ground state Ψ_0 for $\hat{H}$ on $\mathcal{F}_0$, assuming also that there is some way to define $\hat{H}$. [redacted] Ψ_0 can be described

$$\Psi_0 = Z \cdot |0\rangle + \text{states} \perp \text{to} |0\rangle.$$

where $Z = \dfrac{1}{\langle \Psi_0 | 0\rangle}$ is some kind of normalization constant. One of the goals of this perturbation business is to compute the ground energy shift $\Delta E = \langle \Psi_0 | \hat{H} | \Psi_0\rangle$ and the normalization constant Z.

From the finite-dimensional case, where $\mathcal{F}_0 = \Lambda \mathcal{H}$ and $\hat{H}_0$ can be defined without subtracting a constant, we can do the following:

$$\langle 0| e^{\hat{H}_0 T} e^{-\hat{H}T}|0\rangle = e^{+E_0 T} \sum_n \langle 0|\Psi_n\rangle e^{-\tilde{E}_n T} \langle \Psi_n | 0\rangle$$

$$\sim e^{-(\Delta E_0)T} |\langle 0|\Psi_0\rangle|^2 \qquad \text{exp. small error}$$

and so

$$\log \langle 0 | e^{\hat{A}_0 T} e^{-\hat{A} T} | 0 \rangle \sim -\Delta E_0 + \log |\langle \Psi_0 | 0 \rangle|^2 + \underset{\text{error}}{\text{exp. small}}$$

The former can be evaluated using diagrams and is

$$\log \det (1 + G_0 V_T) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \operatorname{Tr} (G_0 V_T)^n$$

where $G_0 = \left(\frac{d}{dt} + H_0\right)^{-1}$ with boundary conditions determined by $|0\rangle$, i.e.

$$G_0(t_1, t_2) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega(t_1 - t_2)}}{i\omega + H_0}$$

Also V_T is the time-dependent operator $V_T(t) = \begin{cases} V & t \in [0,T] \\ 0 & t \notin [0,T] \end{cases}$

Let do $n = 2$

$$\operatorname{Tr} (G_0 V_T)^2 = \int_0^T \int_0^T dt_1 \, dt_2 \; \operatorname{tr} \left(G_0(t_1,t_2) V G_0(t_2,t_1) V \right)$$

$$= \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \operatorname{tr} \left(\frac{1}{i\omega_1 + H_0} V \frac{1}{i\omega_2 + H_0} V \right) \underbrace{\int_0^T \int_0^T dt_1 \, dt_2 \, e^{i\omega_1(t_1 - t_2) + i\omega_2(t_2 - t_1)}}$$

$$= \int_0^T dt_1 \, e^{i(\omega_1 - \omega_2)t_1} \int_0^T dt_2 \, e^{-i(\omega_1 - \omega_2)t_2}$$

$$= \frac{1 - e^{i(\omega_1 - \omega_2)T}}{-i(\omega_1 - \omega_2)} \frac{1 - e^{-i(\omega_1 - \omega_2)T}}{i(\omega_1 - \omega_2)} = 2 \frac{1 - \cos(\omega_1 - \omega_2)T}{(\omega_1 - \omega_2)^2}$$

Now

$$h(T) = \int_{-\infty}^{\infty} f(\omega) \frac{1 - \cos \omega T}{\omega^2} \frac{d\omega}{2\pi}$$

satisfies $h''(T) = \int f(\omega) \cos \omega T \, \frac{d\omega}{2\pi} \in \mathcal{S}$, hence

$$h(T) = aT + b \quad (\text{mod } \mathcal{S})$$

where a and b depend on f.