February 19, 1980

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Situation: You have a perturbation $H = H_0 + H'$ of a self-adjoint operator on $\mathcal{H}$. One has a second quantization of $H_0$ which is a Fock space $\mathcal{F}$ which is roughly the exterior algebra of $\mathcal{H}$, equipped with the extension of $H_0$ as a derivation, and equipped with a ground state $\underline{\Phi}$ which is element corresponding to the negative eigenspace of $H_0$. Thus $\underline{\Phi}$ is the state with all negative energy levels filled. If the perturbation is sufficiently weak, then the negative eigenspace for $H$ should project nicely on that for $H_0$, so Fock space should contain a ground state $\underline{\Psi}$ for $H$ such that $\langle \underline{\Psi} | \underline{\Phi} \rangle \neq 0$. I want to understand two numbers:

$$\Delta E = \langle \underline{\Psi} | H' | \underline{\Psi} \rangle \qquad \text{assuming } \|\underline{\Psi}\| = 1 \text{ and } H_0 \underline{\Psi} = 0$$

$$|\langle \underline{\Psi} | \underline{\Phi} \rangle|^2$$

The formulas for these are messy in the case of discrete spectrum and maybe become simpler for continuous spectrum.

Example: Take $H_0 = \frac{p^2}{2}$ on $\mathcal{H} = L^2(\mathbb{R})$ and take $H = \frac{p^2}{2} + V(x)$. Now fix a $\mu = \varepsilon_F =$ Fermi-energy. Then we get a splitting

$$\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$$

where $\mathcal{H}^-$ is spanned by $e^{ikx}$ with $\frac{k^2}{2} < \varepsilon_F$ or $|k| < k_F$ where $\frac{1}{2} k_F^2 = \varepsilon_F$. (Normally one works in a box $[-L/2, L/2]$ of volume $L$ with periodic bdy conditions. Then $k \in \frac{2\pi}{L}\mathbb{Z}$ and the number $N$ of $k$ with $|k| < k_F$ is $N \sim 2k_F / (\frac{2\pi}{L})$ or $\boxed{N/L = k_F/\pi}$ so

$k_F$ is essentially the particle density.)

Let's compute the ground energy shift. We do this by figuring out the formulas in the discrete case and then extending them formally. We consider $H_\lambda = H_0 + \lambda V$ and let $E_\lambda = \langle \Psi_\lambda | H_\lambda | \Psi_\lambda \rangle$ be the ground energy. One has

$$\frac{dE_\lambda}{d\lambda} = \langle \Psi_\lambda | V | \Psi_\lambda \rangle = tr\left(P_\lambda^- V\right)$$

where $P_\lambda^-$ is the projection onto the eigenspaces for $H_\lambda$ belonging to eigenvalues $< \mu$. Now

$$P_\lambda^- = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \frac{ds}{s - H_\lambda}$$

or better the contour is $\mathrm{Re}(s) = \mu$.

Thus

$$\frac{dE_\lambda}{d\lambda} = \frac{1}{2\pi i} \int tr\left(\underbrace{\frac{1}{s - H_\lambda} V}\right) ds$$

$$\frac{1}{s-H_0} V + \frac{1}{s-H_0} \lambda V \frac{1}{s-H_0} V + \cdots$$

and so integrating from $\lambda = 0$ to $\lambda = 1$ gives.

$$\Delta E = \frac{1}{2\pi i} \int tr\left(\underbrace{\frac{1}{s-H_0} V + \frac{1}{2}\left(\frac{1}{s-H_0} V\right)^2 + \cdots}_{-\log\left(1 - \frac{1}{s-H_0} V\right)}\right) ds$$

or

$$\boxed{+\Delta E = \frac{1}{2\pi i} \int -\log \det\left(1 - \frac{1}{s-H_0} V\right) ds}$$

$\mathrm{Re}(s) = \mu$

In the case $H_0 = \frac{p^2}{2} = -\frac{1}{2}\frac{d^2}{dx^2}$, recall that for $\operatorname{Im} k > 0$ we have

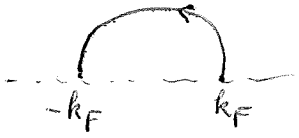
$$\det\left(1 - \frac{1}{k^2+D^2} 2V\right) = A(k)$$

where $A(k)$ is determined by the scattering for the equation $(k^2 + D^2 - 2V)\psi = 0$ :

$$e^{-ikx} \longleftrightarrow A(k)e^{-ikx} + B(k)e^{ikx}.$$

So let's substitute $s = k^2/2$ and we get

$$\Delta E = \frac{1}{2\pi i} \int -\log\left(A(k)\right)\, k\,dk$$




and the contour should enclose the bound states at $k = ik_j$. Suppose there are no bound states. Then we have

$$\Delta E = \frac{1}{2\pi i} \int_{-k_F}^{k_F} \log\left(\frac{A(k)}{A(-k)}\right)\, k\,dk$$

and $\dfrac{A(k)}{A(-k)}$ is essentially the determinant of the scattering matrix.

Recall the basic asymptotic behavior of

$$\det\left(1 - \frac{1}{k^2+D^2} q\right) = A(k)$$

One has

$$\log A(k) = -\left[\operatorname{tr}\left(\frac{1}{k^2+D^2}q\right) + \frac{1}{2}\operatorname{tr}\left(\frac{1}{k^2+D^2}q\right)^2 + \cdots\right]$$

$$= -\iint \frac{e^{ik|x-x'|}}{2ik} q(x)\,dx + \frac{1}{2}\int \frac{e^{2ik|x-x'|}}{(2ik)^2} q(x)q(x')\,dx\,dx' + \cdots$$

so $$\log A(k) \sim -\frac{1}{2ik}\int_{-\infty}^{\infty} g(x)\,dx + O\left(\frac{1}{k^3}\right) \quad \text{as } k \to \pm\infty.$$

(see p. 543). Consequently provided $\int g(x)\,dx = 0$ one sees $\Delta E$ remains finite as $k_F \longrightarrow \infty$.

February 20, 1980

We are considering a gas of independent fermions, which we think of as electrons in a metal, and then we introduce an impurity and try to see what happens. We describe a single fermion by $H_0 = \frac{p^2}{2}$ on $L^2(\mathbb{R})$, and to get the gas we have to give a chemical potential or Fermi energy $\varepsilon_F$. The impurity is represented by a perturbation $H = H_0 + U(x)$. As the impurity potential is turned on: $H_\lambda = H_0 + \lambda U$ the density of the gas changes, the total charge, energy changes, and we want to compute these quantities.

The way to visualize things is to start off in a box so as to have discrete spectra, but this really shouldn't be necessary. Let's work on the line in a box $-L/2 \le x \le L/2$ with $H_0 = -\Delta = -\frac{d^2}{dx^2}$.

Basic eigenfunctions are

$$u_k(x) = \frac{1}{\sqrt{L}} e^{ikx} \qquad k \in \frac{2\pi}{L}\mathbb{Z}$$

and $Hu_k = k^2 u_k$. Extend $H_0$ to a derivation of the exterior algebra of $\mathcal{H} = L^2([-L/2, L/2])$ and let $N$ be the number of particles belonging to the Fermi energy $\varepsilon_F = k_F^2$. Thus

$$N = \text{card}\{k \mid k < k_F\}$$

$$N \cdot \frac{2\pi}{L} \sim 2k_F \qquad \text{or} \qquad \frac{N}{L} \sim \frac{1}{\pi} k_F$$

where this last relation becomes exact as $N, L \to \infty$.

So now we have this $N$-particle ground state which

is the filled Fermi sea. Lets compute the particle density at a given point $x$. What does this mean? Answer: It is $\langle \Psi | \psi^*(x)\psi(x) | \Psi \rangle$ where $\psi(x)$ is the interior product on the exterior algebra associated to the linear functional $\psi(x)f = f(x)$ on $\mathcal{H}$. So if the constants $c_k$ are such that

$$\psi(x) = \sum_k{}' c_k a_k$$

we have $c_k = \psi(x) u_k = u_k(x) = \frac{1}{\sqrt{L}} e^{-ikx}$, so

$$\psi(x) = \sum{}' \frac{1}{\sqrt{L}} e^{-ikx} a_k$$

Hence

$$\langle \Psi | \psi(x)^* \psi(x) | \Psi \rangle = \sum_{k,k'} \frac{1}{L} e^{-ik'x + ikx} \underbrace{\langle \Psi | a_{k'}^* a_k | \Psi \rangle}_{\begin{cases} 0 & \text{unless } k' = k < k_F \\ 1 & \text{otherwise} \end{cases}}$$

$$= \frac{N}{L}$$

which checks.

Lets carry out the analogous construction for the perturbed operator $H = H_0 + U$. Again work in a box so the spectrum is discrete. Let $u_\alpha(x)$ be an orthonormal basis of eigenfunctions $Hu_\alpha = \varepsilon_\alpha u_\alpha$. Then

$$\psi(x) = \sum{}' c_\alpha a_\alpha \qquad c_\alpha = \psi(x) u_\alpha = u_\alpha(x)$$

$$\psi(x) = \sum{}' u_\alpha(x) a_\alpha$$

So the particle density in the ground state $\Psi$ corresponding to the Fermi energy $\varepsilon_F$ is

$$\langle \Psi | \psi(x)^* \psi(x) | \Psi \rangle = \sum_{\alpha,\alpha'}{}' \overline{u_{\alpha'}(x)}\, u_\alpha(x) \underbrace{\langle \Psi | a_{\alpha'}^* a_\alpha | \Psi \rangle}_{\begin{cases} 0 & \text{unless } \alpha = \alpha' \\ & \text{and } \varepsilon_\alpha < \varepsilon_F \\ 1 & \text{otherwise} \end{cases}}$$

$$= \sum_{\varepsilon_\alpha < \varepsilon_F} \overline{u_\alpha(x)}\, u_\alpha(x)$$

This is the diagonal of the kernel for othogonal projection $P^-$ on the eigenspaces of $H$ for energies $< \varepsilon_F$.

$$P^- = \frac{1}{2\pi i} \int \frac{1}{s - H} \, ds$$

$\circlearrowleft$ $\mathrm{Re}(s) = \varepsilon_F$

Recall that

$$G(x,t') = \begin{cases} e^{-H(t-t')} P^+ & t > t' \\ e^{-H(t-t')} (-P^-) & t < t' \end{cases}$$

$$\psi(xt) = \sum u_\alpha(x) e^{-\varepsilon_\alpha t} a_\alpha$$

$$\psi^*(x't') = \sum \overline{u_\alpha(x')} \, e^{\varepsilon_\alpha t'} a_\alpha^*$$

$$G(xt, x't') = \langle \Psi | \psi(xt) \psi^*(x't') | \Psi \rangle \qquad t > t'$$

$$= \sum u_\alpha(x) e^{-\varepsilon_\alpha t} \overline{u_{\alpha'}(x')} \, e^{\varepsilon_{\alpha'} t'} \langle \Psi | a_\alpha a_{\alpha'}^* | \Psi \rangle$$

$$G(xt, x't') = \begin{cases} \sum_{\varepsilon_\alpha > \varepsilon_F}' u_\alpha(x) \overline{u_\alpha(x')} \, e^{-\varepsilon_\alpha (t-t')} & t > t' \\ -\sum_{\varepsilon_\alpha < \varepsilon_F}' u_\alpha(x) \overline{u_\alpha(x')} \, e^{-\varepsilon_\alpha (t-t')} & t < t' \end{cases}$$

The particle density is

$$\langle \Psi | \psi^*(x) \psi(x) | \Psi \rangle = \lim_{t \to 0-} -G(xt, x0)$$

February 22, 1980

Consider $H = -\Delta + V$ on $\mathcal{H} = L^2(\mathbb{R})$ where $V \in C_0^\infty$.

Fix a Fermi energy $\varepsilon_F$. Then we get a splitting $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$, where $H < \varepsilon_F$ on $\mathcal{H}^-$ and $H > \varepsilon_F$ on $\mathcal{H}^+$. We then form the fermion Fock space of $\mathcal{H}$ with ground state $\Psi$ representing the Fermi sea with all the states in $\mathcal{H}^-$ filled:

$$\mathcal{F} = \wedge \mathcal{H}^- \otimes \wedge \mathcal{H}^+$$

We have a Green's function

$$G(xt, x't') = \langle \Psi | T[\psi(xt)\, \psi^*(x't')] | \Psi \rangle$$

which we worked out yesterday in the case where $\mathbb{R}$ is replaced by a box, or where $V$ grows fast enough so that the spectrum is discrete. Recall this: Let $u_\alpha$ be an orthonormal basis of eigenfunctions for $H$, $Hu_\alpha = \varepsilon_\alpha u_\alpha$. Then

$$\psi(x) = \text{"interior mult. by } f \longmapsto f(x) \text{ on } \wedge \mathcal{H}\text{"}$$
$$= \sum u_\alpha(x)\, a_\alpha$$

so $$\psi(xt) = \sum u_\alpha(x)\, a_\alpha(t) \qquad a_\alpha(t) = e^{Ht} a_\alpha e^{-Ht} = e^{-\varepsilon_\alpha t} a_\alpha$$

$$\psi(xt) = \sum u_\alpha(x) e^{-\varepsilon_\alpha t} a_\alpha$$

$$\psi^*(x't') = \sum \overline{u_\alpha(x')}\, e^{\varepsilon_\alpha t'} a_\alpha^*$$

$t > t'$ $$G(xt, x't') = \sum u_\alpha(x) \overline{u_{\alpha'}(x')}\, e^{-\varepsilon_\alpha t + \varepsilon_{\alpha'} t'} \langle \Psi | a_\alpha a_{\alpha'}^* | \Psi \rangle$$

$t < t'$ $$\langle \Psi | -a_{\alpha'}^* a_\alpha | \Psi \rangle$$

So

$$G(xt,x't') = \sum_{\varepsilon_\alpha > \varepsilon_F} u_\alpha(x)\,\overline{u_\alpha(x')}\; e^{-\varepsilon_\alpha(t-t')} \qquad t>t'$$

$$-\sum_{\varepsilon_\alpha < \varepsilon_F} u_\alpha(x)\,\overline{u_\alpha(x')}\; e^{-\varepsilon_\alpha(t-t')} \qquad t<t'$$

In the finite temperature case one has

$$\langle a_\alpha a^*_{\alpha'} \rangle = \delta_{\alpha\alpha'}\,\frac{1}{1+e^{-\beta\varepsilon_\alpha}}$$

$$\langle a^*_{\alpha'} a_\alpha \rangle = \delta_{\alpha\alpha'}\,\frac{e^{-\beta\varepsilon_\alpha}}{1+e^{-\beta\varepsilon_\alpha}}$$

The reason is that

$$\langle a_\alpha a^*_{\alpha'} \rangle = \frac{tr\left(e^{-\beta H} a_\alpha a^*_{\alpha'}\right)}{tr\left(e^{-\beta H}\right)}$$

and because $H$ is diagonal, the exterior algebra can be written as a tensor product of the $\alpha, \alpha'$ part and the rest of $\mathcal{H}$. Thus the finite temperature Green's function is

$$G(xt,x't') = \sum_\alpha u_\alpha(x)\overline{u_\alpha(x')}\; e^{-\varepsilon_\alpha(t-t')} \begin{cases} \dfrac{1}{1+e^{-\beta\varepsilon_\alpha}} & t>t' \\[2ex] \dfrac{-e^{-\beta\varepsilon_\alpha}}{1+e^{-\beta\varepsilon_\alpha}} & t<t' \end{cases}$$

The problem now is to understand what happens to these formulas as the box gets infinite and the spectrum is continuous

Note: The above formulas for finite temperature case should have $H$ replaced by $H - \varepsilon_F$. In the sequel let's assume this has been done, so that $\varepsilon_F = 0$.

Formulas for the finite temperature Green's function

1) $$G(t,t') = e^{-H(t-t')} \begin{cases} \dfrac{1}{1+e^{-\beta H}} & t>t' \\ \dfrac{-e^{-\beta H}}{1+e^{-\beta H}} & t<t' \end{cases}$$

Note that $0 \le t,t' \le \beta$ and that

$$G(\beta,t') = e^{-H(\beta-t')} \frac{1}{1+e^{-\beta H}} = -G(0,t')$$

so that $G$ is the inverse of $\frac{d}{dt}+H$ on $[0,\beta]$ with anti-periodic boundary conditions. Hence we can compute it by Fourier series and get

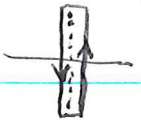
2) $$G(t,t') = \sum_{k \in \frac{2\pi}{\beta}(\frac{1}{2}+\mathbb{Z})} \frac{1}{\beta} \frac{e^{-ik(t-t')}}{-ik+H}$$

Next let us use contour integration to evaluate the above sum. The summation takes place over those $z=ik$ with $k \in \frac{2\pi}{\beta}(\frac{1}{2}+\mathbb{Z})$, i.e. where $e^{\beta z}+1=0$. At such a zero, one has $\frac{d}{dz}(e^{\beta z}+1) = \beta e^{\beta z} = -\beta$. Hence by residues:

3) $$G(t,t') = \frac{-1}{2\pi i} \int \frac{e^{z(t-t')}\,dz}{(z+H)(e^{\beta z}+1)}$$

The contour is a thin rectangle enclosing the zeroes $ik$ of $e^{\beta z}+1$, thin enough so at avoid the spectrum of $H$ which we think of as being seperated by $\varepsilon_F = 0$. Another formula is

3') $$G(t,t') = \frac{-1}{2\pi i} \int \frac{e^{z(t-t')}}{z+H} \left( \frac{1}{e^{\beta z}+1} - 1 \right) dz$$

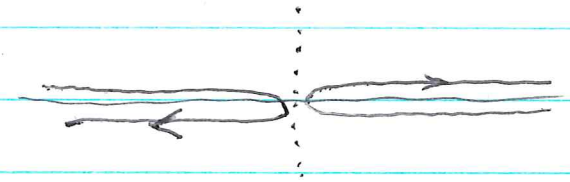


$$\frac{-e^{\beta z}}{1+e^{\beta z}}$$

If $t-t' > 0$, then at least when $t' \in (0,\beta)$ one has $0 < t-t' < \beta$ and so

$$\frac{e^{z(t-t')}}{e^{z\beta}+1}$$

decays as either $Re(z) \to +\infty$ or $-\infty$. Hence the contour (in 3) can be deformed to



which gives

$$G(t,t') = \frac{e^{-H(t-t')}}{1+e^{-\beta H}} \qquad t > t'$$

On the other hand if $t-t' < 0$, then $-\beta < t-t' < 0$ so

$$e^{z(t-t')} \frac{(-e^{\beta z})}{1+e^{\beta z}}$$

decays as either $Re(z) \to +\infty$ or $Re(z) \to -\infty$ so the contour in (3') can be deformed in the same way to yield

$$G(t,t') = - e^{-H(t-t')} \frac{e^{-\beta H}}{1+e^{-\beta H}} \qquad t < t'$$

which checks the formula 1).

Yesterday we worked out the Green's function at non-zero temperature for a system of independent fermions governed by a 1-particle Hamiltonian $H$ and with a given Fermi energy. We found (for $\varepsilon_F = 0$)

$$G(t,t') = e^{-H(t-t')} \begin{cases} \frac{1}{1+e^{-\beta H}} & t > t' \\ \frac{-e^{-\beta H}}{1+e^{-\beta H}} & t < t' \end{cases}.$$

It follows that the density matrix is

$$-G(0^-,0) = \frac{e^{-\beta H}}{1+e^{-\beta H}} = \frac{1}{e^{\beta H}+1}.$$

Thus the density of particles at a point $x$

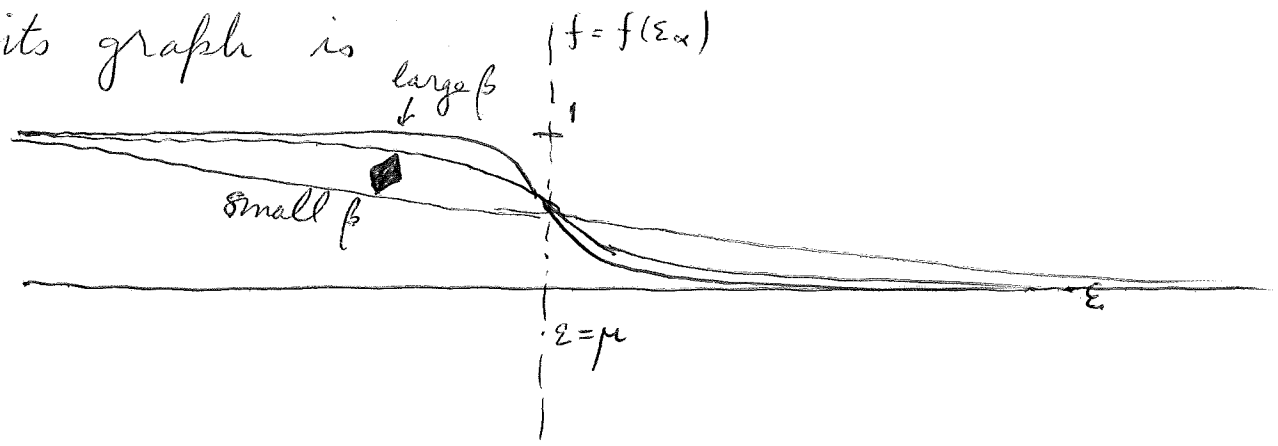
$$\langle \psi^*(x)\psi(x) \rangle = -G(x0^-, x0)$$

$$= \sum_\alpha{}' |u_\alpha(x)|^2 \frac{1}{e^{\beta\varepsilon_\alpha}+1}$$

if $u_\alpha$ is an orthonormal basis of eigenfunctions. The last function is the Fermi function; in general is would be

$$f_\alpha = \langle a_\alpha^* a_\alpha \rangle = \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)}+1}$$

and its graph is




The above extends to the case where $H$ has

continuous spectrum:

$$\langle x | \frac{1}{e^{\beta H}+1} | x \rangle = \int \frac{1}{e^{\beta \varepsilon}+1} \, d E_\varepsilon(x,x)$$

The next project will be to take $H = -\Delta + V$ with $V \in C_0^\infty$ and see if we can compute the particle density well enough so as to understand Friedel oscillations.

---

We need to review eigenfunctions for $H$. Notation: $\psi(x,k)$ ($\phi(x,k)$) is the solution of $(H - k^2)\psi = 0$ or

1) $$(k^2 + \Delta)\psi = V\psi$$

such that

$$\psi(x,k) \sim e^{-ikx} \quad \text{as} \quad x \longrightarrow +\infty$$
$$\phi(x,k) \sim e^{-ikx} \quad \text{as} \quad x \longrightarrow -\infty$$

where we think of $k$ as being in the UHP. The scattering is given by

$$e^{-ikx} \longleftrightarrow A(k)e^{-ikx} + B(k)e^{ikx}$$

or $$\phi(x,k) = A(k)\psi(x,-k) + B(k)\psi(x,k)$$

and the Green's function is

$$G_k(x,x') = \frac{\phi(x_<,k)\,\psi(x_>,k)}{A(k)\,2ik}$$

Let's go over physics notation: $\psi_k^+(x)$ denotes the solution of the Schroedinger equation 1) which differs from $e^{ikx}$ by an outgoing wave. Recall how this

goes: Consider a solution of $i\partial_t \psi = H\psi$ of the form

$$\psi(x,t) = \int e^{ikx - ik^2 t/2m} f(k)\,dk \qquad x \text{ large}$$

where $f$ has support close to $k_0$. Then for large $(x,t)$, $\psi(x,t)$ is negligible unless the exponent is stationary near $k = k_0$. But

$$\frac{d}{dk}\left(kx - \frac{k^2 t}{2m}\right) = x - \frac{k}{m}t = 0$$

means that $\psi$ represents a wave packet travelling with velocity $k/m$, hence momentum $k$. Thus $e^{ikx}$ represents a wave travelling in the direction $k$. It follows that for $k > 0$

$$\psi_k^+ : \tilde{R}(k)e^{-ikx} + e^{ikx} \longleftrightarrow T(k)e^{-ikx}$$

and for $k < 0$

$\psi_k^+$ :

$$T(k)e^{ikx} \longleftrightarrow e^{ikx} + R(k)e^{-ikx}$$

Unfortunately if we give $k$ a small positive imaginary part: $k + i\eta$, then

$$e^{i(k+i\eta)x} = e^{-\eta x + ikx}$$

decays as $x \to +\infty$ which is not what you expect.

?

---

The problem is that $\psi_k^+$ is defined via Lippmann-Schwinger

$$\psi_k^+ = \varphi_k + \frac{1}{k^2 + i\varepsilon + \Delta} V \psi_k^+$$

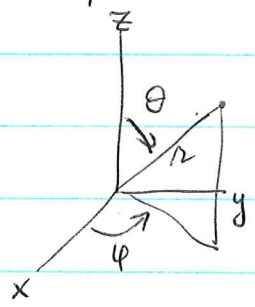
so that one wants to push $k$ so that $k^2 \in$ UHP. Thus we want $k \mapsto k+i\eta$, where $(k+i\eta)^2 = k^2 - \eta^2 + 2ik\eta \in$ UHP so we want $k\eta > 0$. The same formula will work in 3 dims. Note that

$$e^{i(k+i\eta)x} = e^{ikx} e^{-\eta x}$$

decays in the direction $\eta$ which since $k \cdot \eta > 0$ means this function decays in the $k$-direction. Thus we look at complex $\vec{k}$ with $\mathrm{Im}(k^2) > 0$ and define $\psi_k^+$ to be the eigenfunction for $H$ with eigenvalue $k^2$ which differs from $e^{ikx}$ by an $L^2$ function.

So the moral is that the physics way of thinking about the scattering on $\mathbb{R}$ differs from the way I am familiar with. Hence one should perhaps work with a symmetrical situation.

Let's try to do the scattering in 3 dimensions with a spherically symmetric potential $V(r)$. Recall the formulas




$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

$$(1 + r^2 - 2r\cos\theta)^{-1/2} = \sum_{\ell=0} r^\ell P_\ell(\cos\theta)$$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \ell(\ell+1)\right] P_\ell(\cos\theta) = 0$$

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2}{2\ell+1}\delta_{\ell\ell'} \qquad P_\ell(x) = \frac{(2\ell-1)!!}{\ell!}x^\ell + O(x^{\ell-2})$$

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} (2\ell+1) i^\ell j_\ell(kr) P_\ell(\cos\theta)$$

$$j_\ell(r) \sim \frac{\sin(r - \ell\pi/2)}{r} \quad \text{as } r \to \infty \qquad j_\ell(r) \sim \frac{1}{(2\ell+1)!!} r^\ell \quad \text{as } r \to 0$$

February 29, 1980

Given a free eigenfunction $e^{i\underline{k}\cdot\underline{x}}$ the perturbed eigenfunction asymptotic to it as $t \to -\infty$ in some sense is obtained by solving the Lippmann-Schwinger equation:

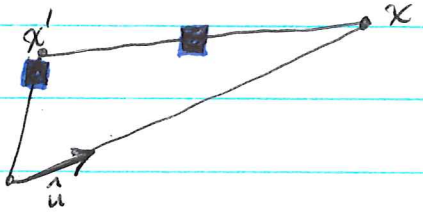
$$\psi_{\underline{k}}^{+}(x) = e^{i\underline{k}\cdot\underline{x}} + \frac{1}{k^2 + i0^+ - H_0} V \psi_k^+ \qquad \text{where}$$

$$\langle x | \frac{1}{k^2 + i0^+ + \Delta} | x' \rangle = \frac{e^{ikr}}{-4\pi r} \qquad r = |x - x'| \quad \text{and} \quad k > 0.$$

Thus

$$\psi_{\underline{k}}^{+}(x) = e^{i\underline{k}\cdot x} + \int \frac{e^{ik|x-x'|}}{-4\pi|x-x'|} V(x') \psi_{\underline{k}}^{+}(x')\, d^3x'$$

Let's look at this for $x = r\hat{u}$, $\hat{u}$ a fixed unit vector and $r \to +\infty$.




$$|x - x'| = \sqrt{r^2 + |x'|^2 - 2r\,\hat{u}\cdot x'}$$

$$= r\left(1 - \frac{\hat{u}\cdot x'}{r}\right) + O\left(\frac{1}{r}\right)$$

$$= r - \hat{u}\cdot x' + O\left(\frac{1}{r}\right)$$

$$\psi_{\underline{k}}^{+}(x) - e^{i\underline{k}\cdot x} = \int \frac{e^{ikr}}{r} \frac{e^{-ik\hat{u}\cdot x'}}{-4\pi} \left(1 + O\left(\frac{1}{r}\right)\right) V(x') \psi_{\underline{k}}^{+}(x')\, dx'$$

Thus we see that provided we let $x \to \infty$ radially: $x = r\hat{u}$ with $r \to \infty$, we have

$$\psi_{\underline{k}}^{+}(x) = e^{i\underline{k}\cdot x} + \frac{e^{ikr}}{r} f(k\hat{u}, \underline{k}) + O\left(\frac{1}{r^2}\right)$$

$$f(\underline{k}', \underline{k}) = -\frac{1}{4\pi} \int e^{-i\underline{k}'\cdot x} V(x) \psi_{\underline{k}}^{+}(x)\, d^3x$$

Consider the situation where $V = V(r)$. Then

we use partial wave expansions:

$$e^{ikz} = \sum_\ell i^\ell (2\ell+1)\, \underbrace{j_\ell(kr)}\, P_\ell(\cos\theta)$$

$$\sim \frac{\sin(kr - \frac{1}{2}\ell\pi)}{kr} = \frac{i^{-\ell} e^{ikr} - i^\ell e^{-ikr}}{2ikr}$$

Let $\psi_k^+(x) = \psi_{\underline{k}}^+(x) =$ [redacted] $e^{-ikz}$ + scattered wave, i.e. $\underline{k} = (0,0,k)$ and $k > 0$ always. Partial waves:

$$\psi_k^+(r,\theta) = \sum_\ell [\text{redacted}]\, R_\ell(r,k)\, P_\ell(\cos\theta)$$

where $R_\ell(r,k)$ satisfies

$$\left\{ -\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} + \frac{\ell(\ell+1)}{r^2} + V(r) - k^2 \right\} R_\ell = 0$$

$$R_\ell \sim \text{const.}\, r^\ell \quad \text{as } r \to 0.$$

For large $r$ one has

$$R_\ell \sim \text{const.}\, \frac{\sin\left(kr - \frac{1}{2}\ell\pi + \delta_\ell(k)\right)}{r}$$

where $\delta_\ell(k)$ is the so-called phase shift. Since $\psi_k^+ - e^{ikz}$ [redacted] $\sim A\frac{e^{ikr}}{r}$, we expect in

$$R_\ell \sim \text{const}\, \frac{e^{ikr} i^{-\ell} e^{i\delta_\ell} - e^{-ikr} i^\ell e^{-i\delta_\ell}}{r}$$

that the constant makes the $\frac{e^{-ikr}}{r}$ term agree with that for $e^{ikz}$. Thus

$$R_\ell(r,k) \sim (2\ell+1) \frac{e^{2i\delta_\ell(k)} e^{ikr} - (-1)^\ell e^{-ikr}}{2ikr}$$

and so we get

$$\psi_k^+(r,\theta) - e^{ikz} \sim A(k,\theta)\frac{e^{ikr}}{r} \qquad \text{as } r \to \infty$$

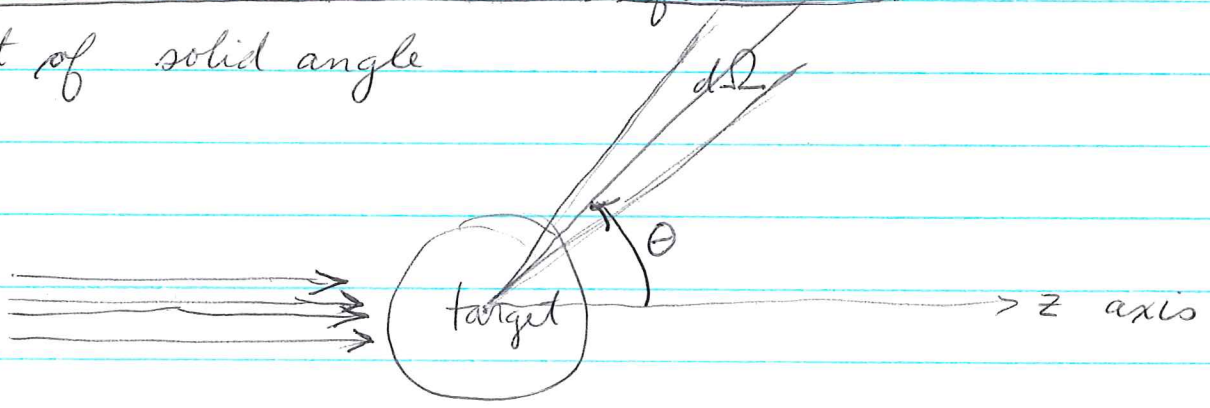
where $$A(k,\theta) = \sum_\ell (2\ell+1) \frac{S_\ell(k)-1}{2ik} P_\ell(\cos\theta)$$ scattering amplitude

$$S_\ell(k) = e^{2i\delta_\ell(k)}$$

The differential cross-section is the quantity

$$\frac{d\sigma}{d\Omega} = |A(k,\theta)|^2$$

and it has the following interpretation. One thinks of $e^{ikz}$ as representing a beam of particles with momentum $k$, $V(r)$ is the potential of many sufficiently separated identical atoms which form the target. $d\Omega$ is an element of solid angle




and $d\sigma$ measures the fraction of the beam scattered into $d\Omega$. The total scattering cross-section is

$$\sigma = \int A(k,\theta)^2 \underbrace{d\Omega}_{2\pi \sin\theta\, d\theta} = \sum_\ell \left|\frac{S_\ell(k)-1}{2ik}\right|^2 (2\ell+1)^2 \frac{2\pi\cdot 2}{2\ell+1}$$

$$\boxed{\sigma = \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2\delta_\ell(k)}$$

$$\text{Im } A(k,\theta) = \sum_\ell (2\ell+1) \frac{1-\cos 2\delta_\ell(k)}{2k} P_\ell(\cos\theta)$$

$$\boxed{\text{Im } A(k,\theta) = \frac{1}{k}\sum_{\ell} (2\ell+1)\sin^2\delta_\ell(k)\cdot P_\ell(\cos\theta)}$$

Since $P_\ell(1) = 1$ we get the optical thm.

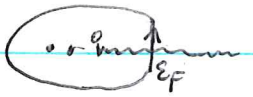
$$\boxed{\sigma = \frac{4\pi}{k}\,\text{Im } A(k,0)}$$

$A(k,0)$ = forward scattering amp.

---

Let's now return to the problem of computing the particle density in the fermi gas given by a Fermi energy $\varepsilon_F = k_F^2$ and the 1-particle Hamiltonian $H = -\Delta + V$. We want the diagonal of the kernel for the projection $P^-$ on states of energy $< \varepsilon_F$.

$$P^- = \frac{1}{2\pi i}\int \frac{ds}{s-H}$$




In the case of three dims with $V = V(r)$, this will be a sum over $\ell$, and the spectrum of angular momentum $\ell$ will be $(2\ell+1)$-fold degenerate. So we should first look at the radial case

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = k^2\psi \qquad \text{on } [0,\infty)$$

with some boundary condition at $x=0$. Then the resolvent is

$$G_k(x,x') = \left\langle x\left|\frac{1}{k^2+\Delta-V}\right|x'\right\rangle = \frac{\phi(x_<,k)\,\psi(x_>,k)}{W(\phi,\psi)}$$

and we have

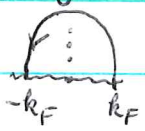
$$\phi(x,k) \sim A(k)e^{-ikx} + A(-k)e^{ikx} \qquad x \gg 0$$

$$\phi(x,k) = A(k)\psi(x,-k) + A(-k)\psi(x,k)$$

so $W(\phi, \psi) = A(k)\, 2ik$

Then

$$\langle x|P^-|x'\rangle = \frac{1}{2\pi i}\int G_k(x,x')\, 2k\,dk$$




$$= \sum_{\substack{\text{bdd}\\ \text{states}}} u_\alpha(x)\overline{u_\alpha(x')} + \frac{i}{2\pi}\int_{-k_F}^{k_F} G_k(x,x')\, 2k\,dk$$

If $k_F = \infty$, we get the completeness relation. The last term can be written

$$\frac{i}{2\pi}\int_0^{k_F} \phi(x_<,k)\left\{\frac{\psi(x_>,k)}{2ikA(k)} + \frac{\psi(x_>,-k)}{2ikA(-k)}\right\} 2k\,dk = \frac{1}{2\pi}\int_0^{k_F} \phi(x_<,k)\phi(x_>,k)\frac{dk}{|A(k)|^2}$$

Thus

$$\langle x|P^-|x'\rangle = \sum_{\substack{\text{bd}\\ \text{states}}} u_\alpha(x)\overline{u_\alpha(x')} + \frac{1}{2\pi}\int_0^{k_F} \phi(x,k)\phi(x',k)\frac{dk}{|A(k)|^2}$$

For $x$ outside the support of $V$ we have

$$\phi(x,k) = A(k)e^{-ikx} + A(-k)e^{ikx} = |A(k)|\left(e^{-ikx+i\delta_0} + e^{ikx-i\delta_0}\right)$$

$$= 2|A(k)|\sin(kx+\delta_0(k))$$

where $\delta_0(k)$ is essentially the argument of $A(k)$. Hence

$$\int_0^{k_F} \phi(x,k)^2\frac{dk}{|A(k)|^2} = 4\int_0^{k_F}\sin^2(kx+\delta_0(k))\,dk$$

Now compare with $V=0$, where $\phi(x,k) = \sin(kx) = \frac{e^{ikx}-e^{-ikx}}{2i}$ so that $A(k) = \frac{i}{2}$ and $|A(k)|^2 = \frac{1}{4}$, and so the density is

$4\int_0^F \sin^2(kx)\,dk$. Difference oscillates.

February 27, 1980

Consider $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^{n-p}$ and let $e_W = e_1 \wedge \cdots \wedge e_p$ denotes the element of $\Lambda^p(\mathbb{C}^n)$ associated to the subspace $W = \mathbb{C}^p$. Given $A \in Gl_n(\mathbb{C})$ we can write it in block form

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{with } \alpha \text{ a } p \times p \text{ matrix}$$

and we have $\langle e_W | \tilde{A} | e_W \rangle = \det(\alpha)$, where $\tilde{A}$ denotes the functorial extension of $A$ to the exterior algebra. Now

$$\| \tilde{A} e_W \|^2 = \langle e_W | \tilde{A}^* \tilde{A} | e_W \rangle$$

If $\tilde{A}^* = \widetilde{A^*}$, then since $A^*A = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^*\alpha + \gamma^*\gamma & \cdot \\ \cdot & \cdot \end{pmatrix}$ we get

$$\| \tilde{A} e_W \|^2 = \det(\alpha^*\alpha + \gamma^*\gamma).$$

To verify $\tilde{A}^* = \widetilde{A^*}$ take decomposable vectors;

$$\langle v_1 \wedge \cdots \wedge v_p | \tilde{A} ( w_1 \wedge \cdots \wedge w_p ) \rangle = \det \langle v_i | A w_j \rangle \langle v_1 \wedge \cdots \wedge v_p | A w_1 \wedge \cdots \wedge A w_p \rangle$$

$$= \det \langle v_i | A w_j \rangle = \det \langle A^* v_i | w_j \rangle = \langle \tilde{A}^* (v_1 \wedge \cdots \wedge v_p) | w_1 \wedge \cdots \wedge w_p \rangle$$

March 2, 1980

Review Fredholm formulas for $(1-\lambda K)^{-1} = \frac{\text{adj}(1-\lambda K)}{\det(1-\lambda K)}$

Put $\det(1-\lambda K) = \sum \lambda^n D_n$, $\text{adj}(1-\lambda K) = \sum \lambda^n A_n$. Then

$$(1-\lambda K)\sum \lambda^n A_n = \sum \lambda^n D_n$$

$$\boxed{A_n - K A_{n-1} = D_n}$$

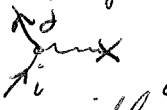
Also $\frac{d}{d\lambda}\det(1-\lambda K) = -\text{tr}(K\,\text{adj}(1-\lambda K))$ or

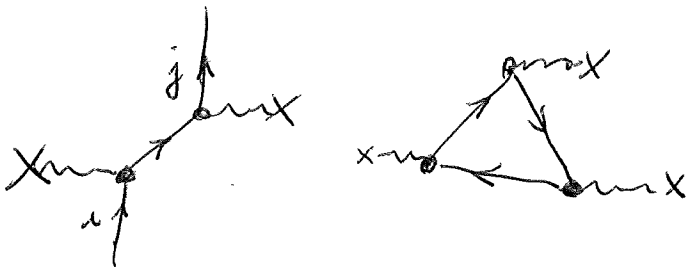
$$\boxed{n\,D_n = -\text{tr}(K A_{n-1})}$$

There's a diagrammatic version of these recursion formulas using

$$\frac{\int e^{-\psi^*(1-\lambda K)\psi}\psi_i\psi_j^*}{\int e^{-\psi^*\psi}} = \frac{\int e^{-\psi^*(1-\lambda K)\psi}\psi_i\psi_j^*}{\int e^{-\psi^*(1-\lambda K)K}} \cdot \frac{\int e^{-\psi^*(1-\lambda K)\psi}}{\int e^{-\psi^*\psi}}$$

$$= (1-\lambda K)^{-1}_{ij}\cdot \det(1-\lambda K) = \left[\text{adj}(1-\lambda K)\right]_{ij}$$

Expanding in powers of the interaction $\lambda K$ gives $\left[\text{adj}(1-\lambda K)\right]_{ij}$ as a sum over diagrams with directed edges labelled by a variable $i$ and vertices of the form  $\lambda K_{ji}$. Thus $(A_n)_{ij}$ is the sum over $n$-vertex diagrams with an entering arrow labelled by $j$ and an exiting arrow labelled by $i$. A typical one is



$D_n$ is similar without external lines. The formula

$$A_n = D_n + K A_{n-1}$$

results from dividing the diagrams according to whether the interaction occurs on the $j-i$ path. The formula

$$n D_n = -\mathrm{tr}(K A_{n-1})$$

results from breaking a contribution to $D_n$ at any of its $n$-vertices; the minus sign is due to the fact we have cut a fermion loop.

Suppose $K = |v\rangle\langle\bar{v}|$ is of rank 1. Then

$$\frac{1}{1-\lambda K} = 1 + \lambda K + \lambda^2 K^2 + \cdots$$

$$= 1 + \lambda |v\rangle\langle\bar{v}| + \lambda^2 |v\rangle\langle\bar{v}|v\rangle\langle\bar{v}| + \cdots$$

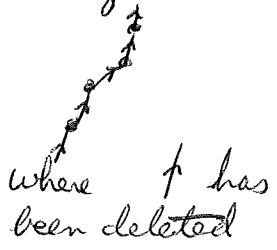
$$= 1 + \frac{\lambda |v\rangle\langle\bar{v}|}{1-\lambda\langle\bar{v}|v\rangle}$$

$$= \frac{1 + \lambda[|v\rangle\langle\bar{v}| - \langle\bar{v}|v\rangle]}{1-\lambda\langle\bar{v}|v\rangle}$$

---

Alternative version which leads to the denominator $\sum_{n\geq 0} \tilde{D}_n \lambda^n = \det(1-\lambda K)\, e^{\lambda \mathrm{tr} K}$, where one has "pulled the poison tooth". The numerator will be

$$\sum_{n\geq 0} \tilde{N}_n \lambda^n = \underbrace{\left[(1-\lambda K)^{-1} - 1\right]}_{\text{diagrams}} \underbrace{\det(1-\lambda K)\, e^{\lambda \mathrm{tr} K}}_{\text{vacuum diagrams where } \circlearrowleft \text{ are deleted.}}$$



Recursion formulas are

$$\begin{cases} n\tilde{D}_n = -\mathrm{tr}(K\tilde{N}_{n-1}) \\ \tilde{N}_n = K(\tilde{N}_{n-1} + \tilde{D}_{n-1}) \end{cases}$$

Diagrams contributing to $\tilde{N}_n$ involve incoming and outgoing external lines, together with $n$ vertices of interaction, at least one of which occurs on the path connecting the external lines.

$$\tilde{N}_0 = 0 \qquad \tilde{N}_1 = K \qquad \tilde{N}_2 = K^2$$

$$\tilde{D}_0 = 1 \qquad \tilde{D}_1 = 0 \qquad \tilde{D}_2 = -\tfrac{1}{2}\mathrm{tr}(K^2)$$

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March 3, 1980

Weinberg's analysis of a N-body system (in Brandeis Summer Institute in Theor. Physics 1964 Vol. II - Lectures on particles and field theory).

We work with N-distinguishable particles, so the Hilbert space is $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and $H = H_0 + V$ where

$$H_0 |p_1, \ldots, p_N\rangle = \sum \frac{p_i^2}{2m_i} |p_1, \ldots, p_N\rangle$$

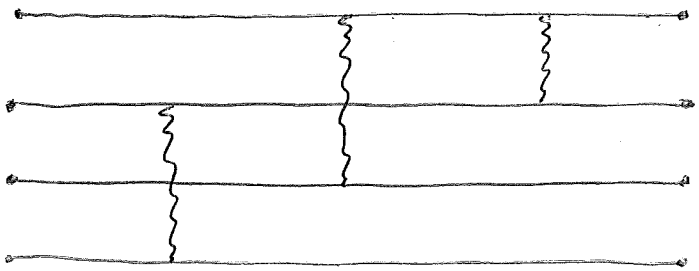
$$V = \sum_{i<j} V_{ij}$$

The problem is to understand matrix elements of

$$G(W) = [W-H]^{-1} = G_0(W) + G_0(W) V G_0(W) + \cdots\cdots$$

$$= G_0(W) + G_0(W) \sum_{i<j} V_{ij} G_0(W) + G_0(W) \sum_{i<j} V_{ij} G_0(W) \sum_{k<\ell} V_{k\ell} + \cdots$$

We draw diagrams as follows:



d c b a

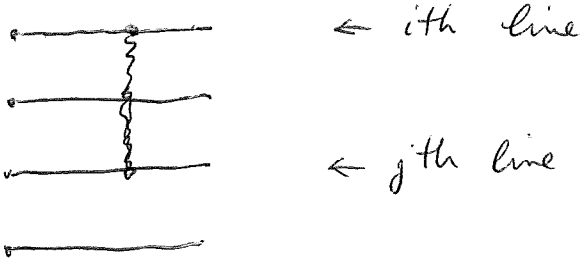
a represents the initial state, d the final state; a vertical wavy line between i th and j th line represents $V_{ij}$. The above diagram represents the contribution to $\langle d|G(W)|a\rangle$ coming from the term $G_0(W) V_{24} G_0(W) V_{13} G_0(W) V_{12} G_0(W)$, that is

$$\langle d|G|a\rangle = \langle d|G_0 V_{24} G_0 V_{13} G_0 V_{12} G_0|a\rangle$$

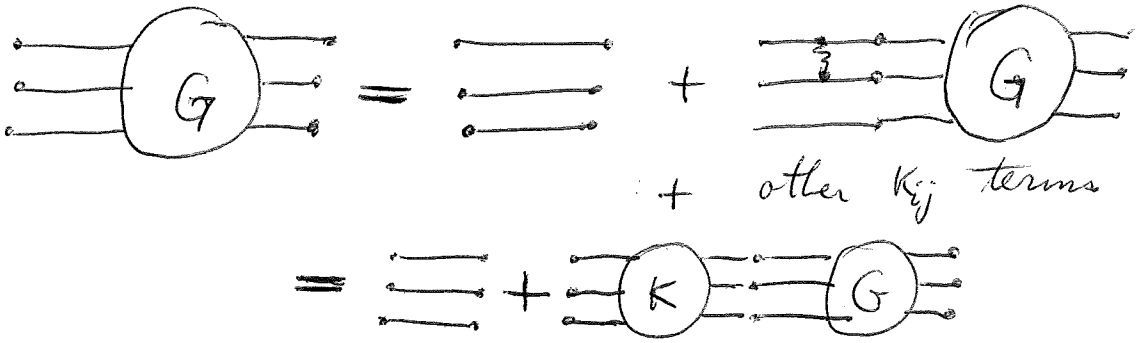
$$= \int dc \int db \, \frac{\langle d|V_{24}|c\rangle \langle c|V_{13}|b\rangle \langle b|V_{12}|a\rangle}{(W-E_d)(W-E_c)(W-E_b)(W-E_a)}$$

Put $K = G_0 V$ so that $G = [1-K]^{-1} G_0$.

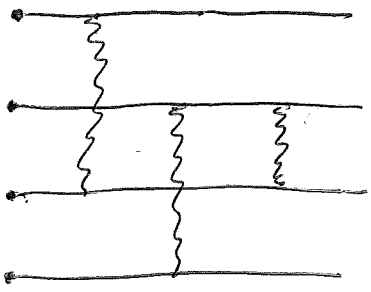
One has $K = \sum_{i<j} G_0 V_{ij}$ where $G_0 V_{ij}$ is represented by the diagram




The dots missing on the right mean that $G_0$ doesn't appear. The equation $G = G^0 + KG$ has the obvious diagram interpretation of looking at the left-most interaction

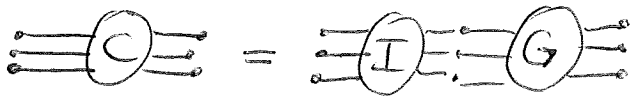



It's clear what is meant by a connected graph. An irreducible graph is one of the form



such that if the ▨ right-most interaction is dropped then it becomes disconnected. Any connected graph has a unique irreducible leading (from the left) part, so we have the equation

$$C = I \cdot G$$



An irreducible graph splits into 2 components when the right-most interaction is dropped:

$$I = \frac{1}{2!} \sum_{\substack{\text{partitions} \\ \{1,\dots,N\} = S_1 \amalg S_2}} C_{S_1} * C_{S_2} \underbrace{V_{S_1 S_2}}_{\substack{\text{Sum of all interactions} \\ \text{between particles of subsystem } S_1 \\ \text{and particles of subsystem } S_2}}$$

can be omitted if ordering of $S_1, S_2$ is not counted

Finally any contribution to $G$ can be built up from its connected components:

$$G = \sum_m \frac{1}{m!} \sum_{S_1 \amalg \cdots \amalg S_m = \{1,\dots,N\}}' C_{S_1} * \cdots * C_{S_m}$$

can be omitted if ordering of $S_1, \dots, S_m$ is not counted.

Convolution is defined by

$$f(W) * g(W) = \frac{1}{2\pi i} \oint f(z)\, g(W-z)\, dz$$

contour enclosed singularities of $f$

and has the effect that

$$\frac{1}{W-a} * \frac{1}{W-b} = \frac{1}{W-a-b}.$$

March 5, 1980

Return to a many electron system and try to understand dielectric response, Kubo formula, polarization propagator, etc. An electron gas with $N$-particles is described classically by the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i<j}' V(r_i - r_j) \qquad V(r) = \frac{e^2}{r}$$

The particle density is

$$\rho(r) = \sum_{i=1}^{N} \delta(r - r_i)$$

Now

$$\iint dx\, dx'\ \rho(x)\, V(x-x')\, \rho(x') = \sum_{i,j}' V(r_i - r_j)$$

except that this has problems when $r_i = r_j$.

Consider next the quantum situation. If we regard the particles as distinguish, the Hilbert space is $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, whose elements are wave functions $\psi(r_1, \cdots, r_N)$. Electrons are fermions, so wave functions are skew-symmetric. The particle density $\rho(r)$ becomes an operator; more generally any symmetric function of $r_1, \cdots, r_n$ gives a multiplication operator on skew-symmetric wave functions. Since $\rho(r)$ is a sum of the one-particle functions $r_i \longmapsto \delta(r - r_i)$, its effect on $\Lambda^N \mathcal{H}_1$ is that of a derivation. Let $u_\alpha$ be an orthonormal basis for $\mathcal{H}_1$. Then on $\mathcal{H}_1$ one has

$$\langle u_\beta | \rho(r) | u_\alpha \rangle = \overline{u_\beta(r)}\, u_\alpha(r)$$

and so on the Fock space

$$\rho(r) = \sum_{\beta,\alpha} a_\beta^*\, \overline{u_\beta(r)}\, u_\alpha(r)\, a_\alpha = \psi(r)^*\, \psi(r)$$

where $\psi(r) = \sum u_\alpha(r)\, a_\alpha$ is the interior multiplication by $f \longmapsto f(r)$.

Assuming $V(0) = 0$ the potential energy can be written

$$\frac{1}{2}\int\int dr\,dr'\ \rho(r)\,V(r-r')\,\rho(r')$$

$$= \frac{1}{2}\int\int dr\,dr'\ \psi(r)^*\psi(r)\,V(r-r')\,\psi(r')^*\psi(r')$$

$$= \frac{1}{2}\int\int dr\,dr'\ \psi(r)^*\psi(r')^*\,V(r-r')\,\psi(r')\psi(r) \quad \Big]\ \text{correct potential energy in general}$$

$$+ \frac{1}{2}\int\int dr\,dr'\ \psi(r)^*V(r-r')\,\underbrace{\{\psi(r),\psi(r')^*\}}_{\delta(r-r')}\,\psi(r')$$

The 2nd term is $V(0)\frac{1}{2}\int dr\,\psi(r)^*\psi(r) = \frac{V(0)}{2}N$. Thus one sees that the potential energy is essentially

$$\frac{1}{2}\int\int dr\,dr'\ \rho(r)\,V(r-r')\,\rho(r') \ +\ \text{constant}$$

---

Thomas-Fermi model for an atom. One has a nucleus with charge $Ze$. The first model (due to Pauli) is treat all the electrons as independent particles subject only to the exclusion principle. Then one puts in $Z$ electrons filling one shell at a time. The Thomas-Fermi and Hartree-Fock models attempt to understand the effects of the Coulomb repulsion between electrons by introducing an average potential due to the nucleus and the other electrons. Let $\varphi(r)$ denote the average or effective potential; by spherical symmetry it depends only on $r = |\underline{r}|$.

With Thomas-Fermi one treats the electrons as a fermi gas with density $\rho(r)$ (here $r = |\underline{r}|$). Once $\varphi(r)$ is given one can solve Schroedinger's equation to get an orthonormal

basis of eigenfunctions $u_\alpha(x_m)$ and then one gets the density

$$\rho(r) = \sum_{\varepsilon_\alpha < \varepsilon_F} |u_\alpha(x_m)|^2$$

by giving a Fermi energy. But there is a way to get at this density semi-classically: Recall:

$$\text{(*)} \qquad \begin{matrix}\text{number of eigenvalues} \\ \text{of } \frac{p^2}{2m} + \varphi(r) < \varepsilon_F\end{matrix} \sim 2 \int_{\frac{\hbar^2 k^2}{2m} + \varphi(r) < \varepsilon_F} \frac{d^3k\, d^3r}{(2\pi)^3} \overset{(\text{spins})}{=} 2 \int_{\frac{p^2}{2m} + \varphi(r) < \varepsilon_F} \frac{d^3p\, d^3r}{(2\pi\hbar)^3}$$

Thus the number of (semi-classical) states belonging to a volume $d^3p\, d^3r$ in phase space is $2\frac{d^3p\, d^3r}{(2\pi\hbar)^3}$. So in a small volume $d^3r$ the number of electrons, namely $\rho(r)\, d^3r$, should have their momenta distributed in such a way as to fill up all the possible states with position $r$ and energy $< \varepsilon_F$. Thus

$$\rho(r) = 2 \frac{\text{volume of } \frac{p^2}{2m} + \varphi(r) < \varepsilon_F}{(2\pi\hbar)^3} = 2 \frac{1}{(2\pi\hbar)^3} \frac{4}{3}\pi \left(2m(\varepsilon_F - \varphi(r))\right)^{3/2}$$

Finally $\varphi(r)$ is determined from $\rho(r)$ by Poisson's equation

$$\Delta\varphi = 4\pi e \rho \qquad -e = \text{charge of electron}$$

together with the boundary condition that

$$\varphi(r) \sim -\frac{Ze^2}{r} \qquad \text{as } r \to 0$$

---

In connection with the "Weyl" formula (*), recall Victor checked it for the harmonic oscillator. One needs:

$$\text{Vol unit ball in } \mathbb{R}^{2n} = \frac{\pi^n}{n!}$$

Is this related to the $\pi^{-3/2}\Gamma(3/2)$ in the fnl. eqn. for $\rho$?